Noncommutation and finite-time correlations with propagating quantum microwave states

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Chapter 1

Introduction

Since its conception, in the early 20\textsuperscript{th} century, quantum mechanics has remained one of the most counterintuitive, yet most accurate, descriptions of nature ever discovered. Over recent years, substantial progress has been made in applying this esoteric theory to the storage and manipulation of information. Quantum information processing, as this new branch of physics has been named, is an umbrella term for a range of research avenues, including, but not limited to; quantum computing \cite{1, 2}, quantum communication \cite{3, 4}, quantum teleportation \cite{5, 6}, and quantum key distribution \cite{7–9}. The advantages of utilising quantum effects in information processing protocols, are clearly evident in the idea of a quantum computer, first theorised by Richard Feynman in 1982 \cite{10}. Classical computing, that is computing that can be entirely described without quantum mechanics, relies on binary bits having either a value of one or zero. One consequence of quantum mechanics is that systems can coexist in multiple eigenstates simultaneously. This allows the classical bit to be replaced with a quantum bit, or qubit, which can be in eignestates $|0\rangle$, $|1\rangle$ or a superposition of both, $\alpha |0\rangle + \beta |1\rangle$. This extends the computational power exponentially, for certain problems, by allowing all inputs to be evaluated simultaneously \cite{11, 12}.

As with classical computers, quantum computation exists in two forms. Discrete quantum computation involves the consideration of discrete variables such as the energy spectra of a system. On the other hand, continuous variables protocols study continuous spectra, for example, amplitude and phase of a propagating electromagnetic wave. In this view the quantization of the field is introduced through the unavoidable quantum uncertainties, due to the Heisenberg uncertainty principle \cite{13}. Two operations which form the backbone of quantum information protocols, realised with continuous variables, are squeezing and displacement. Squeezing, $\hat{S}(\xi)$, involves the reduction of noise in one phase space quadrature, at the expense of a corresponding increase in the other. Displacement, $\hat{D}(\alpha)$, is associated with the geometrical shift of a localised state in phase space. These operations have been widely studied in both the optical and microwave regimes \cite{14–18}.

One area of intense research for both fundamental purposes, and for applications in quantum information processing, is quantum optics. An interesting subdiscipline of quantum optics, known as cavity quantum electrodynamics (QED), involves placing atoms
in an optical cavity [19]. The atom acts as a two level system, fulfilling the requirements necessary for a qubit [20]. This allows information to be stored in the qubit whilst being transferred via photons. The necessity for a cavity is due to the low interaction strength between the qubits and an electromagnetic field, a cavity enables photons pass the qubit multiple times, dramatically increasing the probability of an interaction.

Despite the inclusion of a resonator, Cavity QED continues to suffer from limited interaction strengths between the optical photons and the single atom. This stems from the low electric dipole moments of the atom and the large mode volumes of optical cavities. A solution to these problems can be found by replacing the real atom with an artificial atom, constructed from superconducting Josephson junctions, and the optical cavity with an on-chip microwave resonator [21]. Analogous to the quantum optics experimental protocols, this new field was named Circuit QED [22]. The use of superconducting circuits to construct artificial atoms circumvents the limitations mentioned above, but introduces its own set of experimental drawbacks. As the entire setup is constructed from superconducting components, it requires cooling below the critical temperature of the materials used. Additionally superconducting atoms and resonators operate in the microwave regime, involving frequencies on the order of a few GHz, five orders of magnitude smaller than optical photons. The use of GHz photons sets the base operating temperature of Cavity QED protocols far lower than the transition temperature of the superconducting components. The low energy of GHz photons also precludes the use of traditional single photon detectors and thus the signals must be amplified, with linear rf amplifiers, necessarily adding noise to the signal [23].

In this work we generate quantum microwave signals using a flux-driven Josephson parametric amplifier (JPA), consisting of a $\lambda/4$ coplanar microwave resonator terminated by a direct current superconducting interference device (dc-SQUID). This augmentation allows the resonant frequency of the resonator to be modulated via the application of an external magnetic flux. Correct modulation of the JPA resonant frequency gives rise to parametric amplification and, subsequently, to the generation of squeezed microwave states [24, 25]. We implement the displacement operation either via a coherent signal to the JPA or by a cryogenic directional coupler. To overcome the noise associated with the amplification of microwave signals we employ a bespoke state reconstruction method, developed at the WMI [26, 27].

One of the most pervasive and striking phenomena in quantum mechanics is the idea of noncommutation. In the classical world the ability to measure multiple features of a system simultaneously is trivial, in the quantum realm, however, this is not so. Depending on whether pairs of operators, representing certain quantum mechanical quantities, commute provides information on whether or not they can be simultaneously known [28]. Another, more obvious, consequence is that the order of application of the respective operations will have an effect on the outcome of the experiment [29]. Squeezing and displacement are two such operations which do not commute. As these operations are of upmost
importance in the realisation of quantum information protocols with continuous variables, a full understanding of how they operate together is vital. Furthermore, investigation of noncommutation between non-Hermitian compound operators, constructed from the creation and annihilation operators, $\hat{a}$ and $\hat{a}^\dagger$, are lacking in the current literature. To this end, we demonstrate how the commutation relation between $\hat{S}(\xi)$ and $\hat{D}(\alpha)$ can be experimentally accessed using superconducting quantum circuits.

The temporal dynamics of propagating quantum signals are of fundamental interest for both quantum computation and quantum communication protocols with continuous variables. Determining how correlations, in a propagating quantum signal, are affected by a temporal delay, $\tau$, provides one with an idea of the robustness of the experimental scheme. To this end, we also study the finite time correlations for both single- and two-mode squeezed states. We determine the temporal shape, first, of the second order correlation function, $g^{(2)}(\tau)$, for the single-mode squeezed vacuum. This quantity represents a standard in the quantum optics community [30, 31], allowing comparison of our experiment to others in the optical domain. It also allows us to determine an effective coherence time for continuous variable protocols.

Due to the requirement that an incoming signal be split into two paths, the Hanbury Brown-Twiss interferometer [32], used for determining $g^{(2)}(\tau)$ of single-mode squeezed states, cannot be used for investigating $g^{(2)}(\tau)$ for two-mode squeezing on spatially distinct modes, as employed in this work. This requires the introduction of a new means of characterising quantum correlations. To this end, we observe how the entanglement, quantified using the negativity kernel as an entanglement witness, between the two modes is affected by a temporal delay in one of the modes.

The structure of this thesis is as follows. Chapter 2 introduces the necessary theoretical background to understand the later experimental results. We start with a basic description of the flux-driven JPA, starting with an introduction to superconductivity and the Josephson effect, before introducing the operational principle of the JPA. We then move on to the more important facets of quantum optics, including the various quantum microwave states used throughout and how one can employ a JPA to generate squeezed states of light. The third section introduces the ideas of noncommutation in quantum mechanics, we discuss the commutator in general terms, before moving on to how superconducting quantum circuits can be used to generate both permutations of the squeezing and displacement operators. We then discuss two ways to simplify complicated commutation relations, namely the Bogoliubov transformation and covariance matrix representation. Finally we look at a method for quantifying the overlap between two arbitrary Gaussian states. Chapter 3, subsequently, discusses the experimental setup used throughout and how one can calibrate and characterise the setup. Once this is completed, Chapter 4 discusses the experimental results gained in the investigation of the commutator between $\hat{S}(\xi)$ and $\hat{D}(\alpha)$. We introduce two methods for probing the commutator, one involving a Bogoliubov transformation to simplify the situation and the other developing a phenomenological model to describe the commutator. Both methods are characterised
across the whole phase space. Chapter 5 examines the finite time correlations between single- and two-mode squeezed states. First we analyse the $g^{(2)}(\tau)$ for the single mode squeezed vacuum before moving on to investigating how entanglement in a two-mode squeezed vacuum is affected by a temporal delay. Finally, chapter 6, will give a summary of the main results and a short outlook to future experiments.
Chapter 2

Physical foundations

In this chapter we introduce the necessary physical ideas needed to put later experimental results into context. We start by describing the physics underlying the Josephson parametric amplifier (JPA), a key component in all of the later experiments. In order to gain the required understanding of how this device operates, we will first present the general concepts underlying superconductivity and the Josephson effect. We will then discuss the JPA itself, including its components and operational principles.

Once we have an understanding of the JPA, we turn our attention towards the areas of quantum optics needed to interpret the later results. We first consider a variety of different quantum microwave states, their possible representations and how one can generate such states of light using superconducting quantum circuit elements.

The third theoretical section concerns itself with the idea of noncommuting operators in quantum mechanics, which will play a large role in later sections. After a general introduction, we will describe the covariance matrix representation of quantum states and operators. We then turn the discussion towards the Bogoliubov transformation, an important transformation in the context of noncommuting operators. Finally, we will derive a method for quantitatively comparing two Gaussian states.

2.1 Josephson parametric amplifier

The Josephson parametric amplifier (JPA) is a nonlinear microwave device, where amplification is achieved by modulation of a system parameter. The JPA is constructed from superconducting elements, namely a coplanar superconducting microwave resonator with one end terminated to ground via a direct current superconducting quantum interference device (dc-SQUID). The addition of the dc-SQUID allows one to modulate the resonant frequency of the resonator through the application of a magnetic flux. Under certain conditions this modulation results in a parametric amplification of an input microwave signal. Both major components of the JPA are constructed from superconducting materials, and the dc-SQUID relies on the Josephson effect. We, therefore, begin by discussing superconductivity and the Josephson effect, before moving on to the operational principle of the JPA.


2.1.1 Josephson effect

Many of the components used throughout this work employ superconducting elements. The first discovered feature of superconductivity was that, when cooled below a critical temperature, certain materials exhibit a sharp drop of their resistance, to zero. However, arguably, a more important phenomenon is known as the Meißner effect whereby, below this critical temperature, the superconducting material expels any magnetic fields from inside the bulk of the superconductor. An interesting effect of electrons moving in a material with zero resistance is that it is possible to set up currents within a superconductor which do not dissipate and theoretically would continue indefinitely.

The Josephson effect, predicted by Brian D. Josephson in 1962 [33], describes how these supercurrents behave between two weakly coupled superconductors, conventionally called a Josephson junction. There are multiple ways to realise a Josephson junction but here we will use the same scheme that Josephson considered, specifically two superconductors separated by a tunneling barrier, illustrated in Fig. 2.1. For two normal metals separated by a sufficiently thin insulating layer it is well known that electrons can tunnel through the barrier from one bulk to another. The question Josephson asked was whether Cooper pairs, the bound pairs of electrons which give rise to superconductivity, could produce a similar effect. Each superconductor is described by a macroscopic wave function $\psi_{1,2} = \sqrt{n_{1,2}} e^{i\theta_{1,2}}$, (2.1) where the indices 1 and 2 denote the first and second superconductor, respectively, $n_i$ is...
2.1 Josephson parametric amplifier

the density of Cooper pairs, and $\theta$ represents the global phase of each superconductor. Assuming the electrons constituting a Cooper pair tunnel independently results in a vanishingly small tunneling probability. Contrary to this reasoning, the tunneling probability for Cooper pairs was found to be the same as that of single electrons. Josephson argued that this process could not be thought of as two incoherent electrons tunneling individually but as a coherent tunneling of both electrons simultaneously.

The Josephson effect is described by two equations. The first of which describes the supercurrent, $I_s$, flowing through the Josephson junction [35]

$$I_s(\phi) = I_c \sin \phi,$$

where $\phi = \theta_1 - \theta_2$ is the phase difference across the barrier and $I_c$ is the maximum Josephson current. This maximum current depends on the barrier thickness as well as $n_1$ and $n_2$. Equation (2.2) states that the supercurrent through the Josephson junction varies sinusoidally with the relative phase difference across the junction. The second Josephson equation is given as

$$\frac{\partial \phi}{\partial t} = \frac{2\pi}{\Phi_0} \int_{\Omega} E(r,t) \, dl,$$

where $\Phi_0 = h/2e$. Recognising $\int_{\Omega} E(r,t) \, dl$ as a voltage drop over the junction, Eq. (2.3) relates the rate of change of the relative phase to a voltage drop across the Josephson junction. If the voltage drop is constant this becomes

$$\frac{\partial \phi}{\partial t} = \frac{2\pi}{\Phi_0} V.$$

This results in a phase difference with a linear time dependence

$$\phi(t) = \phi_0 + \frac{2\pi}{\Phi_0} V \cdot t.$$

With this result and Eq. (2.2) we can see that application of a constant voltage to a Josephson junction results in a temporally oscillating supercurrent. The Josephson current oscillates at the Josephson frequency, given as

$$\frac{\nu}{V} = \frac{\omega}{2\pi V} = \frac{1}{\Phi_0} \approx 483.598 \frac{\text{MHz}}{\mu \text{V}}.$$

This shows we can think of a Josephson junction as a voltage controlled oscillator capable of generating high frequencies with relatively low voltages, on the order of GHz for a bias voltage of a few $\mu$V. This is known as the alternating current (AC) Josephson effect (Eq. (2.4)). The direct current (DC) counterpart can be achieved when there is no voltage drop across the junction. Equation (2.4) illustrates that $V = 0$ corresponds to a constant phase difference, $\phi$. From Eq. (2.2), we see that a constant phase difference corresponds to a constant supercurrent across the junction. In addition to the description given by
Eq. (2.2) and Eq. (2.3) another quantity exists which is useful in describing a Josephson junction, namely the Josephson coupling energy. This energy can be thought of as a binding energy resulting from a finite overlap of the macroscopic wave functions and can be expressed as

\[
E_J = E_{J0} (1 - \cos \phi),
\]

(2.7)

where \(E_{J0} = \Phi_0 I_c / 2\pi\). The potential energy of the Josephson junction is given as [35]

\[
U(\phi) = E_{J0} \left(1 - \cos \phi - \frac{I}{I_c} \phi \right) + c,
\]

(2.8)

where \(I/I_c\) is the normalised supercurrent flowing through the Josephson junction and \(c\) is a constant of integration. The potential, \(U(\phi)\), takes on the shape of a tilted washboard potential when \(I \neq 0\), as shown in Fig. 2.2.

Figure 2.2: Illustration of a tilted washboard potential, shown for 2 periods. A and B represent minimum and maximum stationary points, respectively. Here A is a stable fixed point and B is unstable. The nonlinear effects, to be discussed later, can be thought of as a phase particle moving in this nonlinear potential.

2.1.2 Flux-driven Josephson parametric amplifier

The Josephson effect has many practical applications [35, 36]; in particular, we are interested in how Josephson junctions can be employed to create exotic quantum microwave states. To this end we now discuss the flux-driven Josephson parametric amplifier (JPA). A JPA is constructed by terminating one end of a microwave resonator with a dc-SQUID. The dc-SQUID consists of a superconducting loop, with each arm interrupted by a Josephson
2.1 Josephson parametric amplifier

junction. Figure 2.3 shows a dc-SQUID; $I_a$ and $I_b$ represent the individual currents flowing across each Josephson junction and $\Phi_{\text{ext}}$ is the external magnetic flux in the loop.

![Figure 2.3: Schematic of a dc-SQUID, showing a superconducting loop in grey and two tunneling barriers, forming Josephson junctions, in blue. $I$ denotes a total transport current biasing the loop which subsequently splits into $I_a$ and $I_b$, the individual currents flowing across each Josephson junction. $\Phi_{\text{ext}}$ is the external magnetic flux threading the loop.]

For simplicity we consider the case where the kinetic and geometric inductances of the loop can be ignored. With this caveat, the maximum supercurrent in the dc-SQUID is given by \[ I_{\text{c,max}}(\Phi_{\text{ext}}) = 2I_c \left| \cos \left( \frac{\Phi_{\text{ext}}}{\Phi_0} \right) \right|. \] (2.9)

This demonstrates that a dc-SQUID can be described as a single Josephson junction with a critical current modulated by an external magnetic flux. This phenomenon, resulting from interference between the macroscopic wave functions, is where the dc-SQUID derives its name from. The inductance of a Josephson junction is

\[ L_c = \frac{\Phi_0}{2\pi I_c \cos \phi}. \] (2.10)

Similarly one can write the inductance of the dc-SQUID, $L_{\text{SQUID}}$, as

\[ L_{\text{SQUID}}(\Phi_{\text{ext}}) = \frac{\Phi_0}{4\pi I_c \cos \left( \frac{\Phi_{\text{ext}}}{\Phi_0} \right)} \] (2.11)

Here we can see that the dc-SQUID also acts as a flux tuneable inductor.

The second component of the JPA is a microwave resonator. The flux-driven JPA utilises a quarter-wavelength superconducting resonator in a coplanar waveguide geometry illustrated in Fig. 2.4. The physics of microwave resonators is analogous to LC resonators in circuit theory. An LC circuit consists of a combination of resistances, inductances and capacitances. These circuits can operate as resonant electrical oscillators. Due to the analogy with LC resonators, we can write the resonant frequency of a microwave resonator
as \[ \omega_0 = \frac{1}{\sqrt{L_0 C_0}}, \] (2.12)

where \( L_0 \) and \( C_0 \) are inductance and capacitance of the resonator, respectively. A systematic description of microwave resonators is given in Refs. [39, 40].

![Illustration of a quarter-wavelength coplanar waveguide resonator consisting of an inner stripline (grey) sandwiched between two grounded planes (grey) all sputtered onto a dielectric substrate (orange). The inner stripline and both ground planes are fabricated from a superconducting material, such as niobium. The coupling capacitance, \( C_{\text{ext}} \), is implemented by a break in the inner stripline at one end. The inner stripline is shorted to ground at other end. Distance between shorted end and \( C_{\text{ext}} \) equal to \( \lambda/4 \).](image)

Microwave resonator quality can be described by two numbers, an internal and external quality factor. The internal quality factor characterises the losses from inside the resonator and the external quality factor describes how well the resonator couples to the input port. There are several dominant loss channels, at cryogenic temperatures, that effect the internal quality factor. In the low power limit, two-level systems (TLSs) at the various interfaces dominate the internal losses. At higher powers the TLSs are saturated, diluting their impact [41]. In this regime other loss mechanisms take over. Operating at a finite temperature allows for the generation of quasiparticles within the superconductor. The microwave signals involved can interact with these, providing one loss mechanism at higher powers. Another mechanism arises due to the screening current established to expel magnetic fields from the bulk of the superconductor. This screening current creates surface currents, and, therefore results in a finite surface resistance. In addition to these mechanisms, as the resonator is not completely isolated from the environment, other loss channels, such as radiative loss, also play a part. The external quality factor is mainly defined by the coupling capacitor. Whereas a half-wavelength resonator would have a
coupling capacitor at each end, a quarter-wavelength resonator is constructed with one coupling capacitor and one end shorted directly to ground. In a JPA the inner stripline is grounded via a dc-SQUID, this has the effect of adding an extra flux-tuneable inductance to the system. Figure 2.5 shows a circuit diagram of a JPA.

![Circuit diagram for Josephson parametric amplifier. Resonator is shown in red. A dc-SQUID, with Josephson junctions denoted by crosses, (blue) is inductively coupled to a pump line (green). The incident signal, to be amplified, is denoted by $a_{\text{in}}$, and $a_{\text{out}}$ represents an output signal consisting of amplified signal and idler modes.](image)

In order to describe this effect we first define the total inductance of the JPA

$$L_{\text{tot}} = L_0 + L_{\text{SQUID}},$$  \(2.13\)

where $L_0$ is the inductance of the bare resonator, without the presence of a dc-SQUID, and $L_{\text{SQUID}}$ is the dc-SQUID inductance as defined in Eq. (2.11). Replacing $L_0$ with $L_{\text{SQUID}}$ in Eq. (2.12) gives

$$\omega(\Phi_{\text{ext}}) = \frac{\omega_0}{\sqrt{1 + \frac{L_{\text{SQUID}}(\Phi_{\text{ext}})}{L_0}}},$$  \(2.14\)

where $\omega_0 = 1/\sqrt{L_0C_0}$ is the resonant frequency of the bare resonator. One limitation of the above derivation is that it uses a so-called lumped element model. This assumes all elements, such as the inductance and capacitance, of the system are localised, with their respective geometrical sizes being much less than the resonant wavelength, $L_{\text{geo}} \ll \lambda_{\text{res}}$. A more realistic model would take into account that the capacitance and inductance are spatially distributed. A detailed investigation of this distributed element model can be found in Refs. [38, 42–44]. In this approach one considers a wave equation with appropriate boundary conditions to extract the relevant dispersion relation. This can be then used to derive an expression for the resonant frequency of the JPA [44]

$$\omega(\Phi_{\text{ext}}) = \frac{\omega_0}{1 + \frac{L_{\text{SQUID}}(\Phi_{\text{ext}})+L_{\text{loop}}/4}{L_0}},$$  \(2.15\)
where the geometric loop inductance, $L_{\text{loop}}$, has also been included\(^1\). Both methods provide the key concept required to understand the operational principle of a JPA, namely that the addition of a dc-SQUID allows the resonant frequency of the JPA to be tuned via the application of an external magnetic flux.

A tuneable resonant frequency allows the JPA to function as a parametric amplifier. Parametric amplification is the process whereby a periodic modulation of a system parameter leads to amplification of an input signal. An intuitive example of a parametric amplifier is a children’s swing. Here, the child modulates the swing’s centre of gravity by extending or contracting their legs. Amplification of the swing’s amplitude is achieved only if the centre of gravity is modulated at twice the frequency of the swing’s oscillation. In a similar manner, a JPA can act as a phase-insensitive amplifier which, in theory, will only add the minimum necessary noise to obey the Heisenberg uncertainty principle. The mechanism behind this effect is the coupling of four light fields; an incoming signal, a pump field, the amplified signal, and an idler mode. First we will discuss the case where the incoming signal is not present, a process known as parametric down-conversion. This can be understood as one pump photon being split into two, named signal and idler, fulfilling the energy conservation condition $f_{\text{pump}} = f_{\text{signal}} + f_{\text{idler}}$.

\[ \beta_L = \frac{2L_{\text{loop}}I_c}{\Phi_0}. \tag{2.16} \]

\(^1\)For the case of finite $L_{\text{loop}}$, the dependence of $L_{\text{SQUID}}$ on $\Phi_{\text{ext}}$ is, in general, hysteretic and needs to be obtained from numerical calculations.

Figure 2.6: Top-down view of the dc-SQUID potential. Potential shown for a screening factor, $\beta_L = 0.6$ and dc flux $\Phi_{\text{dc}} = 0$. The green point shows a phase particle resting in a local metastable minimum. The red points illustrate adjacent metastable minima.

