

Control of quantum two-level systems

6.3 Control of quantum two-level systems

General concept – How to control a qubit?

Does the answer depend on specific realization? → No, only its implementation

Qubit is **pseudo spin**

- **General concept exists**
- Independent of qubit realization
- Methods from nuclear magnetic resonance (NMR)

Nuclear magnetic resonance

- Method to explore the magnetism of nuclear spins
- Important application → **Magnetic resonance imaging (MRI)** in medicine
- MRI exploits the different nuclear magnetic signatures of different tissues

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Brief MRI history

early suggestions → **H. Carr** (1950) and **V. Ivanov** (1960)



1972 → MRI imaging machine proposed by **R. Damadian** (SUNY)



1973 → 1st MRI image by **P. Lauterbur** (Urbana-Champaign)



Late 1970ies → Fast scanning technique proposed by **P. Mansfield** (Nottingham)



2003 Nobel Prize in Medicine → for P. Lauterbur and P. Mansfield

6.3 Control of quantum two-level systems

Overview: Important NMR techniques

Basic idea

- Rotate spins by static or oscillating magnetic fields
- **Static fields** parallel to quantization axis
 - **Free precession**
 - Changes φ on Bloch sphere
- **Oscillating fields** perpendicular to quantization axis
 - **Change population**
 - Changes θ on Bloch sphere

Important protocols

- Rabi → Population oscillations → Controlled excitations
- Relaxation measurement → T_1
- Ramsey fringes → T_2^*
- Spin echo (Hahn echo) corrects T_2^* for reversible dephasing → T_2

NMR reminder

- Spin-lattice (longitudinal) relaxation time τ_1
- Spin-spin (transversal) relaxation time $\tau_2 \leq 2\tau_1$
- Free induction decay (FID) in presence of magnetic field inhomogeneities → $\tau_2^* \leq \tau_2$
- In traditional NMR Hahn echo does not always yield τ_2 → Multiple echoes (CPMG)

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The Hamiltonian of a quantum two-level system (TLS)

- Arbitrary TLS**
- $\hat{H} = \frac{1}{2} \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$ is Hermitian matrix
 - $H_{11}, H_{22} \in \mathbb{R}$ and $H_{21} = H_{12}^*$
 - we choose $H_{11} > H_{22}$

- Symmetrize**
- Subtract global energy offset $\frac{H_{11}+H_{22}}{4} \times \hat{1}$
 - $\hat{H} = \frac{1}{2} \begin{pmatrix} \epsilon & \Delta - i\tilde{\Delta} \\ \Delta + i\tilde{\Delta} & -\epsilon \end{pmatrix} = \frac{\epsilon}{2} \hat{\sigma}_z + \frac{\tilde{\Delta}}{2} \hat{\sigma}_y + \frac{\Delta}{2} \hat{\sigma}_x$
 - $\epsilon \equiv \frac{H_{11}-H_{22}}{2} > 0$ and $\Delta, \tilde{\Delta} \in \mathbb{R}$
 - Without loss of generality, we usually assume $\tilde{\Delta} = 0$

Natural or physical basis $\{|\varphi_+\rangle, |\varphi_-\rangle\}$

$$\rightarrow \hat{H}_0 = \frac{1}{2} \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix} = \frac{\epsilon}{2} \hat{\sigma}_z$$

$$\rightarrow \hat{H}|\varphi_{\pm}\rangle = \pm \frac{\epsilon}{2}$$

A quantum TLS is called „qubit“
in information processing!

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R. Gross, A. Marx, F. Deppe, and K. Fedorov © Walther-Meißner-Institut (2001 - 2018)

$$\hat{H} = \frac{1}{2} \begin{pmatrix} \epsilon & \Delta - i\tilde{\Delta} \\ \Delta + i\tilde{\Delta} & -\epsilon \end{pmatrix}$$

Diagonalization of \hat{H}

Eigenvalues

$$\rightarrow \det \begin{pmatrix} \epsilon - 2E_{\pm} & \Delta - i\tilde{\Delta} \\ \Delta + i\tilde{\Delta} & -\epsilon - 2E_{\pm} \end{pmatrix} = 0$$

$$\rightarrow (\epsilon - 2E_{\pm})(-\epsilon - 2E_{\pm}) - (\Delta - i\tilde{\Delta})(\Delta + i\tilde{\Delta}) = 0$$

$$\rightarrow E_{\pm} = \pm \frac{1}{2} \sqrt{\epsilon^2 + \Delta^2 + \tilde{\Delta}^2} \equiv \pm \frac{1}{2} \hbar \omega_q$$

Transition energy conveniently expressed in frequency units

Eigenvectors

$$\rightarrow \hat{H} |\Psi_{\pm}\rangle = E_{\pm} |\Psi_{\pm}\rangle$$

$$\rightarrow |\Psi_{-}\rangle = -e^{-i\varphi/2} \sin \frac{\theta}{2} |\varphi_{+}\rangle + e^{+i\varphi/2} \cos \frac{\theta}{2} |\varphi_{-}\rangle$$

$$\rightarrow |\Psi_{+}\rangle = +e^{-i\varphi/2} \cos \frac{\theta}{2} |\varphi_{+}\rangle + e^{+i\varphi/2} \sin \frac{\theta}{2} |\varphi_{-}\rangle$$

Recovers Bloch sphere picture!

$$\tan \theta \equiv \frac{\sqrt{\Delta^2 + \tilde{\Delta}^2}}{\epsilon} \quad \text{with } 0 \leq \theta \leq \pi$$

$$\tan \varphi \equiv \frac{\tilde{\Delta}}{\Delta} \quad \text{with } 0 \leq \varphi \leq 2\pi$$

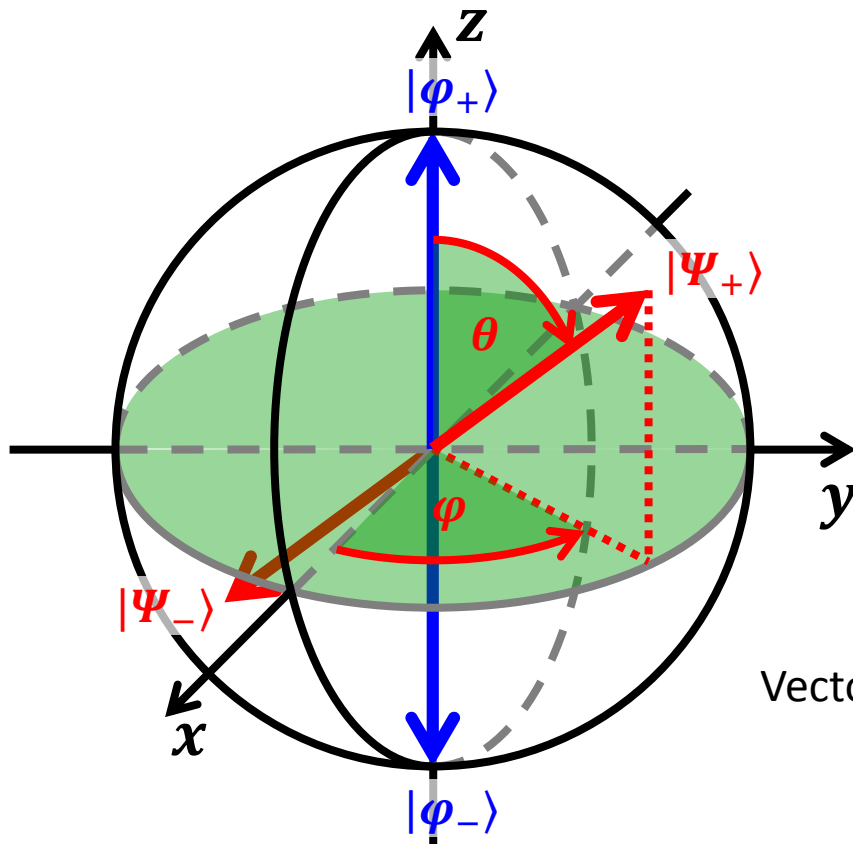
Bloch angles!

C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics*, Volume One (Wiley-VCH)

Basis $\{|\Psi_{+}\rangle, |\Psi_{-}\rangle\} \rightarrow$ **Energy eigenbasis** (because \hat{H} has energy units)

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Visualization on the Bloch sphere



$$|\Psi_+\rangle = +e^{-i\varphi/2} \cos\frac{\theta}{2} |\varphi_+\rangle + e^{i\varphi/2} \sin\frac{\theta}{2} |\varphi_-\rangle$$

$$|\Psi_-\rangle = -e^{-i\varphi/2} \sin\frac{\theta}{2} |\varphi_+\rangle + e^{i\varphi/2} \cos\frac{\theta}{2} |\varphi_-\rangle$$

$$\tan\theta \equiv \frac{\sqrt{\Delta^2 + \tilde{\Delta}^2}}{\epsilon} \quad \text{with } 0 \leq \theta \leq \pi$$

$$\tan\varphi \equiv \frac{\tilde{\Delta}}{\Delta} \quad \text{with } 0 \leq \varphi \leq 2\pi$$

Basis rotation on Bloch sphere!

Vectors $|\Psi_-\rangle$ and $|\Psi_+\rangle$ define new quantization axis

$$\rightarrow \hat{H} = \frac{\sqrt{\epsilon^2 + \Delta^2 + \tilde{\Delta}^2}}{2} \hat{\sigma}_z \text{ in basis } \{|\Psi_-\rangle, |\Psi_+\rangle\}$$

$\rightarrow |\Psi_-\rangle$ and $|\Psi_+\rangle$ become new poles

Practical relevance

$\rightarrow \epsilon$ or Δ (and therefore the rotation angles) often depend on an external control parameter!

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Analogy to spin 1/2 in static magnetic field

Fictitious spin 1/2 in fictitious magnetic field B

$$\rightarrow \hat{H}_\uparrow = -\gamma \hbar \mathbf{B} \cdot \mathbf{S} = -\frac{\gamma \hbar}{2} \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$$

$\rightarrow \gamma$ is the gyromagnetic ratio

$\rightarrow B = (B_x, B_y, B_z)^T$ is the magnetic field vector

Fictitious spin in fictitious B -field

$|\uparrow\rangle$

$|\downarrow\rangle$

$|\uparrow\rangle_u$

$|\downarrow\rangle_u$

(u denotes the quantization axis along which \hat{H}_\uparrow is diagonal)

$\hbar|\gamma||B|$

Polar angles of B

$-\gamma \hbar B_z$

$-\gamma \hbar B_x$

$-\gamma \hbar B_y$

\rightarrow

\rightarrow

\rightarrow

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quantum TLS

$|\varphi_+\rangle$

$|\varphi_-\rangle$

$|\Psi_+\rangle$

$|\Psi_-\rangle$

$E_+ - E_- = \hbar\omega_q$

θ, φ

ϵ

Δ

$\tilde{\Delta}$

Field orientation with respect to the quantization axis depends on $\epsilon, \Delta, \tilde{\Delta}$

Any quantum TLS has a "built-in" static field
 \downarrow
NMR situation!

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Dynamics of the qubit state

Analogy to spin $\frac{1}{2}$ in static magnetic field \rightarrow Dynamics as known from NMR!

