Applied Superconductivity:

Josephson Effect and Superconducting Electronics

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Chapter F

Quantum Mechanical Two-Level Systems

We have seen that quantum bits can be represented by every two-level quantum system. There are numerous cases in physics, which can be in first order approximation treated simply as such kind of system. For example, a system with two states whose energies are close and differ very much from those of all other states of the system can be view as a two-level system. Therefore, we briefly summarize here the basic properties of quantum mechanical two-level systems. In particular we address the effect of an external perturbation as well as an internal interaction on the two states. The general treatment of a two-level system will provide some general and important ideas such as quantum resonance, oscillation between two levels etc..

I Introduction to the Problem

We consider a system with a two-dimensional state space. As an orthonormal basis we choose the system of the two eigenstates $|\phi_1\rangle$ and $|\phi_2\rangle$ (cf. (I.2) and I.3)) of the Hamiltonian $\mathscr{H}_0$, whose eigenvalues are E_1 and E_2, respectively:

$$\mathscr{H}_0|\phi_1\rangle = E_1|\phi_1\rangle \tag{I.1}$$

$$\mathscr{H}_0|\phi_2\rangle = E_2|\phi_1\rangle \ . \tag{I.2}$$

We further take into account an external perturbation or interactions internal to the system, which are not contained in $\mathscr{H}_0$, which is called the unperturbed Hamiltonian. The total Hamiltonian then becomes

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{W} \tag{I.3}$$

with the perturbation or coupling $\mathscr{W}$. The eigenvalues of $\mathscr{H}$ are denoted by $|\Psi_+\rangle$ and $|\Psi_-\rangle$ with the corresponding eigenvalues E_+ and E_-:

$$\mathscr{H}|\Psi_+\rangle = E_+|\Psi_+\rangle \tag{I.4}$$

$$\mathscr{H}|\Psi_-\rangle = E_-|\Psi_-\rangle \ . \tag{I.5}$$

For simplicity, we will assume that $\mathscr{W}$ is time-independent. In the basis of $\{|\phi_1\rangle, |\phi_2\rangle\}$ of the unperturbed eigenstates of $\mathscr{H}_0$, the perturbation $\mathscr{W}$ is represented by a Hermitian matrix

$$\mathscr{W} = \begin{pmatrix} \mathscr{W}_{11} & \mathscr{W}_{12} \\ \mathscr{W}_{21} & \mathscr{W}_{22} \end{pmatrix} \ . \tag{I.6}$$

$\mathscr{W}_{11}$ and $\mathscr{W}_{22}$ are real and moreover $\mathscr{W}_{12} = \mathscr{W}_{21}^{\star}$.

In the absence of any perturbation or coupling the possible eigenenergies of the system are E_1 and E_2 and the states $|\phi_1\rangle$ and $|\phi_2\rangle$ are stationary states, i.e. if the system is prepared in one of these states it stays there forever.

We now have to evaluate what happens if we are introducing a finite coupling $\mathscr{W}$. The consequences of the coupling are the following:

- E_1 and E_2 are no longer the possible eigenstates of the system.

  If we are measuring the energy of the system only the two values E_+ and E_- are possible, which generally differ from E_1 and E_2. Therefore, we first have to calculate the new eigenenergies E_+ and E_- in terms of E_1 and E_2 and the matrix elements $\mathscr{W}_{ij}$ of the coupling $\mathscr{W}$. That is, we have to study the effect of the coupling on the position of the energy levels.

- $|\phi_1\rangle$ and $|\phi_2\rangle$ are no longer stationary states.

  Since $|\phi_1\rangle$ and $|\phi_2\rangle$ are in general no longer eigenstates of the total Hamiltonian $\mathscr{H}$, they are no longer stationary states. If the system stays in the state $|\phi_1\rangle$ at the time $t = 0$, there is a certain probability $P_{12}(t)$ for finding the system in the state $|\phi_2\rangle$ at time t. That is, $\mathscr{W}$ introduces transitions between the two unperturbed states. This justifies the name "coupling" for $\mathscr{W}$. The dynamic aspect of the effect of $\mathscr{W}$ is the second problem we have to address.

