Chapter 8

Microwave Applications

wide application field

- ***passive microwave devices***
  - filters
  - resonators

  → based on the very small losses due to the small microwave surface resistance of superconducting materials
  → improved performance and/or smaller device size

- ***active microwave devices***
  - microwave sources
  - mixers

  → Josephson junctions as voltage controlled oscillators: $\frac{f}{V} = 483\ 597.9$ GHz/V
  → Josephson junctions as nonlinear elements

8.1 High Frequency Properties of SCs

- we make use of the two-fluid model of superconductivity

total carrier density $n = n_n + n_s/2$ (factor ½ due to Cooper pairs)

- transport properties of ***normal fluid***:

Ohm‘s law $J_n = \sigma_n E = \frac{1}{\rho_n} E$

$$\sigma_n = \frac{n_n e^2 \tau}{m}$$ (normal conductivity)

- transport properties of ***superfluid***:

1. London equation (linearized) $$\frac{\partial}{\partial t}(\Lambda \mathbf{J}_s) = \frac{\partial}{\partial t}(\mu_0 \lambda_L^2 \mathbf{J}_s) = \mathbf{E}$$

$$\Lambda \equiv \frac{m_s}{n_s {q_s}^2}$$ (London coefficient)

$$\lambda_L \equiv \sqrt{\frac{m_s}{\mu_0 n_s q^2}}$$ (London penetration depth)

8.1.1 AC Conductivity

normal fluid component

- we assume a harmonic current with angular frequency ω:

with $\mathbf{J}_n = n_n e \mathbf{v}_n$ and $m_n \left(\frac{d\mathbf{v}_n}{dt} + \frac{\mathbf{v}_n}{\tau} \right) \;=\; e\mathbf{E}$ we obtain

$$\mathbf{J}_n \;=\; \left(\frac{n_n e^2}{m_n} \right) \frac{\tau}{1 + \imath\omega\tau} \mathbf{E} \;=\; \sigma_n \mathbf{E}$$

➔ complex normal conductivity:

$$\sigma_n \;=\; \sigma_{n1} - \imath\sigma_{n2} \;=\; \left(\frac{n_n e^2 \tau}{m_n} \right) \frac{1 - \imath\omega\tau}{1 + (\omega\tau)^2} \;=\; \sigma_0 \frac{n_n}{n} \frac{1 - \imath\omega\tau}{1 + (\omega\tau)^2}$$

with $\sigma_0 = \frac{ne^2\tau}{m}$ (normal state conductivity: $n_n = n$)

8.1.1 AC Conductivity

superfluid component

- we assume a harmonic supercurrent with angular frequency ω:

from $\frac{\partial}{\partial t}(\Lambda \mathbf{J}_s) = \frac{\partial}{\partial t}(\mu_0 \lambda_L^2 \mathbf{J}_s) = \mathbf{E}$ we obtain

$$\imath\omega\Lambda\, \mathbf{J}_s = \imath\omega\mu_0\lambda_L^2\, \mathbf{J}_s = \frac{1}{\sigma_s}\mathbf{J}_s = \mathbf{E}$$

➔ purely complex conductivity of superfluid:

$$\sigma_s = \frac{n_s q_s^2}{\imath\omega m_s} = \frac{1}{\imath\omega\Lambda} = \frac{1}{\imath\omega\mu_0\lambda_L^2}$$

