Characterisation and analysis of RF-squid mediated tunable resonator coupling

Master's Thesis
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1. INTRODUCTION

In the past decade, circuit quantum electrodynamics (cQED)\cite{1, 2} has emerged as a highly vivid and interesting field, allowing for the investigations of the fundamental light-matter interaction in quantum mechanical systems\cite{3, 4}. It is also a very promising candidate for implementing quantum information processing algorithms\cite{5, 6}. In cQED experiments, it is essential to control the coupling between subsystems of the experiment. Basic building blocks in cQED are microwave resonators coupled to solid-state artificial atoms, similar to optical cavities interacting with trapped atoms. In experiments on fundamental light-matter interaction or in quantum information applications, resonators can be used, for example, as bus systems coupling multiple qubits or as long-lived quantum memories. Hence, the controlled transfer of excitation between two resonators via a tunable coupling is an important tool. In addition, tunable inter resonator coupling aims beyond traditional quantum optics experiments – such as creation and detection of entanglement and squeezing operations –, at making use of the scalability of experiments in cQED, to perform quantum simulation. For example, by arranging multiple coupled resonators in an array, with tunable nearest-neighbour interactions, arbitrary quasi-local and harmonic Hamiltonians can be simulated\cite{7}.

In recent work, tunable coupling between two resonators with static coupling via frequency tunable ancilla qubits, was demonstrated\cite{8, 9}. In this work, a tunable coupling between fixed resonators is presented, using a shared RF-SQUID (radio frequency superconducting quantum interference device) that acts as a mediator for second order coupling – as proposed in\cite{10, 7}. By externally applying a magnetic field, the coupling strength between the resonators can be controlled. To study the behaviour of our system, methods to describe it are developed – including a Gaussian flux-noise model. By applying them to data acquired in measurements of the transmission spectra, the properties of our system are determined and its adequacy for cQED-experiments is assessed.
This work is structured as follows: In Chapter 2, the system Hamiltonian is derived in order to study its spectrum in the framework of an input-output formalism. The latter describes the relationship between a coherent input (probe) field and the output of the device. Chapter 3.2 describes the measurement setup to record the transmission spectra of the system. In Chapter 4, the measurement data is presented and compared with the model developed in the second chapter. Finally, the performance of the device in the context of using it for photon swapping operations is discussed. In Chapter 5 the work is summarised and an outlook for possible further work on the subject is given.
2. Theory

In this chapter, the theoretical concepts used in this work are presented. First, the system as a whole is presented and described. Later, the components and interaction it consists of are described in more detail. Starting with the microwave resonators and a direct coupling, followed by the RF-SQUID, its interactions with the resonators and the mediated, indirect coupling between the resonators.

2.1. Basic interaction mechanisms

![Diagram](image)

Figure 2.1.: The basic interaction scheme in the studied system. Two LC-microwave resonators interact via two different coupling mechanisms. A first order direct coupling (orange) and a second order indirect coupling (green) mediated by a shared RF-SQUID (magenta). In good approximation the latter acts as a mutual inductance. The Josephson junction is depicted as a cross in the SQUID loop.

The system studied in this thesis consists of two microwave resonators that interact with each other. The interaction takes place through two different forms of coupling: On
the one hand, a direct exchange of energy (first order coupling) and on the other hand, each resonator exchanges energy with a mediator that intermediately stores parts of the resonator field energy in its magnetic field and transfers it to the other resonator (second order coupling). It is important to note, that this energy transfer happens off resonance, so that the RF-SQUID itself never is excited. The interaction strength between the mediator and the resonators is externally tunable.

The two resonators are realised as coplanar transmission line resonators. When approaching each other, their electromagnetic-field overlap leads to a direct coupling, called geometric coupling. Also, both resonators share an RF-SQUID to which they couple inductively (see Figure 2.1). The RF-SQUID acts as a mutual inductance through which both transmission lines can couple to each other. The mutual inductance can be tuned by an externally applied magnetic field threading the RF-SQUID area.

### 2.2. CPW and CSL transmission line resonators

CPW (coplanar waveguide) and CSL (coplanar slotline) transmission lines are coplanar structures of parallel (superconducting) conductors that are able to guide propagating electromagnetic waves. Because of their geometry, the wave guides support QTEM (quasi transversal electromagnetic) eigenmodes that can be described as one-dimensional waves. In the case of a coplanar waveguide (CPW) there is one inner conductor enclosed by two outer conductors (ideally on the same potential) sitting on a substrate. On the other hand, a coplanar slotline waveguide (CSL) only has a single outer conductor.

To obtain the dynamics of such a system, it can be modelled as an equivalent electrical circuit consisting of inductances (of the conductors) and capacitances (between the conductors). A lumped circuit is assumed to be much smaller than the wavelength of the voltage and current signals. In contrast the transmission line is modelled as a distributed element network of inductor-capacitor circuits (LC-circuits) [11] with some inductance and capacitance per unit length (see Figure 2.2). In the continuum limit, where each element is assumed to be of infinitesimal size, classically the telegrapher equations emerge (Equation (A.2)) [11, 12].
2.2. CPW and CSL transmission line resonators

Figure 2.2.: Equivalent distributed element circuit diagram for a coplanar transmission line with inner and outer conductor(s). Every infinitesimal segment of the conductors is modelled as a LC-circuit. The inner conductor with inductance density $l_0$ is capacitively ($c_0$) connected to the outer conductor(s) at ground potential. The whole transmission line is terminated by coupling capacitors.

To obtain a resonating structure – a transmission line resonator – the transmission line is terminated by coupling capacitors. Because of interference standing waves with wavelengths of integer multiples of the resonator length establish. A transmission line resonator becomes «quantum» at low temperatures, when the system relaxes to the ground state, i.e. below an energy of $\hbar \omega \ll k_B T$.

As shown in Appendix A, a coplanar transmission line, that is cooled to its ground state, can be described as a quantum mechanical resonator. Simplified, a constant inductance $l_0$ and capacitance $c_0$ per unit length is assumed. This leads to the harmonic Hamiltonian[11, 13]:

$$\mathcal{H}_{\text{tl}} = \hbar \sum_n \omega_n a_n^\dagger a_n ,$$  

(2.1)

with the annihilation (creation) operators $a_n$ ($a_n^\dagger$) of the $n$-th mode with frequency $\omega_n$. These operators can be expressed in terms of charge $\hat{\theta}_n$ and flux $\hat{\phi}_n$ operators:

$$a_n = \sqrt{\frac{1}{2\hbar}} \left( \sqrt{\omega_n C_T} \hat{\phi}_n + \frac{i}{\sqrt{\omega_n C_T}} \hat{\theta}_n \right) ,$$

$$a_n^\dagger = \sqrt{\frac{1}{2\hbar}} \left( \sqrt{\omega_n C_T} \hat{\phi}_n - \frac{i}{\sqrt{\omega_n C_T}} \hat{\theta}_n \right) ,$$

for a transmission line of length $2\ell$ and the total capacitance $C_T = 2\ell c_0$. The node charge $\hat{\theta}_n$ (node flux $\hat{\phi}_n$) is defined as the time integral over the current (voltage) measured along the path that connects the node $n$ with the ground node [14].

With Equation (2.1) the dynamics of a transmission line resonator can be described. The spatial profile of the ground mode can also be calculated and is plotted in Figure 2.3 in terms of voltage and current. In the middle of the transmission line the voltage has a
2.3. Coupling mechanisms

![Figure 2.3: Normalised spatial profile of current $i(x)$ and voltage $v(x)$ of the ground mode of a transmission line cavity. $x$ defines the position on the transmission line of length $2\ell$, with $x = 0$ the middle of the transmission line.](image)

node and the current is maximal. Thus, the RF-SQUID, as described in later sections, is placed at the middle of the transmission line resonators, to maximise the (inductive) interaction strength between them and the RF-SQUID.

2.3. COUPLING MECHANISMS

2.3.1. GEOMETRIC COUPLING

When transmission line resonators approach each other, their fields overlap. Due to this fact, they begin to interact with each other. This is illustrated in Figure 2.4, where the transmission lines of length $2\ell$ approach each other over a length $2\ell_c$.

The interaction can be described using a distributed equivalent circuit model, where two transmission lines are coupled inductively and capacitively (compare Figure B.1b). The capacitive coupling can be neglected in the current setup, because the transmission lines only approach each other at a voltage node (compare with Figure 2.3). As derived in
2.3. Coupling mechanisms

Appendix B, the interaction can be expressed by the following interaction Hamiltonian in the ground mode

\[ H_{\text{geo}} = -\hbar g_{\text{geo}} (a_1^\dagger + a_1)(a_2^\dagger + a_2), \]  

(2.2)

with the annihilation operator \( a_j(t) = \hat{a}_j e^{-i\omega_j t} \) and creation operator \( a_j^\dagger(t) = \hat{a}_j^\dagger e^{i\omega_j t} \) of the intra cavity field of resonator \( j \), introducing the geometric coupling constant \( g_{\text{geo}} \) [7, 14, 15]. As described in Equation (A.6), \( \phi_j \propto a_j^\dagger + a_j \) in Equation (2.2) can be understood as the flux generated by the transmission line \( j \). Due to this flux one transmission line induces a current inside the other. Thus, the Hamiltonian expresses an interaction between the resonator fluxes with a coupling strength \( g_{\text{geo}} \).

2.3.2. Dynamic coupling

Both transmission line resonators are galvanically coupled to an RF-SQUID as sketched in Figure 2.5, that is they share a segment of their inner conductor with the RF-SQUID loop. This leads to a tunable second order (dynamic) coupling. In this section, a model for this mechanism is derived.

RF-SQUID

A Josephson junction is composed of two superconductors interrupted by a thin insulating layer. In the superconducting state, Cooper pairs are able to tunnel through the insulating barrier. As the tunnelling of Cooper pairs is a coherent phenomenon, it can be described...
2.3. Coupling mechanisms

by a macroscopic wave function. The tunnel current only depends on the phase difference across the junction – the gauge invariant phase difference \( \varphi \). Josephson junctions can be described with the well-known Josephson equations, that also can be derived from this macroscopic model. The current-phase relation

\[
I_J = I_c \sin \varphi
\]  

(2.3)
describes the supercurrent as it flows through the junction. At the critical current, \( I_c \), the Cooper pairs begin to break up and there is an additional normal conducting tunnel current. Similar to an inductance, the junction can store energy reversibly. This energy is called Josephson coupling energy [16],

\[
E_J = \frac{\Phi_0 I_c}{2\pi} (1 - \cos \varphi) \equiv E_{J0} (1 - \cos \varphi),
\]  

(2.4)
with the magnetic flux quantum \( \Phi_0 = \hbar/2e \). To stress the resemblance to a conventional inductance, the nonlinear Josephson inductance,

\[
L_J = \frac{\Phi_0}{2\pi I_c \cos \varphi} \left[ \left( \frac{2\pi}{\Phi_0} \right)^2 \frac{\partial^2 E_J}{\partial \varphi^2} (\varphi) \right]^{-1}.
\]  

(2.5)
can be defined, which is valid for a current smaller than the critical current \( I_J \).

An RF-SQUID is a superconducting loop interrupted by a single Josephson junction. The gauge invariant phase \( \varphi \) can now be expressed in terms of the total (magnetic) flux \( \Phi \) threading the loop area,

\[
\varphi = \frac{2\pi}{\Phi_0} \Phi,
\]  

(2.6)
leading to

\[
I_J = -I_c \sin \left( \frac{2\pi}{\Phi_0} \Phi \right)
\]  

(2.7)
In turn, the total flux \( \Phi \), threading the RF-SQUID loop area, is the sum of an externally applied field \( \Phi_{\text{ext}} \) and of the flux \( L_s I_J \). The ladder is generated by a screening current in the loop with inductance \( L_s \). Thus,

\[
\Phi = \Phi_{\text{ext}} + L_s I_c \sin \left( \frac{2\pi}{\Phi_0} \Phi \right)
\]  

\[
= \Phi_{\text{ext}} - \frac{\Phi_0}{2\pi} \beta_L \sin \left( \frac{2\pi}{\Phi_0} \Phi \right),
\]  

(2.8)
introducing the screening parameter $\beta_L = \frac{2\pi}{\Phi_0} L_s I_c$. The screening parameter is a measure of how strong the RF-SQUID reacts to an externally applied field with a screening current.

In Equation (2.8), the total flux is defined only implicitly. Hence, a closed analytical form of $\Phi(\Phi_{\text{ext}})$ does not exist. However, it is straightforward to find solutions numerically.

**Interaction Hamiltonian**

![Diagram](image)

Figure 2.5.: Both transmission lines each share a segment of their inner conductor with a branch of the RF-SQUID between them. By varying the externally applied magnetic field, the RF-SQUID flux is biased and consequently makes the mediated coupling tunable.

In the following, the interaction between two transmission line resonators sharing a line segment with an RF-SQUID in the middle, is described. By coupling to an RF-SQUID, a resonator can deposit a part of the field energy it creates into the RF-SQUID according to its susceptibility. This energy now can be injected into the second resonator, enabling an indirect exchange of energy between both resonators. This susceptibility depends on the externally applied flux $\Phi_{\text{ext}}$ and in that way the coupling strength between the resonators can be controlled. For $\beta_L \approx 1$, an additional screening current in the ring has to be taken into account.

In order to derive the interaction Hamiltonian for the dynamic coupling, the RF-SQUID coupler is modelled as a passive mutual inductance. However, because the capacity of the Josephson junction, combined with its inductance and that of the loop, an RF-SQUID itself is a (nonlinear) resonator and can be excited. As a consequence this semi-classical model is only valid for off-resonant coupling between the transmission lines and the RF-
2.3. Coupling mechanisms

SQUID. In this case, the RF-SQUID remains in the ground state and follows the interaction adiabatically. The full interaction Hamiltonian[17] can be written as

\[ H_{\text{int}} = E_{\text{cap}} + E_J + E_{\text{ind}}, \]

where the first term represents the capacitive energy of the Josephson junction, which in this case, for an off-resonant treatment of the RF-SQUID, can be neglected. The second term is the Josephson coupling energy. The third term corresponds to the magnetic field energy of the loop.

