Chapter 2

Physics of Josephson Junctions:

The Zero Voltage State
2.1 Basic properties of lumped Josephson junctions

Small spatial dimensions:

→ Gauge invariant phase diff. & current density are uniform

→ variations of supercurrent density on length scale larger than $\lambda_L$

$$\lambda_L = \sqrt{\frac{m_s}{\mu_0 n_s q_s^2}} \approx 10^{-2} \mu m - 1 \mu m$$

Josephson junction: $n_s$ strongly reduced:

$$\lambda_L \rightarrow \lambda_J \approx 10 \mu m - 100 \mu m \text{ (Josephson penetration depth)}$$
2.1 Basic properties of lumped Josephson junctions

2.1.1 The Lumped Josephson Junction

Spatially homogeneous supercurrent density and phase difference → Lumped element JJ

\[ I_s = \int_S J_s \cdot ds \]  
Region of integration is junction area S

Current-phase relation

\[ I_s(t) = I_c \sin \varphi(t) \]

Gauge invariant phase difference

\[ \varphi(t) = \theta_2(t) - \theta_1(t) - \frac{2\pi}{\Phi_0} \int_1^2 A(r, t) \cdot dl \]

2-nd Josephson relation

\[ \frac{\partial \varphi}{\partial t} = \frac{2\pi}{\Phi_0} \int_1^2 E(r, t) \cdot dl \Rightarrow \frac{d\varphi}{dt} = \frac{2\pi}{\Phi_0} V \]

Uniform phase difference → Total derivative

„0“-junction

\[ I = I_c \sin \varphi \]

\[ V = \frac{\Phi_0}{2\pi} \frac{d\varphi}{dt} \]

„π“-junction

\[ I = I_c \sin (\varphi + \pi) \]

\[ V = \frac{\Phi_0}{2\pi} \frac{d\varphi}{dt} \]
2.1.2 The Josephson coupling energy

Finite energy stored in JJ: overlap of macroscopic wave functions $\rightarrow$ Binding energy

Initial current & phase difference $\rightarrow$ Zero

Increase junction current from zero to a finite value
$\rightarrow$ Phase difference has to change
$\rightarrow$ 2-nd Josephson relation: finite-voltage state in a junction
$\rightarrow$ External source has to supply energy (to accelerate the superelectrons)
$\rightarrow$ Stored kinetic energy of moving superelectrons
$\rightarrow$ Integral of the power $= I_s V$ (voltage during current increase)

$$E_J = \int_0^{t_0} I_s V \, dt = \int_0^{t_0} (I_c \sin \bar{\phi}) \left( \frac{\Phi_0}{2\pi} \frac{d\bar{\phi}}{dt} \right) \, dt = \frac{\Phi_0 I_c}{2\pi} \int_0^\varphi \sin \bar{\phi} \, d\bar{\phi}$$

Integration $\rightarrow$

$$E_J = \frac{\Phi_0 I_c}{2\pi} (1 - \cos \varphi) = E_{J0} (1 - \cos \varphi)$$

Josephson coupling energy
2.1.2 The Josephson coupling energy

Josephson coupling energy

\[ E_J = \frac{\Phi_0 I_c}{2\pi} (1 - \cos \varphi) = E_{J0} (1 - \cos \varphi) \]

Order of magnitude

Traditional applications: \( I_c \approx 1 \text{ mA} \) \( \Rightarrow \) \( E_{J0} \approx 3 \times 10^{-19} \text{ J} \approx k_B \times 20000 \text{ K} \)

Quantum circuits: \( I_c \approx 1 \mu \text{ A} \) \( \Rightarrow \) \( E_{J0} \approx 3 \times 10^{-16} \text{ J} \approx k_B \times 20 \text{ K} \)
2.1.3 The superconducting state

\[ |I| < I_c \Rightarrow \text{Constant phase difference } \varphi = \bar{\varphi}_n = \arcsin \frac{I}{I_c} + 2\pi n \]

\[ \Rightarrow \text{Zero junction voltage } \Rightarrow \text{Zero-voltage state / ordinary (S) state} \]

In practice \rightarrow \text{Junction + current source} \rightarrow \text{Stability analysis}

\[ E_{\text{pot}} = E - F \cdot x \Rightarrow \text{Potential energy} \text{ of the system under action of external force} \]

\[ E \Rightarrow \text{Intrinsic free energy of the junction} \]

\[ F \leftrightarrow I \Rightarrow \text{Generalized force} \]

\[ x \Rightarrow \text{Generalized coordinate} \]

\[ F \cdot \frac{\partial x}{\partial t} \leftrightarrow I \cdot V \Rightarrow \text{Power flowing into subsystem} \]

\[ x \leftrightarrow \int V dt = \frac{\hbar}{2e} \varphi + c = \frac{\Phi_0}{2\pi} \varphi + c \]

\[ \Rightarrow \text{Potential energy}: \]

\[ E_{\text{pot}}(\varphi) = E_J - I \left( \frac{\Phi_0}{2\pi} \varphi + c \right) = E_{J0} \left( 1 - \cos \varphi - \frac{I}{I_c} \varphi \right) + \tilde{c} \]

\[ \varphi = \bar{\varphi}_n = \pi - \arcsin \left( \frac{I}{I_c} \right) + 2\pi n \]

\[ \Rightarrow \text{Tilted washboard potential} \]

\[ \text{Stable minima } \varphi_n \]

\[ \text{Unstable maxima } \bar{\varphi}_n \]

\[ \text{Equivalent states for different } n \]
2.1.3 The superconducting state

Properties of the washboard potential

\[ U_0 \equiv E_{\text{pot}}(\phi_{n+1}) - E_{\text{pot}}(\tilde{\phi}_{n+1}) \]

\[ = 2E_{J0} \left[ \sqrt{1 - \left( \frac{l}{I_c} \right)^2} - \frac{l}{I_c} \arccos\left( \frac{l}{I_c} \right) \right] \]

\[ k \equiv \frac{\partial^2 E_{\text{pot}}}{\partial \phi^2} = E_{J0} \sqrt{1 - \left( \frac{l}{I_c} \right)^2} \]