I.1 Relation to Spin-1/2 Systems

It can be shown that the Hamiltonian $\mathscr{H}$ has the same form as that of a spin $1/2$ placed in a static magnetic field **B**, whose components B_x, B_y and B_z are expressed in terms of E_1 and E_2 and the matrix elements $\mathscr{W}_{ij}$. That means that we can associate with every two-level system a spin $1/2$ placed in a static field **B** and described by a Hamiltonian of identical form. The spin is then called a ***fictitious spin***. All results we are deriving in the following can then be interpreted in a simple geometric way in terms of a magnetic moment, Larmor precession and other concepts used for spin $1/2$ systems. This geometrical interpretation often helps to get a helpful illustration of what is going on. For a discussion of the spin-1/2-system, see Appendix G.

II Static Properties of Two-Level Systems

II.1 Eigenstates and Eigenvalues

We first write the Hamiltonian $\mathscr{H}$ in the $\{|\phi_1\rangle, |\phi_2\rangle\}$ basis of the unperturbed eigenstates:

$$\mathscr{H} = \begin{pmatrix} \mathscr{H}_0 + \mathscr{W}_{11} & \mathscr{W}_{12} \\ \mathscr{W}_{21} & \mathscr{H}_0 + \mathscr{W}_{22} \end{pmatrix} . \qquad \text{(II.7)}$$

With $|\Psi\rangle = a|\phi_1\rangle + b|\phi_2\rangle$ we obtain the eigenvalue equation

$$\begin{pmatrix} E_1 + \mathscr{W}_{11} - E & \mathscr{W}_{12} \\ \mathscr{W}_{12}^{\star} & E_2 + \mathscr{W}_{22} - E \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 . \qquad \text{(II.8)}$$

Upon diagonalization of the matrix we find the eigenvalues

$$E_+ = \frac{1}{2}(E_1+W_{11}+E_2+W_{22})+\frac{1}{2}\sqrt{(E_1+W_{11}-E_2-W_{22})^2+4|W_{12}|^2} \quad \text{(II.9)}$$
$$E_- = \frac{1}{2}(E_1+W_{11}+E_2+W_{22})-\frac{1}{2}\sqrt{(E_1+W_{11}-E_2-W_{22})^2+4|W_{12}|^2} \ . \quad \text{(II.10)}$$

We immediately see that E_+ and E_- are identical to E_1 and E_2 for $W=0$. The corresponding eigenvectors can be written as

$$|\Psi_+\rangle = \cos\frac{\theta}{2}e^{-\imath\varphi/2}|\phi_1\rangle+\sin\frac{\theta}{2}e^{+\imath\varphi/2}|\phi_2\rangle \quad \text{(II.11)}$$
$$|\Psi_-\rangle = -\sin\frac{\theta}{2}e^{-\imath\varphi/2}|\phi_1\rangle+\cos\frac{\theta}{2}e^{+\imath\varphi/2}|\phi_2\rangle \ , \quad \text{(II.12)}$$

where the angle θ and φ are given by

$$\tan\theta = \frac{2|W_{12}|}{E_1+W_{11}-E_2-W_{22}} \quad \text{(II.13)}$$
$$W_{21} = |W_{21}|\,e^{\imath\varphi} \ . \quad \text{(II.14)}$$

II.2 Interpretation

In order to discuss the above results we first will do a graphical representation of the effect of coupling. The most interesting effect of the perturbation $\mathscr{W}$ is the fact that it possesses off-diagonal matrix elements $\mathscr{W}_{12}=\mathscr{W}_{21}^\star$. If the off-diagonal terms would vanish, the eigenstates of $\mathscr{H}$ would be the same as those of $\mathscr{H}_0$ and the new eigenenergies would be E_1+W_{11} and E_2+W_{22}. Since the diagonal terms of the perturbation are not very interesting, we will assume $W_{11}=W_{22}=0$ in the following. With this assumption the expression for the eigenenergies simplify to

$$E_+ = \frac{1}{2}(E_1+E_2)+\frac{1}{2}\sqrt{(E_1-E_2)^2+4|W_{12}|^2} \quad \text{(II.15)}$$
$$E_- = \frac{1}{2}(E_1+E_2)-\frac{1}{2}\sqrt{(E_1-E_2)^2+4|W_{12}|^2} \quad \text{(II.16)}$$

with

$$\tan\theta = \frac{2|W_{12}|}{E_1-E_2} \qquad 0\le\theta<\pi \quad \text{(II.17)}$$
$$W_{12} = |W_{21}|\,e^{\imath\varphi} \ . \quad \text{(II.18)}$$