As derived in references [17, 10, 18] (for the coupling of two qubits), the interaction Hamiltonian of the dynamic coupling in the adiabatic approximation can be treated like an effective inductive coupler. The differential inductance \(L_{RF}(\Phi_{\text{ext}})\) corresponds to the response of the RF-SQUID screening current circulating in the ring,

\[ I_J(\Phi_{\text{ext}}) = -I_c \sin \left[ \frac{2\pi}{\Phi_0} \Phi(\Phi_{\text{ext}}) \right], \]

to changes in the external applied flux \(\Phi_{\text{ext}}\) [19, 17]. The inductance can be written as

\[ \frac{1}{L_{RF}(\Phi_{\text{ext}})} = \frac{dI_J}{d\Phi_{\text{ext}}}(\Phi_{\text{ext}}) \]

\[ = - \frac{2\pi I_c}{\Phi_0} \cos \left[ \frac{2\pi}{\Phi_0} \Phi(\Phi_{\text{ext}}) \right] \frac{d\Phi}{d\Phi_{\text{ext}}}(\Phi_{\text{ext}}). \]

(2.9)

As mentioned above, \(\Phi(\Phi_{\text{ext}})\) cannot be expressed in a closed analytical form. However, as stated in the implicit function theorem, the derivative \(\frac{d\Phi}{d\Phi_{\text{ext}}}\) can be calculated explicitly using \(\Phi_{\text{ext}}(\Phi) = \Phi + \frac{\Phi_0}{2\pi} \beta_L \sin \left( \frac{2\pi}{\Phi_0} \Phi \right)\):

\[ \frac{d\Phi}{d\Phi_{\text{ext}}}(\Phi_{\text{ext}}) = \frac{1}{\frac{d\Phi_{\text{ext}}}{d\Phi}(\Phi(\Phi_{\text{ext}}))} \]

\[ = \frac{1}{1 + \beta_L \cos \left( \frac{2\pi}{\Phi_0} \Phi(\Phi_{\text{ext}}) \right)}. \]

(2.10)
2.3. Coupling mechanisms

Substituting (2.10) into (2.9) eventually yields

\[
\frac{1}{L_{\text{RF}}(\Phi_{\text{ext}})} = \frac{1}{L_s} \frac{\beta_L \cos \left( \frac{2\pi}{\Phi_0} \Phi(\Phi_{\text{ext}}) \right)}{1 + \beta_L \cos \left( \frac{2\pi}{\Phi_0} \Phi(\Phi_{\text{ext}}) \right)}.
\]  

(2.11)

Note that unlike lumped element inductances, \( L_{\text{RF}} \) also can be negative.

The Hamiltonian describing the effective inductive energy, can be written as:

\[
\mathcal{H}_{\text{int}} = \frac{1}{2L_{\text{RF}}(\Phi_{\text{ext}})}(\Phi_2 - \Phi_1)^2,
\]  

(2.12)

where \( \Phi_j (j = 1, 2) \) denote the fluxes generated by the transmission line branches (illustrated in Figure 2.5) in the ground mode and correspond to the flux coupled into the RF-SQUID. This resembles a perturbation ansatz: The fluxes generated by the transmission lines are treated as small perturbations of the field inside the RF-SQUID. Therefore, this model is valid only for \(|\Phi_2|, |\Phi_1| \ll \Phi_0\). The flux generated by the branches of the transmission line is given by

\[
\int_{x_1}^{x_2} dx l_0 \cdot i_0(x) \approx \Delta x l_0 \cdot i_0(x_1),
\]

with the position \( x_1 \) and \( x_2 \), where the arms of the RF-SQUID touch the transmission lines, \( \Delta x l_0 \) the inductance of the branch shared with the RF-SQUID of length \( \Delta x \) and \( i_0(x_1) \) the current at position \( x_1 = 0 \). The branch fluxes \( \Phi_j \) can also be expressed in terms of the node fluxes \( \phi_j(x, t) \) (compare with Appendix A) at the positions \( x_1 \) and \( x_2 \). By using a separation ansatz, \( \phi_j(x, t) = \phi_{j,0}(t) u_0(x) \) (compare with Equation (A.3)) and concentrating on the ground mode, \( n = 0 \), they can be approximated by

\[
\Phi_j = \phi_j(x_2, t) - \phi_j(x_1, t)
\]

\[
\approx \phi_{j,0}(t) \Delta x \partial_x u_0(x_1)
\]

\[
= \phi_{j,0}(t) \Delta x l_0 \cdot i_0(x_1).
\]

This is a good approximation as long \( \Delta x \ll \lambda \), with \( \lambda \) the wavelength of the ground mode. The spatial mode, as derived in Appendix A in Equation (A.5), can be written as \( u_0(x) = \sqrt{2} \sin(\sqrt{l_0 c_0} \omega_0 x) \) – where \( l_0 \) and \( c_0 \) are the inductance and capacitance per unit.
length and $\omega_0$ the ground mode frequency – and is plotted in Figure 2.3. At this point the Hamiltonian reads

$$H_{RF} = \frac{(\Delta x \partial_x u_0(0))^2}{2L_{RF}(\Phi_{ext})}(\phi_{2,0} - \phi_{1,0})^2.$$  

After quantising $\phi_{j,0}(t) = \sqrt{\frac{\hbar}{2\omega_0 C_r}} \left( a_j(t) + a_j(t)^\dagger \right)$ (approximating it with the unperturbed flux, see Equation (A.6)) and introducing the dynamic coupling coefficient,

$$g_{RF}(\Phi_{ext}) = \frac{(\Delta x \partial_x u_0(0))^2}{\omega_0 C_r} \frac{1}{2L_{RF}(\Phi_{ext})} \cdot$$

$$= -\frac{l_0 \Delta x^2 \omega_0}{2\ell L_s} \frac{\beta_L \cos \left( \frac{2\pi}{\Phi_0} \Phi(\Phi_{ext}) \right)}{1 + \beta_L \cos \left( \frac{2\pi}{\Phi_0} \Phi(\Phi_{ext}) \right)} \equiv g_{RE0},$$

$$= g_{RE0} \frac{\beta_L \cos \left( \frac{2\pi}{\Phi_0} \Phi(\Phi_{ext}) \right)}{1 + \beta_L \cos \left( \frac{2\pi}{\Phi_0} \Phi(\Phi_{ext}) \right)}, \quad (2.13)$$

the interaction Hamiltonian for the RF-SQUID mediated coupling can finally be written as

$$H_{RF} = \hbar g_{RF}(\Phi_{ext}) \left( (a_2 + a_2)^\dagger - (a_1 + a_1)^\dagger \right)^2. \quad (2.14)$$

It describes the energy stored intermediately in the RF-SQUID loop, due to superimposing fluxes, generated by the transmission lines, inside the loop area.

In Figure 2.6, the dynamic coupling $g_{RF}(\Phi_{ext})$ is plotted versus the external flux $\Phi_{ext}$ for various values of the screening parameter $\beta_L$. Depending on the strength of the flux inside the RF-SQUID, the circulating current creates a field that either enhances or reduces the external flux, leading to a positive or a negative coupling.
2.4. Total system Hamiltonian

After deriving the individual contributions in the previous sections and allowing for small\(^1\) detuning of the resonators \(\Delta \equiv \omega_1 - \omega_2 \ll \omega_1 + \omega_2\), the total system Hamiltonian is

\[
\mathcal{H} = \mathcal{H}_{cl} + \mathcal{H}_{geo} + \mathcal{H}_{RF} = \hbar \omega_1 a_1^\dagger a_1 + \hbar \omega_2 a_2^\dagger a_2 - \hbar g_{geo}(a_1^\dagger + a_1)(a_2^\dagger + a_2) + \frac{\hbar}{2} g_{RF}(\Phi_{ext}) \left( (a_2^\dagger + a_2) - (a_1^\dagger + a_1) \right)^2.
\]

\(1\)The resonator-SQUID coupling depends on the resonator resonance frequency. But for small detuning, this is negligible, as the error made depends on \(\omega_1/\omega_2\).
2.4. Total system Hamiltonian

For spectroscopy experiments where the RF-SQUID is not driven and $|g| \ll \omega_j$, all fast rotating terms average to zero. In this case, the Hamiltonian in the rotating wave approximation (RWA),

$$
\mathcal{H}_{\text{RWA}} = \hbar \sum_{i=1,2} \omega_i a_i^\dagger a_i - \hbar g_{\text{geo}} (a_1^\dagger a_2 + a_1 a_2^\dagger) + \hbar g_{\text{RF}}(\Phi_{\text{ext}}) \left( \sum_{i=1,2} a_i^\dagger a_i - a_1^\dagger a_2 - a_1 a_2^\dagger \right)
$$

$$
= \hbar \sum_{i=1,2} (\omega_i + g_{\text{RF}}(\Phi_{\text{ext}})) a_i^\dagger a_i - \hbar (g_{\text{geo}} + g_{\text{RF}}(\Phi_{\text{ext}})) \left( a_1^\dagger a_2 + a_1 a_2^\dagger \right),
$$

(2.16)

is obtained. This Hamiltonian can be interpreted as a Hamiltonian of two harmonic oscillators with shifted frequencies of $\omega'_i = \omega_i + g_{\text{RF}}(\Phi_{\text{ext}})$ and an interaction term with coupling strength $g(\Phi_{\text{ext}}) = g_{\text{geo}} + g_{\text{RF}}(\Phi_{\text{ext}})$. Depending on the values of $g_{\text{geo}}$, $g_{\text{RF}}$, and $\beta_L$ the combined coupling strength $g(\Phi_{\text{ext}})$ can be positive, zero or negative. In other words, it is possible to compensate for the static coupling and effectively switch off the total coupling. The interaction term can also be interpreted as photon hopping from one resonator to the other at a rate of $g$.

2.4.1. Diagonalisation of the harmonic Hamiltonian

In the small photon number limit ($n \hbar \omega \ll E_j$), where the harmonic treatment of the RF-SQUID is valid, it is instructive to look at the harmonic Hamiltonian. Equation (2.16) can be cast into a sum of harmonic (single-particle) operators [20]. By spanning the two dimensional Hilbert space with the basis $|1\rangle \equiv (1, 0)$ (photon is in resonator 1) and $|2\rangle \equiv (0, 1)$ (photon is in resonator 2), the single photon Hamiltonian in the rotating wave approximation can be expressed as

$$
\mathcal{H}_{\text{RWA}} = \hbar (\omega_1 + g_{\text{RF}}) a_1^\dagger a_1 + \hbar (\omega_2 + g_{\text{RF}}) a_2^\dagger a_2
- \hbar (g_{\text{geo}} + g_{\text{RF}}) \left( a_1^\dagger a_2 + a_1 a_2^\dagger \right)
$$

$$
\Rightarrow \hat{\mathcal{H}}_{\text{RWA}} = \hbar \begin{pmatrix}
\omega_1 + g_{\text{RF}} & -(g_{\text{geo}} + g_{\text{RF}}) \\
-(g_{\text{geo}} + g_{\text{RF}}) & \omega_2 + g_{\text{RF}}
\end{pmatrix}.
$$

(2.17)

\footnote{A phase factor of $\exp(i\Delta t)$ ($\exp(-i\Delta t)$) on the minor diagonal is omitted, because it is irrelevant for the calculations in this chapter. This factor would also appear in the first component of the eigenvectors, but drops out when looking at their components absolute squared.}

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Due to the coupling of the resonators, the system cannot be seen as the sum of two distinct systems. Instead, a holistic view is required. This can be seen when calculating the frequency eigenvalues of (2.17) in the ground state,

\[ \Omega_{1,2} = \frac{\omega_1 + \omega_2}{2} + g_{RF} \mp \frac{1}{2} \sqrt{\left(4(g_{geo} + g_{RF})^2 + \Delta^2\right)^2 + \delta^2} \equiv \delta \]  

(2.18)

Figure 2.7.: The eigenvalues of the coupled system split symmetrically around the arithmetic mean of the resonator eigenfrequencies. The level splitting is determined by the resonator detuning \( \Delta < 0 \) and the coupling strength \( g \).

The energy levels of the coupled system split symmetrically around the arithmetic mean of the dressed levels of the individual resonators (see Figure 2.7). The splitting depends on the detuning \( \Delta \) of both resonators and the coupling strength \( g(\Phi_{ext}) \).

By applying an external magnetic field, the level spacing can be tuned. In Figure 2.8, the energy eigenvalues are plotted versus the geometric coupling \( g_{geo} \). The detuning determines the minimal distance between the energy eigenvalues. A change of the sign of the geometric coupling strength makes the eigenvectors switch positions. In Figure 2.9, the same plot, but for varying \( g_{RF} \) is depicted.

In the experiment, the dynamic coupling strength \( g_{RF}(\Phi_{ext}) \) can be tuned in-situ via an externally applied magnetic field. In Figure 2.10, the energy eigenvalues are plotted versus the external flux \( \Phi_{ext} \) (setting \( \Delta = g_{geo} = 0 \)) for \( \beta_1 = 0.9 \).

The eigenvalues in Equation (2.18) are similar to the eigenvalues of a Jaynes-Cumming-Hamiltonian [21], but feature two different coupling strengths and a dressing through...
2.4. Total system Hamiltonian

Figure 2.8.: Normalised eigenvalues \( \tilde{\Omega}_{1,2} \) plotted versus geometric coupling for \( g_{\text{RF}} = 0 \) and \( \tilde{\Delta} = -0.003 \). At zero coupling \( \tilde{g}_{\text{geo}} = 0 \) the degeneracy is neutralized by the decoupling. The dashed lines represents the eigenvalues for no detuning, but with an offset of \( -\Delta/2 \). The corresponding eigenvectors for negative and positive coupling are added in their respective colour. All frequencies and rates are normalised to \( \omega_1 \), i.e., \( \tilde{\cdot} = \cdot / \omega_1 \).