→ 0 for \( l \rightarrow I_c \)

No minima for \( l > I_c \)

Close to \( I_c \) \( \rightarrow \) \( \alpha \equiv 1 - \frac{l}{I_c} \ll 1 \rightarrow \) Simplified approximations

\[ \varphi_0 = \frac{\pi}{2} - \sqrt{2\alpha} \quad \tilde{\varphi}_0 = \frac{\pi}{2} + \sqrt{2\alpha} \quad U_0 = \frac{2}{3}E_{J0} (2\alpha)^{2/3} \quad k = E_{J0} (2\alpha)^{1/2} \]

Washboard potential extremely useful in describing junction dynamics for \( l > I_c \)
2.1.4 The Josephson inductance

Energy storage in JJ $\to$ Nonlinear reactance

$$\frac{dl_s}{dt} = l_c \cos \varphi \frac{d\varphi}{dt} \quad \Rightarrow \quad \frac{dl_s}{dt} = l_c \cos \varphi \frac{2\pi}{\Phi_0} V$$

For small variations near $I_s = I_c \sin \varphi \to$ JJ equivalent to inductance

$$L_s = \frac{\Phi_0}{2\pi I_c \cos \varphi} = L_c \frac{1}{\cos \varphi}$$

with $L_c = \frac{\hbar}{2eI_c}$

Josephson inductance

Properties of the Josephson inductance:

Negative for $\pi/2 + 2\pi n < \varphi < 3\pi/2 + 2\pi n$

($V > 0 \to$ Oscillating Josephson current)
2.1.5 Mechanical analogs

The pendulum analog

- Plane mechanical pendulum in uniform gravitational field
- Mass \( m \), length \( \ell \), deflection angle \( \theta \)
- Torque \( D \) parallel to rotation axis
- Restoring torque: \( mg\ell \sin \theta \)

Equation of motion \( D = \Theta\ddot{\Theta} + \Gamma \dot{\Theta} + mg\ell \sin \Theta \)

\( \Theta = m\ell^2 \) Moment of inertia
\( \Gamma \) Damping constant

Analogies

\[
\begin{align*}
I & \leftrightarrow D \\
I_c & \leftrightarrow mg\ell \\
\frac{\Phi_0}{2\pi R} & \leftrightarrow \Gamma \\
\frac{C\Phi_0}{2\pi} & \leftrightarrow \Theta \\
\varphi & \leftrightarrow \Theta
\end{align*}
\]

For \( D = 0 \) \( \rightarrow \) Oscillations around equilibrium with

\[
\omega = \sqrt{\frac{g}{\ell}} \quad \leftrightarrow \text{Plasma frequency } \omega_p = \sqrt{\frac{2\pi I_c}{\Phi_0 C}}
\]

Finite torque \( (D > 0) \) \( \rightarrow \) Finite \( \theta_0 \) \( \rightarrow \) Finite, but constant \( \varphi_0 \) \( \rightarrow \) Zero-voltage state
### The washboard potential

Particle moving in tilted washboard potential \( E_{\text{pot}}(\phi) = E_{J0} \left( 1 - \cos \phi - \frac{l}{l_c} \phi \right) \)

→ Analogies

<table>
<thead>
<tr>
<th>Coordinate ( x )</th>
<th>( \phi )</th>
<th>( \frac{d\phi}{dt} \propto V )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity ( v )</td>
<td>( \frac{d\phi}{dt} )</td>
<td>( V )</td>
</tr>
<tr>
<td>Mass ( m )</td>
<td>Junction capacitance ( C )</td>
<td></td>
</tr>
</tbody>
</table>
2.2 Short Josephson Junctions

So far: zero-dimensional JJ (lumped elements)
→ Homogeneous supercurrent density and phase difference

Now: extended junctions
→ Spatial variations $J_s(r)$ and $\varphi(r)$
→ Consider magnetic field generated by the Josephson current itself ("self-field")

Short Josephson junctions
→ Self-field small compared to external field

Long Josephson junctions
→ Self-field no longer negligible

Relevant length scale for transition from short to long Josephson junction:

Josephson penetration depth $\lambda_J = \sqrt{\frac{m_s}{\mu_0 n_s q^2_s}} \gg \lambda_L$

density in weak coupling region

JJ at finite voltage → Temporal interference → Oscillation of Josephson current

JJ at finite phase gradient → Spatial interference → Magn. field dep. of Josephson current
2.2.1 Quantum interference effects - Short JJ in applied field

External magnetic field

→ **Spatial** change of gauge invariant phase difference \( \phi(r) \)

→ **Spatial** interference of macroscopic wave functions in JJ

**Specific geometry**

- Insulating barrier thickness \( d \)
- Junction area \( A = L \times W \)
- Edge effects small: \( W, L \gg d \)
- Electrode thickness \( > \lambda_L \)
- Ext. magnetic field \( B_e = (0, B_y, 0) \)
- Magnetic thickness \( t_B = d + \lambda_{L,1} + \lambda_{L,2} \)

Effect of \( B_e \) on \( J_s \)

→ Phase shift \( \phi(P) - \phi(Q) \) between two points \( P \) and \( Q \) separated by \( dz \)

