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Exercise 11

Exercise to the Lecture

Superconductivity and Low Temperature Physics I
WS 2022/2023

5 Josephson Effect

5.1 Superconductor-Insulator-Superconductor (SIS) Josephson Junction

Exercise:

Use the time-dependent Schrödinger equation to derive the Josephson equations for a Superconductor-Insulator-Superconductor (SIS) junction. Make use of the fact that the strength of the tunnel coupling between the two superconducting electrodes is usually small and can be treated as a weak perturbation.

Solution:

We start from the time-dependent Schrödinger equation

$$\imath\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = E\psi(\mathbf{r},t)\,. \tag{1}$$

and describe the two superconducting electrodes 1 and 2 of the SIS junction by the macroscopic wave functions

$$\psi_1(\mathbf{r},t) = \psi_{0,1}(\mathbf{r},t)e^{\imath\theta_1(\mathbf{r},t)}\,, \qquad \psi_2(\mathbf{r},t) = \psi_{0,2}(\mathbf{r},t)e^{\imath\theta_2(\mathbf{r},t)}\,. \tag{2}$$

Here, $|\psi_{0,1}|^2 = n_{s1}$ and $|\psi_{0,2}|^2 = n_{s2}$ are the densities of the paired electrons in the two junction electrodes. Due to a finite coupling of the two wave functions via the tunneling barrier, the wave function $\psi_1(\mathbf{r},t)$ changes at a rate proportional to the strength of the tunnel coupling K_{12} to superconductor 2. Conversely, the wave function $\psi_2(\mathbf{r},t)$ changes at a rate proportional to the strength of the tunnel coupling K_{21} to superconductor 1. If the coupling strength is small compared to the stationary energy eigenvalues E_1 and E_2 of both superconductors, we can treat the coupled system in analogy to two weakly coupled harmonic oscillators. The time-dependent Schrödinger equations describing the coupled system read as

$$\imath\hbar\frac{\partial\psi_1(\mathbf{r},t)}{\partial t} = E_1\psi_1(\mathbf{r},t)+K_{12}\psi_2(\mathbf{r},t) = +e\Delta\phi\psi_1(\mathbf{r},t)+K_{12}\psi_2(\mathbf{r},t) \tag{3}$$

$$\imath\hbar\frac{\partial\psi_2(\mathbf{r},t)}{\partial t} = E_2\psi_2(\mathbf{r},t)+K_{21}\psi_1(\mathbf{r},t) = -e\Delta\phi\psi_2(\mathbf{r},t)+K_{21}\psi_1(\mathbf{r},t)\,. \tag{4}$$

Here we have assumed that the energy difference $E_1 - E_2 = |q_s|\Delta\phi = 2e\Delta\phi$ is given by the difference $\Delta\phi$ of the electrochemical potentials and we have chosen zero energy symmetrically between E_1 and E_2. By inserting the macroscopic wave functions (2) (this formally corresponds to a Madelung transformation) and separating into real and imaginary part we obtain for the imaginary part the expression

$$\frac{\partial\theta_1}{\partial t} = -\frac{K}{\hbar}\sqrt{\frac{n_{s2}}{n_{s1}}}\cos\varphi - \frac{e\Delta\phi}{\hbar} \tag{5}$$

$$\frac{\partial\theta_2}{\partial t} = -\frac{K}{\hbar}\sqrt{\frac{n_{s1}}{n_{s2}}}\cos\varphi + \frac{e\Delta\phi}{\hbar}\,, \tag{6}$$

where we have assumed $K_{12} = K_{21} = K$ and used the phase difference $\varphi = \theta_2 - \theta_1$ between the two superconductors. We see that the time variation of the phases θ_1 and θ_2 is given by the difference $\Delta\phi$ of the electrochemical potential and the phase difference φ.