The eigenvectors \( \tilde{v}_{1,2} \) that correspond to the eigenfrequencies \( \Omega_{1,2} \) are

\[
\tilde{v}_{1,2} = \left[ 4 + \left( \frac{-\Delta \pm \delta}{g(\Phi_{\text{ext}})} \right)^2 \right]^{-\frac{1}{2}} \left( \frac{-\Delta \pm \delta}{g(\Phi_{\text{ext}})} \right) \frac{1}{2}.
\]  

(2.19)

The ratio of the total coupling strength \( g \) and the resonator detuning \( \Delta \) determines the behaviour of the system, which can be classified through the limits \( g \gg \Delta \) and \( g \ll \Delta \).
Figure 2.9.: Normalised eigenvalues $\tilde{\Omega}_{1,2}$ plotted versus dynamic coupling for $g_{\text{geo}} = 0$ and $\tilde{\Delta} = -0.003$. All frequencies and rates are normalised to $\omega_1$.

In the first case, where the coupling is much stronger than the detuning, the system is described with collective eigenmodes, that are maximally delocalised ($g > 0$):

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (g \gg \Delta),$$

and swapped for $g < 0$. In the second case, where the detuning is much higher than the coupling strength, the »dispersive limit«, the system is described by perfectly localised eigenstates ($\Delta < 0$):

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (g \ll \Delta),$$

and swapped for $\Delta > 0$. Between these extremes, intermediate eigenstates exist that are delocalised, but »prefer« one or the other resonator state. This can be shown formally by projecting the eigenstates $|v_i\rangle$ onto the original basis $|j\rangle$. This yields

$$p_{ij} = \langle j|v_i\rangle^2 = |v_{ij}|^2.$$

The occupation probability of the first eigenstate ($i = 1$) of being in resonator 1 ($j = 1$) is just given by the first component $v_{11}$ of $\vec{v}_1$ squared,

$$p_{11} = |v_{11}|^2 = \frac{1}{2} - \frac{\Delta}{2\tilde{\delta}}.$$
This is plotted versus the coupling strength \( g \) in Figure 2.11. By rearranging Equation (2.23), it can be shown that it solely depends on the ratio \( \vartheta = \frac{g}{\Delta} \) of the total coupling strength \( g = g_{\text{geo}} + g_{\text{RF}} \) and the detuning \( \Delta = \omega_1 - \omega_2 \) between the resonators (for \( \Delta \neq 0 \)),

\[
p_{11}(\vartheta) = \frac{1}{2} - \frac{1}{2} \frac{1}{4 \vartheta^2 + 1}. \tag{2.24}
\]

Because for a quantum state

\[
\bar{v} = \alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

with \(|\alpha_1|^2 + |\alpha_2|^2 = 1\) must be true, the other probabilities are given by

\[
\begin{align*}
p_{12} &= 1 - p_{11} \\
p_{21} &= 1 - p_{11} \\
p_{22} &= p_{11}.
\end{align*}
\]
2.4. Total system Hamiltonian

Figure 2.11.: Occupation probability of first eigenstate plotted versus total coupling strength $\tilde{g}$ for different (normalised) detunings $\Delta$. The resonators decouple at $\tilde{g} = 0$. In this case, the photon is localized in one resonator. A detuning of $\Delta \neq 0$ leads to a mixing ratio different from 1:1 ($p_{11} = p_{12} = .5$). All frequencies and rates are normalised to $\omega_1$, i.e., $\tilde{\cdot} = \cdot / \omega_1$.

The mixing ratio (resonator 1/resonator 2) for the eigenstates $|v_1\rangle$ and $|v_2\rangle$ is then given by

$$\frac{p_{11}}{p_{12}} = \frac{\delta - \Delta}{\delta + \Delta} \quad \text{and} \quad \frac{p_{21}}{p_{22}} = \frac{\delta + \Delta}{\delta - \Delta}. \quad (2.25)$$

The occupation probabilities $p_{11}$ and $p_{12}$ are plotted in Figure 2.12 for different detunings $\Delta$. The decoupling, when $g = g_{geo} + g_{RF} = 0 \Rightarrow \vartheta = 0$, is reflected in the occupation probabilities. Here the state mixing vanishes and a photon in one resonator remains in that resonator (for example $p_{12}(\vartheta = 0) = p_{21}(\vartheta = 0) = 1$). The occupation probabilities are also relevant, when deploying the system as a component in a quantum information processing (QIP) setup, where the behaviour of a single photon is important.
2.4. Total system Hamiltonian

Figure 2.12: The occupation probability $|v_{1j}|^2$ for the resonator $j$ in the first eigenstate $|v_1\rangle$ ((c) and (d)) for different detunings $\tilde{\Delta} = -0.001$ ((a) and (c)) and $\Delta = -0.03$ ((b) and (d)) is plotted versus the external flux, with the corresponding eigenvalue plots ((a) and (b), $\Omega_1$ in blue and $\Omega_2$ in red) to aid orientation, for $g_{RF0} = 0.03$, $g_{geo} = 0.03$ and $\beta_L = 0.9$. For small detunings, compared to the coupling strength, the probability to find a photon in one or the other resonator is equal. For growing detunings the occupation probability for resonator 2 $|v_{12}|^2$ increases towards one. At $\Phi_{ext} = 0.5 \Phi_0$ the coupling strength compensates for the detuning ($\Delta \ll g$) and the occupation probability approaches $1/2$ again. For $g = g_{geo} + g_{RF} = 0$ the occupation probability to find a photon in resonator 2 for the first eigenstate is one. This is the point of decoupling, so no photon hopping takes place. All frequencies and rates are normalised to $\omega_1$, i.e., $\tilde{\cdot} = \cdot / \omega_1$.
2.5. **Input-output formalism**

Spectroscopy is a valuable experimental tool for analysing the energy levels of this system. A coherent electromagnetic signal of a specific frequency that is fed into one side of a cavity, excites the system. After some time, it relaxes again and produces an outgoing electromagnetic field at the ports, which is probed. The response is larger the closer the probe frequency is to the resonance frequency, since the cavity mode is excited.

In theory, this behaviour can be modelled via input-output formalism. The discrete cavity modes are coupled with a continues coherent field outside the cavity. This formalism also introduces a decay rate, \( \gamma \), which is a measure of the lifetime of a photon in the cavity. Input-output theory is a well known tool in quantum optics (cavity QED), but can also be applied to circuit QED.

The field outside the cavity is modelled using input field and output field operators \( b_{m}^{\text{IN}}(t) \) and \( b_{m}^{\text{OUT}}(t) \). The index \( m \) denotes the port number as defined in Figure 2.13.

![Figure 2.13.](image)

Figure 2.13.: Definition of device port numbering. Resonator 1 owns ports 1 and 2 and resonator 2 owns ports 3 and 4. In agreement with the experimental setup 2 and 4 are input ports and 1 and 3 are output ports. The RF-SQUID is sketched in the middle of both resonators.

For low power (low photon) spectroscopy, without additional drive of the RF-SQUID, the rotating wave approximation is a very good description. When this is the case, the
2.5. Input-output formalism

The equation of motion for the annihilation operators \( a_1(t) \) and \( a_2(t) \) of the internal fields of the resonators, using \([a_j, a_j^\dagger] = 1\) can be written as

\[
\frac{da_{1,2}}{dt}(t) = -\frac{i}{\hbar} \left( a_{1,2}(t), \mathcal{H}_{\text{RWA}} \right) - \frac{\gamma_1 + \gamma_2}{2} a_{1,2}(t) + \sqrt{\gamma_{1,3}} b_{1,3}^{\text{IN}}(t) + \sqrt{\gamma_{2,4}} b_{2,4}^{\text{IN}}(t) \tag{2.26}
\]

\[
= -i(\omega_1 + g_{\text{RF}}) a_{1,2}(t) + i(g_{\text{geo}} + g_{\text{RF}}) a_2(t) - \frac{\gamma_1 + \gamma_2}{2} a_{1,2}(t) + \sqrt{\gamma_{1,3}} b_{1,3}^{\text{IN}}(t) + \sqrt{\gamma_{2,4}} b_{2,4}^{\text{IN}}(t) \tag{2.27}
\]

where \( \gamma_m \) denotes the decay rate out of port \( m \) [22]. After Fourier transformation, this equation can be written in algebraic form,

\[
\left[ \mathcal{F} \left( \frac{da_{1,2}}{dt}(t) \right) \right](\omega) = -i\omega a_{1,2}(\omega)
\]

\[
a_{1,2}(\omega) = \frac{\sqrt{\gamma_{1,3}} b_{1,3}^{\text{IN}}(\omega) + \sqrt{\gamma_{2,4}} b_{2,4}^{\text{IN}}(\omega) + i(g_{\text{geo}} + g_{\text{RF}}) a_2(\omega)}{\frac{\gamma_1 + \gamma_2}{2} - i(\omega - \omega_1 + g_{\text{RF}})} \tag{2.28}
\]

With the four boundary conditions (one for every port)

\[
b_{1,2}^{\text{IN}} + b_{1,2}^{\text{OUT}} = \sqrt{\gamma_{1,2}} a_1
\]

\[
b_{3,4}^{\text{IN}} + b_{3,4}^{\text{OUT}} = \sqrt{\gamma_{3,4}} a_2,
\]

the equation can be solved for the output fields \( b_j^{\text{OUT}} \) (and also for \( a_{1,2} \)).

The S-parameters, that relate the outgoing to the incoming signal of the device in a spectroscopy measurement, can now be identified with

\[
S_{ij} = \frac{b_j^{\text{OUT}}}{b_i^{\text{IN}}} \tag{2.29}
\]
2.5. Input-output formalism

In this work, the input signal is incident either in port 2 or port 4 and the transmission out of port 1 and 3 is measured. This means $i_{2,4}^{IN} = 0$ and $i_{1,3}^{OUT}$ are measured. Considering this, only four S-parameters are of interest, namely

$$S_{12}(\Phi, \omega) = \frac{\sqrt{\gamma_1 \gamma_2} \left( \frac{\gamma_3 + \gamma_4}{2} - i (\omega - \omega_2) + ig_{RF}(\Phi) \right)}{g^2(\Phi) + \left( \frac{21 + \gamma_2}{2} - i (\omega - \omega_1) + ig_{RF}(\Phi) \right) \left( \frac{23 + \gamma_4}{2} - i (\omega - \omega_2) + ig_{RF}(\Phi) \right)}$$

$$S_{34}(\Phi, \omega) = \frac{\sqrt{\gamma_3 \gamma_4} \left( \frac{\gamma_1 + \gamma_2}{2} - i (\omega - \omega_1) + ig_{RF}(\Phi) \right)}{g^2(\Phi) + \left( \frac{21 + \gamma_2}{2} - i (\omega - \omega_1) + ig_{RF}(\Phi) \right) \left( \frac{23 + \gamma_4}{2} - i (\omega - \omega_2) + ig_{RF}(\Phi) \right)}$$

$$S_{14}(\Phi, \omega) = \frac{i \sqrt{\gamma_1 \gamma_4} g(\Phi)}{g^2(\Phi) + \left( \frac{21 + \gamma_2}{2} - i (\omega - \omega_1) + ig_{RF}(\Phi) \right) \left( \frac{23 + \gamma_4}{2} - i (\omega - \omega_2) + ig_{RF}(\Phi) \right)}$$

$$S_{32}(\Phi, \omega) = \frac{i \sqrt{\gamma_3 \gamma_2} g(\Phi)}{g^2(\Phi) + \left( \frac{21 + \gamma_2}{2} - i (\omega - \omega_1) + ig_{RF}(\Phi) \right) \left( \frac{23 + \gamma_4}{2} - i (\omega - \omega_2) + ig_{RF}(\Phi) \right)}$$

Near the dressed resonance frequencies $\omega'_{j} = \omega_j + g_{RF}$, their absolute values squared have approximately a Lorentzian shape. In the experiment, the intensity $|S_{ij}(\Phi_{ext}, \omega)|^2$ and the phase $\varphi = \arg(S_{ij}(\Phi_{ext}, \omega))$ are measured. The intensity of the transmission S-parameters is plotted in Figure 2.14 for equal decay rates at every port. Not only does the input-output model introduce a finite line width (decay rates $\gamma_m$), but it also predicts antiresonances at the original dressed resonance frequencies $\omega'_{j} = \omega_j + g_{RF}(\Phi_{ext})$ of the unexcited resonator. Most interestingly, an antiresonance generally appears at the resonance frequency of the system that is missing the exact component that is being driven [23, 24]. In this case, when driving resonator 1, the dressed resonance frequency of resonator 2 manifests as an antiresonance in the transmission spectrum and vice versa.

In Figure 2.15, the transmission spectrum for both resonators is plotted at the point of decoupling. The decoupling occurs when the dynamic coupling $g_{RF}(\Phi_{ext})$ exactly compensates for the geometric coupling $g_{geo}$ and the total coupling $g$ is zero. A photon exchange between the resonators no longer takes place and thus only the resonator frequency can be excited. But nonetheless, the resonator frequencies are still dressed due to interactions with the RF-SQUID.

