Chapter 2

Physics of Josephson Junctions:

The Zero Voltage State
2.1 Basic Properties of Lumped Josephson Junctions

- **small** spatial dimensions:
  - gauge invariant phase diff. & current density are **uniform**
  - variations of supercurrent density on length scale larger than \( \lambda_L \)
  \[
  \lambda_L = \sqrt{\frac{m_s}{\mu_0 n_s q_s^2}} \approx 10 \text{ nm} - 1 \text{ \( \mu \text{m} \)}
  \]
  - Josephson junction: \( n_s \) strongly reduced:
  \[
  \lambda_L \rightarrow \lambda_J \approx 10 - 100 \text{ \( \mu \text{m} \)}
  \]

2.1.1 The Lumped Josephson Junction

- JJ with spatially homogeneous supercurrent density and phase difference: **lumped element**

\[
I_s = \int_S J_s \cdot ds \quad \text{region of integration: junction area } S
\]

- **current-phase relation:**
  \[I_s(t) = I_c \sin \varphi(t)\]

- **gauge invariant phase difference:**
  \[
  \varphi(t) = \theta_2(t) - \theta_1(t) - \frac{2\pi}{\Phi_0} \int_1^2 A(r, t) \cdot d\mathbf{l}
  \]

- **voltage-phase relation:**
  \[
  \frac{\partial \varphi}{\partial t} = \frac{2\pi}{\Phi_0} \int_1^2 E(r, t) \cdot d\mathbf{l} \quad \Rightarrow \quad \frac{d\varphi}{dt} = \frac{2\pi}{\Phi_0} \nabla
  \]

- uniform phase difference \( \rightarrow \) total derivative
2.1.1 The Lumped Josephson Junction

- "0"-junction:
  \[ I = I_c \sin \varphi \]
  \[ V = \frac{\Phi_0}{2\pi} \frac{d\varphi}{dt} \]

- "\pi"-junction:
  \[ I = I_c \sin (\varphi + \pi) \]
  \[ V = \frac{\Phi_0}{2\pi} \frac{d\varphi}{dt} \]

2.1.2 The Josephson Coupling Energy

- finite energy stored in JJ: overlap of macroscopic wave functions → binding energy
- initial current & phase difference: zero
- increase junction current from zero to finite value
  → phase difference has to change
  → voltage-phase relation: finite junction voltage
  → external source has to supply energy (to accelerate the superelectrons)
  → stored in kinetic energy of moving superelectrons
  → integral of the power = \( I_s \cdot V \)

  (voltage during increase of current):
  \[ E_J = \int_{t_0}^{t_0} I_s V \, dt = \int_{t_0}^{t_0} (I_c \sin \tilde{\varphi}) \left( \frac{\Phi_0}{2\pi} \frac{d\tilde{\varphi}}{dt} \right) \, dt \]
2.1.2 The Josephson Coupling Energy

with \( \varphi(0) = 0 \) and \( \varphi(t_0) = \varphi \):

\[
E_J = \frac{\Phi_0 I_c}{2\pi} \int_0^\varphi \sin \tilde{\varphi} \, d\tilde{\varphi}
\]

integration:

\[
E_J = \frac{\Phi_0 I_c}{2\pi} (1 - \cos \varphi) = E_{J0} (1 - \cos \varphi)
\]

**Josephson coupling energy**

- **order of magnitude:**
  - typical \( I_c \): 1 mA \( \rightarrow \) \( E_{J0} \approx 3 \times 10^{-19} \) J
  - corresponds to thermal energy \( k_B T \) around \( k_B \times 20 \, 000 \) K
  - junction with very small critical current \( I_c \approx 1 \mu\text{A} \rightarrow \) thermal energy \( \approx k_B \times 20 \) K
2.1.3 The Superconducting State

\(-l_c < I < l_c \rightarrow \textit{constant phase difference}:\)

\(\varphi = \varphi_n = \arcsin \left( \frac{l}{l_c} \right) + 2\pi n\)

\(\varphi = \tilde{\varphi}_n = \pi - \arcsin \left( \frac{l}{l_c} \right) + 2\pi n\)

\(\varphi = \varphi_n + 2\pi n, \ \tilde{\varphi}_n = \pi - \varphi_n + 2\pi n\)

→ zero junction voltage:

zero-voltage state / ordinary (S) state

• analysis of \textit{stability} of (junction + current source) – system:

\textit{potential energy} \(E_{\text{pot}}\) of the system under action of external force: \(E = F \cdot x\)

\(E:\) intrinsic free energy of the subsystem junction

\(F:\) generalized force: \(F = I\)

\(x:\) generalized coordinate: \(F \cdot \frac{\partial x}{\partial t} \rightarrow \text{power flowing into subsystem} (I \cdot V):\)

\(x = \int V \, dt = \frac{\hbar}{2e} \varphi + c = \frac{\Phi_0}{2\pi} \varphi + c\)

→ \textit{potential energy}:

\(E_{\text{pot}}(\varphi) = E_J(\varphi) - I \left( \frac{\Phi_0}{2\pi} \varphi + c \right)\)

\(= E_J(0) \left[ 1 - \cos \varphi - \frac{I}{l_c} \varphi \right] + \tilde{c}\)

\textit{tilted washboard potential}

stable minima \(\varphi_n\), unstable maxima \(\tilde{\varphi}_n\)

states for different \(n\): equivalent
2.1.3 The Superconducting State

- properties of the washboard potential

\[ U_0 \equiv E_{\text{pot}}(\varphi_{n+1}) - E_{\text{pot}}(\tilde{\varphi}_{n+1}) = 2E_{J_0} \left[ \sqrt{1 - \left(\frac{l}{I_c}\right)^2} - \frac{l}{I_c} \arccos \left(\frac{l}{I_c}\right) \right] \]

\[ k \equiv \frac{\partial^2 E_{\text{pot}}}{\partial \varphi^2} = E_{J_0} \sqrt{1 - \left(\frac{l}{I_c}\right)^2} \]

\[ \rightarrow 0 \text{ for } l/I_c \rightarrow 1 \]