For the real part we obtain

$$\frac{\partial n_{s1}}{\partial t} = +\frac{2K}{\hbar}\sqrt{n_{s1}n_{s2}}\,\sin\varphi \tag{7}$$

$$\frac{\partial n_{s2}}{\partial t} = -\frac{2K}{\hbar}\sqrt{n_{s1}n_{s2}}\,\sin\varphi\,. \tag{8}$$

We immediately see that eqs. (7) and (8) result in

$$\frac{\partial n_{s1}}{\partial t} = -\frac{\partial n_{s2}}{\partial t}\,, \tag{9}$$

expressing the conservation of the particle density. All particles leaving superconductor 1 have to enter superconductor 2 and vice versa, since there is neither a sink nor a source for particles at the interface. The particle numbers are obtained from the densities by multiplying with the volume V. If we multiply the temporal change of the particle numbers $V\frac{\partial n_{s1}}{\partial t}$ and $V\frac{\partial n_{s2}}{\partial t}$ by the charge $q_s = -2e$ of the superconducting charge carriers and divide by the junction area A, we obtain the supercurrents flowing from superconductor 1 to 2 and from 2 to 1, respectively:

$$J_s^{1\to 2} = 2e\frac{V}{A}\frac{\partial n_{s1}}{\partial t}\,, \qquad J_s^{2\to 1} = 2e\frac{V}{A}\frac{\partial n_{s2}}{\partial t}\,. \tag{10}$$

Note that these expressions denote the technical current densities. If, for example, $\partial n_{s1}/\partial t > 0$, i.e. paired electrons are flowing from 2 to 1 resulting in an increase of n_{s1}, the technical current direction describing the flow of positive charges is directed from 1 to 2.

Inserting eqs. (7) and (8) into eq. (10) yields the ***1. Josephson equation***

$$J_s = \frac{4eK}{\hbar}\frac{V}{A}\sqrt{n_{s1}n_{s2}}\,\sin\varphi = J_c\,\sin\varphi\,, \tag{11}$$

also denoted as ***current-phase relation***. The magnitude of the critical current density J_c depends on the densities of the paired electrons in both superconductors and the coupling strength K, which in turn depends on the details of the tunneling barrier such as barrier height V_0 and width d. For thick and high tunneling barriers ($\kappa d \ll 1$) it can be determined to (see lecture notes)

$$J_c \;=\; \frac{-q_s\hbar\kappa}{m_s}\, 2\sqrt{n_{s1}n_{s2}}\; \exp(-2\kappa d) \;=\; \frac{e\hbar\kappa}{m}\, 2\sqrt{n_{s1}n_{s2}}\; \exp(-2\kappa d)\,, \tag{12}$$

where we have used $q_s = -2e$ and $m_s = 2m$ and

$$\kappa \;=\; \sqrt{\frac{2m_s(V_0 - E_0)}{\hbar^2}} \tag{13}$$

is the decay constant of the wave function in the tunneling barrier. Since $J_c = I_c/A \propto \exp(-2\kappa d)$ and at the same time the normal state resistance times area product $\rho_n = R_n A \propto \exp(+2\kappa d)$, the product $J_c\rho_n = I_c R_n$ is independent of the junction area A. For SIS junctions consisting of two equal superconducting materials with energy gap Δ the $I_c R_n$-product is given by[1]

$$I_c(T)R_n \;=\; \frac{\pi\Delta(T)}{2e}\,\tanh\left(\frac{\Delta(T)}{k_B T}\right)\,. \tag{14}$$

For $T \to 0$, we obtain $I_c R_n = \pi\Delta(0)/2e$, i.e., the product of critical current and normal resistance of an SIS junction is determined by the energy gap of the superconductor.

For a bulk superconductor the supercurrent density $J_s(\mathbf{r},t)$ is related to the gauge invariant phase gradient $\gamma(\mathbf{r},t)$ by

$$J_s(\mathbf{r},t) \;=\; \frac{q_s n_s \hbar}{m_s}\left[\nabla\theta(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\mathbf{A}(\mathbf{r},t)\right] \;=\; \frac{q_s n_s \hbar}{m_s}\,\gamma(\mathbf{r},t) \tag{15}$$