In Figure 2.16 the relative phase of the outgoing signal to the ingoing signal is plotted changes by $\pi$ at each resonance and at the antiresonance.
2.5. Input-output formalism

Figure 2.14.: Absolute value squared of S-parameters for a specific coupling strength plotted versus normalised excitation frequency $\tilde{\omega} = \omega/\omega_1$ in logarithmic scale. Resonance maxima at the eigenfrequencies are clearly visible. In this figure, all decay rates are set to be equal. Besides introducing a finite line width the input-output formalism also predicts an antiresonance for $S_{12}$ and $S_{34}$ at the (dressed) original resonance frequency of the other (not driven) cavity. The frequency difference of the antiresonances of $|S_{12}|^2$ and $|S_{34}|^2$ corresponds to the detuning of the resonators [23, 24]. Note that the resonance peaks are not exactly at $\Omega_j$ because of some overlap of both peaks due to the finite line width. The line width (FWHM, $2\tilde{\gamma}$) is indicated by the red line.
2.5. Input-output formalism

Figure 2.15.: Spectrum of decoupled resonators (i.e. $g_{RF}(\Phi_{ext}) = -g_{geo}$). Resonance peaks at the original dressed eigenfrequencies can be observed. The frequency difference is again given by the detuning $\Delta$ of the resonators. All frequencies and rates are normalised to the resonance frequency of resonator 1, so that $\tilde{\omega}_1 = 1$.

Figure 2.16.: In this figure, the phase of $S_{12}$ is plotted versus the frequency for the non-degenerate case. Similar to a classical resonator, below the resonance frequencies (here at $\tilde{\omega} = 0.9$ and 1.1) the phase is $\frac{\pi}{2}$ ahead and above the same amount behind. Vice versa for the antiresonance (at $\tilde{\omega} = 1.0$).
2.5. Input-output formalism

2.5.1. Photon numbers

Equation (2.28) can also be solved for the annihilation operators \( a_{1,2}(\omega) \) of the intra resonator fields:

\[
a_1(\omega) = \frac{\left( \frac{\gamma_1 + \gamma_2}{2} - i(\omega - \omega_2) + ig_{\text{RF}}(\Phi) \right) \sqrt{\gamma_1 b_{\text{IN,2}}} - ig(\Phi) \sqrt{\gamma_2 b_{\text{IN,4}}} \}{g^2(\Phi) + \left( \frac{\gamma_1 + \gamma_2}{2} - i(\omega - \omega_1) + ig_{\text{RF}}(\Phi) \right) \left( \frac{\gamma_1 + \gamma_2}{2} - i(\omega - \omega_2) + ig_{\text{RF}}(\Phi) \right)},
\]

\[
a_2(\omega) = \frac{\left( \frac{\gamma_1 + \gamma_2}{2} - i(\omega - \omega_1) + ig_{\text{RF}}(\Phi) \right) \sqrt{\gamma_1 b_{\text{IN,4}}} - ig(\Phi) \sqrt{\gamma_2 b_{\text{IN,2}}} \}{g^2(\Phi) + \left( \frac{\gamma_1 + \gamma_2}{2} - i(\omega - \omega_1) + ig_{\text{RF}}(\Phi) \right) \left( \frac{\gamma_1 + \gamma_2}{2} - i(\omega - \omega_2) + ig_{\text{RF}}(\Phi) \right)}. \tag{2.30}
\]

This provides the opportunity to derive a relative photon occupation number inside the resonators, comparable to Equation (2.22), by calculating the photon number operator \( a_j^{\dagger}a_j \) in relation the total photon number in the system \( a_1^{\dagger}a_1 + a_2^{\dagger}a_2 \),

\[
\frac{\langle a_1^{\dagger}a_1 \rangle (\omega)}{\langle a_1^{\dagger}a_1 \rangle (\omega) + \langle a_2^{\dagger}a_2 \rangle (\omega)}.
\]

For an input field into resonator 1, that is \( b_{\text{IN,2}} \neq 0 \) and \( b_{\text{IN,4}} = 0 \), this ratio only depends on the characteristic values of the second resonator, namely \( \omega_2 \) and \( \gamma_{r,2} \),

\[
\frac{\langle a_1^{\dagger}a_1 \rangle (\omega)}{\langle a_1^{\dagger}a_1 \rangle (\omega) + \langle a_2^{\dagger}a_2 \rangle (\omega)} = \frac{\gamma_{r,2}^2 + (\omega - \omega_2 - g_{\text{RF}}(\Phi_{\text{ext}}))^2}{\gamma_{r,2}^2 + \left( (g_{\text{geo}} + g_{\text{RF}}(\Phi_{\text{ext}}))^2 - 2g_{\text{RF}}(\Phi_{\text{ext}})(\omega - \omega_2) + (\omega - \omega_2)^2 \right)}
\]

\[
\frac{\langle a_2^{\dagger}a_2 \rangle (\omega)}{\langle a_1^{\dagger}a_1 \rangle (\omega) + \langle a_2^{\dagger}a_2 \rangle (\omega)} = \frac{1 - \langle a_1^{\dagger}a_1 \rangle (\omega)}{\langle a_1^{\dagger}a_1 \rangle (\omega) + \langle a_2^{\dagger}a_2 \rangle (\omega)} \tag{2.32}
\]

When evaluating Equation (2.32) at the eigenvalues \( \Omega_{1,2} \) it yields

\[
\frac{\langle a_1^{\dagger}a_1 \rangle (\Omega_{1,2})}{\langle a_1^{\dagger}a_1 \rangle (\Omega_{1,2}) + \langle a_2^{\dagger}a_2 \rangle (\Omega_{1,2})} = \frac{2g^2 + 2\gamma_{r,2}^2 \mp \Delta \left( \sqrt{\Delta^2 + 4g^2} \mp \Delta \right)}{4g^2 + 2\gamma_{r,2}^2 \mp \Delta \left( \sqrt{\Delta^2 + 4g^2} \mp \Delta \right)}, \tag{2.33}
\]

which becomes Equation (2.23) for \( \gamma_{r,2} \to 0 \) and thus constitutes a generalisation of it for continuous excitation.
2.6. Input-output formalism with Gaussian flux noise

In the experiment, the dynamic coupling strength $g_{RF}(\Phi_{ext})$ can be tuned by externally applying a magnetic field perpendicular to the RF-SQUID. The supercurrent that tunnels through the Josephson junction of the RF-SQUID is $\Phi_0$-periodic. Because a SQUID is inherently extremely flux-sensitive, only small strengths and variations of the external field $\Phi_{ext}$ are necessary to tune the current, and thus the coupling strength $g_{RF}$, through its complete period. $\Phi_0$ corresponding to a magnetic field of $B \approx 1 \times 10^{-7}$ T. That is comparable to a hundredth of the earth magnetic field. $\partial \Phi_{ext} g_{RF}$ can be in the order of 10 GHz/$\Phi_0$. As a consequence, this also means the setup is prone to flux noise from numerous sources (environment, pumps, measurement devices, etc.). Due to finite measurement times, a flux-sensitive measurement now averages over different flux values that fluctuate around a mean. This might blur features of the measurement, because records belonging to distinct flux values are superimposed on one another.

A general approach to incorporating flux noise into the model is to assume Gaussian flux noise. That means the flux is assumed to be normally distributed around a mean value with a variance $\sigma^2$. In other words, when applying an external flux of $\Phi_{ext}$, the probability of actually finding a value $\Phi_{ext}'$ is

$$p(\Phi_{ext}'|\Phi_{ext}, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(\Phi_{ext}' - \Phi_{ext})^2}{2\sigma^2} \right).$$

(2.34)

Now, when recording the S-parameter, the expectation value under the probability distribution $p$ is measured due to averaging:

$$\langle S_{ij}(\Phi_{ext}, \omega) \rangle = \int_{\mathbb{R}} d\Phi_{ext}' p(\Phi_{ext}'|\Phi_{ext}, \sigma) \cdot S_{ij}(\Phi_{ext}', \omega)$$

$$= \int_{\mathbb{R}} d\Phi_{ext}' N_\sigma(\Phi_{ext}' - \Phi_{ext}) \cdot S_{ij}(\Phi_{ext}', \omega)$$

$$= (S_{ij} * N_\sigma)(\Phi_{ext}, \omega).$$

(2.35)

The expectation value can be obtained by convolving the S-parameter $S_{ij}$ in $\Phi_{ext}$ with a normal distribution $N_\sigma$ with variance $\sigma^2$. 

![Image of signal waveform]
3. **Experimental Techniques**

After laying out the theoretical foundation in the previous chapter, in this chapter the experimental techniques to probe the features of the coupled resonator system are described. To begin with, the sample design and fabrication are briefly described. Afterwards, the measurement setup and principle is depicted.

### 3.1. Sample design and fabrication

The samples are fabricated at the WMI by Friedrich Wulschner using thin film deposition and lithography techniques. They consist of two coplanar transmission slotline $\lambda/2$-resonators with conductors made out of niobium deposited on a silicon wafer. The silicon wafer is covered by a thin silicon oxide layer. A profile of the structure is shown in Figure 3.1.

In this thesis, measurement data from four different samples are shown. The main sample is that with codename Z10C. Sample Y1E is fabricated in an attempt to reduce the screening parameter $\beta_L$ by reducing the Josephson junction size. Sample Y2C is the first attempt to use an Al/Al-AlO$_x$/Al-junctions of smaller size, because they can be fabricated with higher precision with the help of electron beam lithography, as opposed to Nb/Al-AlO$_x$/Nb junctions fabricated with optical lithography and used in the other samples.

The transmission lines are capacitively coupled to the ports via finger-capacitors in the inner conductor. A CPW transmission line connects the sample ports with the resonator. That is at the coupling capacitors the CPW transmission line directly changes over to a slotline geometry. In the middle of the resonator lines, two conducting lines connect the inner conductors of both resonators with each other, building a ring (compare with
3.2. Measurement setup

Figure 3.1.: Profile of the CSL transmission lines. From left to right and top to bottom the outer conductor (that is the ground plane), the inner conductor – both made of niobium, a thin silicon oxide layer on top of a silicon wafer.

The left inset in Figure 3.2). One of those connections is interrupted by a Josephson junction. This loop, interrupted by a single Josephson junction, is an RF-SQUID. It has dimensions of $\Delta_x = 200\,\mu m$ and $\Delta y = 100\,\mu m$. The chip layout is shown in Figure 3.2. The Josephson junction of Z10C, Y1E, and Z10D is made out of Nb/Al-AlO\textsubscript{x}/Nb layers. By design it has an area of approximately $2\,\mu m^2$. The Al/Al-AlO\textsubscript{x}/Al-junction of Y2C has an area of approximately $500\,nm \times 200\,nm$.

3.2. Measurement setup

To investigate the energy spectrum of the system consisting of two coupled resonators, it has to be cooled to its ground state. To prevent thermal excitations, temperatures below approximately $100\,mK$ are necessary, which can be reached inside a cryostat. For this setup, a $^3$He/$^4$He dilution refrigerator is used. The spectrum is probed with a 4-port vector network analyser (VNA) built by Rohde & Schwarz. Through signal lines, which reach into the fridge to connect the sample, a coherent signal of small bandwidth and a specific power is fed into the resonator system, in order to excite it. These excitations decay, are emitted out of the system and can in turn be detected with the VNA. The combination of a continuous excitation and decay lead to a dynamic equilibrium of the system. A special feature of the setup is the dynamic coupling, which can be tuned \textit{in-situ}.
3.2. Measurement setup

Figure 3.2.: Chip layout of two transmission slotline resonators that are accessible via ports. The transmission lines are terminated with coupling capacitors (red rectangles and inset on the right). The coupling capacitors also mark the transition from CPW to CSL transmission lines (from two outer conductors to one). In the middle the RF-SQUID shares segments of the inner conductors of both transmission lines. The RF-SQUID is magnified in the inset on the left. The ground-plane is drawn in blue.

via a coil mounted on top of the sample box, in which the sample resides. By sending a DC-current through the coil, the magnetic field threading the RF-SQUID can be changed, influencing the resonator interaction. A current of approximately \( I_{DC} \approx 1 \mu A \) to 10\( \mu A \) is needed to generate a magnetic field on the order of \( B = 10^7 \) T.

In Figure 3.3, the signal flow through the individual components of the setup is illustrated. In order to improve the signal to noise ratio, several attenuators are integrated into the input lines on different temperature stages of the fridge. The reason is, that now a high power signal, compared to the temperature dependent thermal noise power, can be fed into the input line. The required attenuation is delayed to lower temperatures, where the noise has lower power as well. On the other hand, the input signal also transports energy into the system that thermalises at the attenuators and introduces an additional
temperature load. A reasonable compromise is to reduce the signal power step by step at different temperature stages, so that the thermal load is distributed, and at the same time a high signal to noise power is preserved.

After the signal passes through the sample, two microwave circulators in the output line (acting like an electric diode) prevent thermal noise to leak in from higher temperature states. A cold amplifier at 4 K and two amplifiers at room temperature amplify the signal before it is analysed by the VNA. Our chain of amplifiers is designed such that the first stage dominates the overall noise. In other words, the signal-to-noise ratio is limited by the noise added by the cold amplifier, which is on the order of a few kelvin.

To connect the circuit chip to the measurement setup, it is encased by a box with appropriate connectors to the outside. The chip ports are connected the box connectors by silver glue.

A temperature controlled 50 Ω-resistor can be switched in place of the output of the sample. In very good approximation this acts as a black body radiator with well known temperature dependent spectrum [25]. This resistor can be used to calibrate the output line with all elements, in order to remove line specific attenuations from the output signal. This is done by varying its temperature and recording the power inside a frequency band for both output lines. The power scales exponentially with the temperature. However, a cold calibration of the input line without the samplebox is not possible with the current setup.
Figure 3.3.: Signal flow chart of a typical measurement setup. The input signal from the vector network analyser (VNA) is attenuated on several temperature stages to reduce thermal noise. After it passes through the sample it is amplified with a cold and two warm amplifiers (triangles). Two circulators prevent thermal noise from the output lines from entering the sample. Optionally, two temperature controlled resistors acting as thermal noise sources can be hooked up in place of the sample, in order to calibrate the output lines.
4. RESULTS

In this chapter, the results of the transmission measurements are presented and discussed. To probe the energy levels of the coupled two resonator system, one of the resonators is excited by a coherent electromagnetic field of a specific wavelength and power. The excited modes interact through the coupling with the modes of the other resonator and decay out of one or the other resonator after some time. The probe-signal is fed into one side of the resonators and it is detected on the other side, like shown in the diagram of Figure 2.13.