- close to \( I_c \): \( \alpha \equiv 1 - l/I_c \ll 1 \), we get the approximations:

\[ \varphi_0 = \frac{\pi}{2} - \sqrt{2\alpha} \quad \tilde{\varphi}_0 = \frac{\pi}{2} + \sqrt{2\alpha} \quad U_0 = \frac{2}{3} E_{J_0} (2\alpha)^{2/3} \quad k = E_{J_0} (2\alpha)^{1/2} \]

- washboard potential extremely useful in describing junction dynamics for \( I > I_c \)
2.1.4 The Josephson Inductance

- Energy storage in JJ → **nonlinear reactance**

\[
\frac{dl_s}{dt} = l_c \cos \varphi \frac{d\varphi}{dt} \quad \Rightarrow \quad \frac{dl_s}{dt} = l_c \cos \varphi \frac{2\pi}{\Phi_0} V
\]

- For small variations around \( l_s = l_c \sin \varphi \): JJ equivalent to inductance

\[
L_s = \frac{\Phi_0}{2\pi l_c \cos \varphi} = L_c \frac{1}{\cos \varphi}
\]

with \( L_c = \frac{\hbar}{2eI_c} \)

- Josephson inductance:

\( L_s \) is **negative** for \( \pi/2 + 2\pi n < \varphi < 3\pi/2 + 2\pi n \)

(for \( V > 0 \): **oscillating** Josephson current)
2.1.4 Junction Fabrication

Cross-sectional views of two kinds of junctions: R and RC-type. JJ: upper Nb pattern of the Nb/AlO_x/Nb junction, RC: contact between the junction and the resistor, JC: contact between the junction and the M4 layer.

S. Nagasawa et al., 
Physica C: Superconductivity 
Volumes 426–431, Part 2, 
pp. 1525–1532 (2005)
2.1.5 Mechanical Analogs

- **plane mechanical pendulum** in uniform gravitational field:
  
  \[ E_{\text{pot}}(\varphi) = E_{J0} \left[ 1 - \cos \varphi - \frac{l}{l_c} \varphi \right] \]

  phase difference \( \varphi \): angle of pendulum with respect to equilibrium
  supercurrent \( I_s \): torque
  voltage \( V \): angular velocity of pendulum

- **particle** moving in tilted washboard potential coordinate
  \[ x \propto \varphi \]
  velocity
  \[ v \propto \frac{d\varphi}{dt} \propto V \]
2.2 Short Josephson Junctions

• so far: zero-dimensional JJ (lumped elements) → spatially homogeneous supercurrent density and phase difference

• now: extended junctions → spatial variations $J_s(r)$ and $\phi(r)$
  → consider magnetic field generated by the Josephson current itself (self-field):

• short Josephson junctions:
  → self-field small compared to external field

• long Josephson junctions:
  → self-field no longer negligible

• relevant length scale for transition from short to long Josephson junction:

  \[ \lambda_J = \sqrt{\frac{m_s}{\mu_0 n_s q_s^2}} \gg \lambda_L \]

  Josephson penetration depth (density in weak coupling region)

• JJ at finite voltage → temporal interference → oscillation of Josephson current

• JJ at finite phase gradient → spatial interference → magnetic field dependence of Josephson current
2.2.1 Quantum Interference Effects - Short JJ in an Applied Field

- external magnetic field
  - spatial change of gauge invariant phase difference $\varphi(r)$
  - spatial interference of macroscopic wave functions in JJ

- insulating barrier thickness: $d$
- junction area: $A = L \cdot W$
- $W, L >> d$ (edge effects small)
- electrode thickness > $\lambda_L$
- magnetic field: $B_e = (0, B_y, 0)$
- magnetic thickness:
  $$t_B = d + \lambda_{L1} + \lambda_{L2}$$

- effect of $B_e$ on $J_s$:
  - phase shift $\varphi(P) - \varphi(Q)$ between point $P$ and $Q$ separated by $dz$
  - line integral along red contour yields total phase change along closed contour: $2\pi n$
2.2.1 Quantum Interference Effects - Short JJ in an Applied Field

\[ \vec{n} \theta \cdot d\vec{l} = (\theta_{Q_b} - \theta_{Q_a}) + (\theta_{P_c} - \theta_{Q_b}) + (\theta_{P_d} - \theta_{P_c}) + (\theta_{Q_a} - \theta_{P_d}) = 0 \]

1 & 3 are differences across the junction:

\[ \theta_{Q_b} - \theta_{Q_a} = +\varphi(Q) + \frac{2\pi}{\Phi_0} \sum_{Q_a} A \cdot d\vec{l} \]

\[ \theta_{P_d} - \theta_{P_c} = -\varphi(P) + \frac{2\pi}{\Phi_0} \sum_{P_c} A \cdot d\vec{l} \]

1 & 3 are differences acrosss the junction:

\[ \nabla \theta = \frac{2\pi}{\Phi_0} (\wedge J_s + A) \]

\[ \varphi = \theta_2 - \theta_1 - \frac{2\pi}{\Phi_0} \int_1^2 A \cdot d\vec{l} \]

2 & 4 differences in the bulk, supercurrent equation for \( \nabla \theta \):

\[ \theta_{P_c} - \theta_{Q_b} = \int_{Q_b}^{P_c} \nabla \theta \cdot d\vec{l} = \frac{2\pi}{\Phi_0} \int_{Q_b}^{P_c} \wedge J_s \cdot d\vec{l} + \frac{2\pi}{\Phi_0} \sum_{Q_b} A \cdot d\vec{l} \]

\[ \theta_{Q_a} - \theta_{P_d} = \int_{P_d}^{Q_a} \nabla \theta \cdot d\vec{l} = \frac{2\pi}{\Phi_0} \int_{P_d}^{Q_a} \wedge J_s \cdot d\vec{l} + \frac{2\pi}{\Phi_0} \sum_{P_d} A \cdot d\vec{l} \]
2.2.1 Quantum Interference Effects - Short JJ in an Applied Field

- substitution: \[ \varphi(Q) - \varphi(P) = -\frac{2\pi}{\Phi_0} \oint_C A \cdot d\mathbf{l} - \frac{2\pi}{\Phi_0} \int_{Q_b}^{P_d} \mathbf{J}_s \cdot d\mathbf{l} - \frac{2\pi}{\Phi_0} \int_{Q_a}^{P_c} \mathbf{J}_s \cdot d\mathbf{l} \]

