1 Basic Properties of Superconductors

1.3 Surface Impedance of Normal Metals and Superconductors

Exercise:

In the discussion of high frequency properties of normal metals and superconductors the surface impedance $Z_s$ is a useful quantity.

(a) Consider a normal metal with carrier density $n$, momentum relaxation time $\tau_n$ and frequency dependent conductivity $\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau_n}$ with $\sigma_0 = ne\tau_n/m$ the Drude conductivity. Calculate the surface impedance in the limit $\omega\tau_n \ll 1$.

(b) Consider a superconductor with frequency dependent conductivity $\sigma(\omega)$.

Solution:

We start by introducing the concept of the surface impedance. We know that for an ideal conductor in an electromagnetic field, the tangential component $E_t$ of the electric field at the surface has to vanish. However, a current flows in a thin sheet on the surface, as required to support the magnetic field $H_t$ tangential to the surface. In this short-circuit boundary condition, all fields are excluded from the interior of the ideal conductor. In a real conductor, the situation is slightly different. Here, fields extend into the conductor, but decay rapidly with distance from the surface. To avoid to be forced to solve Maxwell’s equations inside conductors, one makes use of the concept of the surface impedance. The surface impedance

$$Z_s = R_s + iX_s \equiv \frac{E_t}{H_t} \quad (1)$$

provides the boundary condition for fields outside the conductor. It also accounts for the dissipation and energy stored inside the conductor. Here, $R_s$ is the surface resistance and the $X_s$ the
surface reactance. The physical reasoning is that the magnetic field $H_t$ causes a surface current flowing in a $\delta$-layer with the sheet current density $J_\square = J \cdot \delta = H_t$, so that $Z = E_t / H_t = E_t / J_\square$.

For a semi-infinite conductor (cf. Fig. 1), the internal fields decay exponentially with distance from the surface with a characteristic decay length $\delta = 1/k$. If we assume that the surface of the semi-infinite conductor extends in the $xy$-plane, we can write the tangential electric field $E_x(z)$ as

$$E_x(z) = E_x(0) e^{ikz} = E_x(0) e^{ik'z} e^{-k''z}, \quad (2)$$

where we have separated the complex decay constant $k = k' + ik''$ into real and imaginary part. With the Maxwell equation $\nabla \times E = -\partial B / \partial t$ we obtain for a harmonic field

$$-i\omega \mu_0 H_y(z) = -i k E_x(z) = -i E_x(z)(k' + ik'') \quad (3)$$

resulting in the surface impedance

$$Z_s = \frac{E_x}{H_y} = \frac{\mu_0 \omega}{k} = \frac{\mu_0 \omega}{k' + ik''}. \quad (4)$$

(a) For normal conductors with conductivity $\sigma(\omega)$, this decay constant (cf. exercise 1.2) is given by

$$k^2(\omega) = \frac{1}{\delta^2(\omega)} = -i \omega \sigma(\omega) \mu_0. \quad (5)$$

The frequency dependent conductivity of a normal metal is given by

$$\sigma(\omega) = \frac{n^2 e^2}{m (1 - i \omega \tau_n)} = \frac{\sigma_0}{1 - i \omega \tau_n}, \quad (6)$$

with the Drude conductivity $\sigma_0 = \frac{n^2 e^2 \tau_n}{m}$ and the momentum relaxation time $\tau_n$. Since $\tau_n$ is typically less than $10^{-12}s$, the limit $\omega \tau \ll 1$ applies up to the GHz regime. In this case the metal has an about frequency independent conductivity $\sigma(\omega) \approx \sigma_0 = ne^2 \tau_n / m$. Since the conductivity is purely real, we obtain

$$k^2 = \frac{1}{\delta^2} \approx -i \omega \sigma_0 \mu_0 \quad (7)$$

$$k = \frac{1}{\delta} \approx \sqrt{\frac{\omega \sigma_0 \mu_0}{2}} (1 - i).$$

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1. We make the Ansatz $E_x(z) = E_0 \exp[i(kz - \omega t)]$ with the complex decay constant $k = k' + ik''$ related to the skin depth by $k^2 = 1/\delta^2$.  

