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Exercise 2

Exercises to the Lecture

Superconductivity and Low Temperature Physics I
WS 2014/2015

1 Basic Properties of Superconductors

1.3 Surface Impedance of Normal Metals and Superconductors

Exercise:

In the discussion of high frequency properties of normal metals and superconductors the surface impedance Z_s is a useful quantity.

(a) Consider a normal metal with carrier density n, momentum relaxation time τ_n and frequency dependent conductivity $\sigma(\omega) = \frac{\sigma_0}{1-\imath\omega\tau_n}$ with $\sigma_0 = ne\tau_n/m$ the Drude conductivity. Calculate the surface impedance in the limit $\omega\tau_n \ll 1$.

(b) Consider a superconductor with frequency dependent conductivity $\sigma(\omega)$.

Solution:

We start by introducing the concept of the surface impedance. We know that for an ideal conductor in an electromagnetic field, the tangential component E_t of the electric field at the surface has to vanish. However, a current flows in a thin sheet on the surface, as required to support the magnetic field H_t tangential to the surface. In this short-circuit boundary condition, all fields are excluded from the interior of the ideal conductor. In a real conductor, the situation is slightly different. Here, fields extend into the conductor, but decay rapidly with distance from the surface. To avoid to be forced to solve Maxwell's equations inside conductors, one makes use of the concept of the surface impedance. The surface impedance

$$Z_s \;=\; R_s + \imath X_s \;\;\equiv\;\; \frac{E_\mathrm{t}}{H_\mathrm{t}} \tag{1}$$

provides the boundary condition for fields outside the conductor. It also accounts for the dissipation and energy stored inside the conductor. Here, R_s is the surface resistance and the X_s the

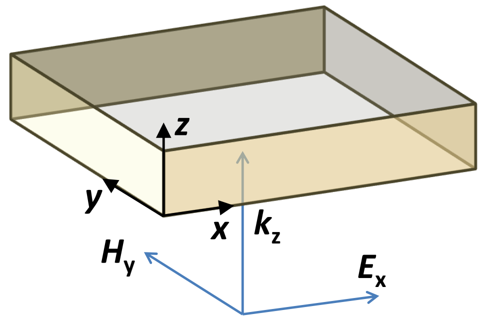

Figure 1: Zur Definition der Oberflächenimpedanz: Eine elektromagnetische Welle mit E-Feld E_x und H-Feld H_y breitet sich in z-Richtung aus und trifft bei $z = 0$ auf einen halbunendlich ausgedehnten, elektrisch leitenden Festkörper.

surface reactance. The physical reasoning is that the magnetic field H_t causes a surface current flowing in a δ-layer with the sheet current density $J_\square = J \cdot \delta = H_\mathrm{t}$, so that $Z_s = E_\mathrm{t}/H_\mathrm{t} = E_\mathrm{t}/J_\square$.

For a semi-infinite conductor (cf. Fig. 1), the internal fields decay exponentially with distance from the surface with a characteristic decay length $\delta = 1/k$. If we assume that the surface of the semi-infinite conductor extends in the xy-plane, we can write the tangential electric field $E_x(z)$ as[1]

$$E_x(z) \;=\; E_x(0)\, \mathrm{e}^{\imath kz} \;=\; E_x(0)\, \mathrm{e}^{\imath k' z}\, \mathrm{e}^{-k'' z} \quad , \tag{2}$$

where we have separated the complex decay constant $k = k' + \imath k''$ into real and imaginary part. With the Maxwell equation $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ we obtain for a harmonic field

$$-\imath\omega\mu_0 H_y(z) \;=\; -\imath k E_x(z) \;=\; -\imath E_x(z)(k' + \imath k'') \tag{3}$$

resulting in the surface impedance

$$Z_s \;=\; \frac{E_x}{H_y} \;=\; \frac{\mu_0\omega}{k} \;=\; \frac{\mu_0\omega}{k' + \imath k''} \quad . \tag{4}$$