The origins of parametric down-conversion in a JPA can be thought of by considering a phase particle oscillating about a local minimum in the JPA potential. This potential is a 3D version of the potential illustrated in Fig. 2.2, shown here in Fig. 2.6. We note that the shape of this plot is affected by the screening factor, defined as
This quantity expresses the ratio between the maximum flux induced by a circulating current in the loop and $\Phi_0/2$. At low pump intensities the oscillations remain harmonic and thus, modeled in time, correspond to a single frequency. However, at higher pump intensities the larger oscillations become nonlinear, i.e. the anharmonic terms in the potential become comparable to the harmonic terms. This nonlinearity allows different frequencies to interact resulting in frequency converting nonlinear processes such as parametric down-conversion. Figure 2.7 illustrates the process of parametric amplification.

![Diagram of parametric amplifier](image)

**Figure 2.7:** a) Mechanism for parametric amplification: energy is transferred from the pump mode into the signal mode. The JPA resonant frequency, $f_0$, is controlled by the dc flux applied to the dc-SQUID. Switching between operational modes is achieved by appropriate selection of $\Delta f$. b) Spectrum showing all modes involved: The dotted red line illustrates unamplified input signal, the solid red line represents the amplified signal, the green line is the idler mode, generated to ensure energy conservation. Finally the blue line represents the pump tone, modulating the resonant frequency at $2f_0$. The Lorentzian gain profile of the amplifier is shown in grey.

For this section we will focus on the nondegenerate case, where $f_{\text{signal}} \neq f_{\text{idler}}$. The degenerate case, which allows for the generation of squeezed microwave states, will be discussed in a later section. One can think of the nondegenerate amplification process as the down-conversion process adding photons into the input signal mode.

In a Josephson parametric amplifier the parametric modulation is achieved through application of external magnetic flux. This can be divided into two parts, $\Phi_{\text{ext}} = \Phi_{\text{dc}} + \Phi_{\text{rf}}$. The dc flux, $\Phi_{\text{dc}}$, is, typically, applied by an external coil and selects the resonant frequency of the JPA resonator, $f_0$. The modulation flux, $\Phi_{\text{rf}}$, is applied by an additional microwave pump tone, $f_{\text{pump}}$. As discussed earlier, for parametric amplification, the resonant frequency needs to be modulated at a frequency twice the frequency of the resonator, $f_{\text{pump}} = 2f_0$. For nondegenerate amplification the input signal frequency should be detuned from the resonator frequency $f_{\text{signal}} = f_0 + \Delta f$. This will result in the creation of an emergent idler mode, at a frequency of $f_{\text{idler}} = f_0 - \Delta f$, in order to ensure energy conservation. If we assume that the resonant frequency is periodically modulated at
$\omega_0 \rightarrow \omega_0[1 + \delta \cos(\alpha \omega_0 t)]$, with $\delta$ being the modulation amplitude, one arrives at the parametrically modulated Hamiltonian of the system [37, 38]

$$\mathcal{H} = \hbar \omega_0 \left[ \hat{a}^\dagger \hat{a} + 2\delta \cos(\alpha \omega_0 t)(\hat{a} + \hat{a}^\dagger)^2 + \frac{1}{2} \right],$$  \hspace{1cm} (2.17)

where terms quadratic in $\delta$ have been ignored, due to the assumption that modulation amplitudes are small. We have expressed this Hamiltonian in terms of the angular frequency, $\omega_0 = 2\pi f_0$. A more detailed theoretical description can be found in Refs. [24, 45, 46].

2.2 Quantum optics

The following section concerns itself with the various facets of quantum optics required to understand the later experimental results. We begin with an introduction to a collection of the more important quantum microwave states and their representations. We then move on to how these states can be experimentally generated, specifically using superconducting quantum circuits.

2.2.1 Quantum microwave states

Quantum microwaves states will play a crucial role in this thesis; consequently it is necessary to discuss how these states can be represented and describe some of the more important examples. A classical electromagnetic wave can be represented as

$$A(t) = A_0 \cos(\omega t + \phi(t)).$$  \hspace{1cm} (2.18)

Applying the identity $\cos(\alpha + \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$, we arrive at

$$A(t) = A_0 \cos(\phi(t)) \cos(\omega t) + A_0 \sin(\phi(t)) \sin(\omega t).$$  \hspace{1cm} (2.19)

This shows an electromagnetic wave can be decomposed into two terms, 90° out of phase with each other. We define two new functions, the in-phase and out-of-phase quadratures

$$I(t) = A_0 \cos(\phi(t)), \hspace{1cm} (2.20)$$

$$Q(t) = A_0 \sin(\phi(t)). \hspace{1cm} (2.21)$$

This allows us to rewrite Eq. (2.19)

$$A(t) = I(t) \cos(\omega t) + Q(t) \sin(\omega t).$$  \hspace{1cm} (2.22)

We can imagine a phase space diagram, with axes of $I$ and $Q$, which covers the whole space of possible states. In the classical representation each state has definite values
for each quadrature and, therefore, states are represented as points in phase space. For the quantum case, as we will see, this is not true. We start by introducing Hermitian quadrature operators \( \hat{p} \) and \( \hat{q} \), the quantum counterpart of \( I \) and \( Q \) \([47–49]\)

\[
\hat{q} = \frac{\hat{a} + \hat{a}^\dagger}{2} \quad \text{and} \quad \hat{p} = \frac{i}{2}(\hat{a}^\dagger - \hat{a}).
\] (2.23)

Here we express the quadrature operators in terms of the bosonic creation and annihilation operators, which obey the canonical commutation relationship \([\hat{a}, \hat{a}^\dagger] = 1\). This results in the quadrature operators also being noncommutative, with the commutator \([\hat{q}, \hat{p}] = i/2\), and requires that they obey a corresponding Heisenberg uncertainty relation

\[
\Delta q \Delta p \geq \frac{1}{4}.
\] (2.24)

One consequence of Eq. (2.24) is that it precludes exact measurement of both operators simultaneously. This means that states can no longer be represented as points in phase space, rather they must now be described by some distribution centred around an expectation value. There are many ways to represent the phase space of quantum mechanical states, a large class of these are referred to as quasi-probability distributions. Akin to a real probability distributions, these allow expectation values of a certain state to be calculated. However, they do have various features which prohibit these distributions from truly being classed as probability distributions. One of the most striking of these features is that quasi-probability distributions allow for negative values. There are a number of quasi-probability distributions that are generally used for the description of quantum states \([47, 50]\), the one used in this body of work is the distribution introduced by Eugene Wigner in 1932 \([51]\)

\[
W(q, p) = \frac{1}{\pi \hbar} \int \langle q - y | \hat{\rho} | q + y \rangle e^{2ipy/\hbar} dy,
\] (2.25)

where \( \hat{\rho} \) is the density matrix of the quantum state under inspection and \( p, q \) are the eigenvalues of the quadrature operators. As with classical probability distributions a Wigner function can also be expressed in terms of the moments of the creation and annihilation operator \([52]\]

\[
W(\alpha) = \sum_{n,m} \frac{\langle (\hat{a}^\dagger)^n \hat{a}^m \rangle}{\pi^2 n! m!} \int \lambda^n (-\lambda^*)^m \exp \left[ -\frac{1}{2} |\lambda|^2 + \alpha \lambda^* - \alpha^* \lambda \right] d^2 \lambda,
\] (2.26)

where \( \int d^2 \lambda \) refers to integrating over the whole phase space. This result shows that complete knowledge of all moments \( \langle (\hat{a}^\dagger)^n \hat{a}^m \rangle \) is equivalent to knowledge of the Wigner function. Most of the states that are to be considered in this thesis belong to a class known as Gaussian states. This is a group of states characterised by a Wigner function with Gaussian marginal distributions, described by the general formula
\[ f(x) = \frac{1}{\sqrt{2\sigma^2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right), \]  

(2.27)

where \( \mu \) is the mean, \( \sigma^2 \) is the variance and \( x \) denotes which quadrature is the marginal variable. We note that for symmetric states \( \sigma^2 \) is the same for both marginal distributions, however for non-symmetric states this is not so. In the same way as a classical Gaussian distribution is completely described by moments up to the second order, the Wigner function of an arbitrary Gaussian state can be completely described by \( \langle \hat{a}^{\dagger n}\hat{a}^m \rangle \) with \( m + n \leq 2 \). We now move the discussion towards various examples of important Gaussian states.

**Vacuum states**

![Figure 2.8: Example of Wigner functions for a vacuum state a) Top view of the Wigner function. Photon number, \( n = \langle \hat{n} \rangle = 0 \) and variances, \( (\Delta p)^2 = (\Delta q)^2 = 1/4 \). This representation will be used later for visualising theoretical and experimental results. b) Top panel shows the 3D representation of the Wigner function of the same state. Lower panel shows the temporal dependence of the signal amplitude, \( E(t) \), for the vacuum state, consisting of the expectation value (red) at 0 and the quantum noise in blue.](image)

The vacuum state is the quantum state with the lowest possible energy, analogous to the ground state of a harmonic oscillator. The application of the annihilation operator on the vacuum state gives 0. This can be expressed as
where we represent the vacuum state as $|0\rangle$. The Hamiltonian of an electromagnetic field is given by

$$
\mathcal{H} = \sum_k \hbar \omega_k (\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}),
$$

(2.29)

where $\sum_k$ represents a sum over the different modes of the field. This Hamiltonian has eigenvalues $\hbar \omega_k (n_k + 1/2)$, where $n_k$ is an integer, and eigenstates $|n_k\rangle$. These energy eigenstates are known as Fock states and are also eigenstates of the photon number operator, $\hat{n} = \hat{a}_k^\dagger \hat{a}_k$

$$
\hat{a}_k^\dagger \hat{a}_k |n_k\rangle = n_k |n_k\rangle.
$$

(2.30)

From this we can see Eq. (2.29) represents the sum of the number of photons in each mode, multiplied by the energy of one photon in that mode plus an extra contribution of $\hbar \omega_k / 2$. This contribution represents the vacuum fluctuations in each mode, arising from the Heisenberg uncertainty relation.

In the Wigner space representation the vacuum is represented by a Gaussian distribution centered at the origin, shown in Fig. 2.8. The vacuum state is also a symmetric minimum uncertainty state. This means the Heisenberg uncertainty relation is fulfilled with an equal signs and the quadrature variances are given by

$$
(\Delta p)^2 = (\Delta q)^2 = \frac{1}{4}.
$$

(2.31)

**Thermal states**

Thermal states, created from black body emitters, are states of light with no phase coherence. Thermal states obey Bose-Planck statistics and therefore have a mean photon number given by

$$
\langle \hat{n}_{th} \rangle = \langle \hat{a}_k^\dagger \hat{a}_k \rangle = 1 \exp\left(\frac{hf}{k_B T}\right) - 1,
$$

(2.32)

where $f$ is the mode frequency, $k_B$ is the Boltzmann constant and $h$ is the Planck constant. We note that Eq. (2.32) only describes the thermal photon population and does not include the zero-point vacuum fluctuations which are always present. This shows that a true vacuum state can only be obtained at $T = 0$; operation at finite temperature necessarily adds thermal photons. The Wigner function of a thermal state has a Gaussian profile with quadrature variances given by
Figure 2.9: Example of Wigner functions for a thermal state. a) Top view of the Wigner function. Photon number, $\langle \hat{n}_{th} \rangle = 1$ and $(\Delta p)^2 = (\Delta q)^2 = \langle \hat{n}_{th} \rangle / 2 + 1/4$. b) Top panel shows the 3D representation of the Wigner function of the same state. Lower panel shows the temporal dependence of the signal amplitude for a thermal state, consisting of the expectation value (red) at 0 and the quantum noise in blue.

$$(\Delta p)^2 = (\Delta q)^2 = \frac{\langle \hat{n}_{th} \rangle}{2} + \frac{1}{4}.$$  \hspace{1cm} (2.33)

From this expression, we can see that the variances increase linearly with mean photon number. Figure 2.9 illustrates the Wigner function of a thermal state. Comparing this to the vacuum state, as shown in Fig. 2.8, one sees a broadening of the Wigner function, and an increase in the quantum noise in the time trace.

**Coherent states**

Coherent states are an important class of states, as they represent a close analogue to the classical wave represented in Eq. (2.18). They find many applications throughout physics, as they are easily generated in lasers or masers (the microwave equivalent of a laser). A coherent state can be viewed as a vacuum state displaced from the origin of phase space and, therefore, are also minimum uncertainty states with variances given by Eq. (2.31). Another way to think of a coherent state comes from the Heisenberg relation
between the photon number, \( n = \langle \hat{n} \rangle \), and phase uncertainties

\[
\Delta n \Delta \phi \geq 1. \tag{2.34}
\]

This allows definition of two types of states. Fock states, with a well defined photon number and undefined phase, and coherent states, with a well defined phase but an undefined photon number. Coherent states, in the limit of large \( n \), are the closest analog to a classical electromagnetic wave.

![Figure 2.10: Example of Wigner functions for a coherent state. a) Top view of the Wigner function. Photon number, \( \langle \hat{n} \rangle = 3 \) and \( \theta = 135^\circ \). A coherent state is a vacuum state displaced in phase space and therefore \( (\Delta p)^2 = (\Delta q)^2 = 1/4 \). b) Top panel shows the 3D representation of the Wigner function of the same state. Lower panel shows the temporal dependence of the signal amplitude for a coherent state. The red dashed line illustrates the expectation value and the blue represents the quantum noise.](figure)

Mathematically coherent states are generated by a displacement operator, \( \hat{D}(\alpha) \), acting on the vacuum

\[
|\alpha\rangle = \hat{D}(\alpha) |0\rangle. \tag{2.35}
\]

We define the displacement operator as

\[
\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}), \tag{2.36}
\]

where \( \alpha = |\alpha|e^{i\Theta} \) is a complex displacement amplitude. We see from Eq. (2.36) that the displacement operator is a unitary operator, \( \hat{D}^\dagger(\alpha)\hat{D}(\alpha) = \hat{D}(\alpha)\hat{D}^\dagger(\alpha) = 1 \). We define
the displacement angle as the angle between the p-axis and the displacement direction. This is related to the phase of the complex displacement amplitude by $\theta = \pi/2 - \Theta$. The displacement operator transforms the creation and annihilation operators as [48]

$$\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha, \quad (2.37)$$

$$\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha) = \hat{a}^\dagger + \alpha^*. \quad (2.38)$$

Coherent states are eigenstates of the annihilation operator

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \quad (2.39)$$

The average number of photons in a coherent state is given by

$$\langle \hat{a}^\dagger\hat{a} \rangle = \langle \alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2. \quad (2.40)$$

**Squeezed states**

Squeezed states provide another, extremely important, subclass of minimum uncertainty states. However, unlike coherent states, they do not necessarily have symmetric quadrature variances. Together with the vacuum state and coherent states, squeezed states complete the class of Gaussian minimum uncertainty states. Combinations of squeezed and coherent states (squeezed coherent states), which will be discussed later, are also minimum uncertainty states. For squeezed states it is possible for the variance of one quadrature to be below that of the vacuum. However due to Eq. (2.24) this needs to be compensated by a proportional increase in the variance in the other quadrature. We will discuss squeezed states in two forms, single- and two-mode. A single-mode squeezed state (SMSS) can be represented, similarly to Eq. (2.35), by a unitary operator acting on the vacuum state

$$|\xi\rangle = \hat{S}(\xi)|0\rangle. \quad (2.41)$$

Here the operator $\hat{S}(\xi)$ is defined as

$$\hat{S}(\xi) = \exp\left(\frac{1}{2}\xi^*\hat{a}^2 - \xi(\hat{a}^\dagger)^2\right), \quad (2.42)$$

where $\xi = re^{i\phi}$ is a complex squeezing amplitude. The angle $\phi$ dictates which quadrature variance will be squeezed and $r$ determines the level of squeezing. We note the following useful transformation properties of the squeezing operator [47, 48]

$$\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi) = \hat{a}\cosh r + \hat{a}^\dagger e^{i\phi}\sinh r, \quad (2.43)$$
Figure 2.11: Example of a Wigner function for a squeezed vacuum. a) Top view of the Wigner function with a quadrature squeezing of 4.5 dB below vacuum. Whilst remaining a minimum uncertainty state a squeezed state has uneven quadrature variances. Squeezing angle, \( \beta \), is defined as the angle between the antisqueezed quadrature and the \( p \) axis. In this case \( \beta = 45^\circ \). Inset shows the \( 1/e \) contours of the squeezed vacuum (blue) and the vacuum (orange) b) Top panel shows the 3D representation of the Wigner function for the same state. Lower panel shows the temporal dependence of the signal amplitude for a squeezed vacuum. As a squeezed vacuum is shown, the expectation value (red) is 0, however, one can see reductions in the quantum noise (blue) at certain points of the trace. Temporal shape of variance, \( \text{Var}(E(t)) \), depends on the squeezing angle and the squeezing level.

\[
\hat{S}^\dagger(\xi)\hat{a}^\dagger \hat{S}(\xi) = \hat{a}^\dagger \cosh r + \hat{a} e^{-i\phi} \sinh r. \tag{2.44}
\]

The variances of the squeezed and antisqueezed quadratures, for an ideal squeezed state, are \( e^{-2r}/4 \) and \( e^{2r}/4 \), respectively. We characterise the squeezing level of our states in decibels, a logarithmic scale comparing the squeezed quadrature variance to the variance of the vacuum state

\[
S_{\text{dB}} = -10 \log_{10}[(\Delta X_{\text{sq}})^2/0.25]. \tag{2.45}
\]

Here a positive squeezing, \( S_{\text{dB}} > 0 \), is equivalent to squeezing below the vacuum level, i.e \((\Delta X_{\text{sq}})^2 < (\Delta X_{\text{vac}})^2\). For a perfect squeezer, \((\Delta X_{\text{sq}})^2 = e^{-2r}/4\), then \( S_{\text{dB,ideal}} = 20r \log_{10}(e) \). We also note that we define our squeezing angle, \( \beta \), as the angle between the antisqueezed quadrature and the \( p \)-axis. The phase of the complex squeezing amplitude, \( \xi \), is related to the squeezing angle, \( \beta \), as \( \beta = -\phi/2 \).

As alluded to earlier, it is also possible to generate multimode squeezed states. To
clarify this we now turn the discussion towards two-mode squeezing. Rather than just suppressing the quantum noise in one quadrature of a single mode, the two-mode case involves squeezing in correlations between the quadratures of two different modes. Various modes can be used, including distinct frequency modes, entangled by some nonlinear optical process, or spatial modes produced by a beam splitter. A two-mode squeezed state (TMSS) is generated by allowing the two-mode squeezing operator to act on a two-mode vacuum

$$|\xi\rangle_{1,2} = \hat{S}_{1,2}(\xi)|0,0\rangle.$$  (2.46)

The two-mode squeezing operator is given by

$$\hat{S}_{1,2}(\xi) = \exp\left(\frac{1}{2}\left(\xi^*\hat{a}_1\hat{a}_2 - \xi\hat{a}_1^\dagger\hat{a}_2^\dagger\right)\right),$$  (2.47)

where $\hat{a}_1^\dagger$, $\hat{a}_2^\dagger$, $\hat{a}_1$, $\hat{a}_2$ are the creation and annihilation operators of the two modes. Clearly, the two-mode squeezed state is not simply a product of two single-mode squeezed states. In fact if each mode is observed individually they are simply thermal states with variances [16, 37]

$$\langle \hat{n}_i \rangle = \langle \hat{n}_2 \rangle = \sinh^2 r.$$  (2.48)

where $\langle \hat{n}_1 \rangle = \langle \hat{n}_2 \rangle = \sinh^2 r$. To see the effects of two-mode squeezing more clearly one can introduce the following superposition quadrature operators, which are linear combinations of those given in Eq. (2.23) [53]

$$\hat{Q}_{1,2} = \frac{1}{\sqrt{2}}(\hat{q}_1 + \hat{q}_2), \quad \text{and} \quad \hat{P}_{1,2} = \frac{1}{\sqrt{2}}(\hat{p}_1 + \hat{p}_2).$$  (2.49)

These superposition quadrature operators now behave in much the same way as the quadrature operators, introduced in Eq. (2.23), behave in the single mode case. With variances, again, for the squeezed and antisqueezed quadratures of $e^{-2r}/4$ and $e^{2r}/4$, respectively. This illustrates how the squeezing now acts on correlations between the quadratures as opposed to on the individual quadratures themselves. An interesting limiting case is as $r \to \infty$; in this case, the wavefunctions of the two-mode squeezed state, $q$ and $p$ bases respectively, become [16]

$$\Psi(q_1, q_2) \propto \delta(q_1 - q_2),$$  (2.50)

$$\Psi(p_1, p_2) \propto \delta(p_1 + p_2).$$  (2.51)

This relationship implies $\hat{q}_1 - \hat{q}_2 = 0$ and $\hat{p}_1 + \hat{p}_2 = 0$. This suggests that, while the individual quadratures are completely uncertain, any measurement on $\hat{q}_1$ instantly decides
2.2 Quantum optics

Figure 2.12: Wigner function illustration of two-mode squeezing. a) and b) represent the two modes considered individually. The Wigner functions are identical to those of a thermal state with $\langle \hat{n} \rangle = \sinh^2 r$. c) and d) show the cross-correlated subspaces demonstrating squeezing on correlated and anticorrelated observables.

the value of $\hat{q}_2$, likewise for the $\hat{p}_i$ quadratures. This illustrates the idea of nonlocal correlations, called quantum entanglement, where a measurement of one mode instantly affects the state of the second mode. This paradoxical result was used by Einstein, Podolsky and Rosen as the basis for the famous EPR paradox [54].