Figure 1: Zur Definition der Oberflächenimpedanz: Eine elektromagnetische Welle mit $E$-Feld $E_x$ und $H$-Feld $H_y$ breitet sich in $z$-Richtung aus und trifft bei $z = 0$ auf einen halbunendlich ausgedehnten, elektrisch leitenden Festkörper.
where we have used \( \sqrt{-i} = \frac{1}{\sqrt{2}}(1 - i) \). Obviously, in this case the magnitude of the real and imaginary part of \( k \) are the same and given by \( k_0 = 1/\delta_0 = \sqrt{\omega \sigma_0 / \mu_0 / 2} \). Then, according to eq. (4) we obtain

\[
Z_s = \frac{\mu_0 \omega}{k} = \frac{\mu_0 \omega}{\sqrt{\mu_0 \omega \sigma_0 / (2(1 - i))}} = \sqrt{\frac{\mu_0 \omega}{2 \sigma_0}} (1 + i) = \frac{\mu_0 \omega \delta_0}{2} (1 + i) .
\] (8)

We see that the surface resistance \( R_s \) and reactance \( X_s \) are equal and given by

\[
R_s = X_s = \sqrt{\frac{\mu_0 \omega}{2 \sigma_0}} = \frac{\mu_0 \omega \delta_0}{2} .
\] (9)

For normal metals both \( R_s \) and \( X_s \) are proportional to \( \sqrt{\omega} \). For example, in Au or Cu at 100 GHz and room temperature, \( \delta_0 \simeq 0.25 \mu m \) and \( Z_s \simeq 0.1(1 + i) \Omega / \square \).

(b) We next consider a superconducting metal with complex ac conductivity \( \sigma(\omega) = \sigma'(\omega) + i \sigma''(\omega) \) given by

\[
\sigma'(\omega) = \frac{1}{\Lambda_s} \left[ \pi \delta(\omega) + \frac{n_n}{n_s} \frac{\tau_n}{1 + (\omega \tau_n)^2} \right]
\] (10)

\[
\sigma''(\omega) = \frac{1}{\Lambda_s} \left[ \frac{1}{\omega} + \frac{1}{n_n} \frac{(\omega \tau_n)^2}{1 + (\omega \tau_n)^2} \right] .
\] (11)

Here we have used the London coefficient \( \Lambda_s \equiv m_s / \tilde{n}_n q_s^2 = \mu_0 \lambda_s^2 / n_s e^2 = \mu_0 \lambda_L^2 \) (\( \lambda_L \) is the London penetration depth). For \( \omega \tau_n \ll 1 \), we can use the approximations

\[
\sigma'(\omega) \simeq \frac{1}{\omega \Lambda_s} \frac{n_n}{n_s} \tau_n = \frac{n_n}{n_s} \sigma_0
\] (12)

\[
\sigma''(\omega) \simeq \frac{1}{\omega \Lambda_s} = \frac{1}{\omega \mu_0 \lambda_L^2} .
\] (13)

By inserting this into the expression (4) for the surface impedance and using \( k^2 = -i \omega \mu_0 \sigma(\omega) \) we obtain

\[
Z_s = \frac{\mu_0 \omega}{k} = \frac{\mu_0 \omega}{\sqrt{-i \mu_0 \omega \sigma_0 \frac{n_n}{n} + \frac{\mu_0 \omega}{\mu_0 \lambda_L^2}}} = \frac{\mu_0 \omega \lambda_L}{\sqrt{1 + \frac{1}{2} \mu_0^2 \omega^2 \lambda_L^4 \sigma_0^2 \left( \frac{n_n}{n} \right)^2 - i \sqrt{\frac{1}{2} \mu_0^2 \omega^2 \lambda_L^4 \sigma_0^2 \left( \frac{n_n}{n} \right)^2}}} .
\] (14)

To evaluate this further we use \( \sqrt{z} = \sqrt{(|z| + \Re(z))/2 + i \sgn(\Im(z)) \sqrt{(|z| - \Re(z))/2}} \). We obtain

\[
Z_s = \frac{\mu_0 \omega \lambda_L}{\sqrt{1 + \frac{1}{2} \mu_0^2 \omega^2 \lambda_L^4 \sigma_0^2 \left( \frac{n_n}{n} \right)^2 - i \sqrt{\frac{1}{2} \mu_0^2 \omega^2 \lambda_L^4 \sigma_0^2 \left( \frac{n_n}{n} \right)^2}}} .
\] (15)

We multiply the denominator by \( \left( \sqrt{\ldots} + i \sqrt{\ldots} \right) \) and make use of the fact that \( \frac{1}{2} \mu_0^2 \omega^2 \lambda_L^4 \sigma_0^2 \left( \frac{n_n}{n} \right)^2 \) is about the square of the ratio of the conductivities of the normal and...
Table 1: Conductivity $\sigma$, penetration depth $\delta_0$ due to the normal skin effect, London penetration depth $\lambda_L$, surface resistance $R_s$ and surface reactance $X_s$ of normal conductors and superconductors for $\omega\tau_n \ll 1$ and temperatures $T \ll T_c$.