→ **Line integral along red contour** yields total phase change along closed contour
2.2.1 Quantum interference effects - Short JJ in applied field

\[ \int_C \nabla \theta \cdot d\mathbf{l} = (\theta_{Q_b} - \theta_{Q_a}) + (\theta_{P_c} - \theta_{Q_b}) + (\theta_{P_d} - \theta_{P_c}) + (\theta_{Q_a} - \theta_{P_d}) = 0 \]

Gauge invariant phase gradient in the bulk superconductor:

\[ \nabla \theta = \frac{2\pi}{\Phi_0} (\wedge \mathbf{J}_s + \mathbf{A}) \]

Gauge invariant phase difference across the barrier:

\[ \varphi = \theta_2 - \theta_1 - \frac{2\pi}{\Phi_0} \int_1^2 \mathbf{A} \cdot d\mathbf{l} \]

1 & 3 are differences across the junction:

\[ \theta_{Q_b} - \theta_{Q_a} = +\varphi(Q) + \frac{2\pi}{\Phi_0} \int_{Q_a}^{Q_b} \mathbf{A} \cdot d\mathbf{l} \]
\[ \theta_{P_d} - \theta_{P_c} = -\varphi(P) + \frac{2\pi}{\Phi_0} \int_{P_c}^{P_d} \mathbf{A} \cdot d\mathbf{l} \]

2 & 4 differences in the bulk, supercurrent equation for \( \nabla \theta \):

\[ \theta_{P_c} - \theta_{Q_b} = \int_{Q_b}^{P_c} \nabla \theta \cdot d\mathbf{l} = \frac{2\pi}{\Phi_0} \int_{Q_b}^{P_c} \wedge \mathbf{J}_s \cdot d\mathbf{l} + \frac{2\pi}{\Phi_0} \int_{P_c}^{Q_b} \mathbf{A} \cdot d\mathbf{l} \]
\[ \theta_{Q_a} - \theta_{P_d} = \int_{P_d}^{Q_a} \nabla \theta \cdot d\mathbf{l} = \frac{2\pi}{\Phi_0} \int_{Q_a}^{P_d} \wedge \mathbf{J}_s \cdot d\mathbf{l} + \frac{2\pi}{\Phi_0} \int_{P_d}^{Q_a} \mathbf{A} \cdot d\mathbf{l} \]
2.2.1 Quantum interference effects - Short JJ in applied field

Substitution \[ \varphi(Q) - \varphi(P) = \frac{2\pi}{\Phi_0} \oint_A \mathbf{A} \cdot d\mathbf{l} - \frac{2\pi}{\Phi_0} \oint_{P_c} \mathbf{J}_s \cdot d\mathbf{l} - \frac{2\pi}{\Phi_0} \oint_{P_d} \mathbf{J}_s \cdot d\mathbf{l} \]

Integration of \( \mathbf{A} \) around closed contour \( \Rightarrow \) Enclosed flux \( \Phi \)
Integration of \( \mathbf{J}_s \) excludes insulating barrier \( \Rightarrow \) Incomplete contour \( C' \)

\[ \oint_{C'} \mathbf{J}_s \cdot d\mathbf{l} = \oint_{P_c} \mathbf{J}_s \cdot d\mathbf{l} + \oint_{P_d} \mathbf{J}_s \cdot d\mathbf{l} \]

Difference of gauge invariant phase differences \( \varphi(Q) - \varphi(P) \)

\[ \varphi(Q) - \varphi(P) = -\frac{2\pi \Phi}{\Phi_0} - \frac{2\pi}{\Phi_0} \oint_{C'} \mathbf{J}_s \cdot d\mathbf{l} \]

Line integral of supercurrent density \( \mathbf{J}_s \)
\( \Rightarrow \) Segments in \( x \)-direction cancel (separation: \( dz \to 0 \))
\( \Rightarrow \) Segments in \( z \)-direction: deep inside SC (\( > \lambda_L \)) \( \Rightarrow \mathbf{J}_s \) exponentially small

\[ \varphi(P) - \varphi(Q) = \frac{2\pi \Phi}{\Phi_0} \quad \frac{\varphi(P) - \varphi(Q)}{2\pi} = \frac{\Phi}{\Phi_0} \]

Total flux enclosed by the loop \( \Rightarrow \Phi = B_y (d + \lambda_{L1} + \lambda_{L2}) dz = B_y t_B dz \)

Magnetic thickness \( t_B \)
Similar argument for $P$ and $Q$ separated by $dy$ in $y$-direction

\[ \nabla \varphi(r, t) = \frac{2\pi}{\Phi_0} t_B [B(r, t) \times \hat{x}] \]

Integration gives:

\[ \varphi(z) = \frac{2\pi}{\Phi_0} B_y t_B z + \varphi_0 \]

Extended 1-st Josephson (current-phase) relation:

\[ J_s(y, z, t) = J_c(y, z) \sin \left( \frac{2\pi}{\Phi_0} t_B B_y z + \varphi_0 \right) = J_c(y, z) \sin (kz + \varphi_0) \]

with

\[ k = \frac{2\pi}{\Phi_0} t_B B_y \]

$J_s$ varies periodically with period $\Delta z = \frac{2\pi}{k} = \frac{\Phi_0}{t_B B_y}$

Flux through the junction within one period: $\Phi_0$
2.2.2 The Fraunhofer diffraction pattern

How does \( I_s = \int \int J_s(y,z) \, dy \, dz \) depend on the applied field \( B_e = (0, B_y, 0) \)?

