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Superconductivity and Low Temperature Physics I

Lecture Notes
Winter Semester 2022/2023

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Chapter 3

Phenomenological Models of Superconductivity

3. Phenomenological Models of Superconductivity

3.1 London Theory

Fritz Wolfgang London (1900 – 1954)

* 7 March 1900 in Breslau
† 30 March 1954 in Durham, North Carolina, USA

study:	Bonn, Frankfurt, Göttingen, Munich and Paris.
Ph.D.:	1921 in Munich
1922-25:	Göttingen and Munich
1926/27:	Assistent of Paul Peter Ewald at Stuttgart, studies at Zurich and Berlin with Erwin Schrödinger.
1928:	Habilitation at Berlin
1933-36:	London
1936-39:	Paris
1939:	Emigration to USA, Duke Universität at Durham

3.1 London Theory

Heinz and Fritz London

3.1 London Theory

1935 **Fritz and Heinz London describe the Meißner-Ochsenfeld effect and perfect conductivity within phenomenological model**

➔ ***they assume a homogeneous pair condensate***

3.1.1 London Equations

- starting point is equation of motion of charged particles with mass m_s and charge q_s

$$m_s \frac{\mathrm{d}\mathbf{v_s}}{\mathrm{d}t} + \frac{m_s}{\tau}\mathbf{v_s} = q_s\mathbf{E}$$

(τ = momentum relaxation time)

- ***two-fluid model***:
  - normal conducting electrons with charge q_n and density n_n
  - superconducting electrons with charge q_s density n_s

- *normal state:* $n_n = n, \;\; n_s = 0$

- *superconducting state* $n_n \rightarrow 0, n_s \rightarrow max$ for $T \rightarrow 0, \tau \rightarrow \infty$, $\mathbf{J}_s = n_s q_s \mathbf{v}_s$

$$\frac{\partial(\Lambda \mathbf{J}_s)}{\partial t} = \mathbf{E}$$

1st London equation

$$\Lambda = \frac{m_s}{n_s q_s^2}$$

London coefficient

BCS theory:

$m_s = 2m_e, q_s = -2e$

$n_s = n/2$

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3.1.1 London Equations

- take the curl of 1st London equation $\frac{\partial(\Lambda \mathbf{J}_s)}{\partial t} = \mathbf{E}$ and use $\boldsymbol{\nabla} \times \mathbf{E} = -\partial \mathbf{B}/\partial t$

$$\Rightarrow \quad \frac{\partial}{\partial t}\underbrace{[\boldsymbol{\nabla} \times (\Lambda \mathbf{J}_s) + \mathbf{B}]} = 0$$

flux $\boldsymbol{\Phi}$ through an arbitrary area inside a sample with infinite conductivity stays constant

e.g. flux trapping when switching off the external magnetic field

- Meißner-Ochsenfeld effect tells us: *not only $\dot{\boldsymbol{\Phi}}$ but $\boldsymbol{\Phi}$ itself must be zero*

  → expression in brackets must be zero

$$\boxed{\boldsymbol{\nabla} \times (\Lambda \mathbf{J}_s) + \mathbf{B} = \mathbf{0}}$$

2nd London equation

- use Maxwell's equation $\boldsymbol{\nabla} \times \mathbf{B} = -\mu_0 \mathbf{J}_s$ $\quad \rightarrow \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{B} = -\mu_0 \boldsymbol{\nabla} \times \mathbf{J}_s \;\Rightarrow\; \mathbf{B} = -\left(\frac{\Lambda}{\mu_0}\right) \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{B}$

  with $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{a} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{a}) - \boldsymbol{\nabla}^2 \mathbf{a}$, we obtain with $\boldsymbol{\nabla} \cdot \mathbf{B} = \mathbf{0}$

$$\boxed{\boldsymbol{\nabla}^2 \mathbf{B} - \frac{\mu_0}{\Lambda} \mathbf{B} = \boldsymbol{\nabla}^2 \mathbf{B} - \frac{1}{\lambda_{\mathrm{L}}^2} \mathbf{B} = \mathbf{0}}$$

$$\lambda_{\mathrm{L}} = \sqrt{\frac{\Lambda}{\mu_0}} = \sqrt{\frac{m_s}{\mu_0 n_s q_s^2}}$$

London penetration depth

3.1.1 London Equations

- example: $B_{\text{ext}} = B_z$

$$\frac{\mathrm{d}^2 B_z}{\mathrm{d}x^2} = \frac{B_z}{\lambda_\mathrm{L}^2}$$

- solution:

$$B_z(x) = B_z(0)\ \exp\left(-\frac{x}{\lambda_\mathrm{L}}\right)$$

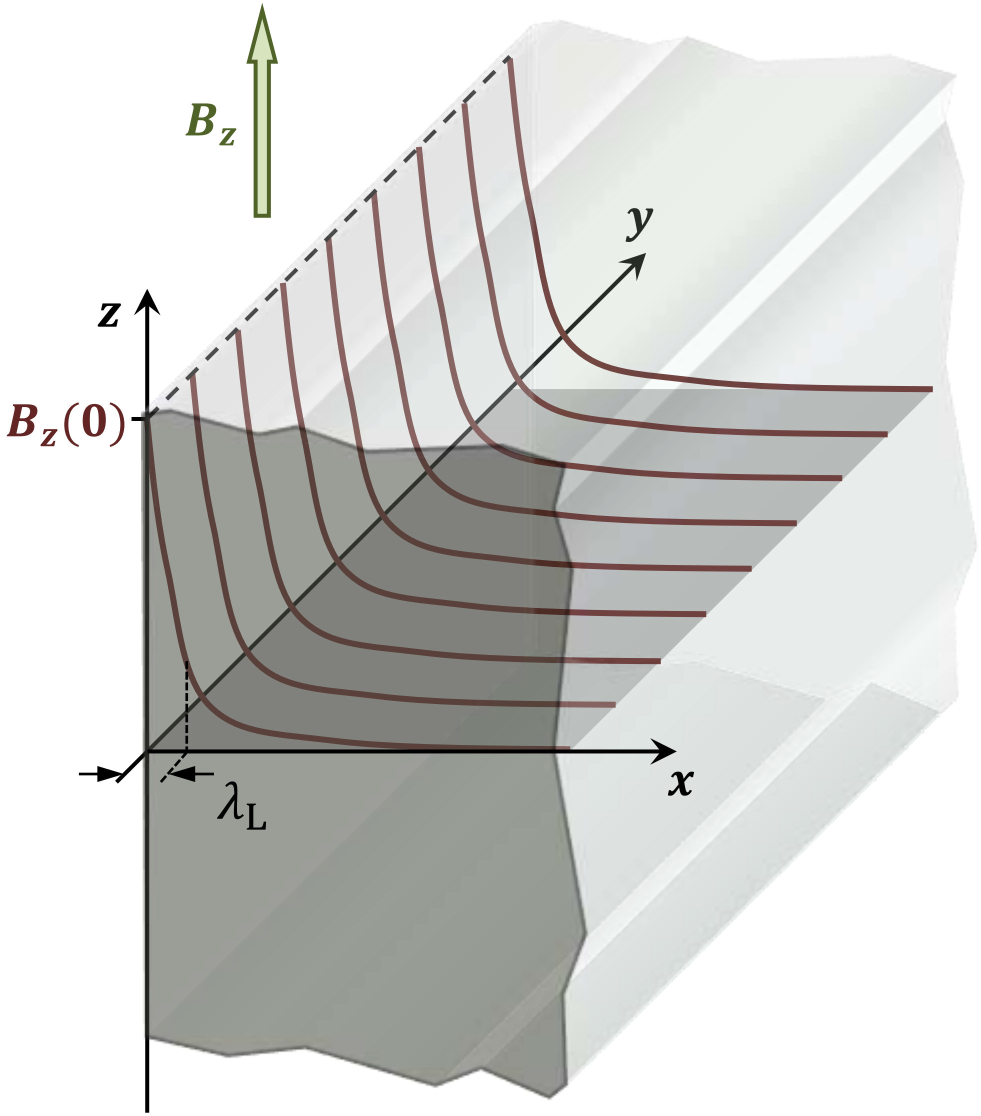
- B_z decays exponentially with characteristic decay length λ_L

$$\lambda_\mathrm{L} = \sqrt{\frac{m_s}{\mu_0 n_s q_s^2}} \sim 10\ - 100\ \mathrm{nm}$$

- T dependence of λ_L

  empirical relation:

$$\lambda_\mathrm{L}(T) = \frac{\lambda_\mathrm{L}(0)}{\sqrt{1-(T/T_C)^4}}$$



3.1.1 London Equations

- with 2nd London equation

$$\boxed{\nabla \times (\Lambda \mathbf{J}_s) + \mathbf{B} = \mathbf{0}}$$

we obtain for J_s:

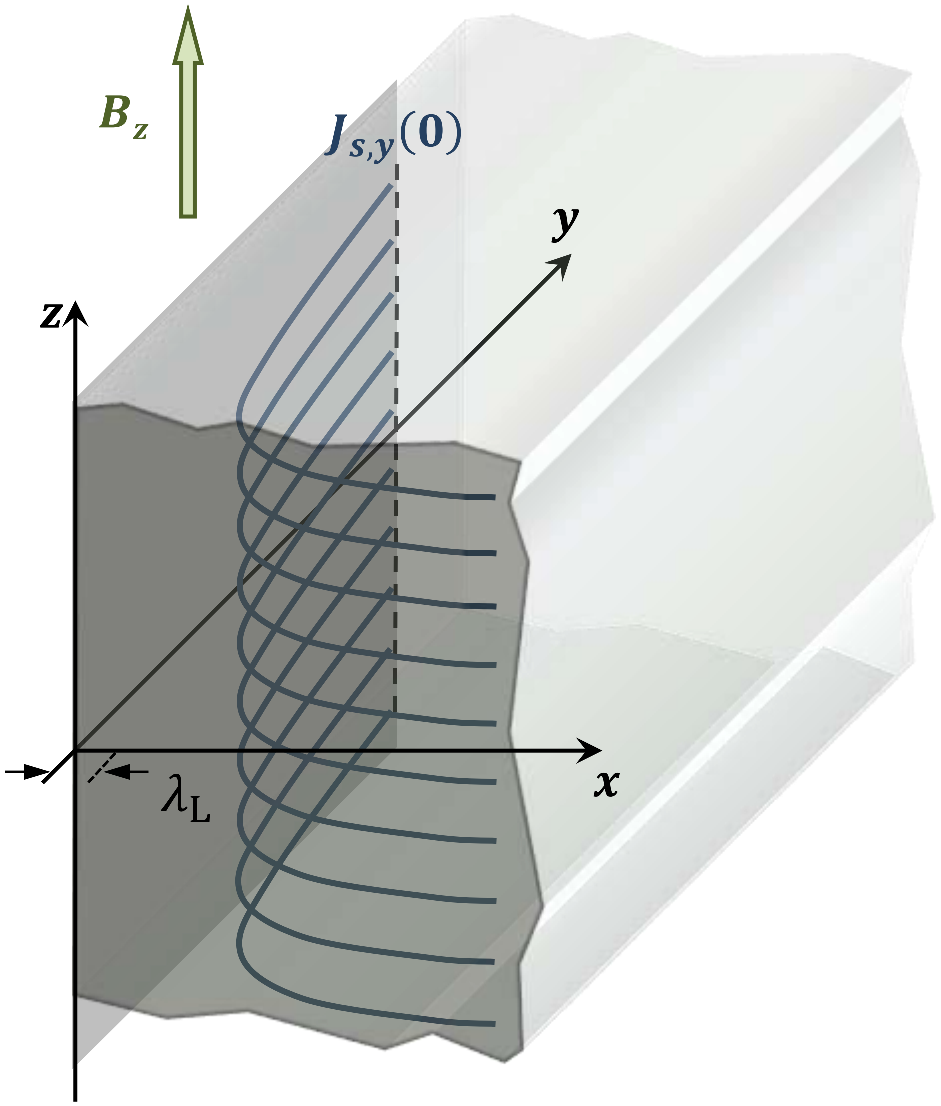
$$\frac{\partial J_{s,y}(x)}{\partial x} - \frac{\partial J_{s,x}(x)}{\partial y} = -\frac{1}{\Lambda}\, B_z(0)\ \exp\left(-\frac{x}{\lambda_\mathrm{L}}\right)$$

integration yields

$$J_{s,y}(x) = \frac{\lambda_\mathrm{L}}{\Lambda}\, B_z(0)\ \exp\left(-\frac{x}{\lambda_\mathrm{L}}\right) \qquad \Lambda = \mu_0 \lambda_\mathrm{L}^2$$

$$J_{s,y}(x) = \frac{H_z(0)}{\lambda_\mathrm{L}}\ \exp\left(-\frac{x}{\lambda_\mathrm{L}}\right)$$

$$\boxed{J_{s,y}(x) = J_{s,y}(0) \exp\left(-\frac{x}{\lambda_\mathrm{L}}\right)}$$



3.1.1 London Equations

- **example**: thin superconducting sheet of thickness d with $B \parallel$ sheet

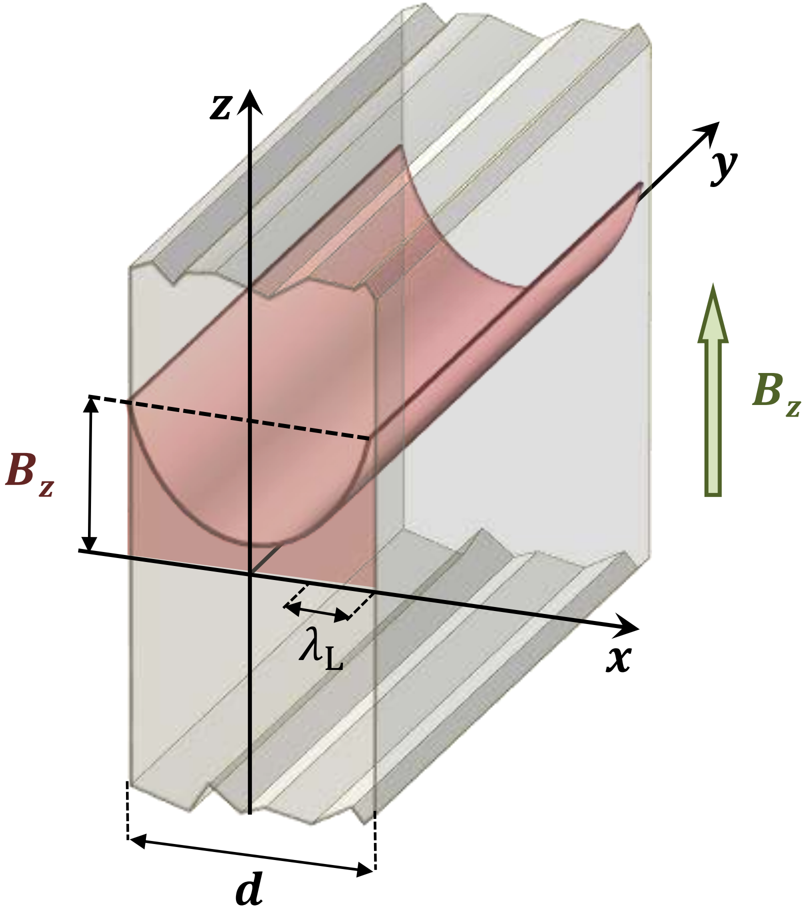
- Ansatz: $$B_z(x) = B_z\ \exp\left(-\frac{x}{\lambda_\mathrm{L}}\right) + B_z\ \exp\left(+\frac{x}{\lambda_\mathrm{L}}\right)$$

- boundary conditions:

$$B_z(-d/2) = B_z(+d/2) = B_z$$

- solution:

$$B_z(x) = B_z \frac{\cosh\left(\frac{x}{\lambda_\mathrm{L}}\right)}{\cosh\left(\frac{d}{2\lambda_\mathrm{L}}\right)}$$



3.1.1 London Equations

- Summary:

$$\frac{\partial(\Lambda \mathbf{J}_s)}{\partial t} = \mathbf{E}$$ 1st London equation

$$\Lambda = \frac{m_s}{n_s q_s^2}$$ London coefficient

$$\nabla \times (\Lambda \mathbf{J}_s) + \mathbf{B} = \mathbf{0}$$ 2nd London equation

$$\lambda_\mathrm{L} = \sqrt{\frac{\Lambda}{\mu_0}} = \sqrt{\frac{m_s}{\mu_0 n_s q_s^2}}$$ London penetration depth

- remarks to the London model:

1. normal component is completely neglected

   → not allowed at finite frequencies !

2. we have assumed a local relation between $\mathbf{J}_s, \mathbf{E}$ and $\mathbf{B}$

   ➢ $\mathbf{J}_s$ is determined by the local fields for every position $\mathbf{r}$

   ➢ this is problematic since mean free path $\ell \rightarrow \infty$ for $\tau \rightarrow \infty$

   → nonlocal extension of London theory by ***A.B. Pippard*** (1953)

3.2 SC as Macroscopic Quantum Phenomenon

- more solid derivation of London equations by assuming that superconductor can be desrcribed by a ***macroscopic wave function***

  - ***Fritz London (> 1948)***
    derived London equations from basic quantum mechanical concepts

- basic assumption of macroscopic quantum model of superconductivity:

  complete entity of all superconducting electrons can be described by macroscopic wave function

$$\boxed{\psi(\mathbf{r},t) = \psi_0(\mathbf{r},t)\ \mathrm{e}^{\imath\theta(\mathbf{r},t)}}$$

amplitude ↗ ↖ phase

- hypothesis can be proven by BCS theory (discussed later)

- **normalization condition**:
  volume integral over $|\psi|^2$ is equal to the number N_s of superconducting electrons

$$\int \psi^\star(\mathbf{r},t)\psi(\mathbf{r},t)\ \mathrm{d}V = N_s \qquad |\psi(\mathbf{r},t)|^2 = \psi^\star(\mathbf{r},t)\psi(\mathbf{r},t) = n_s(\mathbf{r},t)$$

3.2 SC as Macroscopic Quantum Phenomenon

- **revision**: general relations in electrodynamics

electric field: $$\mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} - \boldsymbol{\nabla}\phi_{\mathrm{el}}(\mathbf{r},t)$$

flux density: $$\mathbf{B}(\mathbf{r},t) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r},t)$$

$\mathbf{A}(\mathbf{r},t)$ = vector potential

$\phi_{\mathrm{el}}(\mathbf{r},t)$ = scalar potential

- electrical current is driven by gradient of ***electrochemical potential*** $\phi(\mathbf{r},t) = \phi_{\mathrm{el}}(\mathbf{r},t) + \mu(\mathbf{r},t)/q$:

$$-\boldsymbol{\nabla}\phi(\mathbf{r},t) = -\boldsymbol{\nabla}\phi_{\mathrm{el}}(\mathbf{r},t) - \frac{\boldsymbol{\nabla}\mu(\mathbf{r},t)}{q}$$

- canonical momentum: $$\mathbf{p}(\mathbf{r},t) = m\mathbf{v}(\mathbf{r},t) + q\mathbf{A}(\mathbf{r},t)$$

$p_x = \partial\mathcal{L}/\partial\dot{x}$

$\mathcal{L}$ = Lagrange function

- kinematic momentum: $$m\mathbf{v}(\mathbf{r},t) = \frac{\hbar}{\imath}\boldsymbol{\nabla} - q\mathbf{A}(\mathbf{r},t)$$

3.2 SC as Macroscopic Quantum Phenomenon

- Schrödinger equation for charged particle:

$$\frac{1}{2m}\left(\frac{\hbar}{\imath}\nabla - -q\mathbf{A}(\mathbf{r},t)\right)^2 \Psi(\mathbf{r},t) + [q\phi_{\mathrm{el}}(\mathbf{r},t) + \mu(\mathbf{r},t)]\,\Psi(\mathbf{r},t) = \imath\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}$$

$|\Psi(\mathbf{r},t)|^2$ = probability to find particle at postion r at time t

- ***Madelung transformation***
  insert macroscopic wave function $\psi(\mathbf{r},t) = \psi_0(\mathbf{r},t)\,\mathrm{e}^{\imath\theta(\mathbf{r},t)}$ into Schrödinger equation

$|\psi(\mathbf{r},t)|^2$ = probability to find superfluid density at postion r at time t

replacements:

$\Psi \rightarrow \psi = \psi_0(\mathbf{r},t)\,\mathrm{e}^{\imath\theta(\mathbf{r},t)}$

$q \rightarrow q_s$

$m \rightarrow m_s$

- calculation yields – after splitting up into real and imaginary part and assuming a spatially homogeneous amplitude $\psi_0(r,t) = \psi_0(t)$ of the macroscopic wave function (London approximation) – two fundamental equations

  - ***current-phase relation***: connects supercurrent density with gauge invariant phase gradient
  - ***energy-phase relation***: connects energy with time derivative of the phase

2.2 Special Topic: Madelung Transformation

- we start from Schrödinger equation:

$$\underbrace{\frac{1}{2m_s}\left(\frac{\hbar}{\imath}\boldsymbol{\nabla} - q_s\mathbf{A}(\mathbf{r},t)\right)^2}_{\text{II}} \psi(\mathbf{r},t) + [\underbrace{q_s\phi_{\mathrm{el}}(\mathbf{r},t) + \mu(\mathbf{r},t)}_{\textit{electro-chemical potential}}]\,\psi(\mathbf{r},t) = \underbrace{\imath\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t}}_{\text{I}}$$

- we use the definition $S = \hbar\theta$ and obtain with $\psi(\mathbf{r},t) = \psi_0(\mathbf{r},t)\,\mathrm{e}^{\imath\theta(\mathbf{r},t)}$

$$\text{I} \quad = \imath\hbar\frac{\partial\psi}{\partial t} = \left[\imath\hbar\frac{\partial\psi_0}{\partial t} - \psi_0\frac{\partial S}{\partial t}\right]\mathrm{e}^{\imath S/\hbar}$$

$$\text{II} \quad = \frac{1}{2m_s}\left(\frac{\hbar}{\imath}\boldsymbol{\nabla} - q_s\mathbf{A}\right)^2\psi = \frac{1}{2m_s}\,[\underbrace{-\hbar^2\boldsymbol{\nabla}^2}_{1} + \underbrace{\imath\hbar q_s\,\boldsymbol{\nabla}\cdot\mathbf{A}}_{2} + \underbrace{\imath\hbar q_s\,\mathbf{A}\cdot\boldsymbol{\nabla}}_{3} + \underbrace{q_s^2\mathbf{A}^2}_{4}]\;\psi_0\mathrm{e}^{\imath S/\hbar}$$

2.2 Special Topic: Madelung Transformation

II $$= \frac{1}{2m_s}\left(\frac{\hbar}{\imath}\boldsymbol{\nabla} - q_s\mathbf{A}\right)^2 \psi = \frac{1}{2m_s}\,[\underbrace{-\hbar^2\boldsymbol{\nabla}^2}_{①} + \underbrace{\imath\hbar q_s\,\boldsymbol{\nabla}\cdot\mathbf{A}}_{②} + \underbrace{\imath\hbar q_s\,\mathbf{A}\cdot\boldsymbol{\nabla}}_{③} + \underbrace{q_s^2\mathbf{A}^2}_{④}]\;\psi_0 e^{\imath S/\hbar}$$

① $$= -\frac{\hbar^2\boldsymbol{\nabla}^2}{2m_s}\Psi_0 e^{\imath S/\hbar} = \frac{1}{2m_s}\left[-\hbar^2\boldsymbol{\nabla}^2\Psi_0 + \Psi_0(\boldsymbol{\nabla}S)^2 - 2\imath\hbar\boldsymbol{\nabla}\Psi_0(\boldsymbol{\nabla}S) - \imath\hbar\Psi_0\boldsymbol{\nabla}^2 S\right] e^{\imath S/\hbar}$$

② $$= \frac{1}{2m_s}\imath\hbar q_s\Psi_0(\boldsymbol{\nabla}\cdot\mathbf{A})e^{\imath S/\hbar} + \text{term } 3$$

③ $$= \frac{1}{2m_s}\left[\imath\hbar q_s\mathbf{A}\cdot(\boldsymbol{\nabla}\Psi_0) - q_s\Psi_0\mathbf{A}(\boldsymbol{\nabla}S)\right] e^{\imath S/\hbar}$$

②+③ $$= \frac{1}{2m_s}\left[\imath\hbar q_s\Psi_0(\boldsymbol{\nabla}\cdot\mathbf{A}) + 2\imath\hbar q_s\mathbf{A}\cdot(\boldsymbol{\nabla}\Psi_0) - 2q_s\Psi_0\mathbf{A}(\boldsymbol{\nabla}S)\right] e^{\imath S/\hbar}$$

④ $$= \frac{1}{2m_s}q_s\Psi_0\mathbf{A}^2 e^{\imath S/\hbar}$$

II $$= \left[\Psi_0\frac{(\boldsymbol{\nabla}S - q_s\mathbf{A})^2}{2m_s} - \frac{\hbar^2\boldsymbol{\nabla}^2}{2m_s}\Psi_0 - \frac{\imath}{2m_s}\underbrace{(2\hbar\boldsymbol{\nabla}\Psi_0 + \hbar\Psi_0\boldsymbol{\nabla})(\boldsymbol{\nabla}S - q_s\mathbf{A})}_{=\frac{\hbar}{\Psi_0}\boldsymbol{\nabla}\cdot\left[\Psi_0^2(\boldsymbol{\nabla}S - q_s\mathrm{A})\right]}\right] e^{\imath S/\hbar}$$

$$= \left[\Psi_0\frac{(\boldsymbol{\nabla}S - q_s\mathbf{A})^2}{2m_s} - \frac{\hbar^2\boldsymbol{\nabla}^2}{2m_s}\Psi_0 - \imath\frac{\hbar}{2\Psi_0}\boldsymbol{\nabla}\cdot\left(\frac{\Psi_0^2}{m_s}(\boldsymbol{\nabla}S - q_s\mathbf{A})\right)\right] e^{\imath S/\hbar}$$

2.2 Special Topic: Madelung Transformation

$$\text{(I)} = \imath\hbar\frac{\partial\Psi}{\partial t} = \left[\imath\hbar\frac{\partial\Psi_0}{\partial t} - \Psi_0\frac{\partial S}{\partial t}\right] e^{\imath S/\hbar}$$

$$\text{(II)} = \left[\Psi_0\frac{(\nabla S - q_s\mathbf{A})^2}{2m_s} - \frac{\hbar^2\nabla^2}{2m_s}\Psi_0 - \imath\frac{\hbar}{2\Psi_0}\nabla\left(\frac{\Psi_0^2}{m_s}(\nabla S - q_s\mathbf{A})\right)\right] e^{\imath S/\hbar}$$

- equation for real part:

$$\left[\Psi_0\left(\frac{(\nabla S - q_s\mathbf{A})^2}{2m_s} + q_s\phi\right) - \frac{\hbar^2\nabla^2}{2m_s}\Psi_0\right] e^{\imath S/\hbar} = -\Psi_0\frac{\partial S}{\partial t} e^{\imath S/\hbar}$$

⇨ $$\frac{\partial S}{\partial t} + \underbrace{\frac{(\nabla S - q_s\mathbf{A})^2}{2m_s}}_{=\frac{1}{2}m_s v_s^2 = \frac{1}{2n_s}\Lambda J_s^2} + q_s\phi = \frac{\hbar^2\nabla^2\Psi_0}{2m_s\Psi_0}$$

$\Lambda = \frac{m_s}{q_s^2 n_s} =$ London-Koeffizient

$S \equiv \hbar\theta =$ action

⇨ $$\hbar\frac{\partial\theta(\mathbf{r},t)}{\partial t} + \frac{1}{2n_s}\Lambda\mathbf{J}_s^2(\mathbf{r},t) + q_s\phi_{\mathrm{el}}(\mathbf{r},t) + \mu(\mathbf{r},t) = \underbrace{\frac{\hbar^2\nabla^2\psi_0(\mathbf{r},t)}{2m_s\psi_0(\mathbf{r},t)}}$$

the term on the rhs is called the quantum or Bloch potential, dissappears for spatially homogeneous systems

2.2 Special Topic: Madelung Transformation

$$\hbar\frac{\partial\theta}{\partial t}+\frac{1}{2n_s}\Lambda\mathbf{J}_s^2+q_s\phi_{\mathrm{el}}+\mu=\cancel{\frac{\hbar^2\nabla^2\psi_0}{2m_s\psi_0}}$$

the London theory takes the quasi-classical limit ($\hbar\to 0$) by neglecting the Bohm potential

- ➢ this is in the spirit of the WKB approximation to quantum mechanics, in which terms $\propto\hbar$ are kept and those $\propto\hbar^2$ are omitted

- consequence of the ***London approximation*** is a spatially homogeneous density of the superconducting electrons:

$$\psi_0(\mathbf{r},t)=\psi_0(t) \qquad\qquad n_s(\mathbf{r},t)=|\psi_0(\mathbf{r},t)|^2=|\psi_0(t)|^2=n_s(t)$$

- London approximation results in ***energy-phase relation***

$$\boxed{\hbar\frac{\partial\theta(\mathbf{r},t)}{\partial t}=-\underbrace{\left\{\frac{1}{2n_s}\Lambda\mathbf{J}_s^2(\mathbf{r},t)+q_s\phi_{\mathrm{el}}(\mathbf{r},t)+\mu(\mathbf{r},t)\right\}}_{\text{total energy}}}$$

energy-phase relation since $\partial\theta/\partial t\propto$ total energy

- interpretation of energy-phase relation:

with action $S(\mathbf{r},t)\equiv\hbar\theta(\mathbf{r},t)$ we obtain $\partial S(\mathbf{r},t)/\partial t=-\mathcal{H}(\mathbf{r},t)$

➔ ***energy-phase relation is equivalent to the Hamilton-Jacobi equation in classical physics***

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2.2 Special Topic: Madelung Transformation

I $$= i\hbar\frac{\partial\Psi}{\partial t} = \left[i\hbar\frac{\partial\Psi_0}{\partial t} - \Psi_0\frac{\partial S}{\partial t}\right] \mathrm{e}^{iS/\hbar}$$

II $$= \left[\Psi_0\frac{(\boldsymbol{\nabla}S - q_s\mathbf{A})^2}{2m_s} - \frac{\hbar^2\boldsymbol{\nabla}^2}{2m_s}\Psi_0 - i\frac{\hbar}{2\Psi_0}\boldsymbol{\nabla}\left(\frac{\Psi_0^2}{m_s}(\boldsymbol{\nabla}S - q_s\mathbf{A})\right)\right] \mathrm{e}^{iS/\hbar}$$

- equation for imaginary part:

$$i\hbar\frac{\partial\Psi_0}{\partial t}\mathrm{e}^{iS/\hbar} = -i\frac{\hbar}{2\Psi_0}\boldsymbol{\nabla}\cdot\left(\frac{\Psi_0^2}{m_s}(\boldsymbol{\nabla}S - q_s\mathbf{A})\right)\mathrm{e}^{iS/\hbar}$$

⇒ $$2\Psi_0\frac{\partial\Psi_0}{\partial t} = -\boldsymbol{\nabla}\cdot\left(\frac{\Psi_0^2}{m_s}(\boldsymbol{\nabla}S - q_s\mathbf{A})\right)$$

⇒ $$\underbrace{\frac{\partial\psi_0^2(\mathbf{r},t)}{\partial t}}_{=\,\partial n_s/\partial t} = -\boldsymbol{\nabla}\cdot\underbrace{\left(\psi_0^2\left[\frac{\hbar}{m_s}\boldsymbol{\nabla}\theta(\mathbf{r},t) - \frac{q_s}{m_s}\mathbf{A}(\mathbf{r},t)\right]\right)}_{=\,n_s\mathbf{v}_s\,=\,\mathbf{J}_\rho}$$

continuity equation for probability density $\rho = |\psi_0|^2 = n_s$ and probability current density $\mathbf{J}_\rho$

$\frac{\partial n_s}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J}_\rho = 0$: *conservation law for probability density*

2.2 Special Topic: Madelung Transformation

- we define ***supercurrent density*** $\mathbf{J}_s = q_s \mathbf{J}_\rho$ by multiplying $\mathbf{J}_\rho$ with charge q_s of superconducting electrons :

$$\mathbf{J}_s(\mathbf{r},t) = q_s n_s(\mathbf{r},t) \underbrace{\left\{\frac{\hbar}{m_s}\nabla\theta(\mathbf{r},t) - \frac{q_s}{m_s}\mathbf{A}(\mathbf{r},t)\right\}}_{\mathbf{v}_s \,\rightarrow\, \mathbf{J}_s = n_s q_s \mathbf{v}_s}$$

current-phase relation

- expression for ***supercurrent density*** $\mathbf{J}_s$ is gauge invariant (see below):

$$\mathbf{J}_s(\mathbf{r},t) = \frac{q_s n_s(\mathbf{r},t)\hbar}{m_s} \underbrace{\left\{\nabla\theta(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t)\right\}}$$

gauge invariant phase gradient $\gamma = \nabla\theta' - \frac{q_s}{\hbar}\mathbf{A}' = \nabla\theta - \frac{q_s}{\hbar}\mathbf{A}$

$\mathbf{A}' = \mathbf{A} + \nabla\chi$
$\theta' = \theta + \frac{q_s}{\hbar}\chi$
$\chi =$ scalar function

➢ ***supercurrent density is proportional to gauge invariant phase gradient*** $\mathbf{J}_s \propto \boldsymbol{\gamma}$

for normal conductor $\mathbf{J}_n \propto -\nabla\phi_{\mathrm{el}} = \mathbf{E}$

2.2 Special Topic: Madelung Transformation

- canonical momentum: $\mathbf{p} = m_s \mathbf{v}_s + q_s \mathbf{A}$

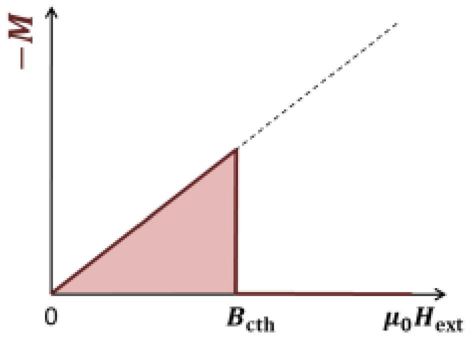
$$\mathbf{p} = m_s \underbrace{\left(\frac{\hbar}{m_s} \nabla\theta(\mathbf{r},t) - \frac{q_s}{m_s} \mathbf{A}(\mathbf{r},t) \right)}_{\mathbf{v}_s} + q_s \mathbf{A}$$

$$\mathbf{p} = \hbar\, \nabla\theta(\mathbf{r},t)$$

➔ **zero total momentum state for vanishing phase gradient: Cooper pairs $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$**

Summary of Lecture No. 3 (1)