At this point it is important to recognise the difference between so-called classical and quantum states. Whilst both can be described quantum mechanically, states with a close classical analogue, such as the coherent states are often referred to as classical states. Conversely states without a classical analog are referred to as quantum. There exist some conditions which are used to define a difference between quantum and classical states more quantitatively. Examples of these include a Wigner function with negative values or sub-Poissonian photon statistics (see Chapter 5). These are, however, not definitive, as not all quantum states fulfill these conditions. Squeezed states, for example, have both a positive Wigner function and super-Poissonian photon statistics but, despite this, are classed as quantum states. This can be seen in their ability to generate quantum entanglement between modes. Entanglement is a phenomenon with no classical analog and can only be generated with quantum states. SMSSs can be used to generate quantum entanglement using a symmetric beam splitter. This device combines two inputs and distributes them evenly between two outputs. A beam splitter will only create entanglement with at least one quantum input state. The need for a quantum input at the hybrid ring to create entanglement was shown theoretically in Ref. [55] and experimentally in Refs. [56, 57]. With a squeezed state incident to the beam splitter we do indeed see path entanglement [27, 58, 59], proving the quantum nature of squeezed states. As we will see in Sec. 2.2.2, this quantum nature is imperative in the generation of the necessary correlations for two-mode squeezing.
2.2.2 Generation of squeezed light

As previously mentioned, if operated in the nondegenerate mode, a JPA can operate as a phase insensitive amplifier. However, if operated in its degenerate mode, with \( f_{\text{signal}} = f_{\text{idler}} \), the JPA will amplify one quadrature of the signal whilst deamplifying the other, leading to a squeezed state, as described above. Qualitatively this can be thought of as stemming from destructive interference between signal an idler modes, resulting from phase relationships setup due to momentum conservation requirements. To quantitatively understand the squeezing process one needs to consider the interaction Hamiltonian between the signal and pump modes. Solving the Heisenberg equations of motion one arrives at the transformations given in Eq. (2.43) and (2.44). Repeating the process for the nondegenerate mode results in the equivalent two-mode transformations, with correlations between signal and idler modes [47, 48].

The squeezed state can be achieved in two guises; with or without a signal incident to the JPA, leading either to a squeezed coherent state or a squeezed vacuum state, respectively. The following discussion holds for both. The advent of squeezing can be seen from the description of noise added by a phase-sensitive amplifier [38]

\[
A_1 A_2 \geq \frac{1}{16} \left| 1 - \frac{1}{\sqrt{G_1 G_2}} \right|^2,
\]

where the subscripts 1 and 2 denote the different quadratures and \( A_i \) and \( G_i \) represent the added noise and gain, respectively, for each quadrature. As the JPA is a quantum limited amplifier no extra noise photons are added, this implies \( G_1 G_2 = 1 \). Acting as a phase sensitive amplifier \( G_1 \neq G_2 \) and therefore if one quadrature is amplified, the other needs to be deamplified.

With the setup employed in this thesis we can generate two-mode squeezing in two ways. Firstly if one operates the JPA in its nondegenerate mode, as described in Sec. 2.1.2, two distinct frequency modes are generated. As the signal and idler photons are generated simultaneously, and if they fulfill the necessary conservation constraints, the parametric amplification process establishes nonlocal correlations between the modes [60, 61]. This results in squeezing in the correlations setup between the signal and idler modes. We recall that individual modes of TMSS do not necessarily exhibit squeezing. Due to the fact that, when operated in its nondegenerate mode, only the amplified signal mode is of interest, this operational mode is classified as a phase-insensitive amplifier. The method employed in this thesis involves operating two JPAs in their degenerate mode, each generating mutually orthogonal SMSSs. Overlapping these modes on a symmetric beam splitter entangles the previously separate modes into a two-mode squeezed state. A full description of this method can be found in Ref. [16].
2.3 Commutation

To conclude our introduction to the theoretical and physical principles underlying this thesis, we will describe some of the pertinent features of commutation and noncommuting operators. We will begin our discussion with a general description of commutation and its importance in quantum mechanics. Following this we consider the different permutations of the squeezing and displacement operators. We will then describe how quantum mechanical operators can be represented as covariance matrices. We will also discuss the Bogoliubov transformation and finally we will derive a method for measuring the fidelity between arbitrary Gaussian states.

2.3.1 Commutator in quantum mechanics

An important feature of quantum mechanics, which lucidly demonstrates its segregation from the classical world, is the idea of noncommuting quantities. Whilst there are examples of classical operations which do not commute, rotations around different axes for example (see Fig. 2.13), the implications of noncommutation in quantum mechanics are far more profound.

Figure 2.13: An intuitive view of noncommutation. $\hat{O}_i(\theta)$ represents a clockwise rotation of $\theta$ around the $i^{th}$ axis. Rotating first around the $y$-axis and then $x$ leads to a different outcome than if one rotates first around $x$ and then $y$, we can express this as $\hat{O}_y \hat{O}_x - \hat{O}_x \hat{O}_y \neq 0$, or in analogy to Eq. (2.54), $[\hat{O}_y, \hat{O}_x] \neq 0$. One can see that it is possible to move from $\hat{O}_x \hat{O}_y$ to $\hat{O}_y \hat{O}_x$ by applying a suitable set of rotations. This can be expressed as $[\hat{O}_y, \hat{O}_x] \propto \hat{O}_{x,y,z}$ where $\hat{O}_{x,y,z}$ is some combination of $\hat{O}_x$, $\hat{O}_y$ and $\hat{O}_z$. This illustrates that the result of Eq. (2.54) need not be simply a number but can also be an operator.

Whether pairs of operators, describing certain quantum quantities, commute provides us with information that is not attainable through any classical model. Two well-known examples of this phenomenon have already been alluded to in Eq. (2.24) and Eq. (2.34). If the quantum operators are Hermitian, meaning that their eigenvalues are associated with
some physical observable, then their commutator leads to a corresponding Heisenberg relation

\[ \Delta A \Delta B \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle|, \tag{2.53} \]

where \( \Delta O = \sqrt{\langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2} \) is the standard deviation and \( \langle \cdot \rangle = \langle \Psi | \cdot | \Psi \rangle \) is the expectation value. Equation (2.53) expresses that a pair of noncommuting, \( [\hat{A}, \hat{B}] \neq 0 \), Hermitian operators correspond to a conjugate pair of observable that cannot be simultaneously measured with arbitrary precision. There is always some intrinsic uncertainty, arising not from some technical limitations, but rather from a fundamental physical limit. The commutator between two operators is defined as

\[ [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}. \tag{2.54} \]

Commutators between Hermitian operators have been extensively investigated \([28, 62]\), utilising the fact that the operators cannot be simultaneously measured. A version of Eq. (2.53), generalised for non-Hermitian operators, is \([63]\)

\[ \langle \{\hat{A}^\dagger, \hat{A}\} \rangle \langle \{\hat{B}^\dagger, \hat{B}\} \rangle \geq |\langle [\hat{A}^\dagger, \hat{B}] \rangle|^2, \tag{2.55} \]

where the anticommutator, \( \{\hat{A}, \hat{B}\} \), is defined as \( \hat{A}\hat{B} + \hat{B}\hat{A} \). However, generally, investigations of commutators between non-Hermitian operators use a fact evident from Eq. (2.54), namely that \( [\hat{A}, \hat{B}] \neq 0 \) leads to \( \hat{A}\hat{B} \neq \hat{B}\hat{A} \). Remembering that, in quantum mechanics, operators act on states resulting in new states, this is then equivalent to \( \hat{A}\hat{B}|\Psi\rangle \neq \hat{B}\hat{A}|\Psi\rangle \). This illustrates that the order of application of operators will have an effect on the experimental outcome \([64]\). Therefore, by comparing contrasting ordering of the respective operators, one can experimentally investigate Eq. (2.54).

### 2.3.2 Squeezed coherent states

Due to the noncommutativity of the squeezing and displacement operators, there exists two ways to generate squeezed coherent states, depending on the order of operator application.

\[ |\alpha, \xi\rangle_{DS} = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle, \tag{2.56} \]

\[ |\alpha, \xi\rangle_{SD} = \hat{S}(\xi)\hat{D}(\alpha)|0\rangle. \tag{2.57} \]

If we first consider the state which is initially squeezed and then displaced, Eq. (2.56), the final displacement amplitude is independent of the squeezing operator. If the state is displaced first and subsequently squeezed, Eq. (2.57), this is not the case. In this scenario the squeezing operation affects the displacement amplitude in the same manner as it affects the quadrature variances. Depending on whether the displacement direction
2.3 Commutation

Figure 2.14: Illustration of the effect of squeezing on coherent states. a) and b) show $\hat{D}(\alpha)\hat{S}(\xi)|0\rangle$ and $\hat{S}(\xi)\hat{D}(\alpha)|0\rangle$, for $\theta = 135^\circ$ and $\beta = 45^\circ$, respectively. In this case the displacement is along the squeezed (deamplified) quadrature and thus the displacement amplitude is also deamplified. c) and d) show $\hat{D}(\alpha)\hat{S}(\xi)|0\rangle$ and $\hat{S}(\xi)\hat{D}(\alpha)|0\rangle$, for $\theta = 45^\circ$ and $\beta = 45^\circ$. Here the displacement is along the antisqueezed (amplified) quadrature resulting in an amplification of the displacement amplitude.

is along the squeezed or antisqueezed quadrature the initial displacement amplitude is deamplified or amplified, respectively (see Fig. 2.14). We note that it is also possible to displace not exactly along either quadrature, but rather somewhere in between. In this scenario the squeezing would have a more complicated affect on the displacement amplitude. A more detailed theoretical description of this is given in Refs. [48, 58].

2.3.3 Bogoliubov transformation

Bogoliubov transformations are unitary transformations first developed independently by Nikolay Bogoliubov and John Valatin in 1958 [65, 66]. Developed firstly for applications in the microscopic, BCS, theory of superconductivity, they now find uses throughout physics [67–69]. A Bogoliubov transformation creates composite operators whilst preserving canonical commutator relations. We consider an example relevant to the later sections of this thesis. We define a new operator which is a unitary transformation of $\hat{a}$

$$\hat{b} = \hat{U}^\dagger\hat{a}\hat{U} = \mu\hat{a} + \nu\hat{a}^\dagger + \delta.$$  \hspace{1cm} (2.58)

In order to preserve the canonical commutation relation we consider the commutator

$$[\hat{b}, \hat{b}^\dagger] = [(\mu\hat{a} + \nu\hat{a}^\dagger + \delta),(\nu^*\hat{a}^\dagger + \nu^*\hat{a} + \delta^*)] = (|\mu|^2 - |\nu|^2)[\hat{a}, \hat{a}^\dagger].$$  \hspace{1cm} (2.59)

Equation (2.58) is clearly only a canonical transformation under the condition

$$|\mu|^2 - |\nu|^2 = 1.$$  \hspace{1cm} (2.60)

We note that $\delta$ present in Eq. (2.58) vanishes in the calculation of the commutator in Eq. (2.59). We have already seen some of these transformations in Sec. 2.2.1, in how the
squeezing and displacement operators transform the creation and annihilation operators. Among their many uses throughout physics, we will see in Chapter 4 how the Bogoliubov transform can be used to simplify experimental investigation of noncommutation in quantum mechanics.

2.3.4 Covariance matrix representation

In certain situations it can be hard to directly calculate the relevant commutator with the standard representation of the operators, in these cases a different representation of the operators is needed. We demonstrate here how quantum operators can be written in terms of their covariance matrix. The covariance matrix describes how sets of operators are correlated. The covariance between two arbitrary operators can be defined as

$$\text{Cov}(\hat{A}, \hat{B}) = \frac{1}{2} \langle \{\hat{A}, \hat{B}\} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle. \tag{2.61}$$

To fully describe the covariance between two operators one needs to examine the correlations between all combinations of the operators. For two operators this forms a two by two matrix

$$\sigma = \begin{pmatrix} \text{Cov}(\hat{A}, \hat{A}) & \text{Cov}(\hat{A}, \hat{B}) \\ \text{Cov}(\hat{B}, \hat{A}) & \text{Cov}(\hat{B}, \hat{B}) \end{pmatrix}. \tag{2.62}$$

When describing Gaussian states it is useful to express Eq. (2.62) in terms of the creation and annihilation operators, i.e. replacing \(\hat{A}\) and \(\hat{B}\) with \(\hat{a}\) and \(\hat{a}^\dagger\)

$$\sigma(\hat{a}, \hat{a}^\dagger) = \Omega \sigma(\hat{a}, \hat{a}^\dagger) \Omega^T, \tag{2.64}$$

where

$$\Omega = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \tag{2.65}$$

and \(\Omega^T\) represents the transpose of matrix \(\Omega\). We can now express various Gaussian states, in terms of Eq. (2.63) and Eq. (2.64), by calculating the expectation values of \(\hat{a}\) and \(\hat{a}^\dagger\), in the corresponding state basis. We now turn our attention to how one can define the covariance matrix of a Gaussian unitary operator, such as \(\hat{S}(\xi)\) or \(\hat{D}(\alpha)\). It can be shown that any Gaussian unitary operator is equivalent to the symplectic map acting on the phase space \([49]\). In this scenario the symplectic map illustrates how a Gaussian operation transforms the quadrature operators and is defined as
\[
\begin{bmatrix}
\hat{q} \\
\hat{p}
\end{bmatrix} = \begin{bmatrix}
\Xi_{11} & \Xi_{12} \\
\Xi_{21} & \Xi_{22}
\end{bmatrix}
\begin{bmatrix}
\hat{q} \\
\hat{p}
\end{bmatrix} + \begin{bmatrix}
d_1 \\
d_2
\end{bmatrix},
\] (2.66)

where the matrix \( \Xi \) is the covariance matrix of the Gaussian operator. To analyse this further we recall the single mode Bogoliubov transformation illustrated in Eq. (2.58). The symplectic map can then be written as [49]

\[
\begin{bmatrix}
\hat{q} \\
\hat{p}
\end{bmatrix} = \begin{bmatrix}
\text{Re}(\mu + \nu) & \text{Im}(-\mu + \nu) \\
\text{Im}(\mu + \nu) & \text{Re}(\mu - \nu)
\end{bmatrix}
\begin{bmatrix}
\hat{q} \\
\hat{p}
\end{bmatrix} + \begin{bmatrix}
\text{Re}(\delta) \\
\text{Im}(\delta)
\end{bmatrix}. (2.67)
\]

This allows a Gaussian quantum operator to be expressed in its covariance matrix form by calculating how the operator in question transforms the annihilation operator. A detailed theoretical analysis can be found in Ref. [49].

### 2.3.5 Fidelity between Gaussian states

Several key results in the following sections require a measure of similarity between Gaussian states, for this reason we now turn towards finding a fidelity measure between arbitrary Gaussian states. We start from the Uhlmann transition probability [70], which can be interpreted as a fidelity between mixed quantum states [71]

\[
F(\rho_1, \rho_2) = \left( \text{Tr} \sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}} \right)^2, (2.68)
\]

where the subscripts 1 and 2 represent the two states and \( \rho_i \) is the density matrix. As will be seen in Sec. 3.2.1, our experimental setup extracts the moments of the signal. Therefore, to use Eq. (2.68) in conjunction with our setup we will need to describe this fidelity in terms moments of \( \hat{a} \) and \( \hat{a}^\dagger \). We first recall the covariance matrix for operators \( \hat{q} \) and \( \hat{p} \), as defined in Eq. (2.64). The elements of this matrix can then be expressed in terms of the moments as

\[
\sigma_{11} = \frac{1}{2}(\langle \hat{a}^2 \rangle + \langle (\hat{a}^\dagger)^2 \rangle - \langle \hat{a} \rangle^2 - \langle \hat{a}^\dagger \hat{a} \rangle + 1 - 2 \langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle), (2.69)
\]

\[
\sigma_{22} = \frac{1}{2}(-\langle \hat{a}^2 \rangle - \langle (\hat{a}^\dagger)^2 \rangle + \langle \hat{a} \rangle^2 + \langle \hat{a}^\dagger \hat{a} \rangle + 1 - 2 \langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle), (2.70)
\]

\[
\sigma_{12} = \sigma_{21} = \frac{i}{4}(\langle \hat{a}^2 \rangle - \langle (\hat{a}^\dagger)^2 \rangle + \langle \hat{a} \rangle^2 - \langle \hat{a}^\dagger \hat{a} \rangle). (2.71)
\]

We can see, from Refs. [72–75], that, for arbitrary single-mode Gaussian states, Eq. (2.68) can be expressed as

\[
F(\rho_1, \rho_2) = \frac{1}{2(\sqrt{\Delta} + \Lambda - \sqrt{\Lambda})} \exp \left( -\frac{1}{2} \delta \alpha^T (\sigma_1 + \sigma_2)^{-1} \delta \alpha \right), (2.72)
\]

where
\[ \Delta = \det(\sigma_1 + \sigma_2) \quad \Lambda = 16 \left( \det(\sigma_1) - \frac{1}{16} \right) \left( \det(\sigma_2) - \frac{1}{16} \right), \quad (2.73) \]

and \( \sigma_1, \sigma_2 \) are the covariance matrices, \( \delta \alpha = (\langle \hat{q}_1 - \hat{q}_2 \rangle, \langle \hat{p}_1 - \hat{p}_2 \rangle) \) are the differences between the average quadratures for the two states.
Chapter 3

Experimental techniques

In the following chapter we will discuss the experimental techniques employed throughout this work. We will first direct the discussion towards the cryogenic part of our experimental setup. The low photon energy of the microwave photons involved in the operation of a JPA preclude the use of standard detection methods employed in the optical domain. The lack of efficient single photon detectors for microwave signals results in the need to amplify the low energy signals, which necessarily adds noise to the signal. To combat this, we will then discuss a state reconstruction method which circumvents the noise added by linear amplification. We then direct the discussion towards the finer details of the data acquisition and processing methods used. Finally, we will outline the experiments required to correctly characterise and calibrate the measurement setup.

3.1 Cryogenics

Due to the superconducting elements involved in the construction of a JPA, cryogenic temperatures are imperative. Additionally, typical frequencies involved with JPA operation are on the order of a few Gigahertz, corresponding to a characteristic temperature of a few hundred millikelvin. Therefore, it is important to reduce thermal fluctuations to below this temperature.

3.1.1 Cryogenic setup

We employ an in-house made cryostat, built at the Walther-Meißner-Institute [76, 77]. The cryostat is a $^3$He/$^4$He dry dilution refrigerator with six temperature stages from room temperature down to approximately 15 mK. The cooling power of a $^3$He/$^4$He dilution refrigerator is provided by the endothermic mixing of two isotopes of helium. As we employ a dry dilution refrigerator, meaning no cryogenic fluids are used for pre-cooling, the first two stages are cooled by a pulse tube cryocooler (PTC). This cools the first stage to 50 K and the second stage to 3 K. After these stages a $^4$He loop is incorporated, cooling the third stage (named the 1K pot) to $\approx 1.2$ K. A separate $^3$He/$^4$He dilution circuit cools the fourth stage (still) to 600 mK, the fifth stage to
100 mK, and finally the sixth stage (mixing chamber) to approximately 15 mK. More information on the exact specifications of the fridge can be found in Refs. [38, 76, 77]. Our sample stage is mounted below the mixing chamber and housed within a cryoperm shield, to prevent unwanted magnetic fields from impacting results, as can be seen in Fig. 3.1. Included on each output line, outside the cryoperm shield, are two magnetically shielded circulators. These prevent signals reflected from the cryogenic amplification chain from entering the sample box through the output lines.

Due to silver’s high thermal conductivity, at cryogenic temperatures, components are attached to a silver rod, which is itself attached to the mixing chamber plate. This allows the components to thermalise to the temperature of the mixing chamber. Figure 3.2 illustrates how the sample stage is organised. The sample stage includes two JPAs, housed in aluminium containers. As the JPAs are measured in reflection, a measurement circulator is connected in front of each JPA, allowing incoming (typically vacuum) and outgoing (typically squeezed vacuum) signals to be separated. The output of each JPA is connected to a symmetric microwave beam splitter, realised by a hybrid ring, via superconducting NbTi/NbTi rigid coaxial cables. The output line of one JPA includes a cryogenic directional coupler, with a coupling of $-20$ dB and insertion losses of $-0.2$ dB, as seen in Fig. 3.3. Each JPA also has a 30 dB heatable attenuator included on its input line. As will be described

Figure 3.1: Photograph of the cryogenic setup used in this thesis, showing the insert of the $^3$He/$^4$He dilution refrigerator and the respective temperature stages.
3.1 Cryogenics

in Sec. 3.4.2, this acts as a controllable noise source which can be used to extract the conversion factor between the measured voltages and the number of photons at the input of the hybrid ring. Thermalisation of all microwave components to the temperature of the mixing chamber is either achieved by attaching the component directly to the silver sample rod or by means of annealed silver wires, connected to the component and the sample rod. Using silver wires allows us to tailor the thermal coupling by adjusting the wire’s length and cross-section.

Throughout the setup a variety of coaxial microwave cables are used. For input lines, cables with both inner and outer conductors constructed from stainless steel are used. Due to their relatively high losses in the GHz regime, these cables are only used for the input lines, as here power dissipation is of little importance. The input lines are connected to copper plates at each temperature stage to ensure thermalisation. Thermal noise is reduced through attenuators included on the input lines at each stage. For the pump lines to the JPAs 14 dB less attenuation is used than for the input signal and displacement lines, this is due to the higher pump powers required. In the main experimental section, at the lowest temperatures, cable losses become more relevant due to the delicate quantum states used. For this reason superconducting NbTi/NbTi cables are used. These cables have much lower losses at cryogenic temperatures, ≈ 0.15 dB per meter, than the stainless steel cables used for the input lines. This has the advantage of allowing the 30 dB attenuator to be heated, producing thermal states, without heating the fridge. These cables are also used for the output lines, up until the first stage of amplification, after this stage stainless steel cables with a silver plated stainless steel inner conductor are used. Different cable types are indicated in Fig. 3.3 via different colours.

The JPA samples are encased within an aluminium box to shield out any external magnetic flux. This box contains the JPA sample with a coil to apply dc magnetic flux.

Figure 3.2: Photograph of the cryogenic setup used in this thesis, illustrating the sample stage inside the cryoperm shield with important microwave components indicated.
Figure 3.3: Schematic of the cryogenic and room temperature setup. Coaxial cables described as 'XX/YY' where 'XX' represents the material of the inner conductor and 'YY' that of the outer conductor. In this figure 'SS' denotes a stainless steel conductor and 'SSS' is silver-plated stainless steel. Attenuators on the input lines, at each temperature stage, are used to equilibrate thermal noise to the corresponding cryogenic temperature. A simplified view of the pulsing mechanism is shown. Detailed pulsing schematic is shown in Sec. 3.4.3 (Fig. 3.10)
3.1 Cryogenics

Figure 3.4: (a) Photograph of JPA setup inside aluminium box. Shielding box contains a superconducting coil for applying dc magnetic flux, a heater and temperature sensor for precise stabilisation of the sample temperature. (b) Optical micrograph of the JPA sample, dc-SQUID shown in a green box and coupling capacitor in a red box, respectively. Inductively coupled microwave pump tone applied from right (indicated by 'P'), signal input and amplified output enter and leave from left (indicated by 'S').