By introducing the two parameters

$$E_m \equiv \frac{1}{2}(E_1+E_2) \quad \text{(II.19)}$$
$$\Delta \equiv \frac{1}{2}(E_1-E_2) \quad \text{(II.20)}$$

we obtain

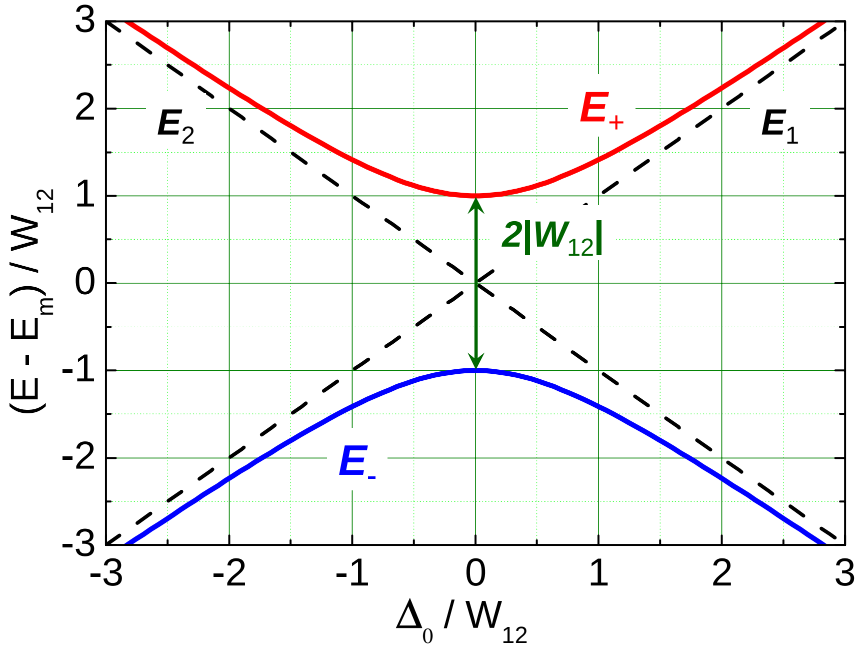

Figure F.1: Variation of the eigenenergies E_+ and E_- as a function of the parameter $\Delta = (E_1 - E_2)/2$. Also shown are the energies E_1 and E_2 (dashed lines).

$$E_+ = E_m + \frac{1}{2}\sqrt{\Delta^2 + 4|W_{12}|^2} \tag{II.21}$$

$$E_- = E_m - \frac{1}{2}\sqrt{\Delta^2 + 4|W_{12}|^2} \ . \tag{II.22}$$

We see that a variation of E_m corresponds to a shift of the eigenenergies E_+ and E_- along the energy axis. It can be further seen from (F.II.11) to (F.II.14) that the eigenstates $|\Psi_+\rangle$ and $|\Psi_-\rangle$ are not affected by changes of E_m. We therefore are not interested in the effect of E_m. In the following we will set the origin of the energy scale such that $E_m = 0$.

The influence of the parameter Δ is more interesting. In Fig. F.1 we have plotted the variation of the eigenenergies E_+, E_-, E_1 and E_2 as a function of the parameter $\Delta = (E_1 - E_2)/2$. It is evident that for E_1 and E_2 two straight lines are obtained with slopes $+1$ and -1, respectively. According to (F.II.21) and (F.II.22), E_+ and E_- describe two branches of a hyperbola, which is symmetrical with respect to the $E = E_m$ and $\Delta = 0$ axis. The asymptotes of the hyperbola are the two straight lines associated with the unperturbed levels. The minimum separation between the two branches is $2|W_{12}|$. We immediately see that $E_+ \to E_1$ and $E_- \to E_2$ for $E_1 > E_2$ as well as $E_+ \to E_2$ and $E_- \to E_1$ for $E_1 < E_2$.

Discussing the effect of the coupling on the position of the energy levels we see the following: First, in the absence of any coupling the levels (E_1 and E_2) cross at the position $(E = E_m, \Delta = 0)$. Under the effect of the off-diagonal coupling the two perturbed levels E_+ and E_- repel each other, i.e. the energy values move further apart from each other, and we obtain the typical ***anti-crossing behavior***. We also see that for any Δ we have

$$|E_+ - E_-| \quad > \quad |E_1 - E_2| \ . \tag{II.23}$$

This result is well known from other fields of physics. For example, in electronic circuit theory the coupling separates the normal frequencies.