- **type-I superconductor in an external magnetic field: free enthalpy density**



  - ➢ for $p, T = const.$: $\mathrm{d}G_s = \frac{V}{\mu_0} B_{\rm ext} \mathrm{d}B_{\rm ext}$ $\quad \mathrm{d}g_s = \mathrm{d}G_s/V$
  - ➢ integration yields $\quad g_s(B_{\rm ext}, T) - g_s(0,T) = \frac{1}{\mu_0}\int_0^{B_{\rm ext}} B'\, dB' = \frac{B_{\rm ext}^2}{2\mu_0}$

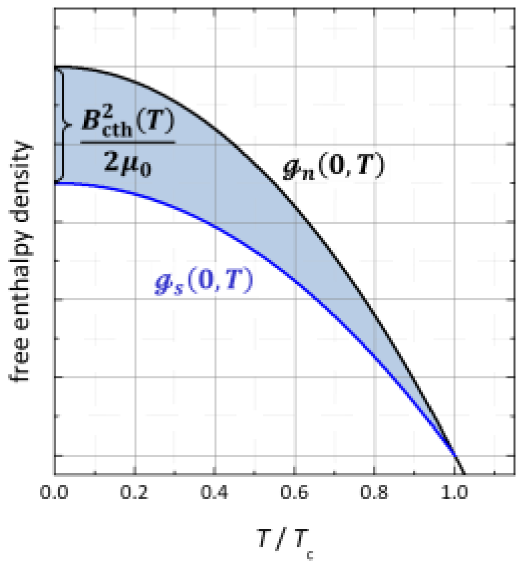
$$@\ B_{\rm ext} = B_{\rm cth}:\ \ g_s(B_{\rm cth}, T) = g_n(B_{\rm cth}, T) \simeq g_n(0,T)$$

$$\Delta g(T) = g_n(0,T) - g_s(0,T) = g_s(B_{\rm cth}, T) - g_s(0,T) = \frac{B_{\rm cth}^2(T)}{2\mu_0} \Longrightarrow \boxed{\Delta g(T) = \frac{B_{\rm cth}^2(T)}{2\mu_0}}$$

condensation energy

- **temperature dependence of the free enthalpy densities g_n and g_s**

$$\boxed{g_s(T) = g_n(T) - \frac{B_{\rm cth}^2(T)}{2\mu_0}}$$



with $\quad B_{\rm cth}(T) = B_{\rm cth}(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right]$ $\quad$ (empirical relation, calculation within BCS theory)

$$g_n(T) = -\int_0^T s_n(T')dT' \propto -T^2$$

Summary of Lecture No. 3 (2)

- **entropy density $\mathscr{s}_s = S_s/V$**

with $-\left(\frac{\partial G}{\partial T}\right)_{p,B_{\rm ext}} = S$ and $\mathscr{s}_s = \frac{S_s}{V} = -\left(\frac{\partial \mathscr{g}_s}{\partial T}\right)_{p,B_{\rm ext}}$, $\mathscr{s}_n = \frac{S_n}{V} = -\left(\frac{\partial \mathscr{g}_n}{\partial T}\right)_{p,B_{\rm ext}} \propto T$ as $c_p = T\,(\partial \mathscr{s}_n/\partial T)_{B_{\rm ext},p}$ and $c_p = \gamma T$ (free electron gas

$$\Delta\mathscr{s}(T) = \mathscr{s}_n(T) - \mathscr{s}_s(T) = -\left(\frac{\partial \Delta \mathscr{g}(T)}{\partial T}\right)_{p,B_{\rm ext}} \Longrightarrow \boxed{\Delta\mathscr{s}(T) = -\frac{B_{\rm cth}}{\mu_0}\frac{\partial B_{\rm cth}}{\partial T}}$$

with $B_{\rm cth}(T) = B_{\rm cth}(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right]$

- **specific heat $\mathbf{c_p}$**

with $C_p = T\left(\frac{\partial S}{\partial T}\right)_{p,B_{\rm ext}} = -T\left(\frac{\partial^2 G}{\partial T^2}\right)_{p,B_{\rm ext}}$ and $\Delta\mathscr{g} = \mathscr{g}_n(T) - \mathscr{g}_s(T) = \frac{B_{\rm cth}^2(T)}{2\mu_0}$

$$\boxed{\Delta c(T) = c_n(T) - c_s(T) = - -T\left(\frac{\partial^2 \Delta\mathscr{g}}{\partial T^2}\right)_{p,B_{\rm ext}} = -\frac{T}{\mu_0}\left[B_{\rm cth}\frac{\partial^2 B_{\rm cth}}{\partial T^2} + \left(\frac{\partial B_{\rm cth}}{\partial T}\right)^2\right]}$$

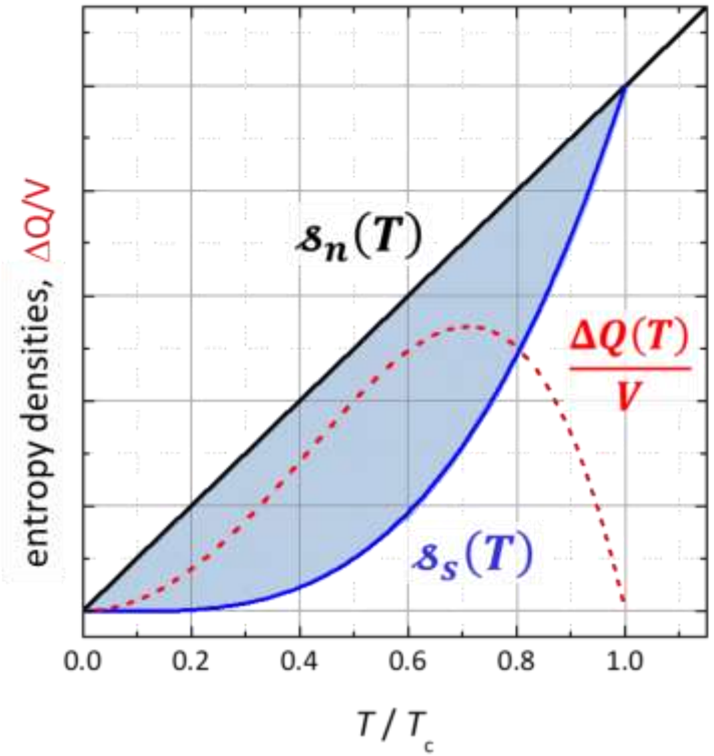
- ➢ jump of specific heat at $T = T_c$: $\boxed{\Delta c_{T=T_c} = -\frac{T_c}{\mu_0}\left(\frac{\partial B_{\rm cth}}{\partial T}\right)^2_{T=T_c} = -\frac{8}{T_c}\frac{B_{\rm cth}^2(0)}{2\mu_0}}$

- ➢ determination of Sommerfeld coefficient for $T \ll T_c$:

$$\boxed{\gamma = \frac{\Delta c_{T\ll T_c}}{T} = \frac{4}{T_c^2}\frac{B_{\rm cth}^2(0)}{2\mu_0}}$$

free electron gas:

$$\Leftrightarrow \gamma = \frac{\pi^2}{3}k_{\rm B}^2\frac{D(E_{\rm F})}{V}$$



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Summary of Lecture No. 3 (3)

- **London theory**
  - simplistic derivation of London equations, starting from equation of motion of charged particles with mass m_s and charge q_s

$$m_s \frac{\mathrm{d}\mathbf{v_s}}{\mathrm{d}t} + \frac{m_s}{\tau}\mathbf{v_s} = q_s\mathbf{E}$$

τ = momentum relaxation time

superconducting state: $n_n \to 0, n_s \to max$ for $T \to 0$, $\tau \to \infty$, $\mathbf{J}_s = n_s q_s \mathbf{v}_s$

$$\frac{\partial(\Lambda\mathbf{J}_s)}{\partial t} = \mathbf{E}$$

1st London equation
(perfect conductivity)

$$\nabla \times (\Lambda\mathbf{J}_s) + \mathbf{B} = \mathbf{0}$$

2nd London equation
(Meißner-Ochsenfeld effect)

$$\Lambda = \frac{m_s}{n_s q_s^2}$$

London coefficient

$$\lambda_{\mathrm{L}} = \sqrt{\frac{\Lambda}{\mu_0}} = \sqrt{\frac{m_s}{\mu_0 n_s q_s^2}}$$

London penetration depth

- **macroscopic quantum model of superconductivity**
  - basic assumption: ***complete entity of all superconducting electrons can be described by macroscopic wave function***

$$\psi(\mathbf{r},t) = \psi_0(\mathbf{r},t)\,\mathrm{e}^{\imath\theta(\mathbf{r},t)}$$

with $|\psi(\mathbf{r},t)|^2 = n_s(\mathbf{r},t)$

  - Madelung transformation (insertion of $\psi(\mathbf{r},t) = \psi_0(\mathbf{r},t)\,\mathrm{e}^{\imath\theta(\mathbf{r},t)}$ into Schrödinger equation) yields :

current-phase relation

$$\mathbf{J}_s(\mathbf{r},t) = q_s n_s(\mathbf{r},t)\underbrace{\left\{\frac{\hbar}{m_s}\nabla\theta(\mathbf{r},t) - \frac{q_s}{m_s}\mathbf{A}(\mathbf{r},t)\right\}}$$

$\mathbf{J}_s = n_s q_s \mathbf{v}_s$

gauge invariant phase gradient:
$\gamma = \nabla\theta' - \frac{q_s}{\hbar}\mathbf{A}' = \nabla\theta - \frac{q_s}{\hbar}\mathbf{A}$

energy-phase relation

$$\hbar\frac{\partial\theta(\mathbf{r},t)}{\partial t} = -\left\{\frac{1}{2n_s}\Lambda\mathbf{J}_s^2(\mathbf{r},t) + q_s\phi_{\mathrm{el}}(\mathbf{r},t) + \mu(\mathbf{r},t)\right\}$$

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Superconductivity and Low Temperature Physics I

Lecture No. 4

R. Gross

3. Phenomenological Models of Superconductivity

3.2 SC as Macroscopic Quantum Phenomenon

key results of Madelung transformation:

① $$-\hbar\frac{\partial\theta(\mathbf{r},t)}{\partial t}=\frac{1}{2n_s}\Lambda\mathbf{J}_s^2(\mathbf{r},t)+q_s\phi_{\mathrm{el}}(\mathbf{r},t)+\mu(\mathbf{r},t)$$ *energy-phase relation*

② $$\mathbf{J}_s(\mathbf{r},t)=\frac{q_s n_s(\mathbf{r},t)\hbar}{m_s}\left\{\nabla\theta(\mathbf{r},t)-\frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t)\right\}$$ *supercurrent density-phase relation*

$$\Lambda\mathbf{J}_s(\mathbf{r},t)=-\left\{\mathbf{A}(\mathbf{r},t)-\frac{\hbar}{q_s}\nabla\theta(\mathbf{r},t)\right\}$$

$\Lambda=\frac{m_s}{q_s^2 n_s}=$ London-Koeffizient

- equations (1) and (2) have ***general validity for charged and uncharged superfluids***

$$q_s=k\cdot q \qquad m_s=k\cdot m \qquad n_s=n/k$$

– $q=-e,\ k=2$: classical superconductor with Cooper pairs with $q_s=-2e, m_s=2m$ und $n_s=n/2$
– $q=0,\ k=1$: neutral Bose superfluid with $n_s=n, m_s=m$ (e.g. superfluid ^{4}He)
– $q=0,\ k=2$: neutral Fermi superfluid with $n_s=n/2, m_s=2m$ (superfluid ^{3}He)

note that in $\Lambda=\frac{m_s}{q_s^2 n_s}=\frac{k\cdot m}{(n/k)\,(kq)^2}$ the factor k drops out ➔ ***k cannot be determined by measuring Λ***

➔ ***we can use equations ① and ② to derive London equations and other important relations!***

3.2.1 Derivation of London Equations

2nd London equation and the Meißner-Ochsenfeld effect:

$$\Lambda \mathbf{J}_s(\mathbf{r},t) = -\left\{\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q_s}\nabla\theta(\mathbf{r},t)\right\}$$

- taking the curl yields

$$\nabla \times \Lambda \mathbf{J}_s(\mathbf{r},t) + \nabla \times \mathbf{A}(\mathbf{r},t) = \nabla \times \left\{\frac{\hbar}{q_s}\nabla\theta(\mathbf{r},t)\right\} = 0$$

with Maxwell´s equations:
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_s$
$\nabla \times \nabla \times \mathbf{B} = \nabla \times \mu_0 \mathbf{J}_s$
$\nabla \times \nabla \times \mathbf{B} = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$
$\nabla \cdot \mathbf{B} = \mathbf{0}$
$\nabla \times \mu_0 \mathbf{J}_s = -\nabla^2 \mathbf{B}$

$$\nabla \times (\Lambda \mathbf{J}_s) + \mathbf{B} = \mathbf{0}$$

2nd London equation

or

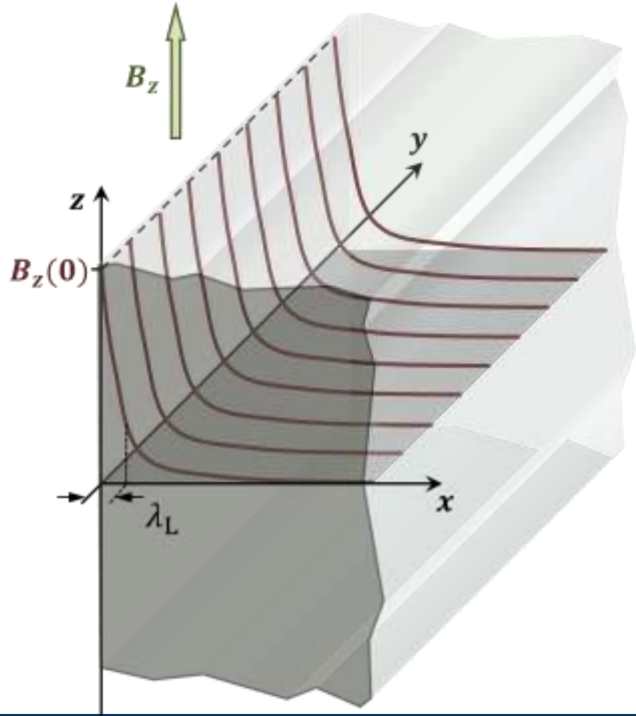
$$\nabla^2 \mathbf{B} - \frac{\mu_0}{\Lambda}\mathbf{B} = \nabla^2 \mathbf{B} - \frac{1}{\lambda_\mathrm{L}^2}\mathbf{B} = \mathbf{0}$$

- describes ***Meißner-Ochsenfeld effect***:
  applied field decays exponentially inside superconductor

decay length

$$\lambda_\mathrm{L} = \sqrt{\frac{m_s}{\mu_0 n_s q_s^2}}$$

London penetration depth



3.2.1 Derivation of London Equations

$$\Lambda \mathbf{J}_s(\mathbf{r},t) = -\left\{\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q_s}\nabla\theta(\mathbf{r},t)\right\}$$

1st London equation and perfect conductivity:

- take the time derivative → $$\frac{\partial}{\partial t}\left(\Lambda \mathbf{J}_s(\mathbf{r},t)\right) = -\left\{\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} - \frac{\hbar}{q_s}\nabla\left(\frac{\partial \theta(\mathbf{r},t)}{\partial t}\right)\right\}$$
- inserting $-\hbar\frac{\partial\theta(\mathbf{r},t)}{\partial t} = \frac{1}{2n_s}\Lambda \mathbf{J}_s^2(\mathbf{r},t) + q_s\phi_{\mathrm{el}}(\mathbf{r},t) + \mu(\mathbf{r},t)$

  and substituting $\mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} - \nabla\phi_{\mathrm{el}}(\mathbf{r},t)$ yields (for $\mu(\mathbf{r},t) = const.$)

$$\boxed{\frac{\partial}{\partial t}\left(\Lambda \mathbf{J}_s(\mathbf{r},t)\right) = \mathbf{E} - \frac{1}{n_s q_s}\nabla\left(\frac{1}{2}\Lambda \mathbf{J}_s^2\right)}$$

1st London equation

  – neglecting 2nd term yields: (see below)

$$\boxed{\frac{\partial}{\partial t}\left(\Lambda \mathbf{J}_s(\mathbf{r},t)\right) = \mathbf{E}}$$

linearized 1st London equation

- **interpretation**:
  for a *time-independent* supercurrent the ***electric field*** inside the superconductor **vanishes**
  → ***dissipationless dc current***

3.2.1 Derivation of London Equations – Summary

- *energy-phase relation*

① $$-\hbar\frac{\partial\theta(\mathbf{r},t)}{\partial t}=\frac{1}{2n_s}\Lambda\mathbf{J}_s^2(\mathbf{r},t)+q_s\phi_{\text{el}}(\mathbf{r},t)+\mu(\mathbf{r},t)$$

- *supercurrent density-phase relation*

② $$\Lambda\mathbf{J}_s(\mathbf{r},t)=-\left\{\mathbf{A}(\mathbf{r},t)-\frac{\hbar}{q_s}\nabla\theta(\mathbf{r},t)\right\}$$

$\Lambda=\frac{m_s}{q_s^2n_s}=$ London-Koeffizient

- *2nd London equation and the Meißner-Ochsenfeld effect:*
  - take the curl →

$$\nabla\times(\Lambda\mathbf{J}_s)=\nabla\times\mathbf{A}=-\mathbf{B}$$

or

$$\nabla^2\mathbf{B}-\frac{\mu_0}{\Lambda}\mathbf{B}=\nabla^2\mathbf{B}-\frac{1}{\lambda_{\text{L}}^2}\mathbf{B}=\mathbf{0}$$

2nd London equation

- *1st London equation and perfect conductivity:*
  - take the time derivative →

$$\frac{\partial}{\partial t}\left(\Lambda\mathbf{J}_s(\mathbf{r},t)\right)=-\left\{\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}-\frac{\hbar}{q_s}\nabla\left(\frac{\partial\theta(\mathbf{r},t)}{\partial t}\right)\right\}$$

what leads to:

$$\frac{\partial}{\partial t}\left(\Lambda\mathbf{J}_s(\mathbf{r},t)\right)=\mathbf{E}-\frac{1}{n_sq_s}\nabla\left(\frac{1}{2}\Lambda\mathbf{J}_s^2\right)$$

1st London equation

3.2.1 Derivation of London Equations – Summary

- **the assumption that the superconducting state can be described by a macroscopic wave function leads to a general expression for the supercurrent density $\mathbf{J}_s$**

- **London equations can be directly derived from the general expression for the supercurrent density $\mathbf{J}_s$ for spatially constant $n_s(\mathrm{r}, t) = n_s(t)$**

  → ***London approximation***

- **London equations together with Maxwell's equations describe the behavior of superconductors in electric and magnetic fields**

- **London equations cannot be used for the description of spatially inhomogeneous situations**

  → ***Ginzburg-Landau theory***

- **London equations can be used for the description of time-dependent situations**

  → ***Josephson equations***

3.2.1 Derivation of London Equations – Summary

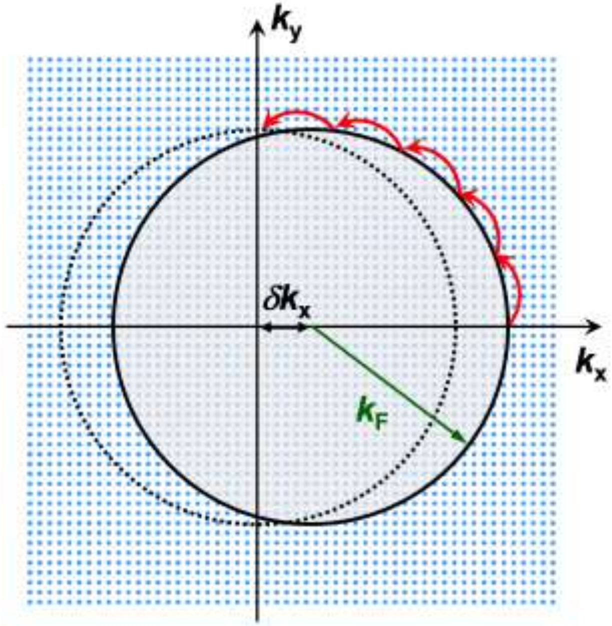
Processes that could cause a decay of $\mathbf{J}_s$ (plausibility consideration)

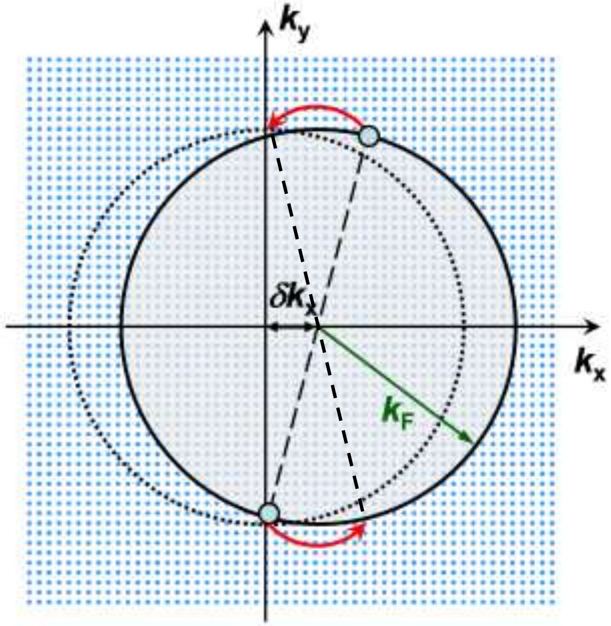
example: consider two-dimensional Fermi circle in $k_x k_y$– plane

- $T = 0$: all states inside the Fermi circle are occupied
- electric field in x-direction → shift of Fermi circle along k_x by $\pm\delta k_x$

- **normal state**: relaxation into states with lower energy (obeying Pauli principle) → centered Fermi circle, current relaxes if E_x is switched off
- **superconducting state**: Cooper pairs with the same center of mass moment (discussion later) → only scattering around the sphere → no decay of supercurrent

normal state





superconducting state

3.2.1 Additional Topic: Linearized 1. London Equation

- the 1. London equation can be linearized in most cases

$$\frac{\partial}{\partial t}\left(\Lambda \mathbf{J}_s(\mathbf{r},t)\right) = \mathbf{E} - \underbrace{\cancel{\frac{1}{n_s q_s}\nabla\left(\frac{1}{2}\Lambda \mathbf{J}_s^2\right)}}_{\text{when can we neglect this term?}}$$

→ we show that this is allowed for $|\mathbf{E}| \gg |\mathbf{v_s}|\,|\mathbf{B}|$ and that this condition is valid in most situations
(force on charge carriers by electric field large compared to Lorentz force due to magnetic field)

- in order to discuss the origin of the extra term (nonlinearity) we use the vector identity

$$\mathbf{a} \times (\nabla \times \mathbf{a}) = \frac{1}{2}\nabla(\mathbf{a}\cdot\mathbf{a}) - (\mathbf{a}\cdot\nabla)\mathbf{a} \quad \text{to write} \quad \frac{1}{2}\nabla\,\mathbf{J}_s^2 = \mathbf{J_s} \times (\nabla \times \mathbf{J_s}) + (\mathbf{J_s}\cdot\nabla)\mathbf{J_s}$$

- then, by using the second London equation, we can rewrite the 1. London equation as

$$\frac{\partial}{\partial t}\left(\Lambda \mathbf{J}_s(\mathbf{r},t)\right) = \mathbf{E} - \frac{1}{n_s q_s}(\mathbf{J_s}\cdot\nabla)\Lambda\mathbf{J_s} + \frac{1}{n_s q_s}(\mathbf{J_s}\times\mathbf{B})$$

- with $\frac{\mathrm{d}}{\mathrm{d}t}\left(\Lambda \mathbf{J}_s(\mathbf{r},t)\right) = \frac{\partial}{\partial t}\left(\Lambda \mathbf{J}_s(\mathbf{r},t)\right) + (\mathbf{v_s}\cdot\nabla)\left(\Lambda \mathbf{J}_s(\mathbf{r},t)\right)$ and $\mathbf{J}_s(\mathbf{r},t) = n_s q_s \mathbf{v}_s(\mathbf{r},t)$ we obtain

$$m_s \frac{\mathrm{d}\mathbf{v}_s}{\mathrm{d}t} = q_s\mathbf{E} + q_s\mathbf{v}_s \times \mathbf{B} \qquad \text{(Lorentz law)}$$

3.2.1 Additional Topic: Linearized 1. London Equation

important conclusion:

- the **nonlinear first London equation** results from the Lorentz's law and the **second London equation**
  → exact form of the expression describing the phenomenon of **zero dc resistance** in superconductors
- the first London equation is derived by using the second London equation
  → ***Meißner-Ochsenfeld effect is the more fundamental property*** of superconductors than the vanishing dc resistance
- we can neglect the nonlinear term if $|\mathbf{E}| \gg \left| \frac{1}{n_s q_s} \boldsymbol{\nabla} \left(\frac{1}{2} \Lambda \mathbf{J}_s^2 \right) \right|$
- as variations of $\mathbf{J}_s$ occur on length scale $\sim \lambda_\mathrm{L}$, we have $\boldsymbol{\nabla}\mathbf{J}_s \sim \mathbf{J}_s/\lambda_\mathrm{L}$ and obtain the condition

$$|\mathbf{E}| \gg |\mathbf{v}_s| \left| \frac{\Lambda \mathbf{J}_s}{\lambda_\mathrm{L}} \right| \Rightarrow \boxed{|\mathbf{E}| \gg |\mathbf{v_c}|\, |\mathbf{B_{cth}}|}$$

with 2. London equation: $\boldsymbol{\nabla} \times (\Lambda \mathbf{J}_s) = \boldsymbol{\nabla} \times \mathbf{A} = -\mathbf{B}$, $J_\mathrm{c} = n_s q_s v_c \simeq H_\mathrm{cth}/\lambda_\mathrm{L}$ and $\Lambda = \mu_0 \lambda_L^2$

typically, $v_c < 1$ m/s even at very high J_c values of the order of 10^{10} A/cm² due to the large n_s values

→ $|\mathbf{E}| \gg 0.01$ V/m @ $B_\mathrm{cth} \simeq 0.1$ T

3.2.1 Additional Topic: Gauge Invariance

gauge invariance of the current-phase relation

$$\Lambda \mathbf{J}_s(\mathbf{r},t) = -\left\{\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q_s}\boldsymbol{\nabla}\theta(\mathbf{r},t)\right\}$$

- physical variables such as $\mathbf{A}, \phi$ or θ are no observable quantities
  - they can be transformed without any influence on observable quantities such as $\mathbf{E}, \mathbf{B}$ or $\mathbf{J}_s$
  - we call such transformations **gauge transformations**

- we see that the observable quantity $\mathbf{J}_s$is determined by $\mathbf{A}$ and θ, that is, by two quantities that are no observables
- since $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \boldsymbol{\nabla} \times (\mathbf{A} + \boldsymbol{\nabla}\boldsymbol{\chi}) = \boldsymbol{\nabla} \times \mathbf{A}'$ for any scalar function χ, there is an infinite number of possible vector potentials giving the correct flux density $\mathbf{B}$

- solution:
  - there is a fixed relation between θ and $\mathbf{A}$ such that we can measure $\mathbf{J}_s$ without being able to measure θ and $\mathbf{A}$
  - we have to demand that the expression for $\mathbf{J}_s$ is independent of the special choice of $\mathbf{A}$
    ➔ ***gauge invarant expression***

3.2.1 Additional Topic: Gauge Invariance

$$\Lambda \mathbf{J}_s(\mathbf{r},t) = -\left\{\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q_s}\boldsymbol{\nabla}\theta(\mathbf{r},t)\right\}$$

gauge invariance of the current phase relation

- we define $\mathbf{A}'(\mathbf{r},t) \equiv \mathbf{A}(\mathbf{r},t) + \boldsymbol{\nabla}\chi(\mathbf{r},t)$

  correspondingly, the electrical field is given by $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\phi = -\frac{\partial \mathbf{A}'}{\partial t} - \boldsymbol{\nabla}\phi'$ $\Rightarrow$ $\phi'(\mathbf{r},t) \equiv \phi(\mathbf{r},t) - \frac{\partial \chi(\mathbf{r},t)}{\partial t}$

- Schrödinger equation for new potentials (with $\psi'(\mathbf{r},t) = \psi_0\, \mathrm{e}^{\iota\theta'(\mathbf{r},t)}$)

$$\frac{1}{2m_s}\left(\frac{\hbar}{\iota}\boldsymbol{\nabla} - q_s\mathbf{A}'(\mathbf{r},t)\right)^2 \psi'(\mathbf{r},t) + [q_s\phi'(\mathbf{r},t) + \mu(\mathbf{r},t)]\, \psi'(\mathbf{r},t) = \iota\hbar\frac{\partial \psi'(\mathbf{r},t)}{\partial t}$$

$$\Rightarrow \quad \Lambda \mathbf{J}_s(\mathbf{r},t) = -\left\{\mathbf{A}'(\mathbf{r},t) - \frac{\hbar}{q_s}\boldsymbol{\nabla}\theta'(\mathbf{r},t)\right\} = -\left\{\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q_s}\boldsymbol{\nabla}\theta(\mathbf{r},t)\right\}$$

$$\mathbf{A}'(\mathbf{r},t) - \frac{\hbar}{q_s}\boldsymbol{\nabla}\theta'(\mathbf{r},t) = \mathbf{A}(\mathbf{r},t) + \boldsymbol{\nabla}\chi(\mathbf{r},t) - \frac{\hbar}{q_s}\boldsymbol{\nabla}\theta'(\mathbf{r},t) = \mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q_s}\boldsymbol{\nabla}\theta(\mathbf{r},t)$$

$$\boldsymbol{\nabla}\theta'(\mathbf{r},t) = \boldsymbol{\nabla}\theta(\mathbf{r},t) + \frac{q_s}{\hbar}\,\boldsymbol{\nabla}\chi(\mathbf{r},t)$$

$$\Rightarrow \quad \psi'(\mathbf{r},t) = \psi(\mathbf{r},t)\, \mathrm{e}^{\iota(q_s/\hbar)\chi(\mathbf{r},t)}$$

$\Rightarrow$ **gauge invariant phase gradient**

$$\gamma(\mathbf{r},t) = \boldsymbol{\nabla}\theta'(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}'(\mathbf{r},t) = \boldsymbol{\nabla}\theta(\mathbf{r},t) + \frac{q_s}{\hbar}\boldsymbol{\nabla}\chi(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t) - \frac{q_s}{\hbar}\boldsymbol{\nabla}\chi(\mathbf{r},t) = \boldsymbol{\nabla}\theta(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t)$$

3.2.1 Additional Topic: The London Gauge

- in some cases it is convenient to choose a special gauge

  → often used: ***London Gauge***

- if the macroscopic wavefunction is single valued (this is the case for a simply connected superconductor containing no flux) we can choose $\chi(\mathbf{r},t)$ such that

$$\theta(\mathbf{r},t) = \theta'(\mathbf{r},t) - \frac{q_s}{\hbar}\,\nabla\chi(\mathbf{r},t) = 0 \quad \text{everywhere}$$

⇨ $$\Lambda\mathbf{J}_s(\mathbf{r},t) = -\left\{\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q_s}\nabla\theta(\mathbf{r},t)\right\} = -\mathbf{A}(\mathbf{r},t)$$

- frequently, we have no conversion of $\mathbf{J}_s$ in $\mathbf{J}_n$ at interfaces or no supercurrent flow throuh sample surface

⇨ $$\nabla\cdot\mathbf{J}_s(\mathbf{r},t) = 0 \quad \Rightarrow \quad \nabla\cdot\mathbf{A}(\mathbf{r},t) = 0$$

a vector potential that satisfies $\nabla\cdot\mathbf{A}(\mathbf{r},t) = 0$ is said to be in the ***London gauge***

- 1. London equation: $$\frac{\partial}{\partial t}\left(\Lambda\mathbf{J}_s(\mathbf{r},t)\right) = \mathbf{E}(\mathbf{r},t) = -\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}$$

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi = \frac{\partial\mathbf{A}}{\partial t}$$

⇨ $$\nabla\phi = 0$$

3.2.2 Fluxoid Quantization

derivation of fluxoid quantization from current-phase relation $\Lambda \mathbf{J}_s(\mathbf{r},t) = -\left\{\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q_s}\nabla\theta(\mathbf{r},t)\right\}$

- integration of expression for supercurrent density around a closed contour

$$\oint_C \Lambda \mathbf{J}_s \cdot \mathrm{d}\ell + \oint_C \mathbf{A} \cdot \mathrm{d}\ell = \frac{\hbar}{q_s}\oint_C \nabla\theta(\mathbf{r},t) \cdot \mathrm{d}\ell$$

$\Lambda = \frac{m_s}{q_s^2 n_s} =$ London-Koeffizient

- ***Stoke's theorem***
  (path C in simply or multiply connected region)

$$\oint_C \mathbf{A} \cdot \mathrm{d}\ell = \int_S (\nabla \times \mathbf{A}) \cdot \hat{\mathbf{n}}\ \mathrm{d}S = \int_S \mathbf{B} \cdot \hat{\mathbf{n}}\ \mathrm{d}S = \Phi$$

- integral of phase gradient:

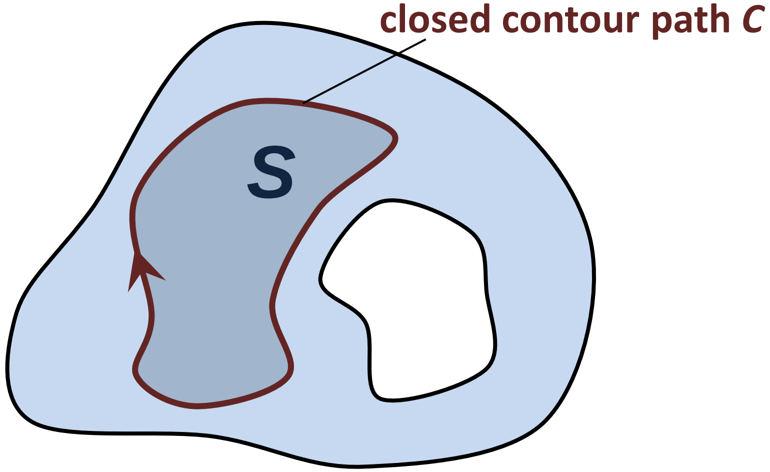
$$\oint_C \nabla\theta(\mathbf{r},t) \cdot \mathrm{d}\ell = \lim_{r_2 \to r_1} [\theta(\mathbf{r}_2,t) - \theta(\mathbf{r}_1,t)] = 2\pi \cdot n$$

$$\Rightarrow \underbrace{\oint_C \Lambda \mathbf{J}_s \cdot \mathrm{d}\ell + \int_S \mathbf{B} \cdot \hat{\mathbf{n}}\ \mathrm{d}S}_{\text{fluxoid}} = n \cdot \frac{h}{q_s} = n \cdot \Phi_0$$

fluxoid quantization

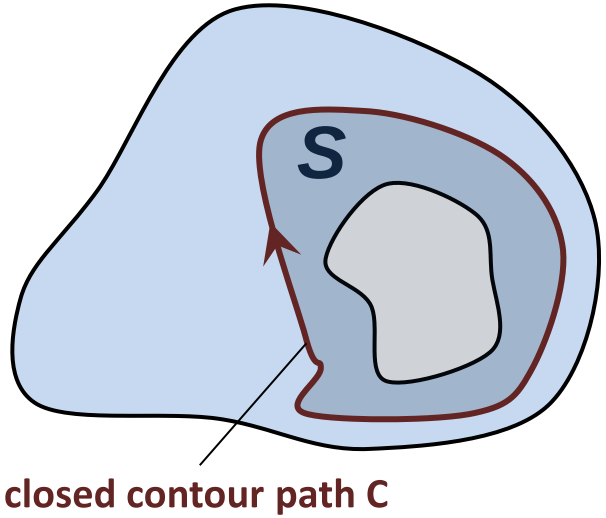
flux quantum: $\Phi_0 = h/|q_s| = h/2e = 2.067\,833\,831(13) \times 10^{-15}$ Vs