- note that the conductivities σ_n and σ_s are strongly temperature dependent

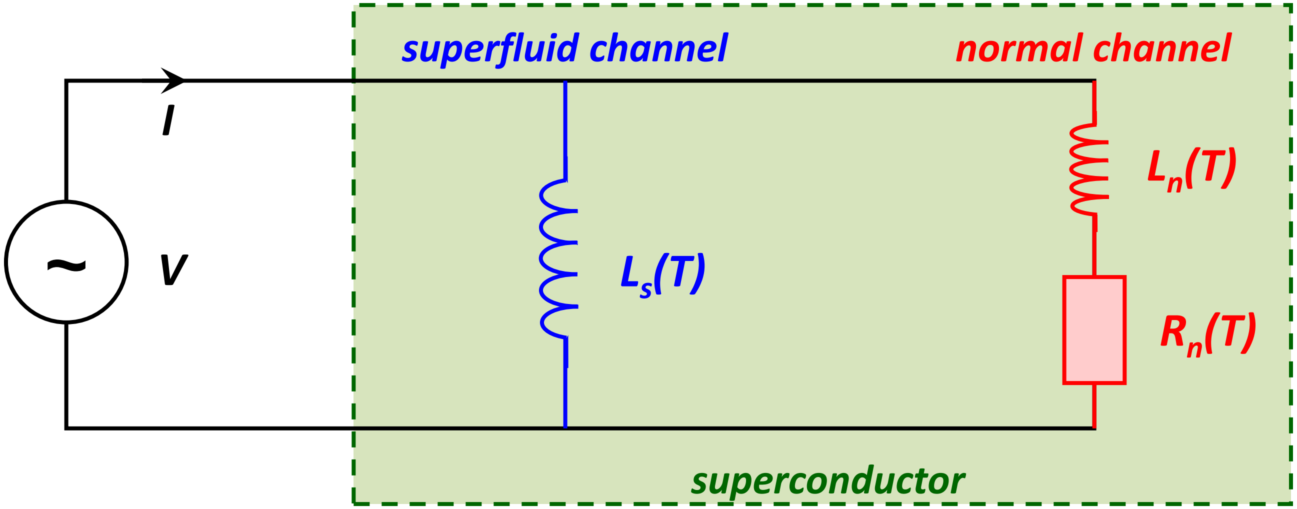
$$\frac{n_n}{n} = \left(\frac{T}{T_c}\right)^4$$

$$\frac{n_s}{2n} = 1 - \left(\frac{T}{T_c}\right)^4$$

8.1.1 AC Conductivity

Two-fluid picture in electrotechnical language

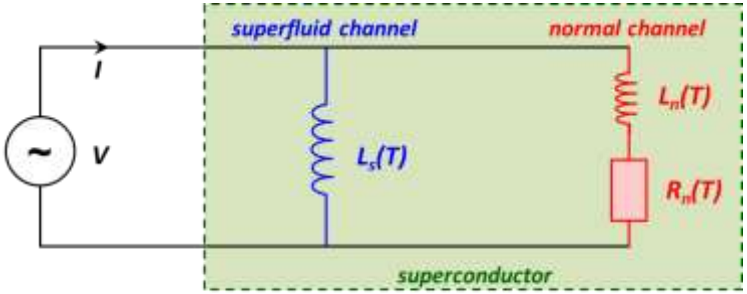
two parallel conduction channels



no resistive part (no loss) purely inductive part (inertia) $L_s(T)$	both resistive (loss) and inductive part (inertia) $L_n(T), R_n(T)$