Near the asymptotes we have $|\Delta| \gg |W_{12}|$ and the expressions (F.II.21) and (F.II.22) can be expanded into a power series in $|W_{12}/\Delta|$:

$$E_+ = E_m + \Delta\left(1 + \frac{1}{2}\left|\frac{W_{12}}{\Delta}\right|^2 + \ldots\right) \tag{II.24}$$

$$E_- = E_m - \Delta\left(1 + \frac{1}{2}\left|\frac{W_{12}}{\Delta}\right|^2 + \ldots\right) \quad . \tag{II.25}$$

On the other hand, for Δ close to zero we obtain

$$E_+ = E_m + |W_{12}| \tag{II.26}$$

$$E_- = E_m - |W_{12}| \quad . \tag{II.27}$$

From this we immediately see that the effect of coupling is more important when the two unperturbed levels have about the same energy. The effect is then of first order as seen from (F.II.26) and (F.II.27), whereas according to (F.II.24) and (F.II.25) it is of second order for $|\Delta| \gg |W_{12}|$.

We next have to discuss the effect of the coupling on the eigenstates. With the parameters E_m and Δ we can rewrite (F.II.17) as

$$\tan\theta = \frac{|W_{12}|}{\Delta} \quad . \tag{II.28}$$

That is, for strong coupling, i.e. $\Delta \ll |W_{12}|$, we have $\theta \simeq \pi/2$. In contrast, for weak coupling, i.e. $\Delta \gg |W_{12}|$, we have $\theta \simeq 0$. Then, at the center of the hyperbola when $E_1 = E_2$, $(\Delta = 0)$ we have

$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}\left[\mathrm{e}^{-\imath\varphi/2}|\phi_1\rangle + \mathrm{e}^{+\imath\varphi/2}|\phi_2\rangle\right] \tag{II.29}$$

$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}\left[-\mathrm{e}^{-\imath\varphi/2}|\phi_1\rangle + \mathrm{e}^{+\imath\varphi/2}|\phi_2\rangle\right] \quad . \tag{II.30}$$

Near the asymptotes, when $|\Delta| \gg |W_{12}|$ (weak coupling), we obtain in first order of $|W_{12}|/\Delta$:

$$|\Psi_+\rangle = \mathrm{e}^{-\imath\varphi/2}\left[|\phi_1\rangle + \mathrm{e}^{+\imath\varphi}\frac{|W_{12}|}{2\Delta}|\phi_2\rangle + \ldots\right] \tag{II.31}$$

$$|\Psi_-\rangle = \mathrm{e}^{+\imath\varphi/2}\left[|\phi_2\rangle - \mathrm{e}^{-\imath\varphi}\frac{|W_{12}|}{2\Delta}|\phi_1\rangle + \ldots\right] \quad . \tag{II.32}$$

As expected, for weak coupling $(\Delta \ll |W_{12}|)$ the perturbed states differ only slightly from the unperturbed ones. According to (F.II.31) the state $\Psi_+\rangle$ differs from $\phi_1\rangle$ only by the global phase factor $\mathrm{e}^{-\imath\varphi/2}$ with an additional small contribution of the state $\phi_2\rangle$. According to (F.II.32) the same is true for $\Psi_-\rangle$. On the other hand, for strong coupling $(\Delta \gg |W_{12}|)$ according to (F.II.29) and (F.II.30) the states $|\Psi_+\rangle$ and $|\Psi_-\rangle$ are very different from the unperturbed states $|\phi_1\rangle$ and $|\phi_2\rangle$, since they are linear superpositions of them with coefficients of the same modulus.