3.2.2 Fluxoid Quantization



- quantization condition holds for all contour lines including contour that can be shrunk to single point

→ $r_1 = r_2$: $\int_{r_1}^{r_2} \nabla\theta \cdot \mathrm{d}\ell = 0$



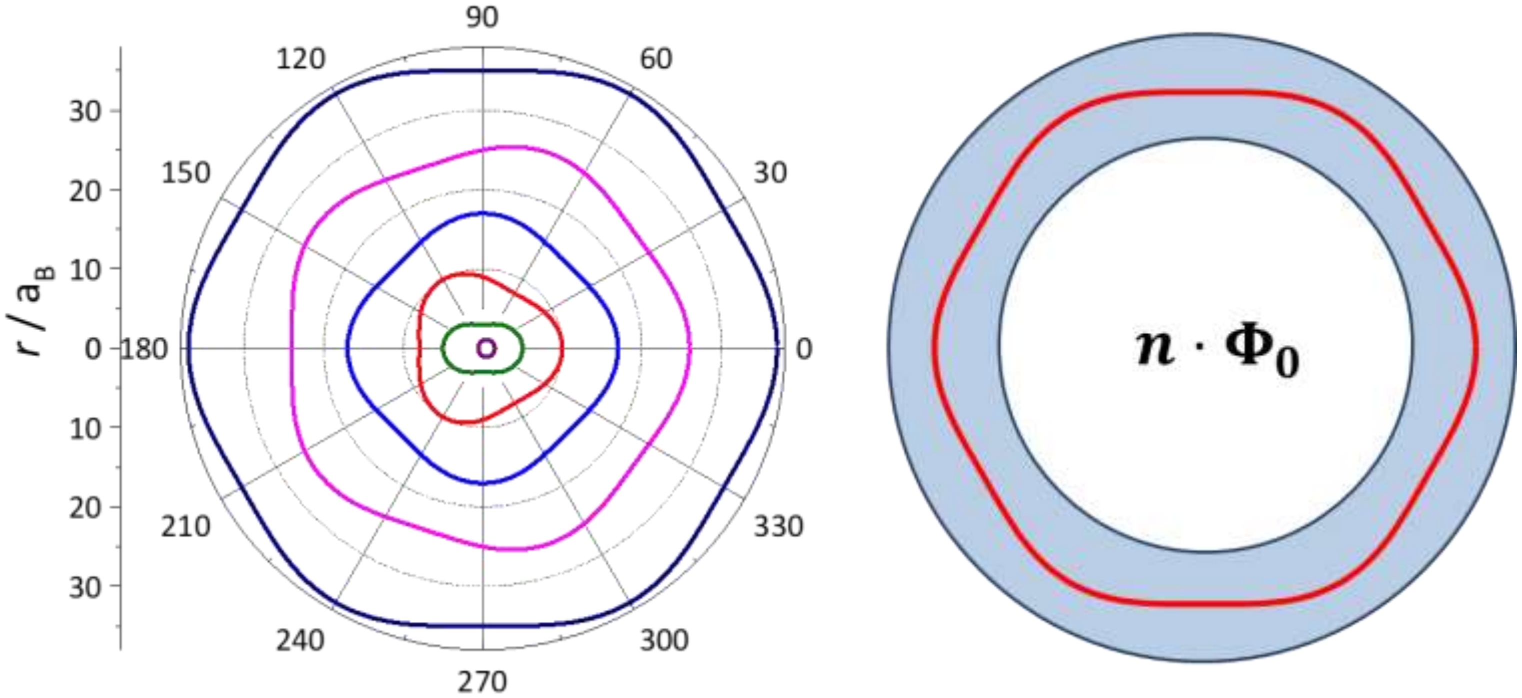
- contour line can no longer be shrunk to single point
  → inclusion of non-superconducting region in contour
  → $r_1 = r_2$: we have built in „memory“ in integration path: $n \neq 0$ possible

→ $r_1 = r_2$: $\int_{r_1}^{r_2} \nabla\theta \cdot \mathrm{d}\ell = n \cdot 2\pi$

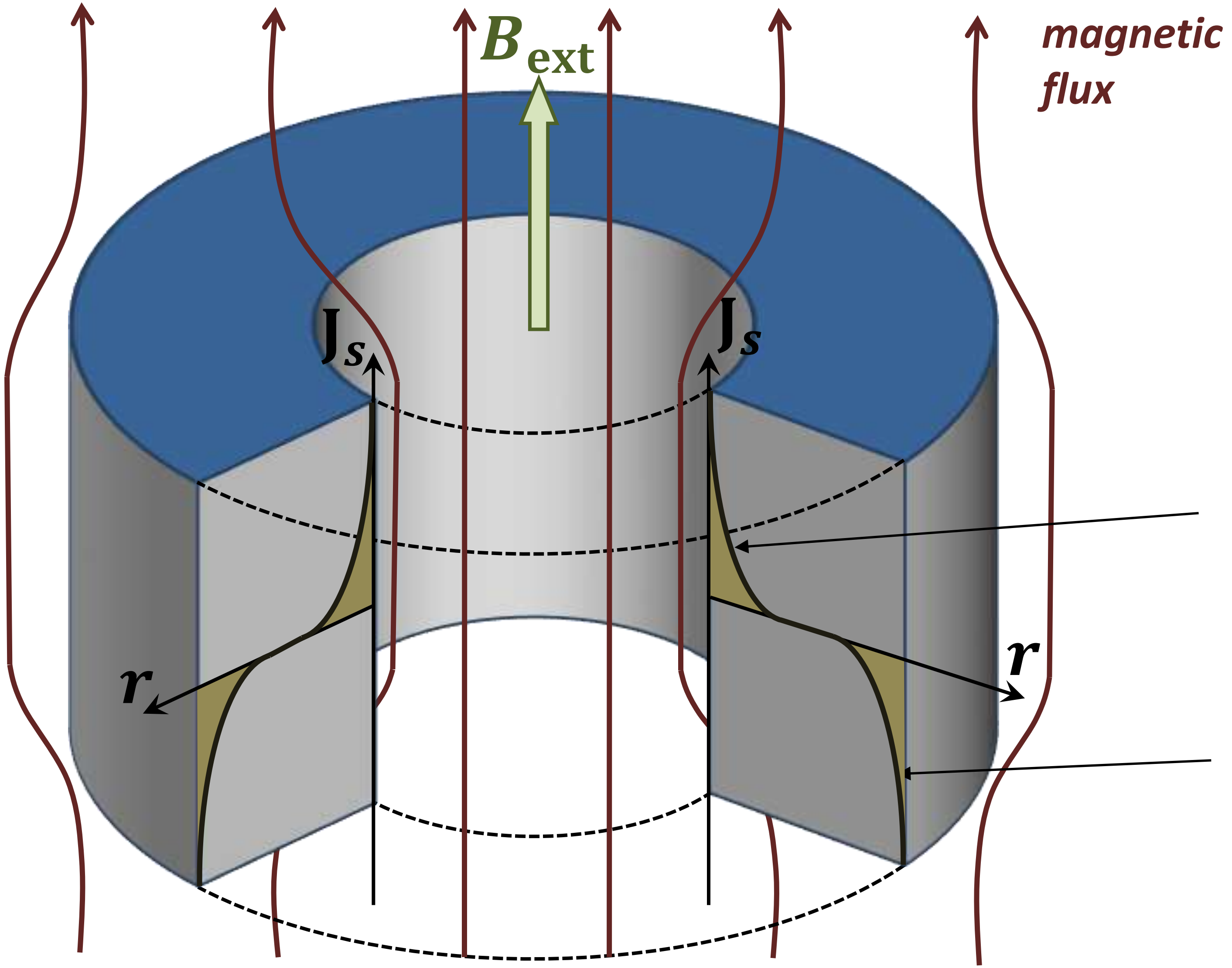
3.2.2 Fluxoid Quantization

physical origin of fluxoid quantization in multiply connected superconductors

- direct consequence of the fact that superconductor can be represented by a ***macroscopic wave function ψ***
    - ➢ phase is allowed to change only by interger multiples of 2π along a closed path in order to obtain a stationary state (constructive interference of the wave funtion)
    - ➢ analogy to Bohr-Sommerfeld quantization in atomic physics



3.2.2 Flux vs. Fluxoid Quantization



screening current on inner wall

screening current on outer wall

3.2.2 Flux vs. Fluxoid Quantization

- ***fluxoid quantization***:
  - $\oint_C \Lambda \mathbf{J}_s \cdot \mathrm{d}\ell + \Phi = n \cdot \Phi_0$ ➔ trapped flux + contribution from $\mathbf{J}_s$ must have **discrete** values $n \cdot \Phi_0$

- ***flux quantization:***
  - superconducting cylinder with wall much thicker than λ_L
  - application of small magnetic field at $T < T_c$

    → screening currents, **no** flux inside

  - application of B_cool **during cool down**: screening current on outer **and** inner wall
  - amount of flux trapped in cylinder: satisfies fluxoid quantization condition
  - wall thickness ≫ λ_L: $\oint_C \Lambda \mathbf{J}_s \cdot \mathrm{d}\ell$ can be taken along closed contour **deep inside** where $J_s = 0$
  - then:

$$\int_S \mathbf{B} \cdot \hat{\mathbf{n}}\ \mathrm{d}S = \Phi = n \cdot \Phi_0$$

⇨ **flux quantization**

  - remove field after cooling down → **trapped flux = integer multiple of flux quantum**

3.2.2 Flux vs. Fluxoid Quantization

- ***flux trapping*:** why is flux not expelled after switching off external field?

$\frac{\partial \mathbf{J}_s}{\partial t} = 0$ according to 1st London equation, since $\mathbf{E} = 0$ in superconductor

$$\frac{\partial}{\partial t}\left(\Lambda \mathbf{J}_s(\mathbf{r},t)\right) = \mathbf{E}$$

- with $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi = 0$ we get:

$$\oint_C \mathbf{E}\cdot \mathrm{d}\ell = -\frac{\partial}{\partial t}\oint_C \mathbf{A}\cdot \mathrm{d}\ell - \oint_C \nabla\phi\cdot \mathrm{d}\ell = -\frac{\partial}{\partial t}\int_S \mathbf{B}\cdot\hat{\mathbf{n}}\ \mathrm{d}S = -\frac{\partial \Phi}{\partial t}$$

Φ: magnetic flux enclosed in loop

contour deep inside the superconductor: $\mathbf{E} = 0$ and therefore $\frac{\partial \Phi}{\partial t} = 0$

→ ***flux enclosed in superconducting cylinder stays constant***

3.2.2 Flux Quantization - Experiment

- discoverd 1961 by
  - *Robert Doll* and *Martin Näbauer* (WMI)
  - *B.S. Deaver* and *W.M. Fairbanks* (Stanford University)

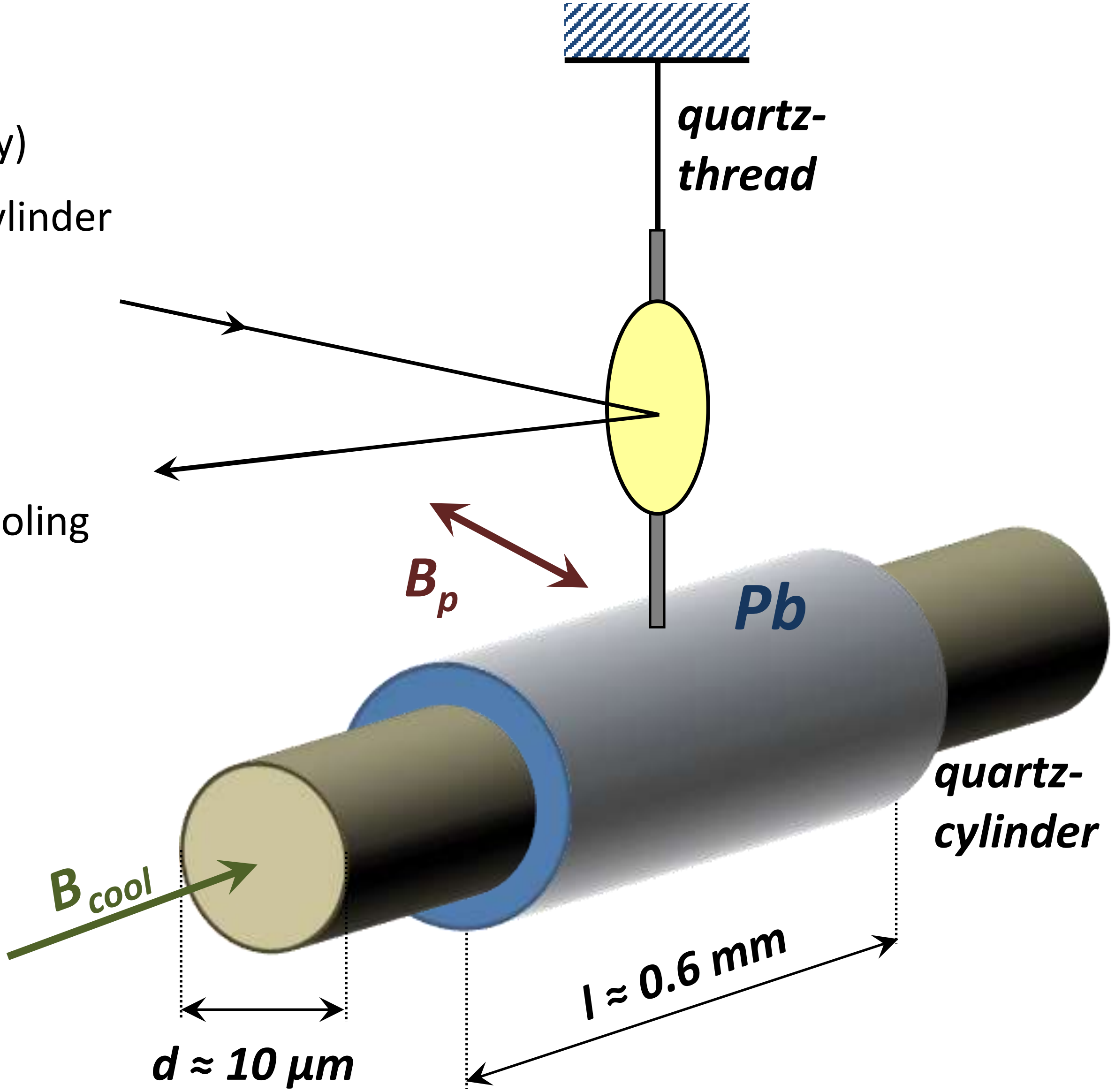
→ ***quantization of magnetic flux*** in a hollow cylinder
→ ***Cooper pairs*** with $q_s = -2e$

- ***experiment by Doll and Näbauer (WMI)***
  - cylinder with wall thickness $\gg \lambda_L$
  - different amounts of flux are frozen in during cooling down in B_{cool}
  - trapping of magnetic flux in hollow cylinder
  - apply torque $\mathbf{D} = \boldsymbol{\mu} \times \mathbf{B_p}$ by probing field $\mathbf{B_p}$
  - increase sensitivity by resonance technique

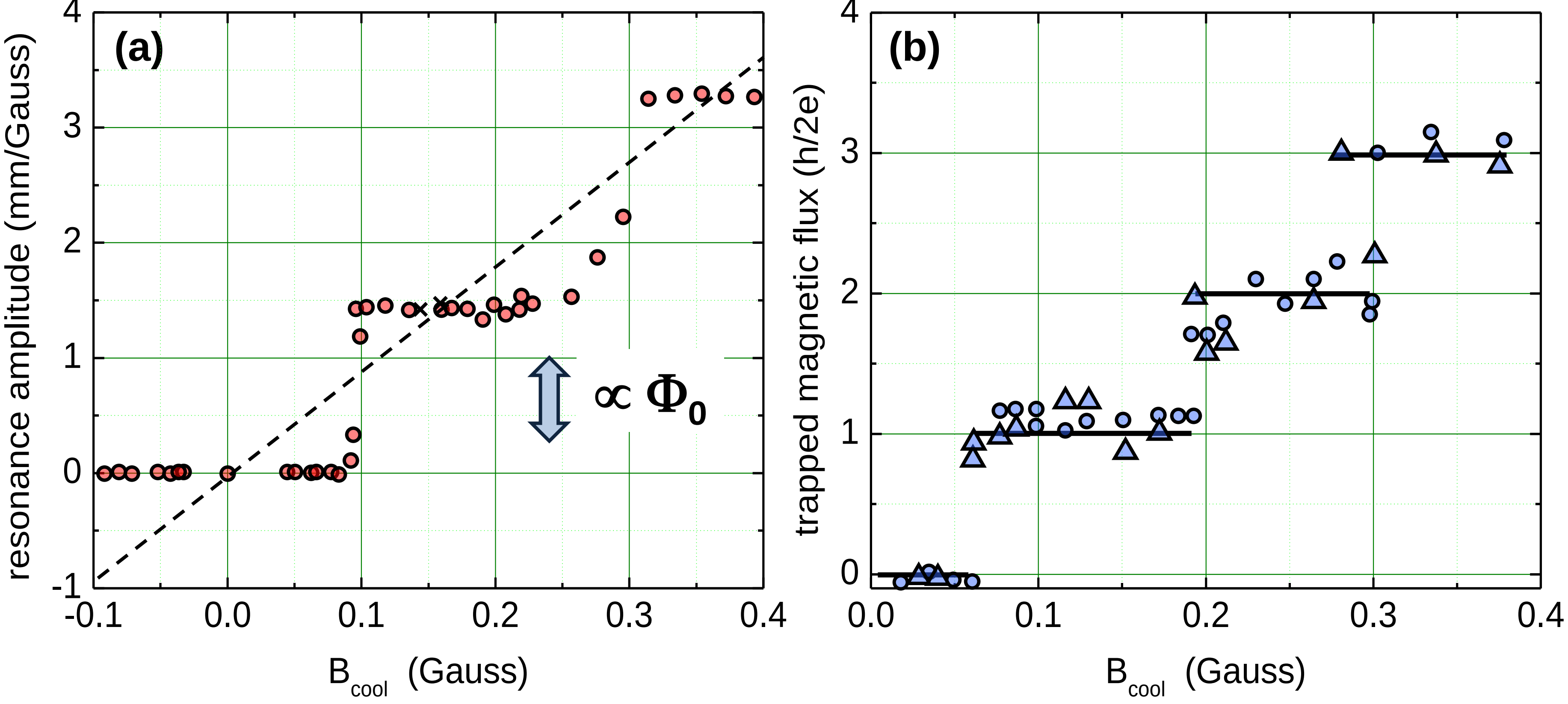
- number of trapped flux quanta:

$$N = B_{\text{cool}}\, \pi(d/2)^2$$

$N \simeq 1$
$@\ B_{\text{cool}} = 10^{-5}\ \text{T}, d = 10\ \mu\text{m}$



3.2.2 Flux Quantization - Experiment



R. Doll, M. Näbauer
Phys. Rev. Lett. **7**, 51 (1961)

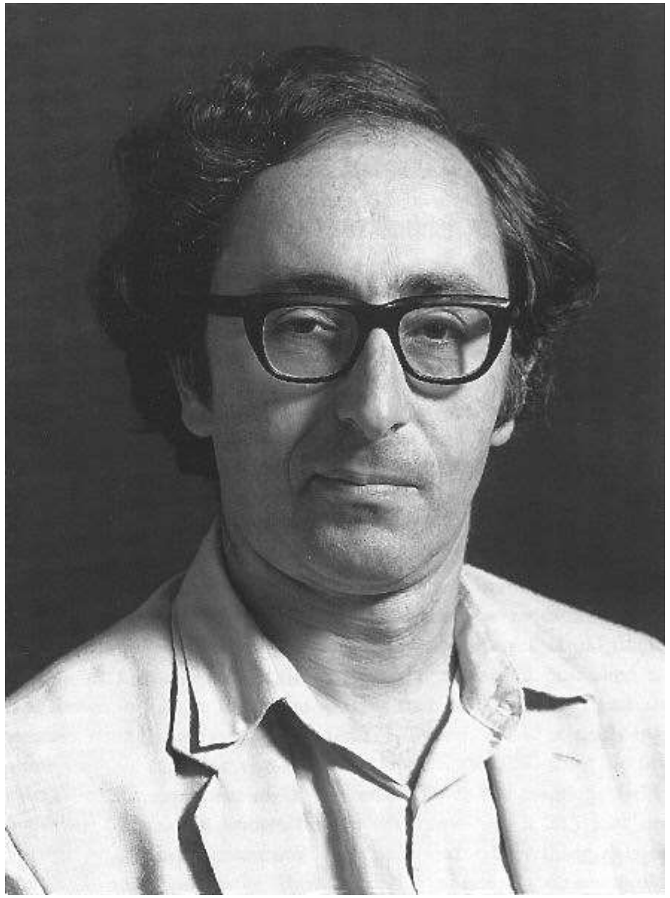
B.S. Deaver, W.M. Fairbank
Phys. Rev. Lett. **7**, 43 (1961)

$$\boxed{\Phi_0 = \frac{h}{2e}}$$

prediction by F. London: h/e
→ ***experimental proof for existence of Cooper pairs***

Paarweise im Fluss
D. Einzel, R. Gross, Physik Journal 10, No. 6, 45-48 (2011)

3.2.3 Josephson Effect (1962)



Brian David Josephson (born 1940)

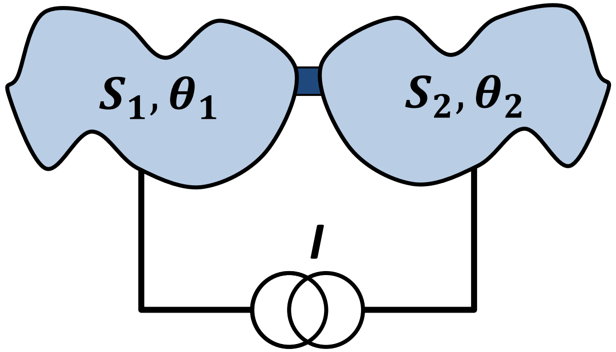
Brian D. Josephson: *Possible New Effects in Superconducting Tunnelling,* Physics Letters **1**, 251–253 (1962), doi:10.1016/0031-9163(62)91369-0.

Nobel Prize in Physics 1973

"for his theoretical predictions of the properties of a supercurrent through a tunnel barrier, in particular those phenomena which are generally known as the Josephson effects"

(together with Leo Esaki and Ivar Giaever)

3.2.3 Josephson Effect (1962)



- ***what happens if we weakly couple two superconductors?***
  - coupling by *tunneling barriers, point contacts, normal conducting layers, etc.*
  - do they form a bound state such as a molecule?
  - if yes, what is the binding energy?

- **B.D. Josephson** in 1962
  (Nobel Prize in physics with Esaki and Giaever in 1973)

→ Cooper pairs can tunnel through thin insulating barrier (T = transmission amplitude for single charge carriers)

expectation: tunneling probability for pairs $\propto (|T|^2)^2$ ➔ extremely small $\sim (10^{-4})^2$

Josephson: tunneling probability for pairs $\propto |T|^2$
coherent tunneling of pairs („***tunneling of macroscopic wave function***“)

predictions:

- ➢ ***finite supercurrent at zero applied voltage***
- ➢ ***oscillation of supercurrent at constant applied voltage***
- ➢ ***finite binding energy of coupled SCs = Josephson coupling energy***

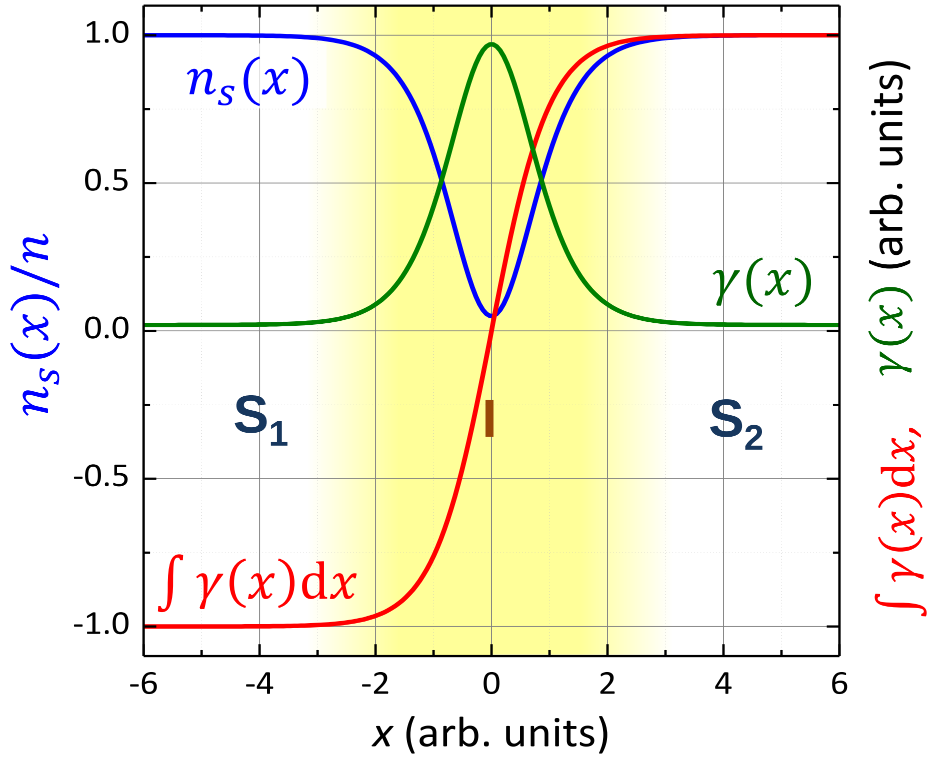
} ***Josephson effects***

3.2.3 Josephson Effect (1962)

- **coupling is weak** → supercurrent density between S_1 and S_2 is small → $|\psi|^2 = n_s$ is not changed in S_1 and S_2
- supercurrent density depends on gauge invariant phase gradient:

$$\mathbf{J}_s(\mathbf{r},t) = \frac{q_s n_s(\mathbf{r},t)\hbar}{m_s}\left\{\nabla\theta(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t)\right\} = \frac{q_s n_s(\mathbf{r},t)\hbar}{m_s}\ \gamma(\mathbf{r},t)$$

- **simplifying assumptions:**
  - current density is spatially homogeneous
  - $\gamma(\mathbf{r},t)$ varies negligibly in S_1 and S_2
  - $\mathbf{J}_s$ is equal in electrodes and junction area → γ in S_1 and S_2 much smaller than in insulator I

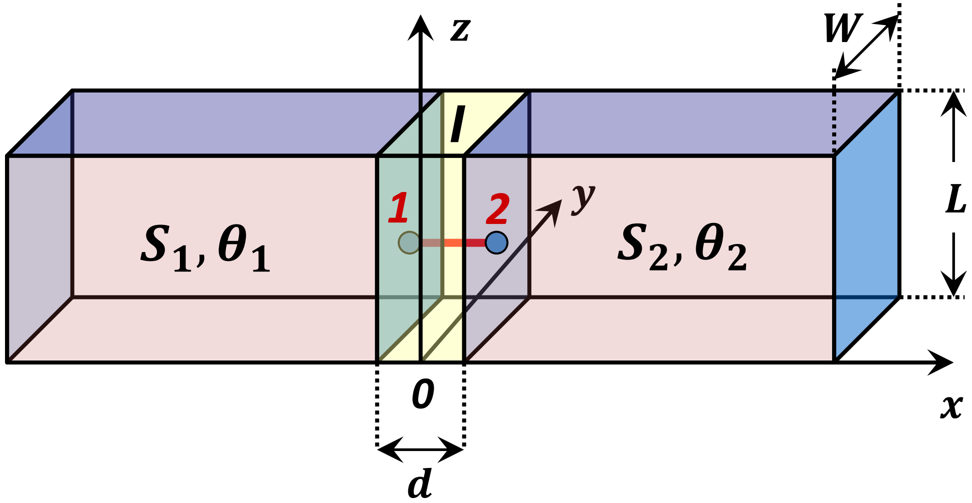


- **approximation:**
  - replace gauge invariant phase gradient γ by ***gauge invariant phase difference φ***:

$$\varphi(\mathbf{r},t) = \int_1^2 \gamma(\mathbf{r},t)\cdot\,\mathrm{d}\ell\ = \int_1^2\left(\nabla\theta(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t)\right)\cdot\mathrm{d}\ell = \theta_2(\mathbf{r},t) - \theta_1(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot\mathrm{d}\ell$$

3.2.3 Josephson Effect (1962)

first Josephson equation:



- we *expect*: $J_s = J_s(\varphi)$

$$J_s(\varphi) = J_s(\varphi + n \cdot 2\pi)$$

- for $J_s = 0$: phase difference must be zero:

$$J_s(0) = J_s(n \cdot 2\pi) = 0$$

$$\Rightarrow \quad \boxed{J_s(\varphi) = J_c \sin\varphi + \sum_{m=2}^{\infty} J_{c,m} \sin(m\varphi)}$$

J_c = crititical or maximum Josephson current density

general formulation of **1st Josephson equation**: ***current-phase relation***

- in most cases: we have to keep only 1st term (especially for weak coupling):

$$\boxed{J_s(\varphi) = J_c \sin\varphi}$$

1. Josephson equation

- *generalization* to ***spatially inhomogeneous*** supercurrent density:

$$J_s(y,z) = J_c(y,z)\ \sin\varphi\,(y,z)$$

derived by Josephson for SIS junctions

supercurrent density J_s varies sinusoidally with phase difference $\varphi = \theta_2 - \theta_1$ w/o external potentials

3.2.3 Josephson Effect (1962)

- other argument why there are only „sin“ contributions to the Josephson current density

$$J_s(\varphi) = J_c \sin\varphi + \sum_{m=2}^{\infty} J_{c,m} \sin(m\varphi) \quad \Rightarrow \quad \textit{time reversal symmetry}$$

- if we reverse time, the Josephson current should flow in opposite direction:

$$t \rightarrow -t \quad \Rightarrow \quad J_s \rightarrow -J_s$$

- the time evolution of the macroscopic wave functions is $\propto \exp[i\theta(t)]$
  - if we reverse time, we have

$$\varphi(\mathbf{r},t) = \theta_2(\mathbf{r},t) - \theta_2(\mathbf{r},t) \xrightarrow{t \rightarrow -t} \varphi(\mathbf{r},-t) = \theta_2(\mathbf{r},-t) - \theta_2(\mathbf{r},-t) = -[\theta_2(\mathbf{r},t) - \theta_2(\mathbf{r},t)] = -\varphi(\mathbf{r},t)$$

- if the Josephson effect stays unchanged under time reversal, we have to demand

$$J_s(\varphi) = -J_s(-\varphi) \quad \Rightarrow$$ satisfied only by sin-terms

3.2.3 Josephson Effect (1962)

second Josephson equation (for spatially homogeneous junction)

- take time derivative of the gauge invariant phase difference $\varphi(t) = \theta_2(t) - \theta_1(t) - \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A}(t)\cdot \mathrm{d}\ell$

$$\frac{\partial\varphi(t)}{\partial t} = \frac{\partial\theta_2(t)}{\partial t} - \frac{\partial\theta_1(t)}{\partial t} - \frac{2\pi}{\Phi_0}\frac{\partial}{\partial t}\int_1^2 \mathbf{A}(t)\cdot \mathrm{d}\ell$$

- substitution of the energy-phase relation $\hbar\frac{\partial\theta(t)}{\partial t} = -\left\{\frac{1}{2n_s}\Lambda\mathbf{J}_s^2(t) + q_s\phi_{\mathrm{el}}(\mathbf{r},t)\right\}$ gives:

$$\frac{\partial\varphi(t)}{\partial t} = -\frac{1}{\hbar}\left(\frac{\Lambda}{2n_s}\left[\mathbf{J}_s^2(2) - \mathbf{J}_s^2(1)\right] + q_s\left[\phi_{\mathrm{el}}(2) - \phi_{\mathrm{el}}(1)\right]\right) - \frac{2\pi}{\Phi_0}\frac{\partial}{\partial t}\int_1^2 \mathbf{A}(t)\cdot \mathrm{d}\ell$$

- supercurrent density across the junction is *continuous* $(\mathbf{J}_s(1) = \mathbf{J}_s(2))$:

$$\frac{\partial\varphi(t)}{\partial t} = \frac{2\pi}{\Phi_0}\int_1^2\left(-\boldsymbol{\nabla}\phi_{\mathrm{el}} - \frac{\partial\mathbf{A}(t)}{\partial t}\right)\cdot \mathrm{d}\ell$$

(term in parentheses = electric field)

$$\Rightarrow\quad \boxed{\frac{\partial\varphi(t)}{\partial t} = \frac{2\pi}{\Phi_0}\underbrace{\int_1^2 \mathbf{E}(t)\cdot \mathrm{d}\ell}_{\text{voltage drop } V} = \frac{2\pi}{\Phi_0}V(t) = \frac{q_sV(t)}{\hbar}}$$

2nd Josephson equation: *voltage – phase relation*

R. Gross and A. Marx , © Walther-Meißner-Institut (2004 - 2021)

3.2.3 Josephson Effect (1962)

- for a constant voltage V across the junction:

$$\frac{\partial\varphi(t)}{\partial t} = \frac{2\pi}{\Phi_0}V = \frac{q_s V}{\hbar}$$

integration yields: $\varphi(t) = \varphi_0 + \frac{2\pi}{\Phi_0}V \cdot t = \varphi_0 + \frac{q_s}{\hbar}V \cdot t$

phase difference increases linearly in time

- supercurrent density J_s oscillates at the Josephson frequency $\nu = V/\Phi_0$:

$$J_s(\varphi(t)) = J_c \sin\varphi(t) = J_c \sin\left(\frac{2\pi}{\Phi_0}V \cdot t\right)$$

$$\frac{\nu}{V} = \frac{\omega/2\pi}{V} = \frac{1}{\Phi_0} = 483.597\,9\ \frac{\mathrm{MHz}}{\mu\mathrm{V}}$$

➔ **Josephson junction = *voltage controlled oscillator***

- **applications:**
  - Josephson voltage standard
  - microwave sources
  - ….