8.1.1 AC Conductivity

Two-fluid picture in electrotechnical language



superfluid channel dominates at low frequencies

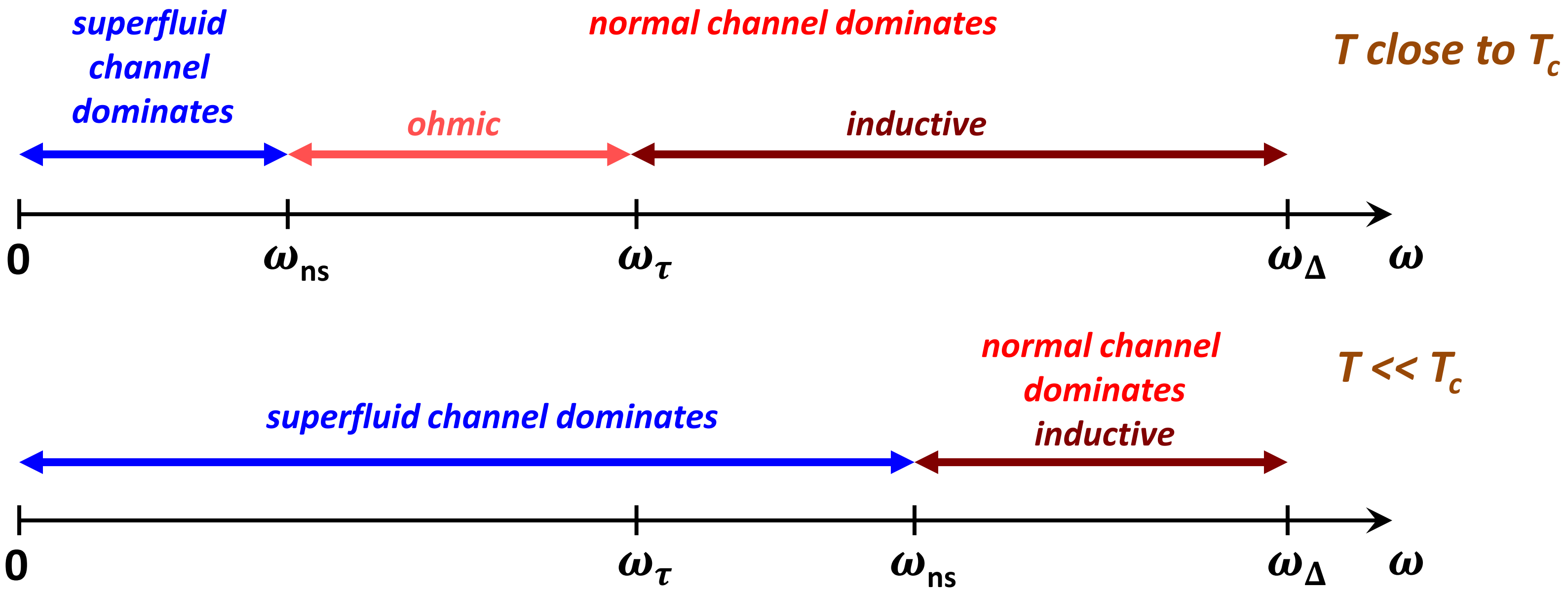
- $\omega = 0$: all current flows through superfluid channel
- $\omega > 0$: J_s decreases and J_n increases
- $\omega = \omega_{ns} = R_n/L_s$: $J_s \simeq J_n$ ➔ crossover frequency

normal channel dominates at high frequencies

- $\omega_{ns} < \omega < \omega_\tau = \frac{R_n}{L_n} = \frac{1}{\tau}$: resistive contribution dominates in normal channel
- $\omega_\tau < \omega < \omega_\Delta$: inductive contribution dominates in normal channel
- $\omega_\Delta < \omega$: above the gap frequency Cooper pairs can be broken up ➔ complicated behavior

8.1.1 AC Conductivity

crossover frequencies depend on temperature since n_n and n_s depend on *T*



8.1.1 AC Conductivity

total conductivity

- normal component $\sigma_n = \sigma_{n1} - \imath\sigma_{n2} = \left(\frac{n_n e^2 \tau}{m_n}\right) \frac{1 - \imath\omega\tau}{1 + (\omega\tau)^2} = \sigma_0 \frac{n_n}{n} \frac{1 - \imath\omega\tau}{1 + (\omega\tau)^2}$

- superfluid component $\sigma_s = \frac{n_s q_s^2}{\imath\omega m_s} = \frac{1}{\imath\omega\Lambda} = \frac{1}{\imath\omega\mu_0\lambda_L^2}$

$$\Rightarrow \quad \sigma = \sigma_s + \sigma_n = \frac{n_n e^2 \tau}{m_n} \frac{1}{1 + (\omega\tau)^2} - \imath \frac{n_n e^2 \tau}{m_n} \frac{\omega\tau}{1 + (\omega\tau)^2} - \imath \frac{1}{\omega\mu_0\lambda_L^2}$$