Since J_s has to be the same in the junction electrodes and the barrier region (there are neither sinks nor sources for the supercurrent), the product $n_s\gamma$ has to be the same. Due to the exponential decay of the macroscopic wave function, $n_{s\mathrm{I}}$ is very small and hence γ_{I} very large in the barrier region. Conversely, n_{s1} and n_{s2} are large and hence γ_1 and γ_2 negligible in the junction electrodes. Therefore, we can use $\gamma_{1,2} \simeq 0$ in very good approximation and obtain for the integral of the gauge invariant phase gradient γ across the junction

$$\begin{aligned}\varphi(\mathbf{r},t) \;=\; \int_1^2 \gamma(\mathbf{r},t)\cdot d\boldsymbol{\ell} \;&=\; \int_1^2 \left(\nabla\theta(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\mathbf{A}(\mathbf{r},t)\right)\cdot d\boldsymbol{\ell} \\ &=\; \theta_2(\mathbf{r},t) - \theta_1(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot d\boldsymbol{\ell}\,. \end{aligned} \tag{16}$$

We now use the time derivative of this gauge invariant phase difference

$$\frac{\partial\varphi}{\partial t} = \frac{\partial\theta_2}{\partial t} - \frac{\partial\theta_1}{\partial t} - \frac{2e}{\hbar}\frac{\partial}{\partial t}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot d\boldsymbol{\ell} \tag{17}$$

[1] V. Ambegaokar, A. Baratoff, *Tunneling Between Superconductors*, Phys. Rev. Lett. **10**, 486-489 (1963).

and insert the time derivatives of the phases θ_1 and θ_2 according to (5) and (6). Doing so we write the potential difference $\Delta\phi$ as a line integral of a potential gradient. For $n_{s1} = n_{s2}$, this results in

$$\begin{aligned} \frac{\partial\varphi}{\partial t} &= \frac{2e\Delta\phi}{\hbar} - \frac{2e}{\hbar}\frac{\partial}{\partial t}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot d\boldsymbol{\ell} \\ &= \frac{2e}{\hbar}\underbrace{\int_1^2 \left(-\boldsymbol{\nabla}\phi - \frac{\partial\mathbf{A}}{\partial t}\right)\cdot d\boldsymbol{\ell}}_{=V} \quad . \end{aligned} \tag{18}$$

Here, V is the voltage drop across the tunneling barrier. The **2. *Josephson equation*** then reads as

$$\frac{\partial\varphi}{\partial t} = \frac{2eV}{\hbar} \ . \tag{19}$$

It is also denoted as ***voltage-phase relation***.

5.2 Josephson Junction with Applied AC Voltage

Exercise:

Calculate the current-voltage characteristic (IVC) of a Josephson junction biased by a voltage

$$V(t) \ = \ V_{\mathrm{dc}} + V_1\cos\omega_1 t \tag{1}$$

consisting of a dc voltage V_{dc} and an ac voltage with amplitude V_1 and angular frequency ω_1. Assume for simplicity that the normal resistance R_n of the junction is voltage independent.

Solution:

We start from the 2. Josephson equation (voltage-phase relation)

$$\frac{\partial\varphi(t)}{\partial t} \ = \ \frac{2eV(t)}{\hbar} \quad . \tag{2}$$

Time integration of the the voltage-phase relation yields

$$\varphi(t) \ = \ \varphi_0 + \frac{2\pi}{\Phi_0}V_{\mathrm{dc}}t + \frac{2\pi}{\Phi_0}\frac{V_1}{\omega_1}\sin\omega_1 t \ , \tag{3}$$

where φ_0 is an integration constant. Inserting this into the 1. Josephson equation, $I_s(t) = I_c\sin\varphi(t)$ (current-phase relation), we obtain

$$I_s(t) \ = \ I_c\sin\left\{\varphi_0 + \frac{2\pi}{\Phi_0}V_{\mathrm{dc}}t + \frac{2\pi}{\Phi_0}\frac{V_1}{\omega_1}\sin\omega_1 t\right\} \quad . \tag{4}$$

We see that the frequency of the Josephson current I_s is a superposition of the constant frequency $\omega_{\mathrm{dc}} = \frac{2\pi}{\Phi_0}V_{\mathrm{dc}}$ and a sinusoidally varying phase. Therefore, the frequency of the supercurrent is not the same as that of the driving ac voltage source. The reason for this is the fact that the nonlinear current-phase relation can couple different frequencies with the driving frequency.