3.2.3 Josephson Effect (1962)

Josephson coupling energy E_J: binding energy of two coupled superconductors

$$\frac{E_J}{A} = \int_0^{t_0} J_s\, V\, \mathrm{d}t = \int_0^{t_0} J_c \sin\varphi \left(\frac{\Phi_0}{2\pi}\frac{\partial\varphi}{\partial t}\right) \mathrm{d}t = \frac{\Phi_0 J_c}{2\pi}\int_0^{\varphi} \sin\varphi' \; \mathrm{d}\varphi'$$

with $\varphi(0) = 0$ and $\varphi(t_0) = \varphi$
$A =$ junction area

integration yields:

$$\boxed{\frac{E_J}{A} = \frac{\Phi_0 J_c}{2\pi}(1 - \cos\varphi)}$$

Josephson coupling energy (per junction area)

3.2 Summary

Macroscopic wave function ψ:

describes ensemble of a macroscopic number of superconducting electrons,
$|\psi|^2 = n_s$ is given by density of superconducting electrons

Current density in a superconductor:

$$\mathbf{J}_s(\mathbf{r},t) = \frac{q_s n_s(\mathbf{r},t)\hbar}{m_s}\left\{\nabla\theta(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t)\right\} = \frac{q_s n_s(\mathbf{r},t)\hbar}{m_s}\left\{\nabla\theta(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\mathbf{A}(\mathbf{r},t)\right\}$$

Gauge invariant phase gradient:

$$\gamma(\mathbf{r},t) = \nabla\theta(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t) = \nabla\theta(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\mathbf{A}(\mathbf{r},t)$$

Phenomenological London equations:

$$(1)\quad \frac{\partial}{\partial t}\left(\Lambda\mathbf{J}_s(\mathbf{r},t)\right) = \mathbf{E} \qquad (2)\quad \nabla\times(\Lambda\mathbf{J}_s) + \mathbf{B} = \mathbf{0} \qquad \Lambda = \frac{m_s}{q_s^2 n_s} = \mu_0\lambda_{\mathrm{L}}^2$$

Fluxoid quantization:

$$\oint_C \Lambda\mathbf{J}_s\cdot d\ell + \int_S \mathbf{B}\cdot\hat{\mathbf{n}}\; dS = n\cdot\frac{h}{q_s} = n\cdot\Phi_0$$

3.2 Summary

Josephson equations:

$$\mathbf{J}_s(\mathbf{r},t) = \mathbf{J}_c(\mathbf{r},t)\sin\varphi(\mathbf{r},t)$$

$$\frac{\partial\varphi(t)}{\partial t} = \frac{2\pi}{\Phi_0}V(t) = \frac{q_s V(t)}{\hbar} \qquad \frac{\omega/2\pi}{V} = \frac{1}{\Phi_0} = 483.597\,9\,\frac{\mathrm{MHz}}{\mu\mathrm{V}}$$

Josephson coupling energy:

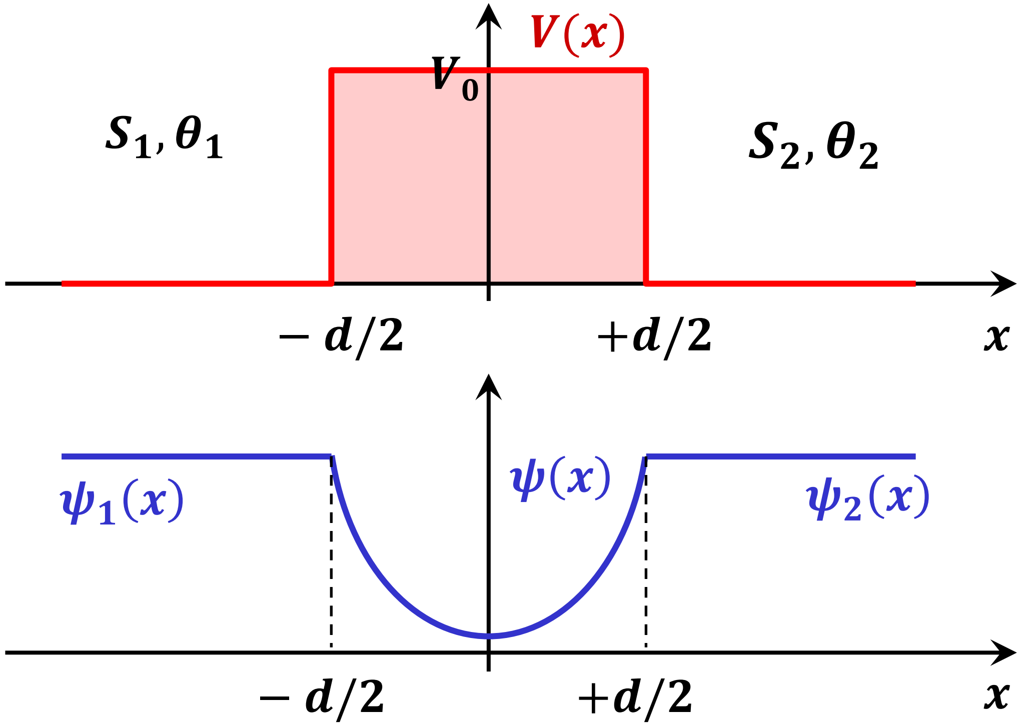
$$\frac{E_J}{A} = \frac{\Phi_0 J_c}{2\pi}(1-\cos\varphi)$$

maximum Josephson current density $\mathbf{J}_c$:

can be calculated by e.g. wave matching method

$$\mathbf{J}_c = -\frac{q_s\hbar\kappa}{m_s}\,2\sqrt{n_{s,1}n_{s,2}}\,\exp(-2\kappa d)$$

➔ more details later



Superconductivity and Low Temperature Physics I

Lecture No. 5

R. Gross

Summary of Lecture No. 4 (1)

- **derivation of 1st and 2nd London equation from current-phase and energy-phase relation**

$$\Lambda \mathbf{J}_s(\mathbf{r},t) = -\left\{\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q_s}\nabla\theta(\mathbf{r},t)\right\}$$

$$-\hbar\frac{\partial\theta(\mathbf{r},t)}{\partial t} = \frac{1}{2n_s}\Lambda \mathbf{J}_s^2(\mathbf{r},t) + q_s\phi_{\mathrm{el}}(\mathbf{r},t) + \mu(\mathbf{r},t)$$

2nd London equation:

$$\nabla \times \Lambda\mathbf{J}_s(\mathbf{r},t) + \nabla \times \mathbf{A}(\mathbf{r},t) = \nabla \times \left\{\frac{\hbar}{q_s}\nabla\theta(\mathbf{r},t)\right\} = 0$$

Meißner-Ochsenfeld effect

⇨ $\nabla \times (\Lambda\mathbf{J}_s) + \mathbf{B} = \mathbf{0}$ or $\nabla^2\mathbf{B} - \frac{1}{\lambda_\mathrm{L}^2}\mathbf{B} = \mathbf{0}$ with $\lambda_\mathrm{L} = \sqrt{\frac{m_s}{\mu_0 n_s q_s^2}}$ ***London penetration depth***

1st London equation

$$\frac{\partial}{\partial t}\left(\Lambda\mathbf{J}_s(\mathbf{r},t)\right) = -\left\{\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t} - \frac{\hbar}{q_s}\nabla\left(\frac{\partial\theta(\mathbf{r},t)}{\partial t}\right)\right\}$$

perfect conductivity

⇨ $\frac{\partial}{\partial t}\left(\Lambda\mathbf{J}_s(\mathbf{r},t)\right) = \mathbf{E} - \frac{1}{n_s q_s}\nabla\left(\frac{1}{2}\Lambda\mathbf{J}_s^2\right)$ or $\frac{\partial}{\partial t}\left(\Lambda\mathbf{J}_s(\mathbf{r},t)\right) = \mathbf{E}$ ***linearized 1st London equation***

London equations together with Maxwell equations describe behavior of superconductors on electromagnetic fields

- **current-phase and energy-phase relations are gauge invariant**

$$\mathbf{J}_s(\mathbf{r},t) = \frac{n_s q_s \hbar}{m_s}\underbrace{\left\{\nabla\theta'(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}'(\mathbf{r},t)\right\}}_{\text{gauge-invariant phase gradient}} = \frac{n_s q_s \hbar}{m_s}\left\{\nabla\theta(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t)\right\}$$

$$\mathbf{A}'(\mathbf{r},t) \Rightarrow \mathbf{A}(\mathbf{r},t) + \nabla\chi(\mathbf{r},t)$$

$$\phi'(\mathbf{r},t) \Rightarrow \phi(\mathbf{r},t) - \frac{\partial\chi(\mathbf{r},t)}{\partial t}$$

$$\nabla\theta'(\mathbf{r},t) \Rightarrow \nabla\theta(\mathbf{r},t) + \frac{q_s}{\hbar}\nabla\chi(\mathbf{r},t)$$

$$\psi'(\mathbf{r},t) \Rightarrow \psi(\mathbf{r},t)\, \mathrm{e}^{\imath(q_s/\hbar)\chi(\mathbf{r},t)}$$

Summary of Lecture No. 4 (2)

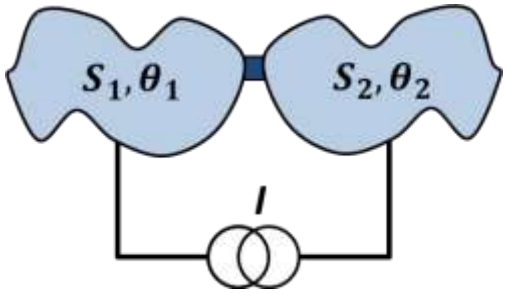
- **derivation of fluxoid quantization from current-phase relation** $\Lambda\mathbf{J}_s(\mathbf{r},t) = -\left\{\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q_s}\nabla\theta(\mathbf{r},t)\right\}$

$$\oint_C \Lambda\mathbf{J}_s\cdot d\ell + \oint_C \mathbf{A}\cdot d\ell = \frac{\hbar}{q_s}\oint_C \nabla\theta(\mathbf{r},t)\cdot d\ell \xrightarrow[\textit{Stoke's theorem}]{} \underbrace{\oint_C \Lambda\mathbf{J}_s\cdot d\ell + \int_S \mathbf{B}\cdot\hat{\mathbf{n}}\; dS}_{\text{fluxoid}} = n\cdot\frac{h}{q_s} = n\cdot\Phi_0$$

flux quantum: $\Phi_0 = h/|q_s| = h/2e = 2.067\,833\,831(13)\times 10^{-15}$ Vs

- **Josephson effects (weakly coupled superconductors)**

replace gauge invariant phase gradient γ by ***gauge invariant phase difference*** $\boldsymbol{\varphi}$:



$$\varphi(\mathbf{r},t) = \int_1^2 \gamma(\mathbf{r},t)\cdot\, d\ell \; = \int_1^2 \left(\nabla\theta(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t)\right)\cdot d\ell = \theta_2(\mathbf{r},t) - \theta_1(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot d\ell$$

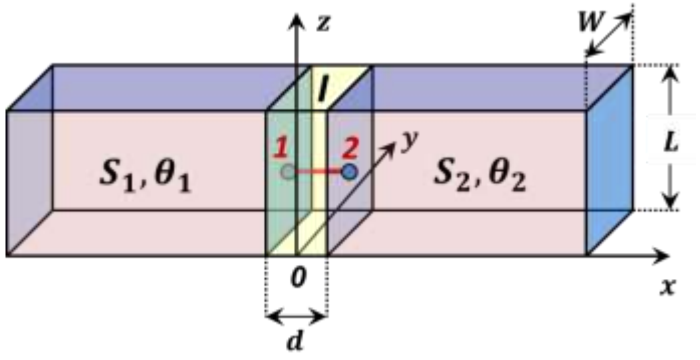
Josephson equations:

$$J_s(\varphi) = J_c\sin\varphi + \sum_{m=2}^{\infty} J_{c,m}\sin(m\varphi)$$

1st Josephson equation:
current – phase relation

$$\frac{\partial\varphi(t)}{\partial t} = \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{E}(t)\cdot d\ell \; = \frac{2\pi}{\Phi_0}V(t)$$

2nd Josephson equation:
voltage – phase relation



Summary of Lecture No. 4 (3)

- **Josephson coupling energy (binding energy of two coupled superconductors)**

$$\frac{E_J}{A} = \int_0^{t_0} J_s\, V\, \mathrm{d}t = \int_0^{t_0} J_c \sin\varphi \left(\frac{\Phi_0}{2\pi}\frac{\partial\varphi}{\partial t}\right) \mathrm{d}t = \frac{\Phi_0 J_c}{2\pi}\int_0^{\varphi} \sin\varphi' \;\mathrm{d}\varphi' \xrightarrow[\textit{integration}]{} \boxed{\frac{E_J}{A} = \frac{\Phi_0 J_c}{2\pi}(1-\cos\varphi)}$$

Josephson coupling energy
(per junction area)

- **Josephson junction biased by constant voltrage**

$$\frac{\partial\varphi(t)}{\partial t} = \frac{2\pi}{\Phi_0}V = \frac{q_s V}{\hbar} \xrightarrow[\textit{integration}]{} \varphi(t) = \varphi_0 + \frac{2\pi}{\Phi_0}V\cdot t = \varphi_0 + \frac{q_s}{\hbar}V\cdot t$$

$$\boxed{J_s(\varphi(t)) = J_c \sin\varphi(t) = J_c \sin\left(\frac{2\pi}{\Phi_0}V\cdot t\right)}$$

J_s oscillates at frequency ν: $\frac{\nu}{V} = \frac{\omega/2\pi}{V} = \frac{1}{\Phi_0} = 483.597\,9\,\frac{\mathrm{MHz}}{\mu\mathrm{V}}$

Josephson junction = *voltage controlled oscillator*

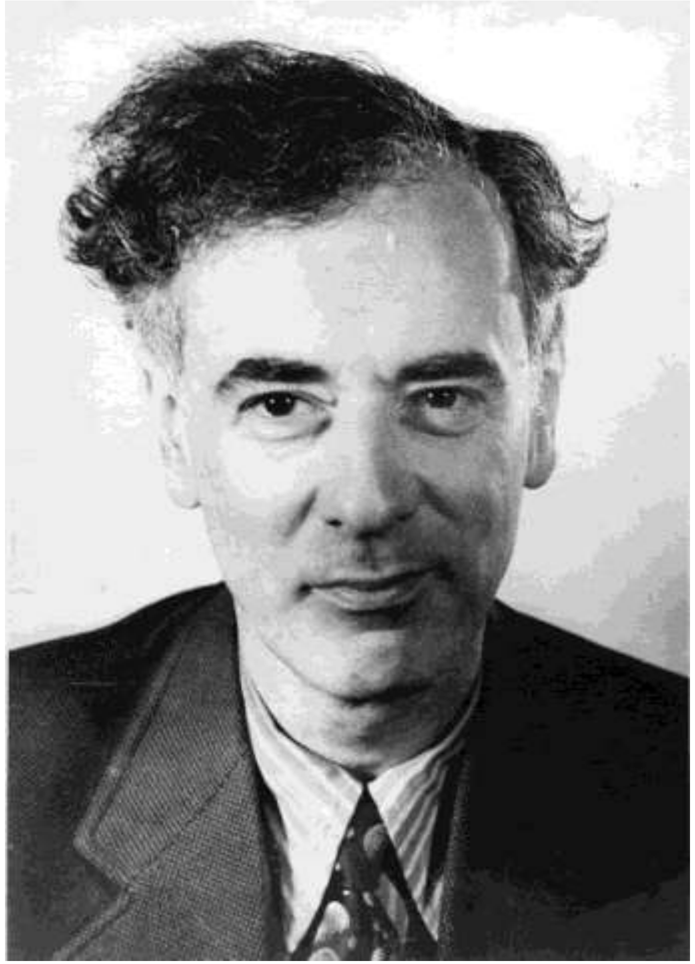
Chapter 3

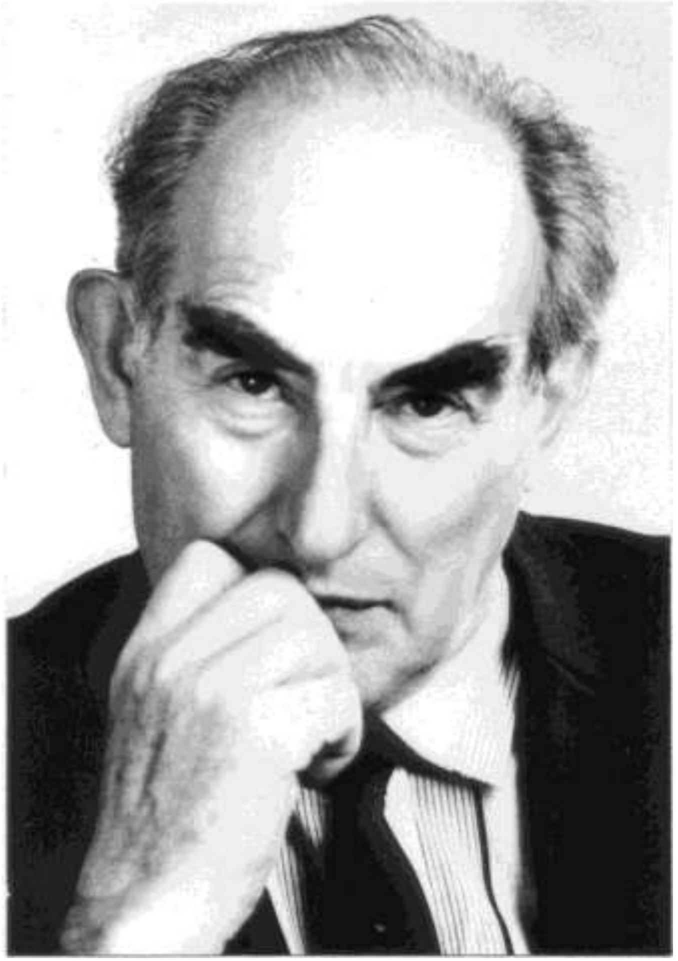
3. Phenomenological Models of Superconductivity

3.3 Ginzburg-Landau Theory

Lev Landau

Nobel Prize 1962





Vitaly Ginzburg

Nobel Prize 2003

- V.L. Ginzburg and L.D. Landau, Zh. Eksp. Teor. Fiz. 20, 1064 (1950). English translation in: L. D. Landau, Collected papers (Oxford: Pergamon Press, 1965) p. 546
- A.A. Abrikosov, Zh. Eksp. Teor. Fiz. 32, 1442 (1957). English translation: Sov. Phys. JETP 5 1174 (1957)
- L.P. Gor'kov, Sov. Phys. JETP 36, 1364 (1959)

3.3 Ginzburg-Landau Theory

- **London theory**: suitable for situations with spatially homogeneous $n_s(\mathbf{r}) = const.$
  → how to treat spatially inhomogeneous systems?

  example: step-like change of wave function at surfaces and interfaces
  → associated with large energy
  → gradual change on characteristic length scale expected

- **Vitaly Lasarevich Ginzburg** and **Lew Davidovich Landau** (1950)

  → **phenomenological description** of superconductor by (based on extension of Landau theory of phase transitions)

  - ***complex, spatially varying order parameter*** $\boldsymbol{\Psi}(\mathbf{r}) = |\boldsymbol{\Psi}(\mathbf{r})|\, \mathbf{e}^{i\theta(\mathbf{r})}$ ***(pair field)***
    with $|\boldsymbol{\Psi}(\mathbf{r})|^2 = \boldsymbol{n_s}(\mathbf{r})$
    $n_s(\mathbf{r})$ = density of superconducting electrons (note that $|\Psi(\mathbf{r})|^2 = n_s(\mathbf{r})/2$, if $|\Psi(\mathbf{r})|^2$ = pair density)
  - ***no time dependence*** (→ GL approach cannot be used to describe Josephson effects)

- **Alexei Alexeyevich Abrikosov** (1957)
  - prediction of flux line lattice for type-II superconductors

- **Lev Petrovich Gor'kov** (1959)
  - Ginzburg-Landau (GL) theory can be inferred from BCS theory for $T \approx T_c$

    → **Ginzburg-Landau- Abrikosov-Gor'kov** (GLAG) theory

3.3 Ginzburg-Landau Theory

A: *Spatially homogeneous superconductor in zero magnetic field*

$$|\Psi(\mathbf{r})|^2 = |\Psi_0(\mathbf{r})|^2 = n_s(\mathbf{r}) = const.$$

describe transition into superconducting state as a phase transition using the complex order parameter $\Psi(\mathrm{r}) = |\Psi_0|\,\mathrm{e}^{i\theta} = const.$

- develop free enthalpy density g_s of superconductor into a power series of $|\Psi|^2$

$$g_s = g_n + \alpha|\Psi|^2 + \frac{1}{2}\beta|\Psi|^4 + \cdots$$

free entalpy density of normal state

higher order terms can be neglected for $T \sim T_c$ as Ψ is very small

- discussion of coefficients α and β:
  - α must change sign at phase transition

    → $T > T_c$: $\alpha > 0$, since $g_s > g_n$
    → $T < T_c$: $\alpha < 0$, since $g_s < g_n$

  - $\beta > 0$, as $\beta < 0$ would always results in $g_s < g_n$ for large $|\Psi|$

    → minimum of g_s always for $|\Psi| \to \infty$

Ansatz:
$\alpha(T) = \bar{\alpha}\left(\frac{T}{T_c} - 1\right) = -\bar{\alpha}\left(1 - \frac{T}{T_c}\right)$ with $\bar{\alpha} > 0$

Ansatz:
$\beta(T) = const. > 0$

3.3 Ginzburg-Landau Theory

A: *Spatially homogeneous superconductor in zero magnetic field*

- the enthalpy density g_s must be minimum in thermal equilibrium

$$\frac{\partial g_s}{\partial|\Psi|} = 0 = 2\alpha(T)|\Psi| + 2\beta|\Psi|^3 + \cdots \;\Rightarrow\; |\Psi_0(T)|^2 = -\frac{\alpha(T)}{\beta}$$

order parameter in thermal equilibrium

$$\alpha(T) = -\bar{\alpha}\left(1 - \frac{T}{T_c}\right)$$

$$\Rightarrow \quad n_s(T) = |\Psi_0(T)|^2 = -\frac{\alpha(T)}{\beta} = \frac{\bar{\alpha}}{\beta}\left(1 - \frac{T}{T_c}\right)$$

describes homogeneous equilibrium state at $T \leq T_c$

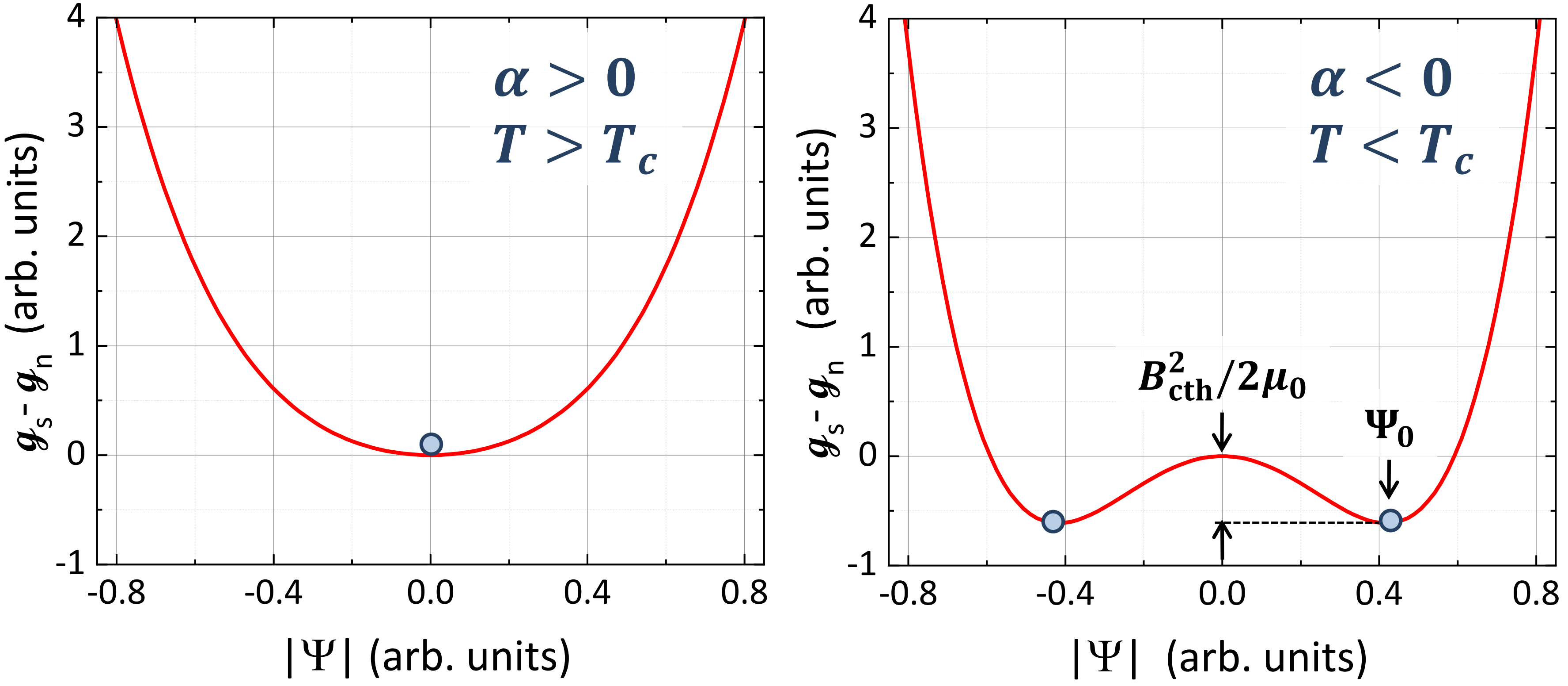
- physical meaning of coefficients α and β

$$g_s - g_n = \underbrace{-\frac{B_{\mathrm{cth}}^2(T)}{2\mu_0}}_{\textbf{condensation energy}} = \alpha(T)|\Psi_0(\mathrm{T})|^2 + \frac{1}{2}\beta|\Psi_0(T)|^4 + \cdots = -\frac{1}{2}\frac{\alpha^2(T)}{\beta} = -\frac{\bar{\alpha}^2}{2\beta}\left(1 - \frac{T}{T_c}\right)^2 = -\frac{n_s(0)}{2}\,\bar{\alpha}\left(1 - \frac{T}{T_c}\right)^2$$

➔ $-\frac{\bar{\alpha}}{2} = -\left[\frac{B_{\mathrm{cth}}^2(0)}{2\mu_0}\right]/n_s(0)$ corresponds to condensation energy per charge carrier at $T = 0$

➔ $\beta = \left[\frac{B_{\mathrm{cth}}^2(T)}{2\mu_0}\right]\frac{2}{n_s^2(T)} \simeq const.$ as B_{cth} and n_s have similar T-dependence close to T_c

3.3 Ginzburg-Landau Theory



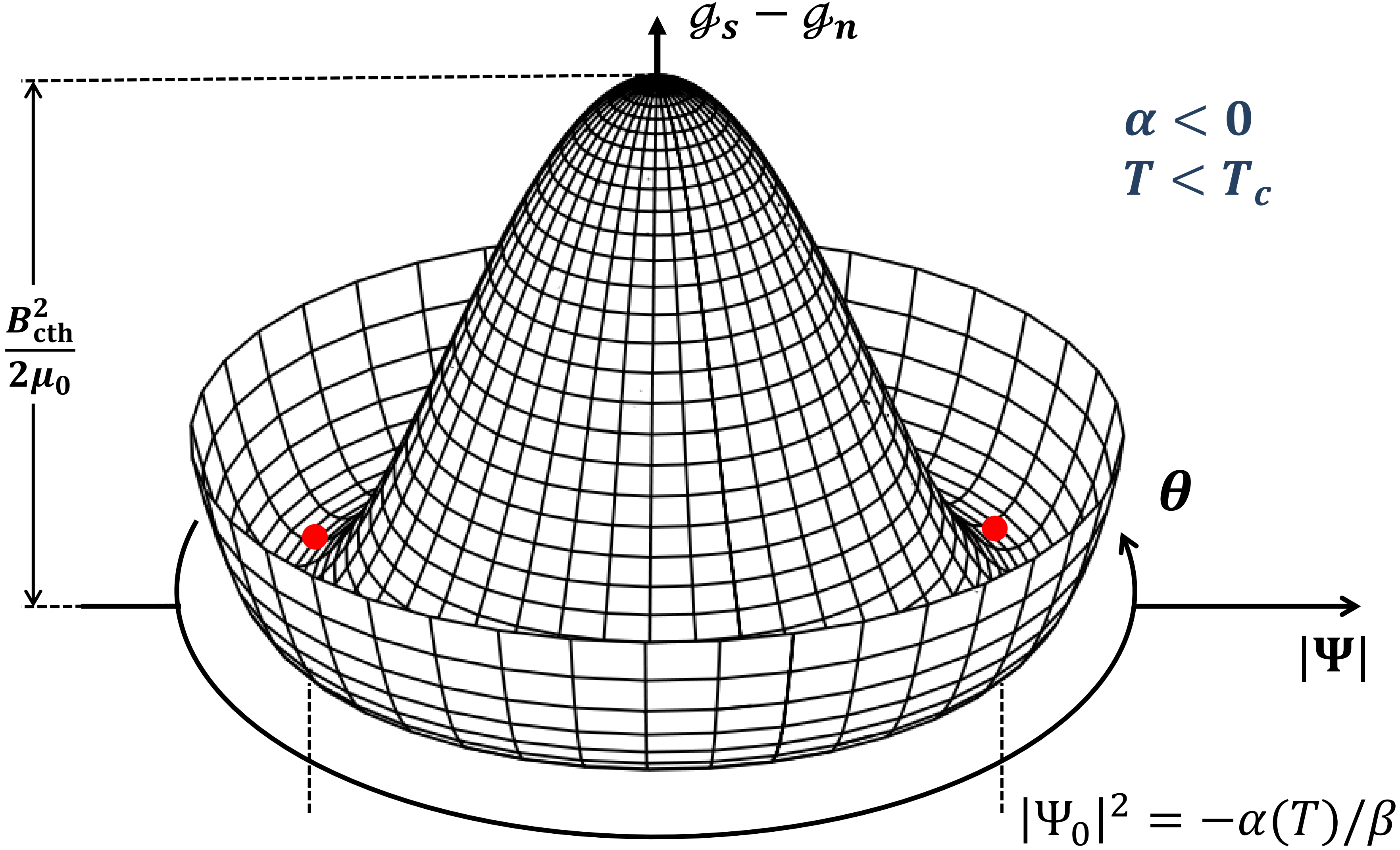
$$g_s - g_n = \alpha(T)|\Psi_0(\mathrm{T})|^2 + \frac{1}{2}\beta|\Psi_0(T)|^4 + \cdots$$

Note:

- only the amplitude $|\Psi|$ is important for finding the minimum and the phase can be chosen arbitrarily
- this changes when $B \neq 0$ and $J_s \neq 0$

3.3 Ginzburg-Landau Theory

- complex order parameter $\Psi(\mathbf{r}) = |\Psi_0(\mathbf{r})|\,\mathrm{e}^{i\theta(\mathbf{r})}$



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3.3 Ginzburg-Landau Theory

- temperature dependence of $\Delta g(T) = g_n(T) - g_s(T)$

$$\Delta g(T) = g_n(T) - g_s(T) = \frac{\bar{\alpha}^2}{2\beta}\left(1 - \frac{T}{T_c}\right)^2 = \frac{n_s(0)}{2}\,\bar{\alpha}\left(1 - \frac{T}{T_c}\right)^2 = \frac{B_{\mathrm{c,GL}}^2(0)}{2\mu_0}\left(1 - \frac{T}{T_c}\right)^2$$

- experimental observation

$$\Delta g(T) = g_n(T) - g_s(T) = \frac{B_{\mathrm{cth}}^2(0)}{2\mu_0}\left[1 - \left(\frac{T}{T_c}\right)^2\right]^2$$

➔ experimental observed temperature dependence does not agree with GLAG prediction, since **GLAG theory is only valid close to T_c**

for $T \simeq T_c$:

$$\Delta g(T) = g_n - g_s(T) = \frac{B_{\mathrm{cth}}^2(0)}{2\mu_0}\left[1 - \left(\frac{T}{T_c}\right)^2\right]^2 \underset{\uparrow}{\simeq} \frac{B_{\mathrm{cth}}^2(0)}{2\mu_0}\left[2\left(1 - \frac{T}{T_c}\right)\right]^2 = \frac{4B_{\mathrm{cth}}^2(0)}{2\mu_0}\left[1 - \frac{T}{T_c}\right]^2$$

$$1 - \left(\frac{T}{T_c}\right)^2 = \left[1 - \frac{T}{T_c}\right]\cdot\left[1 + \frac{T}{T_c}\right] \simeq 2\left[1 - \frac{T}{T_c}\right]$$

➔ good agreement for $T \simeq T_c$ with $B_{\mathrm{c,GL}}(0) = 2B_{\mathrm{cth}}(0)$

3.3 Ginzburg-Landau Theory

entropy density and specific heat for the spatially homogeneous case:

$$g_s(T) = g_n(T) + \alpha(T)|\Psi(T)|^2 + \frac{1}{2}\beta|\Psi(T)|^4$$

$$|\Psi(T)|^2 = -\alpha(T)/\beta$$

$$g_s(T) = g_n(T) - \frac{1}{2}\,\bar{\alpha}\,n_s(0)\left(1 - \frac{T}{T_c}\right)^2$$

$$\alpha(T) = -\bar{\alpha}\left(1 - \frac{T}{T_c}\right)$$

- entropy density $s_{n,s} = -\left(\frac{\partial g_{n,s}}{\partial T}\right)_{B_{\text{ext}},p}$

$$s_s(T) = s_n(T) - \frac{\bar{\alpha}\,n_s(0)}{T_c}\left(1 - \frac{T}{T_c}\right)$$