(a) For normal conductors with conductivity $\sigma(\omega)$, this decay constant (cf. exercise 1.2) is given by

$$k^2(\omega) \;=\; \frac{1}{\delta^2(\omega)} \;=\; -\imath\omega\sigma(\omega)\mu_0 \quad . \tag{5}$$

The frequency dependent conductivity of a normal metal is given by

$$\sigma(\omega) \;=\; \frac{ne^2\tau}{m\,(1 - \imath\omega\tau_n)} \;=\; \frac{\sigma_0}{1 - \imath\omega\tau_n} \tag{6}$$

with the Drude conductivity $\sigma_0 = \frac{ne^2\tau_n}{m}$ and the momentum relaxation time τ_n. Since τ_n is typically less then 10^{-12}s, the limit $\omega\tau \ll 1$ applies up to the GHz regime. In this case the metal has an about frequency independent conductivity $\sigma(\omega) \simeq \sigma_0 = ne^2\tau_n/m$. Since the conductivity is purely real, we obtain

$$\begin{aligned} k^2 &= \frac{1}{\delta^2} \;\simeq\; -\imath\omega\sigma_0\mu_0 \\ k &= \frac{1}{\delta} \;\simeq\; \sqrt{\frac{\omega\sigma_0\mu_0}{2}}\,(1 - \imath) \quad . \end{aligned} \tag{7}$$

[1] We make the Ansatz $E_x(z) = E_0 \exp[\imath(kz - \omega t)]$ with the complex decay constant $k = k' + \imath k''$ related to the skin depth by $k^2 = 1/\delta^2$.

where we have used $\sqrt{-\imath} = \frac{1}{\sqrt{2}}(1-\imath)$. Obviously, in this case the magnitude of the real and imaginary part of k are the same and given by $k_0 = 1/\delta_0 = \sqrt{\omega\sigma_0\mu_0/2}$. Then, according to eq. (4) we obtain

$$Z_s = \frac{\mu_0\omega}{k} = \frac{\mu_0\omega}{\sqrt{\mu_0\omega\sigma_0/2}(1-\imath)} = \sqrt{\frac{\mu_0\omega}{2\sigma_0}}\,(1+\imath) = \frac{\mu_0\omega\delta_0}{2}\,(1+\imath)\ . \tag{8}$$

We see that the surface resistance R_s and reactance X_s are equal and given by

$$R_s = X_s = \sqrt{\frac{\mu_0\omega}{2\sigma_0}} = \frac{\mu_0\omega\delta_0}{2}\ . \tag{9}$$

For normal metals both R_s and X_s are proportional to $\sqrt{\omega}$. For example, in Au or Cu at 100 GHz and room temperature, $\delta_0 \simeq 0.25\,\mu$m and $Z_s \simeq 0.1(1+\imath)\,\Omega/\Box$.

(b) We next consider a superconducting metal with complex ac conductivity $\sigma(\omega) = \sigma'(\omega) + \imath\sigma''(\omega)$ given by

$$\sigma'(\omega) = \frac{1}{\Lambda_s}\left[\pi\delta(\omega) + \frac{n_n}{n_s}\,\frac{\tau_n}{1+(\omega\tau_n)^2}\right] \tag{10}$$

$$\sigma''(\omega) = \frac{1}{\Lambda_s}\left[\frac{1}{\omega} + \frac{1}{\omega}\frac{n_n}{n_s}\,\frac{(\omega\tau_n)^2}{1+(\omega\tau_n)^2}\right]\ . \tag{11}$$