- integration of \( \mathbf{A} \) around closed contour \( \rightarrow \) enclosed flux \( \Phi \)
- integration of \( \mathbf{J}_s \) excludes insulating barrier \( \rightarrow \) incomplete contour \( C' \):

\[
\oint_{C'} \mathbf{J}_s \cdot d\mathbf{l} = \oint_{Q_a}^{P_c} \mathbf{J}_s \cdot d\mathbf{l} + \oint_{Q_b}^{P_d} \mathbf{J}_s \cdot d\mathbf{l}
\]

- difference of gauge invariant phase differences \( \varphi(Q)-\varphi(P) \):

\[ \varphi(Q) - \varphi(P) = -\frac{2\pi \Phi}{\Phi_0} - \frac{2\pi}{\Phi_0} \oint_{C'} \mathbf{J}_s \cdot d\mathbf{l} \]

- line integral of supercurrent density \( \mathbf{J}_s \):
  - segments in \( x \)-direction cancel (separation: \( dz \rightarrow 0 \))
  - segments in \( z \)-direction: deep inside SC \( (\gg \lambda_L) \rightarrow \mathbf{J}_s \) exponentially small

\[ \Rightarrow \text{therefore:} \quad \varphi(P) - \varphi(Q) = \frac{2\pi \Phi}{\Phi_0} \quad \frac{\varphi(P) - \varphi(Q)}{2\pi} = \frac{\Phi}{\Phi_0} \]

- **total flux enclosed** by the loop:

\[ \Phi = B_y (d + \lambda_{L1} + \lambda_{L2}) dz = B_y t_B dz \]
2.2.1 Quantum Interference Effects - Short JJ in an Applied Field

\[ \varphi(P) - \varphi(Q) = \frac{2\pi \Phi}{\Phi_0} \]

\[ \Phi = B_y(d + \lambda_{L1} + \lambda_{L2})dz = B_y t_B dz \]

\[ \Rightarrow \varphi(P) - \varphi(Q) = \frac{2\pi}{\Phi_0} B_y t_B dz = \frac{\partial \varphi}{\partial z} dz \]

\[ \Rightarrow \frac{\partial \varphi}{\partial z} = \frac{2\pi}{\Phi_0} B_y t_B \]

• similar argument for \( P \) and \( Q \) separated by \( dy \) in \( y \)-direction

then

\[ \nabla \varphi(\mathbf{r}, t) = \frac{2\pi}{\Phi_0} t_B \left[ \mathbf{B}(\mathbf{r}, t) \times \hat{\mathbf{x}} \right] \]

• integration gives:

\[ \varphi(z) = \frac{2\pi}{\Phi_0} B_y t_B z + \varphi_0 \]

• current phase relation:

\[ J_s(y, z, t) = J_c(y, z) \sin \left( \frac{2\pi}{\Phi_0} t_B B_y z + \varphi_0 \right) = J_c(y, z) \sin (kz + \varphi_0) \]

with \[ k = \frac{2\pi}{\Phi_0} t_B B_y \]

• \( J_s \) varies periodically with period \( \Delta z = 2\pi/k = \Phi_0/t_B B_y \)

- flux through the junction within one period: \( \Phi_0 \)

\[ \tilde{t}_B = d + \frac{t_1}{\lambda_{L1}} \coth \frac{t_1}{\lambda_{L1}} + \frac{t_2}{\lambda_{L2}} \coth \frac{t_2}{\lambda_{L2}} \]

\( \varphi_0 \): phase difference at \( z = 0 \)

\( \text{nota bene:} \)

if \( t_1 < \lambda_{L1} \) and \( t_2 < \lambda_{L2} \):
Brief Summary

1. Josephson equation: \( J_s(\phi) = J_c \sin \phi \)

2. Josephson equation:
   \[
   \frac{\partial \phi}{\partial t} = \frac{2\pi}{\Phi_0} V
   \]
   - JJ with spatially homogeneous \( J_s \) and \( \phi \): lumped element

Josephson coupling energy
   \[
   E_J = \frac{\Phi_0 I_c}{2\pi} (1 - \cos \phi) = E_{J0} (1 - \cos \phi)
   \]

Tilted washboard potential
   \[
   E_{\text{pot}}(\phi) = E_{J0} \left[ 1 - \cos \phi - \frac{l}{l_c} \phi \right] + \tilde{c}
   \]

Josephson inductance
   \[
   L_s = \frac{\Phi_0}{2\pi I_c \cos \phi} = L_c \frac{1}{\cos \phi}
   \]

- extended junctions: \( J_s = J_s(r) \) and \( \phi = \phi(r) \)
  - **short** JJ: no self-field \( \leftrightarrow \) **long** JJ: self-field
  \( \Rightarrow \) Josephson penetration depth

- **short** junction in external field
  - field induces gradient of phase difference \( \phi \)

  \[
  \frac{\partial \phi}{\partial z} = \frac{2\pi}{\Phi_0} B_y t_B \quad \Rightarrow \quad \phi(z) = \frac{2\pi}{\Phi_0} B_y t_B z + \phi_0
  \]

  \[
  J_s(y, z, t) = J_c(y, z) \sin \left( \frac{2\pi}{\Phi_0} t_B B_y z + \phi_0 \right) = J_c(y, z) \sin (kz + \phi_0)
  \]
• how does \( I_s = \int \int J_s(y, z) \, dy \, dz \) depend on the applied field \( \mathbf{B}_e = (0, B_y, 0) \)?