- specific heat $c_{p,ns} = T\left(\frac{\partial s_{n,s}}{\partial T}\right)_{B_{ext},p}$

$$c_{p,s}(T) = c_{p,n}(T) + \frac{\bar{\alpha}\,n_s(0)}{T_c^2}\,T$$

for $T \rightarrow T_c$: $\Delta c_p = c_{p,s}(T_c) - c_{p,n}(T_c) = \frac{\bar{\alpha}\,n_s(0)}{T_c}$

3.3 Ginzburg-Landau Theory

comparison to BCS result (derived later)

- BCS prediction for specific heat jump at T_c: $$\frac{\Delta c_p(T=T_c)}{c_{n,p}} = 1.43$$
- GLAG result for specific heat jump at T_c: $$\frac{\Delta c_p(T=T_c)}{c_{n,p}} = \frac{\bar{\alpha}\, n_s(0)}{c_{n,p}\; T_c}$$

with $c_{n,p}(T=T_c) = \frac{\pi^2}{3}\frac{D(E_\mathrm{F})}{V}\, k_\mathrm{B}^2 T_c$ we obtain by using BCS result $\frac{\Delta(0)}{k_\mathrm{B}T_c} = 1.764$

$$\frac{\Delta c_p}{c_{n,p}} = \frac{\bar{\alpha}\, n_s(0)}{\frac{\pi^2}{3}\frac{D(E_\mathrm{F})}{V}\, k_\mathrm{B}^2 T_c^2} = \frac{3\cdot 1.764^2}{\pi^2}\;\frac{\bar{\alpha}\, n_s(0)}{\frac{1}{4}\frac{D(E_\mathrm{F})}{V}\,\Delta^2(0)}$$

$\frac{1}{4}\frac{D(E_\mathrm{F})}{V}\,\Delta^2(0)$: BCS condensation energy density

➔ GLAG result agrees with the BCS prediction, if $\frac{\bar{\alpha}\, n_s(0)}{\frac{1}{4}\frac{D(E_\mathrm{F})}{V}\,\Delta^2(0)} = 1.51$ or $\frac{\bar{\alpha}\, n_s(0)/2}{\frac{1}{4}\frac{D(E_\mathrm{F})}{V}\,\Delta^2(0)} = \frac{1.51}{2}$

since $\frac{\bar{\alpha}}{2}\, n_s(0)$ is the GLAG condensation energy density, this is in good approximation the case

3.3 Ginzburg-Landau Theory

Ehrenfest relations for 2$^{\text{nd}}$ order phase transition (see e.g. textbook of Landau & Lifshitz)

$$\Delta\left(\frac{\mathrm{d}V}{\mathrm{d}T}\right) = \frac{\mathrm{d}V_2}{\mathrm{d}T} - \frac{\mathrm{d}V_1}{\mathrm{d}T} = 0 = \Delta\left(\frac{\mathrm{d}V}{\mathrm{d}T}\right)_p + \Delta\left(\frac{\mathrm{d}V}{\mathrm{d}p}\right)_T \frac{\mathrm{d}p}{\mathrm{d}T} \qquad \text{for } T = T_c$$

$$\Delta\left(\frac{\mathrm{d}\mathscr{s}}{\mathrm{d}T}\right) = \frac{\mathrm{d}\mathscr{s}_2}{\mathrm{d}T} - \frac{\mathrm{d}\mathscr{s}_1}{\mathrm{d}T} = 0 = \Delta\left(\frac{\mathrm{d}\mathscr{s}}{\mathrm{d}T}\right)_p + \Delta\left(\frac{\mathrm{d}\mathscr{s}}{\mathrm{d}p}\right)_T \frac{\mathrm{d}p}{\mathrm{d}T} = \Delta\left(\frac{\mathrm{d}\mathscr{s}}{\mathrm{d}T}\right)_p - \Delta\left(\frac{\mathrm{d}V}{\mathrm{d}T}\right)_p \frac{\mathrm{d}p}{\mathrm{d}T} \qquad \text{for } T = T_c$$

with Maxwell relation:
$$\left(\frac{\mathrm{d}S}{\mathrm{d}p}\right)_T = -\left(\frac{\mathrm{d}V}{\mathrm{d}T}\right)_p$$

- Ehrenfest relations connect the discontinuities in

specific heat: $\Delta c_p = T\left(\frac{\mathrm{d}\mathscr{s}}{\mathrm{d}T}\right)_p$

thermal expansion: $\Delta\alpha_p = \left(\frac{\mathrm{d}V}{\mathrm{d}T}\right)_p$

compressibility: $\Delta\kappa_T = \left(\frac{\mathrm{d}V}{\mathrm{d}p}\right)_T$

$$0 = \Delta\left(\frac{\mathrm{d}V}{\mathrm{d}T}\right)_p + \Delta\left(\frac{\mathrm{d}V}{\mathrm{d}p}\right)_T \frac{\mathrm{d}p}{\mathrm{d}T} \Rightarrow \boxed{\Delta\alpha_p\Big|_{T_c} = -\frac{\mathrm{d}p}{\mathrm{d}T}\Big|_{T_c} \Delta\kappa_T\Big|_{T_c}}$$

$$0 = \Delta\left(\frac{\mathrm{d}\mathscr{s}}{\mathrm{d}T}\right)_p - \Delta\left(\frac{\mathrm{d}V}{\mathrm{d}T}\right)_p \frac{\mathrm{d}p}{\mathrm{d}T} \Rightarrow \boxed{\frac{\Delta c_p}{T_c}\Big|_{T_c} = -\frac{\mathrm{d}p}{\mathrm{d}T}\Big|_{T_c} \Delta\alpha_p\Big|_{T_c}}$$

since $\frac{\Delta c_p}{T_c}$ and $\Delta\alpha_p\big|_{T_c}$ are experimentally accessible, we can determine the pressure dependence of T_c

3.3 Ginzburg-Landau Theory

B: *Spatially inhomogeneous superconductor in external magnetic field* $\mathbf{B}_{\text{ext}} = \mu_0 \mathbf{H}_{\text{ext}}$

- as soon as there are finite currents and fields, we have to take into account the **kinetic energy of the superelectrons** and the **field energy**; furthermore, spatial variations of order parameter increase energy: **stiffness**

- kinetic energy density $$\frac{1}{2} n_s m_s v_s^2 = \frac{1}{2} |\Psi(\mathbf{r})|^2 m_s \left(\frac{\hbar}{m_s} \boldsymbol{\nabla} \theta(\mathbf{r}, t) - \frac{q_s}{m_s} \mathbf{A}(\mathbf{r}) \right)^2$$

$$\mathbf{v}_s(\mathbf{r}) = \frac{\hbar}{m_s} \boldsymbol{\nabla} \theta(\mathbf{r}) - \frac{q_s}{m_s} \mathbf{A}(\mathbf{r})$$
$$n_s = |\Psi(\mathbf{r})|^2$$

- stiffness energy of OP $$n_s \frac{\hbar^2 k^2}{2 m_s} = |\Psi(\mathbf{r})|^2 \frac{\hbar^2 (\boldsymbol{\nabla} |\Psi| / |\Psi|)^2}{2 m_s} = \frac{\hbar^2 (\boldsymbol{\nabla} |\Psi|)^2}{2 m_s}$$

- field energy density $$\frac{[\mathbf{b}(\mathbf{r}) - \mathbf{B}_{\text{ext}}]^2}{2\mu_0}$$

$\frac{[\mathbf{b}(\mathbf{r}) - \mathbf{B}_{\text{ext}}]^2}{2\mu_0} = \frac{1}{2} \mu_0 \mathbf{M}^2(\mathbf{r})$ inside SC where $\mathbf{b}(\mathbf{r}) = \mathbf{B}_{\text{ext}} + \mu_0 \mathbf{M}(\mathbf{r})$

- ➢ $\mathbf{b}(\mathbf{r})$ is the local flux density, $\mathbf{B}_{\text{ext}}$ the spatially homogeneous applied flux density
- ➢ in the Meißner state: $\mathbf{b}(\mathbf{r}) = \mathbf{B}_{\text{ext}} + \mu_0 \mathbf{M}(\mathbf{r}) = \mathbf{0}$ inside the superconductor and the integral over the sample volume just gives the additional field expulsion work
- ➢ in normal state: $\mathbf{b}(\mathbf{r}) = \mathbf{B}_{\text{ext}} + \mu_0 \mathbf{M}(\mathbf{r}) = \mathbf{B}_{\text{ext}}$ as $\mathbf{M}(\mathbf{r}) = \mathbf{0}$ and there is no extra energy contribution

3.3 Ginzburg-Landau Theory

B: *Spatially inhomogeneous superconductor in external magnetic field* $\mathbf{B}_{\text{ext}} = \mu_0 \mathbf{H}_{\text{ext}}$

- sum of kinetic energy and stiffness energy

$$\frac{1}{2}|\Psi(\mathbf{r})|^2 m_s \left(\frac{\hbar}{m_s}\nabla\theta(\mathbf{r},t) - \frac{q_s}{m_s}\mathbf{A}(\mathbf{r})\right)^2 + \frac{\hbar^2(\nabla|\Psi|)^2}{2m_s} = \frac{1}{2m_s}\left|\frac{\hbar}{\imath}\nabla\Psi(\mathbf{r}) - q_s\,\mathbf{A}(\mathbf{r})\Psi(\mathbf{r})\right|^2$$

- additional contribution in free enthalpy density

$$\frac{1}{2m_s}\left|\frac{\hbar}{\imath}\nabla\Psi(\mathbf{r}) - q_s\,\mathbf{A}(\mathbf{r})\Psi(\mathbf{r})\right|^2$$

3.3 Ginzburg-Landau Theory

B: *Spatially inhomogeneous superconductor in external magnetic field* $\mathbf{B}_{\text{ext}} = \mu_0 \mathbf{H}_{\text{ext}}$

- additional terms in free enthalpy density for finite $\mathbf{J}_s$ and $\mathbf{B}_{\text{ext}} = \mu_0 \mathbf{H}_{\text{ext}}$

$$g_s = g_n + \alpha|\Psi|^2 + \frac{1}{2}\beta|\Psi|^4 + \cdots + \underbrace{\frac{[\mathbf{b}(\mathbf{r}) - \mathbf{B}_{\text{ext}}]^2}{2\mu_0}} + \underbrace{\frac{1}{2m_s}\left|\frac{\hbar}{\imath}\nabla\Psi(\mathbf{r}) - q_s\,\mathbf{A}(\mathbf{r})\Psi(\mathbf{r})\right|^2}$$

additional field energy density:
e.g. due to work required for
field expulsion
$\propto (\mathbf{b}(\mathbf{r}) - \mathbf{B}_{\text{ext}})^2$

***kinetic energy of the supercurrents**:*
finite gauge invariant phase gradient results in
supercurrent density and increase in kinetic energy

finite stiffness of order parameter:
→ *spatial variations of $|\Psi|$ cost additional energy*

with $\Psi(\mathrm{r}) = |\Psi(\mathrm{r})|\, \mathrm{e}^{i\theta(\mathrm{r})}$ → $$\left[\underbrace{\frac{\hbar^{\mathbf{2}}(\nabla|\Psi|)^2}{2m_s}}_{\text{gradient of amplitude}} + \underbrace{\frac{1}{2}m_s\Bigg(\overbrace{\frac{\hbar}{m_s}\nabla\theta - \frac{q_s}{m_s}\mathbf{A}}^{\mathbf{v}_s}\Bigg)^2|\Psi|^2}_{\text{gradient of phase}}\right]$$

3.3 Ginzburg-Landau Theory

- minimization of free enthalpy $\mathcal{G}_s$:

→ integration of enthalpy density $\mathscr{g}_s$ over whole volume V of superconductor

$$\mathcal{G}_s = \mathcal{G}_n + \int\limits_{\text{sample}} \left\{\alpha|\Psi|^2 + \frac{1}{2}\beta|\Psi|^4 + \cdots + \frac{1}{2m_s}\left|\frac{\hbar}{\imath}\boldsymbol{\nabla}\Psi(\mathbf{r}) - q_s\,\mathbf{A}(\mathbf{r})\Psi(\mathbf{r})\right|^2\right\}\mathrm{d}^3r + \frac{1}{2\mu_0}\iiint\limits_{-\infty}^{\infty}[\mathbf{b}(\mathbf{r}) - \mathbf{B}_{\text{ext}}]^2\mathrm{d}^3r$$

variational calculation:

$$\delta\mathcal{G}_s = \left(\frac{\partial\mathcal{G}_s}{\partial\Psi}\right)\delta\Psi + \left(\frac{\partial\mathcal{G}_s}{\partial\Psi^\star}\right)\delta\Psi^\star = 0$$

$$\delta\mathcal{G}_s = \left(\frac{\partial\mathcal{G}_s}{\partial\mathbf{A}}\right)\delta\mathbf{A} = 0$$

a lot of math

3.3 Ginzburg-Landau Theory

- rewriting the kinetic energy/stiffness contribution using the Gauss (divergence) theorem

$$\mathcal{G}_s = \mathcal{G}_n + \int\limits_{\text{sample}} \left\{ \alpha|\Psi|^2 + \frac{1}{2}\beta|\Psi|^4 + \cdots + \frac{1}{2m_s}\left|\frac{\hbar}{\imath}\nabla\Psi(\mathbf{r}) - q_s\,\mathbf{A}(\mathbf{r})\Psi(\mathbf{r})\right|^2 \right\} \mathrm{d}^3r + \frac{1}{2\mu_0}\iiint\limits_{-\infty}^{\infty} [\mathbf{b}(\mathbf{r}) - \mathbf{B}_{\mathrm{ext}}]^2 \mathrm{d}^3r$$

- Gauss theorem: $\iiint_V [\mathbf{F}\cdot(\nabla g) + g\,(\nabla\cdot\mathbf{F})]\,\mathrm{d}V = \oiint_S g\mathbf{F}\cdot\mathbf{n}\mathrm{d}S$

$$\frac{\hbar^2}{2m_s}\int\limits_{\text{sample}} \left|\nabla\Psi(\mathbf{r}) + \frac{q_s}{\imath\hbar}\,\mathbf{A}(\mathbf{r})\Psi(\mathbf{r})\right|^2 \mathrm{d}^3r$$

$$= \frac{1}{2m_s}\int\limits_{\text{sample}} \Psi^\star(\mathbf{r})\left[\frac{\hbar}{\imath}\nabla - q_s\mathbf{A}(\mathbf{r})\right]^2 \Psi(\mathbf{r})\,\mathrm{d}^3r + \underbrace{\frac{\imath\hbar}{2m_s}\iint\limits_{\text{surface}} \left[\Psi^\star(\mathbf{r})\overbrace{\left(\frac{\hbar}{\imath}\nabla - q_s\mathbf{A}(\mathbf{r})\right)}^{\mathbf{v}_s}\Psi(\mathbf{r})\right]\cdot \overset{\text{surface normal}}{\hat{\mathbf{n}}}\,\mathrm{d}S}_{}$$

takes into account currents flowing through the sample surface

→ vanishes, if there is no current density flowing through surface of superconductor

3.3 Ginzburg-Landau Theory

- minimization of $\mathcal{G}_s$ with respect to variations $\delta\Psi$, $\delta\Psi^\star$ (field term has not to be considered)

$$\delta\mathcal{G}_s = \left(\frac{\partial\mathcal{G}_s}{\partial\Psi}\right)\delta\Psi + \left(\frac{\partial\mathcal{G}_s}{\partial\Psi^\star}\right)\delta\Psi^\star = 0$$

$$\delta\mathcal{G}_s = \int\limits_{\text{sample}} \underbrace{\left\{\left[\alpha\Psi + \beta\Psi|\Psi|^2 + \cdots + \frac{1}{2m_s}\left(\frac{\hbar}{\imath}\boldsymbol{\nabla} - q_s\,\mathbf{A}(\mathbf{r})\right)^2\Psi\right]\delta\Psi^\star + c.c.\right\}}_{=0} \mathrm{d}^3r + \underbrace{\frac{\imath\hbar}{2m_s}\iint\limits_{\text{surface}}\left[\left(\frac{\hbar}{\imath}\boldsymbol{\nabla} - q_s\mathbf{A}(\mathbf{r})\right)\Psi(\mathbf{r})\delta\Psi^\star + c.c.\right]\cdot\hat{\mathbf{n}}\,\mathrm{d}S}$$

since equation must be satisfied for all $\delta\Psi$, $\delta\Psi^\star$

$$\frac{1}{2m_s}\left(\frac{\hbar}{\imath}\boldsymbol{\nabla} - q_s\,\mathbf{A}(\mathbf{r})\right)^2\Psi(\mathbf{r}) + \alpha\Psi(\mathbf{r}) + \frac{1}{2}\beta|\Psi(\mathbf{r})|^2\Psi(\mathbf{r}) = 0$$

1st Ginzburg-Landau equation

SC/insulator interface:

$$\left(\frac{\hbar}{\imath}\boldsymbol{\nabla}_{\hat{\mathbf{n}}} - q_s\mathbf{A}_{\hat{\mathbf{n}}}(\mathbf{r})\right)\Psi(\mathbf{r}) = 0$$

SC/metal interface:

$$\left(\frac{\hbar}{\imath}\boldsymbol{\nabla}_{\hat{\mathbf{n}}} - q_s\mathbf{A}_{\hat{\mathbf{n}}}(\mathbf{r})\right)\Psi(\mathbf{r}) = -\frac{\imath\hbar}{b}\Psi(\mathbf{r})$$

b = real constant

3.3 Ginzburg-Landau Theory

- minimization of $\mathcal{G}_s$ with respect to variation $\delta\mathbf{A}$

$$\delta\mathcal{G}_s = \left(\frac{\partial\mathcal{G}_s}{\partial\mathbf{A}}\right)\delta\mathbf{A} = 0$$

$$\mathcal{G}_s = \mathcal{G}_n + \int\limits_{\text{sample}} \left\{\alpha|\Psi|^2 + \frac{1}{2}\beta|\Psi|^4 + \cdots + \frac{1}{2m_s}\left|\frac{\hbar}{\imath}\nabla\Psi(\mathbf{r}) - q_s\,\mathbf{A}(\mathbf{r})\Psi(\mathbf{r})\right|^2\right\}\mathrm{d}^3r + \frac{1}{2\mu_0}\iiint\limits_{-\infty}^{\infty}[\mathbf{b}(\mathbf{r}) - \mathbf{B}_{\text{ext}}]^2\mathrm{d}^3r$$

- we first derive $\delta g_s(\mathbf{A}) = g_s(\mathbf{r},\mathbf{A}+\delta\mathbf{A}) - g_s(\mathbf{r},\mathbf{A})$ and then calculated $\delta\mathcal{G}_s = \int\delta g_s\mathrm{d}^3r$ (contains only $\mathbf{A}$-dependent part)

$$\delta g_s(\mathbf{A}) = \frac{1}{2\mu_0}\left([\nabla\times(\mathbf{A}+\delta\mathbf{A})]^2 - [\nabla\times\mathbf{A}]^2\right)$$

$$+\frac{1}{2m_s}\left(\left[\frac{\hbar}{\imath}\nabla - q_s\,(\mathbf{A}+\delta\boldsymbol{A})\right]\Psi\right)\left(\left[-\frac{\hbar}{\imath}\nabla - q_s\,(\mathbf{A}+\delta\boldsymbol{A})\right]\Psi^\star\right) - \frac{1}{2m_s}\left(\left[\frac{\hbar}{\imath}\nabla - q_s\mathbf{A}\right]\Psi\right)\left(\left[-\frac{\hbar}{\imath}\nabla - q_s\boldsymbol{A}\right]\Psi^\star\right)$$

$$\delta g_s(\mathbf{A}) = \frac{1}{\mu_0}(\nabla\times\delta\mathbf{A})\cdot(\nabla\times\mathbf{A})$$

$$+\frac{q_s}{2m_s}\left(\frac{\hbar}{\imath}\Psi^\star\nabla\Psi - \frac{\hbar}{\imath}\Psi\nabla\Psi^\star - 2q_s|\Psi|^2\mathbf{A}\right)\cdot\delta\boldsymbol{A}$$

neglecting terms in $\delta\boldsymbol{A}^2$

3.3 Ginzburg-Landau Theory

- integration of the contributions over the sample volume

$$\delta \mathcal{G}_s = \int\limits_{\text{sample}} \delta g_s \, \mathrm{d}^3 r = \int\limits_{\text{sample}} \left\{ \underbrace{\frac{1}{\mu_0} (\boldsymbol{\nabla} \times \delta \mathbf{A})(\boldsymbol{\nabla} \times \mathbf{A})} + \frac{q_s}{2m_s} \left(\frac{\hbar}{\imath} \Psi^\star \boldsymbol{\nabla} \Psi - \frac{\hbar}{\imath} \Psi \boldsymbol{\nabla} \Psi^\star - 2q_s |\Psi|^2 \mathbf{A} \right) \cdot \delta \boldsymbol{A} \right\} \mathrm{d}^3 r$$

$$\frac{1}{\mu_0} \int\limits_{\text{sample}} (\boldsymbol{\nabla} \times \delta \mathbf{A})(\boldsymbol{\nabla} \times \mathbf{A}) \, \mathrm{d}^3 r = \frac{1}{\mu_0} \int\limits_{\text{sample}} \boldsymbol{\nabla}^2 \mathbf{A} \cdot \delta \mathbf{A} \;\; \mathrm{d}^3 r$$

$$\delta \mathcal{G}_s = \int\limits_{\text{sample}} \left\{ \left[\frac{q_s}{2m_s} \left(\frac{\hbar}{\imath} \Psi^\star \boldsymbol{\nabla} \Psi - \frac{\hbar}{\imath} \Psi \boldsymbol{\nabla} \Psi^\star \right) - \frac{q_s^2}{m_s} |\Psi|^2 \mathbf{A} + \frac{1}{\mu_0} \boldsymbol{\nabla}^2 \mathbf{A} \right] \cdot \delta \mathbf{A} \right\} \mathrm{d}^3 r = 0$$

- rewriting of term $\frac{1}{\mu_0} \boldsymbol{\nabla}^2 \mathbf{A}$ making use of Maxwell's equation $\mu_0 \mathbf{J}_s = \boldsymbol{\nabla} \times \mathbf{B}$ and London gauge $\boldsymbol{\nabla} \cdot \mathbf{A} = \mathbf{0}$

$$\mu_0 \mathbf{J}_s = \boldsymbol{\nabla} \times \mathbf{B} = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \boldsymbol{\nabla}^2 \mathbf{A} = -\boldsymbol{\nabla}^2 \mathbf{A} \quad \Rightarrow \quad \frac{1}{\mu_0} \boldsymbol{\nabla}^2 \mathbf{A} = -\mathbf{J}_s$$

$$\delta \mathcal{G}_s = \int\limits_{\text{sample}} \left\{ \underbrace{\left[\frac{q_s}{2m_s} \left(\frac{\hbar}{\imath} \Psi^\star \boldsymbol{\nabla} \Psi - \frac{\hbar}{\imath} \Psi \boldsymbol{\nabla} \Psi^\star \right) - \frac{q_s^2}{m_s} |\Psi|^2 \mathbf{A} - \mathbf{J}_s \right]}_{= 0, \text{ since equation must be satisfied for all } \delta \mathbf{A}} \cdot \delta \mathbf{A} \right\} \mathrm{d}^3 r = 0$$

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3.3 Ginzburg-Landau Theory

- minimization of $\mathcal{G}_s$ with respect to variation $\delta\mathbf{A}$ results in

$$\frac{q_s}{2m_s}\left(\frac{\hbar}{\imath}\Psi^\star\nabla\Psi - \frac{\hbar}{\imath}\Psi\nabla\Psi^\star\right) - \frac{q_s^2}{m_s}|\Psi|^2\mathbf{A} - \mathbf{J}_s = 0$$

$$\Rightarrow \boxed{\mathbf{J}_s = \frac{q_s\hbar}{2m_s\imath}(\Psi^\star\nabla\Psi - \Psi\nabla\Psi^\star) - \frac{q_s^2}{m_s}|\Psi|^2\mathbf{A}}$$

2nd Ginzburg-Landau equation

- **Summary:**
  minimization of $\mathcal{G}_s$ with respect to variation $\delta\Psi$, $\delta\Psi^\star$ and $\delta\mathbf{A}$ results in two differential equations

$$\frac{1}{2m_s}\left(\frac{\hbar}{\imath}\nabla - q_s\,\mathbf{A}(\mathbf{r})\right)^2\Psi(\mathbf{r}) + \alpha\Psi(\mathbf{r}) + \frac{1}{2}\beta|\Psi(\mathbf{r})|^2\Psi(\mathbf{r}) = 0$$

1st Ginzburg-Landau equation

$$\mathbf{J}_s = \frac{q_s\hbar}{2m_s\imath}(\Psi^\star\nabla\Psi - \Psi\nabla\Psi^\star) - \frac{q_s^2}{m_s}|\Psi|^2\mathbf{A}$$

2nd Ginzburg-Landau equation

3.3 GL-Theory vs. Macroscopic Quantum Model

comparison of the results provided by GLAG theory and the macroscopic quantum model

macroscopic quantum model

i. current-phase relation

$$\mathbf{J}_s(\mathbf{r},t) = q_s n_s(\mathbf{r},t)\left\{\frac{\hbar}{m_s}\nabla\theta(\mathbf{r},t) - \frac{q_s}{m_s}\mathbf{A}(\mathbf{r},t)\right\}$$

assumption: $|\psi(\mathbf{r})|^2 = n_s(\mathbf{r}) = const.$

ii. energy-phase relation

$$\hbar\frac{\partial\theta(\mathbf{r},t)}{\partial t} = -\left\{\frac{1}{2n_s}\Lambda\mathbf{J}_s^2(\mathbf{r},t) + q_s\phi_{\mathrm{el}}(\mathbf{r},t) + \mu(\mathbf{r},t)\right\}$$

iii. ……………………………………………..

no corresponding equation as $|\psi(\mathbf{r})|^2 = n_s(\mathbf{r}) = const.$ is assumed

- cannot account for spatially inhomogeneous situations
- can describe time-dependent phenomena (e.g. Josephson effect)

GLAG theory

i. 2nd Ginzburg-Landau equation

$$\mathbf{J}_s = \frac{q_s\hbar}{2m_s i}\left(\Psi^\star\nabla\Psi - \Psi\nabla\Psi^\star\right) - \frac{q_s^2}{m_s}|\Psi|^2\mathbf{A}$$

note that for $|\Psi(\mathbf{r})|^2 = n_s(\mathbf{r}) = const.$ this equation is equivalent to the current-phase relation

ii. ……………………………………………..

no corresponding equation as $\Psi(\mathbf{r})$ is assumed to depend only on $\mathbf{r}$ and not on t

iii. 1st Ginzburg-Landau equation

$$0 = \frac{1}{2m_s}\left(\frac{\hbar}{i}\nabla - q_s\mathbf{A}\right)^2\Psi + \alpha\Psi + \beta|\Psi|^2\Psi$$

- can well describe spatially inhomogeneous situations
- cannot account for time-dependent phenomena

Note: extensions of GLAG theory to describe time-dependent processes have been formulated

3.3 GL Theory: Length Scales

Characteristic length scales – penetration depth:

- **2nd GL equation**:

$$\mathbf{J}_s = \frac{q_s\hbar}{2m_s\imath}(\Psi^\star\nabla\Psi - \Psi\nabla\Psi^\star) - \frac{q_s^2}{m_s}|\Psi|^2\mathbf{A}$$

for $|\Psi(\mathbf{r})|^2 = n_s(\mathbf{r}) = const.$

$$\mathbf{J}_s = \frac{q_s\hbar}{2m_s\imath}(\imath|\Psi|^2\nabla\theta + \imath|\Psi|^2\nabla\theta) - \frac{q_s^2}{m_s}|\Psi|^2\mathbf{A}$$

with $|\Psi|^2 = n_s$

$$\mathbf{J}_s(\mathbf{r},t) = n_s q_s\left(\frac{\hbar}{m_s}\nabla\theta(\mathbf{r},t) - \frac{q_s}{m_s}\mathbf{A}(\mathbf{r},t)\right)$$

exactly corresponds to current-phase relation derived from macroscopic quantum model

allows to derive

- 1st and 2nd London equation
- characteristic screening length for B_{ext} ➔ **GL penetration depth λ_{GL}**

- GL penetration depth agrees with London penetration depth as equilibrium superfluid density is $n_s = |\Psi|^2 = |\alpha|/\beta$

$$\lambda_{\text{GL}} = \sqrt{\frac{m_s}{\mu_0 n_s q_s^2}} = \sqrt{\frac{m_s\beta}{\mu_0|\alpha|q_s^2}}$$

3.3 GL Theory: Length Scales

Characteristic length scales – coherence length:

- **1^{st} GL equation**:

$$0 = \frac{1}{2m_s}\left(\frac{\hbar}{i}\boldsymbol{\nabla} - q_s\mathbf{A}\right)^2 \Psi + \alpha\Psi + \beta|\Psi|^2\Psi$$

normalization $\widetilde{\Psi} = \Psi/|\Psi_0|, \quad n_s = |\Psi|^2 = -|\alpha|/\beta$ ($|\Psi_0| =$ homogeneous value)

and use of 1^{st} GL equation

$$0 = \frac{1}{2m_s}\left(\frac{\hbar}{i}\boldsymbol{\nabla} - q_s\mathbf{A}\right)^2 \Psi + \alpha\Psi + \beta|\Psi|^2\Psi = \frac{1}{2m_s}\left(\frac{\hbar}{i}\boldsymbol{\nabla} - q_s\mathbf{A}\right)^2 \widetilde{\Psi} + \alpha\widetilde{\Psi} + |\alpha|\left|\widetilde{\Psi}\right|^2\widetilde{\Psi}$$

$$0 = \underbrace{\frac{\hbar^2}{2m_s|\alpha|}}\left(\frac{1}{i}\boldsymbol{\nabla} - q_s\mathbf{A}\right)^2 \widetilde{\Psi} + \widetilde{\Psi} + \left|\widetilde{\Psi}\right|^2\widetilde{\Psi}$$

2^{nd} characteristic length scale

$$\boxed{\xi_{\mathrm{GL}} = \sqrt{\frac{\hbar^2}{2m_s|\alpha|}}}$$

GL coherence length

- for $A = 0$ and small deviations $\delta f = |\Psi| - |\Psi_0|$ we obtain (neglecting higher oder terms)

$$\boldsymbol{\nabla}^2\delta f = \frac{1}{\xi_{\mathrm{GL}}^2}\delta f$$

➔ deviations δf from homogeneous state decay exponentially on characteristic scale ξ_{GL}

3.3 GL Theory: Length Scales

Temperature dependence of characteristic length scales:

- Ansatz for α and β: $\alpha(T) = \bar{\alpha}\left(\frac{T}{T_c} - 1\right) = -\bar{\alpha}\left(1 - \frac{T}{T_c}\right)$ with $\bar{\alpha} > 0$; $\beta(T) = \beta = const.$

$$n_s(T) = |\Psi(T)|^2 = -\frac{\alpha(T)}{\beta} = \frac{\bar{\alpha}}{\beta}\left(1 - \frac{T}{T_c}\right) = n_s(0)\left(1 - \frac{T}{T_c}\right)$$

- with $\xi_{\mathrm{GL}} = \sqrt{\frac{\hbar^2}{2m_s|\alpha(T)|}}$ and $\lambda_{\mathrm{GL}} = \sqrt{\frac{m_s\beta}{\mu_0|\alpha(T)|q_s^2}}$ GL theory predicts

$$\lambda_{\mathrm{GL}}(T) = \frac{\lambda_{\mathrm{GL}}(0)}{\sqrt{1 - \frac{T}{T_c}}}$$

$$\lambda_{\mathrm{GL}}(0) = \sqrt{\frac{m_s}{\mu_0 n_s(0) q_s^2}}$$

both length scales diverge for $T \rightarrow T_c$

$$\xi_{\mathrm{GL}}(T) = \frac{\xi_{\mathrm{GL}}(0)}{\sqrt{1 - \frac{T}{T_c}}}$$

$$\xi_{\mathrm{GL}}(0) = \sqrt{\frac{\hbar^2}{2m_s\bar{\alpha}}}$$

3.3 GL Theory: Length Scales

- experimentally measured T-dependence: $$\lambda_{\mathrm{L}}(T) = \frac{\lambda_{\mathrm{L}}(0)}{\sqrt{1-\left(\frac{T}{T_c}\right)^4}}$$ discrepancy expected as GL theory is valid only close to T_c

we use $1-\left(\frac{T}{T_c}\right)^4 = \left[1-\left(\frac{T}{T_c}\right)^2\right]\cdot\left[1+\left(\frac{T}{T_c}\right)^2\right] \simeq 2\left[1-\left(\frac{T}{T_c}\right)^2\right] \simeq 4\left[1-\left(\frac{T}{T_c}\right)\right]$ for $T \simeq T_c$

$$\Rightarrow \quad \lambda_{\mathrm{L}}(T) \simeq \frac{\lambda_{\mathrm{L}}(0)}{2\sqrt{1-\left(\frac{T}{T_c}\right)}} = \frac{\lambda_{\mathrm{GL}}(0)}{\sqrt{1-\left(\frac{T}{T_c}\right)}} = \lambda_{\mathrm{GL}}(T)$$

that is, measured dependence agrees reasonably well with GL prediction close to T_c, but we have to use $\lambda_{\mathrm{GL}}(0) = \lambda_{\mathrm{L}}(0)/2$

3.3 GL Theory: GL Parameter

Ginzburg-Landau parameter:

$$\kappa \equiv \frac{\lambda_{\mathrm{GL}}}{\xi_{\mathrm{GL}}} = \sqrt{\frac{2\beta}{\mu_0}\frac{m_s}{\hbar q_s}} = \frac{\sqrt{2}\,m_s}{\mu_0 q_s \hbar n_s(T)}\,B_{\mathrm{cth}}(T)$$

(weak T dependence via β)

$$\left\{\begin{array}{l} \lambda_{\mathrm{GL}}(T) = \sqrt{\dfrac{m_s}{\mu_0 n_s(T) q_s^2}} = \sqrt{\dfrac{m_s\beta}{\mu_0|\alpha(T)|q_s^2}} \\[2ex] \xi_{\mathrm{GL}}(T) = \sqrt{\dfrac{\hbar^2}{2m_s|\alpha(T)|}} \\[2ex] |\alpha(T)| = \dfrac{B_{\mathrm{cth}}^2(T)}{2\mu_0}\dfrac{2}{n_s(T)} \end{array}\right.$$