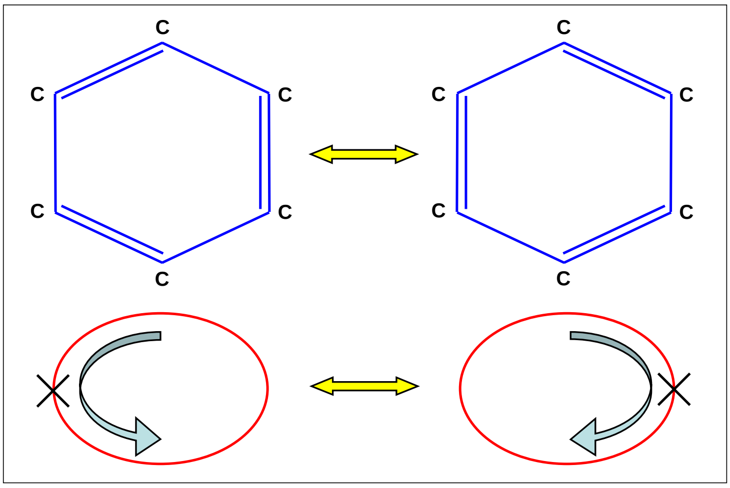

Figure F.2: The two possible configurations of the double bonds in a benzene molecule (top) and of the circulating current in a superconducting loop containing a Josephson junction (bottom).

II.3 Quantum Resonance

We briefly discuss the case where the eigenenergies of $\mathscr{H}_0$ are two-fold degenerate, i.e. $E_1 = E_2 = E_m$. In this case the coupling W_{12} lifts the degeneracy as discussed above giving rise to a level with reduced energy. That means that if the ground state of a physical system is two-fold degenerate and all other levels are sufficiently far away any purely off-diagonal coupling between the corresponding states is causing a reduction of the ground state energy of the system.

There are many examples of this phenomenon such as the resonance stabilization of the benzene C_6H_6 molecule shown in Fig. F.2. The ground state of the molecule includes three double bonds between neighboring carbons. The eigenfunctions $|\phi_1\rangle$ and $|\phi_2\rangle$ correspond to the two possible configurations of the double bonds shown in Fig. F.2. By symmetry reasons we expect that the ground state energy of the system is $\langle\phi_1|\mathscr{H}|\phi_1\rangle = \langle\phi_2|\mathscr{H}|\phi_2\rangle = E_m$ resulting in a two-fold degenerate ground state. However, the off-diagonal matrix element $\langle\phi_1|\mathscr{H}|\phi_2\rangle$ is not zero resulting in a finite coupling between the states $|\phi_1\rangle$ and $|\phi_2\rangle$. This gives rise to two distinct energy levels with one having an energy lower than E_m. Therefore, the benzene molecule is more stable than we would have expected and the true ground state of the molecule is not represented by one of the two configurations shown in Fig. F.2. The true ground state rather is a superposition of the two configurations.

Further examples are the ionized hydrogen molecule H_2^+ consisting of two protons and one electron. Again there are two possible configuration with the electron localized at proton 1 and proton 2 with degenerate energies. By a finite coupling of these two configurations we again obtain a states with reduced energy. In this state the electron is no longer localized at one of the protons but is delocalized. It is this delocalization which is by reducing the potential energy responsible for the chemical bond.

In chapter 9 as a further example we discuss a superconducting loop with an odd number of Josephson junctions. For half of a flux quantum in the loop there are two degenerate states with circulating currents in opposite direction. Again by a finite coupling a state with lowered energy is achieved given by a superposition of the two configurations.

III Dynamic Properties of Two-Level Systems

III.1 Time Evolution of the State Vector

We assume a state vector at the instant t given by the superposition

$$|\Psi(t)\rangle = a(t)|\phi_1\rangle + b(t)|\phi_2\rangle \ . \tag{III.33}$$

The evolution of the state vector is determined by the Schrödinger equation

$$\imath\hbar\frac{d}{dt}|\Psi(t)\rangle = (\mathscr{H}_0 + \mathscr{W})|\Psi(t)\rangle = (\mathscr{H}_0 + \mathscr{W})(a(t)|\phi_1\rangle + b(t)|\phi_2\rangle) \ . \tag{III.34}$$

By projecting this equation onto the basis vectors $|\phi_1\rangle$ and $|\phi_2\rangle$, we obtain (for $W_{11} = W_{22} = 0$):

$$\imath\hbar\frac{d}{dt}a(t) = E_1 a(t) + W_{12} b(t) \tag{III.35}$$

$$\imath\hbar\frac{d}{dt}b(t) = W_{21} a(t) + E_2 b(t) \ . \tag{III.36}$$

For finite coupling ($|W_{12}| \neq 0$) we obtain a linear system of homogeneous coupled differential equations. In order to solve this system we have to look for the eigenvectors $\Psi_+\rangle$ with eigenvalue E_+ and $\Psi_-\rangle$ with eigenvalue E_- of the operator $\mathscr{H} = \mathscr{H}_0 + \mathscr{W}$, whose matrix elements are the coefficients of equations (F.III.35) and (F.III.36). We then have to decompose $\Psi(0)\rangle$ in terms of $\Psi_+\rangle$ and $\Psi_-\rangle$ as