• integration in in y-direction: \( i_c(z) = \int_{-W/2}^{W/2} J_c(y, z) \, dy \)

\[
\Rightarrow I_s(B_y) = \int_{-L/2}^{L/2} i_c(z) \sin(kz + \varphi_0) \, dz = \Im \left\{ e^{i\varphi_0} \int_{-\infty}^{\infty} i_c(z) e^{ikz} \, dz \right\}
\]

• integral: complex, multiplication by \( e^{i\varphi_0} \) does not change magnitude

\( \Rightarrow \text{maximum Josephson current} \): magnitude

\[
l_s^m(B_y) = \left| \int_{-\infty}^{\infty} i_c(z) \, e^{ikz} \, dz \right|
\]

magnetic field dependence of \( I_s^m \)

\( \rightarrow \text{Fourier transform} \) of \( i_c(z) \)

\( \rightarrow \) analogy to optics

• \( J_c(y,z) \) homogeneous \( \rightarrow i_c(z) \) constant for \( \rightarrow \) diffraction pattern of a slit: *Fraunhofer diffraction pattern*:

\[
l_s^m(\Phi) = I_c \left| \frac{\sin \frac{kL}{2}}{\frac{kL}{2}} \right| = I_c \left| \frac{\sin \frac{\pi \Phi}{\Phi_0}}{\frac{\pi \Phi}{\Phi_0}} \right| \quad \Phi = B_y t_B L \quad \text{flux though the junction}
\]

\( I_c = i_c L \)

• experimental observation of \( I_s^m(\Phi) \) \( \rightarrow \) proof of Josephson tunneling of pairs
2.2.2 The Fraunhofer Diffraction Pattern

- spatially homogeneous maximum current density $J_c(y,z)$
  \[ \Rightarrow \text{Fraunhofer diffraction pattern:} \]

- maximum current density integrated along $y$-direction

- experiment: study of the homogeneity of the supercurrent flow in JJ
2.2.2 The Fraunhofer Diffraction Pattern

- interpretation of the shape of $I_s^m(\Phi)$: spatial distribution of $i_s(z) = \int J_s(y,z)dz$ for different applied fields

- zero field, $\Phi = 0$:
  $\rightarrow \varphi(z) = \varphi_0$
  $\rightarrow i_s(z) = \text{const.}$
  $\rightarrow$ Josephson current maximum for $\varphi_0 = -\pi/2$:
  $\rightarrow J_s(y,z) = -J_c(y,z)$

- $\Phi = \Phi_0/2$:
  $\varphi(z) = \frac{2\pi \Phi}{\Phi_0} \frac{z}{L} + \varphi_0 = \frac{\pi z}{L} + \varphi_0$
  $\rightarrow$ sinusoidal variation of supercurrent with $z$ difference between edges:
  $\varphi(L/2) - \varphi(-L/2) = \pi$

- half of a full oscillation period
- Josephson current maximum for $\varphi_0 = -\pi/2$
- linear increase of phase from $-\pi$ at $z = -L/2$ to 0 at $z = L/2$
2.2.2 The Fraunhofer Diffraction Pattern

Josephson current tends to decrease with increasing field

Spatial interference effect of macroscopic wave functions: plane of constant phase in superconductor 2 is tilted by:

\[ \delta \varphi(z) = \frac{2\pi}{\Phi_0} B_y t_B z + \varphi_0 \]

Here: destructive interference
2.2.2 The Fraunhofer Diffraction Pattern

- closed current loop
- no penetration of applied field into electrodes
  - Josephson vortex
- no normal core
- vortex core in barrier region
2.2.2 The Fraunhofer Diffraction Pattern

- arbitrary direction of the applied field within barrier plane:

\[
B_e = B_y \hat{y} + B_z \hat{z}
\]

\[
\Rightarrow I_s^m(\Phi) = I_c \left| \frac{\sin \frac{\pi \Phi_y}{\Phi_0}}{\frac{\pi \Phi_y}{\Phi_0}} \right| \left| \frac{\sin \frac{\pi \Phi_z}{\Phi_0}}{\frac{\pi \Phi_z}{\Phi_0}} \right|
\]

\[
\Phi_y = B_y t_b L \quad \Phi_z = B_z t_b W
\]

\[
\Rightarrow I_s^m(B_e) = \left| \int_S J_c(y, z)e^{ik \cdot r} dS \right|
\]
2.2.3 Determination of the Maximum Josephson Current Density

- **real JJ**: inhomogeneities, e.g. spatially varying barrier thickness
  - experimental determination of $J_c(y,z)$ by measuring $I_s^m(B)$
  - **no** access via inverse Fourier transform
  - **lack of phase** information
- **approximate** $i_c(z)$ under certain assumptions
  e.g.: symmetry to junction midpoint

\[
i_c(z - L/2) = \frac{1}{\pi} \int_0^\infty |I_s^m(k)| \cos(kz)(-1)^{n(k)} dk
\]

\[n: \text{number of zeros of } |I_s^m(k)| \text{ between 0 and } k\]

- information on $J_c(y,z)$ on **small length scale** ⇒ **high fields**
- spatial resolution $\approx 1/B_y$:

\[\frac{2\pi}{k} = \frac{\Phi_0}{t_B B_y} = L \frac{\Phi_0}{\Phi}\]

measurement of $I_s^m(\Phi)$ up to $\Phi/\Phi_0 = 1$

- **spatial resolution**: junction length $L$

- **tailored junctions**:
  - only central maximum desirable (x-ray detectors, suppression of supercurrent)

- **Gaussian current distribution**:

\[i_c(z) = i_c(0) \exp\left(-\frac{z^2}{2\sigma^2}\right)\]
2.2.3 Determination of the Maximum Josephson Current Density

- \( I_s^m(\Phi) \)-dependence without side lobes:

\[
I_s^m(\Phi) = \sqrt{\frac{1}{2\pi}} i_c(0) L \exp(-\sigma k^2) = \sqrt{\frac{1}{2\pi}} i_c(0) L \exp\left(-\sigma \frac{4\pi^2 \Phi^2}{L^2 \Phi_0^2}\right)
\]

- junction shape: should approach a Gauss curve for homogeneous \( J_c(y,z) \)

\[ \rightarrow \text{integrated current density in } y\text{-direction: } i_c(z) = \int J_c(y,z) \, dy \rightarrow \text{Gaussian profile} \]
2.2.3 Determination of the maximum Josephson current density

**Additional topic:**

**supercurrent auto-correlation function**

**Comparison:**
- optical diffraction experiment ↔ field dependence of maximum Josephson current:
  - transmission function $P_0(z) \leftrightarrow i_C(z)$
  - square root of light intensity $P_t$ in focal plane $\leftrightarrow |I_{c_m}(B_y)|$

backtransformation gives $P_i$ (spatial resolution given by # of diffraction orders)
$\leftrightarrow$ phase is lost, backtransformation of intensity $(|I_{c_m}|)^2 (B_y)$

$\rightarrow$ autocorrelation function of supercurrent distribution
2.2.3 Determination of the maximum Josephson current density