Intrinsic dynamics \rightarrow Free evolution \rightarrow Precession about z-axis

Driven evolution \rightarrow Population (Rabi) oscillations

Pulsed driving schemes

\rightarrow Ramsey fringes

\rightarrow Hahn echo

NMR-type control and pulsed control of $\hbar\omega_q \rightarrow$ **Single qubit gates**

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Free precession

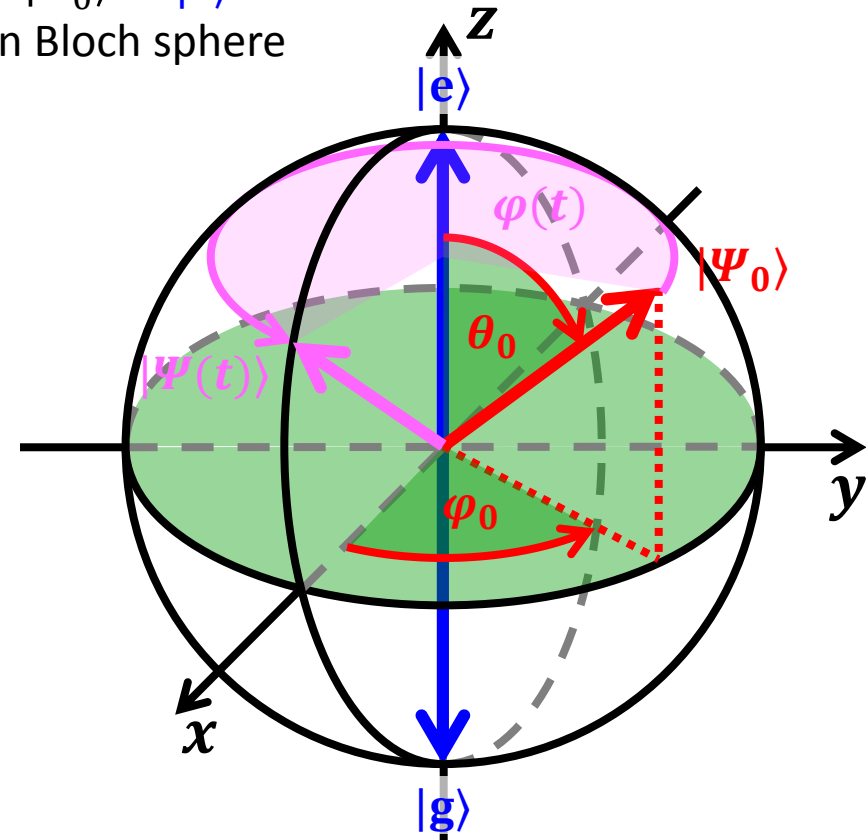
$$|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|e\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|g\rangle$$

In the energy eigenbasis $\{|g\rangle, |e\rangle\}$, the qubit state vector $|\Psi_0\rangle$ is

- parallel to built-in field for $|\Psi_0\rangle = |g\rangle$
- antiparallel to the built-in field for $|\Psi_0\rangle = |e\rangle$
- Built-in field points along z-axis on Bloch sphere

When qubit state vector $|\Psi_0\rangle$ is not parallel or antiparallel to built-in field

- Free evolution corresponds to **free precession about the z-axis**
- Also called Larmor precession
- In absence of decoherence, only $\varphi(t)$ evolves linearly with time t



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Larmor precession – formal calculation

$$|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|e\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|g\rangle$$

Energy eigenbasis $\rightarrow \hat{H} = \frac{\hbar\omega_q}{2}\hat{\sigma}_z \rightarrow$ fictitious field aligned along quantization axis

Time-independent $\hat{H} \rightarrow$ Develop into stationary states

$$|\Psi(t)\rangle = \langle e|\Psi_0\rangle e^{-i\frac{\omega_q t}{2}}|e\rangle + \langle g|\Psi_0\rangle e^{i\frac{\omega_q t}{2}}|g\rangle$$

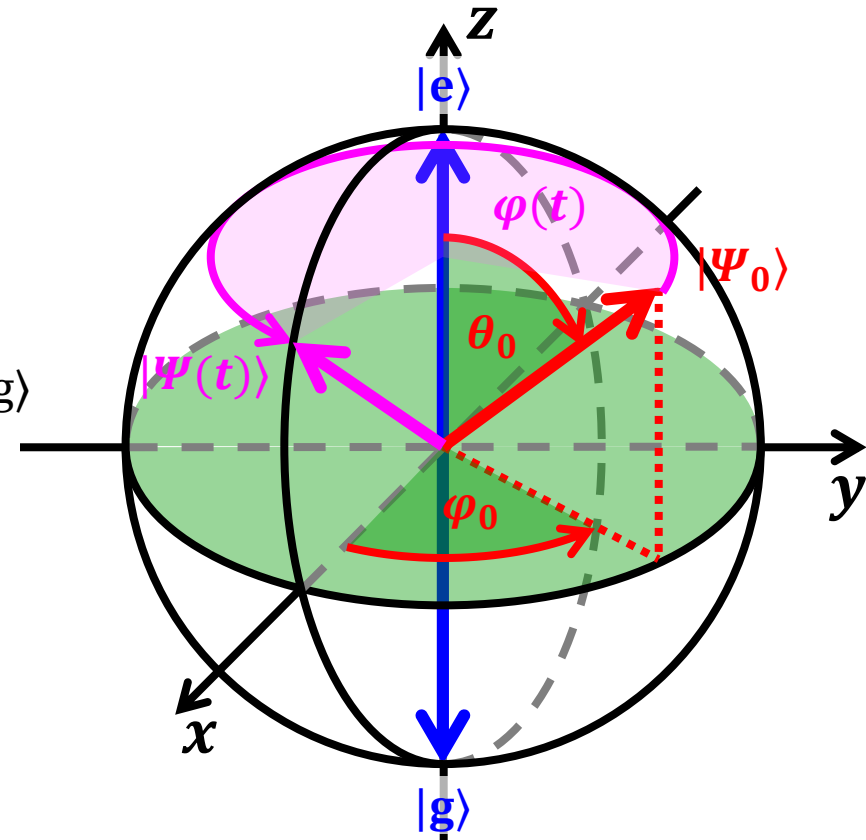
$$|\Psi_0\rangle = \cos\left(\frac{\theta_0}{2}\right)|e\rangle + e^{i\varphi_0}\sin\left(\frac{\theta_0}{2}\right)|g\rangle$$

$$|\Psi(t)\rangle = e^{-i\frac{\omega_q t}{2}}\cos\frac{\theta_0}{2}|e\rangle + e^{i\varphi_0}e^{i\frac{\omega_q t}{2}}\sin\frac{\theta_0}{2}|g\rangle$$

Global phase arbitrary

$$|\Psi(t)\rangle = \cos\frac{\theta_0}{2}|e\rangle + e^{i(\varphi_0 + \omega_q t)}\sin\frac{\theta_0}{2}|g\rangle$$

$\varphi(t)$



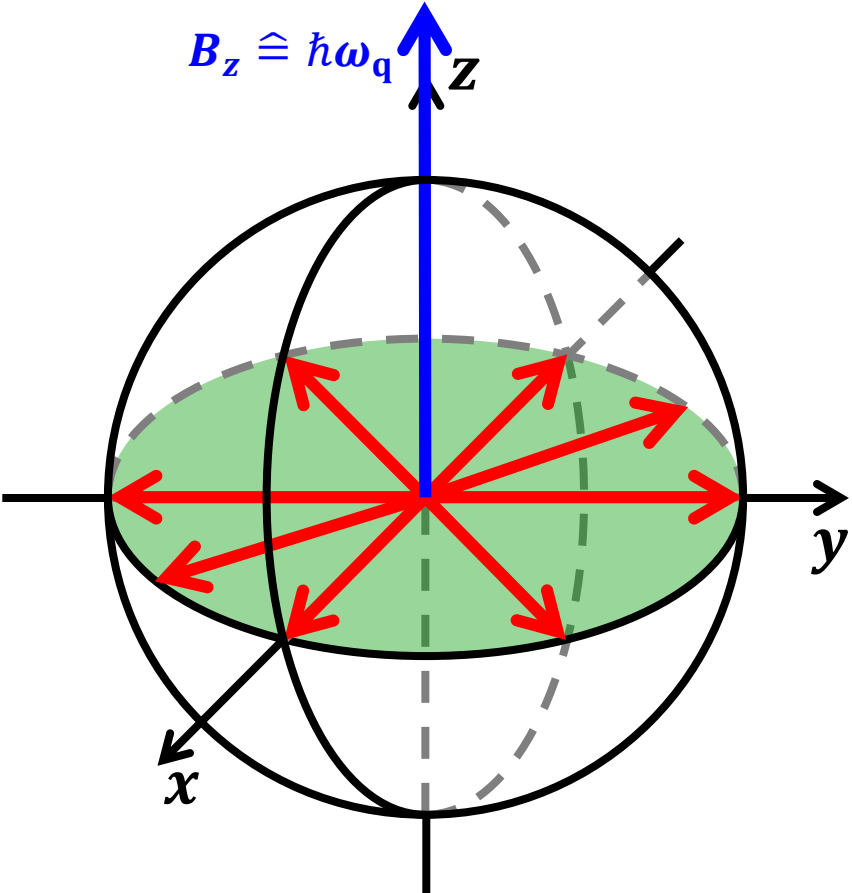
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$$\hat{H}_\uparrow = -\frac{\gamma\hbar}{2} \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$$

Rotating drive field

Consider the qubit state vector $|\Psi\rangle$ expressed in energy eigenbasis $\{|g\rangle, |e\rangle\}$

Apply a drive field with power $\hbar\omega_d$ rotating around the z-axis at frequency ω



$$\rightarrow \hat{H}(t) = \frac{\hbar}{2} \begin{pmatrix} \omega_q & \omega_d e^{-i\omega t} \\ \omega_d e^{i\omega t} & -\omega_q \end{pmatrix}$$

ωt	B_x	B_y
0	ω_d	0
$\pi/4$	$\omega_d/\sqrt{2}$	$\omega_d/\sqrt{2}$
$\pi/2$	0	ω_d
$3\pi/4$	$-\omega_d/\sqrt{2}$	$\omega_d/\sqrt{2}$
π	$-\omega_d$	0
$5\pi/4$	$-\omega_d/\sqrt{2}$	$-\omega_d/\sqrt{2}$
$3\pi/2$	0	$-\omega_d$
$7\pi/4$	$\omega_d/\sqrt{2}$	$-\omega_d/\sqrt{2}$

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Driven quantum TLS

$$\hat{H}(t) = \frac{\hbar}{2} \begin{pmatrix} \omega_q & \omega_d e^{-i\omega t} \\ \omega_d e^{i\omega t} & -\omega_q \end{pmatrix} = \underbrace{\frac{\hbar\omega_q}{2} \hat{\sigma}_z}_{\equiv \hat{H}_0} + \underbrace{\frac{\hbar\omega_d}{2} (\hat{\sigma}_- e^{+i\omega t} + \hat{\sigma}_+ e^{-i\omega t})}_{\equiv \hat{H}_d}$$

Operators $\hat{\sigma}_+ \equiv |e\rangle\langle g|$ and $\hat{\sigma}_- \equiv |g\rangle\langle e|$ create or annihilate an excitation in the TLS

→ Note: $\hat{\sigma}_+$ and $\hat{\sigma}_-$ are unintuitive in the IT convention, because of the unintuitive association of a negative eigenvalue of $\hat{\sigma}_z$ with state $|1\rangle$!