In order to study the coupling mechanism, the dynamic coupling strength is varied. By sending a DC-current to the coil sitting on top of the sample box, the magnetic flux $\Phi_{\text{ext}}$ threading the RF-SQUID can be controlled and, in this way, also the dynamic coupling strength $g_{\text{RF}}(\Phi_{\text{ext}})$. For every value of the external magnetic flux a frequency spectrum of the transmission is recorded. The resonance frequencies of the resonator system manifest themselves as transmission maxima in the spectra. And because the effective resonance frequencies of the system depend on the coupling strength, the influence of the coupling on the energy levels of the system can be measured.

In Figure 4.1 and Figure 4.2, the recorded spectra for the different S-parameters – that is different combinations of excited and measured resonators – of sample Z10C are plotted versus the external flux $\Phi_{\text{ext}}$ and the excitation frequency $\omega/2\pi$. With the input power at the sample of $P_{\text{in}} = -140$ dBm (including the intrinsic signal line attenuation of $\approx 10$ dB), the decay rate of the resonator of approximately $\gamma_1 \approx 6$ MHz and an eigenfrequency of $\omega_1/2\pi \approx 6.45$ GHz (these parameters can be estimated by fitting a Lorentzian to a resonance peak for $\Phi_{\text{ext}} \approx 0$) the number of photons in the system can be approximated by

$$n_p = \frac{1}{\frac{n}{2} \omega_0^2 Q_{\text{ext}}} P_{\text{in}} \approx \frac{P_{\text{in}}}{\hbar \omega \gamma} \lesssim 1,$$

$$\text{where } n = \frac{\hbar \omega}{2 \omega_0^2} P_{\text{in}} \approx \frac{P_{\text{in}}}{\hbar \omega \gamma} \lesssim 1.$$
for $Q_{\text{ext}} \ll Q_{\text{int}}$, with the internal (external) quality factors $Q_{\text{int}}$ ($Q_{\text{ext}}$) of the resonator, and with $Q_{L}^{-1} = Q_{\text{int}}^{-1} + Q_{\text{ext}}^{-1}$.

The first thing to note: the plots are obviously symmetric around $\Phi_{\text{ext}} = 0.5 \Phi_{0}$ in accordance with the model. Furthermore, in the plots, two distinct modes can be seen. Their eigenfrequencies vary with the coupling strength tuned by the applied external flux. The distance between the resonance peaks is approximately proportional to the coupling strength. The plot can be divided into three different areas. For a start, around $\Phi_{\text{ext}} = 0.5 \Phi_{0}$ the total coupling $g$ is negative and can reach some $-100$ MHz. At the border of this area, the dynamic coupling compensates the static geometric coupling and both resonators decouple. This especially becomes visible in the $S_{32}$ spectrum seen in Figure 4.1. Resonator 1 is excited and the output of resonator 2 is recorded. At the distinct points of decoupling, symmetrically around $\Phi_{\text{ext}} = 0.5 \Phi_{0}$, a drop in the transmission out of resonator 2 can be seen as vertical cuts in the plot. This is to be expected, as no coupling means no interaction between the resonators. So by exciting resonator 1, resonator 2 now will not be excited, in contrast to the coupled case. At the left and the right the decoupling points the total coupling strength changes its sign. This region of positive coupling strengths constitutes the third area.

In Figure 4.3, flux cuts through the plots are shown for positive coupling and the point of decoupling. For coupling strengths different from zero two modes are visible, and for the decoupled case only one mode appears. Another feature of the spectra, which stands out, are the different relative peak heights of the two modes. The mode, that is stronger in $S_{12}$ is the weaker one in $S_{34}$.

Unfortunately, the connection between the sample port and the sample box connector was defect for the output port 1, leading to a much weaker signal (by approximately $-10$ dB). The calibration with the thermal noise source resulted only in a mismatch between the output lines of $1.47$ dB, and thus cannot be the only reason of the weaker signal at port 1. We attribute this to a weak electrical contact of the sample box connector to resonator ports on the sample. The connection is made with a contact glued with silver glue onto the chip. This seems to produce unrepeatable results, since it is sensitive to mechanical

---

1The color scale is created based on CIELab colour space transformed to polar coordinates. CIELab emulates the human perception of colours. The encoded value is represented in the luminosity channel. This enables a better objective judgement of the plot without getting distracted by visualisations artefacts [26, 27, 28].
Figure 4.1.: Measured transmission (colour encoded\textsuperscript{1}) $S$-parameters $S_{34}$ (a) and $S_{32}$ (b) of sample Z10C for input ports 4 and 2 and output port 3 plotted versus excitation frequency $\omega/2\pi$ and external flux $\Phi_{\text{ext}}$. The input power at the sample is approximately $-140$ dBm ($\leq 1$ photon on average) at a temperature of $T \leq 50$ mK. In $S_{32}$ the decoupling of the modes is clearly visible in the transmission decrease. A detuning of the resonators of $\Delta = 23.8$ MHz introduces an asymmetry of the transmission peak heights of the two modes in $S_{34}$ and reversed in $S_{12}$ (see Figure 4.2). In $S_{34}$ the antiresonance near the higher mode can be guessed, but is mainly lost in the noise floor.
Figure 4.2.: Measured transmission S-parameter $S_{12}$ (a) and $S_{14}$ (b) of sample Z10C for input ports 2 and 4 and output port 1 plotted versus the excitation frequency $\omega/2\pi$ and external flux $\Phi_{\text{ext}}$ at a temperature of $T \leq 50$ mK. The stronger total signal strength of output port 3 compared to port 1 indicates a weaker electrical connection to port 1 (compare with Figure 4.1). The peak height differences caused by the resonator detuning in $S_{12}$ is clearly visible.
Figure 4.3.: Measured transmission spectra of sample Z10C for $\Phi_{\text{ext}} = 0 \Phi_0$ – column (a) – and $\Phi_{\text{ext}} = 0.47 \Phi_0$ – column (b) –, where the resonators decouple.
stress. Better options, such as wire bonding or soldering, are not feasible with the current sample box layout.

4.1. Tunable coupling

4.1.1. Modulation of coupling strength

By varying the external magnetic flux via the coil current, the RF-SQUID is biased and in effect the dynamic coupling strength \( g_{RF}(\Phi_{ext}) \) can be tuned (compare with Equation (2.13)). For every current value a frequency spectrum is recorded. In Figure 4.1 and Figure 4.2 such a spectrum is plotted for sample with codename Z10C.

In order to convert the coil current into the flux inside the RF-SQUID loop, the flux \( \Phi_0 \)-periodicity of the dynamic coupling strength can be utilised, see Figure 4.4. For convenience, the transmission is scaled to compensate for a different attenuator configuration in the setup, in order to ease comparison.

A straightforward way to determine the coupling parameters \( g_{geo}, g_{RF0}, \) and \( \beta_L \), is to fit every frequency slice – that is for different coupling strength – with a Lorentzian, to track the transmission maxima. These coincide in very good approximation (neglecting small shifts due to the finite line width) with the eigenfrequencies of the Hamiltonian in the RWA (see Equation (2.18)). Neglecting the detuning, the transmission maxima are displaced by approximately \( \Delta \omega = 2g(\Phi_{ext}) \). By only fitting the displacement, the eigenfrequencies need not to be known. Figure 4.5 shows the result of the fit. The parameter best estimates are given by

<table>
<thead>
<tr>
<th>parameter</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_{geo}/2\pi )</td>
<td>((29 \pm 5) \text{ MHz})</td>
</tr>
<tr>
<td>( g_{RF0}/2\pi )</td>
<td>((22 \pm 3) \text{ MHz})</td>
</tr>
<tr>
<td>( \beta_L )</td>
<td>(0.91 \pm 0.13)</td>
</tr>
</tbody>
</table>

This rather simple approach has the major drawback that the resonator frequencies and the decay rates can not be determined this way. Also, for detunings comparable with
Figure 4.4.: High power spectrum of the $S_{32}$-parameter of sample Z10C plotted versus the external flux for more than one period. This can be used to convert the coil current into the external flux in multiples of $\Phi_0$. In this case, the conversion factor is approximately $8.72 \mu\text{A}/\Phi_0$.

the coupling strength, this approximate displacement is no longer valid. Later, a second method without those drawbacks is presented.
Figure 4.5.: Fit (white line) of the resonance peak displacement $\Delta \omega = 2g(\Phi_{\text{ext}})$ to the frequency spectrum of the $S_{34}$-parameter.
4.1. Tunable coupling

Figure 4.6.: High power spectra of the $S_{34}$- and $S_{32}$-parameters. In the upper plot spectra for decoupled resonators are plotted. (a) When exciting resonator 1, the decoupling manifest itself in a drop in the output power ($\approx -40\,\text{dB}$) of resonator 2 ($S_{32}$), because the photon exchange between the resonators is inhibited. (b) The spectra for $\Phi_{\text{ext}} = 0.5 \Phi_0$ are displayed. Since the coupling strength is maximal, the effect of the detuning of the resonators is reduced, and the ideal 1:1 occupation of the resonators is approached. Thus, ideally, the peak heights would be the same. In this case however, presumably different attenuations in the input lines of approximately 2 dB are dominating the result.

Decoupling

Figure 4.6 shows a high power spectrum of the $S_{34}$ and $S_{32}$ transmission parameters, that is, resonator 1 is excited and the output of resonators 1 and 2 are measured. Figure 4.6(a) shows the transmission for a vanishing total coupling strength $g \approx 0$. Compared to the transmission of the excited resonator 1, the transmission of resonator 2 is suppressed by approximately $-40\,\text{dB}$. It clearly indicates, that the photon transfer between the resonators is successfully inhibited and the resonators decouple. In other words, the tunable coupling allows for on-demand switching of the total coupling.
4.1. Tunable coupling

Figure 4.6(b) depicts the transmission spectra for maximal coupling ($\Phi_{\text{ext}} = 0.5 \Phi_0$). Ideally, the peak transmissions would be nearly the same (exactly the same for $\Delta = 0$), but instead a difference of approximately $2 \text{dB}$ can be observed. Most likely, this is caused by different intrinsic attenuations of both input lines.

4.1.2. Phase

In addition to the signal intensity, the phase difference between input- and output-signal is measured. But because it is not possible to calibrate the signal line in the current setup and, more importantly, because of the weak strength for off-resonant signals, it is very difficult to measure the phase difference reliably with enough precision for low power.

In addition to the phase difference caused by the sample, the signal lines add a phase difference proportional to the probe frequency. By incorporating this into the phase of the input-output formalism, it can be fitted to the data. The complex signal provided by the VNA has to be complex conjugated, in order to make it agree to the model. This might be due to a different definitions of the phase. However, further investigations are necessary. The measurement data is modified accordingly, by multiplying a factor $-1$ to the phase. The result is plotted in Figure 4.7.

For excitation frequencies lower than the resonance frequencies the phase of the excitation is ahead by $\pi/2$ of the response and lags behind by the same amount, after passing the resonance frequency. It the other way round for the antiresonance. A zero-crossing marks the positions of the (anti)resonances and could be a nice way to track them, if the measurement becomes feasible at the single photon level.

All in all, our phase data exhibits very good qualitative and even reasonable quantitative agreement with the expected curve.
4.1. Tunable coupling

Figure 4.7.: High power (> 1000 photons) $S_{12}$ transmission (a) and fit to the phase difference for sample $Y2C$ and positive coupling (b). A linear, frequency dependent offset was included in the model, to remove the phase shift caused by the signal line. The fitted model is plotted in red. Because at high powers additional effects can occur that are not included into the model, the results must be interpreted with care. The intersections with the frequency axis mark the resonances and the antiresonance of the transmission spectrum.

4.1.3. Complete device characterisation without noise

The coupling strength is not the only parameter of interest, when assessing the performance of the system. Among others, it is also important to find the original resonance frequencies of the undressed resonators and their decay rates. Especially when they are not the same for both resonators and break symmetry.

With the used measurement setup it is possible to feed a signal into one port of each resonator and detect the output of their respective second port or the output port of the second resonator. The measured transmission S-parameters are: $S_{12}$ and $S_{34}$ for measuring the transmission through resonator 1 and 2, and $S_{14}$ and $S_{32}$ for measuring the output signal of resonator 1 and 2 when exciting the other. These S-parameters are plotted in Figure 4.1 and Figure 4.2. With the input-output-formalism from Section 2.5 the parameters $\omega_1$, $\omega_2$, $\gamma_{r,1}$, $\gamma_{r,2}$, $g_{geo}$, $g_{RF0}$, and $\beta_L$ can be estimated by fitting the model to the data. With the current setup it is not possible to measure the decay rates of every
4.1. Tunable coupling

single port. However, the mean decay rate out of both ports of one resonator can be determined: \( \gamma_{r,1} = (\gamma_1 + \gamma_2)/2 \) and \( \gamma_{r,2} = (\gamma_3 + \gamma_4)/2 \).