- Definition of auto-correlation function:

\[ AC(\delta) = \int_{-\infty}^{\infty} i_c(z) i_c(z + \delta) \, dz \]

\[ \rightarrow \text{overlap of } i_c(z) \text{ and same function shifted by } \delta \]

- Wiener-Khinchine theorem:

Autocorrelation function of \( i_c(z) \):

\[ AC(\delta) = \int_{-\infty}^{\infty} |I_s^m(k)|^2 e^{ik\delta} \, dk \]

\[ k = \frac{2\pi}{\Phi_0} t_B B_y = \frac{1}{L} \frac{2\pi}{\Phi_0} \frac{\Phi}{\Phi_0} \]

- Spatial information of AC-function depends on magnetic field interval

Spatial resolution:

\[ 2\pi / k = L \frac{\Phi_0}{\Phi} \]

- Recording 100 lobes in \( I_s^m(B_y) \) \( \rightarrow \) spatial resolution \( 0.01 \times \) junction width

- Statistical information in envelope of \( |I_s^m(B_y)|^2 \):

E.g. inhomogeneities with probability \( p(a) \propto 1/a \) (\( p \propto \) length scale \( a = \) const.)

\[ |I_s^m(B_y)|^2 \propto 1/B_y \] ("spatial 1/f noise")
2.2.3 Determination of the maximum Josephson current density

- Random distribution of filaments with width $a$:
  - Envelop constant up to $k = \frac{2\pi}{a}$, then $\approx \frac{1}{B_y^2}$

  "Spatial shot noise"

- Analysis of AC gives statistical information on current density inhomogeneities

- $\text{YBa}_2\text{Cu}_3\text{O}_{7-\delta}$ grain boundary JJ
  - Slope of envelop is about -0.65
    - $|I_{s,m}(B_y)|^2 \propto \frac{1}{B_y^{1.3}}$
    - $p(a) \propto \frac{1}{a^{1.5}}$
    - Small scale inhomogeneities are more probable
2.2.4 Additional Topic: Direct Imaging of Supercurrent Distribution

**scanning** of JJ by focused electron / laser beam

→ measure change $\delta I_s^m(y,z)$ as function of beam position $(y,z)$

→ $\delta I_s^m(y,z) \propto J_c(y,z) \rightarrow$ 2-dim. image of $J_c(y,z)$

→ spatial **resolution** $\approx$ thermal healing length ($\approx 1 \, \mu m$)
2.2.5 Additional Topic: Short JJ - Energy Considerations

junction energy: \( E = E_s + E_i \) (electrode and barrier energy)

\[
E_s = \frac{1}{2\mu_0} \int_{V_s} \left( B^2 + \mu_0 \nabla J_s^2 \right) dV
\]

\[
E_I = \frac{1}{2\mu_0} \int_{V_i} B^2 dV + \int_{A_i} \frac{1}{d} \frac{E_J}{dV} dV
\]

\[
= \frac{1}{2\mu_0} \int_{V_i} B^2 dV + \int_{A_i} \frac{\Phi_0 J_c(y, z)}{2\pi} \left[ 1 - \cos \varphi(z) \right] dy dz
\]

\[
\Rightarrow E = \frac{1}{2\mu_0} \int_{V_s+V_i} B^2 dV + \frac{1}{2} \int_{V_s} \nabla J_s^2 dV + \int_{A_i} \frac{\Phi_0 J_c(y, z)}{2\pi} \left[ 1 - \cos \varphi(z) \right] dy dz
\]

\( E_B \): energy due to external field
\( E_J \): Josephson coupling energy

short junction: \( E_B \gg E_J \)
define: short junction: $E_B \gg E_J$

for $t_1, t_2 \gg \lambda_L$ the first integral dominates:

$$E_B = \frac{1}{2\mu_0} \int_{V_s+V_i} B^2 dV$$

integration volume: $W \cdot L \cdot (d + 2\lambda_L) = A_i \cdot t_B$

$$E_B = \frac{1}{2\mu_0} B_y^2 W L t_B = \frac{1}{2\mu_0} \frac{\Phi^2 W}{t_B L} \quad \Phi = B_y L t_B$$

$E_J$ for spatially homogeneous $J_c(y,z)$:

$$E_J = \frac{\Phi_0 I_c}{2\pi} - \int_{-W/2}^{W/2} \int_{-L/2}^{L/2} \frac{\Phi_0 J_c(y,z)}{2\pi} \cos \varphi(z) \, dy \, dz$$

$$= \frac{\Phi_0 I_c}{2\pi} - \int_{-L/2}^{L/2} \frac{\Phi_0 I_c(z)}{2\pi} \cos \varphi(z) \, dz = \frac{\Phi_0 I_c}{2\pi} - \frac{\Phi_0 I_c}{2\pi} \frac{\sin \frac{\pi \Phi}{\Phi_0}}{\frac{\pi \Phi}{\Phi_0}} \cos \varphi(0)$$

for one flux quantum in the junction $E_B \gg E_J$ requires:

$$\frac{1}{2\mu_0} \frac{\Phi_0^2 W}{t_B L} \gg \frac{\Phi_0 I_c}{2\pi}$$

with $J_c = I_c/WL$: $L \ll \tilde{\lambda}_J \equiv \frac{\pi \Phi_0}{\mu_0 J_c t_B}$

($\approx$ Josephson penetration depth)
2.2.6 The Motion of Josephson Vortices

- Josephson vortices: visualize Josephson current density vortices moving in $z$-direction at constant speed $v_z$
- Short junction: self-field negligible
  $\rightarrow$ flux density in junction given by $\mathbf{B}_e = (0, B_y, 0)$
  $\rightarrow$ gauge invariant phase difference:

$$\frac{\partial \varphi}{\partial z} = \frac{2\pi}{\Phi_0} B_y t_B$$

- Passage of 1 vortex changes phase by $2\pi$:

$$\frac{\partial \varphi}{\partial t} = \frac{2\pi}{\Phi_0} \frac{\partial \Phi}{\partial t} = \frac{2\pi}{\Phi_0} B_y t_B \frac{\partial z}{\partial t} = \frac{2\pi}{\Phi_0} B_y t_B v_z$$

$$\Rightarrow \varphi(z, t) = \frac{2\pi}{\Phi_0} B_y t_B (z - v_z t) + \varphi(0) = k(z - v_z t) + \varphi(0)$$

$$\Rightarrow J_s(y, z, t) = J_c(y, z) \sin (k(z - v_z t))$$

- Current density pattern: moves at $v_z$
  Vortex with period $p = L \Phi_0 / \Phi$
  Number of vortices in junction: $N_v = \frac{L}{p} = \frac{\Phi}{\Phi_0}$
2.2.6 The Motion of Josephson Vortices