Matrix representation → $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$\begin{aligned} \hat{\sigma}_+ &= |e\rangle\langle g| \\ \hat{\sigma}_- &= |g\rangle\langle e| \end{aligned}$$

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Time evolution of driven quantum TLS

Qubit state $|\Psi(t)\rangle = a_e(t)|e\rangle + a_g(t)|g\rangle$
obeys **Schrödinger equation**

$$i \frac{d}{dt} a_e(t) = \frac{\omega_q}{2} a_e(t) + \frac{\omega_d}{2} e^{-i\omega t} a_g(t)$$

$$i \frac{d}{dt} a_g(t) = \frac{\omega_d}{2} e^{i\omega t} a_e(t) - \frac{\omega_q}{2} a_g(t)$$

Time-dependent coupled differential equations \rightarrow Difficult to solve

$$b_e(t) \equiv e^{i\omega t/2} a_e(t)$$

\rightarrow Move to rotating frame

$$b_g(t) \equiv e^{-i\omega t/2} a_g(t)$$

\rightarrow Schrödinger equation **loses explicit time dependence**

$$i \frac{d}{dt} b_e(t) = \frac{\omega_q - \omega}{2} b_e(t) + \frac{\omega_d}{2} b_g(t)$$

$$i \frac{d}{dt} b_g(t) = \frac{\omega_d}{2} b_e(t) - \frac{\omega_q - \omega}{2} b_g(t)$$

$$\hat{H}(t) = \frac{\hbar}{2} \begin{pmatrix} \omega_q & \omega_d e^{-i\omega t} \\ \omega_d e^{i\omega t} & -\omega_q \end{pmatrix}$$

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = H |\Psi(t)\rangle$$

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$$\hat{H}(t) = \frac{\hbar}{2} \begin{pmatrix} \omega_q & \omega_d e^{-i\omega t} \\ \omega_d e^{i\omega t} & -\omega_q \end{pmatrix}$$

Interpretation of the rotating frame

$$i \frac{d}{dt} b_e(t) = \frac{\omega_q - \omega}{2} b_e(t) + \frac{\omega_d}{2} b_g(t)$$

$$i \frac{d}{dt} b_g(t) = \frac{\omega_d}{2} b_e(t) - \frac{\omega_q - \omega}{2} b_g(t)$$

The frame rotates at the angular speed ω of the drive

- Driving field appears at rest
- Drive can be in resonance with Larmor precession frequency ω_q
- Away from resonance $|\omega - \omega_q| \gg 0$
 - red terms dominate → no $|g\rangle \leftrightarrow |e\rangle$ transitions induced by drive
- Near resonance $\omega \approx \omega_q$
 - blue terms dominate → $|g\rangle \leftrightarrow |e\rangle$ transitions induced by drive

Formal treatment → Effective Hamiltonian \tilde{H} describing the same dynamics as $\hat{H}(t)$

$$\tilde{H} = \frac{\hbar}{2} \begin{pmatrix} -\Delta\omega & \omega_d \\ \omega_d & \Delta\omega \end{pmatrix} \quad i\hbar \frac{d}{dt} |\tilde{\Psi}(t)\rangle = \tilde{H} |\tilde{\Psi}(t)\rangle$$

with $\Delta\omega \equiv \omega - \omega_q$ $|\tilde{\Psi}(t)\rangle \equiv b_e(t)|e\rangle + b_g(t)|g\rangle$

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Dynamics of the effective Hamiltonian

$$\tilde{H} = \frac{\hbar}{2} \begin{pmatrix} -\Delta\omega & \omega_d \\ \omega_d & \Delta\omega \end{pmatrix}$$

→ Diagonalize →
$$\tilde{H} = \frac{-\hbar \sqrt{\Delta\omega^2 + \omega_d^2}}{2} \hat{\sigma}_z$$

$$\tan \theta = -\frac{\omega_d}{\Delta\omega}$$

$$\tan \varphi = 0$$

→ New eigenstates →

$$|\Psi_+\rangle = +e^{-i\varphi/2} \cos\frac{\theta}{2} |\varphi_+\rangle + e^{+i\varphi/2} \sin\frac{\theta}{2} |\varphi_-\rangle$$

$$|\Psi_-\rangle = -e^{-i\varphi/2} \sin\frac{\theta}{2} |\varphi_+\rangle + e^{+i\varphi/2} \cos\frac{\theta}{2} |\varphi_-\rangle$$

$$\Delta\omega \equiv \omega - \omega_q$$

$$|\tilde{\Psi}_+\rangle = +\cos\frac{\theta}{2} |e\rangle + \sin\frac{\theta}{2} |g\rangle$$

$$|\tilde{\Psi}_-\rangle = -\sin\frac{\theta}{2} |e\rangle + \cos\frac{\theta}{2} |g\rangle$$

Expand into stationary states

$$|\tilde{\Psi}(t)\rangle = \langle\tilde{\Psi}_-|\tilde{\Psi}_0\rangle e^{+\frac{it}{2}\sqrt{\Delta\omega^2+\omega_d^2}} |\tilde{\Psi}_-\rangle + \langle\tilde{\Psi}_+|\tilde{\Psi}_0\rangle e^{-\frac{it}{2}\sqrt{\Delta\omega^2+\omega_d^2}} |\tilde{\Psi}_+\rangle$$

Initial state $|\Psi_0\rangle = |g\rangle$ (energy ground state)

$$|\tilde{\Psi}(t)\rangle = \cos\frac{\theta}{2} e^{+\frac{it}{2}\sqrt{\Delta\omega^2+\omega_d^2}} |\tilde{\Psi}_-\rangle + \sin\frac{\theta}{2} e^{-\frac{it}{2}\sqrt{\Delta\omega^2+\omega_d^2}} |\tilde{\Psi}_+\rangle$$

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Probability P_e to find TLS in $|e\rangle$

$$P_e \equiv |\langle e|\Psi(t)\rangle|^2 = |\langle e|\tilde{\Psi}(t)\rangle|^2 =$$

$$= \left| \sin \frac{\theta}{2} \cos \frac{\theta}{2} \left(e^{-\frac{it}{2}\sqrt{\Delta\omega^2 + \omega_d^2}} - e^{+\frac{it}{2}\sqrt{\Delta\omega^2 + \omega_d^2}} \right) \right|^2$$

$$= \left| \sin \theta \sin \left(\frac{t\sqrt{\Delta\omega^2 + \omega_d^2}}{2} \right) \right|^2$$

$$P_e = \frac{\omega_d^2}{\Delta\omega^2 + \omega_d^2} \sin^2 \left(\frac{t\sqrt{\Delta\omega^2 + \omega_d^2}}{2} \right)$$

$$|\tilde{\Psi}(t)\rangle = \cos \frac{\theta}{2} e^{+\frac{it}{2}\sqrt{\Delta\omega^2 + \omega_d^2}} |\tilde{\Psi}_-\rangle + \sin \frac{\theta}{2} e^{-\frac{it}{2}\sqrt{\Delta\omega^2 + \omega_d^2}} |\tilde{\Psi}_+\rangle$$

$$|\tilde{\Psi}_+\rangle = +\cos \frac{\theta}{2} |e\rangle + \sin \frac{\theta}{2} |g\rangle$$

$$|\tilde{\Psi}_-\rangle = -\sin \frac{\theta}{2} |e\rangle + \cos \frac{\theta}{2} |g\rangle$$

$$\Delta\omega \equiv \omega - \omega_q$$

$$\tan \theta = -\frac{\omega_d}{\Delta\omega}$$

Driven Rabi oscillations

TLS population oscillates with Rabi

frequency $\omega_R \equiv \sqrt{\Delta\omega^2 + \omega_d^2}$

under transversal drive

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Rabi Oscillations on the Bloch sphere

$$|\tilde{\Psi}(t)\rangle = \cos\frac{\theta}{2} e^{+it\sqrt{\Delta\omega^2 + \omega_d^2}} |\tilde{\Psi}_-\rangle + \sin\frac{\theta}{2} e^{-it\sqrt{\Delta\omega^2 + \omega_d^2}} |\tilde{\Psi}_+\rangle$$

On resonance $\omega = \omega_q$

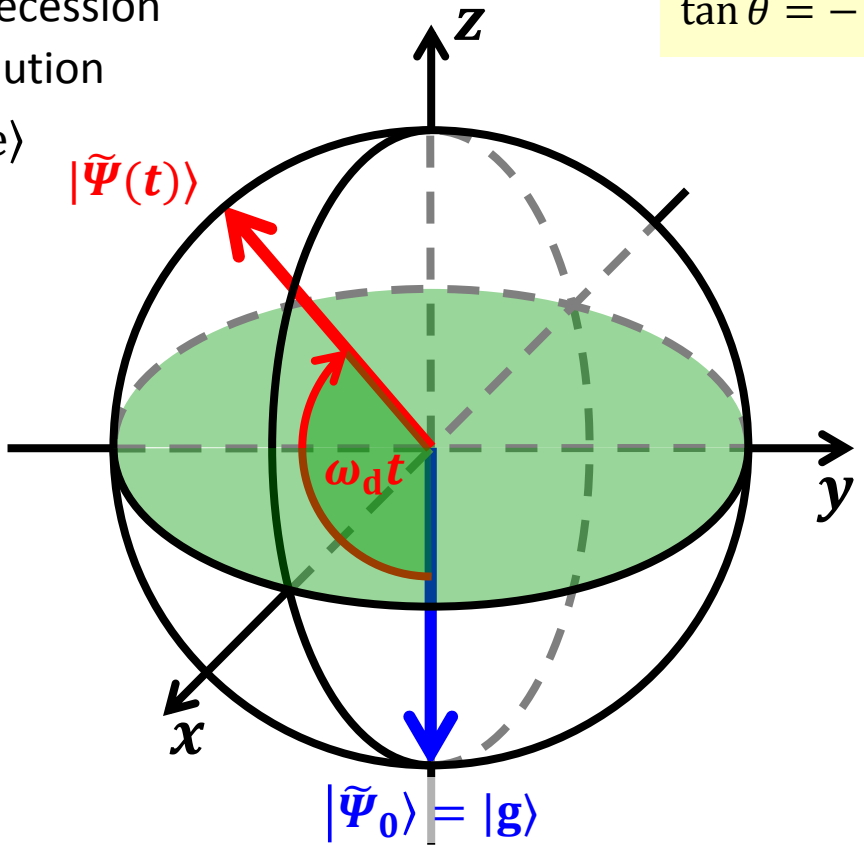
- Rotating frame cancels Larmor precession
- State vector $|\tilde{\Psi}(t)\rangle$ has no φ -evolution
- $|\tilde{\Psi}(t)\rangle = \cos\frac{\omega_d t}{2} |g\rangle + i \sin\frac{\omega_d t}{2} |e\rangle$
- Rotation about x -axis

$$\Delta\omega \equiv \omega - \omega_q$$

$$\tan\theta = -\frac{\omega_d}{\Delta\omega}$$

Finite detuning $|\Delta\omega| > 0$

- Additional precession of at $\Delta\omega$
- Population oscillates faster
- Reduced oscillation amplitude



$$P_e = \frac{\omega_d^2}{\Delta\omega^2 + \omega_d^2} \sin^2\left(\frac{t\sqrt{\Delta\omega^2 + \omega_d^2}}{2}\right)$$

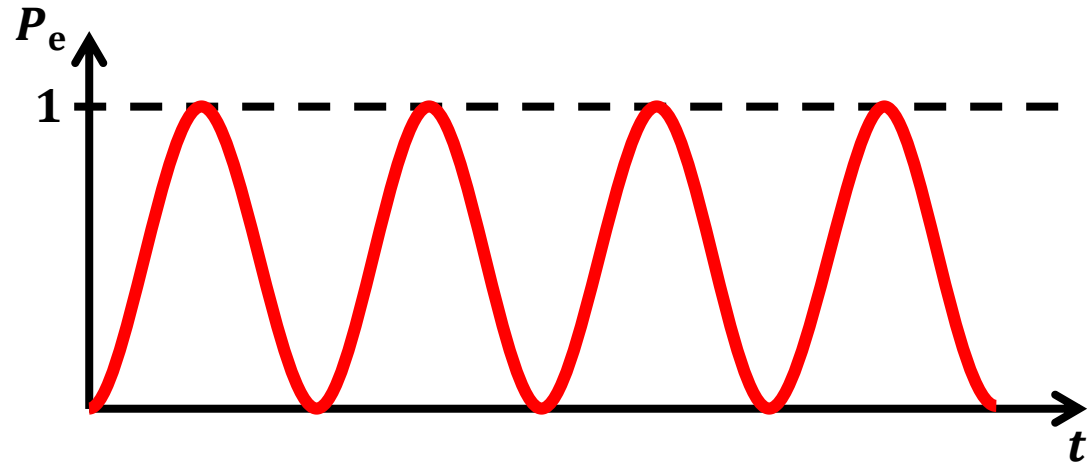
Arbitrary equatorial axis $\rightarrow \hat{H}_d = \frac{\hbar\omega_d}{2} (\hat{\sigma}_- e^{+i(\omega t + \varphi)} + \hat{\sigma}_+ e^{-i(\omega t + \varphi)})$