**Parameter estimation**

Due to unavoidable noise – in general, not to be confused with the Gaussian flux noise model – the measurement data is assumed to be normally distributed around the mean value given by the model with a variance of \( \sigma^2 \). Its probability distribution of every point can be written as

\[
p_{ij}(\omega, \Phi_{\text{ext}}|\vec{\alpha}) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left[ -\frac{1}{2\sigma^2} \left( M_{ij}(\omega, \Phi_{\text{ext}}) - |S_{ij}(\omega, \Phi_{\text{ext}}; \vec{\alpha})|^2 \right)^2 \right],
\]

with given parameters

\[\vec{\alpha} = (\omega_1, \omega_2, \gamma_{r,1}, \gamma_{r,2}, g_{\text{geo}}, g_{\text{RF}}, 0, \beta_L),\]

the measured data point \( M_{ij}(\omega, \Phi_{\text{ext}}) \), and the model \( S_{ij}(\omega, \Phi_{\text{ext}}; \vec{\alpha}) \) at the point \( (\omega, \Phi_{\text{ext}}) \). The total distribution for every measured point \( (\omega, \Phi_{\text{ext}}) \in X \) is given by the product of the probability distribution at these points – which also can be expressed as a sum in the exponent,

\[
L_{ij}(\vec{\alpha}) = \prod_{\vec{x} \in X} p_{ij}(\vec{x}|\vec{\alpha}) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left[ -\frac{1}{2\sigma^2} \sum_{\vec{x} \in X} \left( M_{ij}(\vec{x}) - |S_{ij}(\vec{x}; \vec{\alpha})|^2 \right)^2 \right]. \tag{4.1}
\]

\( L_{ij} \) is called the likelihood [29, 30] for a realisation of parameter \( \vec{\alpha} \). The total likelihood function for all S-parameters is given by their product

\[
L = L_{12} \cdot L_{14} \cdot L_{32} \cdot L_{34}, \tag{4.2}
\]
which can be maximised to calculate the best (most probable) parameter estimates $\vec{\alpha}^\ast$. To ease the calculation, its negative natural logarithm is minimised (by omitting the unimportant normalisation factor from above),

$$\min_{\vec{\alpha}} (-\ln L(\vec{\alpha})) = \frac{1}{2\sigma^2} \sum_{\vec{x} \in X} \sum_{i=1,3,j=2,4} (M_{ij}(\vec{x}) - |S_{ij}(\vec{x}; \vec{\alpha})|^2)^2. \quad (4.3)$$

This method is called »Maximum-Likelihood«-method, originally devised in [31]. In this case, it leads to the well known least mean square-method for the complete two dimensional data set. That is, the whole spectrum of all measured S-parameter is fitted simultaneously. To calculate an estimate of the parameter variance, the covariance matrix of the parameter distribution can be estimated. As shown in [30], this can be achieved by taking the inverse of the Hessian matrix $H_L(\vec{\alpha})$ of $L(\vec{\alpha})$ evaluated at the extremum that is obtained from the fit,

$$\text{Cov}(\vec{\alpha}) = (H_L(\vec{\alpha})|_{\vec{\alpha}^\ast})^{-1}. \quad (4.4)$$

The parameters’ variances are now given by the diagonal entries of the covariance matrix. Since the VNA does not provide the per point variance of the measurement, $\sigma$ in Equation (4.3) is estimated with the standard deviation estimator,

$$\sigma \approx \sqrt{\frac{1}{n-1} \sum_{\vec{x} \in X} \sum_{i=1,3,j=2,4} (M_{ij}(\vec{x}) - |S_{ij}(\vec{x}; \vec{\alpha}^\ast)|^2)^2},$$

with $n$ number of data points.

**Fit**

The actual fit is done using the NMinimize-routine of Wolfram’s Mathematica. This routine tries to find numerically the global minimum. To refine the results, a local minimum search algorithm implemented as FindMinimum in Mathematica is used. The fit is relatively expensive because for every iteration, a number of data points on the order of $10^5$-$10^6$ has to be evaluated. To speed up the evaluation time the Compile-function is used to prevent symbolic evaluation and to translate the Mathematica code into native code. To compare the results, an equivalent fitting algorithm is implemented in Julia, a free programming language for data analysis. For the actual optimisation the free NLopt library, maintained by Steven G. Johnson, is used. From the optimisation algorithms
4.1. Tunable coupling

<table>
<thead>
<tr>
<th>parameter</th>
<th>estimate (MHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1/2\pi$</td>
<td>6482.64 ± 0.06</td>
</tr>
<tr>
<td>$\omega_2/2\pi$</td>
<td>6458.95 ± 0.06</td>
</tr>
<tr>
<td>$\gamma_{r,1}$</td>
<td>5.77 ± 0.03</td>
</tr>
<tr>
<td>$\gamma_{r,2}$</td>
<td>6.53 ± 0.02</td>
</tr>
<tr>
<td>$g_{\text{geo}}$</td>
<td>27.61 ± 0.05</td>
</tr>
<tr>
<td>$g_{\text{RF},0}$</td>
<td>22.36 ± 0.09</td>
</tr>
<tr>
<td>$\beta_L$</td>
<td>0.918 ± 0.002</td>
</tr>
</tbody>
</table>

Table 4.1.: Parameter estimates from fit of the input-output model to the intensity measurement data. $\beta_L$ is unit-less. The errors are given with 4σ-significance.

tested from the NLopt library, *Sbplx* appeared to be most robust for the given problem. Also note that, in order to calculate $g(\Phi(\Phi_{\text{ext}}))$, $\Phi(\Phi_{\text{ext}})$ had to be determined numerically because of its implicit definition. Although root finding algorithms are quite fast for this case, unnecessary evaluations had to be avoided, by caching intermediate results, to optimise the execution time.

To take different port couplings and signal lines into account, scaling parameters $n_{ij}$ are multiplied to the transmission intensity ($n_{ij} \cdot |S_{ij}|^2$) and are also used as degrees of freedom in the fit. In Figure 4.8, Figure 4.9, and in Figure 4.10 the results of the fit are plotted. The best parameter estimates can be found in Table 4.1.

In Figure 4.10 flux cuts for different values of the external flux are plotted versus the driving frequency. The model is in good agreement with the data for areas that are not strongly flux-sensitive. The model correctly predicts the positions of the resonance maxima, the peak height difference caused by detuning, the positions of the antiresonances (as far as it can be judged from the data), and the decoupling of the resonators in addition to other features. For flux-sensitive parts some problems arise (compare with fourth plot at $\Phi_{\text{ext}} = 0.484 \Phi_0$ in Figure 4.10). The peak heights are underestimated for the higher frequency mode and overestimated for the modulation, lower frequency mode. The latter can be partly explained by incorporating Gaussian flux-noise into the model (see next section). In Figure 4.10(c) the nearest cut to the point of decoupling is plotted. As expected, only one Lorentzian shaped mode at the dressed eigenfrequency of the resonator is visible, confirming the decoupling of the resonators.
Figure 4.8.: The fitted input-output formalism in (a) and (b) plotted versus the external flux $\Phi_{\text{ext}}$ and driving frequency $\omega/2\pi$ for output port 3. Corresponding measurement data is plotted in (c) and (d). The detuning of the resonators can be seen from the asymmetric placement of the antiresonance between the resonances. This feature is less visible in the measurement data because of signal-to-noise ratio.
Figure 4.9.: The fitted input-output formalism in (a) and (b) plotted versus the external flux $\Phi_{\text{ext}}$ and driving frequency $\omega/2\pi$ for output port 1. Corresponding measurement data is plotted in (c) and (d). As also can be seen from the fit, the overall signal of port 1 is approximately 4 times weaker than that of port 3. It suggests a connection problem to the chip for this port.
4.1. Tunable coupling

Figure 4.10.: Flux-slices of $S_{34}$ taken from Figure 4.8 for different external fields $\Phi_{\text{ext}}$ plotted versus the excitation frequency. On the right the eigenvalues are plotted for orientation (Equation (2.18), using best parameter estimates from Table 4.1). The flux-positions from which the slices are taken are marked here. The plot in (c) at $\Phi_{\text{ext}} = 0.47 \Phi_0$ is taken at the nearest point to the decoupling. In the flux-insensitive parts ((a) to (c)), the model (red) is in good agreement with the data. But in the flux-sensitive areas, exemplarily shown in the last plot at the bottom, the model cannot explain the attenuation of the peaks. Note, that the misestimation of the resonance peak position in (d) is most likely caused by current to flux calibration errors. This is to be expected, because of the very rapid changing peak positions in this area.
4.1. Tunable coupling

4.1.4. Influence of Gaussian flux noise

An RF-SQUID intrinsically is a highly flux-sensitive device and therefore susceptible to flux-noise. In this experiment, it directly influences the dynamic coupling $g_{RF}(\Phi_{\text{ext}})$. In addition, for $\Phi_{\text{ext}} \approx (0.5 \pm 0.1) \Phi_0$, the change of the dynamic coupling strength is very high, i.e. it is in the order of $\partial_{\Phi_{\text{ext}}} g(\Phi_{\text{ext}}) \approx 10 \text{ GHz}/\Phi_0$ (also compare with Figure 4.18). When recording the transmission signal, the VNA averages for a finite time. In this time the signal varies because of flux noise. This means, that for flux-sensitive areas the resonance peaks will be broadened and shifted towards lower frequencies (because $\partial^2_{\Phi_{\text{ext}}} g(\Phi_{\text{ext}}) > 0$). A Gaussian flux-noise model, presented in Section 2.6, takes this into account.

Implementation

The convolution of the $(\omega, \Phi_{\text{ext}})$-spectrum with a normal distribution in the flux-direction cannot be done analytically because of the implicit definition of the internal flux $\Phi(\Phi_{\text{ext}})$ (see Equation (2.8)). Instead, it is done numerically using a discrete convolution. The integral from Equation (2.35) becomes a sum,

$$\langle S_{ij}(\Phi_{\text{ext},m}, \omega) \rangle \approx \sum_{n=-r}^{r} T(n, t) \cdot S_{ij}(\Phi_{\text{ext},m-n}, \omega),$$

with $\Phi_{\text{ext},m}$ a value at index $m$ from the sampled external flux. In the easiest case $T$ in Equation (4.5) is just a sampled Gaussian kernel,

$$T(n, t) = \frac{1}{N} \exp \left( -\frac{n^2}{2t} \right),$$

with normalisation

$$N = \sum_{n=-r}^{r} T(n, t).$$

The variance $t$ of the discrete Gaussian kernel is given by $t = (\sigma_n/s)^2$, for a step size between the flux value samples $s = \Phi_{\text{ext},m+1} - \Phi_{\text{ext},m}$. The radius of the filter $r$ is chosen to be four times the standard deviation, rounded to the next integer: $r = \lceil 4s \rceil$. This already covers more than 99.99 % of the integral. More involved implementations are
4.1. Tunable coupling

Based on scale space theory [32, 33] but are beyond the scope of this thesis. In either case, the convolution theorem states, that a convolution becomes algebraic after a Fourier transformation. This can be utilized to calculate the discrete convolution rather efficiently using the Fast-Fourier-Transformation (FFT) algorithm.

To improve the result, the flux variable sampling is interpolated by a factor of ten, before applying the Gaussian filter. This means, for every sampled point in the measurement, nine additional points in between each sample point are evaluated. In Figure 4.11 the average error compared to a numerical integration is plotted versus the interpolation scaling factor. With an interpolation factor of 10 and an original sampling step size of $2.5\,\text{m}\Phi_0$, the error made is smaller than 1 per mill on average. Furthermore, the discrete Gaussian filter is several orders of magnitude faster than numerical integration, which makes a simultaneous fit to all S-parameter spectra feasible.

![Graph showing relative error versus interpolation factor](image)

Figure 4.11.: Average relative error plotted on a logarithmic scale versus the interpolation scaling factor compared to a numerical integration of the convolution with a normal distribution, for a typical original step size of $2.5\,\text{m}\Phi_0$. With a scaling factor greater than ten, the average error is already smaller than one per mill.
In addition to the previously introduced parameters above (see section 4.1.3), the Gaussian noise variance $\sigma_n^2$ is introduced and added as an additional dimension to the parameter space of the optimisation. The parameter estimates can be found in Table 4.2. They differ mostly in the estimation of $\beta_L$, which in this case is higher, but still lower than 1 (for $\beta_L > 1$ hysteretic behaviour can be observed, see following section). In Figure 4.12 and Figure 4.13 the results of the fit are plotted as colour coded spectra.

<table>
<thead>
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<th>parameter</th>
<th>estimate (MHz)</th>
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<td>$\gamma_{r1}$</td>
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<tr>
<td>$\beta_L$</td>
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</tr>
<tr>
<td>$\sigma_n$</td>
<td>(1.27 ± 0.03) mΦ₀</td>
</tr>
</tbody>
</table>

Table 4.2.: Parameter estimates from fit of the input-output model including Gaussian flux-noise to the measured transmission spectra. $\beta_L$ is unit-less. The errors are given with 4σ-significance.

A standard deviation of the flux noise of $\sigma_n \approx 1.3 \text{ mΦ₀}$ cannot be ignored. Including it into the model, improves the agreement of the model and the measurement data. In Figure 4.14, flux cuts are plotted versus the excitation frequency for different external flux values. For comparison, the noise-free model is also plotted. For the flux-insensitive parts both models do not differ. But at flux values with a high flux-dependency of the system the peaks are broadened and damped (see Figure 4.14d). Although the model now agrees far better on the peak heights with the data, they still are not predicted perfectly. Partly, this can be caused by a current to flux conversion with too low precision. Already slight deviations can have a considerably effect. The same is true for the peak positions.
Figure 4.12.: The fitted input-output formalism model with Gaussian flux-noise for $S_{34}$ (a) and $S_{32}$ (b) plotted versus the external flux $\Phi_{\text{ext}}$ and excitation frequency $\omega/2\pi$ for output port 3. It includes a flux-noise with a standard deviation of $1.3 m\Phi_0$. By directly comparing the figures to the noise-free model in (c) and (d), it can be seen, that the maxima in the area around $\Phi_{\text{ext}} \approx 0.5 \Phi_0$ are broader. Also from the $S_{32}$-plot it can be seen, that the decoupling-dip is not as pronounced any more.
Figure 4.13.: The fitted input-output formalism model with Gaussian flux-noise for $S_{12}$ (a) and $S_{14}$ (b) plotted versus the external flux $\Phi_{\text{ext}}$ and excitation frequency $\omega/2\pi$ for output port 1. The corresponding noise-free model is plotted in (c) and (d).
Figure 4.14.: Flux-slices of $S_{34}$ taken from Figure 4.12 for different external fields $\Phi_{\text{ext}}$ plotted versus the excitation frequency. On the right the calculated eigenvalues are plotted for orientation. The model that includes the Gaussian flux-noise is plotted in red. The unmodified model is plotted in dashed grey for comparison. As expected, in flux-insensitive areas ((a) to (c)) the models barely differ. However, in flux-sensitive parts (d) distinct differences become visible. Especially the mode attenuation can now partly be explained.
4.1.5. Screening parameter $\beta_L$

The screening parameter $\beta_L$ determines the relationship between the internal and the external field of the RF-SQUID and thus directly influences the maximal reachable coupling strength. If $\beta_L$ is too small, the dynamic coupling could not be tuned to be strong enough to compensate for the geometric coupling, so that the resonators are decoupled. In Figure 4.15 a measured spectrum for sample Y1E is shown, where this is the case. The upper mode modulates with varying coupling strength, but both resonators never decouple. On the other hand, the higher $\beta_L$ the more sensitive the coupling strength becomes to flux variations, and the more it is exposed to flux-noise.