Integration in y-direction:

\[
i_c(z) = \int_{-W/2}^{W/2} J_c(y,z) \, dy
\]

\[
\Rightarrow I_s(B_y) = \int_{-L/2}^{L/2} i_c(z) \sin(kz + \varphi_0) \, dz = \Im \left\{ e^{i\varphi_0} \int_{-\infty}^{\infty} i_c(z)e^{ikz} \, dz \right\}
\]

Integral: complex, multiplication by \( e^{i\varphi_0} \) does not change magnitude

\( \rightarrow \) Magnitude yields maximum Josephson current

\[
I_s^m(B_y) = \left| \int_{-\infty}^{\infty} i_c(z) e^{ikz} \, dz \right|
\]

Magnetic field dependence of \( I_s^m \)

\( \rightarrow \) **Fourier transform** of \( i_c(z) \)

\( \rightarrow \) Analogy to optics

\( J_c(y,z) \) homogeneous \( \rightarrow i_c(z) \) constant \( \rightarrow \) Diffraction pattern of a slit \( \rightarrow \) **Fraunhofer pattern**

\[
I_s^m(\Phi) = I_c \left| \frac{\sin \frac{kL}{2}}{\frac{kL}{2}} \right| = I_c \left| \frac{\sin \frac{\pi\Phi}{\Phi_0}}{\frac{\pi\Phi}{\Phi_0}} \right|
\]

\( \Phi = B_y t_B L \) \hspace{1cm} Flux though the junction

\( I_c = i_c L \)

**Experimental** observation of \( I_s^m(\Phi) \) \( \rightarrow \) Proof of Josephson tunneling of pairs
2.2.2 The Fraunhofer diffraction pattern

Spatially homogeneous maximum current density $J_c(y, z)$

→ Fraunhofer diffraction pattern

Maximum current density integrated along $y$-direction

Experiment → Study the homogeneity of the supercurrent flow in JJ
2.2.2 The Fraunhofer diffraction pattern

Interpretation of the shape of $I_s^m(\Phi)$

→ Spatial distribution $i_s(z) = \int J_s(y, z) dy$
  for different applied fields

$\Phi = 0$

→ $\varphi(z) = \varphi_0$
→ $i_s(z) = \text{const.}$
→ Josephson current maximum for $\varphi_0 = -\frac{\pi}{2}$
→ $J_s(y, z) = -J_c(y, z)$

$\Phi = \Phi_0/2$

$$\varphi(z) = \frac{2\pi \Phi}{\Phi_0} \frac{z}{L} + \varphi_0 = \frac{\pi z}{L} + \varphi_0$$

→ Sinusoidal supercurrent variation with $z$
  difference between edges:

$$\varphi(L/2) - \varphi(-L/2) = \pi$$

→ Half of an oscillation period
→ Josephson current maximum for $\varphi_0 = -\frac{\pi}{2}$
→ Linear increase of the phase from $-\pi$ at $z = -\frac{L}{2}$ to 0 at $z = +\frac{L}{2}$
2.2.2 The Fraunhofer diffraction pattern

Local Josephson current density in negative and positive $x$-direction, but no net total current

→ Josephson current tends to decrease with increasing field
2.2.2 The Fraunhofer diffraction pattern

Spatial interference effect of macroscopic wave functions

→ Plane of constant phase in superconductor 2 is tilted by

$$\delta \varphi(z) = \frac{2\pi}{\Phi_0} B_y t_B z + \varphi_0$$

here: destructive interference
2.2.2 The Fraunhofer diffraction pattern

- Closed current loop
- No penetration of applied field into electrodes
- **Josephson vortex**
- No normal core because vortex core naturally in barrier region
2.2.2 The Fraunhofer diffraction pattern

Arbitrary direction of the applied field within barrier plane

\[ B_e = B_y \hat{y} + B_z \hat{z} \]

\[ \Rightarrow I_s^m(\Phi) = I_c \left| \begin{array}{c|c} \sin \frac{\pi \Phi_y}{\Phi_0} & \sin \frac{\pi \Phi_z}{\Phi_0} \\ \hline \end{array} \right| \]

\[ \Phi_y = B_y t_b L \quad \Phi_z = B_z t_b W \]

\[ \Rightarrow I_s^m(B_e) = \left| \int_S J_c(y, z) e^{ik \cdot r} dS \right| \]
2.2.3 Determination of the maximum Josephson current density

Inhomogeneous junctions

E.g., spatially varying barrier thickness
Experimental determination of $J_c(y, z)$ by measuring $I^m_s(B)$?
→ No access via inverse Fourier transform
   (Lack of phase information)

→ Approximate $I_c(z)$ under certain assumptions
  Example → Symmetry to junction midpoint

\[
i_c(z - L/2) = \frac{1}{\pi} \int_0^\infty |I^m_s(k)| \cos(kz)(-1)^{n(k)} \, dk
\]
\[
k = \frac{2\pi}{\Phi_0} t_B B_y
\]
\[
n \equiv \text{number of zeros of } |I^m_s(k)| \text{ between 0 and } k
\]

Spatial resolution

→ Information on $J_c(y, z)$ on small length scale?
→ Spatial resolution $\propto B_y^{-1}$

→ High B-fields required!
→ Spatial resolution for fields $\Phi \leq \Phi_0$ → Junction length $L$
Tailored junctions

Sometimes Frauenhofer sidelobes not desired (X-ray detectors)

\[ i_c(z) = i_c(0) \exp \left( -\frac{z^2}{2\sigma^2} \right) \]

\[ \text{No side lobes!} \]

→ Junction shape should approach Gauss curve for homogeneous \( J_c(y, z) \)

→ Integrated current density in \( y \)-direction

\[ i_c(z) = \int J_c(y, z) \, dy \]

\[ I_s^m(\Phi) = \sqrt{\frac{1}{2\pi}} i_c(0) L \exp (-\sigma k^2) = \sqrt{\frac{1}{2\pi}} i_c(0) L \exp \left( -\sigma \frac{4\pi^2 \Phi^2}{L^2 \Phi_0^2} \right) \]
2.2.3 Determination of the maximum Josephson current density