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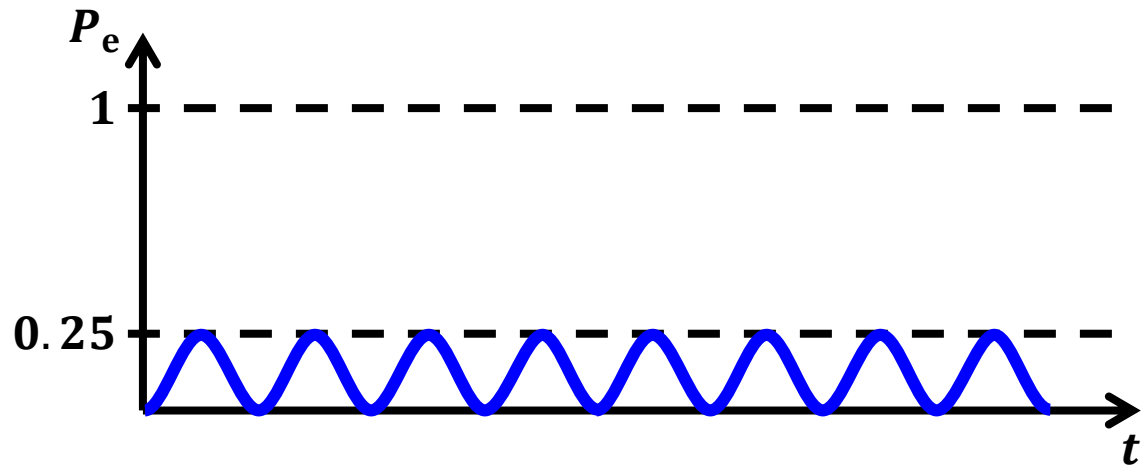
Rabi Oscillations – Graphical representation

$$P_e = \frac{\omega_d^2}{\Delta\omega^2 + \omega_d^2} \sin^2\left(\frac{t\sqrt{\Delta\omega^2 + \omega_d^2}}{2}\right)$$

On resonance $\omega = \omega_q$



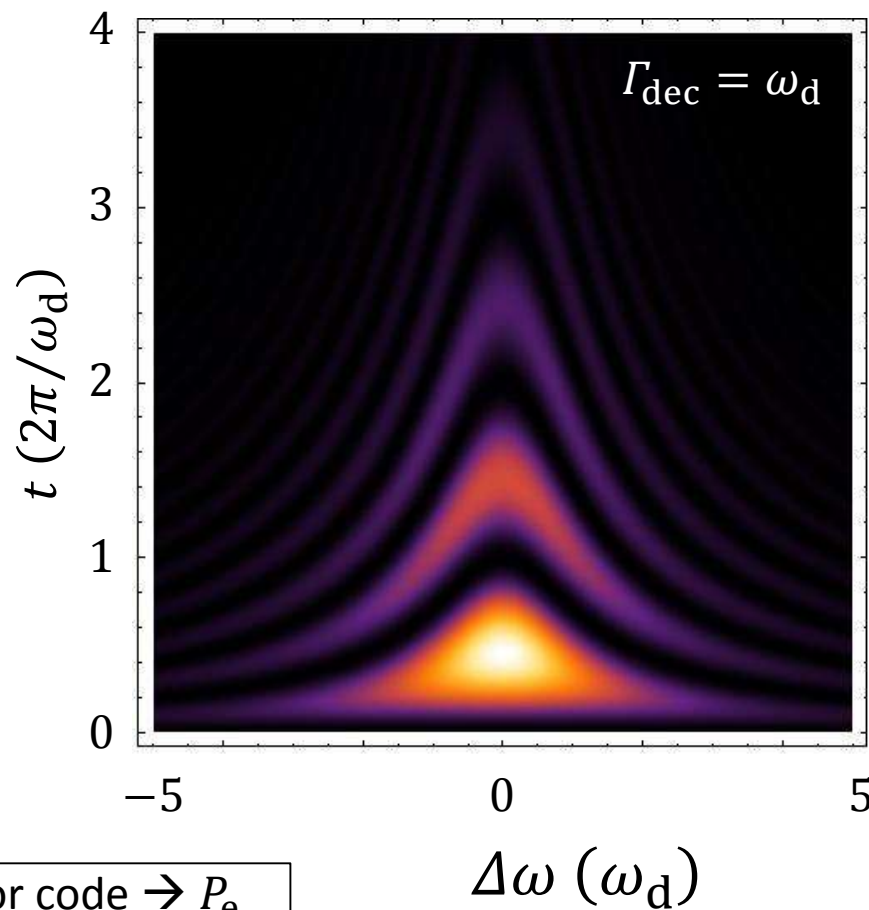
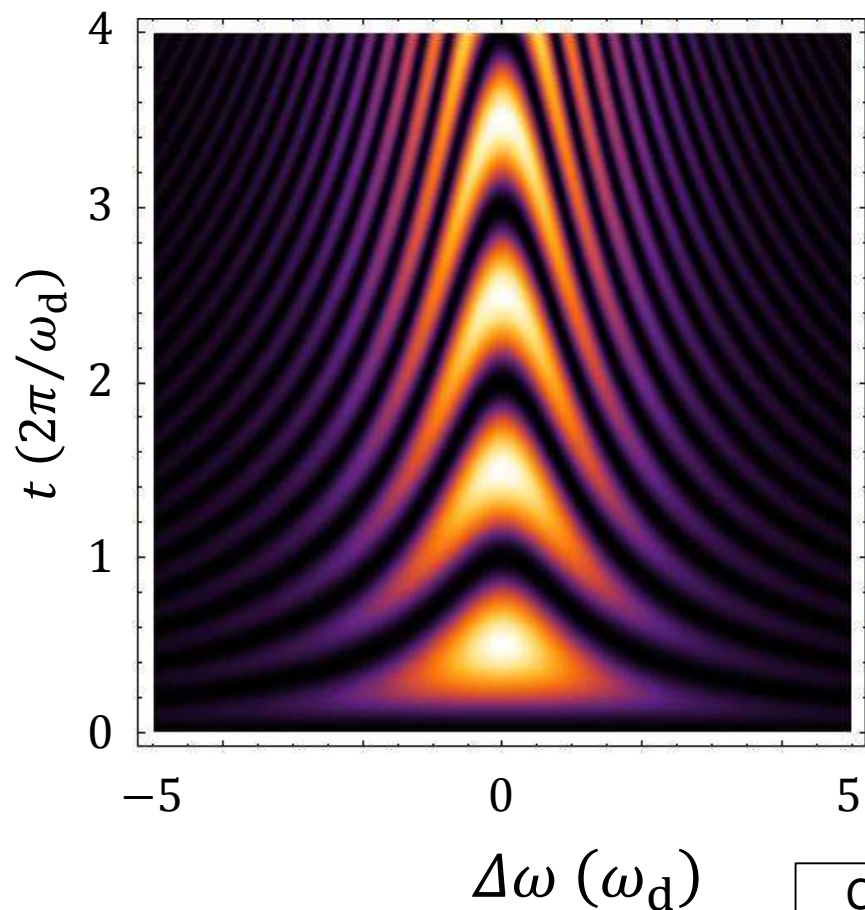
Detuning $|\Delta\omega| = \sqrt{3} \omega_d$



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Rabi Oscillations – Graphical representation

$$P_e = \frac{\omega_d^2}{\Delta\omega^2 + \omega_d^2} \sin^2 \left(\frac{t \sqrt{\Delta\omega^2 + \omega_d^2}}{2} \right)$$

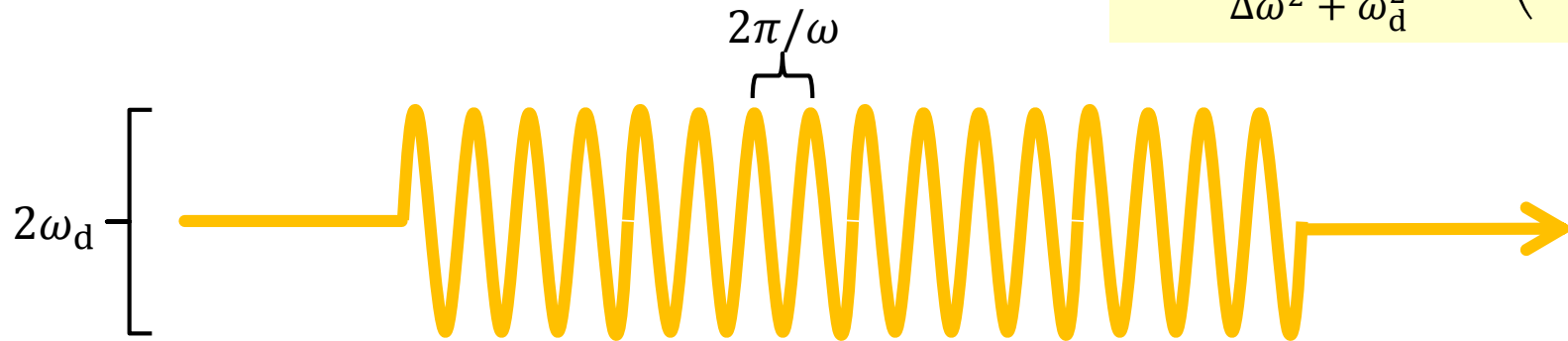


Color code $\rightarrow P_e$
(0 = black, 1 = white)

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Oscillating vs. rotating drive – Microwave pulses

$$P_e = \frac{\omega_d^2}{\Delta\omega^2 + \omega_d^2} \sin^2\left(\frac{t\sqrt{\Delta\omega^2 + \omega_d^2}}{2}\right)$$



Oscillating drive $2\hbar\omega_d \cos \omega t = \hbar\omega_d(e^{+i\omega t} + e^{-i\omega t})$

- In frame rotating with $+\omega$ the $e^{-i\omega t}$ -component rotates fast with -2ω
- For $\omega_d \ll \omega$, this fast contribution averages out on the timescale of the slowly rotating component → **Rotating wave approximation**
- Interaction picture (removing eigenenergies → rotating frame with ω_q → qubit-drive interaction)

$$\rightarrow \hat{\sigma}_{\pm} \rightarrow \hat{\sigma}_{\pm} e^{\pm i\omega_q t} \rightarrow \cos \omega t \hat{\sigma}_x \rightarrow \cos \omega t (\hat{\sigma}_+ e^{+i\omega_q t} + \hat{\sigma}_- e^{-i\omega_q t})$$

$$\rightarrow 2\hbar\omega_d (e^{+i\omega t} + e^{-i\omega t}) (\hat{\sigma}_+ e^{+i\omega_q t} + \hat{\sigma}_- e^{-i\omega_q t})$$

$$= 2\hbar\omega_d (\hat{\sigma}_+ e^{+i(\omega+\omega_q)t} + \hat{\sigma}_- e^{-i(\omega+\omega_q)t} + \hat{\sigma}_+ e^{-i\Delta\omega t} + \hat{\sigma}_- e^{+i\Delta\omega t})$$