- change of gauge-invariant phase difference:
  \[ \Delta \varphi = 2\pi \frac{\Phi}{\Phi_0} = 2\pi N_V \]
  \(2\pi \times \# \text{ of vortices}\)

- rate of vortex passage:
  \[ \frac{dN_V}{dt} = \frac{1}{2\pi} \frac{d\Delta \varphi}{dt} \]

- with the voltage-phase relation:
  \[ \frac{dN_V}{dt} = \frac{V}{\Phi_0} \]
  \[ \frac{d\varphi}{dt} = \frac{2\pi}{\Phi_0} V \]

  constant velocity of vortices
  \(\Rightarrow\) constant junction voltage / vortex rate

  \(\Rightarrow\) application: single flux quantum pump

  @ pump frequency \(f = \frac{dN_V}{dt}\) \(\Rightarrow V = f \cdot \Phi_0\)
2.2. Summary – Short Josephson Junctions

- short: lateral junction dimensions small compared to Josephson penetration depth
- effect of magnetic field parallel to junction electrodes

\[ \nabla \varphi(r, t) = \frac{2\pi}{\Phi_0} t_B [B(r, t) \times \hat{z}] \]

Josephson current density

\[ J_s(y, z, t) = J_c(y, z) \sin \left( \frac{2\pi}{\Phi_0} t_B B_y z + \varphi_0 \right) \]

- spatial distribution of \( i_s(z) = \int J_s(y, z) dz \)
- Josephson vortex


2.2. Summary – Short Josephson Junctions

- magnetic field dependence of maximum Josephson current:

\[ I_{s}^{m}(\Phi) = I_{c} \left| \sin \left( \frac{kL}{2} \right) \right| = I_{c} \left| \frac{\sin \left( \frac{\Phi}{\Phi_{0}} \right)}{\frac{\pi \Phi}{\Phi_{0}}} \right| \]

\[ \Phi = B_{y} t_{b} L \quad \text{flux though junction} \]

\[ I_{c} = i_{c} L \]

\[ k = \frac{2\pi}{\Phi_{0}} t_{B} B_{y} \]

Fraunhofer diffraction pattern

→ analogy to optics (single slit diffraction)

no inverse Fourier transform (missing phase)

autocorrelation function

- motion of Josephson vortices:

motion of single vortex across junction results in phase change of \(2\pi\)

\[ V \propto \frac{\partial \Phi}{\partial t} = \frac{2\pi}{\Phi_{0}} \frac{\partial \Phi}{\partial t} = \frac{2\pi}{\Phi_{0}} B_{y} t_{B} \frac{\partial z}{\partial t} = 2\pi \frac{\Phi}{\Phi_{0}} \frac{v_{z}}{L} \]

constant motion of vortices → constant junction voltage \(\propto\) vortex rate
2.3 Long Josephson Junctions

2.3.1 The Stationary Sine-Gordon Equation

\[ \frac{\partial \varphi}{\partial z} = \frac{2\pi}{\Phi_0} B_y t_B , \quad \nabla \varphi(\mathbf{r}, t) = \frac{2\pi}{\Phi_0} t_B [\mathbf{B}(\mathbf{r}, t) \times \hat{x}] \quad \rightarrow \text{generally valid} \]

- **now:** magnetic flux density given by *external and self-generated field*

  with \( \mathbf{B} = \mu_0 \mathbf{H} \) and \( \mathbf{D} = \varepsilon_0 \mathbf{E} \):

  \[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \]

- **zero-voltage state:**

  \[ \frac{\partial B_y(z)}{\partial z} = -\mu_0 J_x(z) \]

- **spatial derivative:**

  \[ \frac{\partial^2 \varphi(z)}{\partial z^2} = -\frac{2\pi t_B}{\Phi_0} \frac{\partial B_y(z)}{\partial z} = -\frac{2\pi \mu_0 t_B}{\Phi_0} J_x(z) \]

- **assume:** \( J_c(y,z) = \text{const.} \) and use \( J_x(y,z) = J_s(y,z) \) \( \rightarrow J_x(z) = -J_c \sin \varphi \)

\[ \frac{\partial^2 \varphi(z)}{\partial z^2} = \frac{2\pi \mu_0 t_B J_c}{\Phi_0} \sin \varphi(z) = \frac{1}{\lambda_J^2} \sin \varphi(z) \]

**stationary Sine-Gordon equation (SSGE)** (nonlinear differential equation)

**Josephson penetration depth:**

\[ \lambda_J \equiv \sqrt{\frac{\Phi_0}{2\pi \mu_0 t_B J_c}} \]
2.3.1 The Stationary Sine-Gordon Equation

two-dimensional stationary Sine-Gordon equation:

\[ \frac{\partial^2 \varphi(y, z)}{\partial y^2} + \frac{\partial^2 \varphi(y, z)}{\partial z^2} = \frac{1}{\lambda_j^2} \sin \varphi(y, z) \]

- relation between London and Josephson penetration depth

\[ \lambda_L \equiv \sqrt{\frac{m_s}{\mu_0 n_s q_s^2}} \quad \leftrightarrow \quad \lambda_J \equiv \sqrt{\frac{\Phi_0}{2\pi \mu_0 t_B J_c}} \]

with \( J_s = q_s n_s \left\{ \frac{\hbar}{m_s} \nabla \theta(r, t) - \frac{q_s}{m_s} A(r, t) \right\} \)

\[ J_c \simeq q_s n_s^* \frac{\hbar}{m_s} \frac{2\pi}{t_B} \]

insert into expression for Josephson penetration depth

\[ \lambda_J \equiv \sqrt{\frac{\Phi_0}{2\pi \mu_0 t_B J_c}} \simeq \sqrt{\frac{\hbar}{q_s \mu_0 t_B \left[ (q_s n_s^* \hbar/m_s)(2\pi/t_B) \right]}} = \sqrt{\frac{m_s}{2\pi \mu_0 n_s^* q_s^2}} = \lambda_L(n_s^*) \]