**Additional topic: Supercurrent auto-correlation function**

**Comparison**

- Optical diffraction experiment
  - Transmission function $P_0(z)$
  - Square root of light intensity $P_t$ in focal plane
  - BackTransform $\rightarrow P_i$ (spatial resolution given by number of diffraction orders)

- Field dependence of max. Josephson current
  - $i_c(z)$
  - $I_s^m(B_y)$
  - Phase is lost $\rightarrow$ BackTransform of intensity $(I_c^m)^2(B_y)$
  - Autocorrelation function of the supercurrent distribution
2.2.3 Determination of the maximum Josephson current density

Definition of auto-correlation function
→ Overlap of $i_c(z)$ with itself, but shifted by $\delta$

Wiener-Khinchine theorem
→ Autocorrelation function of $i_c(z)$:

$$AC(\delta) = \int_{-\infty}^{\infty} |I_s^m(k)|^2 e^{ik\delta} dk$$

$$k = \frac{2\pi}{\Phi_0} t_B B_y = \frac{1}{L} \frac{2\pi}{\Phi_0} \Phi$$

Spatial information contained in AC-function depends on magnetic field interval

**Spatial resolution**

$$\frac{2\pi}{k} = L \frac{\Phi_0}{\Phi}$$

Record 100 lobes in $I_s^m(B_y)$ → Spatial resolution $0.01 \times$ junction width

→ Statistical information in envelope of $|I_s^m(B_y)|^2$
2.2.3 Determination of the maximum Josephson current density

Prototypical examples:

→ Inhomogeneities with probability $p(a) \propto 1/a \leftrightarrow p \times a = \text{const.}$
  $\Rightarrow |I_s^m(B_y)|^2 \propto B_y^{-1}$
  $\Rightarrow \text{“Spatial 1/f noise”}$

→ Random distribution of filaments with width $a$:
  $\Rightarrow$ Envelope constant up to $k = \frac{2\pi}{a}$
  $\Rightarrow |I_s^m(B_y)|^2 \propto B_y^{-2}$
  $\Rightarrow \text{“Spatial shot noise”}$

$\text{YBa}_2\text{Cu}_3\text{O}_{7-\delta}$ grain boundary JJ
Slope of envelop is $-0.65$

$\Rightarrow |I_s^m(B_y)|^2 \propto B_y^{-1.3}$

$\Rightarrow p(a) \propto \frac{1}{a^{1.5}}$

$\Rightarrow$ Small scale inhomogeneities are more probable

Analysis of autocorrelation function gives statistical information on current density inhomogeneities

![Graph showing current density vs. magnetic field for YBa$_2$Cu$_3$O$_{7-\delta}$ grain boundary JJ](image)
2.2.4 Additional topic:
Direct imaging of the supercurrent distribution

Scanning of JJ by focused electron / laser beam

→ Measure change $\delta I_s^m(y, z)$ as function of beam position $(y, z)$

$\delta I_s^m(y, z) \propto J_c(y, z) \rightarrow$ 2D image of $J_c(y, z)$

→ Spatial resolution $\approx$ thermal healing length ($\approx 1 \, \mu m$)
2.2.6 The motion of Josephson vortices

**Josephson vortices**
- Visualize Josephson current density
- Vortices moving in z-direction at constant speed $v_z$
- Short junction
  - Self-field negligible
  - Flux density in junction given by $\mathbf{B}_e = (0, B_y, 0)$
- Gauge invariant phase difference
  \[
  \frac{\partial \varphi}{\partial z} = \frac{2\pi}{\Phi_0} B_y t_B
  \]
  \[
  \Phi = B_y t_B z
  \]
- Passage of 1 vortex changes phase by $2\pi$
  \[
  \frac{\partial \varphi}{\partial t} = \frac{2\pi}{\Phi_0} \frac{\partial \Phi}{\partial t} = \frac{2\pi}{\Phi_0} B_y t_B \frac{\partial z}{\partial t} = \frac{2\pi}{\Phi_0} B_y t_B v_z
  \]
  \[
  \Rightarrow \varphi(z, t) = \frac{2\pi}{\Phi_0} B_y t_B (z - v_z t) + \varphi(0)
  \]
  \[
  = k (z - v_z t) + \varphi(0)
  \]
2.2.6 The motion of Josephson vortices

\[ J_s(y, z, t) = J_c(y, z) \sin[k(z - v_z t)] \]

- Current density pattern: moves at \( v_z \)
- Vortex with period \( p = \frac{L \Phi_0}{\Phi} \)
  - Number of vortices in junction
    \[ N_V = \frac{L}{p} = \frac{\Phi}{\Phi_0} \]

- Change of gauge-invariant phase difference
  \[ \Delta \varphi = 2\pi \frac{\Phi}{\Phi_0} = 2\pi N_V \]

\[ 2\pi \times \# \text{ of vortices} \]
2.2.6 The motion of Josephson vortices

→ Rate of vortex passage

\[
\frac{dN_V}{dt} = \frac{1}{2\pi} \frac{d\Delta\varphi}{dt}
\]

with the voltage-phase relation

\[
\frac{dN_V}{dt} = \frac{V}{\Phi_0}
\]

Constant velocity of vortices → Constant junction voltage / vortex rate

→ Application
   → Single flux quantum pump
   → Pump frequency \( f = \frac{dN_V}{dt} \) → \( V = f \cdot \Phi_0 \)
2.2 Summary – Short Josephson Junctions

→ Short JJ = lateral junction dimensions small compared to Josephson penetration depth