Also included are a heater and temperature sensor for precise stabilisation of the JPA temperature. Figure 3.4 shows the setup inside the aluminium box as well as an optical micrograph of the JPA sample.

The JPA samples are designed and fabricated at NEC smart energy research laboratories, Japan. Thermally oxidised silicon with a thickness of 300µm is used as a substrate. The resonator and pump line, in a coplanar waveguide geometry, are patterned into a previously sputtered 50 nm thick Nb film. The dc-SQUID is fabricated from aluminium using conventional shadow evaporation. The samples used in this thesis are designed to have an external quality factor of $Q_{\text{ext}} = 200$. The external quality factor is defined by the geometry of the coupling capacitor.

The room temperature section of the experimental setup, shown in the upper portion of Fig. 3.3, is designed to implement the state reconstruction method, to be described in Sec. 3.2.1. This involves the splitting of the signal, via a cryogenic hybrid ring and the amplification of each chain. In practise, the amplification consists of cryogenic and room temperature stages. Initially the signal is amplified by a set of cryogenic high electron mobility transistor (HEMT) amplifiers, which define the noise temperature of the detection chain. The second amplification stage occurs at room temperature via AMT-A0033 and JS3-25-8P amplifiers. These are operated at a controlled temperature of $20\pm0.05^\circ$C. The temperature is stabilised via a Peltier cooler. Once amplification is complete, the signal is mixed with a local oscillator detuned from the signal frequency by
11 MHz. This has the effect of down-converting the signal to an intermediate frequency of $f_{\text{IF}} = f_{\text{LO}} - f_{\text{signal}} = 11$ MHz. Due to the nature of the mixer the sum frequency is also generated but a low pass filter is included to filter this out. Once the signal has been down-converted, further amplification is provided by an AU-1447-R amplifier in each arm. The signal from each chain then enters an analog to digital converter (ADC) to digitise the signal, with a sampling frequency of 400 MHz. The ADC then passes the data to a computer to be processed. We employ a digital, low-pass, finite impulse response (FIR) filter in the data processing code, during the digital down-conversion process, with a fixed cut-off frequency $f_{\text{FIR}} = 400$ kHz. A more detailed description of the experimental setup can be found in Ref. [38].

### 3.1.2 Generation of displacement

There exist separate permutations of the squeezing and displacement operators which yield different states. Implementation of the displacement operator first involves the application of a signal tone to the JPA input port. As a coherent signal is nothing more than the displacement operator acting on the vacuum, using the JPA to squeeze an incoming coherent signal is the same as applying first the displacement operator to the vacuum and then applying the squeezing operator leading to the state $\hat{S}(\xi) \hat{D}(\alpha) |0\rangle$.

Realising the state $\hat{D}(\alpha) \hat{S}(\xi) |0\rangle$, i.e. applying the displacement operator after the squeezing operator, is slightly more complicated. For this the JPA needs to be used, without a signal tone, squeezing only vacuum fluctuations. After that the resulting squeezed vacuum is displaced using a cryogenic directional coupler [14]. This cryogenic element functions much in the same way as a highly asymmetric beam splitter in optics, weakly coupling a strong coherent signal to the squeezed vacuum emitted from the JPA.

In the limit of high transmissivity and a strong coherent signal, the output signal from the directional coupler is

$$\hat{a}_{\text{out}} \approx \hat{a}_{\text{in}} + \sqrt{1 - \tau} \tilde{\alpha},$$

(3.1)

where $\hat{a}_{\text{out}}$ and $\hat{a}_{\text{in}}$ are the output and input states of the directional coupler, respectively; $\tau$ is the transmissivity and $\tilde{\alpha}$ represents the amplitude of the coupled coherent signal. This is analogous to Eq. (2.37), and so the directional coupler has the affect of displacing the input state with an amplitude and phase set by the strong coherent signal.

### 3.2 State tomography

Throughout this work we employ two different methods to reconstruct quantum states. In the following section we will briefly describe each. For a more complete theoretical description we direct the reader towards Refs. [26, 27, 38, 58, 59].
3.2.1 Dual-path state reconstruction

We first turn our attention to the so called dual-path state reconstruction method. This method, developed at the WMI, was designed to measure low-energy quantum microwave signals whilst eliminating the added noise that accompanies linear amplification. The absence of efficient single photon detectors in the microwave regime requires that the signals be amplified, and the linear amplifiers used for this necessarily add a large amount of noise. The dual-path method deals with this by first splitting the signal with a symmetric microwave beam splitter, realised by a hybrid ring. Each path is then individually amplified, adding uncorrelated noise to each path. The signal is then digitized and demodulated, extracting the quadratures, \( I_{1,2} \) and \( Q_{1,2} \). Measuring the cross-correlations between the two paths and averaging allows the reconstruction of the state at the input of the hybrid ring whilst cancelling out the uncorrelated noise added by the amplification chain. A schematic of the dual-path protocol is illustrated in Fig. 3.5.

A beam splitter is a linear device which combines two input signals and distributes them evenly between two outputs, with the relations

\[
\begin{pmatrix}
\hat{c}_1 \\
\hat{c}_2
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix} \begin{pmatrix}
\hat{a} \\
\hat{\nu}
\end{pmatrix},
\]

(3.2)

where \( \hat{c}_{1,2} \) are the output signal operators and \( \hat{a}, \hat{\nu} \) are the input signal operators. This
shows that the beam splitter creates a linear superposition of the input states. The dual-path method assumes full knowledge of Eq. (3.2) and the ancillary state, \( \hat{\nu} \). A natural choice of \( \hat{\nu} \), and the one employed in this thesis, is the vacuum state. It can be shown \([26, 38, 58]\) that one can subsequently extract the signal moments \( \langle (\hat{a}^\dagger)^n \hat{a}^m \rangle \) from the measured cross-correlation moments, \( \langle I_1 I_2^* Q_1^l Q_2^r \rangle \).

### 3.2.2 Reference-state reconstruction

The assumptions made by the dual-path method, i.e full knowledge of the beam splitter relations and the necessity for an ancillary vacuum state at the second input to the hybrid ring, make it unsuitable for certain tasks. In the context of this thesis the main problem is the need for an ancillary vacuum state. In later sections we want to combine two single-mode squeezed vacua in the hybrid ring, in order to create a two-mode squeezed state. The hybrid ring is constructed with two inputs, allowing two signals to be combined. Therefore, having two single-mode squeezed vacua incident to the hybrid ring precludes the use of an ancillary vacuum state, rendering the dual-path method ineffective. To combat this, we now introduce another method of state tomography, known as reference-state reconstruction method \([78, 79]\). This method employs one amplification chain, as opposed to two, and uses a comparison with a known state to eliminate the added noise. For this method first a known state, in our case the vacuum, is sent through the setup, this is then used to determine the noise characteristics of the amplification chain. Once the moments of the added noise are known, they can be calibrated out when reconstructing the moments of the signal. A detailed theoretical description can be found in Refs. \([27, 58]\).

### 3.3 Data acquisition and processing

We use an Acqiris DC440 card for data acquisition and processing. After the signals have been digitised by the ADCs we apply three levels of averaging. The limited internal memory of the Acqiris card requires that part of the averaging is done, externally, on a separate computer. Traces of 64000 samples are recorded and then transferred to a computer. The trace is then digitally demodulated to dc, and \( I \) and \( Q \) quadratures are extracted. Once the quadrature values are obtained the cross-correlations \( \langle I_1^j I_2^k Q_1^l Q_2^r \rangle \) can be calculated up to 4th order, \( j + k + l + r \leq 4 \) where \( j, k, l, r \in \mathbb{N} \). While Gaussian states are fully described by their second order moments, higher order moments can be used to check the Gaussianity of the states. This process is then repeated for \( N \) segments. The whole process of averaging over \( N \) segments is then repeated for \( L \) cycles. The final layer of averaging involves repeating the entire averaging of segments and cycles again over \( M \) sweeps. Each new segment is triggered by a data timing generator, referenced to a 10 MHz rubidium source.

The data acquisition program is a specialised program, written in C++, run through
3.4 Calibration and characterisation

Before the later experiments can be conducted, it is necessary to precisely calibrate the setup and characterise how the squeezing depends on the pump power sent to the JPA and the displacement power.

3.4.1 Flux dependence of the JPA resonant frequency

As discussed in Sec. 2.1.2, the magnetic flux, used to modulate a JPA, comes in two forms. A dc-flux, $\Phi_{dc}$, tunes the resonant frequency of the JPA, before a separate rf-flux, $\Phi_{rf}$, creates a fast modulation of the resonant frequency, generating parametric amplification.

In this section we will examine the selection of $\Phi_{dc}$. To determine the flux dependence of the JPA we employ a Keysight PNA N5224A vector network analyser (VNA) with a frequency range of 10 MHz to 25 GHz. Figure 3.6 shows the experimental scheme for determining the flux dependence of a JPA. The $\lambda/4$ geometry of the JPA resonator requires measurements to be made in reflection. We use a circulator to separate the input.
and output signals. The frequency of the signal from the VNA is swept for various coil currents. The pump signal to the JPA is turned off, as we are interested, at this moment, in how the resonator responds to $\Phi_{dc}$. We define the reflection coefficient of a JPA as the ratio between input and output signals

$$\Gamma = \frac{a_{out}}{a_{in}}. \quad (3.3)$$

The VNA measures a scattering parameter between port one and two (shown in Fig. 3.6), $S_{12}$. This scattering parameter is a transmission coefficient for the entire setup, not only the JPA. Therefore a calibration of the input and output lines is required to extract $\Gamma$ from $S_{12}$. We complete this calibration by tuning the JPA away from the measured frequency window. Making the assumption that the JPA has a reflection coefficient of one when off resonance allows us to determine the combined loss or gain of the input and output lines. This can then be taken into account in the final results, leaving only $\Gamma$ of the JPA. We subsequently record both the magnitude and phase of $\Gamma$. Figure 3.7 shows

![Figure 3.7](image)

**Figure 3.7:** Flux dependence of JPA 1, with a designed $Q_{ext} = 200$. The VNA output power is $-20$ dB. a) Magnitude of $\Gamma$ plotted for a range of coil currents and signal frequencies. b) Phase response of the reflected signal, $\text{arg}(\Gamma)$

typical data for a flux sweep for the JPA1 sample, with a designed external quality factor of $Q_{ext} = 200$. We show both the phase and magnitude plots of $\Gamma$. We note from Fig. 3.7 the characteristic periodic behavior of the flux dependence, as to be expected from the addition of a dc-SQUID to the resonator. We also notice very little magnitude response in Fig. 3.7(a). This is indicative of an overcoupled $\lambda/4$ resonator, i.e. one where $Q_{ext} \ll Q_{in}$. This can be thought of qualitatively that a higher internal quality factor, relative to the external quality factor, means less signal photons are lost in the resonator. This results in an outgoing signal very similar, in absolute power, to the incoming one and therefore very little magnitude change is detected by the VNA. We see, from Fig. 3.7(b), a strong phase response of the reflected signal. This results from the extra distance the signal travels
when it is at the JPA resonant frequency. As the resonator is measured in reflection the signal travels along the resonator twice, resulting in a \( \lambda/2 \) difference between the incoming and outgoing signal, i.e. a 180° phase shift. This phase shift is seen in the phase of the reflection coefficient. We use these plots to determine a working point, in terms of the JPA resonant frequency, corresponding to a specific value of \( \Phi_{dc} \) for further experiments. The selected working point plays a crucial role in the generation of squeezing using a JPA. If the working point is selected at a steep point in the flux dependence graph, small fluctuations in the magnetic flux correspond to large changes in the resonant frequency and, therefore, the JPA will be more susceptible to noise. Conversely if one chooses a working point where the gradient of the flux dependence plot is too shallow then the pump signal will not modulate the resonant frequency enough, and it will not allow for sufficient gain. For a deeper investigation of the flux dependence of a JPA we refer the reader to Refs. [38, 45].

### 3.4.2 Dual-path and PNCF calibrations

Whilst the dual-path state reconstruction does not rely on equal amplification of each path, the precision of the results can be improved by balancing the amplitude of each path. Furthermore, for the dual-path method to function correctly, it requires there to be a 180° phase difference between the two chains. To balance the amplification of each chain the raw ADC outputs from each chain are recorded. The two step attenuators, indicated in Fig. 3.3, allow the attenuation of each path to be individually adjusted. This is used to assure that the total detected rf power in each channel is equal, within a margin of \( < \pm 0.05 \text{ dB} \). The dual-path method assumes a 180° phase shift between the channels, resulting from the geometry of the hybrid ring. To calibrate the phase offset between the channels a strong coherent signal is applied through the signal line to one of the JPAs. Measuring the data with the Acqiris card and fitting the data with a sinusoidal curve allows the phase difference to be extracted. An analog phase shifter, present in one of
the local oscillator arms (evident in Fig. 3.3), is used to manually adjust the relative phase between the two paths. The fit also allows a balance factor to be extracted, this is then used as a voltage pre-factor in later experiments to ensure the quadratures in both channels are on the same order.

Once the dual-path has been appropriately calibrated, we turn our attention towards determining the photon number conversion factor (PNCF). The setup detects the quadrature moments, \( \langle I_1^2 Q_1^2 \rangle \), in units of volts. However, we are interested in the signal moments, \( \langle (\hat{a}^\dagger)^n \hat{a}^m \rangle \), of the state at the input to the hybrid ring, which have units of photon number. Therefore, a conversion factor between the measured moments and the signal moments is required. To correctly calibrate this, we send a well-known signal into the system and measure the resulting voltage. We employ a heatable 30 dB attenuator mounted in the signal line to the JPA (see Fig. 3.3). This acts as a black body radiator, emitting a temperature dependent thermal state, with a well-defined number of photons. The use of superconducting cables between the heatable attenuator and measurement circulator has the effect of thermally decoupling the heatable attenuator from the rest of the system. This means the attenuator can be heated whilst all the other microwave components remain at a stable temperature. A second benefit relies on the low losses associated with superconducting cables, resulting in more precise calibration of the PNCFs.

The total power at the detector of each chain is given by

\[
P_{1,2} = \frac{\langle I_{1,2}^3 \rangle + \langle Q_{1,2}^2 \rangle}{R} = \frac{\kappa G_{1,2}}{R} \left( \frac{1}{2} \coth \left( \frac{\hbar f_0}{2 k_B T_{\text{att}}} \right) + n_{1,2} \right),
\]

where \( R = 50 \Omega \), subscripts 1 and 2 represent the two amplification chains, \( T_{\text{att}} \) is the

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{figure_a.png}
\caption{(a)}
\end{subfigure}
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{figure_b.png}
\caption{(b)}
\end{subfigure}
\caption{Example PNCF fits for a) \( \langle I_1^2 \rangle \) and b) \( \langle I_2^2 \rangle \). In both figures blue crosses are the experimental data and the red line represents the theoretical fit, according to Eq. (3.4). Better quality data for path one over path two, resulting from the different number of noise photons added by the two cryogenic HEMT amplifiers.}
\end{figure}
attenuator temperature, $G$ is the amplification gain, $f_0$ is the center frequency of the detection bandwidth and $n$ is number of added noise photons. The coth term denotes the number of thermal photons emitted by the attenuator. From this we can see that the term $\kappa G_i$ relates units of volts, from the left hand side of Eq. (3.4), to units of photon number, on the right hand side. We sweep the temperature of the attenuator, $T_{\text{att}}$, from $40 \text{ mK}$ to $800 \text{ mK}$ and fit the recorded voltage data with Eq. (3.4). By fitting each of the quadratures $\langle I_1^2 \rangle$, $\langle I_2^2 \rangle$, $\langle Q_1^2 \rangle$, $\langle Q_2^2 \rangle$ individually, we can extract separate PNCFs for each quadrature.

From Fig. 3.9, we notice better quality data for one path, $\langle I_1^2 \rangle$, stemming from the cryogenic HEMT amplifiers. The amplifiers for each path are not identical and one path adds more noise photons. We see from table 3.1 that the noise added by the cryogenic amplifier in channel one is around half of that in channel two.

For both chains we see that below around $70 \text{ mK}$ the heatable attenuator contributes negligibly to the measured signal, and the signal is dominated by noise added by the amplifiers.

### 3.4.3 Characterisation of squeezing and displacement

To characterise the squeezing gained from our setup we investigate how the measured squeezing level is affected by both power sent to the pump line of the JPA and displacement power sent both to the JPA signal line and directional coupler. To observe squeezing the JPA is operated in the degenerate mode, with the signal frequency tuned to half the pump frequency. However squeezing of the vacuum can also be achieved, with no coherent signal incident.

At this point it is useful to introduce the pulsing scheme used throughout our experiments. We employ a scheme with four temporal sections, allowing the measurement of vacuum, coherent, squeezed vacuum and squeezed coherent states in one measurement trace. The pulsing is realised by combining rectangular voltage pulses, generated with a DTG5000 signal generator, with a continuous signal from a microwave source in a harmonic mixer.

We illustrate in Fig. 3.10 the four part pulse scheme, as applied to the generation of squeezed coherent states, displaced via the directional coupler. The same pulse scheme can be used to investigate both permutations of the squeezing and displacement operators, depending on whether the 'Displacement’ pulse train is being used to pulse the signal.

<table>
<thead>
<tr>
<th>Moment</th>
<th>PNCF ($\times10^{-4}V^2$)</th>
<th>Noise photons</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle I_1^2 \rangle$</td>
<td>$4.136 \pm 0.085$</td>
<td>$15.389 \pm 0.331$</td>
</tr>
<tr>
<td>$\langle Q_1^2 \rangle$</td>
<td>$4.185 \pm 0.085$</td>
<td>$15.191 \pm 0.323$</td>
</tr>
<tr>
<td>$\langle I_2^2 \rangle$</td>
<td>$4.067 \pm 0.085$</td>
<td>$28.048 \pm 0.600$</td>
</tr>
<tr>
<td>$\langle Q_2^2 \rangle$</td>
<td>$4.097 \pm 0.085$</td>
<td>$27.837 \pm 0.592$</td>
</tr>
</tbody>
</table>

Table 3.1: PNCF values and added noise photons.
Figure 3.10: Pulse scheme used throughout the following work. Initial four pulse sections allow measurement of four states in one measurement trace. We include a buffer section at the end of the trace. Red line shows pulsing scheme applied to the pump signal. The blue line illustrates the displacement pulsing, which is used to pulse both the directional coupler and the signal line to the JPA. The black line denotes the triggering of the Acqiris card. Pulsing is implemented by mixing a pulsed microwave signal from a DTG signal generator with the continuous microwave signal.

When investigating squeezed coherent states, $\hat{S}(\xi)\hat{D}(\alpha) |0\rangle$ we see an advantage, due to the effect of squeezing on the resultant displacement angle of $\hat{S}(\xi)\hat{D}(\alpha) |0\rangle$ (see Sec. 4.3.2). The displacement angle of $\hat{S}(\xi)\hat{D}(\alpha) |0\rangle$ depends nonlinearly on the phase of the microwave source. This means that small changes in the phase at the source correspond to larger changes in the resultant displacement angle of $\hat{S}(\xi)\hat{D}(\alpha) |0\rangle$. As a result it is not possible to stabilise the phase of $\hat{S}(\xi)\hat{D}(\alpha) |0\rangle$, as desired. To combat this, it is necessary to calculate the phase from the section of the trace with the JPA pump turned off. Later, in the context of two-mode squeezing, the pulse scheme allows us to compare the path entanglement for the single- and two-mode cases.

To characterise the squeezing performance of the JPA, we begin by demonstrating the effect of pumping the JPA on the squeezing level. For this we sweep the pump power and extract the squeezing level from the measured signal moments. We note that the required power for given squeezing level is heavily dependent on the selected working point. We see from Fig. 3.11(a) that above a certain power the squeezing level starts decreasing. This is as, for higher signal gains the JPA enters a bifurcation regime, where higher order effects become relevant [15, 38, 47]. In later experiments, Fig. 3.11(a) will...
allow us to select the power required at the JPA to result in the desired squeezing level. Similarly, to characterise how the squeezing level depends on the displacement power, we plot squeezing as a function of displacement power. We see from Fig. 3.11(b) that the squeezing remains almost constant up to 200 displacement photons. Figure 3.11(b) illustrates results for displacement via the directional coupler. However we see a similar behaviour for displacement via the signal line to the JPA.

Figure 3.11: Squeezing characterisation for a JPA frequency of 5.585 GHz, corresponding to a coil current of 35 µA. The blue line represents measured squeezing level and the orange line illustrates the measured photon number. a) Dependence of the squeezing level on the pump power. b) Dependence of the squeezing level on displacement power.
Chapter 4

Commutator relationship between squeezing and displacement operators

The following chapter will present the experimental results from the investigation of the commutation relation between the squeezing and displacement operators. Analysis of compound operators constructed from $\hat{a}$ and $\hat{a}^\dagger$, such as $\hat{S}(\xi)$ and $\hat{D}(\alpha)$, are lacking in the modern literature. We also note that the squeezing and displacement operations form the core of quantum communication and quantum computation protocols with continuous variables. This makes a deeper understanding of these operations of great interest. First, we look at the theoretical derivation of the commutator itself and then how this result changes in the covariance matrix representation. We then introduce two methods for studying the commutator. First, we apply a Bogoliubov transformation to investigate the commutator. In the final section, we look at developing a phenomenological model for describing the commutators dependence on the displacement amplitude. All experimental results obtained in this chapter use the cryogenic and room temperature setups illustrated in Fig. 3.3 and signal moments are extracted with the dual-path state reconstruction method, as described in Sec. 3.2.1. The JPA temperature is stabilised at 50 mK. Throughout this chapter we employ a DTG5000 pulse generator, generating pulse patterns with a 5 kHz repetition rate. The pulse scheme used is shown in Fig. 3.10, each section of the pulse trace is 40 $\mu$s long, in data analysis we allow for a ring up time of 20 $\mu$s. We record multiple traces of 64000 samples, collected at 2.5 ns intervals, these traces are then subject to 3 stages of averaging. In all measurements we use a squeezing angle, $\beta = 45^\circ$. We use a digital FIR filter with a designed bandwidth of 400 kHz.

