Chapter 8

Microwave Applications
8 Microwave Applications

wide application field

• *passive microwave devices*
  – filters
  – resonators
    → based on the very small losses due to the small microwave surface resistance of superconducting materials
    → improved performance and/or smaller device size

• *active microwave devices*
  – microwave sources
  – mixers
    → Josephson junctions as voltage controlled oscillators: \( \frac{f}{V} = 483\,597.9 \text{ GHz/V} \)
    → Josephson junctions as nonlinear elements
8.1 High Frequency Properties of SCs

- we make use of the two-fluid model of superconductivity

  total carrier density \( n = n_n + n_s / 2 \)  \( \text{(factor} \ ½ \text{ due to Cooper pairs)} \)

- transport properties of \textit{normal fluid}:

  Ohm’s law

  \[
  J_n = \sigma_n E = \frac{1}{\rho_n} E
  \]

  \[
  \sigma_n = \frac{n_n e^2 \tau}{m} \quad \text{(normal conductivity)}
  \]

- transport properties of \textit{superfluid}:

  1. London equation (linearized)

  \[
  \frac{\partial}{\partial t} (\Lambda J_s) = \frac{\partial}{\partial t} (\mu_0 \lambda_L^2 J_s) = E
  \]

  \[
  \Lambda \equiv \frac{m_s}{n_s q_s^2} \quad \text{(London coefficient)}
  \]

  \[
  \lambda_L \equiv \sqrt{\frac{m_s}{\mu_0 n_s q_s^2}} \quad \text{(London penetration depth)}
  \]
8.1.1 AC Conductivity

**normal fluid component**

- we assume a harmonic current with angular frequency $\omega$:

$$\text{with } J_n = n_ne_v \text{ and } m_n\left(\frac{dv_n}{dt} + \frac{v_n}{\tau}\right) = eE$$

we obtain

$$J_n = \left(\frac{n_ne^2}{m_n}\right) \frac{\tau}{1 + i\omega\tau} E = \sigma_n E$$

⇒ complex normal conductivity:

$$\sigma_n = \sigma_{n1} - i\sigma_{n2} = \left(\frac{n_ne^2\tau}{m_n}\right) \frac{1 - i\omega\tau}{1 + (\omega\tau)^2} = \sigma_0 \frac{n_n}{n} \frac{1 - i\omega\tau}{1 + (\omega\tau)^2}$$

with $\sigma_0 = \frac{ne^2\tau}{m}$

(normal state conductivity: $n_n = n$)
8.1.1 AC Conductivity

**Superfluid Component**

- We assume a harmonic supercurrent with angular frequency $\omega$:

  \[
  \frac{\partial}{\partial t} (\wedge \mathbf{J}_s) = \frac{\partial}{\partial t} (\mu_0 \lambda_L^2 \mathbf{J}_s) = \mathbf{E}
  \]

  from

  \[
  i\omega \wedge \mathbf{J}_s = i\omega \mu_0 \lambda_L^2 \mathbf{J}_s = \frac{1}{\sigma_s} \mathbf{J}_s = \mathbf{E}
  \]

  → purely complex conductivity of superfluid:

  \[
  \sigma_s = \frac{n_s q_s^2}{i\omega m_s} = \frac{1}{i\omega \wedge} = \frac{1}{i\omega \mu_0 \lambda_L^2}
  \]

- Note that the conductivities $\sigma_n$ and $\sigma_s$ are strongly temperature dependent

  \[
  \frac{n_n}{n} = \left( \frac{T}{T_c} \right)^4
  \]

  \[
  \frac{n_s}{2n} = 1 - \left( \frac{T}{T_c} \right)^4
  \]
8.1.1 AC Conductivity

Two-fluid picture in electrotechnical language

two parallel conduction channels

\[ I \]
\[ V \]

superfluid channel

no resistive part
(no loss)
purely inductive part
(inertia)
\[ L_s(T) \]

normal channel

both resistive
(loss)
and inductive part
(inertia)
\[ L_n(T), R_n(T) \]

superconductor
8.1.1 AC Conductivity

Two-fluid picture in electrotechnical language

- **superfluid channel** dominates at low frequencies
  - $\omega = 0$: all current flows through superfluid channel
  - $\omega > 0$: $J_s$ decreases and $J_n$ increases
  - $\omega = \omega_{ns} = R_n/L_s$: $J_s \approx J_n \Rightarrow$ crossover frequency