→ Effect of magnetic field parallel to junction electrodes

\[ \nabla \varphi(r, t) = \frac{2\pi}{\Phi_0} t_B [B(r, t) \times \hat{x}] \]

phase gradient

Josephson current density

\[ J_s(y, z, t) = J_c(y, z) \sin \left( \frac{2\pi}{\Phi_0} t_B B_y z + \varphi_0 \right) \]

→ Spatial distribution of \( i_s(z) = \int J_s(y, z) dy \)

→ Josephson vortex
2.2 Summary – Short Josephson Junctions

→ Magnetic field dependence of maximum Josephson current

\[ I_s^m(\Phi) = I_c \left| \frac{\sin \frac{kL}{2}}{\frac{kL}{2}} \right| = I_c \left| \frac{\sin \frac{\pi \Phi}{\Phi_0}}{\frac{\pi \Phi}{\Phi_0}} \right| \]

Fraunhofer diffraction pattern

→ Analogy to single slit diffraction in optics, but no inverse Fourier transform (missing phase)

Autocorrelation function

→ Motion of Josephson vortices

Motion of single vortex across junction results in phase change of \(2\pi\)

\[ V \propto \frac{\partial \phi}{\partial t} = 2\pi \Phi_0 \frac{\partial \Phi}{\partial t} = 2\pi \Phi_0 B_y t_B \frac{\partial z}{\partial t} = 2\pi \frac{\Phi}{\Phi_0} \frac{v_z}{L} \]

Constant motion of vortices \(\rightarrow\) Constant junction voltage / vortex rate
2.3 Long Josephson junctions

2.3.1 The stationary Sine-Gordon equation

\[ \frac{\partial \varphi}{\partial z} = \frac{2\pi}{\Phi_0} B_y t_B, \quad \nabla \varphi(r, t) = \frac{2\pi}{\Phi_0} t_B \left[ \mathbf{B}(r, t) \times \hat{x} \right] \]

→ generally valid

Now → Magnetic flux density given by external and self-generated field

Ampere’s law → \[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \]

Zero-voltage state:

\[ \frac{\partial B_y(z)}{\partial z} = -\mu_0 J_x(z) \]

Spatial derivative:

\[ \frac{\partial^2 \varphi(z)}{\partial z^2} = -\frac{2\pi t_B}{\Phi_0} \frac{\partial B_y(z)}{\partial z} = -\frac{2\pi \mu_0 t_B}{\Phi_0} J_x(z) \]

Assume \( J_c(y, z) = \text{const.} \) and use \( J_x(y, z) = -J_s(y, z) \) → \( J_x(z) = -J_c \sin \varphi(z) \)

\[ \frac{\partial^2 \varphi(z)}{\partial z^2} = \frac{2\pi \mu_0 t_B J_c}{\Phi_0} \sin \varphi(z) = \frac{1}{\lambda_j^2} \sin \varphi(z) \]

Stationary Sine-Gordon equation (SSGE) (nonlinear differential equation)

Josephson penetration depth \( \lambda_j \equiv \sqrt{\frac{\Phi_0}{2\pi \mu_0 t_B J_c}} \)
2.3.1 The stationary Sine-Gordon equation

Two-dimensional stationary Sine-Gordon equation

\[ \frac{\partial^2 \varphi(y, z)}{\partial y^2} + \frac{\partial^2 \varphi(y, z)}{\partial z^2} = \frac{1}{\lambda_J^2} \sin \varphi(y, z) \]

Relation between London and Josephson penetration depth

\[ \lambda_L \equiv \sqrt{\frac{m_s}{\mu_0 n_s q_s^2}} \quad \leftrightarrow \quad \lambda_J \equiv \sqrt{\frac{\Phi_0}{2\pi \mu_0 t_B J_c}} \]

with \( J_s = q_s n_s \left\{ \frac{\hbar}{m_s} \nabla \theta(r, t) - \frac{q_s}{m_s} A(r, t) \right\} \)

\[ J_c \approx q_s n_s^* \frac{\hbar}{m_s} \frac{2\pi}{t_B} \]

insert into expression for Josephson penetration depth

\[ \lambda_J = \sqrt{\frac{\Phi_0}{2\pi \mu_0 t_B J_c}} \approx \sqrt{\frac{\hbar}{q_s \mu_0 t_B} \frac{m_s t_B}{2\pi \hbar q_s n_s^*}} = \sqrt{\frac{m_s}{2\pi \mu_0 n_s^* q_s^2}} \approx \lambda_L(n_s^*) \]

\( \lambda_J \) corresponds to the London penetration depth of the weak coupling region with reduced superelectron density \( n_s^* \)
2.3.1 The stationary Sine-Gordon equation

Additional topic: Analytical solutions of the SSGE

Small $\varphi \rightarrow$ Linearization: $\sin \varphi \approx \varphi \rightarrow \frac{\partial^2 \varphi(z)}{\partial z^2} = \frac{1}{\lambda_J^2} \varphi(z) \Rightarrow \varphi(z) = \varphi(0)e^{-z/\lambda_J}$

Magnetic field along the junction $\Rightarrow B_y(z) = -\frac{\varphi(0)}{2\pi} \frac{\Phi_0}{\lambda_J t_B} e^{-z/\lambda_J}$

$\Rightarrow \lambda_J$ is a decay length

with $\frac{\partial B_y(z)}{\partial z} = -\mu_0 J_x(z) \Rightarrow J_x(z = 0) = \frac{1}{\lambda_J} \frac{B_y(z = 0)}{\mu_0}$

$\Rightarrow$ Current flows at the edges of the junction

$\Rightarrow$ Meißner solution, possible for $J_x < J_c$ or $B_y(z = 0) \leq \mu_0 J_c \lambda_J$