Furthermore, if $\beta_L > 1$, the mapping of the external onto the internal flux is not unambiguously defined any more and a hysteretic behaviour can be observed, as it is shown in Figure 4.17. In this regime, metastable current states of the RF-SQUID exist [34]. This means, that although the external flux $\Phi_{\text{ext}}$ is varied continuously, sudden flux jumps can occur, when reaching a certain threshold. Also, the internal flux value depends on the sweep direction of the external flux. The reason for this behaviour becomes more obvious when looking at the plot in Figure 4.16, where the internal flux is plotted versus the external flux. For $\beta_L \leq 1$, every given external flux value allows only for one specific value for the internal flux. But if $\beta_L > 1$, this will not be true any more. There will be multiple values on different branches that belong to a single value of the external flux. While varying the external flux, a flux jump occurs, when a transition from one branch to another is induced. Trivially this happens, when one branch continuous, but the others do not, and in effect the relationship between the fluxes becomes uniquely defined again. In Figure 4.17 a measurement of sample Y2C with this hysteretic flux-dependency is plotted.
Figure 4.15.: Spectra of sample YIE. The critical current of the sample’s Josephson junction is too small $\beta_L \approx 0.2$, so the dynamic coupling cannot compensate completely for the geometric coupling. This behaviour is also well described by the model.
4.1. Tunable coupling

Figure 4.16.: The calculated internal flux $\Phi$ plotted versus the externally applied flux $\Phi_{\text{ext}}$. For a screening parameter $\beta_L > 1$ hysteretic behaviour can be observed, that is, the value of $\Phi$ depends on the direction from which it was reached. $\Phi$ is not a well defined function, since for an external flux value multiple values of $\Phi$ on multiple branches exist. Transitions between these branches create sudden jumps of the internal flux value and thus, of the dynamic coupling $g_{RF}(\Phi_{\text{ext}})$ as well.
Figure 4.17.: Spectra of Y2C sample. The critical current of the sample’s Josephson junction is too high ($\beta_L > 1$). This has the effect, that the dynamic coupling $g_{RF}(\Phi_{\text{ext}})$ shows a hysteretic behaviour. In (a) the coil current, and by association the external flux, is varied from minus to plus and vice versa for (b). Depending on the direction of the variation, flux jumps occur at different positions.
4.1. Tunable coupling

4.1.6. Discussion of the results

The first important result is, that for an appropriate screening parameter $\beta_L$, the mediated dynamic coupling allows for on-demand decoupling of the resonators. In Figure 4.23 a close-up of the decoupling points for a crossover measurement is shown and compared to the calculated occupation probability.

Furthermore, in order to evaluate the usefulness of the device for quantum information processing experiments, it is important to compare the achievable coupling strength to the decay rates and the resonator detuning. The total coupling strength $g(\Phi_{\text{ext}})$ is plotted versus the external flux in Figure 4.18. The coupling strength varies from $g_{\text{min}} \approx 40 \text{ MHz}$ (positive frequency shift) up to $g_{\text{max}} \approx -250 \text{ MHz}$ (negative frequency shift). But, at the same time, for negative coupling its derivative reaches values higher than $\partial_{\Phi_{\text{ext}}} g \approx 10 \text{ GHz}/\Phi_0$. Thus, being extremely flux sensitive, the coupling strength for negative coupling is prone to flux-noise and hard to control. On the other hand, for $\Phi_{\text{ext}} \approx 0 \Phi_0$ the coupling strength is much less flux dependent, but its value, $g_{\text{min}}$, becomes comparable to the detuning $\Delta \approx 24 \text{ MHz}$ and the decay rates $\gamma_{r,i} \approx 6 \text{ MHz}$ (corresponding to a $Q$-factor of approximately 600).

![Figure 4.18. Total, calculated coupling strength $g(\Phi_{\text{ext}})$ against the external flux using the fit parameters from Table 4.2.](image)

When interpreting the total coupling strength $g$ as a hopping rate, it has to be compared to the decay rates of the system. If $\gamma_{r,i} \geq g$, an excitation in one resonator would relax faster
4.1. Tunable coupling

than it could be transferred to the other resonator in a time $1/g$. For the estimated values of the system the coupling strength is up to six times as high as the decay rates, which is barely high enough to induce a hopping between the resonators reliably. However, the decay rates (the $Q$-factor of the resonator) can be improved in fabrication. Internal quality factors above $10^4$-$10^6$ have been demonstrated [35, 36].

The influence of the detuning between the resonators can be best seen, when looking at the occupation probabilities for a photon being in one or the other resonator. As shown in Section 2.4.1, for an eigenstate of the system, the occupation probability for being in one resonator, depends on the ratio between the coupling strength $g$ and the detuning of the resonators $\Delta$. A detuning increases the occupation probability in one resonator at the expense of the other resonator. The occupation probabilities for the first eigenstate are plotted in Figure 4.19 for $g = -40$ MHz versus the detuning $\Delta$.

![Figure 4.19](image)

Figure 4.19.: Occupation probability of the first eigenstate for total coupling strength $g = -40$ MHz versus the detuning $\Delta$. The measured detuning is marked with a green dashed line. Detuning shifts the occupation probability towards one or the other resonator.

For the fitted device parameters, the relative occupation for the first eigenstate for maximal positive coupling at $\Phi_{\text{ext}} = 0$ is approximately

$$\frac{\delta - \Delta}{\delta + \Delta} = \frac{\sqrt{4g(\Phi_{\text{ext}} = 0)^2 + \Delta^2} - \Delta}{\sqrt{4g(\Phi_{\text{ext}} = 0)^2 + \Delta^2} + \Delta} \approx \frac{0.35}{0.65}.$$
as it can be calculated from Equation (2.25). Thus, for the second eigenstate it is exactly the inverse.

To include the decay rate of the resonators into the calculation, the relative occupation numbers, as derived from the intra cavity field operators in Equation (2.32), can be used. They are plotted in Figure 4.20 for the fitted device parameters. For resonators with

![Figure 4.20.](image)

Figure 4.20.: Calculated relative occupation numbers for both resonators plotted versus the excitation frequency based on the input-output model for the fitted parameters. The input signal is assumed to be fed into resonator 1. The green, dashed lines denote the calculated eigenvalues. For perfectly symmetrical resonators with high $Q$-factors, the eigenvalues would coincide with the intersection of both functions. The detuning $\Delta$ causes more photons to be in resonator 2 than in 1 for the first eigenstate and the other way round for the second. The peak (dip) in between is caused by the antiresonance. For $\gamma = 0$ it would be equal to 1 (0).

no detuning and infinite $Q$-factors, the eigenvalues would perfectly coincide with the intersection of both functions, leading to a 1:1 mixing ratio. However, for a finite detuning and decay rate, this is no longer the case.

When exciting resonator 1, the decay rate $\gamma_{r2}$ of resonator 2 introduces a small asymmetry in the mixing ratios between both eigenstates, not addressed by the simpler model above:

$$\frac{\langle a_1^\dagger a_1 \rangle (\Omega_1)}{\langle a_2^\dagger a_2 \rangle (\Omega_1)} \approx 0.364 \quad \text{and} \quad \frac{\langle a_1^\dagger a_1 \rangle (\Omega_2)}{\langle a_2^\dagger a_2 \rangle (\Omega_2)} \approx 0.651.$$
and accordingly for an excitation of resonator 2. In Figure 4.21 the calculated resonator relative occupation numbers are plotted versus the detuning $\Delta$, for the fitted best estimate parameters, when exciting resonator 1. Due to the decay out of resonator 2, their intersection of a 1:1 mixing ratio does not happen at zero detuning.

![Graph](image)

Figure 4.21.: Plot of model resonator occupation at first calculated eigenvalue, based on the best estimates from the fit for an input field fed into resonator 1. In dashed green, the actual resonator detuning is plotted. Note, that the lines do not intersect at $\Delta = 0$. This is caused by the decay rate $\gamma_{r,2}$. The same plot but for the second eigenvalue is given by mirroring it along the $\Delta$-axis, i.e. $\Delta \to -\Delta$.

To realize photon swapping experiments (similar to [37]), it is required to transfer a photon from one resonator to the other. One possibility is to make use of Rabi-oscillations. When turning on the resonator coupling, the resonator occupation probability oscillates continuously between both states. By switching off the coupling after half the round trip time, the probability for a photon being transferred to the other resonator is maximal.

By calculating the time evolution of a state $|\psi(t)\rangle$,

$$|\psi(t)\rangle = e^{-i \hat{H}_{\text{RWA}} t}|\Psi(0)\rangle$$
which is initially prepared to be in resonator 1 $|\Psi(t = 0)\rangle = |1\rangle = (1, 0)$, it can be seen that a detuning of the resonators decreases the probability for a photon hopping into the other resonator. This is plotted for the estimated parameters from above in Figure 4.22.

Figure 4.22.: Calculated state evolution for initial state in resonator 1 for detuning $\Delta \approx 24$ MHz and $g(0 \Phi_0) \approx -40$ MHz in terms of resonator occupation probability $|\langle j \psi(t) \rangle|^2$, with $j = 1, 2$). The dashed line corresponds to an exponential decay $\exp(-\gamma t)$ with rate $\gamma \approx 6$ MHz. The detuning regulates the maximal possible probability for a photon hopping to resonator 2.

Experimentally, a small resonator detuning of the resonators could be approached by inserting a DC-SQUID [38, 39, 40] or a transmon-qubit [41] into the transmission line of one resonator, to change in-situ its resonance frequency. Opposed to the broader tuning range of a DC-SQUID, with the dispersive shift of a transmon the resonator frequency can be tuned by only a few MHz.

Interesting to note, is that the flux-noise model also predicts an influence of the noise on the positions of the antiresonance. This is depicted in Figure 4.24. Because of the non-linear flux dependence of the dynamic coupling strength, the antiresonances are shifted towards higher frequencies and not as pronounced as before.
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Figure 4.23.: High power spectrum of $S_{32}$-parameter for different coupling strengths tuned by the external flux. In (a) it displays the output of resonator 2 while driving resonator 1. When the dynamic coupling compensates for the static geometric coupling, the resonators decouple and thus do not exchange photons any more. This is reflected in the transmission drop, seen in the frequency cut in (b) at $\omega/2\pi = 6.439$ GHz. At the decoupling points the transmission of resonator 2 decreases by at least 40 dB. In (c) the calculated occupation probability (Equation (2.23), for values from fit, Table 4.2) for a photon in the resonator for the first eigenvalue is plotted. At the decoupling points it is zero. Due to the detuning a probability of $(0.5 \approx -3 \text{ dB})$ is only nearly reached for $\Phi_{\text{ext}} = 0.5 \Phi_0$, where $g \gg \Delta$. 

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Figure 4.24.: Comparison of both models – without (grey) vs. with flux-noise (blue) for negative couplings, i.e. a mode shift towards lower frequencies. Because of the non-linear dependence on the external flux, the flux-noise influences not only the peak width and height, but also their positions. This is also true for the anti-resonances. Compared to the noiseless case they are shifted towards higher frequencies (on the order of 1 MHz to 10 MHz).
5. **Summary & Outlook**

In this thesis, the characteristics of a two-transmission line resonator system with an RF-SQUID-mediated, tunable coupling are studied. First the system Hamiltonian is derived, including the static geometric coupling energy and the externally tunable coupling energy, mediated by a mutually shared RF-SQUID. The eigenvalues and eigenvectors of the Hamiltonian are calculated and studied. Because of the coupling, the system in the ground state can be described by two collective, delocalised modes. The energy levels depend on the coupling strength, and thus can be tuned. Based on the system Hamiltonian in the rotating wave approximation, an input-output formalism is developed, to enable investigations of the system using spectral analysis. Since an RF-SQUID is intrinsically highly sensitive to noise, the input output model is later extended to include a Gaussian flux noise model.

The two-coupled-resonator system is studied experimentally, measuring its spectrum while continuously exciting it with a coherent probe signal. A coupling strength dependent mode splitting could be observed, and it is proved that an on-demand decoupling of the resonators is possible. The implementation of the fitting algorithm using a numerical model is described and discussed. This allows the recovery of the characteristic parameters of the coupled resonator system. The total coupling strength range is $-40$ MHz to $250$ MHz, but at the same time, for negative coupling, feature blurring due to high flux-sensitivity in this area is prominent and not negligible. The standard deviation of the flux-noise $\sigma_n$ is approximately $1.3\, m\Phi_0$. Also, the impact of a frequency detuning of the resonators and their decay rates is discussed. It is found that a detuning comparable to the coupling strength leads to a notable asymmetric resonator occupation probability. In this case, a detuning of approximately $24$ MHz compared to a total coupling strength of $-40$ MHz leads to a ratio between the resonator photon occupation numbers of approximately 3:2. When deploying the device for photon swapping operations, this might be an unwanted feature, and the resonator detuning would have to be minimized. Also, the
resonator decay rates of approximately 6 MHz should be optimised to take advantage of the whole potential of the device. Furthermore, devices with various values of the screening parameter of the RF-SQUID $\beta_L$ are studied and hysteretic behaviour is found in the case of a sample with $\beta_L > 1$.