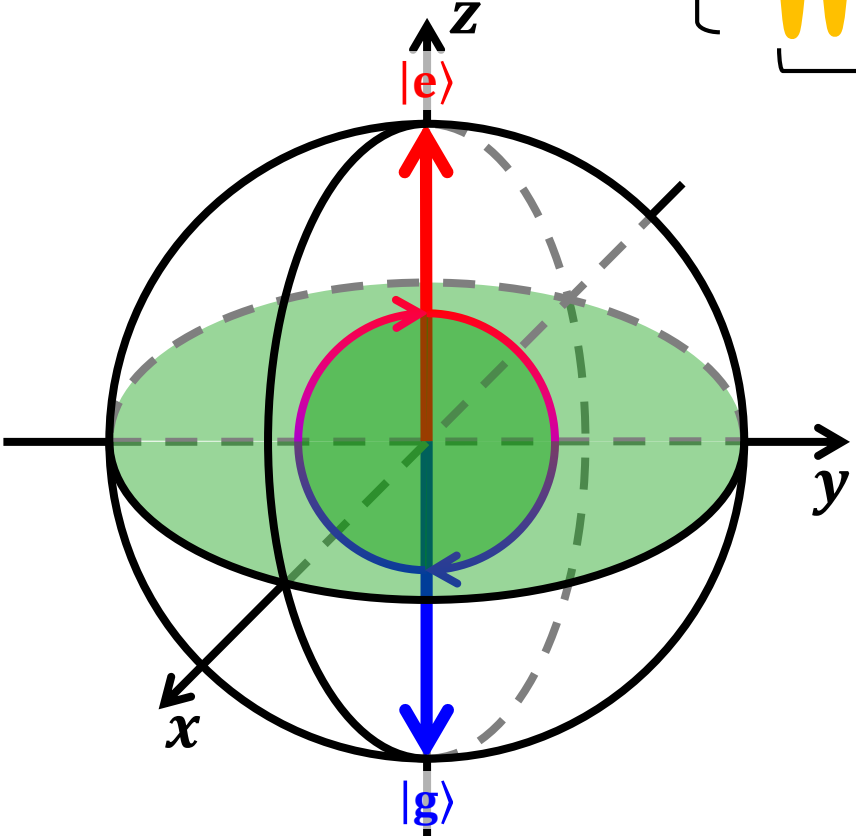
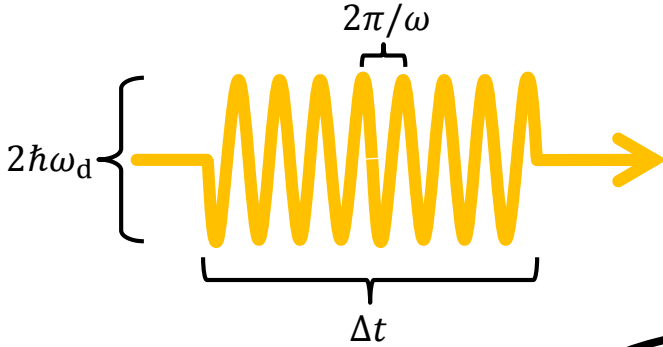
$$\approx 2\hbar\omega_d (\hat{\sigma}_+ e^{-i\Delta\omega t} + \hat{\sigma}_- e^{+i\Delta\omega t}) \quad (\text{if } 2\omega_d \ll \omega, \omega_q)$$

$$\Delta\omega \equiv \omega - \omega_q$$

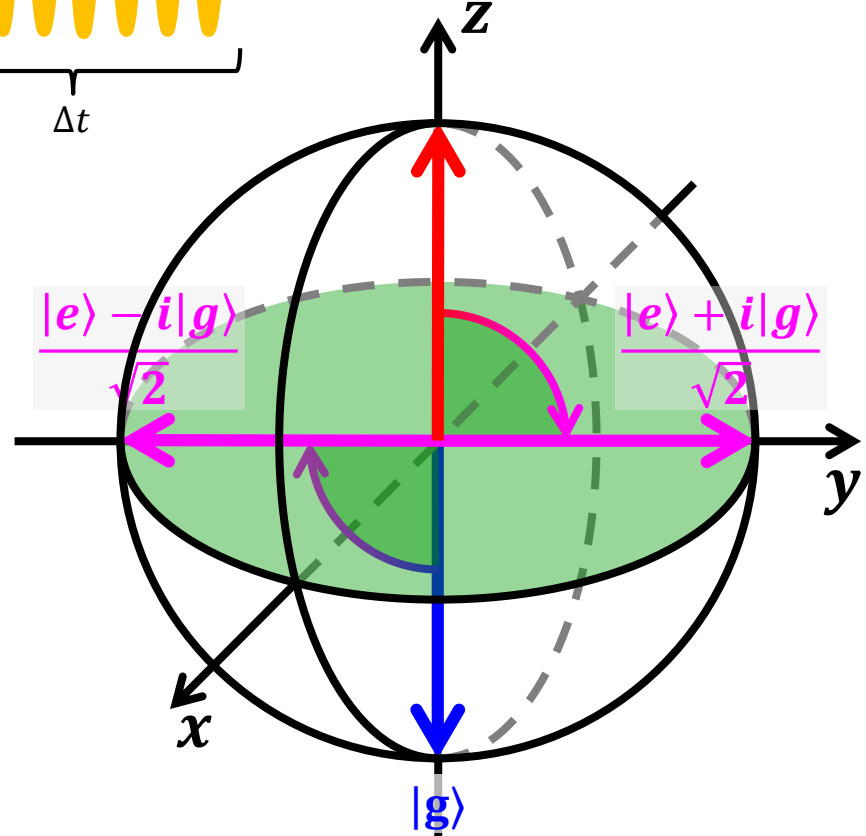
→ **Rabi oscillations just as for rotating drive**

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Important drive pulses on the Bloch sphere



π -pulse ($\omega_d \Delta t = \pi$)
 $|g\rangle \leftrightarrow |e\rangle$ flips, refocus phase evolution



$\pi/2$ -pulse ($\omega_d \Delta t = \pi/2$)
 Rotates into equatorial plane and back

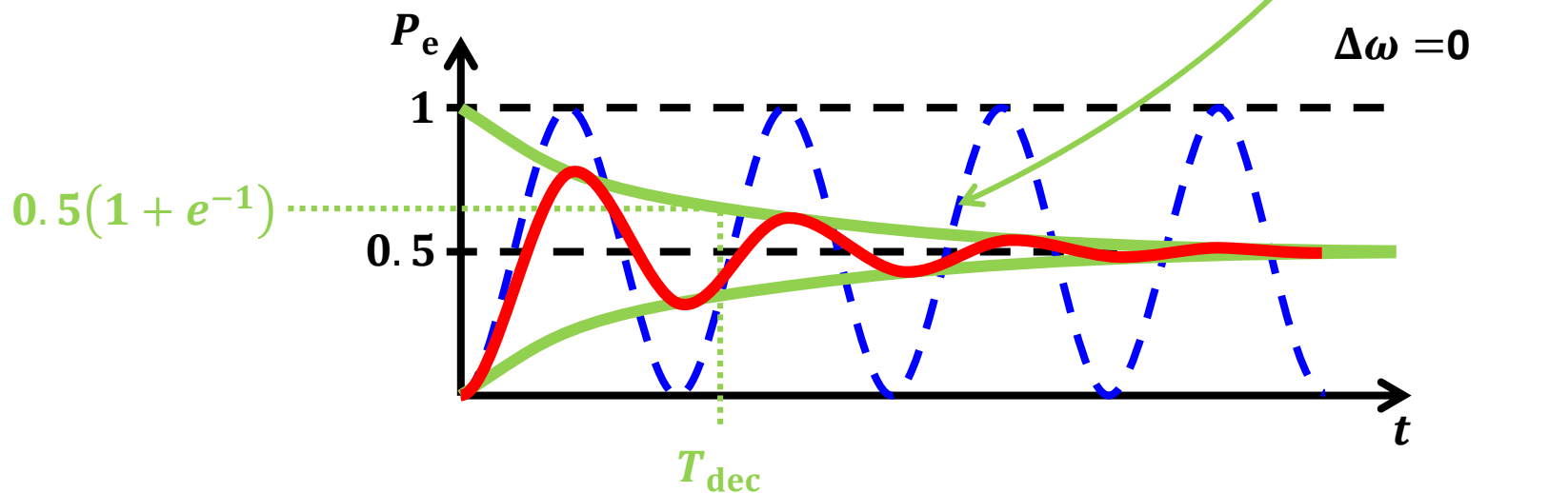
6.3 Control of quantum two-level systems

Rabi Oscillations in presence of decoherence

$$P_e = \frac{\omega_d^2}{\Delta\omega^2 + \omega_d^2} \sin^2 \left(\frac{t \sqrt{\Delta\omega^2 + \omega_d^2}}{2} \right)$$

Effect of decoherence (qualitative approach)

- Loss of coherent properties to environment at a constant rate $\Gamma_{\text{dec}} = 2\pi/T_{\text{dec}}$
- “Change of coherence” (time derivative) proportional to “amount of coherence”
- **Exponential decay** of coherent property with factor $e^{-\frac{\Gamma_{\text{dec}} t}{2\pi}} = e^{-\frac{t}{T_{\text{dec}}}}$
- Argument holds well for population decay (energy relaxation, T_1)
- Loss of phase coherence more diverse depending on environment (exponential, Gaussian, or power law)
- Experimental timescales range from few ns to 100 μs



6.3 Control of quantum two-level systems

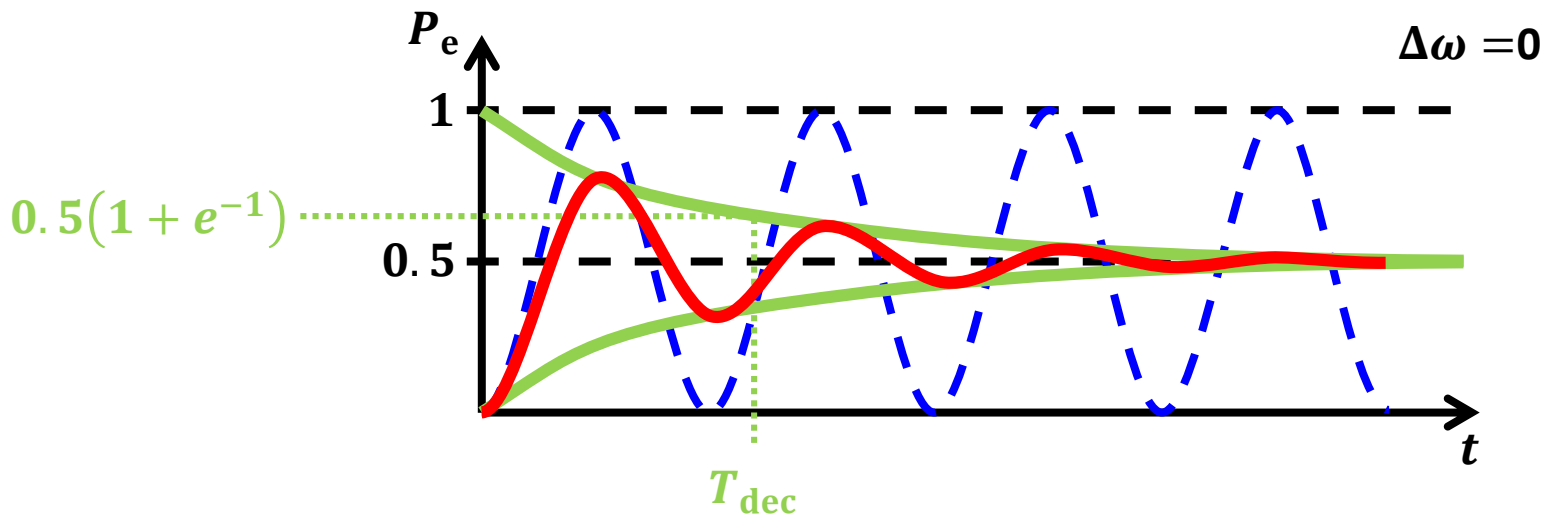
Rabi decay time

$$P_e = \frac{\omega_d^2}{\Delta\omega^2 + \omega_d^2} \sin^2 \left(\frac{t \sqrt{\Delta\omega^2 + \omega_d^2}}{2} \right)$$

Complicated interplay between T_1 , T_2^* , and the drive

- At long times, small oscillations persist for oscillatory drive
- Nevertheless useful **order-of-magnitude check** for T_{dec}
- Important tool for single-qubit gates

- To determine T_1 , T_2^* , T_2 correctly, more sophisticated protocols are required
- **Energy relaxation measurements, Ramsey fringes, spin echo**