In order to analyze the resulting complex time dependence of the Josephson current we rewrite eq. (4) as a Fourier series. In order to do so we use the Fourier-Bessel series identity

$$e^{\imath b \sin x} = \sum_{n=-\infty}^{+\infty} \mathcal{J}_n(b)\, e^{\imath n x} \quad . \tag{5}$$

Here, $\mathcal{J}_n$ is the n^{th} order Bessel function of first kind. It is evident from (4) that the argument of the sine function is of the form $(a + b\sin x)$. Hence, in order to use the identity (5) we write

$$\sin(a + b\sin x) = \Im\left\{e^{\imath(a+b\sin x)}\right\} \quad . \tag{6}$$

The Fourier-Bessel series together with the fact that $\mathcal{J}_{-n}(b) = (-1)^n \mathcal{J}_n(b)$ allows us to write

$$e^{\imath(a+b\sin x)} = \sum_{n=-\infty}^{+\infty} \mathcal{J}_n(b)\, e^{\imath(a+nx)} = \sum_{n=-\infty}^{+\infty} (-1)^n \mathcal{J}_n(b)\, e^{\imath(a-nx)} \quad . \tag{7}$$

Finally, the imaginary part of (6) then gives

$$\sin(a + b\sin x) = \sum_{n=-\infty}^{+\infty} (-1)^n \mathcal{J}_n(b)\, \sin(a - nx) \quad . \tag{8}$$

With $x = \omega_1 t$, $b = \frac{2\pi V_1}{\Phi_0 \omega_1}$ and $a = \varphi_0 + \omega_{\rm dc} t = \varphi_0 + \frac{2\pi}{\Phi_0} V_{\rm dc} t$, we can rewrite eq. (4) as

$$I_s(t) = I_c \sum_{n=-\infty}^{+\infty} (-1)^n \mathcal{J}_n\left(\frac{2\pi V_1}{\Phi_0 \omega_1}\right) \sin\left[(\omega_{\rm dc} - n\omega_1)t + \varphi_0\right] \quad . \tag{9}$$

We see that due to the nonlinear current-phase relation we obtain a current response to the applied voltage, in which the frequency $\omega_{\rm dc}$ couples to multiples of the driving frequency ω_1.

The most interesting aspect of eq. (9) is the fact that the ac voltage driving the junction can result in a dc current (denoted as Shapiro steps), if the argument of the sine function becomes zero. That is, we obtain a dc current response for

$$\omega_{\rm dc} = n\omega_1 \qquad \text{or} \qquad V_{\rm dc} = V_n = n\,\frac{\Phi_0}{2\pi}\,\omega_1 \quad . \tag{10}$$

For a specific n the amplitude of the time-averaged supercurrent is

$$|\langle I_s \rangle_n| = I_c \left| \mathcal{J}_n\left(\frac{2\pi V_1}{\Phi_0 \omega_1}\right)\right| \tag{11}$$

with the detailed value depending on the initial value φ_0 (see inset of Fig. 1).

For all other voltages $V_{\rm dc} \neq V_n$ we have a series of sinusoidally time dependent terms with a vanishing dc component of the supercurrent. Thus, for $V_{\rm dc} \neq V_n$ we have a finite current only due to quasiparticle tunneling. With $I_{\rm qp} = V(t)/R_n$ we obtain

$$\langle I \rangle = \underbrace{\langle I_s \rangle}_{=0} + \langle I_{\rm qp} \rangle = \frac{V_{\rm dc}}{R_n} + \underbrace{\langle \frac{V_1}{R_n}\cos\omega_1 t \rangle}_{=0} = \frac{V_{\rm dc}}{R_n} \quad . \tag{12}$$