- $\omega\tau \ll 1$ (low-frequency approximation):

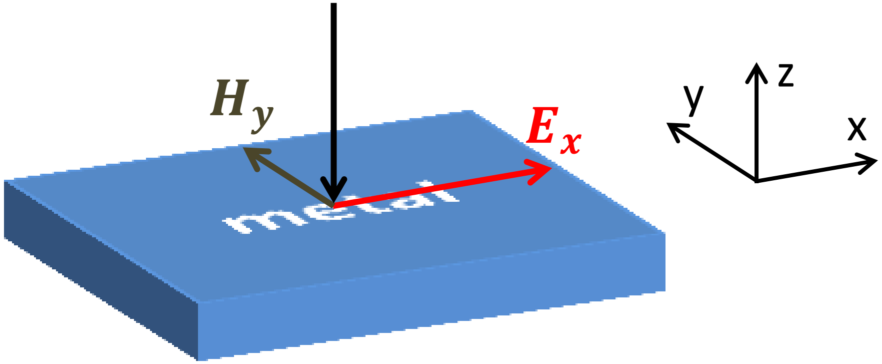
$$\sigma = \sigma_1 - \imath\sigma_2 = \frac{n_n e^2 \tau}{m_n} - \imath \frac{1}{\omega\mu_0\lambda_L^2}$$

resistive losses in normal component

inductive response due to inertia of Cooper pairs

8.1.2 Surface Impedance

- ***surface impedance***
  - characteristic impedance seen by a plane wave incident perpendicular upon a flat surface of a conductor
  - given by the ratio of the electric and the magnetic field at the surface



$$Z_s = R_s + \imath X_s \equiv \frac{E_t}{H_t}$$

$$E_x(z) = E_x(0) \mathrm{e}^{-z/\delta} \, \mathrm{e}^{-\imath z/\delta}$$

$$\delta(\omega) = \sqrt{\frac{2}{\mu_0 \omega \sigma(\omega)}}$$ (skin depth)

with $\nabla \times E = \partial B / \partial t$

$$-\imath \omega \mu_0 H_y(z) = -\frac{E_x(z)}{\delta} - \imath \frac{E_x(z)}{\delta} \Rightarrow Z_s = \frac{E_x}{H_y} = \frac{\mu_0 \omega \delta}{2}(1+\imath) = \sqrt{\frac{\mu_0 \omega}{2\sigma}}(1+\imath)$$

8.1.2 Surface Impedance

- *surface impedance of a normal metal*

for $\omega\tau \ll 1$, we have $\sigma(\omega) \simeq \frac{ne^2\tau}{m} = \sigma_0$ → real number

$$R_s = X_s = \sqrt{\frac{\mu_0\omega}{2\sigma_0}} = \sqrt{\frac{\mu_0\omega\delta_0}{2}}$$

➔ R_s and X_s are proportional to $\sqrt{\omega}$!!

example:

Au or Cu @t 100 GHz and room temperature
$\delta \simeq 0.25\ \mu m$ and $Z_s \simeq 0.1\ (1+i)\ \Omega/\boxdot$

8.1.2 Surface Impedance

- *surface impedance of a superconductor*

use $\sigma = \sigma_s + \sigma_n = \frac{n_n e^2 \tau}{m_n}\frac{1}{1+(\omega\tau)^2} - i\frac{n_n e^2 \tau}{m_n}\frac{\omega\tau}{1+(\omega\tau)^2} - i\frac{1}{\omega\mu_0\lambda_L^2}$

$$Z_s = \sqrt{\frac{\mu_0\omega}{2\sigma}}(1+i)$$

$$Z_s = R_s + iX_s = \sqrt{\frac{i\mu_0\omega}{\sigma}} = \left[\frac{\sigma_n\frac{1}{1+(\omega\tau)^2} - i\sigma_n\frac{\omega\tau}{1+(\omega\tau)^2} - i\frac{1}{\omega\mu_0\lambda_L^2}}{i\omega\mu_0}\right]^{-1/2}$$