6.3 Control of quantum two-level systems

Pure and mixed states on the Bloch sphere

No decoherence → Unitary evolution

→ State vector $|\Psi\rangle$ always well defined → **Pure state**

→ Describe via **density matrix** $\rho \equiv |\Psi\rangle\langle\Psi|$

→ Hermitian → $\rho^\dagger = \rho$

→ Expectation value $\langle\Psi|\hat{A}|\Psi\rangle = \text{Tr}[\rho\hat{A}]$

→ $\text{Tr}\rho = 1$ → Bloch vector length 1

Decoherence → Loss of unitary evolution

→ State vector $|\Psi\rangle$ only known probabilistically

→ **Mixed state** $\rho \equiv \sum_{i=1}^n p_i |\Psi_i\rangle\langle\Psi_i|$

→ $\text{Tr}\rho^2 < 1$ for $n > 1$

→ Bloch vector length < 1

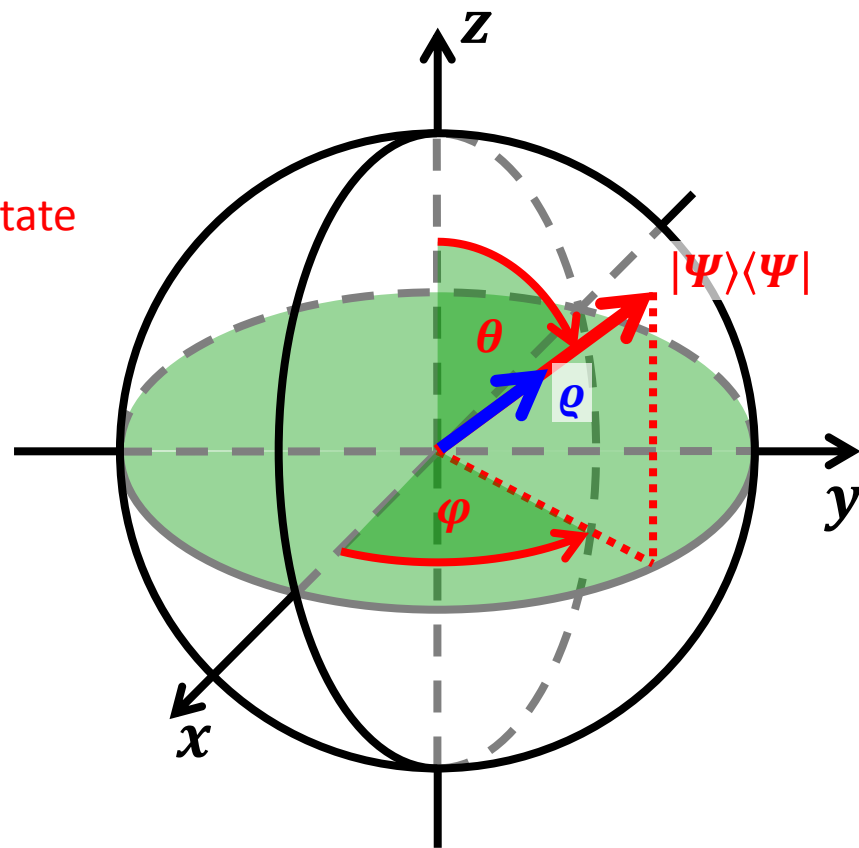
→ p_i are classical probabilities!

→ Center of Bloch sphere → Completely depolarized state $\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

→ General state → $\rho = \frac{1}{2}(1 + \mathbf{a} \cdot \boldsymbol{\sigma})$ with $\boldsymbol{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)^T$

→ **Bloch vector** $\mathbf{a} \equiv (a_x, a_y, a_z)^T$ with $|\mathbf{a}| \leq 1$

→ Eigenvalues of ρ are $\frac{1}{2}(1 \pm |\mathbf{a}|)$



6.3 Control of quantum two-level systems

Example: Pure state density matrix on the Bloch sphere

Goal → Get intuition for the Bloch vector \mathbf{a}

$$|\Psi\rangle = \cos\frac{\theta}{2}|e\rangle + e^{i\varphi}\sin\frac{\theta}{2}|g\rangle$$

$$\begin{aligned} \rho &= |\Psi\rangle\langle\Psi| = \left(\cos\frac{\theta}{2}|e\rangle + e^{i\varphi}\sin\frac{\theta}{2}|g\rangle \right) \left(\cos\frac{\theta}{2}\langle e| + e^{-i\varphi}\sin\frac{\theta}{2}\langle g| \right) = \\ &= \cos^2\frac{\theta}{2}|e\rangle\langle e| + \sin^2\frac{\theta}{2}|g\rangle\langle g| + e^{i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}|e\rangle\langle g| + e^{-i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}|g\rangle\langle e| \end{aligned}$$

$$\begin{aligned} \hat{\sigma}_+ &= |e\rangle\langle g| \\ \hat{\sigma}_- &= |g\rangle\langle e| \end{aligned}$$

$$= \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & 1 - \cos\theta \end{pmatrix}$$

$$\begin{aligned} \sin\theta &= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ 1 + \cos\theta &= \cos^2\frac{\theta}{2} \\ 1 - \cos\theta &= \sin^2\frac{\theta}{2} \end{aligned}$$

Bloch vector → $\rho = \frac{1}{2}(1 + \mathbf{a} \cdot \boldsymbol{\sigma}) = \frac{1}{2} \begin{pmatrix} 1 + a_z & a_x - ia_y \\ a_x + ia_y & 1 - a_z \end{pmatrix}$

Compare coefficients → $\mathbf{a} = \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix}$ → Vector on surface of a sphere
Unit length & polar angles (θ, φ)

6.3 Control of quantum two-level systems

Dynamics of the state vector on the Bloch sphere with decoherence

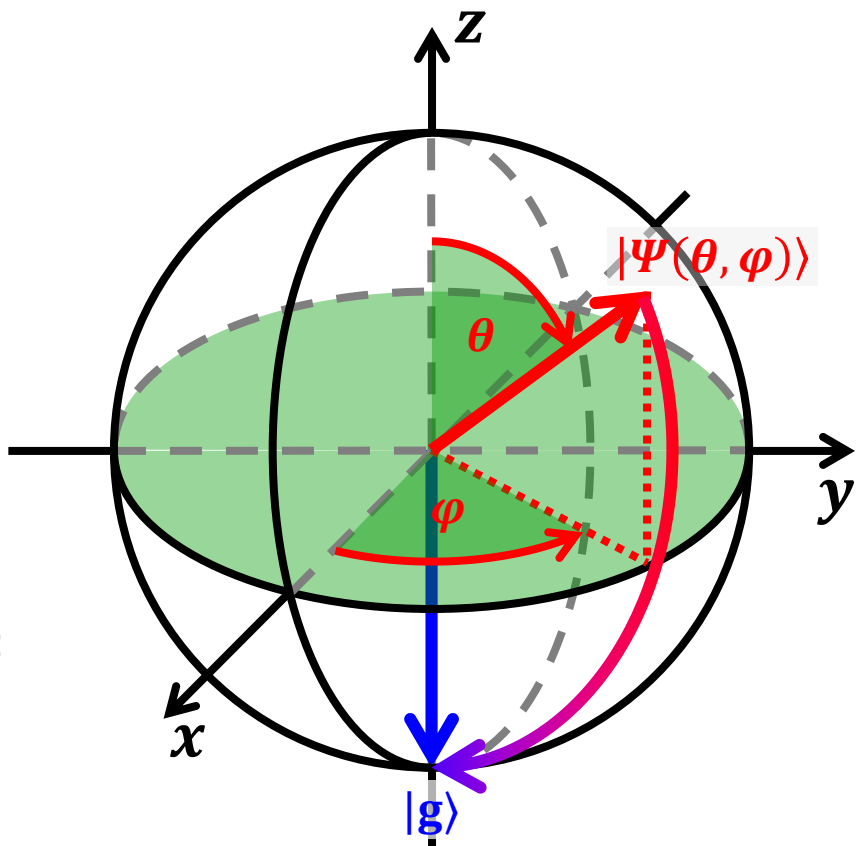
Equations of motion for spin $\frac{1}{2}$ under influence of the control fields $\mathbf{B}(t)$

$$\frac{d\mathbf{r}(t)}{dt} = -\mathbf{B}(t) \times \mathbf{r}(t) - \frac{1}{T_1} [r_z(t) - r_z(0)] \hat{z} - \frac{1}{T_2} [r_x(t) \hat{x} + r_y(t) \hat{y}]$$

- $\mathbf{r}(t)$ is state vector on the Bloch sphere
- Introduced in 1946 by Bloch for NMR systems
- Describe simple exponential decays with time constants T_1 and T_2
- In simple scenarios confirmed by quantum master equations

6.3 Control of quantum two-level systems

Energy relaxation on the Bloch sphere



Environment induces energy loss

- State vector collapses to $|g\rangle$
- Implies also loss of phase information
- Intrinsically irreversible
- T_1 -time → rate $\Gamma_1 = \frac{2\pi}{T_1}$

Golden Rule argument

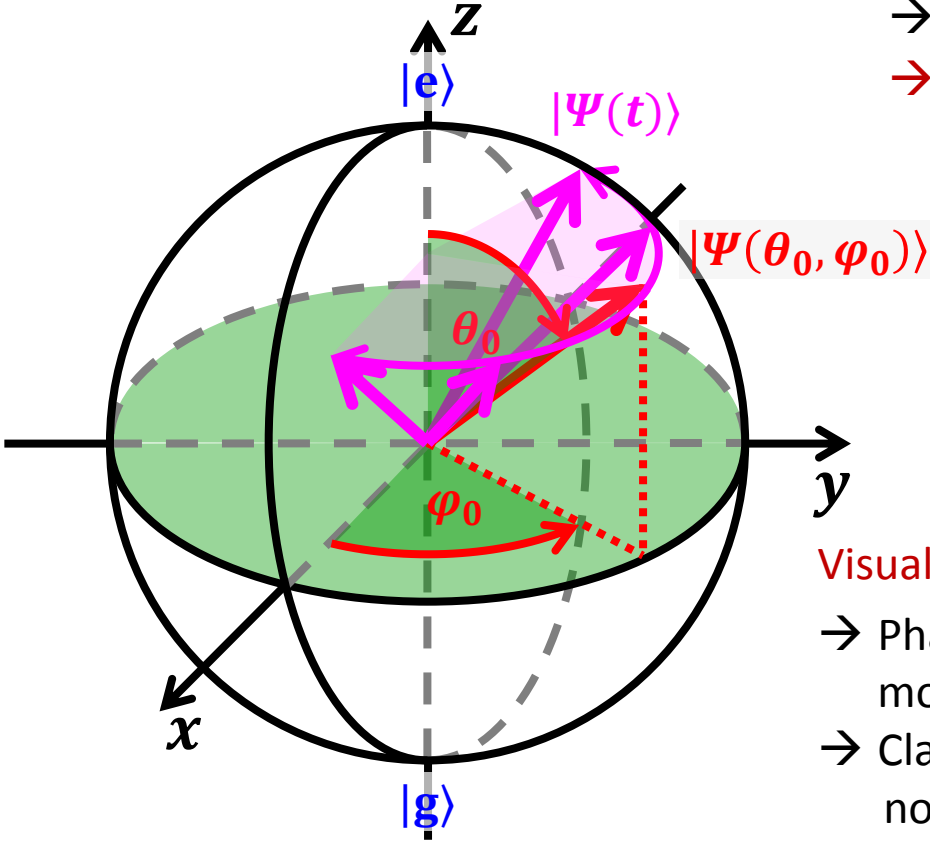
- $\Gamma_1 \propto S(\omega_q)$
- $S(\omega)$ is noise spectral density
- High frequency noise
- Intuition: Noise induces transitions

Quantum jumps

- Single-shot, quantum nondemolition measurement yields a discrete jump to $|g\rangle$ at a random time
- Probability is equal for each point of time
- Exponential decay with $e^{-\frac{t}{T_1}}$