- **normal channel** dominates at high frequencies
  - $\omega_{ns} < \omega < \omega_\tau = \frac{R_n}{L_n} = \frac{1}{\tau}$: resistive contribution dominates in normal channel
  - $\omega_\tau < \omega < \omega_\Delta$: inductive contribution dominates in normal channel
  - $\omega_\Delta < \omega$: above the gap frequency Cooper pairs can be broken up $\Rightarrow$ complicated behavior
8.1.1 AC Conductivity

crossover frequencies depend on temperature since $n_n$ and $n_s$ depend on $T$
8.1.1 AC Conductivity

**total conductivity**

- normal component

\[ \sigma_n = \sigma_{n1} - i\sigma_{n2} = \left( \frac{n_ne^2\tau}{m_n} \right) \frac{1 - i\omega\tau}{1 + (\omega\tau)^2} = \sigma_0 \frac{n_n}{n} \frac{1 - i\omega\tau}{1 + (\omega\tau)^2} \]

- superfluid component

\[ \sigma_s = \frac{n_sq_s^2}{i\omega m_s} = \frac{1}{i\omega\Lambda} = \frac{1}{i\omega\mu_0\lambda_L^2} \]

\[ \sigma = \sigma_s + \sigma_n = \frac{n_ne^2\tau}{m_n} \frac{1}{1 + (\omega\tau)^2} - i \frac{n_ne^2\tau}{m_n} \frac{\omega\tau}{1 + (\omega\tau)^2} - i \frac{1}{\omega\mu_0\lambda_L^2} \]

- \( \omega\tau \ll 1 \) (low-frequency approximation):

\[ \sigma = \sigma_1 - i\sigma_2 = \frac{n_ne^2\tau}{m_n} - i \frac{1}{\omega\mu_0\lambda_L^2} \]

resistive losses in normal component

inductive response due to inertia of Cooper pairs
8.1.2 Surface Impedance

- **surface impedance**
  - characteristic impedance seen by a plane wave incident perpendicular upon a flat surface of a conductor
  - given by the ratio of the electric and the magnetic field at the surface

\[ Z_s = R_s + iX_s \equiv \frac{E_t}{H_t} \]

\[ E_x(z) = E_x(0)e^{-z/\delta} e^{-iz/\delta} \quad \delta(\omega) = \sqrt{\frac{2}{\mu_0 \omega \sigma(\omega)}} \quad \text{(skin depth)} \]

with \( \nabla \times E = \partial B / \partial t \)

\[ -i \omega \mu_0 H_y(z) = -\frac{E_x(z)}{\delta} - i \frac{E_x(z)}{\delta} \quad Z_s = \frac{E_x}{H_y} = \frac{\mu_0 \omega \delta}{2} (1 + i) = \sqrt{\frac{\mu_0 \omega}{2\sigma}} (1 + i) \]
8.1.2 Surface Impedance

- *surface impedance of a normal metal*

for $\omega \tau \ll 1$, we have $\sigma(\omega) \approx \frac{ne^2 \tau}{m} = \sigma_0 \rightarrow$ real number

$$R_s = X_s = \sqrt{\frac{\mu_0 \omega}{2\sigma_0}} = \sqrt{\frac{\mu_0 \omega \delta_0}{2}}$$

$\Rightarrow R_s$ and $X_s$ are proportional to $\sqrt{\omega}$ !!

example:

Au or Cu @ 100 GHz and room temperature
$\delta \approx 0.25 \, \mu m$ and $Z_s \approx 0.1 \ (1 + i) \, \Omega/\square$
8.1.2 Surface Impedance

- **Surface impedance of a superconductor**

\[
\sigma = \sigma_s + \sigma_n = \frac{n_v e^2 \tau}{m_n} \frac{1}{1 + (\omega \tau)^2} - \frac{1}{m_n} \frac{\omega \tau}{1 + (\omega \tau)^2} - \frac{1}{\omega \mu_0 \lambda_L^2}
\]

\[
Z_s = \sqrt{\frac{\mu_0 \omega}{2\sigma}} (1 + i)
\]

\[
Z_s = R_s + iX_s = \sqrt{\frac{i\mu_0 \omega}{\sigma}} = \left[ \frac{\sigma_n}{1+(\omega \tau)^2} - i\sigma_n \frac{\omega \tau}{1+(\omega \tau)^2} - i\frac{1}{\omega \mu_0 \lambda_L^2} \right]^{-1/2}
\]

\[
\sqrt{2i} = (1 + i)
\]

\[
= i\omega \mu_0 \left[ i\omega \mu_0 \sigma_n \frac{1}{1 + (\omega \tau)^2} + \omega \mu_0 \sigma_n \frac{\omega \tau}{1 + (\omega \tau)^2} + \frac{1}{\lambda_L^2} \right]^{-1/2}
\]