- solve for B_{cth} ⇨ $$B_{\mathrm{cth}}(T) = \frac{\Phi_0}{2\pi\sqrt{2}\,\xi_{\mathrm{GL}}(T)\lambda_{\mathrm{GL}}(T)}$$

relation between GL and BCS coherence length:

$$\xi_{\mathrm{GL}} = \sqrt{\frac{\hbar^2}{2m_s|\alpha(T)|}}$$

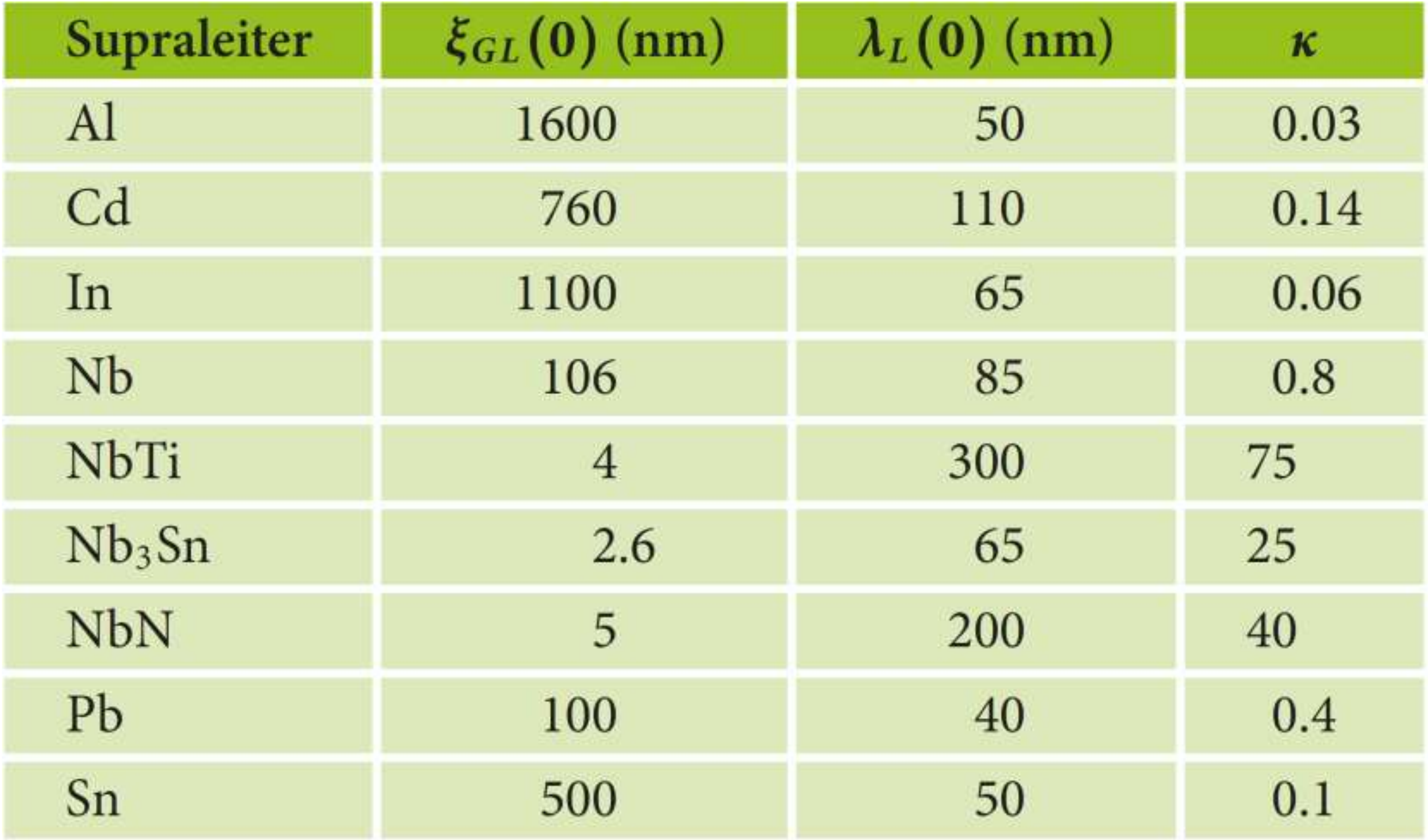
- $\alpha/2$ = condensation energy per superconducting electron
- BCS: average condensation energy per superconducting electron at $T = 0$:
  $\simeq \frac{1}{4}D(E_{\mathrm{F}})\Delta^2(0)/N = 3\Delta^2(0)/8E_{\mathrm{F}}$ with $E_{\mathrm{F}} = 3N/2D(E_{\mathrm{F}})$

→ α corresponds to $\approx -3\Delta^2(0)/4E_{\mathrm{F}}$

➔ $$\xi_{\mathrm{GL}}(0) = \sqrt{\frac{4\hbar^2 E_{\mathrm{F}}}{6m_s\Delta^2(0)}} \underbrace{=}_{E_{\mathrm{F}}=\frac{1}{2}m\,v_{\mathrm{F}}^2=\frac{1}{4}\,m_s v_{\mathrm{F}}^2} \sqrt{\frac{\hbar^2 v_{\mathrm{F}}^2}{6\Delta^2(0)}} = \frac{\hbar v_{\mathrm{F}}}{\sqrt{6}\,\Delta(0)}$$

agrees well with correct BCS result: $\xi_0 = \hbar v_{\mathrm{F}}/\pi\Delta(0)$

3.3 GL Theory: Length Scales



Supraleiter	$\xi_{GL}(0)$ (nm)	$\lambda_L(0)$ (nm)	κ
Al	1600	50	0.03
Cd	760	110	0.14
In	1100	65	0.06
Nb	106	85	0.8
NbTi	4	300	75
Nb_3Sn	2.6	65	25
NbN	5	200	40
Pb	100	40	0.4
Sn	500	50	0.1

3.3 GL Theory: S/N Interface

Superconductor-normal metal interface:

- assumptions: superconductor extends in x-direction from $x > 0$, no applied magnetic field: $\mathbf{A} = 0$

$$0 = \frac{\hbar^2}{2m_s\alpha}\left(\frac{1}{i}\nabla - \frac{q_s}{\hbar}\mathbf{A}\right)^2 \widetilde{\Psi} + \widetilde{\Psi} + \left|\widetilde{\Psi}\right|^2\widetilde{\Psi} \Rightarrow 0 = \xi_{\mathrm{GL}}^2 \frac{\partial^2\widetilde{\Psi}}{\partial x^2} + \widetilde{\Psi} + \left|\widetilde{\Psi}\right|^2\widetilde{\Psi} \qquad (\widetilde{\Psi} = \Psi/|\Psi_0|, \text{ with } |\Psi_0| = |\Psi_\infty|)$$

- boundary conditions:

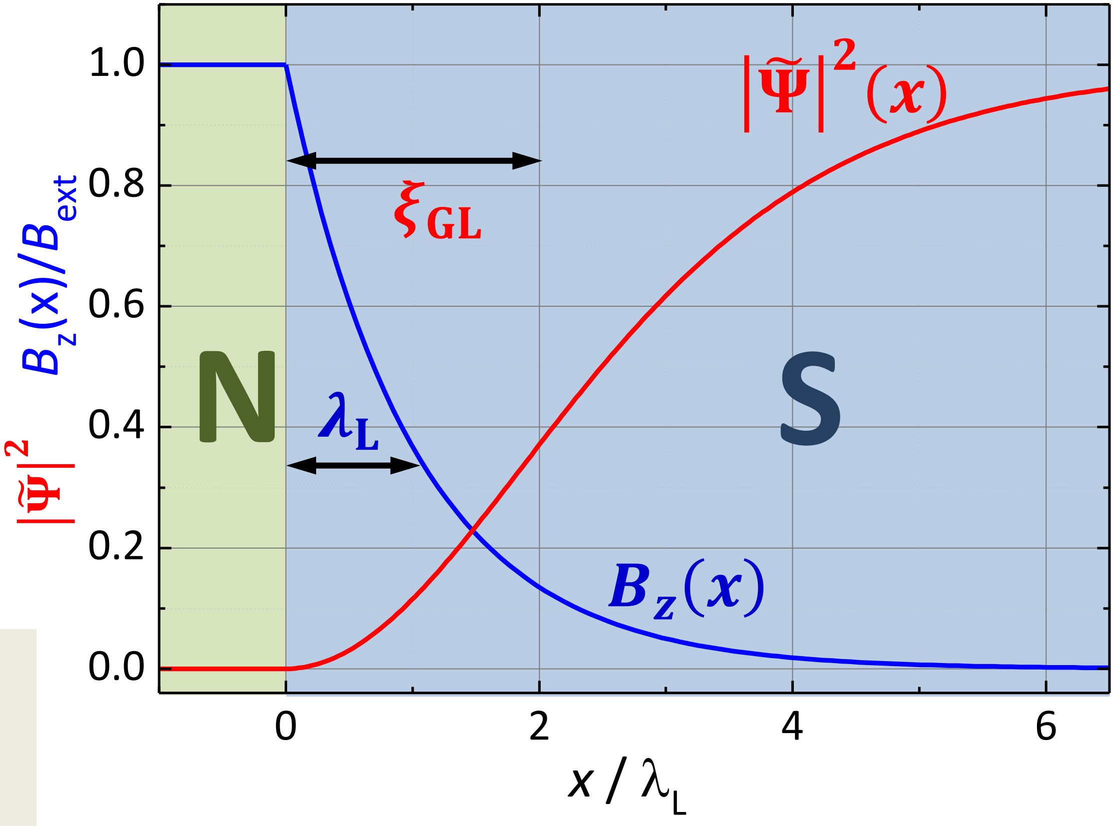
$$\widetilde{\Psi}(x=0) = 0, \qquad \widetilde{\Psi}(x \to \infty) = 1$$

$$\lim_{x\to\infty} \partial\widetilde{\Psi}/\partial x = 0$$

- solution:

$$\widetilde{\Psi}(x) = \tanh\left(\frac{x}{\sqrt{2}\,\xi_{\mathrm{GL}}}\right)$$

$$\left|\widetilde{\Psi}(x)\right|^2 = \frac{n_s(x)}{n_s(\infty)} = \tanh^2\left(\frac{x}{\sqrt{2}\,\xi_{\mathrm{GL}}}\right)$$



important:

$\left|\widetilde{\Psi}(x)\right|$ increases on characteristic length scale ξ_{GL} from 0 to 1 (for $B_{\mathrm{ext,z}} = 0$)

$B_{\mathrm{ext,z}}$ decays in SC on characteristic length scale λ_{GL} (for $\left|\widetilde{\Psi}(x)\right| = const.$)

Superconductivity and Low Temperature Physics I

Lecture No. 6

R. Gross

Summary of Lecture No. 5 (1)

- **Ginzburg-Landau Theory** (1950)

  → **phenomenological description** of superconductor by a ***complex, spatially varying order parameter*** $\mathbf{\Psi(r) = |\Psi(r)|\, e^{i\theta(r)}}$ ***with*** $\mathbf{|\Psi(r)|^2 = n_s(r)}$ (based on extension of Landau theory of phase transitions)

- **Ginzburg-Landau Theory: spatially homogeneous case, no applied magnetic field** $(|\Psi(\mathbf{r})|^2 = n_s(\mathbf{r}) = const.)$

  develop free enthalpy density g_s of superconductor into a power series of $|\Psi|^2$

$$g_s = g_n + \alpha|\Psi|^2 + \frac{1}{2}\beta|\Psi|^4 + \cdots$$

Ansatz: $\alpha(T) = \bar{\alpha}\left(\frac{T}{T_c} - 1\right) = -\bar{\alpha}\left(1 - \frac{T}{T_c}\right)$ with $\bar{\alpha} > 0$

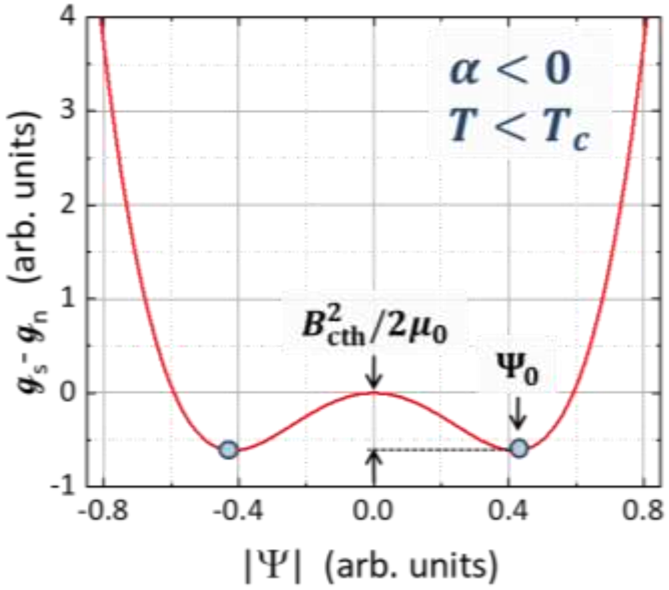
$\beta(T) = const. > 0$

minumum of g_s for

$$n_s(T) = |\Psi_0(T)|^2 = -\frac{\alpha(T)}{\beta} = \frac{\bar{\alpha}}{\beta}\left(1 - \frac{T}{T_c}\right)$$

$$\frac{\bar{\alpha}}{2} = \left[\frac{B_{\text{cth}}^2(0)}{2\mu_0}\right] / n_s(0) =$$

condensation energy per charge carrier at $T = 0$



Summary of Lecture No. 5 (2)

- **Ginzburg-Landau Theory: spatially inhomogeneous case ($|\Psi(\mathbf{r})|^2 = n_s(\mathbf{r}) \neq const.$), finite magnetic field $\mathbf{B}_{\text{ext}} = \mu_0 \mathbf{H}_{\text{ext}}$**

  additional terms in free enthalpy density due to finite $\mathbf{J}_s$ and $\mathbf{B}_{\text{ext}} = \mu_0 \mathbf{H}_{\text{ext}}$

$$g_s = g_n + \alpha|\Psi|^2 + \frac{1}{2}\beta|\Psi|^4 + \cdots + \underbrace{\frac{[\mathbf{b}(\mathbf{r}) - \mathbf{B}_{\text{ext}}]^2}{2\mu_0}} + \underbrace{\frac{1}{2m_s}\left|\frac{\hbar}{\imath}\nabla\Psi(\mathbf{r}) - q_s\,\mathbf{A}(\mathbf{r})\Psi(\mathbf{r})\right|^2}$$

additional field energy density:
e.g. due to work required for
field expulsion
$\propto (\mathbf{b}(\mathbf{r}) - \mathbf{B}_{\text{ext}})^2$

kinetic energy of the supercurrents*:*
finite gauge invariant phase gradient results in
supercurrent density and increase in kinetic energy

finite stiffness of order parameter:
→ *spatial variations of $|\Psi|$ cost additional energy*

- **minimization of total free enthalpy by variational approach yields Ginzburg-Landau equations**

$$\frac{1}{2m_s}\left(\frac{\hbar}{\imath}\nabla - q_s\,\mathbf{A}(\mathbf{r})\right)^2 \Psi(\mathbf{r}) + \alpha\Psi(\mathbf{r}) + \frac{1}{2}\beta|\Psi(\mathbf{r})|^2\Psi(\mathbf{r}) = 0$$

1st GL equation ⇨ $\xi_{\text{GL}} = \sqrt{\dfrac{\hbar^2}{2m_s|\alpha|}}$ **coherence length**

$$\mathbf{J}_s = \frac{q_s\hbar}{2m_s\imath}(\Psi^\star\nabla\Psi - \Psi\nabla\Psi^\star) - \frac{q_s^2}{m_s}|\Psi|^2\mathbf{A}$$

2nd GL equation ⇨ $\lambda_{\text{GL}} = \sqrt{\dfrac{m_s\beta}{\mu_0|\alpha|q_s^2}}$ **field screening length**

$$\xi_{\text{GL}}(T) = \xi_{\text{GL}}(0)/\sqrt{1 - \frac{T}{T_c}} \qquad \lambda_{\text{GL}}(T) = \lambda_{\text{GL}}(0)/\sqrt{1 - \frac{T}{T_c}}$$

Summary of Lecture No. 5 (3)

Supraleiter	$\xi_{GL}(0)$ (nm)	$\lambda_L(0)$ (nm)	κ
Al	1600	50	0.03
Cd	760	110	0.14
In	1100	65	0.06
Nb	106	85	0.8
NbTi	4	300	75
Nb_3Sn	2.6	65	25
NbN	5	200	40
Pb	100	40	0.4
Sn	500	50	0.1

- **Ginzburg-Landau parameter**

$$\kappa \equiv \frac{\lambda_{\mathrm{GL}}}{\xi_{\mathrm{GL}}} = \sqrt{\frac{2\beta}{\mu_0}\frac{m_s}{\hbar q_s}} = \frac{\sqrt{2}\,m_s}{\mu_0 q_s \hbar n_s(T)}\,B_{\mathrm{cth}}(T)$$

⇨

$$B_{\mathrm{cth}}(T) = \frac{\Phi_0}{2\pi\sqrt{2}\,\xi_{\mathrm{GL}}(T)\lambda_{\mathrm{GL}}(T)}$$

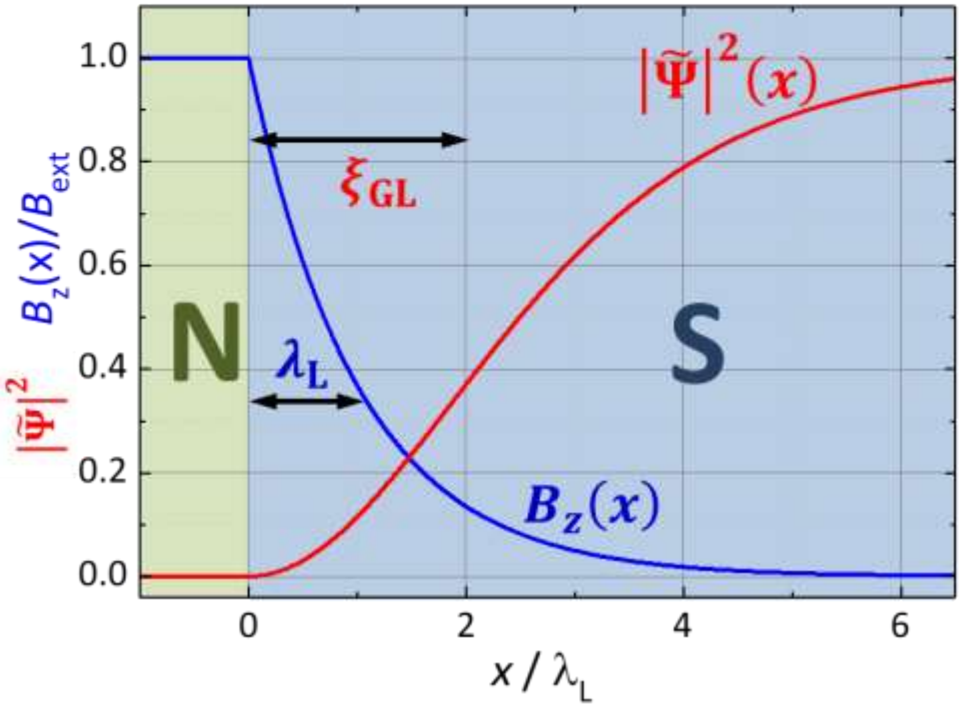
- **application of GL equation: calculate variation of order parameter and flux density at N/S boundary**

$$\left|\widetilde{\Psi}(x)\right|^2 = \frac{n_s(x)}{n_s(\infty)} = \tanh^2\left(\frac{x}{\sqrt{2}\,\xi_{\mathrm{GL}}}\right)$$

calculated for $B_{\mathrm{z}} = 0$

$$B_z(x) = B_z(0)\exp\left(-\frac{x}{\lambda_{\mathrm{GL}}}\right)$$

calculated for $\left|\widetilde{\Psi}(x)\right| = const.$



key result:

$\left|\widetilde{\Psi}(x)\right|$ increases $\propto \tanh^2$ in SC on characteristic length scale ξ_{GL} from 0 to 1

$B_{\mathrm{z}}(x)$ decays exponentially in SC on characteristic length scale λ_{GL}

3. Phenomenological Models of Superconductivity

3.3.1 Type-I and Type-II Superconductors

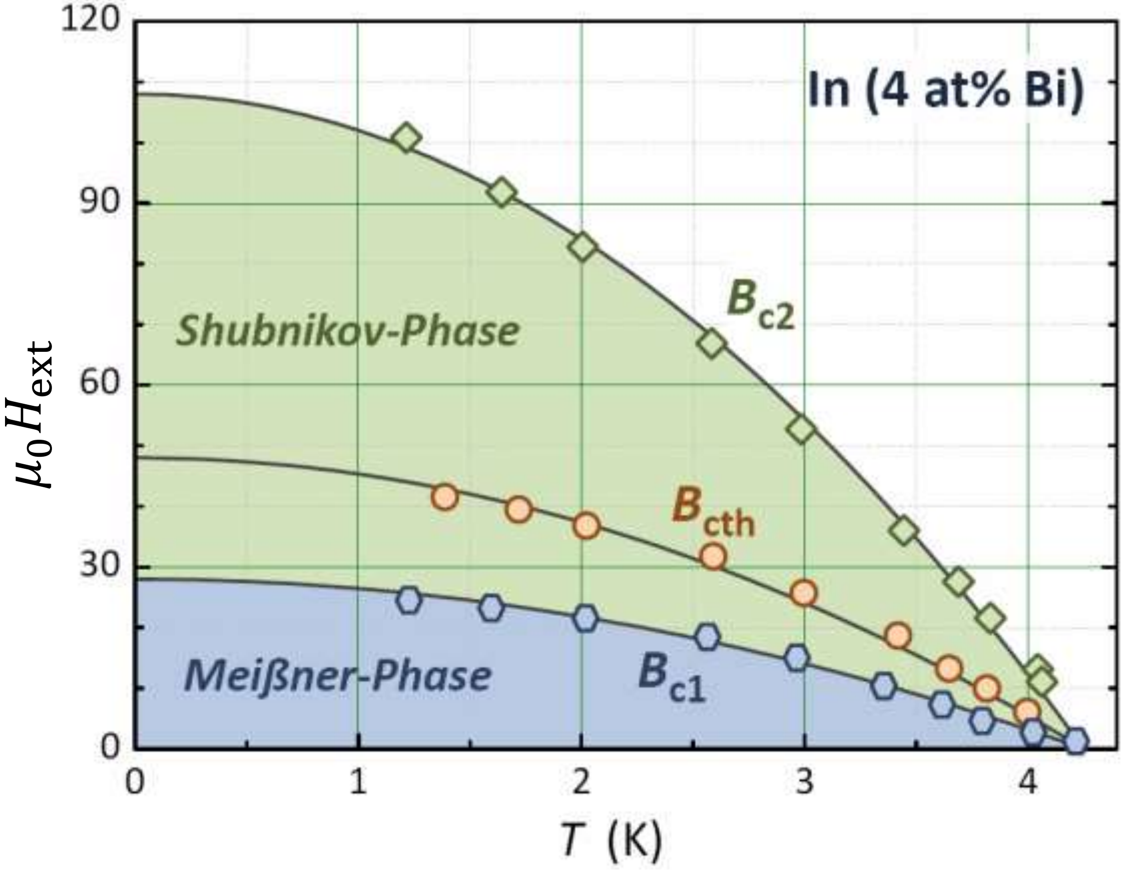
experimental facts:

- **type-I superconductors**:

  expel magnetic field until B_{cth}: $B_i = 0$
  → only Meißner phase
  → ***single critical field*** $\boldsymbol{B_{\mathrm{cth}}}$

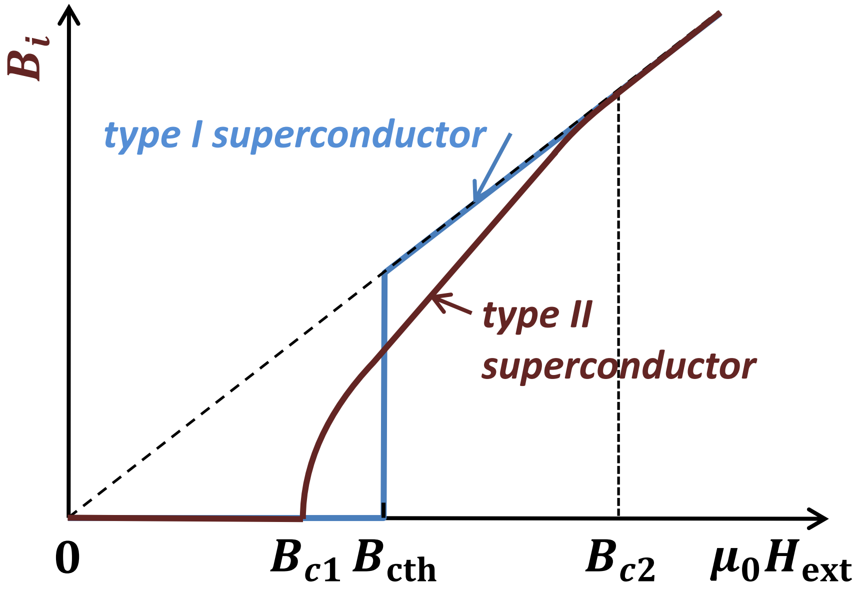
- **type-II superconductors**:

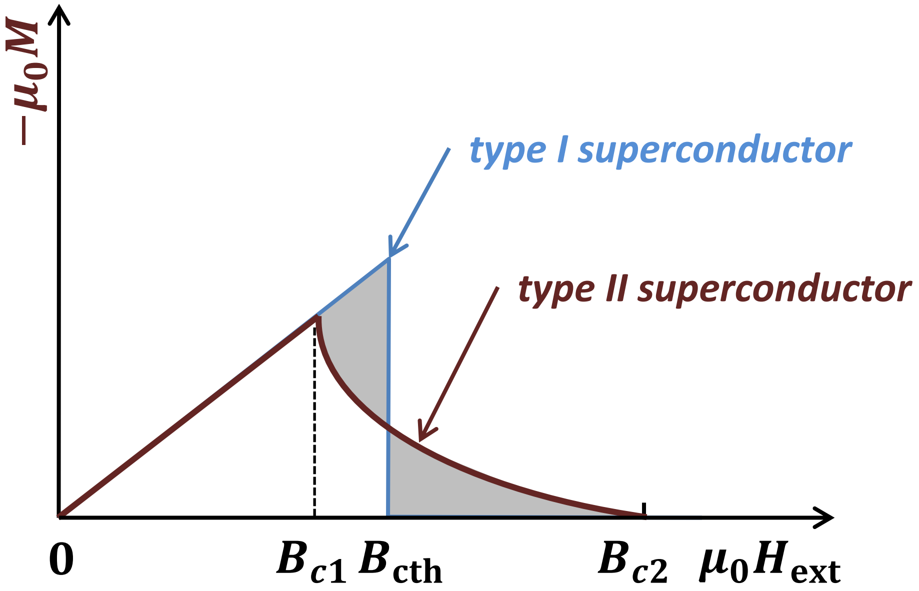
  partial field penetration above B_{c1}
  → $B_i > 0$ for $B_{\mathrm{ext}} > B_{c1}$
  → ***Shubnikov phase*** between $B_{c1} \leq B_{\mathrm{ext}} \leq B_{c2}$
  → ***upper and lower critical fields*** $\boldsymbol{B_{c1}}$ ***and*** $\boldsymbol{B_{c2}}$



R. Gross and A. Marx , © Walther-Meißner-Institut (2004 - 2021)

3.3.1 Type-I and Type-II Superconductors





- thermodynamic critical field defined as:
  (for type-I and type-II superconductors)

$$\underbrace{g_s - g_n}_{\text{condensation energy}} = -\frac{B_{\mathrm{cth}}^2(T)}{2\mu_0}$$

- area under $M(H_{\mathrm{ext}})$ curve is the same for type-I and type-II superconductor with the same condensation energy:

$$g_s(T) - g_n(T) = -\frac{B_{\mathrm{cth}}^2(T)}{2\mu_0} = \int_0^{B_{\mathrm{cth}}} \mathbf{M} \cdot \mathrm{d}\mathbf{B}_{\mathrm{ext}} = \int_0^{B_{c2}} \mathbf{M} \cdot \mathrm{d}\mathbf{B}_{\mathrm{ext}}$$

3.3.1 Type-I and Type-II Superconductors

difference between type-I and type-II superconductors: determined by sign of N/S boundary energy

- lowering of energy due to savings in field expulsion work (per area)

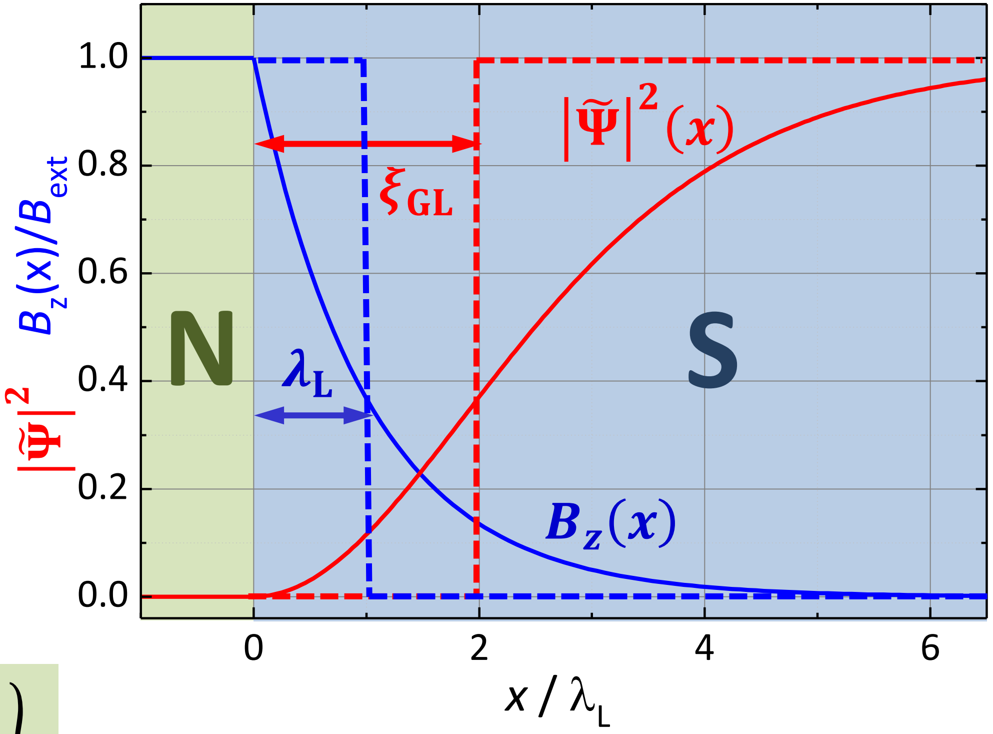
$$\frac{\Delta E_B}{F} = -\int_0^\infty \frac{B_z(x)^2}{2\mu_0}\,\mathrm{d}x \simeq -\frac{B_{\mathrm{ext}}^2}{2\mu_0}\,\lambda_{\mathrm{L}}$$

- increase of energy due to loss in condensation energy (per area)

$$\frac{\Delta E_C}{F} = \frac{B_{\mathrm{cth}}^2}{2\mu_0}\int_0^\infty |\widetilde{\Psi}|^2\,\mathrm{d}x \simeq \frac{B_{\mathrm{cth}}^2}{2\mu_0}\,\xi_{\mathrm{GL}}$$

- resulting boundary energy

$$\frac{\Delta E_C}{F} + \frac{\Delta E_B}{F} \simeq \frac{B_{\mathrm{cth}}^2}{2\mu_0}\,\xi_{\mathrm{GL}} - \frac{B_{\mathrm{ext}}^2}{2\mu_0}\,\lambda_{\mathrm{L}} = \frac{B_{\mathrm{cth}}^2}{2\mu_0}\left\{\xi_{\mathrm{GL}} - \left(\frac{B_{\mathrm{ext}}}{B_{\mathrm{cth}}}\right)^2 \lambda_{\mathrm{L}}\right\}$$



3.3.1 Type-I and Type-II Superconductors

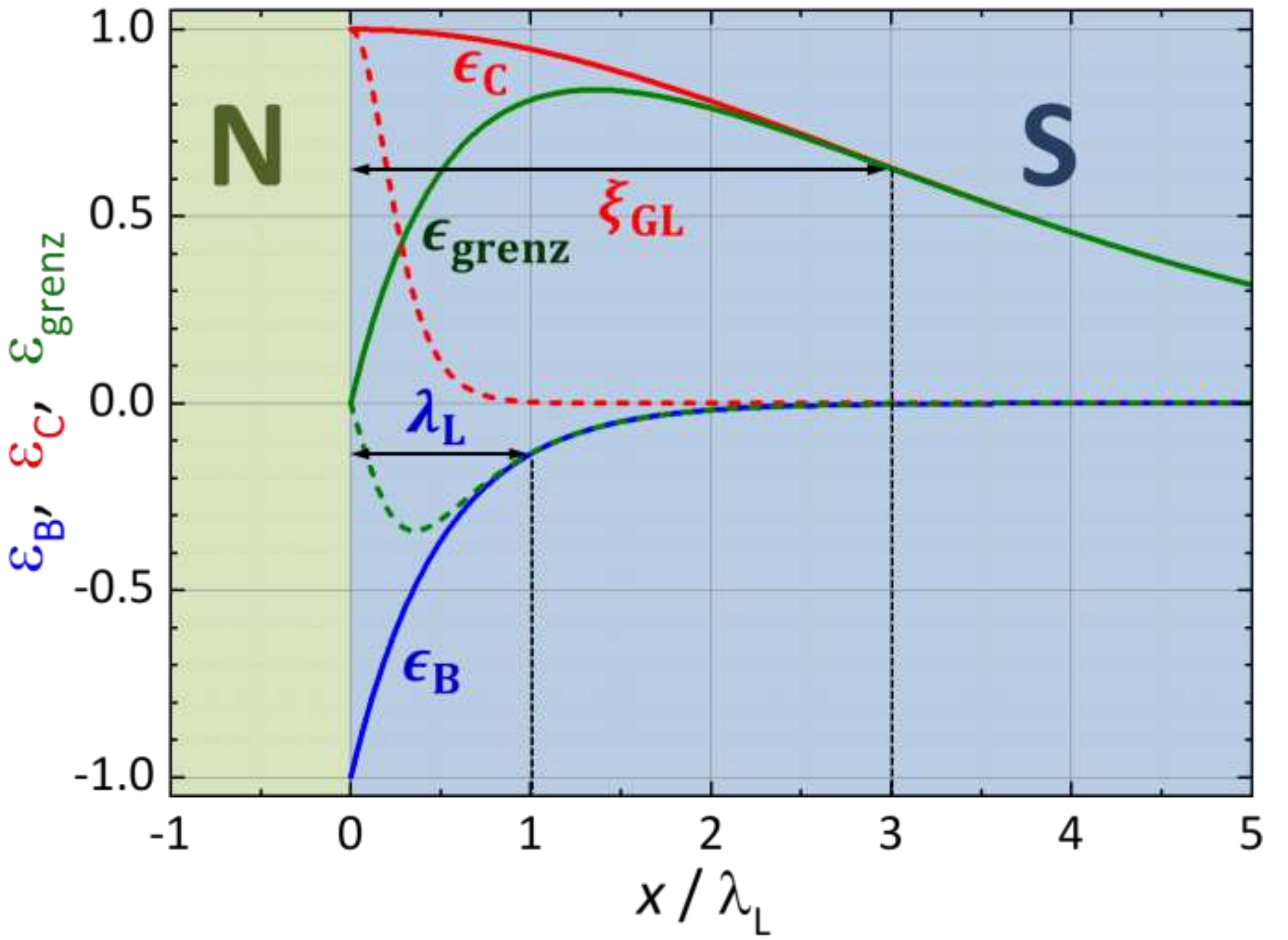
normalized bounday energy per unit length (≡ energy density)

$$\varepsilon_B \simeq -\frac{b^2(x)/2\mu_0}{B_{\mathrm{ext}}^2/2\mu_0} = -\left[\mathrm{e}^{-x/\lambda_\mathrm{L}}\right]^2$$

$$\varepsilon_C \simeq \frac{\left(B_{\mathrm{cth}}^2/2\mu_0\right)\left[n_{s(\infty)} - n_s(x)\right]}{\left(B_{\mathrm{cth}}^2/2\mu_0\right) n_s(\infty)} = 1 - \frac{n_s(x)}{n_s(\infty)}$$

$$\varepsilon_C \simeq 1 - \tanh^2\left(\frac{x}{\sqrt{2}\,\xi_{\mathrm{GL}}}\right)$$

$$\Rightarrow \varepsilon_{\mathrm{Grenz}} \simeq 1 - \tanh^2\left(\frac{x}{\sqrt{2}\,\xi_{\mathrm{GL}}}\right) - \left[\mathrm{e}^{-x/\lambda_\mathrm{L}}\right]^2$$