Small junction $L \ll \lambda_J \rightarrow \frac{\partial^2 \varphi(z)}{\partial z^2} \approx 0 \Rightarrow \frac{\partial \varphi(z)}{\partial z} \approx \text{const} \Rightarrow \text{Short junction result}$
Particular solution of the SSGE:

\[
\varphi(z) = \pm 4 \arctan \left\{ \exp \left( \frac{z - z_0}{\lambda_J} \right) \right\} + 2\pi n
\]

\[
B_y(z) = \pm \frac{\Phi_0}{\pi \lambda_L t_B} \frac{1}{\cosh \left( \frac{z - z_0}{\lambda_J} \right)}
\]

\[
J_x(z) = -J_s(z) = \pm \frac{\Phi_0}{\pi \mu_0 \lambda_L^2 t_B} \frac{\sinh \left( \frac{z - z_0}{\lambda_J} \right)}{\cosh \left( \frac{z - z_0}{\lambda_J} \right)} = \pm 2J_c \frac{\sinh \left( \frac{z - z_0}{\lambda_J} \right)}{\cosh \left( \frac{z - z_0}{\lambda_J} \right)}
\]

General solution: particular + homogeneous solution

→ Important case: junction of infinite length, \( \frac{d\varphi}{dz} \) vanishes for \( z \to \pm \infty \)

→ Particular solution = Complete solution
2.3.2 The Josephson Vortex

Decay length for $J_s$ and $B_y \rightarrow \lambda_J$

Maximum of $J_s$ does not coincide with maximum of $B_y$

Integration

→ Total current = 0
→ Total flux = $\Phi_0$
→ Josephson vortex in an infinitely long junction
2.3.2 The Josephson vortex

Energy per unit length of vortex:

\[ E_{\text{Vortex}} = \frac{E_I}{W} = \frac{4\Phi_0 J_c \lambda_J}{\pi} \]

→ \( E_{\text{Vortex}} > 0 \) → We need external field and/or current to supply energy

Magnetic flux density \( B_{c1} \) for first vortex entrance:

\[ B_{c1} = \frac{\mu_0}{\Phi_0} E_{\text{Vortex}} = \frac{4\mu_0 J_c \lambda_J}{\pi} = \frac{2\Phi_0}{\pi^2 \lambda_J t_B} \]

\( B_{c1} \approx \) Magnetic flux density of a single flux quantum distributed over an area \( t_B \times \lambda_J \)

Here: Infinitely long junction
→ Simple & Prototypical case (boundary conditions not relevant)
→ Reveals relevant physical principles
→ Real junctions → Boundary conditions → Complex vortex dynamics!
2.3.3 Junction types and boundary conditions

Junction geometry determines current flow $\rightarrow$ Boundary conditions of SSGE
$\rightarrow$ Magnetic flux density at junction edges:

$$
\frac{\partial \varphi}{\partial z} \bigg|_{z=0} = \frac{2\pi t_B}{\Phi_0} B_y \bigg|_{z=0}
$$

$$
\frac{\partial \varphi}{\partial z} \bigg|_{z=L} = \frac{2\pi t_B}{\Phi_0} B_y \bigg|_{z=L}
$$

$$
\frac{\partial \varphi}{\partial y} \bigg|_{y=0} = -\frac{2\pi t_B}{\Phi_0} B_z \bigg|_{y=0}
$$

$$
\frac{\partial \varphi}{\partial y} \bigg|_{y=W} = -\frac{2\pi t_B}{\Phi_0} B_z \bigg|_{y=W}
$$

Problem: $\mathbf{B} = \mathbf{B}^{\text{ex(ternal)}} + \mathbf{B}^{\text{el(ectrode)}}$

$\rightarrow \mathbf{B}^{\text{el}}$ not negligible

$\rightarrow$ Junction geometries are complicated

$\rightarrow$ Current distribution in electrodes depends on current distribution in JJ itself

$\rightarrow$ Boundary conditions depend on solution

$\rightarrow$ Numerical iteration method required

Three basic types of junction geometries

$\rightarrow$ Overlap junction

$\rightarrow$ In-line junction

$\rightarrow$ Grain boundary junctions

J. Mannhart, J. Bosch, R. Gross, R. P. Huebener
2.3.3 Junction Types and Boundary Conditions

**Overlap junction**
- Overlap of width $W$
- Junction of length $L$ extends in $z$-direction
- Perpendicular to current flow
- $B^{el} \parallel z \rightarrow \perp$ to the short side
- $\Phi^{el} = B^{el} \times W \times t_B$ negligibly small

**Inline junction**
- Overlap of length $L$
- Junction of length $L$ extends in $z$-direction
- Parallel to current flow
- $B^{el} \perp z \rightarrow \parallel$ to the short side
- $\Phi^{el} = B^{el} \times L \times t_B$ not negligible
2.3.3 Junction Types and Boundary Conditions

Grain boundary junction

→ Mixture of overlap and inline geometry
→ Junction area is perpendicular to electrode currents
→ Junction area extends in $yz$-plane
→ Perpendicular to current flow
→ Both $y$- and $z$-component of $B^el$ are in junction plane
→ $B^el_z$ has negligible impact since $W \ll L$
→ $\Phi^el_z = B^el_z W t_B \ll \Phi^el_y = B^el_y L t_B$
→ Finite inline admixture $s = \frac{W}{L} \ll 1$
2.3.3 Junction types and boundary conditions

\( I_s^m(B^{ex}) \) for different junction geometries

Overlap junction

\[ \rightarrow \text{Highest } I_s^m \propto \text{junction area } A_i = L \times W \text{ at zero field} \]

Inline junction

\[ \rightarrow I_s^m \text{ saturates at } 4J_c W \lambda_j \text{ (Meißner screening, current flow at edges)} \]