**Outlook**

To sharpen the analysis described in this thesis, calibrated time-resolved measurements would be of interest. A promising possibility for calibrating the in- and output lines would be to make use of the mixing chamber $-30$ dB-attenuator as an approximate black body radiator. With a calibrated measurement apparatus, a more detailed and better quantifiable insight into the resonator occupation probabilities would be possible. By introducing a single photon source, for example by adding a transmon qubit to one resonator, time resolved measurements could confirm photon swapping operations using Rabi oscillations, as described above.

This thesis studies the system in the rotating wave approximation. By driving the RF-SQUID directly using an on-chip antenna, it should be possible to access the rotating terms in the full Hamiltonian in Equation (2.15). Terms such as $(a_j^\dagger)^2 + a_j^2$ indicate the possibility of parametric amplification [42], similar to measurements already demonstrated for the Josephson Parametric Amplifier (JPA) [43, 44]. And indeed, the first measurements with the presented chip are very promising. By applying an additional second tone to a resonator input port at the double resonator frequency in order to drive the RF-SQUID, we observed signal amplifications of at least a order of magnitude (not shown).

Also, the ability of the device to squeeze quantum states should be studied with interest, as proposed by Tian et. al. in [10]. Initial theoretical reflections suggest the possibility of two-mode squeezing [45, 46, 47, 48].
Appendices
A. **Coplanar transmission line resonators**

![Equivalent distributed element circuit diagram for a coplanar transmission line with inner and outer conductor(s).](image)

Figure A.1.: Equivalent distributed element circuit diagram for a coplanar transmission line with inner and outer conductor(s). Every infinitesimal segment of the conductors is modelled as an LC-circuit. The inner conductor with inductance density $l_0$ is capacitively ($c_0$) connected to the outer conductor(s) at ground potential. The whole transmission line is terminated by coupling capacitors.

In this chapter, a quantum mechanical treatment of coplanar transmission lines is derived. Following references [11] and [13], the branch flux $\phi(x, t) = \int_{t_0}^{t} d\tau \ v(\tau)$, with the voltage drop $v$ across the branch, at the node between capacitance and inductance at position $x$ is introduced. The voltage drop $v$ across the capacitance can then be written as

$$v(x, t) = \partial_t \phi(x, t) = \dot{\phi}(x, t)$$

and the current $i(x, t)$,

$$\partial_x \phi(x, t) = -l_0 i(x, t).$$

This can be seen by applying Kirchhoff’s voltage law to an LC element of size $dx$ in the distributed circuit,

$$v(x, t) - v(x + dx, t) = -l_0 dx \ \partial_x i(x, t) \xrightarrow{dx \to 0} \ \partial_x v(x, t) = \partial_x \partial_t \phi(x, t) = -l_0 dx \ \partial_t i(x, t).$$

$$\Rightarrow \partial_x \phi(x, t) = -l_0 i(x, t)$$
Therefore, the total Lagrangian is given by the integral over the Lagrangian density of a resonator of length $2\ell$, which is the capacitive and inductive field energy of every infinitesimal element corresponding to kinetic and potential energy,

$$L_{tl} = \int_{-\ell}^{\ell} dx \, L(\phi, \partial_t \phi, \partial_x \phi, t) = \frac{1}{2} \int_{-\ell}^{\ell} dx \left[ c_0 \dot{\phi}^2(x, t) - \frac{(\partial_x \phi)^2(x, t)}{l_0} \right]. \quad (A.1)$$

Using the Euler-Lagrange equations for fields the corresponding equation of motion is

$$\partial_x \left[ \frac{\partial_x \phi(x, t)}{l_0} \right] = c_0 \ddot{\phi}(x, t). \quad (A.2)$$

This is a wave equation. It can be solved in terms of normal modes of the cavity,

$$\phi(x, t) = \sum_n \phi_n(t) u_n(x). \quad (A.3)$$

With $\phi_n(t) \propto e^{-i\omega_n t}$, and thus $\ddot{\phi}_n = -\omega_n^2 \phi_n$, for every mode $\phi_n(t) u_n(x)$ of Equation (A.3)

$$\partial_x^2 u_n(x) = -\omega_n^2 l_0 c_0 u_n(x)$$

holds. Provided a high quality factor of the cavities, it is safe to assume current nodes at the coupling capacitors, i.e. $\partial_x u_n(x = \pm \ell, t) = 0$. Also $u_n$ satisfies the modified orthogonal relation

$$\int_{-\ell}^{\ell} dx \, c_0 u_n(x) u_m(x) = C_r \delta_{n,m} \quad (A.4)$$

with $C_r = 2\ell c_0$. Incorporating the aforementioned boundary condition and orthogonal relation, the solution of the wave equation is

$$u_n(x) = \sqrt{2} \sin(\omega_n x \sqrt{l_0 c_0}) \quad (A.5)$$

with resonator frequencies $\omega_n = (2n + 1)\pi / 2\ell \sqrt{l_0 c_0}$. Substituting Equation (A.3) back into Equation (A.1) and applying Equation (A.4) the total Lagrangian collapses into a sum over eigenmodes,

$$L_{tl} = \sum_n \frac{C_r}{2} \dot{\phi}_n^2 - \frac{C_r}{2} \omega_n^2 \dot{\phi}_n^2.$$
By introducing the charge $\theta_n = C_r \phi_n$ as conjugated momentum of the flux $\phi_n$, the system Hamiltonian is

$$
H_{\text{dl}} = \sum_n \frac{\theta_n^2}{2C_r} + \frac{C_r}{2} \omega_n^2 \phi_n^2 .
$$

Using the correspondence principle of quantum mechanics the charge and flux functions are replaced by their respective Fock operators

$$
\hat{\theta}_n(t) = i \sqrt{\frac{\hbar \omega_n C_r}{2}} \left( a_n^\dagger(t) - a_n(t) \right)
$$

$$
\hat{\phi}_n(t) = \sqrt{\frac{\hbar}{2\omega_n C_r}} \left( a_n^\dagger(t) + a_n(t) \right) ,
$$

(A.6)

with $a_n(t) = \hat{a}_n e^{-i \omega_n t}$.

Finally the system Hamiltonian in the Heisenberg picture is given by a sum over harmonic oscillators.

$$
H_{\text{dl}} = \hbar \sum_n \omega_n a_n^\dagger a_n .
$$

(A.7)

Here $a_n$ and $a_n^\dagger$ are the annihilation and creation operators for the $n$th mode in the cavity. They fulfil $[a_n, a_m^\dagger] = \delta_{n,m}$. In Figure 2.3 the spatial profile of the current and voltage of the ground mode is plotted.
B. Geometrical coupling

In case of spatial proximity of the two transmission lines, they couple to each other directly. In this setup this is true for a finite segment of both lines of length $2\ell_c$ (compare with figure B.1). In the lumped element picture the magnetic field created by the current in one line interacts with the inductance of the other line and vice versa. This can be modelled via mutual inductances $l_m$. They are depicted in red in the picture. Both lines can have a potential difference at opposing points. This introduces capacitive coupling, which can be modelled as mutual capacitances $c_m$ depicted in blue.

The following derivation of the geometric coupling Hamiltonian of two symmetric transmission line resonators is based on [7]. The dynamics of this system can be obtained by applying the Kirchhoff’s current law (KCL) to a node $\phi_{1,n}$ (compare with Figure B.2) [14, 15]. The currents coming from the $c_0$ and $c_m$ branches are

$$i_c \equiv \Delta x c_0 \ddot{\phi}_{1,n} + \Delta x c_m (\ddot{\phi}_{1,n} - \ddot{\phi}_{2,n}). \quad \text{(B.1)}$$

The current $i_{1,l}$ coming from the left branch ($\phi_{1,n-1} \rightarrow \phi_{1,n}$) is composed of its inherent and induced current via the mutual inductance $l_m$, which is proportional to the opposing branch ($\phi_{2,n-1} \rightarrow \phi_{2,n}$) current $i_{2,l}$. This leads to a linear system of equations,

$$\Delta x \cdot \begin{pmatrix} l_0 & l_m \\ l_m & l_0 \end{pmatrix} \begin{pmatrix} i_{1,l} \\ i_{2,l} \end{pmatrix} = \begin{pmatrix} \phi_{1,n} - \phi_{1,n-1} \\ \phi_{2,n} - \phi_{2,n-2} \end{pmatrix} \equiv \begin{pmatrix} \phi_{1,l} \\ \phi_{2,l} \end{pmatrix}. \quad \text{(B.2)}$$

The solution for $i_{1,l}$ is given by

$$i_{1,l} = \frac{1}{\Delta x} \frac{l_0 \varphi_{1,l} - l_m \varphi_{2,l}}{l_0^2 - l_m^2}. \quad \text{(B.3)}$$
Figure B.1.: (a) Scheme of two transmission lines of length $2\ell$ approaching each other over a length $2\ell_c$. (b) Distributed element equivalent circuit for inductive and capacitive coupling of transmission lines due to their spatial proximity over the length $2\ell_c$. Mutual capacitances $c_m(n)$ and inductances $l_m(n)$ are indicated in blue and red. $\phi_{1,n}$ and $\phi_{2,n}$ define the fluxes at the circuit nodes $n$ of resonator 1 and 2.

Equivalently the current leaving to right branch ($\phi_{1,n} \rightarrow \phi_{1,n+1}$) is

$$i_{1,r} = \frac{1}{\Delta x} \frac{l_0 \phi_{1,r} - l_m \phi_{2,r}}{l_0^2 - l_m^2},$$

with the branch fluxes $\varphi_{i,r} \equiv \phi_{i,n+1} - \phi_{i,n}$ and $\varphi_{i,r} \equiv \phi_{i,n+2} - \phi_{i,n}$. Altogether, the KCL becomes

$$i_c + i_l - i_{1,r} = 0.$$

In the continuum limit ($\Delta x \to 0$), using

$$\lim_{\Delta x \to 0} \frac{\phi_{i,n+1} - 2\phi_{i,n} + \phi_{i,n-1}}{(\Delta x)^2} = \partial_x^2 \phi_i(x,t),$$

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Figure B.2.: Extract of the distributed element model from Figure B.1b of the interaction section. The $n$th node flux of resonator $j$ is labeled as $\phi_{j,n}$. The current direction of the individual branches are represented by arrows. The spanning tree [14] is chosen in such a way that each inductance $l_0$ can be reached across the associated capacitance $c_0$ and both have a positive sign.
the equation of motion for resonator \(i = 1\) is
\[
 c_0 \ddot{\phi}_1(x, t) + c_m \left( \ddot{\phi}_1(x, t) - \ddot{\phi}_2(x, t) \right) = \frac{1}{l_0^2 - l_m^2} \left[ l_0 \partial_x^2 \phi_1(x, t) - l_m \partial_x^2 \phi_2(x, t) \right].
\] (B.3)

By defining the tensors \(c \equiv c_0 \mathbb{1} + c_m(x)(\mathbb{1} - \sigma_x)\) and \(l \equiv l_0 \mathbb{1} - l_m(x)\sigma_x\) (with the Pauli matrix \(\sigma_x\) and defining \(c_m(|x| < \ell_c) = c_m\), \(l_m(|x| < \ell_c) = l_m\) and zero else) the total Lagrangian, corresponding to Equation (B.3), can be written compactly as
\[
 L_{\text{tot}} = \int_{-\ell}^{\ell} dx \sum_{i,j=1,2} \left[ \frac{1}{2} c_{ij}(x) \dot{\phi}_i(x, t) \dot{\phi}_j(x, t) - \frac{1}{2} \frac{l_{ij}(x)}{l_0^2 - l_m^2} \partial_x \phi_i(x, t) \partial_x \phi_j(x, t) \right].
\] (B.4)

Equation (B.3) again can be solved by the spectral decomposition with an included separation ansatz similar to Section 2.2,
\[
 \phi_i(x, t) = \sum_n \psi_{j,n}(t) u_n(x).
\] (B.5)

Note that the spatial mode is assumed to be the same for both resonators for symmetry reasons. The coupling also gives rise to a dressed resonance frequency of the resonators. However it can be shown, that these corrections can be neglected in the strong coupling regime [7]. The exact eigenmode structure can be obtained numerically by diagonalisation, as shown in [13].

To derive the Hamiltonian of the ground mode with frequency \(\omega_0\), the fields
\[
 \psi_j = \sqrt{\frac{\hbar a_0}{2}} \left( a_j^{\dagger} + a_j \right)
\] (B.6)
\[
 \dot{\psi}_j = i \sqrt{\frac{\hbar}{2a_0}} \left( a_j^{\dagger} - a_j \right)
\] (B.7)
are quantized in terms of Fock ladder operators \(a_j\) of the specific resonator. \(a_0\) is a geometric normalisation constant depending on the overlap \(\int_{-\ell}^{\ell} dx \, u_0^2(x)\). By absorbing the remaining factors into the coupling strengths \(g_c\) and \(g_i\), the Hamiltonian can be written as
\[
 \mathcal{H} \approx \hbar \omega_0 \sum_{i=1,2} a_i^{\dagger} a_i + h g_c (a_1^{\dagger} - a_1)(a_2^{\dagger} - a_2) - h g_i (a_1^{\dagger} + a_1)(a_2^{\dagger} + a_2).
\]
Effectively this is the Hamiltonian of two transmission lines of the ground mode with a geometric coupling term:

\[ \mathcal{H}_{\text{geo}} = \hbar g_c (a_1^\dagger - a_1)(a_2^\dagger - a_2) - \hbar g_i (a_1^\dagger + a_1)(a_2^\dagger + a_2). \]  

(B.8)
**BIBLIOGRAPHY**


Bibliography


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