3.3.1 Type-I and Type-II Superconductors

discussion of boundary energy at superconductor/normal metal interface

$$\Delta E_{\mathrm{boundary}} = \Delta E_C + \Delta E_B \simeq \frac{B_{\mathrm{cth}}^2}{2\mu_0}\left[\xi_{\mathrm{GL}} - \left(\frac{B_{\mathrm{ext}}}{B_{\mathrm{cth}}}\right)^2 \lambda_{\mathrm{GL}}\right]$$

I. Type I superconductor: $\xi_{\mathrm{GL}} \geq \lambda_{\mathrm{GL}}$

- ➢ boundary energy is always positive for $B_{\mathrm{ext}} \leq B_{\mathrm{cth}}$
  - ➔ formation of boundary is avoided → perfect flux expulsion (Meißner state) up to $B_{\mathrm{ext}} = B_{\mathrm{cth}}$

II. Type II superconductor: $\xi_{\mathrm{GL}} < \lambda_{\mathrm{GL}}$

- ➢ boundary energy is always positive for $B_{\mathrm{ext}} \leq B_{c1} < B_{\mathrm{cth}}$
  - ➔ formation of boundary is avoided → perfect flux expulsion (Meißner state) up to $B_{\mathrm{ext}} = B_{\mathrm{c1}}$
- ➢ boundary energy becomes negative for $B_{\mathrm{ext}} > B_{c1}$
  - ➔ formation of mixed state, as energy can be lowered by formation of N/S-boundaries
  - ➔ N-regions are made as small as possible to maximize bounday → lower limit is set by flux quantization
  - ➔ type II SC can expel field and stay in superconducting state up to $B_{c2} > B_{\mathrm{cth}}$, as field expulsion work is lowered

- exact calculation yields

$$\kappa = \lambda_{\mathrm{GL}}/\xi_{\mathrm{GL}} \leq 1/\sqrt{2} \quad \text{type I superconductor}$$
$$\kappa = \lambda_{\mathrm{GL}}/\xi_{\mathrm{GL}} \geq 1/\sqrt{2} \quad \text{type II superconductor}$$

3.3.1 Type-I and Type-II Superconductors:

Demagnetization Effects and Intermediate State

- ideal $B_i(H_{\mathrm{ext}})$ dependence valid only for vanishing demagnetization effects
  - e.g. for long cylinder or slab with $H_{\mathrm{ext}}||$ cylinder

- for finite demagnetization (characterized by demagnetization factor N)

$$\mathbf{H}_{\mathrm{mac}} = \mathbf{H}_{\mathrm{ext}} - N \cdot \mathbf{M} \qquad \text{(macroscopic field)}$$

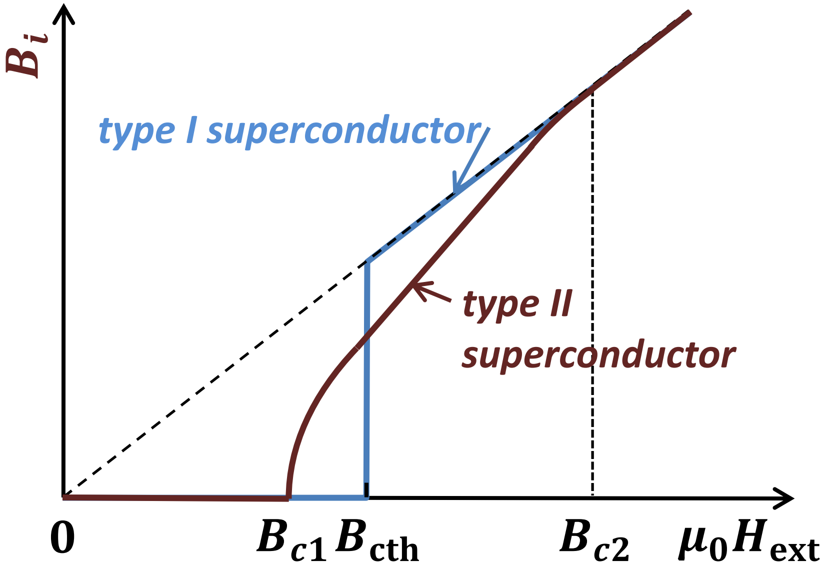
with $\mathbf{M} = \chi\mathbf{H}_{\mathrm{mac}} = -\mathbf{H}_{\mathrm{mac}}$ (perfect diamagnetism)

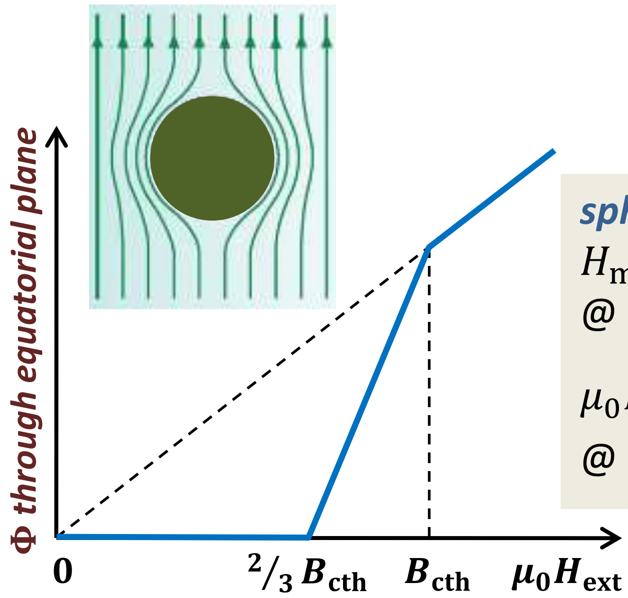
$$\Rightarrow \quad \boxed{\mathbf{H}_{\mathrm{mac}} = \frac{\mathbf{H}_{\mathrm{ext}}}{1-N}}$$

long cylinder:	$N \simeq 0$	$H_{\mathrm{mac}} \simeq H_{\mathrm{ext}}$
flat disk:	$N \simeq 1$	$H_{\mathrm{mac}} \to \infty$
sphere:	$N \simeq 1/3$	$H_{\mathrm{mac}} \to 1.5\, H_{\mathrm{ext}}$

- formation of **intermediate state** in Meißner regime by demagnetization effects

➔ ***intermediate state can have complex structure***



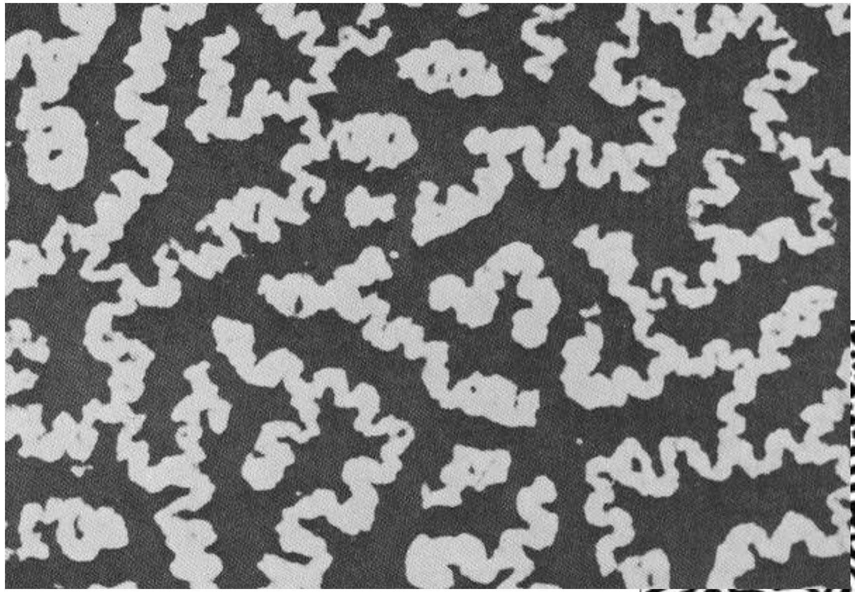


sphere:
$H_{\mathrm{mac}} = 1.5\, H_{\mathrm{ext}}$
@ equator

$\mu_0 H_{\mathrm{mac}} = B_{\mathrm{cth}}$
@ $\mu_0 H_{\mathrm{ext}} = \frac{2}{3} B_{\mathrm{cth}}$

3.3.1 Type-I and Type-II Superconductors:

Demagnetization Effects and Intermediate State



magneto-optical image

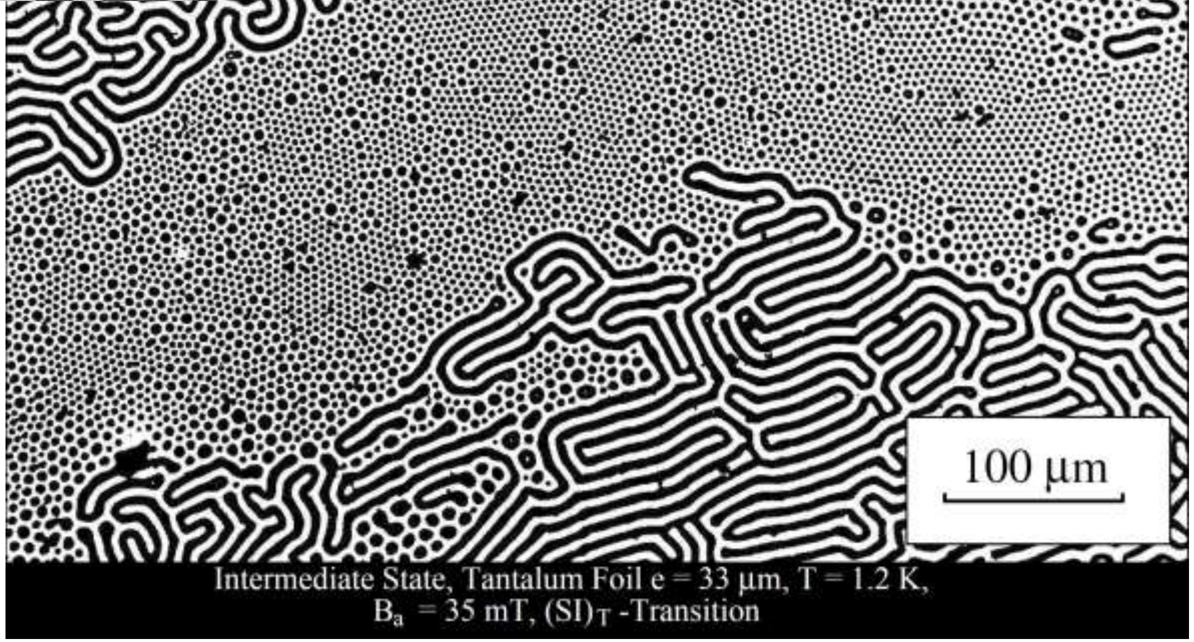
of the intermediate state of In (type I superconductor)

bright: normal regions

(from Buckel)

intermediate state of Al plate

(type I superconductor)



Intermediate State, Tantalum Foil e = 33 µm, T = 1.2 K, B_a = 35 mT, $(SI)_T$ -Transition

intermediate state of Ta foil

(type II superconductor)

3.3.2 Type-II Superconductors: Upper and Lower Critical Field

Task: derive expression for B_{c2} from GLAG-equations (Abrikosov, 1957)

- we use the 1st GL equation and linearize it as $|\Psi(r)|^2 \to 0$ for large $B_{\text{ext}} \to B_{c2}$

$$0 = \frac{1}{2m_s}\left(\frac{\hbar}{i}\boldsymbol{\nabla} - q_s\mathbf{A}\right)^2\Psi + \alpha\Psi + \cancel{\beta|\Psi|^2\Psi} \quad \Rightarrow \quad \boxed{\frac{1}{2m_s}\left(\frac{\hbar}{i}\boldsymbol{\nabla} - q_s\mathbf{A}\right)^2\Psi = -\alpha\Psi}$$

- further approximations:

$\mathbf{B} \simeq \mu_0\mathbf{H}_{\text{ext}}$, since $\mathbf{M} \to 0$ for $\mu_0\mathbf{H}_{\text{ext}} \to \mathbf{B}_{c2}$

$\mathbf{H}_{\text{ext}} = (0,0,H_z) \rightarrow \mathbf{A} = (0, A_y, 0)$ with $A_y = \mu_0 H_z x = B_z x$

$$\Rightarrow \boxed{\frac{\partial^2\Psi}{\partial x^2} + \left(\frac{\partial}{\partial y} - \frac{\imath q_s B_z}{\hbar}x\right)^2\Psi + \frac{\partial^2\Psi}{\partial z^2} = \frac{2m_s\alpha}{\hbar^2}\Psi = -\frac{1}{\xi_{\text{GL}}^2}\Psi}$$

corresponds to Schrödinger equation of free particle with charge q_s, mass m_s and total energy $-\alpha$ in an applied magetic field B_z

→ solution and eigenenergies are known: ***Landau levels***

3.3.2 Type-II Superconductors: Upper and Lower Critical Field

- energy eigenvalues of the Landau levels for motion in plane perpendicular to $B_{\mathrm{ext},z}$:

$$\varepsilon_n = \hbar\omega_c\left(n+\frac{1}{2}\right) = \hbar\frac{q_s B_{\mathrm{ext},z}}{m_s}\left(n+\frac{1}{2}\right) = -\alpha - \frac{\hbar^2 k_z^2}{2m_s} = \frac{\hbar^2}{2m_s}\left(\frac{1}{\xi_{\mathrm{GL}}^2} - k_z^2\right)$$

with $\alpha(T) = -\frac{\hbar^2}{2m_s\xi_{\mathrm{GL}}^2(T)}$

- resolving for $B_{\mathrm{ext},z}$ yields:

$$B_{\mathrm{ext},z} = \frac{\hbar}{2q_s}\left(\frac{1}{\xi_{\mathrm{GL}}^2} - k_z^2\right)\left(n+\frac{1}{2}\right)^{-1}$$

- lowest level for $n = 0, k_z = 0$ yields solution for maximum field:

$$B_{\mathrm{ext},z} = \frac{\hbar}{q_s\xi_{\mathrm{GL}}^2} = \frac{h}{q_s}\frac{1}{2\pi\xi_{\mathrm{GL}}^2} = \frac{\Phi_0}{2\pi\xi_{\mathrm{GL}}^2}$$

$$B_{c2}(T) = \frac{\Phi_0}{2\pi\xi_{\mathrm{GL}}^2(T)} = \frac{\Phi_0}{2\pi\xi_{\mathrm{GL}}^2(0)}\left(1-\frac{T}{T_c}\right)$$

$$B_{c2}(T) = \sqrt{2}\,\kappa\, B_{\mathrm{cth}}(T)$$ with $B_{\mathrm{cth}} = \frac{\Phi_0}{2\pi\sqrt{2}\ \xi_{\mathrm{GL}}\,\lambda_{\mathrm{GL}}}$

➔ $B_{c2} \geq B_{\mathrm{cth}}$ for $\kappa > 1/\sqrt{2}$

interpretation of B_{c2}:

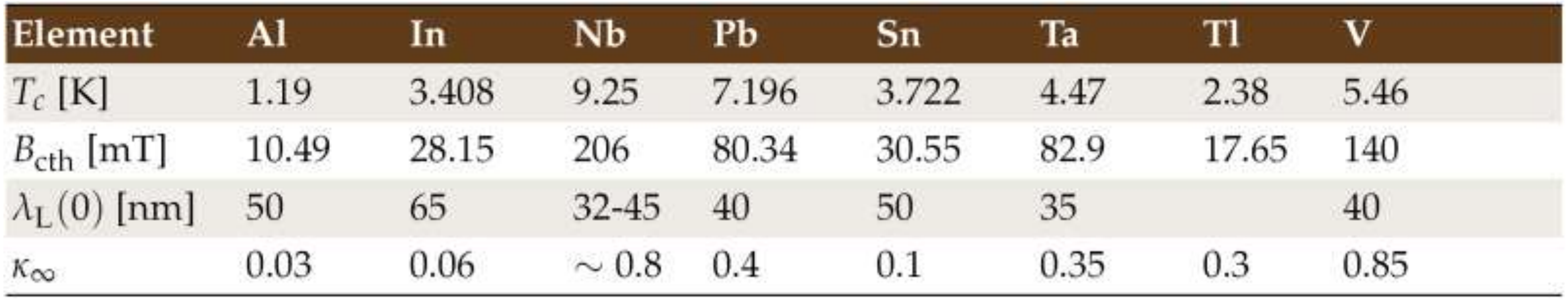
- ➢ as $n_s(r)$ is allowed to vary on length scale not smaller than $r \simeq \xi_{\mathrm{GL}}$, the minimum size of a N-region in the superconductor is $\simeq \pi\xi_{\mathrm{GL}}^2$
- ➢ for $B_{\mathrm{ext}} = B_{c2}$, the areal density of the flux quanta is just $B_{c2}/\Phi_0 \simeq 1/\pi\xi_{\mathrm{GL}}^2$, that is, for $B_{\mathrm{ext}} = B_{c2}$ the N-regions completely fill the superconductor

3.3.2 Type-II Superconductors: Upper and Lower Critical Field

$\kappa = \lambda_{\mathrm{GL}}/\xi_{\mathrm{GL}} \leq 1/\sqrt{2}$ type I superconductor

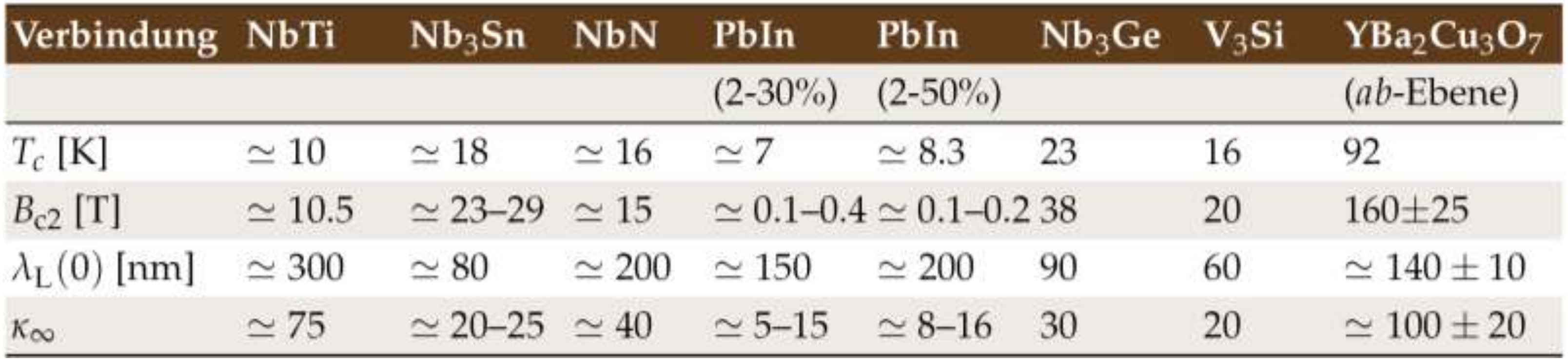
$\kappa = \lambda_{\mathrm{GL}}/\xi_{\mathrm{GL}} \geq 1/\sqrt{2}$ type II superconductor

B_{cth} and λ_{L} of type-I superconductors

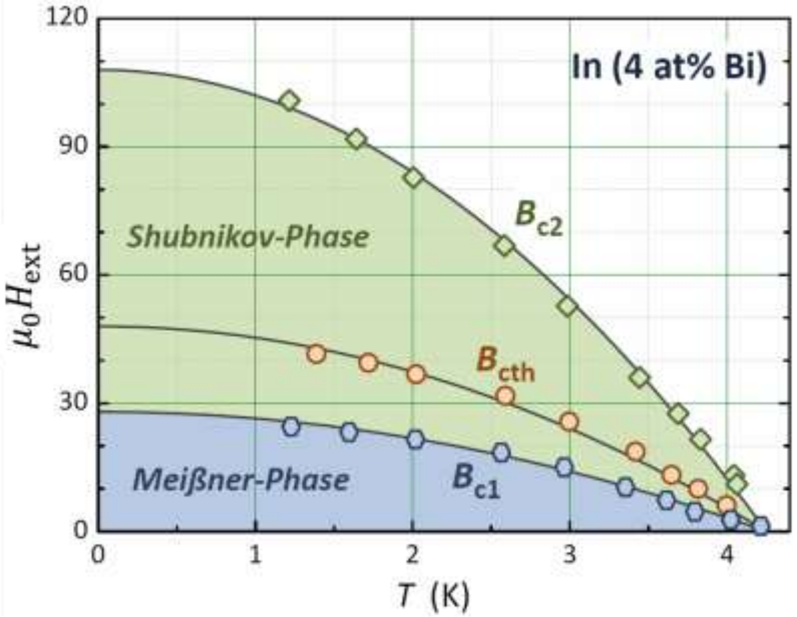


Element	Al	In	Nb	Pb	Sn	Ta	Tl	V
T_c [K]	1.19	3.408	9.25	7.196	3.722	4.47	2.38	5.46
B_{cth} [mT]	10.49	28.15	206	80.34	30.55	82.9	17.65	140
$\lambda_{\mathrm{L}}(0)$ [nm]	50	65	32-45	40	50	35		40
κ_∞	0.03	0.06	~ 0.8	0.4	0.1	0.35	0.3	0.85

B_{c2} and λ_{L} of type-II superconductors

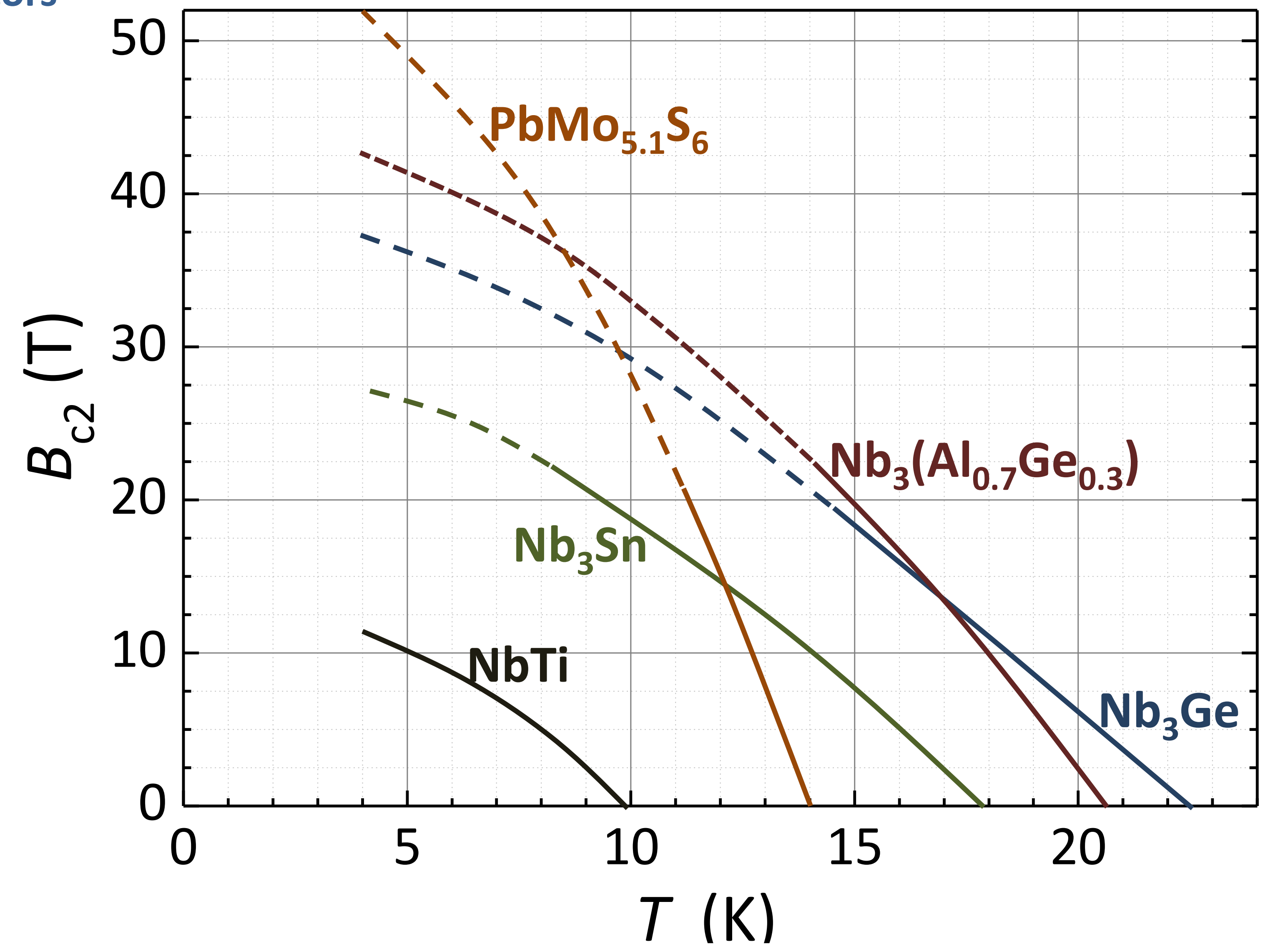


Verbindung	NbTi	Nb_3Sn	NbN	PbIn	PbIn	Nb_3Ge	V_3Si	$YBa_2Cu_3O_7$
				(2-30%)	(2-50%)			(*ab*-Ebene)
T_c [K]	$\simeq 10$	$\simeq 18$	$\simeq 16$	$\simeq 7$	$\simeq 8.3$	23	16	92
B_{c2} [T]	$\simeq 10.5$	$\simeq 23$–29	$\simeq 15$	$\simeq 0.1$–0.4	$\simeq 0.1$–0.2	38	20	160 ± 25
$\lambda_{\mathrm{L}}(0)$ [nm]	$\simeq 300$	$\simeq 80$	$\simeq 200$	$\simeq 150$	$\simeq 200$	90	60	$\simeq 140 \pm 10$
κ_∞	$\simeq 75$	$\simeq 20$–25	$\simeq 40$	$\simeq 5$–15	$\simeq 8$–16	30	20	$\simeq 100 \pm 20$



3.3.2 Type-II Superconductors: Upper and Lower Critical Field

B_{c2} of type II superconductors



3.3.2 Type-II Superconductors: Upper and Lower Critical Field

Task: derive the expression for B_{c1} from GLAG-equations

- derivation of ***lower critical field B_{c1}*** is more difficult (no linearization of GL equations possible)

→ we use simple argument, that flux generated by B_{c1} in area $\pi\lambda_{\mathrm{L}}^2$ must be at least equal to Φ_0

$$\int_0^\infty B_{c1} \exp\left(-\frac{r}{\lambda_{\mathrm{L}}}\right) 2\pi r \, \mathrm{d}r = \Phi_0$$

⇨ $$B_{c1} = \frac{\Phi_0}{2\pi\lambda_{\mathrm{L}}^2}$$

here, we have assumed $|\Psi(r)|^2 = n_s(r) = const.$ (London approximation)

- more precise result based on solution of GL equations:

$$\boxed{B_{c1} = \frac{\Phi_0}{2\pi\lambda_{\mathrm{L}}^2} (\ln\kappa + 0.08)}$$

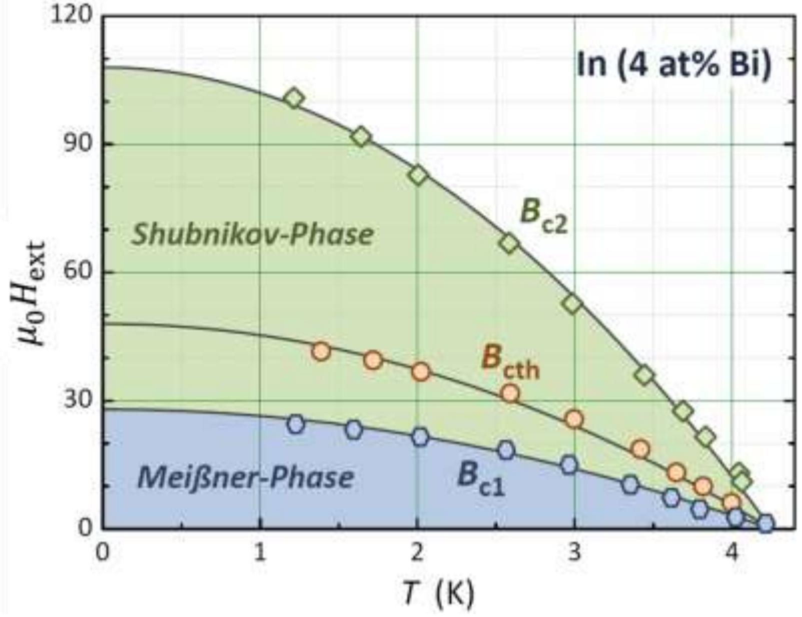
$$\boxed{B_{c1} = \frac{1}{\sqrt{2}\,\kappa} (\ln\kappa + 0.08)\, B_{\mathrm{cth}}}$$

with $B_{\mathrm{cth}} = \dfrac{\Phi_0}{2\pi\sqrt{2}\;\xi_{\mathrm{GL}}\,\lambda_{\mathrm{GL}}}$

➔ $B_{c1} \leq B_{\mathrm{cth}}$ for $\kappa > 1/\sqrt{2}$

3.3.3 Type-II Superconductors: Shubnikov Phase and Flux Line Lattice

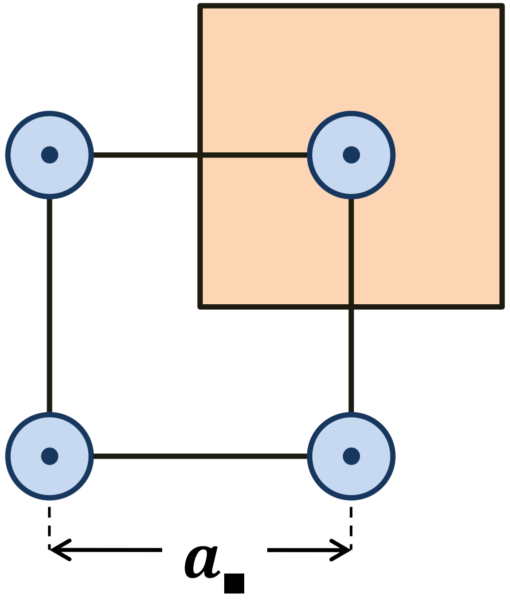
- solution of the GL-equations in the intermediate field regime $B_{c1} < B_{\text{ext}} < B_{c2}$ is in general complicated
  - ➢ linearization of GL-equations is no longer a good approximation
    → ***numerical solotion of GL equations***
  - ➢ here: only qualitative discussion

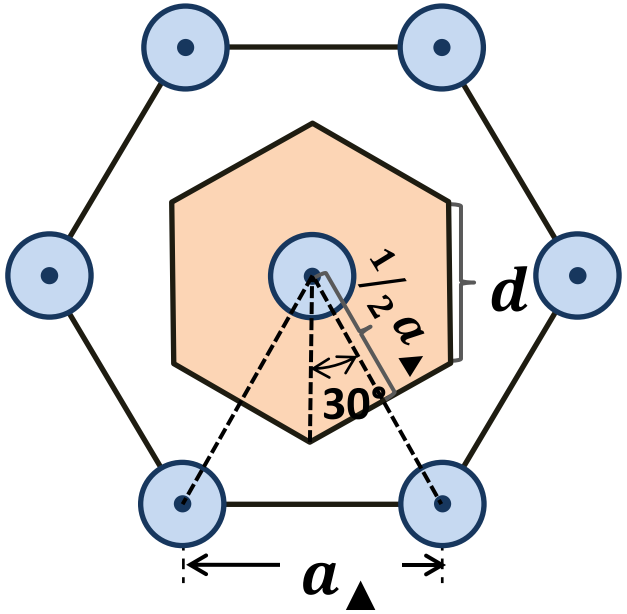


How is the magnetic flux arranged in Shubnikov phase above B_{c1}?