<table>
<thead>
<tr>
<th>property</th>
<th>normal conductor</th>
<th>superconductor</th>
</tr>
</thead>
<tbody>
<tr>
<td>conductivity $\sigma_0$</td>
<td>$\frac{ne^2\tau_n}{m}$</td>
<td>$\sigma' + \sigma'' = \frac{ne^2\tau_n}{m} \left(\frac{n\mu_0}{\pi}\right) + \frac{1}{\omega\mu_0\lambda_L^2}$</td>
</tr>
<tr>
<td>field penetration depth $\delta_0$</td>
<td>$\sqrt{2/\omega\mu_0\epsilon_0}$</td>
<td>$\delta_s = \lambda_L$</td>
</tr>
<tr>
<td>surface resistance $R_s$</td>
<td>$\frac{1}{2}\omega\mu_0\delta_0 = \sqrt{\frac{\omega\mu_0}{2\epsilon_0}}$</td>
<td>$R_s = \frac{1}{2}\omega^2\mu_0\lambda_L^2\sigma_0 \left(\frac{n\mu_0}{\pi}\right)$</td>
</tr>
<tr>
<td>surface reactance $X_s$</td>
<td>$\frac{1}{2}\omega\mu_0\delta_0 = \sqrt{\frac{\omega\mu_0}{2\epsilon_0}}$</td>
<td>$X_s = \omega\mu_0\lambda_L$</td>
</tr>
</tbody>
</table>

the superfluid. Since this ratio is small compared to unity for temperatures not close to the transition temperature $T_c$, we can further approximate (15) to obtain

$$Z_s = R_s + iX_s \simeq \frac{1}{2}\mu_0^2\omega^2\lambda_L^3\sigma_0\frac{n\mu_0}{\pi} + \mu_0\omega\lambda_L.$$ (16)

We see that the surface resistance $R_s$, the real part of $Z_s$, increases proportional to $\omega^2$ in contrast to normal conductors, where $R_s \propto \sqrt{\omega}$. Furthermore, it increases proportional to $\lambda_L^3$ and the conductivity $\sigma_0 n_\| / n$ of the normal fluid component. In Table 1 the most relevant characteristics of superconductors are compared to those of normal metals. For a superconductor at a frequency well below its energy gap frequency and $T \ll T_c$, the London penetration depth $\lambda_L$ is about constant. For niobium at 4.2 K and frequencies below about 700 GHz, $\lambda_L \simeq 0.1 \mu m$ and we obtain $Z_s \simeq \mu_0\omega\lambda_L = i\mu_0\omega\lambda_L$, corresponding to a surface or kinetic inductance $L_k = \mu_0\lambda_L$ [Henry/□], which is independent of frequency. In niobium, $L_k \simeq 0.13 \text{pH/□}$, giving $Z_s \simeq 0.08 \Omega/□$ at 100 GHz.

Fig. 2 shows the theoretically expected surface resistance as a function of frequency for the superconductor Nb and the normal metal Cu. We see that for frequencies below about 100 GHz the surface resistance of Nb is considerably lower than for Cu at 77 K. At high frequencies, there is a cross-over due to the much weaker frequency dependence of
the surface resistance of normal metals. Note that the surface resistance is expected to be further reduced by going to lower temperatures due to the strong decrease of $n_n$. At $T/T_c \ll 1$, $\lambda_L(T) \simeq \text{const}$ and $n_n \propto \exp(-2\Delta_0/k_B T)$. Therefore, an exponential decrease of $R_s$ with decreasing $T$ is expected. However, this behavior is usually not observed in experiments. Rather a temperature independent residual surface resistance is measured at very low $T$, which is attributed to material defects. For Nb this residual resistance is as low as $10^{-9}\Omega/\square$ at 10 GHz, whereas it reaches only about $10^{-5}\Omega/\square$ for high temperature superconducting YBa$_2$Cu$_3$O$_{7-\delta}$ films.

**Kinetic Inductance** The surface reactance $X_s$, the imaginary part of the surface impedance, is purely inductive. The equivalent inductance $L_k$ is denoted as *kinetic inductance*

$$L_k = \mu_0 \lambda_L$$  \hspace{1cm} (17)

The kinetic inductance reflects the inertia or equivalently the kinetic energy of the carriers of the superfluid.