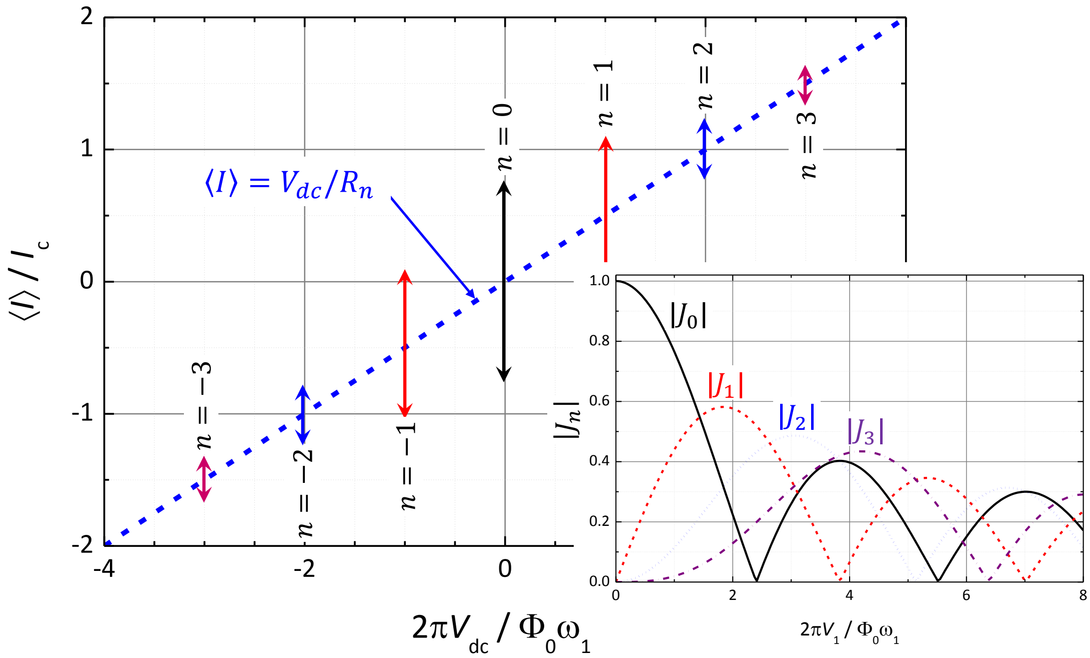

Figure 1: The dc component of the current plotted versus the applied dc voltage for a junction driven by an applied voltage $V(t) = V_{\rm dc} + V_1 \cos\omega_1 t$. At the voltages $V_n = n\frac{\Phi_0}{2\pi}\omega_1$ current steps, the so-called Shapiro steps appear. Their height has a Bessel function dependence on the amplitude V_1 of the ac voltage as shown in the inset.

Obviously, for $V_{\rm dc} \neq V_n$ we obtain a linear (ohmic) IVC if, as assumed, the normal resistance is independent of voltage. Only for $V_{\rm dc} = V_n$ a finite time-average of the Josephson current contributes. In summary, the IVC is an ohmic dependence with sharp current spikes at $V_{\rm dc} = V_n$ (see Fig. 1). The amplitude of the current spikes is given by (11) and depends on the amplitude V_1 of the ac voltage. The appearance of current steps at fixed voltages V_n that already has been predicted by **B. Josephson** is due to the formation of higher harmonics of the signal frequency due to the nonlinearity of the Josephson junction. The n^{th} step corresponds to the phase locking of the junction oscillation by this n^{th} harmonic.

When we are applying for example an ac driving voltage source with $\omega_1/2\pi = 10\,$GHz for various values of the applied dc voltage $V_{\rm dc}$, a constant dc current appears at $V_{\rm dc} = 0$ and $V_n = n\,\frac{\Phi_0}{2\pi}\,\omega_1 \simeq n \cdot 20\,\mu$V. That is, we obtain current steps in the IVC, which have constant spacing $\delta V = \frac{\Phi_0}{2\pi}\,\omega_1 \simeq 20\,\mu$V. Note that the spacing only depends on the frequency of the applied ac voltage and on fundamental constants.