$$|\Psi(0)\rangle = \alpha|\Psi_+\rangle + \beta|\Psi_-\rangle \ , \tag{III.37}$$

where α and β are determined by the initial conditions. We then have

$$|\Psi(t)\rangle = \alpha e^{-\imath E_+ t/\hbar}|\Psi_+\rangle + \beta e^{-\imath E_- t/\hbar}|\Psi_-\rangle \ , \tag{III.38}$$

which enables us to derive $a(t)$ and $b(t)$ by projecting $|\Psi(t)\rangle$ onto the basis states $|\phi_1\rangle$ and $|\phi_2\rangle$.

It can be shown that a system with the basis state given by (F.III.38) oscillates between the two unperturbed states $|\phi_1\rangle$ and $|\phi_2\rangle$. To demonstrate that we assume that $|\Psi(0)\rangle = |\phi_1\rangle$ and calculate the probability $P_{12}(t)$ of finding the system in the basis state $|\phi_2\rangle$ at the time t.

III.2 The Rabi Formula

We first express the state $|\Psi(0)\rangle = |\phi_1\rangle$ on the $\{|\Psi_+\rangle, |\Psi_-\rangle\}$ basis. By inverting the expressions (F.II.11) and (F.II.12) we obtain

$$|\Psi(0)\rangle = |\phi_1\rangle = e^{+\imath\varphi/2}\left[\cos\frac{\theta}{2}|\Psi_+\rangle - \sin\frac{\theta}{2}|\Psi_-\rangle\right] \ . \tag{III.39}$$

Using the time evolution (F.III.38) we then obtain

$$|\Psi(t)\rangle = e^{+\imath\varphi/2}\left[\cos\frac{\theta}{2}e^{-\imath E_+ t/\hbar}|\Psi_+\rangle - \sin\frac{\theta}{2}e^{-\imath E_- t/\hbar}|\Psi_-\rangle\right] \ . \tag{III.40}$$

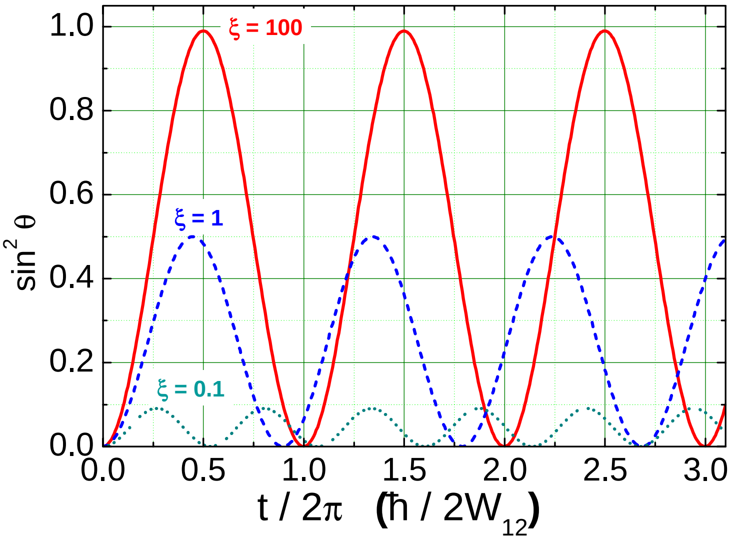

Figure F.3: Variation of the probability P_{12} of finding the system in state $|\varphi_2\rangle$ at time t, when it was in state $|\varphi_1\rangle$ at $t = 0$. $P_{12}(t)$ is shown for three different values of the parameter $\xi = |W_{12}|^2/(E_1 - E_2)^2$ (weak coupling: $\xi \ll 1$, strong coupling: $\xi \gg 1$).