$\sqrt{2i} = (1+i)$

$$= i\omega\mu_0\left[i\omega\mu_0\sigma_n\frac{1}{1+(\omega\tau)^2} + \omega\mu_0\sigma_n\frac{\omega\tau}{1+(\omega\tau)^2} + \frac{1}{\lambda_L^2}\right]^{-1/2}$$

for $\omega\tau \ll 1$, and using $\sigma = \sigma_1 - i\sigma_2$ with $\sigma_1 = \sigma_n$ and $\sigma_2 = 1/\omega\mu_0\lambda_L^2$

$$Z_s = \frac{i\omega\mu_0}{\lambda_L}\left[1 + i\omega\mu_0\lambda_L^2\sigma_n\right]^{-1/2} = i\sqrt{\frac{\omega\mu_0}{\sigma_2}}\left[1 + i\frac{\sigma_1}{\sigma_2}\right]^{-1/2}$$

8.1.2 Surface Impedance

$$Z_s = \frac{\imath\omega\mu_0}{\lambda_\mathrm{L}}\left[1+\imath\omega\mu_0\lambda_\mathrm{L}^2\sigma_n\right]^{-1/2} = \imath\sqrt{\frac{\omega\mu_0}{\sigma_2}}\left[1+\imath\frac{\sigma_1}{\sigma_2}\right]^{-1/2}$$

further simplification at $T \ll T_c$: $\sigma_s \ll \sigma_2$ → approximation $(1+x)^{-1/2} \simeq 1-\frac{1}{2}x$

$$Z_s = \imath\sqrt{\frac{\omega\mu_0}{\sigma_2}}\left(1-\imath\frac{\sigma_1}{2\sigma_2}\right) = \sqrt{\frac{\omega\mu_0\sigma_1^2}{2\sigma_2^3}}+\imath\sqrt{\frac{\omega\mu_0}{\sigma_2}}$$