Asymmetric inline junction

\[ \rightarrow \text{Fields generated in bottom and top electrode point in the same direction} \]
\[ \rightarrow \text{Adds to/subtracts from external field} \]
\[ \rightarrow \text{Increase or decrease of } I_s^m \text{ with field} \]
2.3.4 The Pendulum analog

SSGE equivalent to equation of motion of pendulum (neglecting electrode currents)

\[ z \rightarrow t, \varphi \rightarrow \theta, \frac{1}{\lambda_j^2} \rightarrow \omega_0^2 = \frac{g}{\ell} \]

\[ \theta \quad \text{Angle of the pendulum measured from the top} \]

\[ \omega_0 \quad \text{Natural frequency of the pendulum} \]

Pendulum with very large \( E_{\text{kin}} \)

\[ \rightarrow \text{Gravitational acceleration negligible} \]

\[ \rightarrow \text{Corresponds to limit of } \lambda_j \rightarrow \infty \]

\[ \frac{\partial^2 \varphi}{\partial z^2} = 0 \text{ and } \frac{\partial \varphi}{\partial z} = \text{const.} \]

\[ \rightarrow \text{Sinusoidal variation of } J_S(z) \]

Pendulum with less \( E_{\text{kin}} \), but still nonzero kinetic energy at top

\[ \rightarrow \theta(t) \text{ anharmonic} \]

\[ \rightarrow \text{Corresponds to nonsinusoidal, periodically reversing } J_S(z) \]

\[ \rightarrow \text{Each spatial cycle of the oscillating current contains one flux quantum} \]

\[ \rightarrow \text{In this case, Josephson vortices are localized entities} \]
2.3.4 The Pendulum analog

Pedulum with $E_{\text{kin}}$ just sufficient to go over the top

→ Meißen limit

→ Start at $-\theta_0$ with $\left(\frac{d\theta}{dt}\right)_0$ at time $t$ corresponding to $-\frac{L}{2}$
→ Pendulum moves very slowly for long time while going over the top (interior of junction)
→ Exponential acceleration, recovering initial velocity at $\theta_0$
  (at time $t$ corresponding to $+\frac{L}{2}$)

→ Negligible energy at the top
→ Conservation of energy connects $\theta_0$ and $\left(\frac{d\theta}{dt}\right)_0$

\[
\left(\frac{2\pi B_y t_B}{\Phi_0}\right)^2 = \left(\frac{d\varphi}{dz}\right)_0^2 = \frac{2}{\lambda_J^2}(1 - \cos \varphi_0) \Rightarrow \cos \varphi_0 = 1 - \frac{1}{2} \left(\frac{B_y}{\mu_0 J_c \lambda_J}\right)^2
\]

→ Small fields $\varphi_0 = \frac{B_y}{\mu_0 J_c \lambda_J}$ (Taylor expansion of cosine!)
→ Strongest field that can be screened is $B_{\text{max}} = 2\mu_0 J_c \lambda_J$ (for $\varphi_0 = \pi$)
→ (Screening at $B_{\text{max}}$ is only metastable!)
## Summary (short junctions)

### Josephson penetration depth

\[ \lambda_J \equiv \sqrt{\frac{\Phi_0}{2\pi \mu_0 t_B J_c}} \]

(Short – long junctions)

### Josephson coupling energy

\[ E_J = \frac{\Phi_0 I_c}{2\pi} (1 - \cos \varphi) = E_{J0} (1 - \cos \varphi) \]

### Nonlinear inductance

\[ L_s = \frac{\Phi_0}{2\pi I_c \cos \varphi} = L_c \frac{1}{\cos \varphi} \quad L_c = \frac{\hbar}{2e I_c} \]

### Washboard potential

\[ E_{pot}(\varphi) = E_J(\varphi) - \frac{\Phi_0}{2\pi} I \varphi = E_{J0} \left[ 1 - \cos \varphi - \frac{I}{I_c} \varphi \right] \]

### In-plane magnetic field \( B_y \)

\[ \frac{\partial \varphi}{\partial z} = \frac{2\pi}{\Phi_0} B_y t_B \]

### Spatial oscillations of the Josephson current density:

\[ \Rightarrow J_s(y, z, t) = J_c(y, z) \sin \left( \frac{2\pi}{\Phi_0} t_B B_y z + \varphi_0 \right) = J_c(y, z) \sin (kz + \varphi_0) \]

### Integral Josephson current \( \rightarrow \) FT of \( i_c(z) \):

\[ \Rightarrow I_s^m(B_y) = \left| \int_{-\infty}^{\infty} i_c(z) e^{ikz} dZ \right| \quad \quad i_c(z) = \int_{-W/2}^{W/2} J_c(y, z) dy \]
Summary (long junctions)

SSGE: spatial distribution of gauge invariant phase difference:

\[
\frac{\partial^2 \varphi(y, z)}{\partial y^2} + \frac{\partial^2 \varphi(y, z)}{\partial z^2} = \frac{2\pi \mu_0}{\Phi_0} t_B J_c \sin \varphi(y, z) = \frac{1}{\lambda_j^2} \sin \varphi(y, z)
\]

self-consistent solution: boundary conditions depend on flux density at edges

3 basic types: inline, overlap and grain boundary junctions

particular solution of SSGE: Josephson vortex

\[
\varphi(z) = \pm 4 \arctan \left\{ \exp \left( \frac{z - z_0}{\lambda_J} \right) \right\} + 2\pi n
\]

Insight into the solutions for \( \varphi(z) \) can be found via the pendulum analog

SSGE equivalent to equation of motion of pendulum

\[
z \rightarrow t, \varphi \rightarrow \theta, \frac{1}{\lambda_j^2} \rightarrow \omega_0^2 = \frac{g}{\ell}
\]