Here we have used the London coefficient $\Lambda_s \equiv m_s/\widetilde{n}_s q_s^2 = m/n_s e^2 = \mu_0\lambda_L^2$ (λ_L is the London penetration depth). For $\omega\tau_n \ll 1$, we can use the approximations

$$\sigma'(\omega) \simeq \frac{1}{\omega\Lambda_s}\,\frac{n_n}{n_s}\,\tau_n = \frac{n_n}{n}\,\sigma_0 \tag{12}$$

$$\sigma''(\omega) \simeq \frac{1}{\omega\Lambda_s} = \frac{1}{\omega\mu_0\lambda_L^2}\ . \tag{13}$$

By inserting this into the expression (4) for the surface impedance and using $k^2 = -\imath\omega\mu_0\sigma(\omega)$ we obtain

$$\begin{aligned} Z_s &= \frac{\mu_0\omega}{k} = \frac{\mu_0\omega}{\sqrt{-\imath\mu_0\omega\sigma_0\frac{n_n}{n} + \frac{\mu_0\omega}{\mu_0\omega\lambda_L^2}}} \\ &= \frac{\mu_0\omega\lambda_L}{\sqrt{1-\imath\mu_0\omega\lambda_L^2\sigma_0\frac{n_n}{n}}}\ . \end{aligned} \tag{14}$$

To evaluate this further we use $\sqrt{z} = \sqrt{(|z|+\Re(z))/2} + \imath\,\mathrm{sgn}(\Im(z))\sqrt{(|z|-\Re(z))/2}$. We obtain

$$Z_s = \frac{\mu_0\omega\lambda_L}{\sqrt{1+\frac{1}{2}\mu_0^2\omega^2\lambda_L^4\sigma_0^2\left(\frac{n_n}{n}\right)^2} - \imath\sqrt{\frac{1}{2}\mu_0^2\omega^2\lambda_L^4\sigma_0^2\left(\frac{n_n}{n}\right)^2}}\ . \tag{15}$$

We multiply the denominator by $\left(\sqrt{\ldots} + \imath\sqrt{\ldots}\right)$ and make use of the fact that $\frac{1}{2}\mu_0^2\omega^2\lambda_L^4\sigma_0^2\left(\frac{n_n}{n}\right)^2$ is about the square of the ratio of the conductivities of the normal and

Table 1: Conductivity σ, penetration depth δ_0 due to the normal skin effect, London penetration depth λ_L, surface resistance R_s and surface reactance X_s of normal conductors and superconductors for $\omega\tau_n \ll 1$ and temperatures $T \ll T_c$.

property	normal conductor	superconductor
conductivity	$\sigma_0 = \frac{ne^2\tau_n}{m}$	$\sigma' + \imath\sigma'' = \frac{ne^2\tau_n}{m}\left(\frac{n_n}{n}\right) + \frac{\imath}{\omega\mu_0\lambda_L^2}$
field penetration depth	$\delta_0 = \sqrt{2/\omega\mu_0\sigma_0}$	$\delta_s = \lambda_L$
surface resistance	$R_s = \frac{1}{2}\omega\mu_0\delta_0 = \sqrt{\frac{\omega\mu_0}{2\sigma_0}}$	$R_s = \frac{1}{2}\omega^2\mu_0^2\lambda_L^3\sigma_0\left(\frac{n_n}{n}\right)$
surface reactance	$X_s = \frac{1}{2}\omega\mu_0\delta_0 = \sqrt{\frac{\omega\mu_0}{2\sigma_0}}$	$X_s = \omega\mu_0\lambda_L$

the superfluid. Since this ratio is small compared to unity for temperatures not to close to the transition temperature T_c, we can further approximate (15) to obtain

$$Z_s = R_s + \imath X_s \simeq \frac{1}{2}\mu_0^2\omega^2\lambda_L^3\sigma_0\frac{n_n}{n} + \imath\mu_0\omega\lambda_L \ . \tag{16}$$