\( \lambda_J \) corresponds to London penetration depth of weak coupling region with reduced superelectron density \( n_s^* \)
2.3.1 The Stationary Sine-Gordon Equation

additional topic: analytical solutions of the SSGE

- **small $\varphi \rightarrow$ linearization: $\sin \varphi \approx \varphi$$\quad \frac{\partial^2 \varphi(z)}{\partial z^2} = \frac{1}{\lambda_j^2} \varphi(z) \Rightarrow \varphi(z) = \varphi(0)e^{-z/\lambda_j}$

magnetic field along the junction:

$$\Rightarrow B_y(z) = -\frac{\varphi(0)}{2\pi} \frac{\Phi_0}{\lambda_j t_B} e^{-z/\lambda_j}$$

$\lambda_j$ is a decay length

with

$$\frac{\partial B_y(z)}{\partial z} = -\mu_0 J_x(z) \Rightarrow J_x(z = 0) = \frac{1}{\lambda_j} \frac{B_y(z = 0)}{\mu_0}$$

- for a small junction $L \ll \lambda_j$:

$$\frac{\partial^2 \varphi(z)}{\partial z^2} \approx 0 \quad \Rightarrow \quad \frac{\partial \varphi(z)}{\partial z} \approx \text{const}$$

$\Rightarrow$ short junction result
2.3.2 The Josephson Vortex

**particular** solution of the SSGE:

\[
\varphi(z) = \pm 4 \arctan \left\{ \exp \left( \frac{z-z_0}{\lambda_J} \right) \right\} + 2\pi n
\]

\[
B_y(z) = \pm \frac{\Phi_0}{\pi \lambda_L t_B} \frac{1}{\cosh \left( \frac{z-z_0}{\lambda_J} \right)}
\]

\[
J_x(z) = -J_s(z) = \pm \frac{\Phi_0}{\pi \mu_0 \lambda_J^2 t_B} \frac{\sinh \left( \frac{z-z_0}{\lambda_J} \right)}{\cosh \left( \frac{z-z_0}{\lambda_J} \right)} = \pm 2J_c \frac{\sinh \left( \frac{z-z_0}{\lambda_J} \right)}{\cosh \left( \frac{z-z_0}{\lambda_J} \right)}
\]

**general** solution: **particular** + **homogeneous** solution

important case: junction of infinite length, \( d\varphi/dz \) vanishes at \( z = \pm \infty \)

\( \rightarrow \) particular solution \( \equiv \) complete solution
2.3.2 The Josephson Vortex

- decay length for $J_s$ and $B_y$: $\lambda_J$
- maximum of $J_s$
- integration: total current = 0, total flux = $\Phi_0$
  $\rightarrow$ **Josephson vortex** in a long JJ
2.3.2 The Josephson Vortex

**additional topic: Energy of the Josephson Vortex Solution**

Energy stored in long JJ:

\[
E = \frac{1}{2\mu_0} \int_{V_s+V_i} B^2 dV + \frac{1}{2} \int_{V_s} \mathbf{J}^2_s dV + \int_{A_i} \frac{\Phi_0 J_c(y, z)}{2\pi} [1 - \cos \varphi(z)] dy dz
\]

\[
E_s = \frac{1}{2\mu_0} \int_{V_s} (B^2 + \mu_0 \mathbf{J}^2_s) dV
\]

\[
E_I = \frac{1}{2\mu_0} \int_{V_i} B^2 dV + \int_{A_i} \frac{\Phi_0 J_c(y, z)}{2\pi} [1 - \cos \varphi(z)] dy dz
\]

stored energy for vortex solution, thick electrodes:

\[
E = \frac{1}{2\mu_0} \int_{V_s+V_i} B^2 dV + \int_{A_i} \frac{\Phi_0 J_c(y, z)}{2\pi} [1 - \cos \varphi(z)] dy dz
\]

Using \( \partial \varphi / \partial z = 2\pi B_y t_B / \Phi_0 \) and integration over \( y \) and \( x \):

\[
E = \frac{\Phi_0 J_c W}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{1}{2} \lambda^2_j \left( \frac{\partial \varphi(z)}{\partial z} \right)^2 + [1 - \cos \varphi(z)] \right\} dz
\]

Now we use the vortex solution \( \varphi(z) = 4 \arctan \left\{ \exp \left( \frac{z - z_0}{\lambda_j} \right) \right\} \)
2.3.2 The Josephson Vortex

**additional topic: Energy of the Josephson Vortex Solution**

\[ \varphi(z) = 4 \arctan \left\{ \exp \left( \frac{z}{\lambda_J} \right) \right\} = -2 \sin^{-1} \left( \frac{1}{\cosh(z/\lambda_J)} \right) \]

\[ \sin \frac{\varphi}{2} = -\frac{1}{\cosh(z/\lambda_J)} \]

\[ 1 - \cos \varphi = 2 \sin^2 \left( \frac{\varphi}{2} \right) \]

\[ \Rightarrow 1 - \cos \varphi(z) = 2 \frac{1}{\cosh^2 \left( \frac{z}{\lambda_J} \right)} \]

\[ \frac{1}{2} \lambda_J^2 \left( \frac{\partial \varphi}{\partial z} \right)^2 = 2 \frac{1}{\cosh^2 \left( \frac{z}{\lambda_J} \right)} \]

\[ \Rightarrow E = \frac{2 \Phi_0 J_c W}{\pi} \int_{\infty}^{-\infty} \frac{1}{\cosh^2 \left( \frac{z}{\lambda_J} \right)} \, dz = \frac{2 \Phi_0 J_c W \lambda_J}{\pi} \left[ \tanh \frac{z}{\lambda_J} \right]_{-\infty}^{\infty} = \frac{4 \Phi_0 J_c W \lambda_J}{\pi} \]

Energy per unit length of vortex:

\[ E_{\text{Vortex}} = \frac{E_l}{W} = \frac{4 \Phi_0 J_c \lambda_J}{\pi} \]