- ➢ due to negative N/S boundary energy for $B_{c1} \leq B_{\text{ext}} \leq B_{c2}$, magnetic flux is split into smallest possible portions to maximize N/S interface
- ➢ lower bound for flux portions is flux quantum Φ_0
- ➢ flux quanta behave like permanent magnets with parallel magnetic moment
  - → flux lines repel each other
  - → prefer arrangement with maximum separation between flux quanta
  - → optimum configuration is ***hexagonal flux line lattice*** ➔ ***Abrikosov Vortex Lattice***

3.3.3 Type-II Superconductors: Shubnikov Phase and Flux Line Lattice





$$\tan 30° = \frac{d/2}{a_{\blacktriangle}/2} = \frac{d}{a_{\blacktriangle}} \Rightarrow d = a_{\blacktriangle} \tan 30°$$

$$A_6 = \frac{3\sqrt{3}}{2} d^2 = \frac{3\sqrt{3}}{2} (a_{\blacktriangle} \tan 30°)^2 = \frac{\Phi_0}{B_{\mathrm{ext}}}$$

$$a_{\blacksquare} = \sqrt{\Phi_0 / B_{\mathrm{ext}}}$$

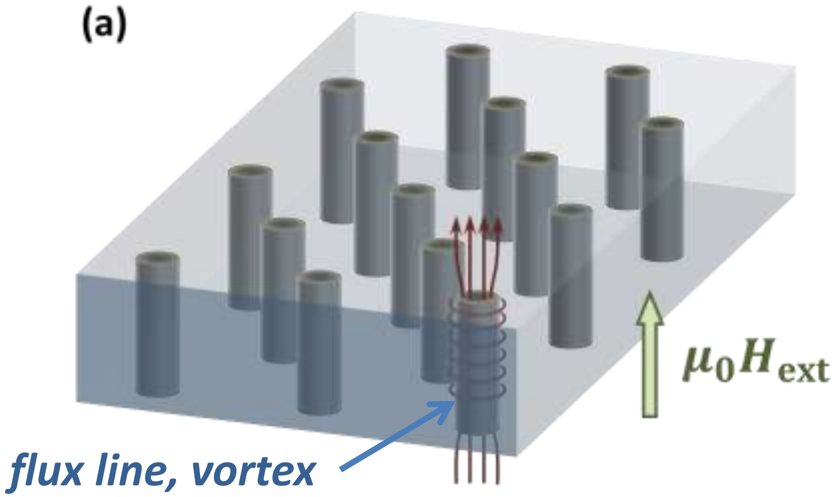
$$a_{\blacktriangle} = 1.075 \sqrt{\Phi_0 / B_{\mathrm{ext}}}$$

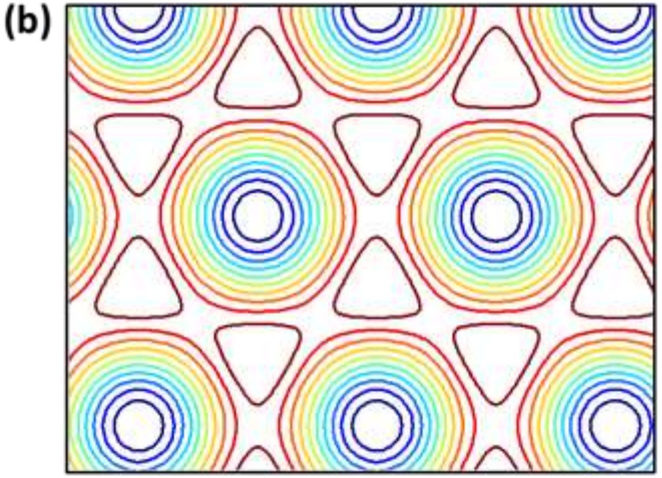
- distance between flux lines is maximum in hexagonal lattice
  - ➔ energetically most favorable state
  - ➔ square lattice also often observed, since other effects (e.g. Fermi surface topology) play a significant role

3.3.3 Type-II Superconductors: Shubnikov Phase and Flux Line Lattice

How does the spatial distribution of the magnetic flux density and the superfluid density look like in the Shubnikov-phase?

sketch of the flux line lattice in a type II SC

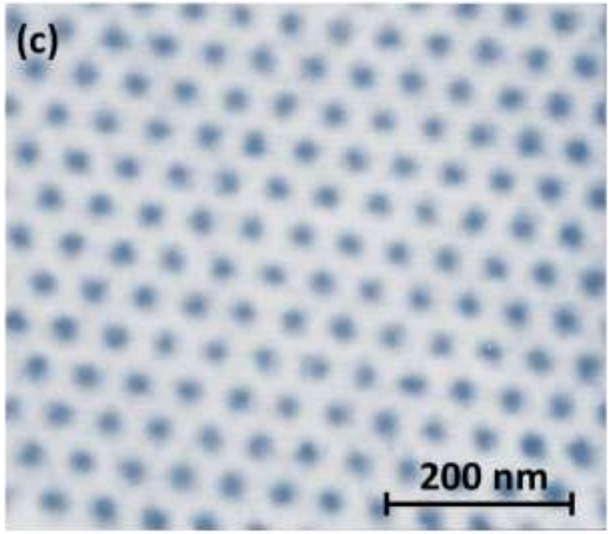


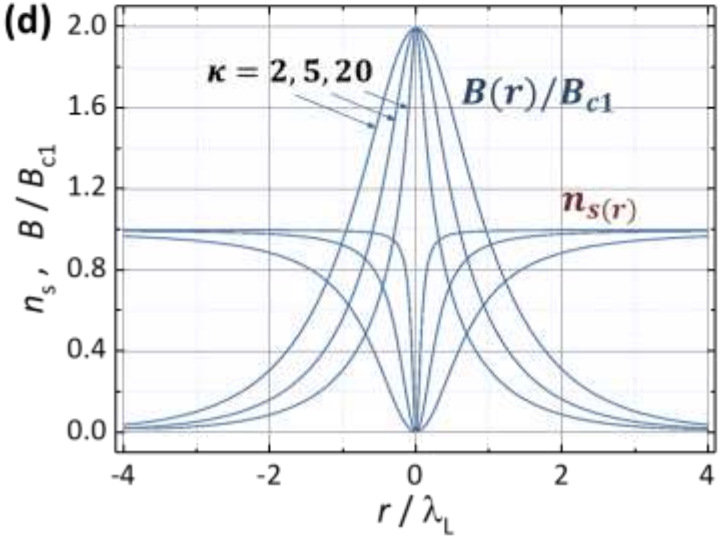


calculated contour lines of $n_s(\mathbf{r}) = |\Psi|^2(\mathbf{r})$ in the hexagonal Abrikosov vortex lattice

image of the flux line lattice in a NbSe$_2$-single crystal (type II SC) obtained by scanning tunneling microscopy @ $B_{\text{ext}} = 1$ T

(H. F. Hess et al., Phys. Rev. Lett. 62, 214 (1989), © (2012) American Physical Society)

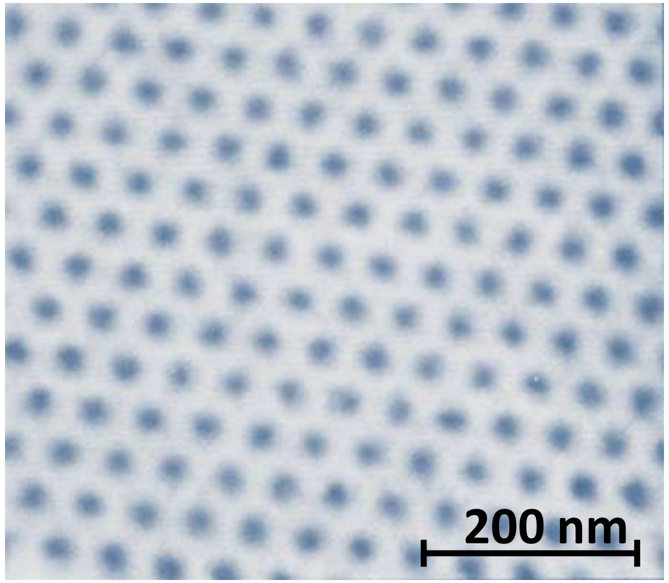




calculated radial distribution of $n_s(r)$ and $B(r)/B_{c1}$ for an isolated flux line

(E. H. Brandt, Phys. Rev. Lett. 78, 2208 (1997))

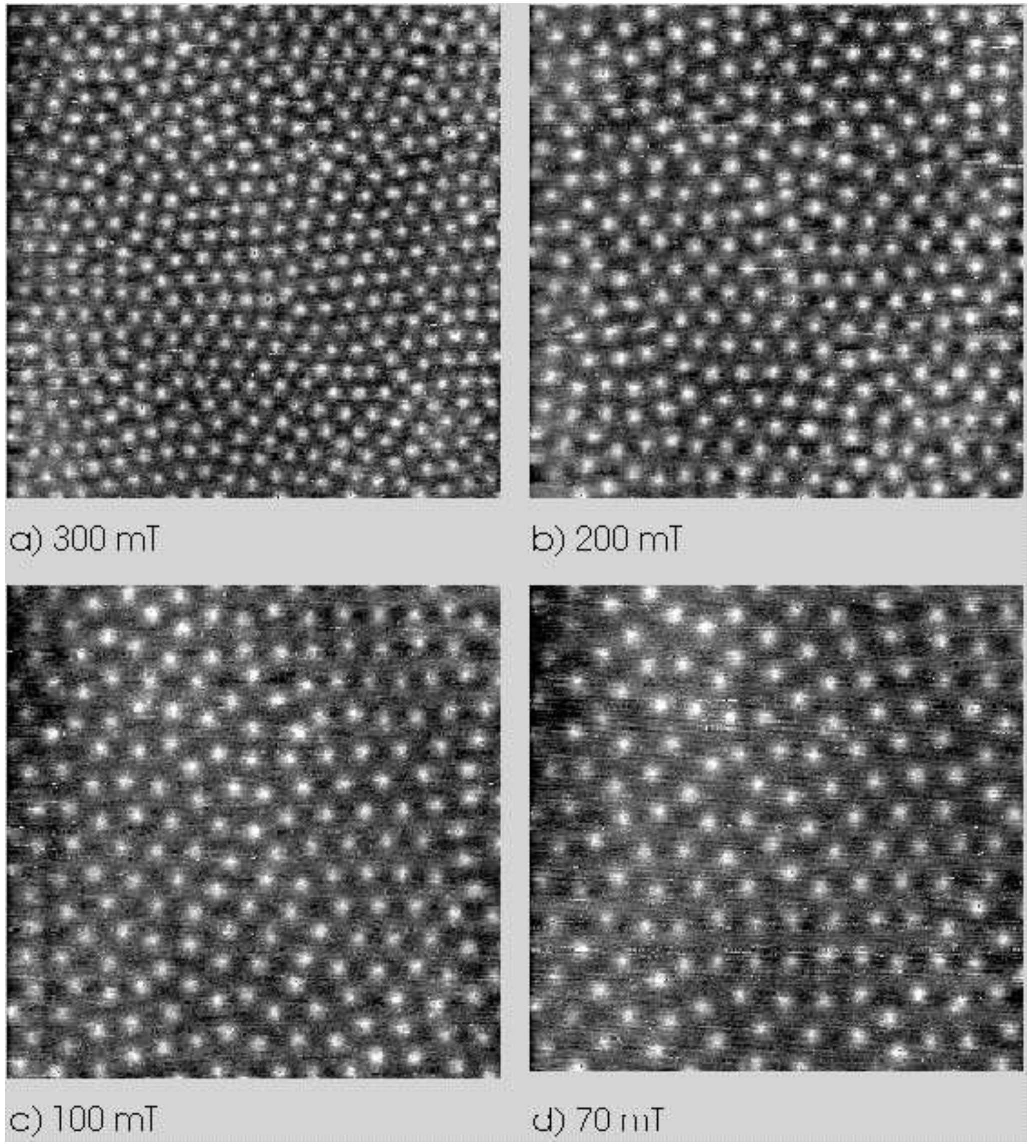
3.3.3 Type-II Superconductors: Shubnikov Phase and Flux Line Lattice



NbSe$_2$: flux-line lattice of non-irradiated single crystal at 1 T

distortion of ideal flux line lattice by defects

→ ***flux line pinning***



Right: STM-images showing the flux line lattice of ion irradiated NbSe$_2$ (T=3 K, I=40 pA, V=0.5 mV) taken during increasing the applied magnetic filed to 70, 100, 200, 300 mT. The images always show the same sample area of 2 x 2 µm (source: University of Basel)

3.3.3 Type-II Superconductors: Shubnikov Phase and Flux Line Lattice

Bitter technique:
decoration of flux-line lattice by „Fe smoke“
→ imaging by SEM

U. Essmann, H. Träuble (1968)
MPI Metallforschung
Nb, $T = 4$ K
disk: 1mm thick, 4 mm ø
$B_{\mathrm{ext}} = 985$ G, $a = 170$ nm

D. Bishop, P. Gammel (1987)
AT&T Bell Labs
YBCO, $T = 77$ K
$B_{\mathrm{ext}} = 20$ G, $a = 1\,200$ nm

similar work:
- L. Ya. Vinnikov, ISSP Moscow
- G. J. Dolan, IBM NY

3.3.4 Type-II Superconductors: Flux Lines

Radial dependence of $n_s(r)$ and $\mathrm{b}(r)$ across a single flux line

- **radial dependence of Ψ** (requires numerical solution of GL equations):

  we use the Ansatz

$$\widetilde{\Psi}(r) = \frac{\Psi(r)}{\Psi_0} = \widetilde{\Psi}_\infty \, f(r) \, \mathrm{e}^{\imath\theta(r)}$$

with $\widetilde{\Psi}_\infty = \widetilde{\Psi}(r \to \infty)$ and the radial function $f(r)$

insertion into the nonlinear GL equations yields equation for $f(r)$:

solution: $$\boxed{f(r) = \tanh\left(c \frac{r}{\xi_{\mathrm{GL}}}\right)}$$

with $c \approx 1$ and $n_s(r) = \left|\widetilde{\Psi}(r)\right|^2 = f^2(r)$

3.3.4 Type-II Superconductors: Flux Lines

- **radial dependence of $\mathbf{b}(r)$**

for simplicity we only calculate the London vortex by using the approximation $|\widetilde{\Psi}(r)| \simeq 1$

➔ good approximation for $\lambda_\mathrm{L} \gg \xi_\mathrm{GL}$ or $\kappa \gg 1$: extreme type II superconductors

2nd London equation $\nabla \times (\Lambda \mathbf{J}_s(r)) + \mathbf{b}(r) = \underbrace{\hat{\mathbf{z}}\, \Phi_0 \delta_2(r)}_{\text{accounts for the presence of vortex core}}$ $\delta_2(r) =$ 2D delta-function

interpretation:

with Maxwell eqn. $\nabla \times \mathbf{b}(r) = \mu_0 \mathbf{J}_s(r)$ we obtain $\lambda_\mathrm{L}^2\, \nabla \times (\nabla \times \mathbf{b}) + \mathbf{b} = \hat{\mathbf{z}}\, \Phi_0 \delta_2(r)$

integration over circular area S with $r \gg \lambda_\mathrm{L}$ perpendicular to $\hat{\mathbf{z}}$ yields

$$\underbrace{\int_S \mathbf{b} \cdot \mathrm{d}S}_{\Phi} + \underbrace{\lambda_\mathrm{L}^2 \oint_{\partial S} (\nabla \times \mathbf{b}) \cdot \mathrm{d}\ell}_{= 0 \text{ since } \nabla \times \mathbf{b} = \mu_0 \mathbf{J}_s \text{ and } \mathbf{J}_s \simeq 0 \text{ for } r \gg \lambda_L} = \hat{\mathbf{z}}\ \Phi_0 \quad \Rightarrow \quad \Phi = \Phi_0$$

➔ $$\boxed{\nabla^2 \mathbf{b}(r) - \frac{1}{\lambda_\mathrm{L}^2} \mathbf{b}(r) = -\frac{\Phi_0}{\lambda_\mathrm{L}^2} \hat{\mathbf{z}}\, \delta_2(r)}$$

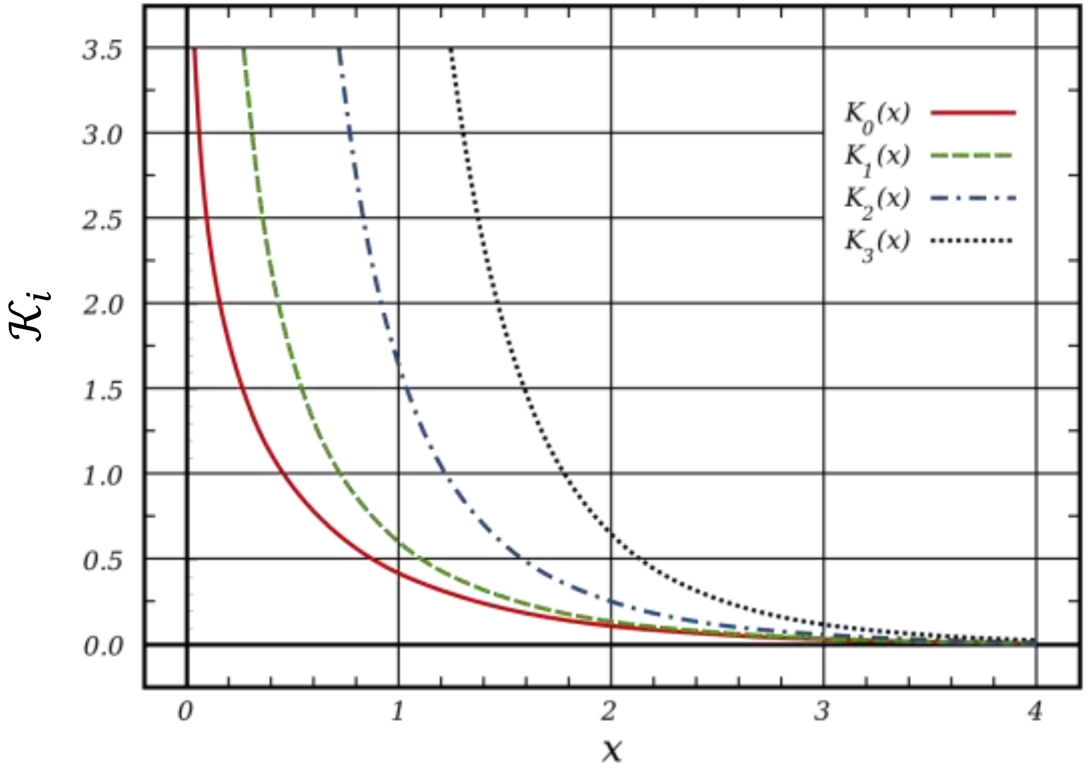
we use
$\nabla \times \nabla \times \mathbf{b} = \nabla(\nabla \cdot \mathbf{b}) - \nabla^2 \mathbf{b}$
$\nabla \cdot \mathbf{b} = \mathbf{0}$

3.3.4 Type-II Superconductors: Flux Lines

- solution of $$\nabla^2 \mathbf{b}(r) - \frac{1}{\lambda_\mathrm{L}^2}\mathbf{b}(r) = -\frac{\Phi_0}{\lambda_\mathrm{L}^2}\hat{\mathbf{z}}\,\delta_2(r)$$

⇨ $$b(r) = \frac{\Phi_0}{2\pi\lambda_\mathrm{L}^2}\mathcal{K}_0\left(\frac{r}{\lambda_\mathrm{L}}\right)$$

is exact result only if we assume $\xi_\mathrm{GL} \to 0$ ➔ **London solution**



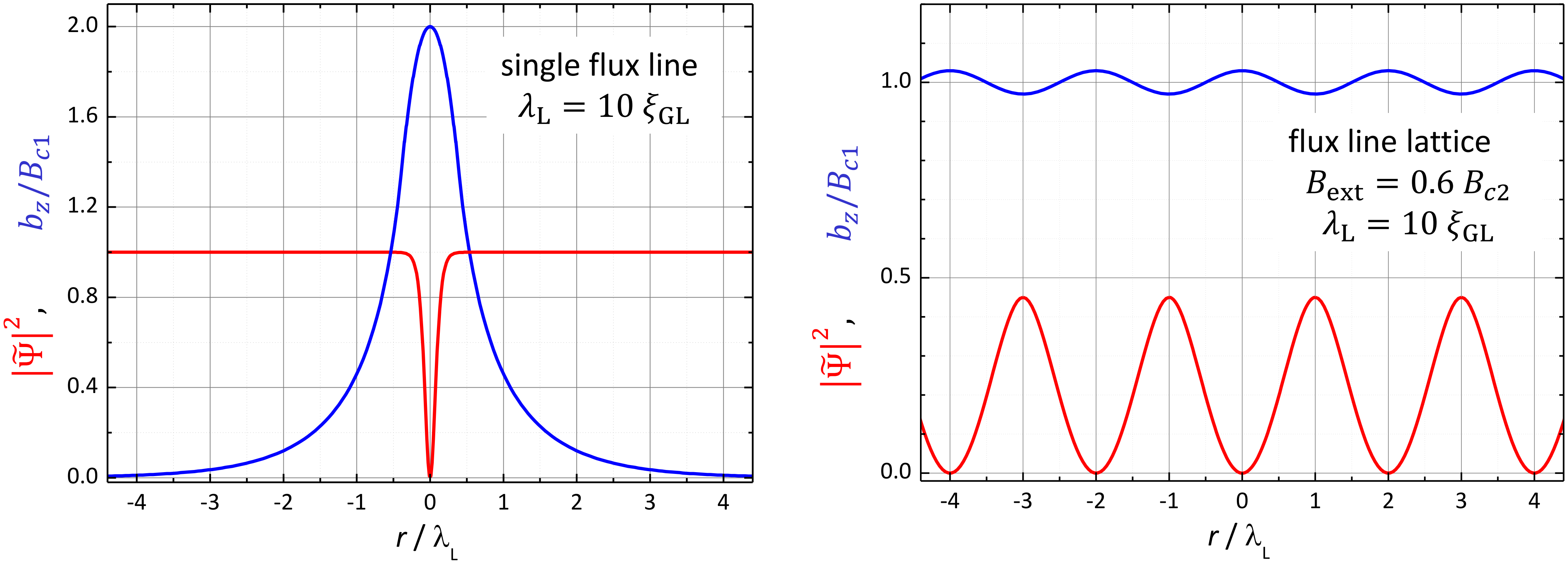
$\mathcal{K}_i$: ith

3.3.4 Type-II Superconductors: Flux Lines

- solution of $\nabla^2 \mathbf{b}(r) - \frac{1}{\lambda_L^2}\mathbf{b}(r) = -\frac{\Phi_0}{\lambda_L^2}\hat{\mathbf{z}}\,\delta_2(r)$ becomes more complicated if we assume finite ξ_{GL}

⇨ we have to take into account spatial variation of $\widetilde{\Psi}(r)$

numerical solution of GL equations



3.3.4 Type-II Superconductors

Further applications of the GL equations

- **calculation of the energy per unit length of a flux line** (London approximation: only field energy and kinetic energy of supercurrents)

$$\epsilon_{\mathrm{L}} = \frac{\Phi_0^2}{4\pi\mu_0\lambda_{\mathrm{L}}^2}\ln\kappa = \frac{B_{\mathrm{cth}}^2}{2\mu_0}\,4\pi\xi_{\mathrm{GL}}^2\ln\kappa = \underbrace{\frac{B_{\mathrm{cth}}^2}{2\mu_0}\,\pi\xi_{\mathrm{GL}}^2}\cdot 4\ln\kappa$$

with $B_{\mathrm{cth}} = \frac{\Phi_0}{2\pi\sqrt{2}\ \xi_{\mathrm{GL}}\,\lambda_{\mathrm{GL}}}$

ϵ_{L} corresponds to $4\ln\kappa$ times the loss of condensation in vortex core

- **calculation of nucleation field at surface of superconductor**

(in finite-size superconductors the boundary conditions at the surface have to be taken into account)

$$B_{c3} = 1.695\ B_{c2}$$

- **depairing critical current density (cf. 6.2.1)** (note that $|\Psi|^2$ decreases with increasing superfluid velocity)

$$J_{c,\mathrm{GL}}(T) = \frac{\Phi_0}{3\pi\sqrt{3}\ \mu_0\ \xi_{\mathrm{GL}}(T)\lambda_{\mathrm{GL}}^2(T)} = 0.544\ \frac{B_{\mathrm{cth}}(T)}{\mu_0\lambda_{\mathrm{L}}(T)}$$

with $B_{\mathrm{cth}} = \frac{\Phi_0}{2\pi\sqrt{2}\ \xi_{\mathrm{GL}}\,\lambda_{\mathrm{GL}}}$

3.3 Summary – GLAG Theory

The Ginzburg-Landau Theory explains:

- all London results
- type-II superconductivity (Shubnikov or vortex state): $\kappa = \frac{\lambda_{\mathrm{L}}}{\xi_{\mathrm{GL}}} > 1/\sqrt{2}$
- behavior at surface of superconductors and interfaces to non-superconducting materials

The Ginzburg-Landau Theory does not explain:

- $q_s = -2e$
- microscopic origin of superconductivity
- not applicable for $T << T_c$
- non-local effects

Literature:

- P.G. De Gennes, Superconductivity of Metals and Alloys
- M. Tinkham, Introduction to Superconductivity
- N.R. Werthamer in *Superconductivity*, edited by R.D. Parks

Summary of Lecture No. 6 (1)

- **normal metal/superconductor interface: boundary energy**

$$\Delta E_{\text{boundary}} = \Delta E_C + \Delta E_B \simeq \frac{B_{\text{cth}}^2}{2\mu_0}\left[\xi_{\text{GL}} - \left(\frac{B_{\text{ext}}}{B_{\text{cth}}}\right)^2 \lambda_{\text{GL}}\right]$$

$$\kappa = \lambda_{\text{GL}}/\xi_{\text{GL}} \leq 1/\sqrt{2} \quad \text{type I superconductor}$$
$$\kappa = \lambda_{\text{GL}}/\xi_{\text{GL}} \geq 1/\sqrt{2} \quad \text{type II superconductor}$$

I. **Type I superconductor: $\xi_{\text{GL}} \gtrsim \lambda_{\text{GL}}$**
   - boundary energy is always positive for $B_{\text{ext}} \leq B_{\text{cth}}$ → Meißner state up to $B_{\text{ext}} = B_{\text{cth}}$

II. **Type II superconductor: $\xi_{\text{GL}} \lesssim \lambda_{\text{GL}}$**
   - boundary energy is always positive for $B_{\text{ext}} \leq B_{c1}$ → Meißner state up to $B_{\text{ext}} = B_{c1}$
   - boundary energy becomes negative for $B_{\text{ext}} > B_{c1}$
     → formation of mixed state
     → type II SC can expel field $B_{c2} > B_{\text{cth}}$, as field expulsion work is lowered

- **formation of intermediate state in type-I and type-II SCs below B_{c1} due to finite demagnetization effects**

- **upper and lower critical field of type-II superconductors**

$$B_{c1} = \frac{\Phi_0}{2\pi\lambda_{\text{L}}^2}(\ln\kappa + 0.08)$$

$$B_{c1} = \frac{1}{\sqrt{2}\,\kappa}(\ln\kappa + 0.08)\,B_{\text{cth}}$$

➔ $B_{c1} \lesssim B_{\text{cth}}$ for $\kappa < 1/\sqrt{2}$

with $B_{\text{cth}} = \frac{\Phi_0}{2\pi\sqrt{2}\;\xi_{\text{GL}}\,\lambda_{\text{GL}}}$

$$B_{c2} = \frac{\Phi_0}{2\pi\xi_{\text{GL}}^2}$$

$$B_{c2}(T) = \sqrt{2}\,\kappa\,B_{\text{cth}}(T)$$

➔ $B_{c2} \geq B_{\text{cth}}$ for $\kappa > 1/\sqrt{2}$

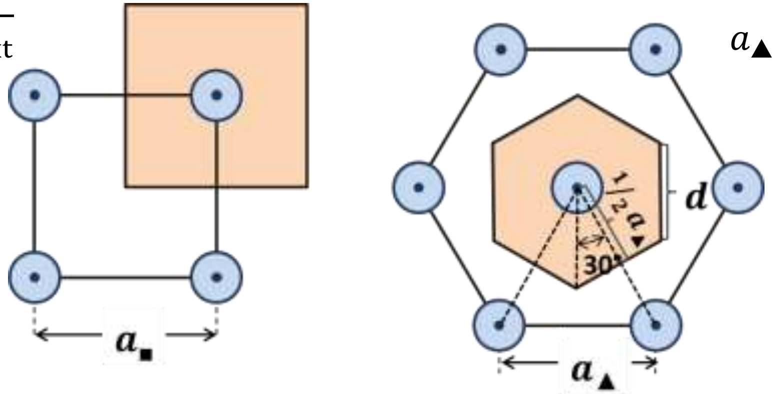
B_{c1}: flux density generates flux Φ_0 in area $\pi\lambda_{\text{L}}^2$, B_{c2}: normal cores of flux lines with area $\pi\xi_{\text{GL}}^2$ fill superconductor completely

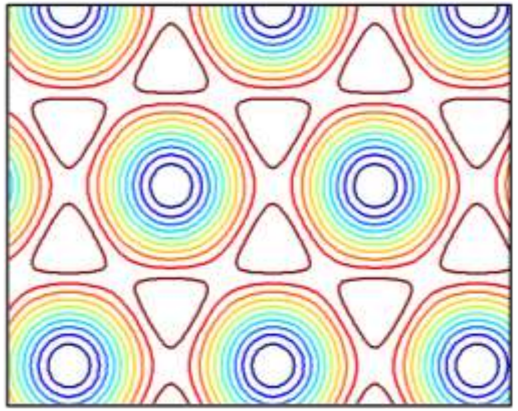
Summary of Lecture No. 6 (2)

- **flux line lattice**
  - flux quanta behave like permanent magnets with parallel magnetic moment
    - → flux lines repel each other
    - → arrangement with maximum separation between flux quanta
    - → optimum configuration is ***hexagonal (Abrikosov) flux line lattice***
  - spatial distribution of flux density $\mathbf{b}(\mathbf{r})$ and order parameter $n_s(\mathbf{r}) = |\Psi|^2(\mathbf{r})$ by numerical solution of GL equations

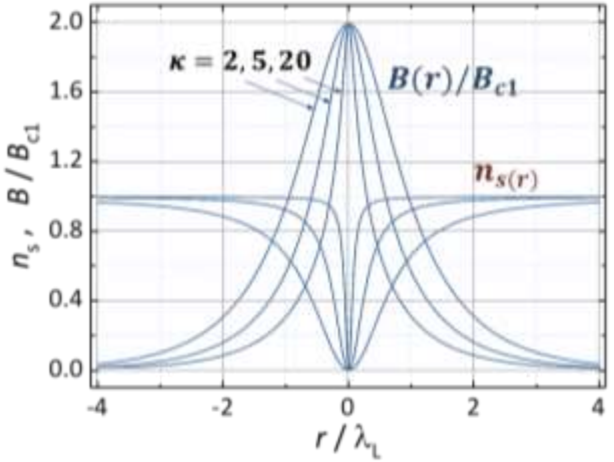
$a_\blacksquare = \sqrt{\Phi_0/B_\mathrm{ext}}$

$a_\blacktriangle = 1.075\sqrt{\Phi_0/B_\mathrm{ext}}$





calculated contour lines of $n_s(\mathbf{r}) = |\Psi|^2(\mathbf{r})$



radial distribution of $n_s(r)$ and $B(r)/B_{c1}$ for an isolated flux line

- **single flux line: radial dependence of $\mathbf{b}(r)$**

$$\nabla^2\mathbf{b}(r) - \frac{1}{\lambda_\mathrm{L}^2}\mathbf{b}(r) = -\frac{\Phi_0}{\lambda_\mathrm{L}^2}\hat{\mathbf{z}}\,\delta_2(r)$$

  - solution (with assumption $\xi_\mathrm{GL} \to 0$ ➔ London approximation)

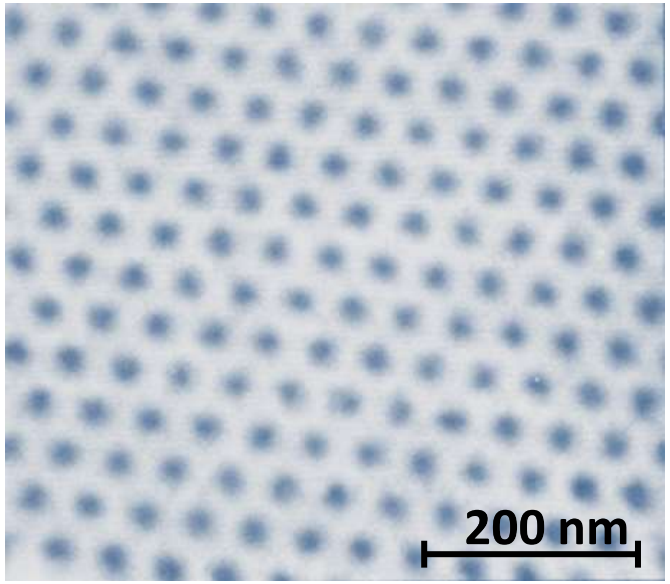
$$\mathbf{b}(r) = \frac{\Phi_0}{2\pi\lambda_\mathrm{L}^2}\mathcal{K}_0\left(\frac{r}{\lambda_\mathrm{L}}\right)$$

$\mathcal{K}_0$: 0th order modified Bessel function of 2nd kind

Summary of Lecture No. 6 (3)

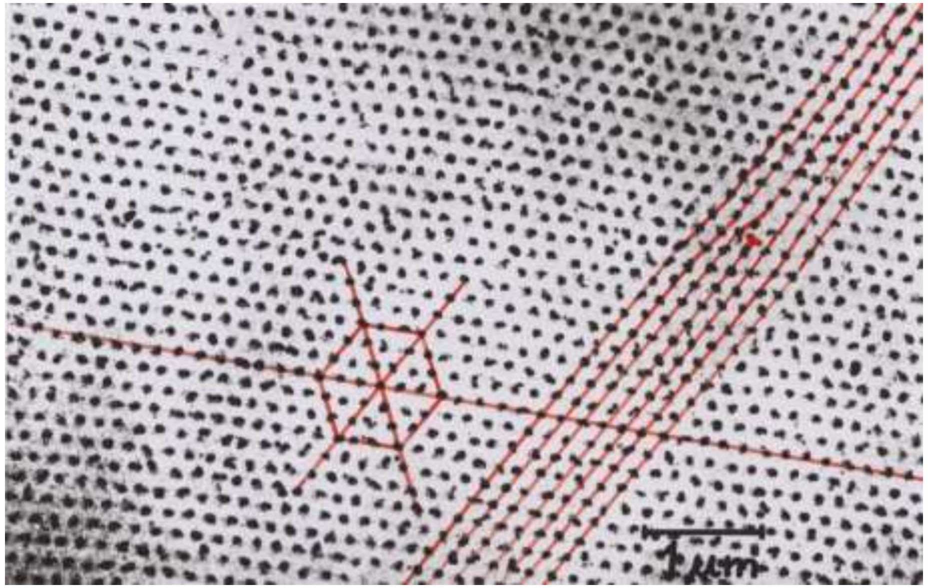
- **imaging of flux line lattice**

➢ scanning tunneling microscopy (Hess, 1989)
contrast by different DOS in vortex cores



NbSe$_2$: flux-line lattice of non-irradiated single crystal at 1 T

➢ Bitter technique (Träuble & Essmann, 1968)
decoration of vortex core by paramegnatic iron smoke (nanoparticles) and imaging by SEM



Nb, $T = 4$ K, disk: 1mm thick, 4 mm ø
$B_{\text{ext}} = 985$ G, $a = 170$ nm