Exercise to the Lecture

Superconductivity and Low Temperature Physics I
WS 2014/2015

4 Microscopic Theory

4.9 Particle Current Density

Exercise:

Within Boltzmann transport theory the particle current density in a metal can be written as

\[
J = \frac{1}{V} \sum \frac{\hbar k}{m} \left[ f(\epsilon_k) - f_0(\epsilon_k) \right] = \frac{1}{V} \sum \frac{\hbar k}{m} \left[ -\frac{\partial f_0(\epsilon_k)}{\partial \epsilon_k} \delta \epsilon_k \right] = n v
\]

with

\[
f_0(\epsilon_k) = \frac{1}{e^{(\epsilon_k - \mu)/k_B T} + 1}.
\]

Here, \( n \) is the particle density, \( v \) the particle drift velocity, \( \epsilon_k = \hbar^2 k^2 / 2m \), \( \delta \epsilon_k = \hbar k \cdot v \), and \( f_0(\epsilon_k) \) the thermal equilibrium Fermi-Dirac distribution.

(a) Use the above definition of the particle current density to derive the particle density in a normal metal.

(b) Calculate the density \( n^{qp} \) of Bogoliubov quasiparticles in a superconductor (normal fluid density).

(c) Calculate the density \( n^s \) of the paired electrons in a superconductor (superfluid density density).

(d) Use the superfluid density to discuss the temperature dependence of the London penetration depth \( \lambda_L \) close to the transition temperature \( T_c \).
Solution:

(a) The $i^{th}$ component ($i = 1, 2, 3$) of the particle current density can be written as

$$ J_i = \frac{1}{V} \sum_{k'} \frac{\hbar k_i}{m} \left[ -\frac{\partial f_0(\epsilon_k)}{\partial \epsilon_k} \hbar k_i v_j \right] = n_i v_j . \quad (1) $$

With this result we can define the particle current density in a normal metal as

$$ n_{ij} = \frac{1}{V} \sum_{k'} y_k \frac{\hbar^2 k k}{m} , \quad (2) $$

where $kk$ is the dyadic product and

$$ y_k = -\frac{\partial f_0(\epsilon_k)}{\partial \epsilon_k} = \frac{1}{2k_B T \cosh(\epsilon_k/k_B T) + 1} = \frac{1}{4k_B T \cosh^2(\epsilon_k/k_B T)} . \quad (3) $$

To evaluate (2) we have to convert the summation into an integration. We replace $\epsilon_k$ by $\xi_k = \epsilon_k - \mu$ in (3) and obtain

$$ n_{ij} = \frac{1}{V} \sum_{k'} \left( -\frac{\partial f_0(\zeta_k + \mu)}{\partial \zeta_k} \right) \frac{\hbar^2 k k}{m} $$

$$ = \frac{1}{V} \frac{\hbar^2 k_F^2}{m} \frac{d\Omega_k}{4\pi} \frac{d\zeta_k}{-\mu} D(\mu + \zeta_k) \left( -\frac{\partial f_0(\mu + \zeta_k)}{\partial \zeta_k} \right) $$

$$ \approx \frac{1}{3} \delta_{ij} \frac{D(E_F)\hbar^2 k_F^2}{m} \frac{1}{4k_B T} \int_{-\mu}^{\infty} \frac{d\zeta_k}{\cosh^2(\zeta_k/2k_B T)} $$

$$ \frac{x = \zeta_k/2k_B T}{x = \zeta_k/2k_B T} \frac{1}{2} n_0 \delta_{ij} \frac{\int_{-\infty}^{\infty} dx}{\tanh(\infty) - \tanh(-\infty)} = n_0 \delta_{ij} . \quad (4) $$

Here, $\hat{k} = k/|k|$ and $|k| \simeq k_F$. Furthermore, we have used $D(\mu + \zeta_k) \simeq D(E_F) = \text{const}$, since the function $\partial f_0^0 / \partial E_k$ is finite only in a narrow energy interval $\sim k_B T$ around the chemical potential $\mu$, and we have set the lower integration limit to $-\infty$, since typically $\mu/2k_B T \gg 1$ for a metal. Obviously we obtain the expected result that the particle density is given by the electron density of the normal metal.