For \(\omega \tau \ll 1\), and using \(\sigma = \sigma_1 - i\sigma_2\) with \(\sigma_1 = \sigma_n\) and \(\sigma_2 = 1/\omega \mu_0 \lambda_L^2\)

\[
Z_s = \frac{i\omega \mu_0}{\lambda_L} \left[ 1 + i\omega \mu_0 \lambda_L^2 \sigma_n \right]^{-1/2} = i \sqrt{\frac{\omega \mu_0}{\sigma_2}} \left[ 1 + i\frac{\sigma_1}{\sigma_2} \right]^{-1/2}
\]
8.1.2 Surface Impedance

\[ Z_s = \frac{i \omega \mu_0}{\lambda_L} \left[ 1 + i \omega \mu_0 \lambda_L^2 \sigma_n \right]^{-1/2} = i \sqrt{\frac{\omega \mu_0}{\sigma_2}} \left[ 1 + i \frac{\sigma_1}{\sigma_2} \right]^{-1/2} \]

Further simplification at \( T \ll T_c \): \( \sigma_s \ll \sigma_2 \) \( \rightarrow \) approximation \((1 + x)^{-1/2} \approx 1 - \frac{1}{2} x\)

\[ Z_s = i \sqrt{\frac{\omega \mu_0}{\sigma_2}} \left( 1 - i \frac{\sigma_1}{2 \sigma_2} \right) = \sqrt{\frac{\omega \mu_0 \sigma_1^2}{2 \sigma_2^3}} + i \sqrt{\frac{\omega \mu_0}{\sigma_2}} \]

with \( \sigma_1 = n_n e^2 \tau / m_n \) and \( \sigma_2 = 1 / \omega \mu_0 \lambda_L^2 \)

\[ Z_s = R_s + i X_s = \frac{\omega^2 \mu_0^2 \lambda_L^3 n_n e^2 \tau}{2 m_n} + i \omega \mu_0 \lambda_L = \frac{1}{2} \frac{\omega^2 \mu_0^2 \lambda_L^3 \sigma_0 \left( \frac{n_n}{n} \right)}{i} + i \omega \mu_0 \lambda_L \]

\( R_s \propto \omega^2 \lambda_L^3 \left( \frac{n_n}{n} \right) \) and \( X_s \propto \omega \lambda_L \) \( \rightarrow \) different from normal metals: \( R_s \propto \sqrt{\omega} \)

Example:

\( \text{Nb} \) @ 100 GHz and \( T \ll T_c \), \( Z_s \approx i \omega \mu_0 \lambda_L \)
\( \lambda_L \approx 0.1 \mu m \) and \( Z_s \approx i 0.08 \Omega/cm \)
8.1.2 Surface Impedance

![Graph showing the relationship between surface resistance and frequency for different materials and temperatures. The graph includes lines for OFHC Cu at 77K, Nb at T/T_c = 0.8, and Nb at T/T_c = 0.4. The x-axis represents frequency (Hz) ranging from 10^8 to 10^11, and the y-axis represents surface resistance (Ω/□) ranging from 10^-6 to 10^-2.]
8.1.2 Surface Impedance

**kinetic inductance**

for $\omega \tau \ll 1$ and $T \ll T_c$:

$$Z_s = R_s + iX_s = \frac{1}{2} \omega^2 \mu_0^2 \lambda_L^3 \sigma_0 \left( \frac{n_n}{n} \right) + i\omega \mu_0 \lambda_L$$

- surface reactance $X_s$ is purely inductive
  - the equivalent inductance, $X_s = i\omega L_k$, is denoted as **kinetic inductance**

$$L_k = \mu_0 \lambda_L$$

$L_k$ reflects the kinetic energy of the superfluid
- (acceleration of superfluid requires energy)