6.3 Control of quantum two-level systems

Dephasing on the Bloch sphere



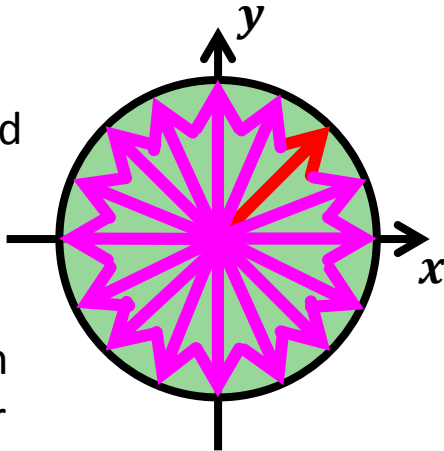
Environment induces **random phase changes**

- $\Gamma_\varphi \propto S(\omega \rightarrow 0), T_\varphi = \frac{2\pi}{\Gamma_\varphi}$, (for Markovian bath)
- Low-frequency noise is detuned from ω_q
- **No energy transfer**

- $1/f$ -noise → $S(\omega) \propto \frac{1}{\omega}$
- Example: two-level fluctuator bath
- To some extent reversible
- Decay laws $e^{-\frac{t}{T_\varphi}}, e^{-\left(\frac{t}{T_\varphi}\right)^2}, \left(\frac{t}{T_\varphi}\right)^\beta$

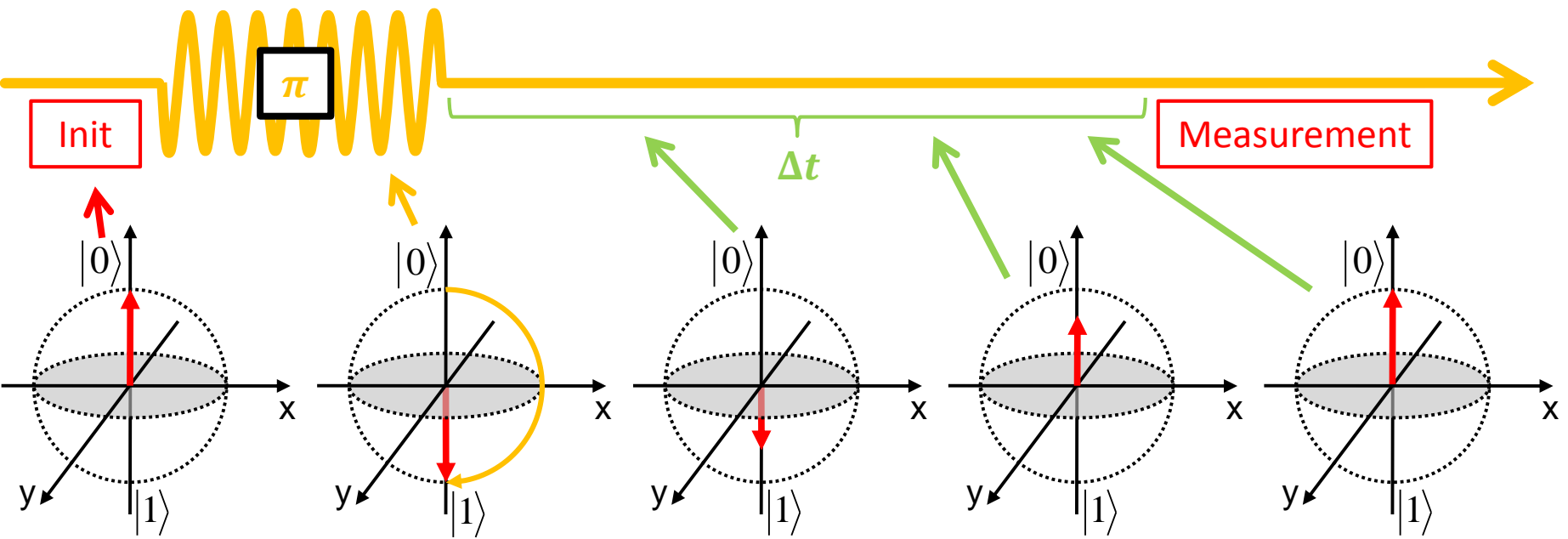
Visualization

- Phase φ becomes more and more unknown with time
- Classical probability, no superposition!
- Phase coherence lost when arrows are distributed over whole equatorial plane

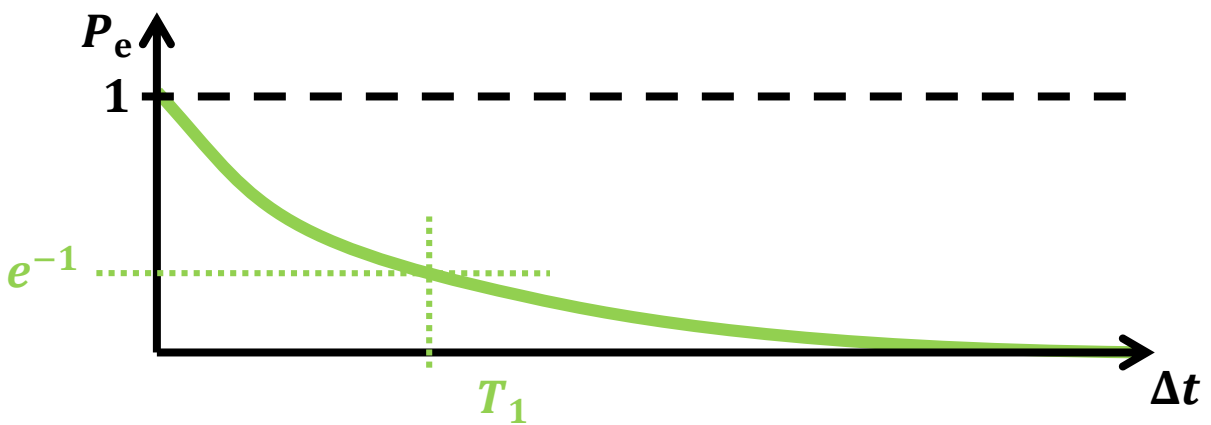


6.3 Control of quantum two-level systems

Qubit dynamics – Relaxation (T_1)



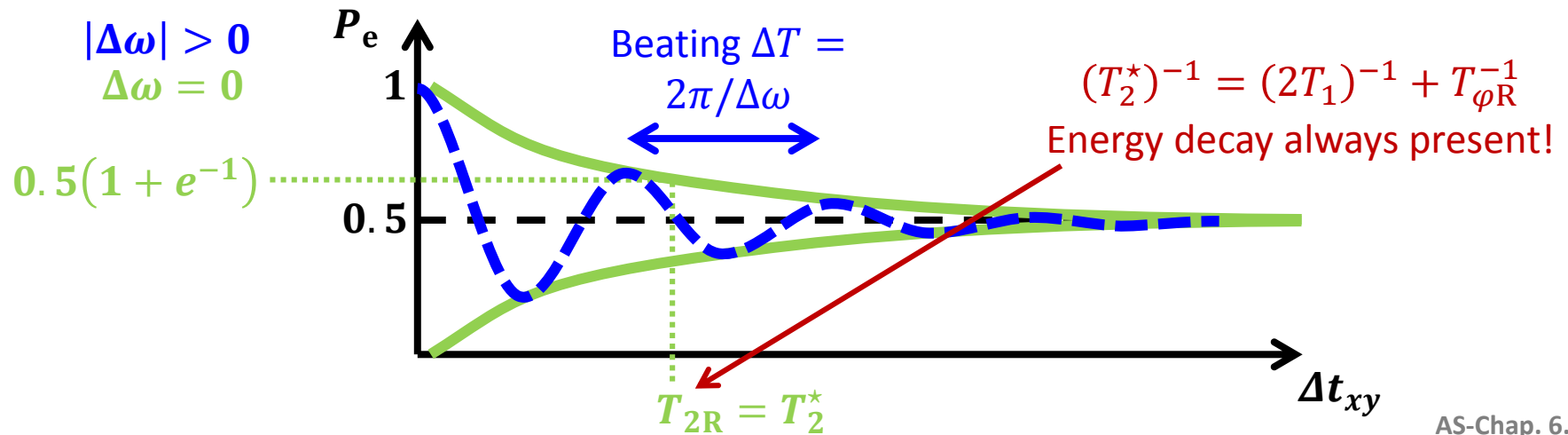
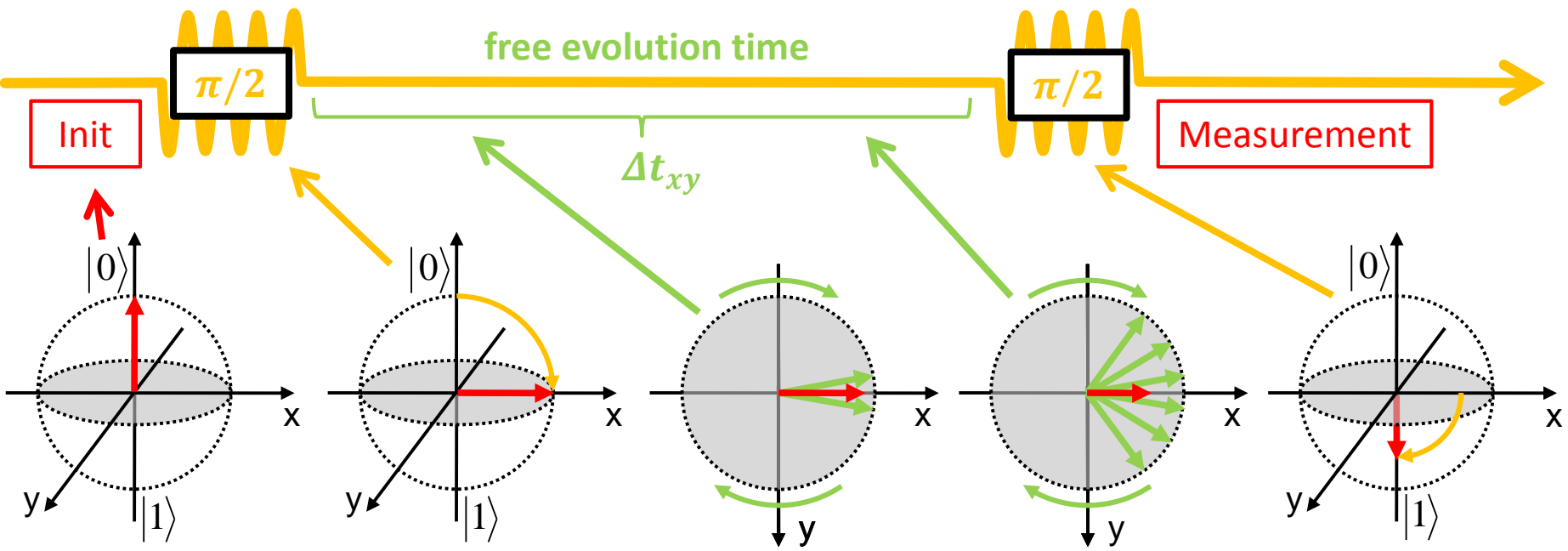
Rotating frame & no detuning ($\Delta\omega = \omega - \omega_q = 0$) \rightarrow no xy -evolution



R. Gross, A. Marx, F. Deppe, and K. Fedorov © Walther-Meißner-Institut (2001 - 2018)

6.3 Control of quantum two-level systems

Qubit dynamics – Ramsey fringes (T_2^*)