Magnetic flux density \( B_{c1} \) for first vortex entrance:

\[ B_{c1} = \frac{\mu_0}{\Phi_0} E_{\text{Vortex}} = \frac{4 \mu_0 J_c \lambda_J}{\pi} = \frac{2 \Phi_0}{\pi^2 \lambda_J t_B} \]

\[ B_{c1} \approx \text{magnetic flux density of a single flux quantum distributed over an area } t_B \cdot \lambda_J \]

\[ \frac{\partial \varphi}{\partial z} = \frac{2 \pi}{\Phi_0} B_y t_B \]

\[ B_y(z) = \frac{\Phi_0}{\pi \lambda_L t_B} \frac{1}{\cosh \left( \frac{z}{\lambda_J} \right)} \]

\[ \frac{\partial \varphi}{\partial z} = \frac{2}{\lambda_J \cosh \left( \frac{z}{\lambda_J} \right)} \]

\[ E_{\text{Vortex}} > 0 \Rightarrow \text{we need: external field and }/\text{or current to supply energy} \]
2.3.3 Junction Types and Boundary Conditions

- **Junction geometry** determines current flow \(\Rightarrow\) **Boundary conditions** of SSGE
- Magnetic flux density at junction edges:

\[
\begin{align*}
\frac{\partial \varphi}{\partial z} \bigg|_{z=0} &= \frac{2\pi t_B}{\Phi_0} B_y \bigg|_{z=0} \\
\frac{\partial \varphi}{\partial y} \bigg|_{y=0} &= -\frac{2\pi t_B}{\Phi_0} B_z \bigg|_{y=0} \\
\frac{\partial \varphi}{\partial z} \bigg|_{z=L} &= \frac{2\pi t_B}{\Phi_0} B_y \bigg|_{z=L} \\
\frac{\partial \varphi}{\partial y} \bigg|_{y=W} &= -\frac{2\pi t_B}{\Phi_0} B_z \bigg|_{y=W}
\end{align*}
\]

- **Problem:** \(B = B^\text{ex} + B^\text{el}\)
  - \(B^\text{el}\) not negligible and **junction geometries are complicated**
  - Current distribution in electrodes depends on current distribution in JJ itself
    - Boundary conditions depend on solution
    - **Numerical iteration method required** !!

- Three basic types of junction geometries:
  - Overlap junction
  - In-line junction
  - Grain boundary junctions
2.3.3 Junction Types and Boundary Conditions

A: overlap junction
- overlap of width $W$
- junction of length $L$ extends in $z$-direction perpendicular to current flow
- $B^e$ is parallel to $z \rightarrow \perp$ to the short side
$\Rightarrow \Phi^e = B^e W t_B$ negligibly small

B: inline junction
- overlap of length $L$
- junction of length $L$ extends in $z$-direction parallel to current flow
- $B^e$ is perpendicular to $z \rightarrow \perp$ to the long side
$\Rightarrow \Phi^e = B^e L t_B$ is not negligible
2.3.3 Junction Types and Boundary Conditions

C: grain boundary junction

- mixture of overlap and inline geometry
- junction area is perpendicular to electrode currents
- junction area extends in yz-plane
  perpendicular to current flow
- both y- and z-component of $B_{el}$ are in junction plane
- $B_{z}^{el}$ has negligible impact since $W \ll L$
  $\Phi_{z}^{el} = B_{el}Wt_{B} \ll \Phi_{y}^{el} = B_{el}Lt_{B}$
- finite inline admixture $s = W/L \ll 1$

HTS bicrystal grain boundary junction
2.3.3 Junction Types and Boundary Conditions

**overlap junction:**

**inline junction:**

**asymmetric inline junction:**

highest zero field $I_s^m$ ($\propto$ junction area $A_i = LW$)

$I_s^m$ saturates at $4J_c W\lambda_j$ (Meißner screening, current flow at edges)

fields generated in bottom and top electrode point in the same direction

→ adds to/subtracts from external field

→ increase or decrease of $I_s^m$ with field
Summary (short junctions)

Josephson penetration depth

$$\lambda_J \equiv \sqrt{\frac{\Phi_0}{2\pi \mu_0 t_B J_c}}$$

(Josephson coupling energy)

$$E_J = \frac{\Phi_0 I_c}{2\pi} (1 - \cos \varphi) = E_{J0}(1 - \cos \varphi)$$

(Nonlinear inductance)

$$L_s = \frac{\Phi_0}{2\pi I_c \cos \varphi} = L_c \frac{1}{\cos \varphi} \quad L_c = \frac{\hbar}{2eI_c}$$

(Washboard potential)

$$E_{pot}(\varphi) = E_J(\varphi) - \frac{\Phi_0}{2\pi} I = E_{J0} \left[ 1 - \cos \varphi - \frac{I}{I_c} \varphi \right]$$

(In-plane magnetic field $B_y$)

$$\frac{\partial \varphi}{\partial z} = \frac{2\pi}{\Phi_0} B_y t_B$$

(Spatial oscillations of the Josephson current density)

$$\Rightarrow J_s(y, z, t) = J_c(y, z) \sin \left( \frac{2\pi}{\Phi_0} t_B B_y z + \varphi_0 \right) = J_c(y, z) \sin (kz + \varphi_0)$$

(Integral Josephson current $\rightarrow$ FT of $i_c(z)$)

$$\Rightarrow I_s^m(B_y) = \left| \int_{-\infty}^{\infty} i_c(z)e^{ikz} \, dz \right| \quad i_c(z) = \int_{-W/2}^{W/2} J_c(y, z) \, dy$$
Summary (long junctions)

SSGE: spatial distribution of gauge invariant phase difference:

\[
\frac{\partial^2 \varphi(y, z)}{\partial y^2} + \frac{\partial^2 \varphi(y, z)}{\partial z^2} = \frac{2\pi \mu_0}{\Phi_0} \frac{t_B J_c}{\lambda_j^2} \sin \varphi(y, z) = \frac{1}{\lambda_j^2} \sin \varphi(y, z)
\]

self-consistent solution: boundary conditions depend on flux density at edges

3 basic types: inline, overlap and grain boundary junctions

particular solution of SSGE: Josephson vortex

\[
\varphi(z) = \pm 4 \arctan \left\{ \exp \left( \frac{z - z_0}{\lambda_j} \right) \right\} + 2\pi n
\]