4.1 Commutation relation

In Sec. 2.3.1 we discussed the commutator in general terms, we will now refine our discussion to the specific case of the squeezing and displacement operators. To simplify notation throughout we use the state definitions from Sec. 2.3.2

$$|\alpha,\xi\rangle_{DS} = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle,$$  \hspace{1cm} (4.1)
We also introduce a naming scheme where we will refer to states displaced using the directional coupler as displaced squeezed states and states displaced using a coherent signal incident to the JPA as squeezed coherent states. To begin our discussion we first recall the general definition of a commutator

\[ [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}, \]

and the mathematical definitions of the squeezing and displacement operators

\[ \hat{S}(\xi) = \exp \left( \frac{1}{2} (\xi^* \hat{a}^2 - \xi (\hat{a}^\dagger)^2) \right), \]
\[ \hat{D}(\alpha) = \exp \left( \alpha \hat{a}^\dagger - \alpha^* \hat{a} \right). \]

Due to the exponential nature of Eqs. (4.4) and (4.5), evaluating the commutator is not straightforward. However, one can decompose both operators into their respective Taylor series, expanded about the points \( \frac{1}{2} \xi^* \hat{a}^2 - \frac{1}{2} \xi (\hat{a}^\dagger)^2 = 0 \) and \( \alpha \hat{a}^\dagger - \alpha^* \hat{a} = 0 \). To the first order, this is given by

\[ \hat{S}(\xi) = 1 + \frac{1}{2} \xi^* \hat{a}^2 - \frac{1}{2} \xi (\hat{a}^\dagger)^2, \]
\[ \hat{D}(\alpha) = 1 + \alpha \hat{a}^\dagger - \alpha^* \hat{a}. \]

Inserting these results into Eq. (4.3), and applying the canonical commutation relation \( [\hat{a}, \hat{a}^\dagger] = 1 \) leaves us with the result

\[ [\hat{S}(\xi), \hat{D}(\alpha)] = \xi^* \hat{a}\hat{a} - \xi \hat{a}^\dagger \hat{a}^\dagger. \] (4.8)

To apply this result to our experimental setup, we look at the overlap between the commutator and the full spectrum of Fock states. The appropriateness of this choice can be seen by expanding the commutator

\[ \sum_{n=0}^{\infty} \langle n | [\hat{S}(\xi), \hat{D}(\alpha)] | 0 \rangle = \sum_{n=0}^{\infty} \langle n | \hat{S}(\xi) \hat{D}(\alpha) | 0 \rangle - \sum_{n=0}^{\infty} \langle n | \hat{D}(\alpha) \hat{S}(\xi) | 0 \rangle. \] (4.9)

This can now be thought of as the difference between the Fock states in \( \hat{S}(\xi) \hat{D}(\alpha) | 0 \rangle \) and \( \hat{D}(\alpha) \hat{S}(\xi) | 0 \rangle \). As we measure with the vacuum as an initial state, this allows the result to be investigated with our experimental setup. Inserting Eq. (4.8) into Eq. (4.9), gives

\[ \sum_{n=0}^{\infty} \langle n | [\hat{S}(\xi), \hat{D}(\alpha)] | 0 \rangle = -\xi \alpha^*. \] (4.10)
The commutator now only depends on the complex squeezing and displacement amplitudes. To continue our analysis we look at the first moments for $|\alpha,\xi\rangle_{SD}$ and $|\alpha,\xi\rangle_{DS}$

$$\langle \hat{a} \rangle_{SD} = \langle 0 | \hat{D}^\dagger \hat{S}^\dagger \hat{a} \hat{S} \hat{D} | 0 \rangle = \alpha \cosh r - \alpha^* e^{i\phi} \sinh r,$$

(4.11)

$$\langle \hat{a} \rangle_{DS} = \langle 0 | \hat{S}^\dagger \hat{D}^\dagger \hat{a} \hat{D} \hat{S} | 0 \rangle = \alpha.$$

(4.12)

These moments represent the displacement amplitudes of the Wigner functions, i.e. the distance to the center points of the states in Wigner space. An interesting remark can be made by calculating the difference between Eq. (4.11) and (4.12) and taking the first order of the Taylor expansions of $\cosh r$ and $\sinh r$

$$\langle \hat{a} \rangle_{SD} - \langle \hat{a} \rangle_{DS} = -\xi \alpha^*.$$

(4.13)

Comparing Eq. (4.13) with Eq. (4.10) one can immediately see that

$$\sum_{n=0}^\infty \langle n | \left[ \hat{S}(\xi), \hat{D}(\alpha) \right] | 0 \rangle = \langle \hat{a} \rangle_{SD} - \langle \hat{a} \rangle_{DS}.$$

(4.14)

This illustrates that the Fock decomposition, described in Eq. (4.9), does have physical significance. Eq. (4.14) claims that, to the first order expansion and with a vacuum state as input, the commutator $\left[ \hat{S}(\xi), \hat{D}(\alpha) \right]$ describes the difference between the center points of $|\alpha,\xi\rangle_{SD}$ and $|\alpha,\xi\rangle_{DS}$ in phase space. We will see that this claim can be verified more concretely, both theoretically and experimentally, in later sections.

### 4.2 Covariance matrix representation

We now turn our attention to the commutator between the squeezing and displacement operators in the covariance matrix representation

$$[\Xi_S,\Xi_D] = \Xi_S \Xi_D - \Xi_D \Xi_S.$$

(4.15)

In this context, this illustrates how the variances differ between $|\alpha,\xi\rangle_{SD}$ and $|\alpha,\xi\rangle_{DS}$. In Sec. 2.3.4 we described how Gaussian operators can be expressed in terms of their covariance matrices. We recall the definition of the symplectic map given in Eq. (2.67)

$$\begin{bmatrix} \hat{q} \\ \hat{p} \end{bmatrix} = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{21} & \Xi_{22} \end{bmatrix} \begin{bmatrix} \hat{q} \\ \hat{p} \end{bmatrix} + \begin{bmatrix} \Re(\delta) \\ \Im(\delta) \end{bmatrix},$$

(4.16)

where

$$\Xi = \begin{bmatrix} \Re(\mu + \nu) & \Im(-\mu + \nu) \\ \Im(\mu + \nu) & \Re(\mu - \nu) \end{bmatrix}.$$

(4.17)
and $\mu$, $\nu$, and $\delta$ come from the unitary Bogoliubov transformation $\hat{U}^\dagger \hat{a} \hat{U} = \mu \hat{a} + \nu \hat{a}^\dagger + \delta$. Substituting $\hat{S}$ and $\hat{D}$ for $\hat{U}$, we arrive at

$$\hat{D}^\dagger(\alpha) \hat{a} \hat{D}(\alpha) = \hat{a} + \alpha,$$

$$\hat{S}^\dagger(\xi) \hat{a} \hat{S}(\xi) = \hat{a} \cosh r + \hat{a}^\dagger e^{i\phi} \sinh r.$$  (4.18) (4.19)

From these one can extract $\mu$, $\nu$ and $\delta$ for each operator.

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$\nu$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{S}(\xi)$</td>
<td>$\cosh r$</td>
<td>$e^{i\phi} \sinh r$</td>
<td>0</td>
</tr>
<tr>
<td>$\hat{D}(\alpha)$</td>
<td>1</td>
<td>0</td>
<td>$\alpha$</td>
</tr>
</tbody>
</table>

Table 4.1: Components of the Bogoliubov transformation for squeezing and displacement operators.

Using Euler’s formula, $e^{i\phi} = \cos \phi + i \sin \phi$, and inserting the values from Table 4.1 into Eq. (4.17), we arrive at formulae for the covariance form of the squeezing and displacement operators.

$$\Xi_D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\Xi_S = \cosh r \cdot I_2 + \sinh r \begin{bmatrix} \cos \phi & \sin \phi \\ \sin \phi & -\cos \phi \end{bmatrix}.$$  (4.20) (4.21)

The effect of the squeezing operator can be clearly seen if we assume a real squeezing amplitude, i.e. $\phi = 0$. In this limit, the covariance matrix for the squeezing operator becomes

$$\Xi_S = \begin{bmatrix} e^r & 0 \\ 0 & e^{-r} \end{bmatrix}.$$  (4.22)

Given that the diagonal elements of a single-mode covariance matrix represent the variances in the respective quadratures, Eq. (4.22) shows that one quadrature variance is deamplified whilst the other is amplified. We also notice that the displacement operator covariance matrix is nothing more than the identity matrix, as to be expected, because displacing a state in phase space has no effect on the quadrature variances. Returning to the commutator and given that the covariance matrix of the displacement operator is the identity matrix, we arrive at

$$[\Xi_S, \Xi_D] = \Xi_S \Xi_D - \Xi_D \Xi_S = 0.$$  (4.23)

This shows that the reversal of operator order has no effect on the quadrature variances.
As all states here are Gaussian, which are fully described by their first two moments, this implies that the only quantity that is effected by the commutator is the displacement amplitude of the Wigner function, adding more weight to the claim made by Eq. (4.14). This holds, however, only for Gaussian states, as for non-Gaussian states higher order moments are also important.

4.3 Bogoliubov transformation applied to $\hat{D}(\alpha)$ and $\hat{S}(\xi)$ ordering

In the following section we will experimentally investigate the claim that the effect of noncommutation, between $\hat{S}(\xi)$ and $\hat{D}(\alpha)$, is only evident in the relative displacement amplitudes. It can be shown, with a Bogoliubov transformation, that one can equate contrasting operator permutations with a correction to the complex displacement amplitude [47]

$$\hat{D}(\alpha)\hat{S}(\xi)|0\rangle = \hat{S}(\xi)\hat{D}(\gamma)|0\rangle,$$

where $\gamma = \alpha \cosh r + \alpha^* e^{i\phi} \sinh r$. We confirm this relationship by replacing $\alpha$ in Eq. (4.11) with $\gamma$ and see that we do, in fact, find Eq. (4.12). Equation (4.24) states that by the choice of an appropriate displacement amplitude, $\gamma$, the different orderings of the operators, $\hat{S}(\xi)$ and $\hat{D}(\alpha)$, yield identical states. In the following experiments we first measure the state $|\alpha,\xi\rangle_{DS}$, from this we extract $\xi$ and $\alpha$. This allows us to calculate a value for $\gamma$, which is subsequently fed back to the microwave source on the signal line and state $|\gamma,\xi\rangle_{SD}$ is measured. For the quantitative estimation of the overlap between $|\alpha,\xi\rangle_{DS}$ and $|\gamma,\xi\rangle_{SD}$, we use the fidelity equation, as defined in Sec. 2.3.5. This allows a direct experimental investigation of Eq. (4.24).

4.3.1 Calibration of the displacement amplitude

Before experimental investigation of the commutator relation can be conducted, appropriate calibration of the setup is required. The first requisite calibration measurement involves ensuring that both displacement methods generate the same displacement amplitude. The microwave tones to the signal port of the JPA and to the directional coupler experience different levels of losses. These losses need to be calibrated so that we know which power levels we need to apply at room temperatures to ensure both displacement methods would provide the same displacement amplitude. For this, we turn the JPA pump off and only measure coherent signals, for a range of displacement powers, generated from both the JPA signal line and from displacing vacuum fluctuations with the directional coupler. Comparing the measured number of photons from each line allows an offset to be calculated. For averaging we use 140 segments, 120 cycles and 10 sweeps, resulting in
Due to the logarithmic nature of the decibel scale it is useful to convert the input displacement into milliwatts via

\[ P_{\text{mW}} = 10^{\frac{P_{\text{dB}}}{10}}. \] (4.25)

In this basis the number of photons depends linearly on the displacement power, with \(|\alpha|^2 = 0\) at \(P_{\text{mW}} = 0\). This allows us to write the dependence as

\[ P_{\text{mW},i} = m_i |\alpha_i|^2, \] (4.26)

where \(i\) denotes whether the signal (S) or directional coupler (D) line is under inspection and \(m_i\) is a constant of proportionality. Comparing the slopes of the two plots allows us to determine the power offset between the two lines. Our aim is that for a given input power the measured number of photons in the final output states is the same for each line. Equating the number of photons and rearranging we arrive at

\[ P_{\text{mW},S} = \frac{m_D}{m_S} P_{\text{mW},D}. \] (4.27)

Applying Eq. (4.25), to convert the above result back into decibels, we obtain

\[ P_{\text{dB},S} = P_{\text{dB},D} + P_{\text{corr}}, \] (4.28)

where the correction power, \(P_{\text{corr}} = 10 \log \left( \frac{m_D}{m_S} \right)\). The losses experienced by the microwave tones express a frequency dependence. This arises from the fact that many components,
4.3 Bogoliubov transformation applied to $\hat{D}(\alpha)$ and $\hat{S}(\xi)$ ordering

4.3.2 Calibration of the displacement angle

The displacement angle is stabilised in the section of the pulse scheme where the pump line is off and only the displacement tone is on, as shown in Fig. 3.10. It became evident that a calibration of the displacement angle, for squeezed coherent states, was required due to a discrepancy between the displacement angle for the coherent state and that of the squeezed coherent state [15]. It is conceivable that this difference arises from the fact that pumping the JPA has a nonlinear effect on the JPA impedance. As in microwave circuits impedance plays the same role as the refractive index in the optics regime, this has the effect of implementing a nonlinear phase shift on a signal traveling through the JPA.

The calibration was achieved by sweeping the phase, $\theta$, of the signal tone around the entire phase space for a given displacement and pump power. We employ averaging of 120 segments, 140 cycles and 15 sweeps, resulting in $1.6128 \times 10^{10}$ raw samples, for each data point. The centre points of the Wigner functions are recorded and should follow an ellipse described by Eq. (4.11). Due to the aforementioned angle discrepancy there will be

Figure 4.2: Example power calibration plot for a signal frequency, $f_{\text{signal}} = 5.323$ GHz. The red data points represent data for coherent states generated by the directional coupler, and the black points show data for coherent states from the signal line. a) Power at the microwave source, in mW. The logarithmic nature causes the points to bunch up on the linear power scale, the inset therefore shows the low power data points. Comparison between the gradients of these two plots allows calculation of the power offset between the two displacement sources. b) Power in dBm. The difference in gradients in a) corresponds, in b), to a relative shift of the two plots along the $y$-axis, due to decibels being a logarithmic scale.
Chapter 4  Commutator relationship between squeezing and displacement operators

Figure 4.3: Schematic for the calibration of displacement angle. Squeezed coherent states are generated for a range of displacement phases. The center points of the Wigner functions are extracted from the measured quadrature moments.

A phase offset. To determine this offset we fit the center points of the measured Wigner functions with Eq. (4.11), augmented with an extra phase offset in $\alpha$

$$
\langle \hat{a} \rangle_{SD} = |\alpha| e^{i(\Theta+\Theta_{corr})} \cosh r - |\alpha| e^{-i(\Theta+\Theta_{corr})} e^{i\phi} \sinh r,
$$

(4.29)

where $\Theta_{corr}$ is the added phase offset. Using $\Theta_{corr}$ as a fitting parameter and fitting Eq. (4.29) to the data allows us to determine the angle offset. This angle offset depends

Figure 4.4: Displacement angle calibration. Red crosses indicate the center points of the experimentally measured Wigner functions. The theoretically fitted curve, from Eq. (4.29), is shown here in green. As this calibration depends on the squeezing level, this example plot is shown for $S_{dB} = 3.93$ dB and a working point $f_0 = 5.323$ GHz.
4.3 Bogoliubov transformation applied to $\hat{D}(\alpha)$ and $\hat{S}(\xi)$ ordering

heavily on the squeezing level and therefore, this calibration must be repeated each time the squeezing level is changed. Figure 4.4 shows the addition of $\Theta_{\text{corr}}$ results in a close fit with the experimental data.

4.3.3 Calculation of the squeezing amplitude

Throughout this chapter the squeezing angle is selected manually for each measurement via adjusting the phase of the pump tone. The squeezing amplitude, however, depends on the amplification gain of the JPA and is therefore harder to accurately measure. Whilst

![Figure 4.5: Schematic for the calculation of the squeezing amplitude. Squeezed vacua are generated for a range of JPA pump powers. The quadrature variances for both squeezed and antisqueezed quadratures are extracted from the measured moments.](image)

the JPA gain is dependent on the pump tone power it is also heavily dependent on the incident noise photons. Therefore, we model the output quadrature variances as an input variance, consisting of the vacuum variance and some thermal component, affected by a phase (quadrature) sensitive gain. This is then used to model both the squeezed and antisqueezed quadratures

$$ (\Delta X_{\text{sq}})^2 = G_{\text{sq}} \left( \frac{n_{\text{th}}}{2} + \frac{1}{4} \right), $$

$$ (\Delta X_{\text{anti}})^2 = G_{\text{anti}} \left( \frac{n_{\text{th}}}{2} + \frac{1}{4} \right), $$

where $G_{\text{sq}} = e^{-2\tau}$ and $G_{\text{anti}} = e^{2\tau}$ are the squeezing and antisqueezing gains of the JPA. The output variances can be extracted from measured moments, leaving a pair of simultaneous equations with two unknowns. This method also provides an estimation of the number of thermal photons incident to the JPA. To confirm the result of this model we note that the total number of photons output from the JPA for a squeezed vacuum,
with a thermal contribution, can be written as \[80\]

\[
\langle \hat{a}^\dagger \hat{a} \rangle = n_{\text{th}} \cosh 2r + \sinh^2 r. \tag{4.32}
\]

Comparing the number of photons calculated from Eq. (4.32) with the number of photon measured from the dual-path setup, for a range of squeezing levels, allows us to determine the accuracy of our model. We make the comparison for a range of different pump powers to the JPA. We see, from Fig. 4.6, a good agreement between theory and experiment, with no degradation for higher pump tone powers, illustrating the reliability of the above model for calculations of the squeezing amplitude. The agreement in Fig. 4.6 is achieved without any fitting parameters. We note that both \( n_{\text{th}} \) and \( r \) are calculated from our model.

<table>
<thead>
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<th>0.5757</th>
<th>0.6576</th>
<th>0.7795</th>
<th>0.9121</th>
<th>1.074</th>
<th>1.2978</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_{\text{th}} )</td>
<td>0.0477</td>
<td>0.0698</td>
<td>0.079</td>
<td>0.115</td>
<td>0.152</td>
<td>0.248</td>
<td>0.425</td>
<td>0.805</td>
</tr>
</tbody>
</table>

**Table 4.2:** Squeezing factor and thermal photons calculated from Eqs. (4.30) and (4.31)

Table 4.2 shows the squeezing parameters and thermal photons for the data plotted in Fig. 4.6.
4.3.4 Fidelity measurements

We can now use the calibration results and the squeezing amplitude model to investigate Eq. (4.24). For these measurements we increase the averaging to 100 segments, 140 cycles and 50 sweeps, resulting in $4.48 \times 10^{10}$ raw samples per data point. The following results use a working point of $f_0 = 5.535 \text{ GHz}$, corresponding to a coil current of $44 \mu \text{A}$. We use a squeezing level $S_{\text{dB}} \approx 4 \text{ dB}$ throughout. We first measure the state $\hat{D}(\alpha)\hat{S}(\xi)|0\rangle$, a displaced squeezed state. We then extract the squeezing amplitude from the measured quadrature variances. The complex displacement amplitude, $\alpha$, can be easily extracted from the moments, as shown in Eq. (4.12). With these quantities one can calculate $\gamma$, as defined above. We convert the displacement amplitude to a power in decibels via

$$P_{\text{dB}}(\gamma) = m_D |\gamma|^2 + P_{\text{corr}}.$$  \hspace{1cm} (4.33)

The power at the signal line is then set to $P_{\text{dB}}(\gamma)$. The displacement stabilisation phase set to

$$\theta = \frac{\pi}{2} - (\arg(\gamma) + \Theta_{\text{corr}}),$$  \hspace{1cm} (4.34)

where $\pi/2$ comes from our definition of the displacement angle, as described in Sec. 2.2.1. Using Eqs. (4.33) and (4.34) we now measure the state $\hat{S}(\xi)\hat{D}(\gamma)|0\rangle$. We express the result as a fidelity, calculated via Eq. (2.72). The above method is repeated for a range of displacement powers and displacement angles.
Figure 4.8: a) Dependence of the fidelity on displacement power for $\theta = 135^\circ$ and $\beta = 45^\circ$. Error bars are of a statistical nature. b) Squeezing level versus displacement power. The average squeezing is $S_{\text{dB}} = 3.93$ dB. c) Comparison of photon numbers. The red data points denote the displaced squeezed states, $|\alpha,\xi\rangle_{\text{DS}}$. The blue data points represent squeezed coherent states, $|\gamma,\xi\rangle_{\text{SD}}$. Squeezed photons are included, as the squeezing is the same for each. Squeezing of $\approx 4$ dB corresponds to approximately 0.3 squeezed photons.

Figure 4.8(a) illustrates how the measured fidelity changes with displacement power. The error bars in Fig. 4.8(a) are of a statistical nature, they are obtained by measuring the fidelity for each sweep which has passed the physicality check, that is sweeps which result in states obeying the Heisenberg uncertainty principle. Any nonphysical states are likely a product of our finite measurement times, which results in reconstructed signal moments which are not Gaussian. We obtain the statistical error, $\epsilon$, via

$$\epsilon_{\pm} = \pm \frac{\sigma}{\sqrt{N}}, \quad (4.35)$$

where $\sigma$ is the standard deviation, $N$ is the number of selected sweeps and $\pm$ denotes the upper and lower error bounds. We employ a squeezing level of $S_{\text{dB}} \approx 4$ dB, corresponding to a mean number of squeezing photons $\langle \hat{n}_{\text{sq}} \rangle \approx 0.3$, in obtaining these results. From Fig. 4.8(b) we see that this squeezing level is stable throughout the displacement power sweep. Figure 4.8(a) shows a finite deterioration of the fidelity for increasing displacement power. The reason for the finite drop in fidelity is multifaceted. As the fidelity can be thought of as an overlap between the Wigner functions it is highly susceptible to phase fluctuations. Phase errors are incorporated into the results in two ways. The first is due to the finite measurement times involved in these experiments. Even with the phase stabilisation scheme, as described in Sec. 3.3, employed, phase fluctuations on the order of $0.5^\circ$ are still present. A second path for phase errors to enter the above results comes
from the finite phase resolution of the microwave sources. These sources are accurate to one decimal place. As described in Sec. 4.3.2, the resulting displacement angle of $|\alpha,\xi\rangle_{SD}$, $\theta_{SD}$ depends nonlinearly on the stabilization angle, $\theta_D$. This means phase fluctuations on smaller scales than the device resolution can have a large effect on the end displacement angle, $\theta_{SD}$, and subsequently on the measured fidelity. We can quantify phase fluctuations by calculating the average deviation of the measured displacement angle, $\theta_{SD}$, from its target phase. Calculating this value for both states we arrive at $0.35^\circ$ for $|\alpha,\xi\rangle_{DS}$ and $-0.98^\circ$ for $|\gamma,\xi\rangle_{SD}$. Whilst these are relatively small deviations, they are in different directions. In addition to this, the effects of phase deviations are amplified at higher displacement powers.