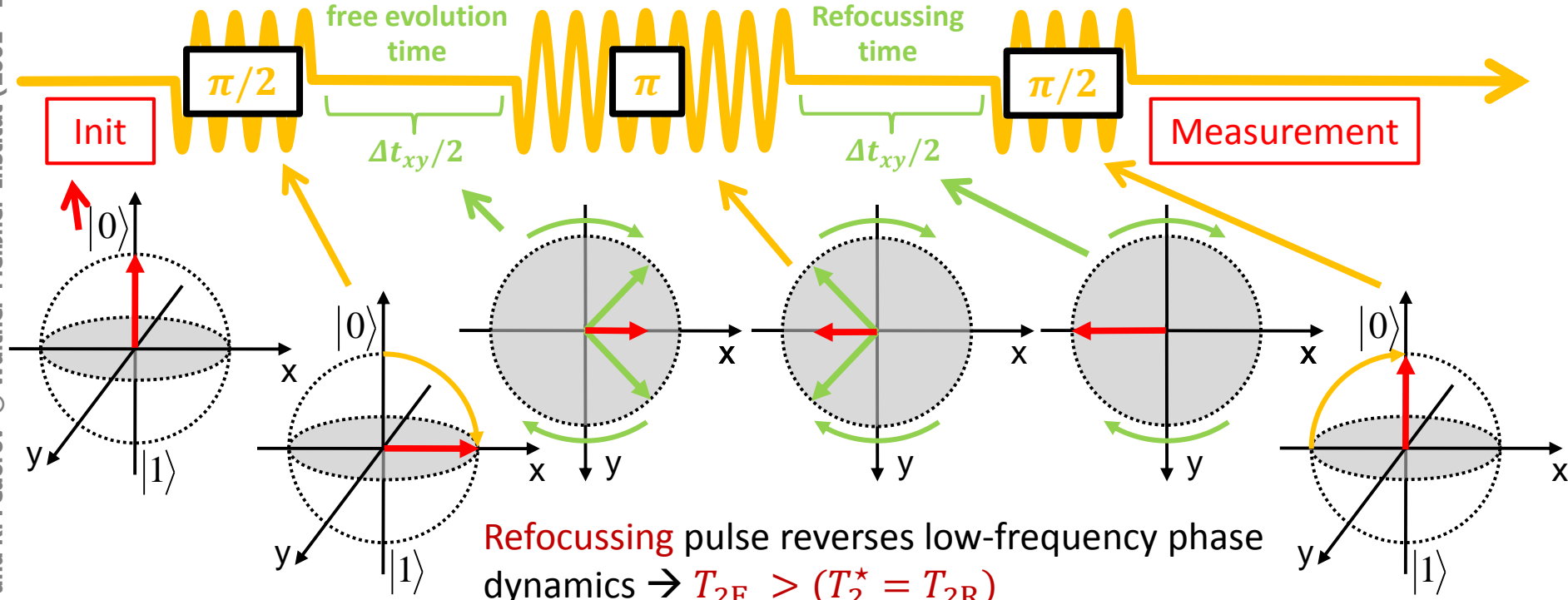


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6.3 Control of quantum two-level systems

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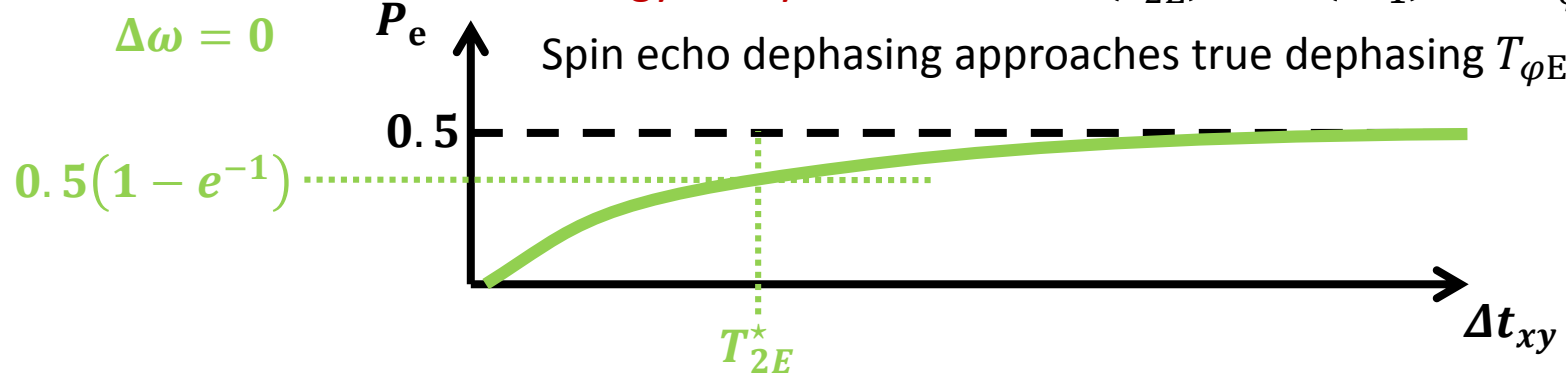
Qubit dynamics – Spin echo (T_{2E}^*)



Refocussing pulse reverses low-frequency phase dynamics $\rightarrow T_{2E} > (T_2^* = T_{2R})$

Energy decay unaffected $\rightarrow (T_{2E})^{-1} = (2T_1)^{-1} + T_{\varphi E}^{-1}$

Spin echo dephasing approaches true dephasing $T_{\varphi E} \approx T_{\varphi}$



6.3 Control of quantum two-level systems

Ramsey vs. spin echo sequence

Simple picture $\rightarrow T_{2E} \simeq T_2$

More precise \rightarrow Spin echo cancels the effect of low-frequency noise

\rightarrow Pulse sequences act as filters!

\rightarrow Environment described by noise spectral density $S(\omega)$

Decay envelope $\propto e^{-t^2 \int_{-\infty}^{+\infty} S(\omega) F_{R,E}(\frac{\omega \Delta t_{xy}}{2}) d\omega}$

$$F_R(\omega \Delta t_{xy}) = \frac{\sin^2 \frac{\omega \Delta t_{xy}}{2}}{\left(\frac{\omega \Delta t_{xy}}{2}\right)^2}$$

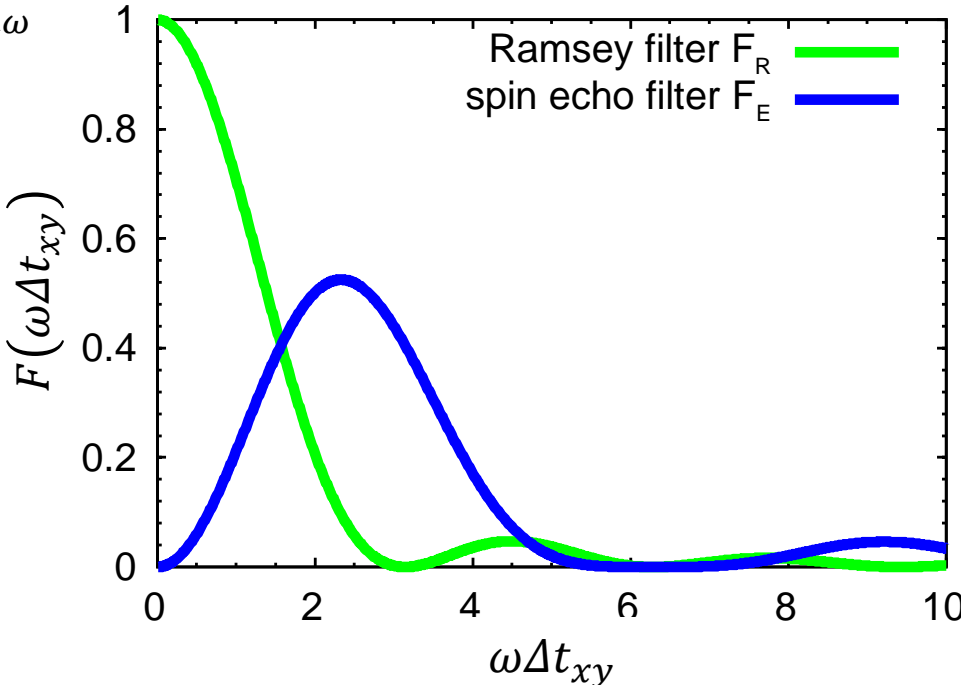
$$F_E(\omega \Delta t_{xy}) = \frac{\sin^4 \frac{\omega \Delta t_{xy}}{4}}{\left(\frac{\omega \Delta t_{xy}}{4}\right)^2}$$

Sequence length Δt_{xy} is important!

Spin echo sequence

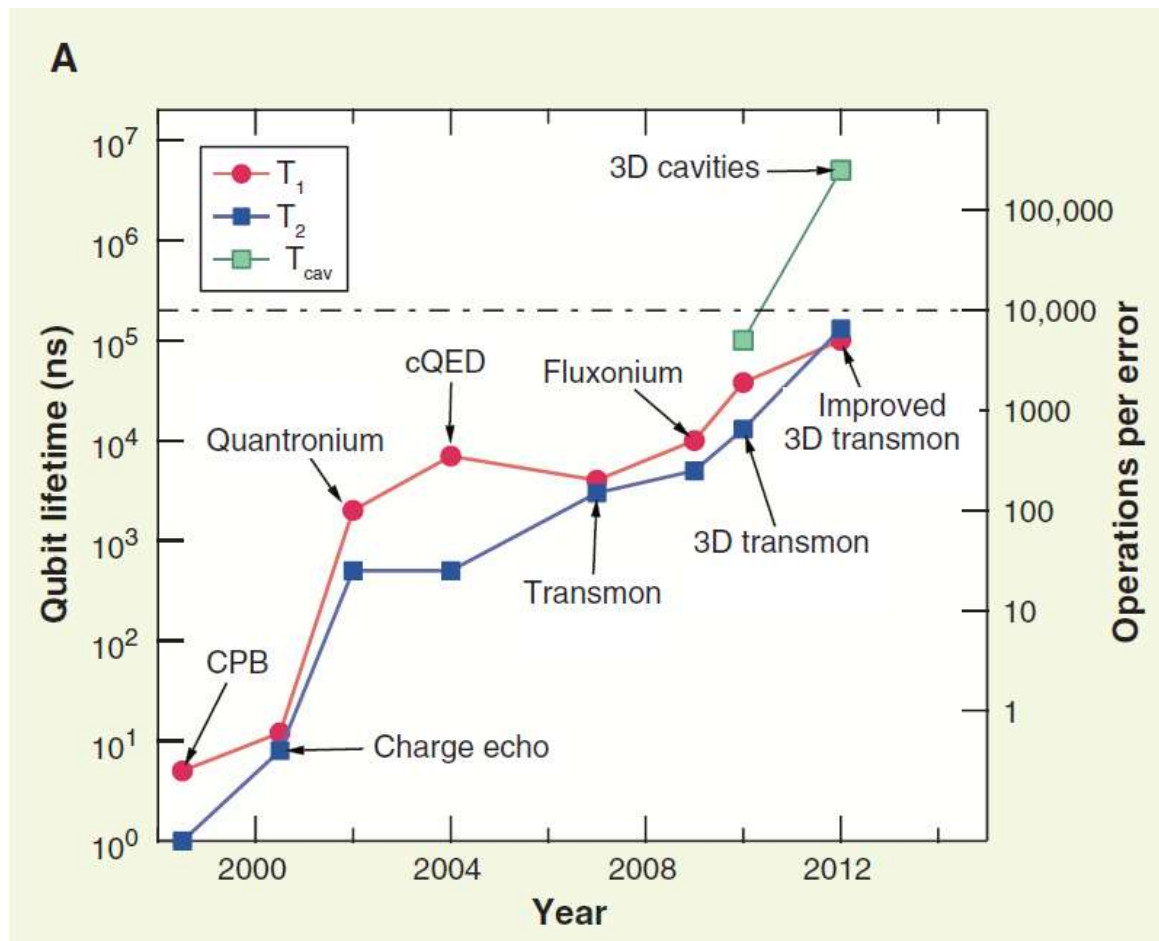
\rightarrow Filters low-frequency noise for $\omega \Delta t_{xy} \rightarrow 0$

$\rightarrow \omega \Delta t_{xy} \approx 2 \rightarrow$ Noise field fluctuates synchronously with π -pulse \rightarrow No effect



6.3 Control of quantum two-level systems

Superconducting qubits – Improvement of decoherence times



M. H. Devoret and R. J. Schoelkopf, *Science* **339**, 1169 (2013); DOI:10.1126/science.1231930