The probability amplitude of finding the system in state $|\phi_2\rangle$ at time t is given by

$$\begin{aligned} \langle\phi_2|\Psi(t)\rangle &= e^{+\imath\varphi/2}\left[\cos\frac{\theta}{2}e^{-\imath E_+ t/\hbar}\langle\phi_2|\Psi_+\rangle - \sin\frac{\theta}{2}e^{-\imath E_- t/\hbar}\langle\phi_2|\Psi_-\rangle\right] \\ &= e^{+\imath\varphi/2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left[e^{-\imath E_+ t/\hbar} - e^{-\imath E_- t/\hbar}\right] . \end{aligned} \quad \text{(III.41)}$$

With this expression we obtain

$$\begin{aligned} P_{12}(t) = |\langle\phi_2|\Psi(t)\rangle|^2 &= \frac{1}{2}\sin^2\theta\left[1 - \cos\left(\frac{E_+ - E_-}{\hbar}t\right)\right] \\ &= \sin^2\theta\,\sin^2\left(\frac{E_+ - E_-}{2\hbar}t\right) . \end{aligned} \quad \text{(III.42)}$$

Using the expression (F.II.15) and (F.II.16) for E_+ and E_- we can rewrite this equation to obtain the so called ***Rabi formula***

$$P_{12}(t) = \frac{2|W_{12}|^2}{4|W_{12}|^2 + (E_1 - E_2)^2}\sin^2\left[\sqrt{4|W_{12}|^2 + (E_1 - E_2)^2}\,\frac{t}{2\hbar}\right] . \quad \text{(III.43)}$$

We see from (F.III.42) and (F.III.43) that $P_{12}(t)$ oscillates with the frequency $(E_+ - E_-)/\hbar$, which is the Bohr frequency of the system. We further see that $P_{12}(t)$ varies between zero and a maximum value equal to $\sin^2\theta$, which is obtained for the times $t = (2n+1)\pi\hbar/(E_+ - E_-)$ with $n = 0, 1, 2, 3, \ldots$ (see Fig. F.3). According to (F.III.43) the value of $\sin^2\theta$ as well as the oscillation frequency are functions of $|W_{12}|$ and $(E_1 - E_2)$.

For $E_1 = E_2$ we have $(E_+ - E_-)/\hbar = 2|W_{12}|/\hbar$. Then, according to (F.III.43) $P_{12}(t)$ has the maximum possible value of unity at the moments $t = (2n+1)\pi\hbar/2|W_{12}|$. That is, the system that is originally in the state $|\phi_1\rangle$ at $t = 0$ is in the state $|\phi_2\rangle$ at $t = \pi\hbar/2|W_{12}|$. Evidently any coupling between two states of equal energy causes the system to oscillate completely between the two states at a frequency proportional to the coupling. This phenomenon is known also for classical systems. For example, when we couple two pendulums of the same frequency by suspending them from the same support and we set only pendulum 1 into motion at $t = 0$, we will have after a certain time pendulum 1 in complete rest whereas pendulum 2 is oscillating with the initial amplitude of pendulum 1.

Fig. F.3 shows that the oscillation period $(E_+ - E_-)/\hbar$ of $P_{12}(t)$ decreases when $(E_1 - E_2)$ increases due to a decrease of the parameter $\xi = |W_{12}|^2/(E_1 - E_2)^2$ at constant $|W_{12}|$. Note that for weak coupling ($|W_{12}| \ll E_1 - E_2$) we have $\xi \ll 1$ and hence $\sin^2\theta$ becomes very small. This is not surprising, since in the case of weak coupling the state $|\phi_1\rangle$ is very close to the stationary state $\Psi_+\rangle$ and therefore the system starting at state $|\phi_1\rangle$ evolves very little over time.

Above we have mentioned the H_2^+ molecule as an example for a two-level system. According to the result (F.III.43) we expect an oscillation of the electron between the two protons of the molecule at a frequency given by the Bohr frequency $(E_+ - E_-)/\hbar$ given by the two stationary states $\Psi_+\rangle$ and $\Psi_-\rangle$ of the molecule. This oscillation corresponds to an oscillation of the mean value of the electric dipole moment of the molecule. Therefore, when the molecule is not in a stationary state, an oscillating dipole field can appear. Such an oscillating dipole can exchange energy with an electromagnetic field of the same frequency. Hence, this frequency must be seen in the absorption and emission spectrum of the molecule. Of course, the same is true for a superconducting flux or charge qubit representing a two-level system. In many experiments the interaction of an electromagnetic field of varying frequency with the qubit has been measured showing absorption/emission features at the characteristic frequency $(E_+ - E_-)/\hbar$.