Figure 4.8(c) compares the measured number of photons for each state, providing an insight into how the displacement amplitude behaves. Here we have included the squeezed photons but as these are approximately equal for both states and small compared to the number of displacement photons they do not impact the results. We see from Fig. 4.8(c) a small deviation in the photon number. The photon number for a squeezed coherent state consists of a contribution from the displacement amplitude, $\langle \hat{n}_{\text{disp}} \rangle = |\alpha|^2$, and from squeezed photons, $\langle \hat{n}_{\text{sq}} \rangle = \sinh^2 r$ as well as a finite thermal contribution. As the squeezing and thermal components are the same for both states, this deviation implies some discrepancy in the displacement amplitudes of $|\alpha,\xi\rangle_{DS}$ and $|\gamma,\xi\rangle_{SD}$. The factors involved in calculating the amplitude of $\gamma$ are the displacement amplitude of $|\alpha,\xi\rangle_{DS}$, $\alpha$ and the squeezing factor, $r$. As $\alpha$ is measured directly from $|\alpha,\xi\rangle_{DS}$, and has been shown to be relatively unaffected by noise and losses, it is a stable quantity and unlikely to be responsible for the photon number deviation [14]. We have shown from Fig. 4.6 that our model for calculating $r$ is also exceedingly stable and, again, is unlikely to be responsible for this discrepancy.

A likely cause for the dissonance between the photon numbers could come from the calibration of the displacement amplitude, as described in Sec. 4.3.1. To increase the speed of the calibration measurements we use less averaging than for the fidelity measurements. We defend this decision by noticing that strong coherent signals are incredibly stable and less susceptible to noise and losses, and therefore, should be robust enough to provide faithful results even with a reduction in the averaging. However, as we sweep the displacement power from low to high, the lower power measurements could be affected by the reduced averaging. We see from Fig. 4.2 that the fit does not intersect the $y$ axis exactly at zero. This shows that there could be some error in the gradients which would translate into an error in the calibration. It is also feasible that the calibration does not remain constant throughout the experiment. Due to the high levels of averaging, the measurements are completed on timescales of hours. It is reasonable that the gains of the HEMT and other amplifiers could fluctuate on these timescales. As can be seen from Eq. (4.33), the gradients calculated from Fig. 4.2 enter the conversion twice. This amplifies even small errors.
Another possible explanation comes from the fact that the displacement direction is along the squeezed quadrature. A consequence of this, as mentioned in Sec. 2.3.2, is that the original displacement amplitude is deamplified. This means that more displacement photons need to be incident to the JPA in order for $|\gamma,\xi\rangle_{SD} = |\alpha,\xi\rangle_{DS}$. This becomes relevant at higher powers when the JPA can reach its 1 dB compression point. The 1 dB compression point of an amplifier is the input power at which the signal gain is 1 dB lower than that which would be expected for a linear amplifier. This would mean that the displacement amplitude would not be deamplified as much as if the JPA was still responding linearly, subsequently affecting the calculated fidelity.

To characterise the fidelity over the whole phase space it is necessary to map its phase dependence. To this end we sweep the displacement angle for a fixed displacement power corresponding to $\langle \hat{n}_{\text{disp}} \rangle \approx 3$. Figure 4.9(a) demonstrates the phase dependence of the fidelity. We see an oscillatory pattern stemming from the dependence of $|\gamma,\xi\rangle_{SD}$ on the relationship between the squeezing and displacement angles. We note peaks in the fidelity for displacement angles, $\theta = 45^\circ$ and $225^\circ$. One possible explanation for this is that as, for these displacement angles, the displacement direction is along the antisqueezed quadrature. In this situation, fluctuations or discrepancies in the displacement amplitude have less effect on the overlap of the Wigner function and, therefore, less effect on the measured fidelity.

![Figure 4.9](image_url)

**Figure 4.9:** a) Dependence of fidelity on the displacement phase, $\beta = 45^\circ$ and $\langle \hat{n}_{\text{disp}} \rangle \approx 3$. Error bars are of a statistical nature. b) Squeezing level versus displacement phase. Average squeezing of $S_{\text{dB}} = 3.96$ dB. Squeezing displayed for $|\gamma,\xi\rangle_{SD}$. c) Comparing number of photons for $|\gamma,\xi\rangle_{SD}$ (blue data points) and $|\alpha,\xi\rangle_{DS}$ (red data points).

Figure 4.9(c) shows a comparison of the photon numbers for the displacement phase sweep. As can be seen, the number of photons follows the same oscillatory pattern as Fig. 4.9(a), with phase values with smaller differences in photon number corresponding
to higher fidelity. As previously mentioned, differences in photon number stem from differences in the displacement amplitude. Changes in the displacement amplitude are introduced via two mechanisms. One are the phase fluctuations, this is due to the fact that the final displacement amplitude of a squeezed coherent state depends not only on the original displacement angle but also on the relationship between the squeezing and displacement angles. The second mechanism, as asserted previously, relies on the fact that deviations in the displacement amplitude are also an effect of a less than perfect power calibration.

### 4.4 Phenomenological commutator model

We now introduce a phenomenological model for the $\alpha$ dependence of the commutator. For the following experiments we employ a working point of $f_0 = 5.323$ GHz, corresponding to a coil current of $66 \mu$A. Again we use a squeezing level of $S_{\text{dB}} \approx 4$ dB. We employ averaging of 100 segments, 140 cycles and 50 sweeps, resulting in $4.48 \times 10^{10}$ raw samples per data point. In order to compare the commutator to experiment we attempt to model the commutator as a fidelity. This would allow us to measure the fidelity between $\hat{D}(\alpha)\hat{S}(\xi)|0\rangle$ and $\hat{S}(\xi)\hat{D}(\alpha)|0\rangle$ and then compare this to our model. Both fidelities should start at unity for zero displacement amplitude and decrease as $|\alpha|$ increases, as at higher displacement amplitudes the effects of the commutator are more visible.

#### 4.4.1 Characterisation of experimental fidelity

![Schematic](image)

**Figure 4.10:** Schematic for the measurement of the fidelity between $\hat{S}(\xi)\hat{D}(\alpha)|0\rangle$ and $\hat{D}(\alpha)\hat{S}(\xi)|0\rangle$. Squeezed coherent states are generated by squeezing a coherent signal incident to the JPA. Displaced squeezed states are generated by squeezing vacuum fluctuations and then displacing with the directional coupler.
This approach involves the measurement of both $\hat{D}(\alpha)\hat{S}(\xi)|0\rangle$ and $\hat{S}(\xi)\hat{D}(\alpha)|0\rangle$. We measure these states for a range of displacement amplitudes and along two displacement angles.

To characterise the experimental fidelity, we plot the fidelity between the two states versus the displacement amplitude and compare this to a theoretically calculated fidelity. This is achieved by using the moment equations, given explicitly in Refs. [38, 58], to theoretically construct $\hat{D}(\alpha)\hat{S}(\xi)|0\rangle$ and $\hat{S}(\xi)\hat{D}(\alpha)|0\rangle$. These theoretically constructed states can then be compared to the experimental fidelity, calculated from Eq. (2.72). Figure 4.11 shows the results of this measurement. The measurements are repeated over

![Figure 4.11](image_url)
4.4 Phenomenological commutator model

We now discuss how one can compare the fidelity plots shown in Fig. 4.11 to the commutator between $\hat{S}(\xi)$ and $\hat{D}(\alpha)$. We develop a phenomenological fidelity which depends on the commutator. For this process we use some requisite boundary conditions. The first two conditions derive from the definition of a fidelity. $F(\alpha) \to 1$ as $\alpha \to 0$ and $F(\alpha) \to 0$ as $\alpha \to \infty$. To understand the final constraint we recall Eq. (2.55), applied to the squeezing and displacement operators

$$\langle \{\hat{D}(\alpha), \hat{D}(\alpha)\} \{\hat{S}(\xi), \hat{S}(\xi)\} \rangle \geq |\langle [\hat{D}(\alpha), \hat{S}(\xi)] \rangle|^2.$$  \hspace{1cm} (4.36)

We remember that both $\hat{S}(\xi)$ and $\hat{D}(\alpha)$ are unitary operators. This is expressed, mathematically, as $\hat{U}^\dagger \hat{U} = \hat{U} \hat{U}^\dagger = 1$. Applying this to the left hand side (LHS) of Eq. (4.36)

$$\langle \{\hat{D}(\alpha), \hat{D}(\alpha)\} \{\hat{S}(\xi), \hat{S}(\xi)\} \rangle = \langle \hat{D}(\alpha)^\dagger \hat{D}(\alpha) + \hat{D}(\alpha) \hat{D}(\alpha)^\dagger \rangle = 2,$$  \hspace{1cm} (4.37)

$$\langle \{\hat{S}(\xi), \hat{S}(\xi)\} \rangle = \langle \hat{S}(\xi)^\dagger \hat{S}(\xi) + \hat{S}(\xi) \hat{S}(\xi)^\dagger \rangle = 2.$$  \hspace{1cm} (4.38)

This leads to an interesting result

$$|\langle [\hat{D}(\alpha), \hat{S}(\xi)] \rangle|^2 \leq 4.$$  \hspace{1cm} (4.39)

We notice that here we have defined a constraint for $[\hat{D}(\alpha), \hat{S}(\xi)]$, whereas we wish to...
measure \([\hat{S}(\xi), \hat{D}(\alpha)]\). We can write the Hermitian adjoint of the displacement operator as

\[\hat{D}^\dagger(\alpha) = \hat{D}(-\alpha) = -\hat{D}(\alpha).\]  \hspace{1cm} (4.40)

This then gives

\[\comm{\hat{D}^\dagger(\alpha)}{\hat{S}(\xi)} = -\comm{\hat{D}(\alpha)}{\hat{S}(\xi)} = \comm{\hat{S}(\xi)}{\hat{D}(\alpha)}.\]  \hspace{1cm} (4.41)

When applied to Eq. (4.39), we arrive at

\[|\langle [\hat{S}(\xi), \hat{D}(\alpha)] \rangle|^2 \leq 4.\]  \hspace{1cm} (4.42)

This states that the commutator has an upper limit, providing a third constraint on our phenomenological model. Combining these three boundary conditions we arrive at a, purely phenomenological, fidelity equation \(F_p\)

\[F_p(\alpha) = \left(1 - \frac{|\langle [\hat{S}(\xi), \hat{D}(\alpha)] \rangle|^2}{4}\right)^\eta,\]  \hspace{1cm} (4.43)

where we have included \(\eta\) as a fitting parameter. We replace \(\langle [\hat{S}(\xi), \hat{D}(\alpha)] \rangle\) with the Fock decomposition, as defined in Eq. (4.10), and investigate \(F_p(\alpha)\) for different orders of the Taylor expansion in Eq. (4.6) and Eq. (4.7).

We note that Eq. (4.10) does not represent a true expectation value and, therefore, should not necessarily abide by the limit given by Eq. (4.42). We defend this substitution by noting the physical significance of this decomposition, illustrated in Eq. (4.14). This makes it likely that Eq. (4.10) also has an upper limit. Throughout this investigation we also attempted to calculate more rigorous expectation values, shown explicitly in Appendix A.1. However, we see improved results when using Eq. (4.10). Due to the need for Taylor expansions the approximations should only be valid in the limit of small squeezing and displacement amplitudes.

We use a squeezing of \(S_{\text{dB}} \approx 4\, \text{dB}\) throughout, corresponding to a squeezing amplitude of \(r \approx 0.5\). We investigate a displacement range of \(|\alpha| = 0.025\) to 0.26. Figure 4.13 illustrates Eq. (4.43) fitted to the experimentally measured fidelity. These results use the commutator as calculated with the first and second order Taylor expansions of \(\hat{S}(\xi)\) and \(\hat{D}(\alpha)\).

We extracted fitting parameters of \(\eta_{135^\circ} = 7.835\) and \(\eta_{45^\circ} = 2.961\) for the first order and \(\eta_{135^\circ} = 2.568\) and \(\eta_{45^\circ} = 2.261\) for the second order, where the subscripts refer to the displacement angle. We notice that the fitting parameters are different for the different displacement angles, this results from the situation illustrated in Fig. 4.12. We see a good agreement, for all plots, between the data and the model, although the fit seems to be better for the first order. We see a similar behaviour for the third and fourth order plots (shown explicitly in Appendix A). Table 4.3 shows the fitting parameter, \(\eta\), extracted for various orders of the Taylor expansion. Due to the exponential increase in complexity with increasing orders, we only investigate up to the fourth order. We see for all orders a
Figure 4.13: Results of the phenomenological model, as given in Eq. (4.43), for small displacement amplitudes and using the sum over Fock states (Eq. (4.10)). The blue circles represent the fit and crosses represent the experimental data. a) and b) Plots show results for first order Taylor expansions of $\hat{S}(\xi)$ and $\hat{D}(\alpha)$. c) and d) Plots for second order Taylor expansions. a) Displacement angle $\theta = 135^\circ$. Fitting parameter $\eta = 7.835$. b) $\theta = 45^\circ$ and $\eta = 2.961$. c) $\theta = 135^\circ$ and $\eta = 2.5679$. c) $\theta = 45^\circ$ and $\eta = 2.2641$.

<table>
<thead>
<tr>
<th>Order of expansion</th>
<th>$\theta = 135^\circ$</th>
<th>$\theta = 45^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>7.8353</td>
<td>2.9613</td>
</tr>
<tr>
<td>2nd</td>
<td>2.5679</td>
<td>2.2641</td>
</tr>
<tr>
<td>3rd</td>
<td>1.5759</td>
<td>1.1793</td>
</tr>
<tr>
<td>4th</td>
<td>10.1067</td>
<td>22.4053</td>
</tr>
</tbody>
</table>

Table 4.3: Fitting parameter, $\eta$, for various Taylor expansion orders. Fitting parameters are shown for both angles $\theta = 135^\circ$ and $\theta = 45^\circ$ and for small displacement amplitude range, $|\alpha| = 0.025 - 0.26$. 

discrepancy between the fitting parameters for different displacement angles. Although, for the second and third orders the fitting parameters for both displacement angles are similar, the fourth order fit parameters exhibit a large dissonance.
Chapter 4  Commutator relationship between squeezing and displacement operators

Figure 4.14: Fitting of Eq. (4.43) to experimentally measured fidelity for first order Taylor expansions and a higher range of displacement amplitudes, $|\alpha| = 0.028$ to 2.69. Blue circles represent the fitted function and the crosses represent experimental data. a) Represents a displacement angle, $\theta = 135^\circ$, with a fitting parameter $\eta = 7.1047$ b) Results for $\theta = 45^\circ$ and a fitting parameter of $\eta = 2.364$.

<table>
<thead>
<tr>
<th>Order of expansion</th>
<th>$\theta = 135^\circ$</th>
<th>$\theta = 45^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1$^{st}$</td>
<td>7.1047</td>
<td>2.3638</td>
</tr>
<tr>
<td>2$^{nd}$</td>
<td>$-0.0205 + 0.3968i$</td>
<td>$-0.0583 + 0.1732i$</td>
</tr>
<tr>
<td>3$^{rd}$</td>
<td>$-0.1207 + 0.2427i$</td>
<td>$-0.0873 + 0.1215i$</td>
</tr>
<tr>
<td>4$^{th}$</td>
<td>$-0.08 + 0.4256i$</td>
<td>$-0.0646 + 0.1787i$</td>
</tr>
</tbody>
</table>

Table 4.4: Fitting parameter, $\eta$, for various Taylor expansion orders for larger alpha values. Fitting parameters are shown for both $\theta = 135^\circ$ and $\theta = 45^\circ$. All values extracted for a sweep of larger displacement amplitudes, $|\alpha| = 0.028$ to 2.69. Complex values indicate the model failed, due to the fidelity becoming negative.

All orders fit the data well and minimal information can be extracted from the fitting parameters, meaning improvements from higher orders are not clearly visible. To extract information on the difference between the orders we now try to fit the same formula to higher displacement amplitudes. We now investigate a range of $|\alpha| = 0.028$ to 2.69. Figure 4.14 illustrates how the first order Taylor expansion fits to the data over a larger displacement range. We see again that the first order fits the data very well. The fitting parameter extracted from this fit, over the large displacement range, are $\eta_{135^\circ} = 7.1047$ and $\eta_{45^\circ} = 2.364$. We see that these values are in good agreement with the fitting parameters extracted from the fits in Fig. 4.13. In an attempt to see if any improvement can be gained from higher orders of the Taylor expansion we again compare the fitting parameters extracted from the various different orders, shown here in table 4.4. We see, as stated above, good agreement between the fitting parameters for the first order for both small and larger displacement amplitudes.
From table 4.4, we see the unexpected result that taking higher orders of the Taylor expansion into account worsens the fit. The failure of the model is evident in the extracted fitting parameters being complex. This stems from whenever the contents of the bracket in Eq. (4.43) becomes negative. In order to reach a real value for the fit, the function must employ a complex fitting parameter. The negative values come from the fact that for these orders the commutator, as described in Eq. (4.10), does not adhere to the limit of four, set by Eq. (4.42).

One reason for the failure of the fit could come from our choice of basis for expressing the commutator, Eq. (4.10). We express the commutator as a Fock state decomposition, which is not strictly an expectation value. We continued using this representation after noticing the result of Eq. (4.14), as this shows some physical significance of this description. However, this representation should not necessarily abide by the limit given by Eq. (4.42). This is also likely to be a reason why the higher orders do not add the corrections that were to be expected.

In the course of this investigation we also attempted to use more rigorous definitions for the expectation value, such in the state $|0\rangle$, $\langle 0| [\hat{S}(\xi), \hat{D}(\alpha)] |0\rangle$, and in the coherent basis, $\langle \alpha| [\hat{S}(\xi), \hat{D}(\alpha)] |\alpha\rangle$. However, when inserted into Eq. (4.43), none fitted the data as well as Eq. (4.10)\(^1\). This could be due to the simplicity of the model or that, due to the complexity of the expressions, only the lower orders were accessible to us. It could be that to acquire a good agreement between theory and experiment one needs to inspect much higher expansion orders. A problem with $\langle 0| [\hat{S}(\xi), \hat{D}(\alpha)] |0\rangle$ is that as we measure at some finite temperature we never truly have vacuum states but rather small thermal states, incident to the JPA. It would, perhaps, be more appropriate to look at the expectation value $\langle \Omega| [\hat{S}(\xi), \hat{D}(\alpha)] |\Omega\rangle$, where $|\Omega\rangle$ represents a weak thermal state. A similar reason could also explain why the coherent basis failed. The quantity $\langle \Psi| [\hat{S}(\xi), \hat{D}(\alpha)] |\Psi\rangle$ expresses the commutator expectation value as applied to the state $|\Psi\rangle$. The coherent basis subsequently fails as we do not measure the fidelity in the coherent basis.

If more consistent results are required whilst still employing the Fock decomposition, as used above, the phenomenological model would need to be revised. This revision would need to remove the constraint, added by Eq. (4.42), and find a new way to normalise Eq. (4.10). A refined model would also need to find some way to account for the differing fitting parameters for different displacement angles.

We have seen how we can generate both permutations of the squeezing and displacement operators, $\hat{D}(\alpha)\hat{S}(\xi)|0\rangle$ and $\hat{S}(\xi)\hat{D}(\alpha)|0\rangle$, with our setup. Measurement of the signal moments allows the extraction of squeezing and displacement amplitudes from the states. We have seen how a Bogoliubov transformation can be used to equate the different operator orders with an appropriate change to the displacement amplitude. We have used our setup to experimentally verify this claim, and have shown high fidelities, $F(\rho_1, \rho_2) \approx 1$, even for

\(^1\)The results found using $\langle 0| [\hat{S}(\xi), \hat{D}(\alpha)] |0\rangle$ and $\langle \alpha| [\hat{S}(\xi), \hat{D}(\alpha)] |\alpha\rangle$ can be found in Appendix A.
high displacement amplitudes. We have shown how one can construct a phenomenological model for the commutators dependence on the displacement amplitude and how this model can be investigated experimentally. Overall these results represent an important contribution in the understanding of the combined behavior of two fundamental operations.
Chapter 5

Finite-time correlations of single- and two-mode squeezed microwave states

This chapter concerns its self with the investigation of finite-time correlations for both single- and two-mode squeezed microwave states (SMSS and TMSS, respectively). These experiments will study intensity correlations of SMSS, and observe the behaviour of quantum correlations of TMSS, subjected to an asymmetric finite-time delay. The time dependence of these correlation functions allows us to define a coherence time in the continuous variables regime, i.e. how long of a delay in one path can still maintain quantum correlations. This is an important quantity for many quantum communication protocols as it defines the range of acceptable delays in respective transmission lines. First, we discuss the results for single-mode squeezed states in the form of intensity correlation measurements. Second, we will continue with the measurements of negativity (used for quantifying quantum entanglement between two spatially separate modes) versus a finite-time delay in one of the transmission lines. Finally, we discuss the obtained results in the framework of quantum communication experiments and consider possible implications.