We see that the surface resistance R_s, the real part of Z_s, increases proportional to ω^2 in contrast to normal conductors, where $R_s \propto \sqrt{\omega}$. Furthermore, it increases proportional to λ_L^3 and the conductivity $\sigma_0 n_n/n$ of the normal fluid component. In Table 1 the most relevant characteristics of superconductors are compared to those of normal metals. For a superconductor at a frequency well below its energy gap frequency and $T \ll T_c$, the London penetration depth λ_L is about constant. For niobium at 4.2 K and frequencies below about 700 GHz, $\lambda_L \simeq 0.1\,\mu$m and we obtain $Z_s \simeq \imath\mu_0\omega\lambda_L = \imath\omega L_k$, corresponding to a surface or kinetic inductance $L_k = \mu_0\lambda_L$ [Henry/□], which is independent of frequency. In niobium, $L_k \simeq 0.13$ pH/□, giving $Z_s \simeq \imath 0.08\,\Omega$/□ at 100 GHz.

Fig. 2 shows the theoretically expected surface resistance as a function of frequency for the superconductor Nb and the normal metal Cu. We see that for frequencies below about 100 GHz the surface resistance of Nb is considerably lower than for Cu at 77 K. At high frequencies, there is a cross-over due to the much weaker frequency dependence of

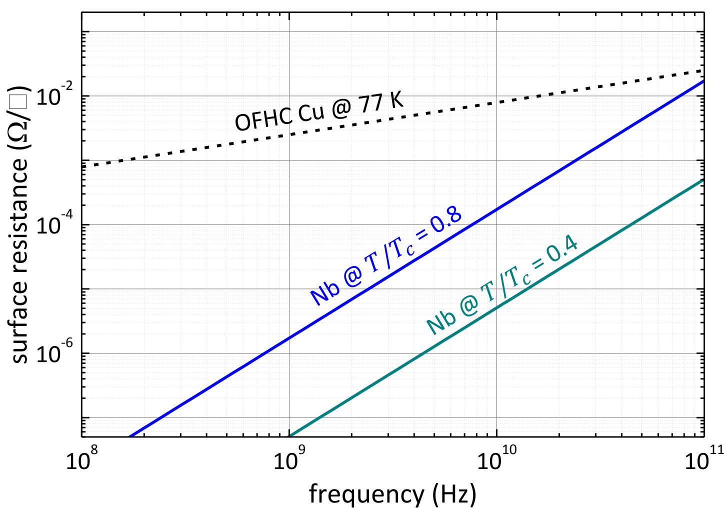

Figure 2: Intrinsic surface resistance versus frequency for Nb and oxygen free high conductance (OFHC) Cu. For Cu, $\sigma_0 = 10^8\Omega^{-1}\text{m}^{-1}$, for Nb, $\lambda_L(0) = 85$ nm and $\sigma_0 = 10^8\Omega^{-1}\text{m}^{-1}$ were used.

the surface resistance of normal metals. Note that the surface resistance is expected to be further reduced by going to lower temperatures due to the strong decrease of n_n. At $T/T_c \ll 1$, $\lambda_{\mathrm{L}}(T) \simeq const$ and $n_n \propto \exp(-2\Delta_0/k_{\mathrm{B}}T)$. Therefore, an exponential decrease of R_s with decreasing T is expected. However, this behavior is usually not observed in experiments. Rather a temperature independent residual surface resistance is measured at very low T, which is attributed to material defects. For Nb this residual resistance is as low as $10^{-9}\Omega/\Box$ at 10 GHz, whereas it reaches only about $10^{-5}\Omega/\Box$ for high temperature superconducting $YBa_2Cu_3O_{7-\mathrm{ffi}}$ films.

Kinetic Inductance The surface reactance X_s, the imaginary part of the surface impedance, is purely inductive. The equivalent inductance L_{k} is denoted as ***kinetic inductance***

$$L_{\mathrm{k}} = \mu_0 \lambda_{\mathrm{L}} \; . \tag{17}$$

The kinetic inductance reflects the inertia or equivalently the kinetic energy of the carriers of the superfluid.