with $\sigma_1 = n_n e^2\tau/m_n$ and $\sigma_2 = 1/\omega\mu_0\lambda_L^2$

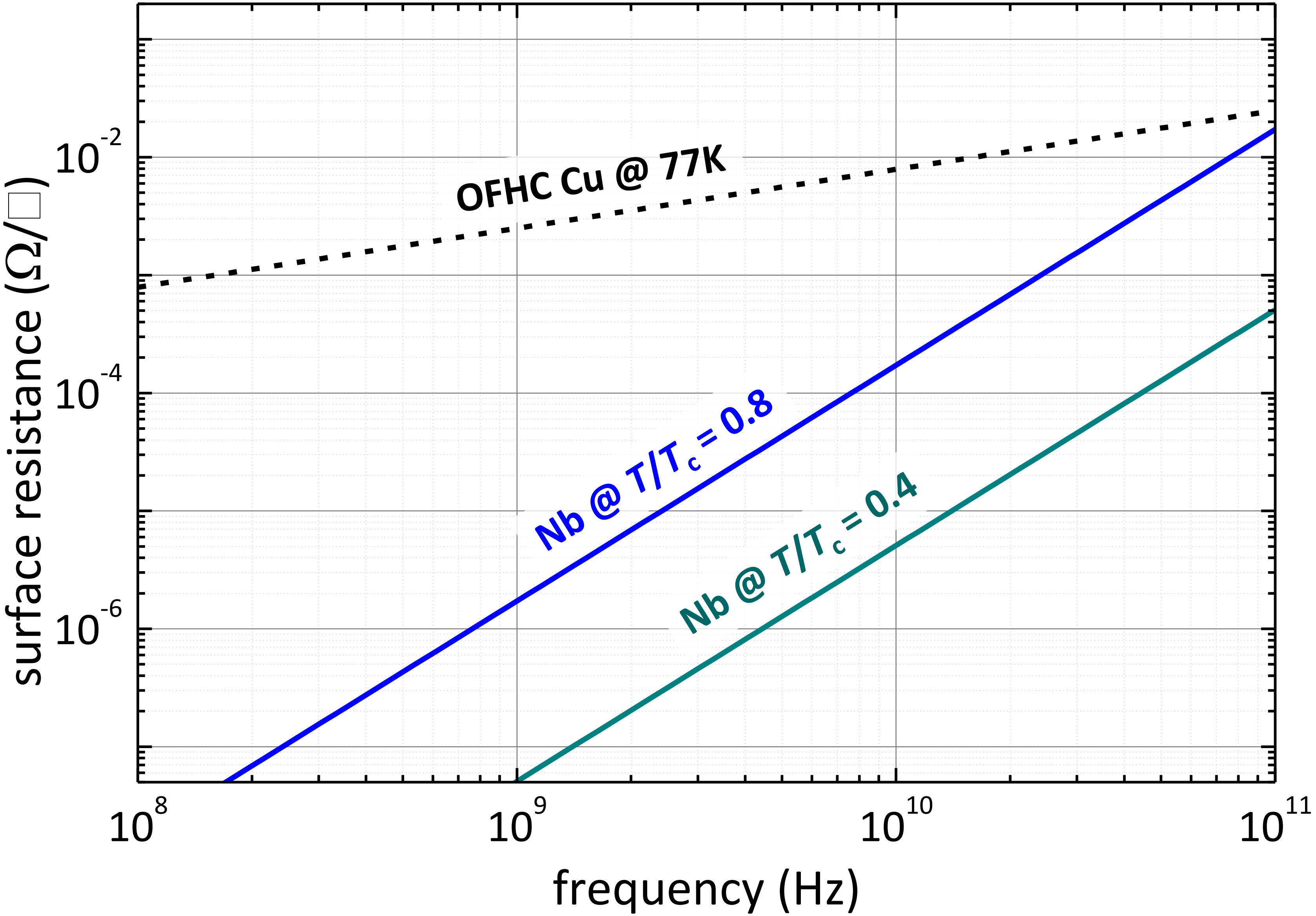
$$Z_s = R_s + \imath X_s = \frac{\omega^2\mu_0^2\lambda_\mathrm{L}^3 n_n e^2\tau}{2m_n} + \imath\omega\mu_0\lambda_\mathrm{L} = \frac{1}{2}\omega^2\mu_0^2\lambda_\mathrm{L}^3\sigma_0\left(\frac{n_n}{n}\right) + \imath\omega\mu_0\lambda_\mathrm{L}$$

➔ $R_s \propto \omega^2\lambda_L^3\left(\frac{n_n}{n}\right)$ and $X_s \propto \omega\lambda_L$!! → different from normal metals: $R_s \propto \sqrt{\omega}$

example:

Nb @ 100 GHz and $T \ll T_c$, $Z_s \simeq i\omega\mu_0\lambda_L$
$\lambda_L \simeq 0.1\ \mu m$ and $Z_s \simeq i\ 0.08\ \Omega/\boxdot$

8.1.2 Surface Impedance



8.1.2 Surface Impedance

kinetic inductance

for $\omega\tau \ll 1$ and $T \ll T_c$:

$$Z_s = R_s + \imath X_s = \frac{1}{2}\omega^2\mu_0^2\lambda_L^3\sigma_0\left(\frac{n_n}{n}\right) + \imath\omega\mu_0\lambda_L$$

- surface reactance X_s is purely inductive
  ➔ the equivalent inductance, $X_s = i\omega L_k$, is denoted as ***kinetic inductance***

$$L_k = \mu_0\lambda_L$$

L_k reflects the kinetic energy of the superfluid
(acceleration of superfluid requires energy)