5.1 Intensity correlations of single-mode squeezed microwave states

There exist various methods of investigating correlations in physics. We characterise single-mode intensity correlations of squeezed microwave states with the second order intensity correlation function. This represents a standard in the quantum optics community and, as such, proves itself as a useful quantity, not only in the localised case of this experiment but also in a more general sense, as it allows direct comparison to similar experiments in the optical domain. We first introduce the theoretical definitions of the second order intensity correlation, both generally and as applied to microwave states. We then present the experimental results acquired using our setup to measure the temporal dependence of these intensity correlations.
5.1.1 Second order intensity correlation function

The second order intensity correlation function in the classical picture represents fluctuations of the intensity in an electromagnetic field and gives a measure for the degree of coherence of a light field. In the quantum picture this translates to a measure of the temporal correlation in photon number and is defined as

\[ g^{(2)}(\tau) = \frac{\langle \hat{a}^\dagger(0)\hat{a}^\dagger(\tau)\hat{a}(0)\hat{a}(\tau) \rangle}{\langle \hat{a}^\dagger(0)\hat{a}(0) \rangle^2}, \]  

(5.1)

where \( \tau \) represents a time delay and the superscript 2 is used to differentiate second and first order correlation functions. This can be thought of as the probability of a photon being detected at time \( \tau \), if one was detected at time 0. This allows the definition of three regimes

\[ g^{(2)}(\tau) = g^{(2)}(0), \]  

(5.2)

\[ g^{(2)}(\tau) < g^{(2)}(0), \]  

(5.3)

\[ g^{(2)}(\tau) > g^{(2)}(0). \]  

(5.4)

Equation (5.2) describes the situation where the probability of finding a photon at time \( \tau \) is the same as at time 0, meaning the photons are detected at even intervals. This regime describes coherent light, such as light emitted from a laser. In the second situation, illustrated by Eq. (5.3), the probability of detecting a photon decreases with increasing time. This results in photons being detected in bunches and is, therefore, known as bunched light. This regime is realised by thermal and squeezed light. Unlike the first two regimes, the third finds no classical analogy. Equation (5.4) describes light where the probability of finding a photon increases with increasing time. This results in photon detections being spread out in time and is called antibunched light. This completely quantum situation describes photons emitted from a single atom. Along side Eqs. (5.2) - (5.4), information can also be gleaned from the \( g^{(2)} \) function at time 0. This number allows us to extract information concerning the photon statistics of the light under investigation. Again we can define three regimes, closely tied to those described above

\[ g^{(2)}(0) = 1, \]  

(5.5)

\[ g^{(2)}(0) > 1, \]  

(5.6)

\[ g^{(2)}(0) < 1. \]  

(5.7)

Equation (5.5) represents photon statistics with Poissonian distribution, and generally describes coherent light. The second case, Eq. (5.6), shows light with super-Poissonian photon statistics, i.e a photon number distribution with a variance larger than that of a Poissonian distribution. This describes bunched light. Equation (5.7) describes light with sub-Poissonian photon statistics, a distribution with a variance lower than that
of a Poissonian distribution, generally this is linked with the phenomenon of photon antibunching\(^1\). It can be shown \([47, 48, 82]\) that all classical fields must obey

\[
\begin{align*}
g^{(2)}(0) & \geq 1, \\
g^{(2)}(\tau) & \leq g^{(2)}(0),
\end{align*}
\]

(5.8)\quad (5.9)

This shows that both antibunching and sub-Poissonian photon statistics are nonclassical phenomena. This suggests that light with sub-Poissonian photon statistics has fluctuations in its photon number lower than is classically allowed. For this reason light fulfilling this condition is often referred to as photon number squeezed light.

Equations (5.8) and (5.9), similar to a Wigner function with negative values, provide a distinction between classical and quantum states of light. However, squeezed light proves to be an exception to the rule. As we will see, squeezed light is generally bunched and described by super-Poissonian photon statistics\(^2\), whilst, nonetheless, being a highly nonclassical state of light.

The second order correlation function can be measured using a Hanbury-Brown and Twiss style intensity interferometer \([32]\). This interferometer can be realised by splitting the desired signal using a symmetric beam splitter, adding a time delay to one path, and then measuring cross-correlations between the two paths. As can be seen from Chapter 3, our setup acts as such an interferometer with \(\tau = 0\). To probe the time dependence of the second order correlation function we digitally add a delay to one of the paths. This schematic is shown in Fig. 5.1. Beyond this we also require a version of Eq. (5.1) expressed in terms of the quadrature moments. It can be shown that the unnormalised version of this, that is, just the numerator of Eq. (5.1), with a delay only in path two can be given as

\[
G^{(2)}(\tau) = \frac{4}{g_1 g_2} \langle \hat{S}_1^\dagger(0)\hat{S}_2(\tau)\hat{S}_2(\tau)\hat{S}_1(0) \rangle - 2 g^{(1)}(0) \langle \hat{V}_2(\tau)\hat{V}_2^\dagger(\tau) \rangle - 2 g^{(1)}(\tau) \langle \hat{V}_1(0)\hat{V}_1^\dagger(0) \rangle - 4 \langle \hat{V}_1(0)\hat{V}_1^\dagger(0) \rangle \langle \hat{V}_2(\tau)\hat{V}_2^\dagger(\tau) \rangle,
\]

(5.10)

where \(g_1\) and \(g_2\) are the photon number conversion factors for each path. This equation is taken from work by Roberto di Candia \(et\ al.\) in the group of Enrique Solano in Bilbao.

We also introduce the first order correlation function which describes fluctuations in the magnitude of the electromagnetic field

\[
G^{(1)}(\tau) = \langle \hat{a}^\dagger(0)\hat{a}(\tau) \rangle = \frac{2}{\sqrt{g_1 g_2}} \langle \hat{S}_1^\dagger(0)\hat{S}_2(\tau) \rangle,
\]

(5.11)

\(^1\)Whilst the ideas of photon antibunching and sub-Poissonian photon statistics are linked at low \(\tau\), one does not presuppose the other. It is also possible to have states of light with \(g^{(2)}(\tau) > g^{(2)}(0)\) and \(g^{(2)}(0) > 1\) or vice versa \([81]\).

\(^2\)We note that, in certain situations, antibunched squeezed light can be generated \([81, 83]\).
For conciseness, we have introduced the envelope functions

\[ \hat{S}_i(t) = \hat{I}_i(t) + i\hat{Q}_i(t), \quad i \in \{1,2\}. \] (5.12)

We have also introduced operators describing the noise in each amplification path

\[ \hat{V}_i = \sqrt{\frac{1}{g_i}} \left( \sqrt{g_i - \hat{h}_i + \hat{\nu}_i} \right), \quad i \in \{1,2\}, \] (5.13)

where \( \hat{h} \) represents thermal noise added by the amplification chain and \( \hat{\nu} \) is the noise added by the IQ-mixer. To fully represent Eq. (5.1) in terms of the quadrature moments Eq. (5.10) is normalised by Eq. (5.11).

\[ g^{(2)}(\tau) = \frac{G^{(2)}(\tau)}{G^{(1)}(0)^2} \] (5.14)
5.1 Intensity correlations of single-mode squeezed microwave states

5.1.2 Time dependent second order correlation function

We now move the discussion towards the experimental results gained in the investigation of the second order correlation function, for the single-mode squeezed vacuum. Figure 5.1 illustrates how our setup can be used as an intensity interferometer. We include a digital delay into one of the paths. This is implemented in the data processing code. After the signal has been amplified, and downconverted, it is digitised by an analog to digital converter (ADC). This hands two large arrays to the computer for processing. We digitally shift one array with respect to the other. We use a sampling frequency of 400 MHz, which corresponds to a step size of 2.5 ns. Adding the delay digitally, thus, allows a smallest time delay of 2.5 ns. Once the delay has been incorporated the two ADC arrays are then used to calculate the moments $\langle I_1^2 I_2^2(\tau)Q_1^2Q_2^2(\tau)\rangle$. Inserting these measured quadrature moments into Eq. (5.14) allows $g^{(2)}(\tau)$ to be calculated. We measure $g^{(2)}$ for a variety of time delays and a number of different squeezing levels. The following experiments use a JPA frequency of $f_0 = 5.4$ GHz, corresponding to a coil current of 66 $\mu$A. The JPA temperature is stabilised to 50 mK. We use the four part pulse scheme, illustrated in Fig. 3.10, with pulse sections of 40 $\mu$s. In the following experiments we only pulse the JPA pump line as no displacement is required. The pulse frequency is 5 kHz. We use record traces of 64000 samples and employ averaging of 50 sweeps, 100 segments and 140 cycles, corresponding to $4.48 \times 10^{10}$ raw samples per data point.

![Intensity correlation function, $g^{(2)}$, as a function of the time delay, $\tau$, for various squeezing levels. Statistical error bars are on the scale of the marker symbols. Measurement data indicates single-mode squeezed states are bunched and described by super-Poissonian photon statistics](image)

We see from Fig. 5.2 that for all squeezing levels $g^{(2)}(0)$ is above 1, indicating that squeezed vacuum states are described by super-Poissonian photon statistics. We also see that $g^{(2)}(\tau)$ decreases for increasing $\tau$. This illustrates that the microwave squeezed vacuum states we generate are bunched. An interesting observation from Fig. 5.2 is that the correlation function decreases with increased squeezing. As a consistency check, we
note that the second order correlation function with no time delay can be defined as \[ g^{(2)}(0) = 1 + \frac{\text{Var}(\hat{n}) - \langle \hat{n} \rangle}{\langle \hat{n} \rangle^2}, \] (5.15)

where \( \text{Var}(\hat{n}) \) is the photon number variance. For a squeezed vacuum \( \text{Var}(\hat{n}) = \langle \hat{n} \rangle (1 - \cosh(2r)) \) and \( \langle \hat{n} \rangle = \sinh^2 r \). Inserting these into Eq. (5.15) and using the identity \( \cosh(2r) = 2 \sinh^2 r + 1 \), leaves us with

\[ g^{(2)}(0) = 3 + \frac{1}{\langle \hat{n} \rangle}. \] (5.16)

We can now compare \( g^{(2)}(0) \) calculated from Eq. (5.16), with the \( g^{(2)}(0) \) measured using the quadrature moments and Eq. (5.10), for \( \tau = 0 \). Figure 5.3 illustrates the results of this check. We see excellent agreement between the two methods allowing us to be confident that our method of extracting \( g^{(2)}(\tau) \) is correct. Equation (5.16) also provides an explanation of why Fig. 5.2 shows lower \( g^{(2)}(\tau) \) values for higher squeezing levels. As the correlation function, \( g^2 \) depends inversely on the photon number, the increased photon number associated with higher squeezing levels results in a lower \( g^2 \) function.

As alluded to earlier, \( g^{(2)}(\tau) \) allows us to determine a coherence time, \( \tau_c \), for continuous variables protocols. We define this as the time it takes for \( g^{(2)}(\tau) \) to reach \( 1/e \) of its initial value. This represents the time delay that can be added to one of the paths whilst still maintaining super-Poissonian behaviour. Table 5.1 shows the coherence times for different squeezing levels. We see that, although \( g^{(2)}(\tau) \) is lower for higher squeezing levels, increasing squeezing level leads to longer coherence times. Although higher squeezing results in longer coherence times, higher squeezing levels also result in states that are more susceptible to phase noise. Quantum communication protocols need to balance these
Table 5.1: Coherence times, extracted from Fig. 5.2, for a range of squeezing levels.

<table>
<thead>
<tr>
<th>$S_{dB}$</th>
<th>$\tau_c$ (µs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.942 dB</td>
<td>1.682</td>
</tr>
<tr>
<td>7.692 dB</td>
<td>1.838</td>
</tr>
<tr>
<td>8.345 dB</td>
<td>1.878</td>
</tr>
<tr>
<td>9.131 dB</td>
<td>1.935</td>
</tr>
</tbody>
</table>

two facts when choosing the desired squeezing level.

We now turn to comparing our results to the theoretical results derived by Urtzi Las Heras et al. at the QUTIS group in Bilbao. These results show that the unnormalised $g^{(2)}$ function, $G^{(2)}(\tau)$, can be described by

$$G^{(2)}(\tau) = 1 + \text{sinc}^2(\omega\tau) \frac{1 + 2\sigma_p^2(\sigma_p^2 - 1) + 2\sigma_q^2(\sigma_q^2 - 1)}{(1 - \sigma_p^2 - \sigma_q^2)^2},$$  \hspace{1cm} (5.17)$$

where $\omega$ being the bandwidth of the digital filter and $\sigma_i$ represents the variance of a quadrature, with the subscript denoting different quadratures

$$\sigma_p^2 = \frac{1}{(2\chi - \kappa - \gamma)^2}[(2\chi + \kappa - \gamma)^2(n_{b_{in}} + 1/2) + 4\kappa\gamma(n_{c_{in}} + 1/2)],$$  \hspace{1cm} (5.18)$$

$$\sigma_q^2 = \frac{1}{(2\chi + \kappa + \gamma)^2}[(2\chi - \kappa + \gamma)^2(n_{b_{in}} + 1/2) + 4\kappa\gamma(n_{c_{in}} + 1/2)],$$  \hspace{1cm} (5.19)$$

where $\chi$ is the pump coupling rate, $\gamma$ is the internal loss rate, $\kappa$ is the external loss rate, $n_{b_{in}}$ and $n_{c_{in}}$ represent the number of externally and internally added thermal photons, respectively. It is interesting to note that the shape of the correlation function is defined by the shape of the FIR filter function used. In our case we use a digital FIR filter, which can be well approximated by a square filter. This results in a $\text{sinc}^2$ like shape of the correlation function, $g^{(2)}(\tau)$. We see from Fig. 5.4 a good agreement between experimental data and theoretical fit. The fit also allows the extraction of parameters such as the filter bandwidth, the pump rate, $n_{b_{in}}$, and $n_{c_{in}}$. The extraction of the measurement bandwidth also gives another means to test the validity of Eq. (5.17). From the fit, the filter bandwidth is calculated to be $\simeq 420$ kHz. This is in good agreement with our designed FIR bandwidth of 400 kHz. The theory also provides an expression for calculating the squeezing amplitude, $r$

$$r = \ln \left( \frac{2\chi + \kappa}{2\chi - \kappa} \right).$$  \hspace{1cm} (5.20)$$

We note that this equation is only valid in the case $\kappa \gg \gamma$, i.e. the external losses are much higher than the internal losses. The fit allows the extraction of $\chi$. The external losses can be calculated by $\kappa = f_0/Q_{\text{ext}}$, where the external quality factor can be determined from the flux dependence of the JPA resonant frequency, as described in Sec. 3.4.1. The calculation
of $r$ from the above theory ($r_{\text{fit}}$) subsequently allows another means of comparison for the semi-phenomenological model used to calculate $r$ in Sec. 4.3.3 ($r_{\text{phenom}}$).

<table>
<thead>
<tr>
<th>$S_{\text{dB}}$</th>
<th>$r_{\text{phenom}}$</th>
<th>$r_{\text{fit}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.942 dB</td>
<td>0.9014</td>
<td>0.7558</td>
</tr>
<tr>
<td>7.692 dB</td>
<td>1.0186</td>
<td>0.8710</td>
</tr>
<tr>
<td>8.345 dB</td>
<td>1.1438</td>
<td>1.1543</td>
</tr>
<tr>
<td>9.131 dB</td>
<td>1.3053</td>
<td>1.3803</td>
</tr>
</tbody>
</table>

**Table 5.2:** Comparison of squeezing amplitude, $r$, as calculated from the model in Sec. 4.3.3 ($r_{\text{phenom}}$) to $r$ as calculated from Eq. (5.20) ($r_{\text{fit}}$).

Overall we see from table 5.2 a reasonable agreement between the two methods for calculating the squeezing amplitude. Due to the agreement between theory and experiment we saw in Fig. 4.6, it is likely that the discrepancies, evident from table 5.2, stem from the fit. Equation (5.20) shows that the calculation of $r$ presupposes precise knowledge of the pumping coupling rate and both internal and external quality factors. The quality factors are determined from the VNA $S_{12}$ measurements, as described in Sec. 3.4.1. We find these
5.2 Time dependence of negativity for two-mode squeezed microwave states

We examined the negativity for two-mode squeezed vacuum states. The quality factors were determined to be $Q_{\text{ext}} = 280$ and $Q_{\text{int}} = 3 \times 10^4$. We observed that the external quality factor is similar to the designed value of 200. These values are determined precisely and are unlikely to contribute largely to the discrepancies seen in Table 5.2. As the squeezing coupling rate, $\chi$, is extracting from the fit, it is susceptible to fluctuations in the measurement data. Another drawback to extracting quantities via fitting the data is that the values depend on the amount of fitting parameters. The fit in our case involves four fitting parameters, $n_{b_{\text{in}}}$, $n_{\text{c}_{\text{in}}}$, $\chi$, and $\omega$. This all suggests that the value of $\chi$ is likely to be the reason for the discrepancies in Table 5.2. This is not to say that the origin of the discrepancy lies in the pumping rate, as this is controlled by an external source with precise power and phase control. The problem lies in our extraction of the value from the fit.

5.2 Time dependence of negativity for two-mode squeezed microwave states

Two-mode squeezed vacuum states are fundamental in quantum communication protocols. Accordingly, we now investigate the temporal characteristics of the correlations between modes of a two-mode squeezed vacuum. We initially discuss how one can quantify quantum entanglement between two spatially separated modes. Second, we characterise our generated two-mode squeezing, before moving on to how one can use the negativity to define the temporal robustness of the established quantum correlations. Figure 5.5 illustrates the scheme used in the generation of two-mode squeezing and the measurements of the temporal response of entanglement between the two modes. Finer details of the cryogenic and room temperature setup are shown in Fig. 3.3. We display results for two FIR filter bandwidths, 400 kHz and 800 kHz. For the 400 kHz filter bandwidth, results the JPAs are stabilised at 50 mK. For the 800 kHz results, the JPAs are stabilised at 40 mK. All results in this section employ a JPA frequency of $f_0 = 5.323$ GHz. This corresponds to a coil current of 66 $\mu$A for JPA 1 and 71.8 $\mu$A for JPA 2. We use the four part pulse scheme, illustrated in Fig. 3.10, with pulse sections of 40 $\mu$s. In the following experiments we pulse the pump lines of JPA 1 and JPA 2. The pulse frequency is 5 kHz. For the 400 kHz measurements, we record traces of 64000 samples and employ averaging of 40 sweeps, 100 segments and 150 cycles, corresponding to $3.84 \times 10^{10}$ raw samples per data point. For the 800 kHz measurements, we record traces of 64000 samples, and use 50 sweeps, 90 segments and 210 cycles, corresponding to $6.048 \times 10^{10}$ raw samples per data point. For the lowest two squeezing levels, measured with an 800 kHz filter bandwidth, we reduce the sweep number to 40, resulting in $4.84 \times 10^{10}$ raw samples per data point.

5.2.1 Negativity as an entanglement witness

The intensity interferometer, shown in Fig. 5.1, is used to study finite-time correlations of a single mode. This cannot be directly applied for studies of correlations between
two spatial modes, requiring the use of a different means of quantifying these quantum correlations (e.g. quantum entanglement). There exist many metrics to quantify the amount of entanglement between two modes. We choose the negativity criterion as our entanglement witness, as first introduced in Ref. [84]. The negativity for a composite system, consisting of subsystems 1 and 2, is defined as [27, 58, 59, 85]

\[
N(\rho) = \sqrt{\text{tr}(\rho^{T_1} \rho) - \text{tr}(\rho)^2},
\]

where \(\rho\) is the density matrix of the composite system, \(\rho^{T_1}\) represents the partial transpose of \(\rho\) with respect to subsystem 1 and \(\|\rho\|_1 = \text{Tr}|\rho|\). This is equivalent to \(\sum_i |\lambda_i|\), or the sum of negative eigenvalues of \(\rho^{T_1}\). Entanglement between two modes is closely linked to the mathematical idea of separability. A composite wavefunction is said to be separable if it can be expressed as the tensor product of two separate wavefunctions

\[
|\Psi\rangle_{AB} = |\Psi\rangle_A \otimes |\Psi\rangle_B.
\]

If a composite wavefunction does not fulfill this condition then the subsystems must be, in some way, correlated, these composite states are referred to as entangled. The negativity relies on the Peres-Horodecki (PH) criterion for separability of a composite quantum system [86]. However, unlike the PH criterion, the negativity is only well suited
5.2 Time dependence of negativity for two-mode squeezed microwave states

...to Gaussian states. The PH criterion states that for a quantum state to be separable all eigenvalues of $\rho^{T_1}$ must be positive. Conversely, any system where $\rho^{T_1}$ has at least one negative eigenvalue must be nonseparable and therefore entangled. This shows that a positive value for $N(\rho)$ indicates an entangled system, with increasing $N(\rho)$ corresponding to increasing entanglement. It can be shown that the moments of the partial transpose can be written as

$$M^{(2)} = \begin{pmatrix} 1 & \langle \hat{a}_1 \rangle & \langle \hat{a}_2 \rangle \\ \langle \hat{a}_1 \rangle & \langle \hat{a}_1^\dagger \hat{a}_1 \rangle & \langle \hat{a}_2^\dagger \hat{a}_1 \rangle \\ \langle \hat{a}_2 \rangle & \langle \hat{a}_1 \hat{a}_2 \rangle & \langle \hat{a}_2^\dagger \hat{a}_2 \rangle \end{pmatrix},$$

(5.23)

where subscript 1 and 2 denote the two modes making up the composite system. In our context this refers to the two outputs of the hybrid ring. The negativity can also be written in terms of the two-mode covariance matrix

$$\sigma = \begin{pmatrix} \alpha & \gamma \\ \gamma^T & \beta \end{pmatrix},$$

(5.24)

where $\alpha$, $\beta$ and $\gamma$ are $2 \times 2$ matrices. Explicit expressions for these can be found in Appendix B. This allows the negativity to be reformulated as

$$N = \max\{0, 1 - \frac{\nu}{2} \} = \max\{0, \tilde{N} \},$$

(5.25)

where $\tilde{N}$ is the negativity kernel, $\nu = \sqrt{(\Delta(\sigma) - \Delta^2(\sigma) - 4\det\sigma)/2}$ and $\Delta(\sigma) = \det\alpha + \det\beta + 2\det\gamma$. We see from Eq. (5.25), that the negativity is always positive, with a state only being separable when $N = 0$. Experimentally, 0 is a difficult number to measure due to experimental fluctuations. To combat this, in the later results, we plot the negativity kernel as a function of a temporal delay. This allows us to define a separable state as one where the upper error bound of the negativity kernel is below 0. A deeper theoretical description of the negativity can be found in Refs. [58, 85, 87].