(b) We next consider the superconducting state of a metal and calculate the normal fluid density $n_{qp}$ of the Bogoliubov quasiparticles. To calculate $n_{qp}$ we can use eqs. (2) and (3) but have to replace $\zeta_k = \epsilon_k - \mu$ by the quasiparticle energy $E_k = \sqrt{\zeta_k^2 + \Delta^2}$ in (3). With
\[ Z(k)d^3k = D(E_k)dE_k = D(\xi_k)d\xi_k \] (conservation of states) we obtain
\[
n_{ij}^{\text{qp}} \approx \frac{1}{V} \sum_{k\sigma} \frac{\hbar^2 \mathbf{k}_k}{m} \int \frac{d\Omega_k}{4\pi} \int_{\mathbb{R}} d\xi_k D(\mu + \xi_k) \left(-\frac{\partial f_k}{\partial E_k}\right)
\]
\[
\approx \frac{1}{N} \delta_{ij} \frac{D(E_F)\hbar^2 k_F^2}{mV} \int_{\mathbb{R}} d\xi_k \frac{1}{4k_B T} \int_{-\mu}^{\infty} d\xi_k D(\mu + \xi_k) \left(-\frac{\partial f_k}{\partial E_k}\right)
\]
\[
x = \xi_k / 2k_B T \approx \frac{1}{2} n \delta_{ij} \int_{-\infty}^{\infty} \frac{dx}{\cosh^2 \sqrt{x^2 + \left(\frac{\Delta(T)}{2k_B T}\right)^2}} = n \delta_{ij} Y(T).
\]

We see that the normal fluid density is given by the normal state particle density multiplied by the Yosida function \(Y(T)\) (cf. Fig. 1). The Yosida function is zero at \(T = 0\) and continuously increases towards one at \(T = T_c\). Therefore, the quasiparticle density decreases from \(n_{ij}^{\text{qp}}(T = n \delta_{ij})\) at \(T = T_c\) to \(n_{ij}^{\text{qp}}(T = 0) = 0\) at \(T = 0\).

(c) Since the total particle number is conserved on going from the normal to the superconducting state, the density of the paired electrons in the superconducting state (superfluid

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Figure 1: Yosida function plotted versus \(\Delta(T) / k_B T\) using (a) a linear and (b) logarithmic scale.
density density) is given by

\[ n_{ij}^s(T) = n - n_{ij}^{qp}(T) = n [1 - Y(T)] \delta_{ij} \]

\[ = n \left[ 1 - \int_0^\infty \frac{dx}{\cosh^2 \sqrt{x^2 + \left( \frac{\Delta(T)}{2k_B T} \right)^2} \delta_{ij} \right]. \quad (6) \]

(d) We can use the temperature dependence of the Yosida function to discuss the temperature dependence of the superfluid density and, in turn, the London penetration depth

\[ \lambda_L(T) = \sqrt{\frac{m_s}{\mu_0 n^s(T) q_s^2}} \frac{\lambda_L(0)}{\sqrt{1 - Y(T)}}. \quad (7) \]

For \( T \approx T_c (\Delta(T) \to 0) \), we can approximate the temperature dependence of the Yosida function by

\[ \lim_{T \to T_c} Y(T) = 1 - 2 \left( 1 - \frac{T}{T_c} \right) \]

and obtain

\[ \lim_{T \to T_c} \lambda_L(T) = \frac{\lambda_L(0)}{\sqrt{2 \left( 1 - \frac{T}{T_c} \right)}} \quad (8) \]

We see that \( \lambda_L(T) \) diverges for \( T \to T_c \). This result is obvious since a normal metal cannot screen stationary magnetic fields.

For \( T \to 0 \), the Yosida function shows a thermally activated behavior

\[ \lim_{T \to 0} Y(T) = \sqrt{\frac{2\pi \Delta(T)}{k_B T}} e^{-\frac{\Delta(T)}{k_B T}}, \]

as can be seen in Fig. 1(b). Since \( Y(T \ll T_c) \ll 1 \), according to (7) the London penetration depth shows a very weak temperature dependence in the temperature regime well below \( T_c \).