5.2.2 Characterisation of two-mode squeezing

Before investigating how a time delay in one of the spatial modes affects the measured entanglement, we first discuss the generation of two-mode microwave squeezing. As mentioned in Sec. 2.2.2, we generate squeezing on two correlated spatially distinct modes. This is achieved by overlapping two orthogonal single-mode squeezed vacuum states on a symmetric beam splitter [16]. The beam splitter acts as an entangler and, therefore, establishes the necessary correlations between the two modes. We quantify the level of
entanglement via the negativity kernel, as described above. As the Wigner function of a two-mode squeezed states are four-dimensional, we plot the marginal distributions, i.e. the Wigner function integrated over two variables. Examining these marginal distributions allows one to gain a qualitative view of two-mode squeezing. In the following discussion we will refer to the Wigner functions of the individual modes as the self-correlated subspaces and the marginal distributions of the two-mode squeezed state as the cross-correlated subspaces. This scenario is illustrated in Fig. 5.6. We see that, as expected, each path

![Hybrid ring](image)

**Figure 5.6:** Marginal distributions of the Wigner function characterising experimental two-mode squeezing results. Two orthogonally squeezed vacuum states impinge on the hybrid ring. Squeezing levels of 7.968 dB for JPA 1 and 8.089 dB for JPA 2 and a JPA frequency of $f_0 = 5.323 \text{ GHz}$ are used. Tomography of the output states shows thermal Wigner functions in the self-correlated subspaces (green borders) and two-mode squeezing in the cross-correlated subspaces (red borders). Phases for the input states, $\beta_{\text{JPA1}} = 45^\circ$ and $\beta_{\text{JPA2}} = 135^\circ$, result in squeezing in the $p_1 - q_2$ and $q_1 - p_2$ correlations.

considered individually results in the Wigner function of a thermal state. However when considering the cross-correlated subspaces we see the reduction in noise, characteristic of a two-mode squeezed state. From Fig. 5.6, we see there are four possible cross-correlated subspaces. Which of these exhibit squeezing is determined by the choice of phases for the input squeezed vacua. For our case $\beta_{\text{JPA1}} = 45^\circ$ and $\beta_{\text{JPA2}} = 135^\circ$, where the subscripts JPA1 and JPA2 refer to JPA 1 and 2, respectively. In this configuration the squeezing is seen on cross-correlated subspaces of $p_1 - q_2$ and $q_1 - p_2$. If we were to use values of $\beta_{\text{JPA1}} = 90^\circ$ and $\beta_{\text{JPA2}} = 0^\circ$, for example, the squeezing would be seen in the correlations between $p_1 - p_2$ and $q_1 - q_2$, as is simulated in Fig. 2.12. We see from this that rotating the phase of both input states, while maintaining the $90^\circ$ phase shift between them has the effect of transferring the two-mode squeezing between different quadrature correlations.

Another interesting situation is where the squeezing angles of both input states are parallel. Figure 5.7 illustrates the experimental Wigner functions for input squeezed
states with parallel squeezing angles. We now see the situation where each channel, when considered individually, exhibits squeezing. However, the cross-correlated subspaces display thermal state Wigner functions. Figure 5.7, combined with Fig. 5.6, shows that changing the relative phase between the input squeezed states has the effect of transferring squeezing between the individual paths and the cross-correlations [88].

5.2.3 Negativity versus temporal delay

After confirming above that we are generating two mode squeezed states, we can now investigate how the entanglement between the spatially separated modes is affected by including a temporal delay into one of the paths. As with the second order correlation function measurements, the time delay, \( \tau \), is included digitally, before the calculation of moments and subsequent averaging takes place. For this method we use the reference-state reconstruction method [78, 79], as described in Sec. 3.2.2, to extract the signal moments. We investigate the temporal response of the measured negativity for a range of squeezing levels for the input states. Figure 5.8 plots negativity versus \( \tau \) for a range of squeezing levels and for two difference FIR filter bandwidths, 400 kHz and 800 kHz. We notice that, for zero time delay, higher input squeezing leads to higher negativity, and, therefore, higher entanglement. This fits well with a result outlined in Ref. [89]. The authors argue that there exists a conservation law between the single-mode nonclassicality, at the inputs to the hybrid ring, and the entanglement between the output modes. We also see the interesting behaviour that for higher squeezing level the negativity decays faster than for
Another interesting observation is that an increase in squeezing level is accompanied by a change in the shape of the negativity plot. This can be clearly seen by comparing the \( \tau \) dependence of the negativity for the highest and lowest squeezing levels, shown in Fig. 5.8. We illustrate these two plots side by side below. Figure 5.9 compares the temporal dependence of the negativity for high and low squeezing levels. We see negative values for the negativity kernels in Fig. 5.9. Due to statistical fluctuations, we class a state as entirely separable when its upper error bar is below zero. We see an immediate difference in the shapes of the plots. For higher squeezing we see a clear exponential decay of the negativity, whereas for a lower squeezing level the plot shows a sinc like dependence. Whilst the temporal shape of the single mode correlations, in the form of the \( g^{(2)} \) function, are entirely dictated by the shape of the digital filter function used, it seems that for the two-mode correlations this is not the case. The exact squeezing level for this transition is yet to be determined. As yet, there exists no theoretical reasoning for either the change in the shape of the negativity plot or the decrease in robustness of the correlations for higher squeezing levels. However, one possible theory is that the natural bandwidth of the JPA is heavily influenced by the pump power. It is likely that this bandwidth has an effect on the temporal shape of the negativity and thus a change in this bandwidth could be responsible for the change seen in Fig. 5.9. A second reason could be due to the fact that at higher squeezing levels phase noise becomes more relevant, it is conceivable that this then results in a change of shape for higher squeezing levels.
5.2 Time dependence of negativity for two-mode squeezed microwave states

Figure 5.9: Dependence of negativity on temporal delay, $\tau$, for low and high squeezing levels.  
(a) $S_{\text{dB}} = 7.22$ dB and FIR bandwidth of 400 kHz  
(b) $S_{\text{dB}} = 2.4$ dB and FIR bandwidth of 400 kHz  
(c) $S_{\text{dB}} = 8.0$ dB and FIR bandwidth of 800 kHz  
(d) $S_{\text{dB}} = 3.2$ dB and FIR bandwidth of 800 kHz.  
a)/c) show an exponential dependence of the negativity on the time delay whereas b)/d) shows a sinc-like shaped plot.

In this chapter, we have seen how our dual-path receiver can act as an intensity interferometer. With the addition of a digital time delay, this allows us to measure the time response of the second order correlation function. These results can then be used to determine that the squeezed vacua generated by a JPA are bunched and are described by super-Poissonian photon statistics. We have seen that higher squeezing levels lead to lower initial intensity correlations, but that these correlations are more robust than for lower squeezing levels. Theory from the QUTIS group shows that the temporal behaviour of the second order correlation function is described by the shape of the FIR filter used.

We have also shown how our setup can be used to generate two-mode squeezed microwave states and how one can use the negativity criterion to describe the quantum entanglement between two spatially separate modes. We have shown experimentally, how the entanglement is affected by a temporal delay in one mode and that higher squeezing
levels result in initially higher entanglement which decays faster than for lower squeezing levels. We have demonstrated that a change in the temporal shape takes place between high and low squeezing levels.
Chapter 6

Conclusions and outlook

Throughout this work we have employed flux-driven JPAs for the generation of squeezed microwave states. The JPA consists of a $\lambda/4$ coplanar microwave resonator, terminated by a dc-SQUID. Modulation of the external flux applied to the dc-SQUID at twice the JPA resonant frequency results in parametric amplification. Operated in its degenerate mode, i.e. with the frequencies of the generated signal and idler photons being equal, leads to the generation of single-mode squeezed states. Squeezed coherent states were generated either by squeezing a coherent signal incident to the JPA, or by displacing squeezed vacuum states with a cryogenic coupler. We saw stable squeezing for displacement powers of 200 displacement photons.

The ability of our setup to generate both permutations of $\hat{S}(\xi)$ and $\hat{D}(\alpha)$ permitted the investigation of the commutation relation between these operators. Two approaches were used in tackling this investigation. First, we noticed that the application of a Bogoliubov transformation allows the separate operator order to be equated via a correction to the displacement amplitude. We characterised this method across the entire phase space and saw little degradation of the fidelity over a range of displacement powers and phases. The finite decrease in fidelity is likely down to phase fluctuations and imperfect calibration of the displacement power and phase offsets, which have more impact at higher displacement powers. Secondly, we introduced a phenomenological model describing how fidelity between the two states depends on the displacement amplitude. We extracted fitting parameters for our model and saw they differed for different displacement phases. We saw the strange behaviour, that for higher orders of the Taylor expansions of $\hat{S}(\xi)$ and $\hat{D}(\alpha)$, the model failed. This is likely due to our choice of basis for calculating the commutator.

The bespoke state reconstruction method used throughout this work involves the splitting of the signal under investigation, and the measurement of quadrature cross-correlations. This allowed our setup to function as a Hanbury-Brown and Twiss intensity interferometer, providing a means to investigate the second order correlation function, $g^{(2)}(\tau)$, of squeezed states. This correlation function, which describes correlations in photon number, acts as a standard in the field of quantum optics, and therefore provides a metric to compare our setup to related experiments. Beyond this, it also allows the definition of an effective coherence time for protocols using continuous variables.
We measured the time dependence of $g^{(2)}(\tau)$ for a range of squeezing levels. We saw that $g^{(2)}(\tau)$ is lower for higher squeezing levels. This is due to our normalisation of $g^{(2)}(\tau)$ with the number of photons, as the squeezing level increases so does the contribution of thermal photons, resulting in a decrease in $g^{(2)}(\tau)$. We did, however, see that $g^{(2)}(\tau)$ decays slower for higher squeezing levels. This means higher squeezing levels allow larger time delays to be introduced whilst still maintaining reasonable correlations. This is an important feature for quantum communication protocols. We calculated the coherence time for a squeezing level of $S_{dB} = 9.131 \text{ dB}$ to be $\tau_c = 1.935 \mu s$. We used theoretical results developed by Urtzi Las Heras et al. at the QUTIS group in Bilbao, to fit the unnormalised $g^{(2)}(\tau)$. We noticed from the theory, that the shape of $g^{(2)}(\tau)$ is defined by the shape of the digital filter function used. The fit was then used to calculate a value for the real squeezing amplitude $r$, allowing direct comparison to $r$ values calculated for the commutator measurements. We saw some discrepancy between the two, likely due to the large number of fitting parameters necessary.

Combining two orthogonal squeezed vacua, generated from two distinct JPAs, in a symmetric microwave beam splitter creates a two-mode squeezed state. In this situation, squeezing is generated in correlations established between two spatially separated modes. We characterised the two-mode squeezing by investigating how the outputs of the hybrid ring are affected by, first, changing the global phase whilst keeping the relative 90° phase shift between them and, second, changing the relative phase between the squeezing angles. We saw that changing the global phase, but maintaining orthogonality between the states, has the effect of transferring the squeezing between different quadrature correlations. Each output of the hybrid ring, when considered individually, was a thermal state. We saw that changing the relative phase of the input states transfers squeezing between the individual paths (parallel squeezing angles), and correlated phase spaces (orthogonal squeezing angles).

As we generated two-mode squeezed states in spatially separated modes, we could no longer employ the $g^{(2)}(\tau)$ to describe the temporal correlations of our states. This is due to the fact that the Hanbury Brown-Twiss intensity interferometer necessarily involves splitting a single signal into two paths. To combat this we investigated how the nonlocal correlations established between the modes were affected by the inclusion of a temporal delay in one mode. We used the negativity kernel as an entanglement witness and measured its temporal dependence for a range of squeezing levels and for two FIR filter bandwidths. We saw that higher input squeezing levels resulted in higher initial entanglement but a faster decay. We also observed a change in the temporal shape of the negativity for high and low squeezing levels. For higher squeezing levels we saw a clear exponential dependence of the negativity on $\tau$. However, for lower squeezing levels, the negativity had a sinc like shape. This change was more clearly evident when using the wider 800 kHz FIR filter bandwidth.

Current efforts are directed towards understanding the results gained in the investigation
of the temporal dependence of the negativity kernel. So far this involves repeating the measurements for a range of different filter functions. The QUTIS group are also working on developing a theoretical description of the negativity’s dependence on a temporal delay.

We are also working to refine the phenomenological commutator model introduced in Sec. 4.4.2. This could be tackled in two ways. One method would involve finding a more appropriate basis for the commutator expectation value, e.g. a weak thermal state. Another method would be to keep the Fock decomposition but adapt the model. This would involve removing the constraint added by Eq. (4.42) and finding a more appropriate way to normalise the commutator. A new model would also need to incorporate the difference in fitting parameters for different displacement phases. It would also be instructive to repeat the measurements for different squeezing levels:

The generation of entanglement via the hybrid ring opens a range of experiments accessible to our scheme. Remote state preparation (RSP) is a process whereby quantum information can be transmitted from one location to another. This is similar to quantum teleportation protocols, however in RSP the state to be transferred is known. In the context of our experiment, this process involves making a projection measurement on one mode of a two-mode squeezed state. This result can then be fed forward and used to prepare the second mode into the required state.

This work has expanded the knowledge of the squeezing and displacement operations and further shown the suitability and aptness of propagating quantum microwave states for applications in quantum information processing and quantum communications.
Appendix A

Additional plots for phenomenological commutator model

We show the full set of plots generated in the development of the phenomenological commutator model

\[ F_p(\alpha) = \left( 1 - \frac{|\langle [\hat{S}(\xi), \hat{D}(\alpha)] \rangle|^2}{4} \right)^\eta. \]  \hspace{1cm} (A.1)

We fit Eq. (A.1) to the experimentally measured fidelity between \( \hat{S}(\xi)\hat{D}(\alpha)|0\rangle \) and \( \hat{D}(\alpha)\hat{S}(\xi)|0\rangle \), using \( \eta \) as a fitting parameter. All results displayed in this appendix use a working point of 66\( \mu \)A corresponding to a JPA frequency of \( f_0 = 5.323 \text{ GHz} \). Figs. A.1, A.2, and A.3 all show results for small displacement amplitudes \( |\alpha| = 0.025 - 0.27 \) and displacement angles of \( \theta = 135^\circ \) and \( \theta = 45^\circ \).

We first illustrate the results for the commutator expressed as a Fock decomposition, \( \langle [\hat{S}(\xi), \hat{D}(\alpha)] \rangle = \sum_{n=0}^{\infty} \langle n | [\hat{S}(\xi), \hat{D}(\alpha)] |0\rangle \). We show plots for the first four orders of the Taylor expansions of \( \hat{S}(\xi) \) and \( \hat{D}(\alpha) \). These results are shown in Fig. A.1.

We next illustrate results where the commutator expectation value is calculated in the coherent basis, \( \langle [\hat{S}(\xi), \hat{D}(\alpha)] \rangle = \langle \alpha | [\hat{S}(\xi), \hat{D}(\alpha)] |\alpha\rangle \). We show results for the first three orders of the Taylor expansions. These results are shown in Fig. A.2.

The third method for calculating the commutator expectation value is in the vacuum basis, \( \langle [\hat{S}(\xi), \hat{D}(\alpha)] \rangle = \langle 0 | [\hat{S}(\xi), \hat{D}(\alpha)] |0\rangle \). We show results for the first four orders of the Taylor expansions. We have omitted the first order as to the first order expansion the commutator is 0. Figure A.3 illustrates these results.

We also examined the model for higher displacement amplitudes, \( |\alpha| = 0.025 - 2.7 \). Again we show results for both displacement angles, \( \theta = 135^\circ \) and \( \theta = 45^\circ \). Figure A.4 shows the Fock decomposition method for higher displacement amplitudes. Figure A.5 shows the results with the commutator expectation value calculated in the coherent basis. Finally, Fig. A.6 illustrates the results with the commutator expectation value calculated in the vacuum basis.

We see that the Fock decomposition provides the best fits to the data. We also notice that none of the bases show improvements for higher expansion orders.
Figure A.1: Fidelity plots for commutator calculated via Fock decomposition. Fitting parameter, $\eta$, displayed for each plot. Left column shows results for $\theta = 135^\circ$ and right column for $\theta = 45^\circ$. a) and b) show results for first order of expansion. c) and d) show results for second order of expansion. e) and f) show results for third order of expansion. g) and h) show results for fourth order of expansion.
Figure A.2: Fidelity plots for commutator expectation value calculated in coherent basis. Fitting parameter, $\eta$, displayed for each plot. Left column shows results for $\theta = 135^\circ$ and right column for $\theta = 45^\circ$. a) and b) show results for first order of expansion. c) and d) show results for second order of expansion. e) and f) show results for third order of expansion.
Figure A.3: Fidelity plots for commutator expectation value calculated in the vacuum state. Fitting parameter, $\eta$, displayed for each plot. Left column shows results for $\theta = 135^\circ$ and right column for $\theta = 45^\circ$. a) and b) show results for second order of expansion. c) and d) show results for third order of expansion. e) and f) show results for fourth order of expansion.
Figure A.4: Fidelity plots for commutator calculated via Fock decomposition. Plots shown for displacement amplitude range of $|\alpha| = 0.025 - 2.7$. Fitting parameter, $\eta$, displayed for each plot. Left column shows results for $\theta = 135^\circ$ and right column for $\theta = 45^\circ$. a) and b) show results for first order of expansion. c) and d) show results for second order of expansion. e) and f) show results for third order of expansion. g) and h) show results for fourth order of expansion.
Figure A.5: Fidelity plots for commutator expectation value calculated in coherent basis. Plots shown for displacement range of $|\alpha| = 0.025 - 2.7$. Fitting parameter, $\eta$, displayed for each plot. Left column shows results for $\theta = 135^\circ$ and right column for $\theta = 45^\circ$. a) and b) show results for first order of expansion. $\eta$ is undefined as the fitting program was unable to complete in the given parameter range. c) and d) show results for second order of expansion. c) and d) show results for third order of expansion.
Figure A.6: Fidelity plots for commutator expectation value calculated in the vacuum state. Plots shown for displacement range of $|\alpha| = 0.025 - 2.7$. Fitting parameter, $\eta$, displayed for each plot. Left column shows results for $\theta = 135^\circ$ and right column for $\theta = 45^\circ$. a) and b) show results for second order of expansion. c) and d) show results for third order of expansion. e) and f) show results for fourth order of expansion.
Appendix B

Negativity

We use this space to clarify the two mode covariance matrix shown in Eq. (B.1)

\[ \sigma \equiv \begin{pmatrix} \alpha & \gamma \\ \gamma^T & \beta \end{pmatrix}. \quad (B.1) \]

Here we have introduced the 2×2 matrices \( \alpha, \beta \) and \( \gamma \), defined as

\[ \alpha \equiv \begin{pmatrix} \alpha_1 & \alpha_3 \\ \alpha_3 & \alpha_2 \end{pmatrix}, \quad \beta \equiv \begin{pmatrix} \beta_1 & \beta_3 \\ \beta_3 & \beta_2 \end{pmatrix}, \quad \gamma \equiv \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}. \quad (B.2) \]

We can define the components in terms of the creation and annihilation operators of the two modes \( \hat{a}_{1,2} \) and \( \hat{a}_{1,2}^\dagger \)

\[ \alpha_1 = \langle \hat{a}_{1}^2 \rangle + \langle (\hat{a}_{1}^\dagger)^2 \rangle + 2\langle \hat{a}_{1}^\dagger\hat{a}_{1} \rangle - \langle \hat{a}_{1} + \hat{a}_{1}^\dagger \rangle^2 + 1, \quad (B.3) \]

\[ \alpha_2 = -\langle \hat{a}_{1}^2 \rangle - \langle (\hat{a}_{1}^\dagger)^2 \rangle + 2\langle \hat{a}_{1}^\dagger\hat{a}_{1} \rangle + \langle \hat{a}_{1} - \hat{a}_{1}^\dagger \rangle^2 + 1, \quad (B.4) \]

\[ \alpha_3 = i\left( -\langle \hat{a}_{1}^2 \rangle + \langle (\hat{a}_{1}^\dagger)^2 \rangle + \langle \hat{a}_{1} \rangle^2 - \langle \hat{a}_{1}^\dagger \rangle^2 \right), \quad (B.5) \]

\[ \beta_1 = \langle \hat{a}_{2}^2 \rangle + \langle (\hat{a}_{2}^\dagger)^2 \rangle + 2\langle \hat{a}_{2}\hat{a}_{2} \rangle - \langle \hat{a}_{2} + \hat{a}_{2}^\dagger \rangle^2 + 1, \quad (B.6) \]

\[ \beta_2 = -\langle \hat{a}_{2}^2 \rangle - \langle (\hat{a}_{2}^\dagger)^2 \rangle + 2\langle \hat{a}_{2}\hat{a}_{2} \rangle + \langle \hat{a}_{2} - \hat{a}_{2}^\dagger \rangle^2 + 1, \quad (B.7) \]

\[ \beta_3 = i\left( -\langle \hat{a}_{2}^2 \rangle + \langle (\hat{a}_{2}^\dagger)^2 \rangle + \langle \hat{a}_{2} \rangle^2 - \langle \hat{a}_{2}^\dagger \rangle^2 \right), \quad (B.8) \]
\[ \gamma_{11} = \frac{1}{2} \left( \langle \hat{a}_1 \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger + \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle / 2 ight) 
+ \langle \hat{a}_2 \hat{a}_1 + \hat{a}_2 \hat{a}_1^\dagger + \hat{a}_2^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_1^\dagger \rangle / 2 
- \langle \hat{a}_1 + \hat{a}_1^\dagger \rangle \langle \hat{a}_2 + \hat{a}_2^\dagger \rangle, \tag{B.9} \]

\[ \gamma_{12} = \frac{1}{2i} \left( \langle \hat{a}_1 \hat{a}_2 - \hat{a}_1 \hat{a}_2^\dagger + \hat{a}_1^\dagger \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle / 2i 
+ \langle \hat{a}_2 \hat{a}_1 + \hat{a}_2 \hat{a}_1^\dagger - \hat{a}_2^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_1^\dagger \rangle / 2i 
+ i \langle \hat{a}_1 + \hat{a}_1^\dagger \rangle \langle \hat{a}_2 - \hat{a}_2^\dagger \rangle, \tag{B.10} \]

\[ \gamma_{21} = \frac{1}{2i} \left( \langle \hat{a}_1 \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger - \hat{a}_1^\dagger \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle / 2i 
+ \langle \hat{a}_2 \hat{a}_1 - \hat{a}_2 \hat{a}_1^\dagger + \hat{a}_2^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_1^\dagger \rangle / 2i 
+ i \langle \hat{a}_1 - \hat{a}_1^\dagger \rangle \langle \hat{a}_2 + \hat{a}_2^\dagger \rangle, \tag{B.11} \]

\[ \gamma_{22} = \frac{1}{2} \left( \langle -\hat{a}_1 \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger + \hat{a}_1^\dagger \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle / 2 
+ \langle -\hat{a}_2 \hat{a}_1 + \hat{a}_2 \hat{a}_1^\dagger + \hat{a}_2^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_1^\dagger \rangle / 2 
+ \langle \hat{a}_1 - \hat{a}_1^\dagger \rangle \langle \hat{a}_2 - \hat{a}_2^\dagger \rangle. \tag{B.12} \]
Bibliography


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