Chapter 5

5. Josephson Effect

5.1 Josephson Equations
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5.3 Applications of the Josephson Effect
What happens if we weakly couple two superconductors?
5.1 Josephson Effect (cf. 2.2.3)

• **what happens if we weakly couple two superconductors?**
  - coupling by *tunneling barriers, point contacts, normal conducting layers, etc.*
  - do they form a bound state such as a molecule?
  - if yes, what is the binding energy?

• **B.D. Josephson** in 1962
  (nobel prize with Esaki and Giaever in 1973)

  ➔ Cooper pairs can tunnel through thin insulating barrier
  *
  *naive expectation:*
  - tunneling probability for pairs \( \propto (|T|^2)^2 \)
  ➔ extremely small \( \approx (10^{-4})^2 \)

  *Josephson:*
  - tunneling probability for pairs \( \propto |T|^2 \)
  - coherent tunneling of pairs (*tunneling of macroscopic wave function*)

  ➔ **finite supercurrent at zero applied voltage**
  ➔ **oscillation of supercurrent at constant applied voltage**
  ➔ **finite binding energy of coupled SCs = Josephson coupling energy**
5.1 Josephson Effect (cf. 2.2.3)

- coupling is weak → supercurrent density is small → $|\Psi|^2 = n_s$ is not changed
- supercurrent density depends on gauge invariant phase gradient $\gamma$:

$$J_s(r, t) = \frac{q_s n_s \hbar}{m_s} \left[ \nabla \theta(r, t) - \frac{2\pi}{\Phi_0} A(r, t) \right] = \frac{q_s n_s \hbar}{m_s} \gamma(r, t)$$

- **simplifying assumptions:**
  - current density is homogeneous
  - $\gamma$ varies negligibly in electrodes
  - $J_s$ same in electrodes and junction area
    - $\gamma$ in superconducting electrodes much smaller than in insulator I
- **then:**
  - replace gauge invariant phase gradient $\gamma$ by **gauge invariant phase difference**:

$$\varphi(r, t) = \int_1^2 \gamma(r, t) = \int_1^2 \left( \nabla \theta - \frac{2\pi}{\Phi_0} A \right) \cdot dl$$

$$= \theta_2(r, t) - \theta_1(r, t) - \frac{2\pi}{\Phi_0} \int_1^2 A(r, t) \cdot dl$$
5.1 Josephson Effect (cf. 2.2.3)

**first Josephson equation:**

- we expect:
  \[ J_s = J_s(\varphi) \]
  \[ J_s(\varphi) = J_s(\varphi + n2\pi) \]
- for \( J_s = 0 \): phase difference \( \varphi \) must be zero:
  \[ J_s(0) = J_s(n \cdot 2\pi) = 0 \]
therefore:

\[
J_s(\varphi) = J_c \sin \varphi + \sum_{m=2}^{\infty} J_m \sin(m\varphi)
\]

**general formulation** of 1st Josephson equation: *current-phase relation*

- in most cases: keep only 1st term (especially for weak coupling):

1. Josephson equation: \[ J_s(\varphi) = J_c \sin \varphi \]

- **generalization** to **spatially inhomogeneous** supercurrent density:

\[
J_s(y, z, t) = J_c(y, z) \sin \varphi(y, z, t)
\]

\( J_c \): critical/maximum Josephson current density

derived by Josephson for SIS junctions

- supercurrent density varies **sinusoidally** with \( \varphi = \theta_2 - \theta_1 \) w/o ext. potentials
5.1 Josephson Effect (cf. 2.2.3)

- other argument why there are only sin contributions to Josephson current

\[
J_s(\varphi) = J_c \sin \varphi + \sum_{m=2}^{\infty} J_m \sin(m\varphi)
\]

*time reversal symmetry*

- if we reverse time, the Josephson current should flow in opposite direction
  - \( t \rightarrow -t, \ J_s \rightarrow -J_s \)

- the time evolution of the macroscopic wave functions is \( \propto \exp[i\theta(t)] = \exp[i\omega t] \)
  - if we reverse time, we have

\[
\varphi(t) = \theta_2(t) - \theta_1(t) \quad \xrightarrow{t \rightarrow -t} \quad \varphi(-t) = \theta_2(-t) - \theta_1(-t) = -[\theta_2(t) - \theta_1(t)] = -\varphi(t)
\]

- if the Josephson effect stays unchanged under time reversal, we have to demand

\[
J_s(\varphi) = -J_s(-\varphi) \quad \text{satisfied only by sin-terms}
\]
5.1 Josephson Effect (cf. 2.2.3)

**second Josephson equation:**

- time derivative of the gauge invariant phase difference:

  \[
  \frac{\partial \varphi}{\partial t} = \frac{\partial \theta_2}{\partial t} - \frac{\partial \theta_1}{\partial t} - \frac{2\pi}{\Phi_0} \frac{\partial}{\partial t} \int_1^2 A(r, t) \cdot dl
  \]

- substitution of the energy-phase relation gives:

  \[
  -\hbar \frac{\partial \theta}{\partial t} = \frac{1}{2n_s} \Lambda J_s^2 + q_s \phi
  \]

- supercurrent density across the junction is continuous (\(J_s(1) = J_s(2)\)):

  \[
  \frac{\partial \varphi}{\partial t} = \frac{2\pi}{\Phi_0} \int_1^2 \left( -\nabla \phi - \frac{\partial A}{\partial t} \right) \cdot dl
  \]

  (term in parentheses = electric field)

**2. Josephson equation:**

\[
\frac{\partial \varphi}{\partial t} = \frac{2\pi}{\Phi_0} \int_1^2 E(r, t) \cdot dl
\]

**voltage – phase relation**

**voltage drop**
5.1 Josephson Effect (cf. 2.2.3)

• for a constant voltage across the junction:

\[
\frac{\partial \varphi}{\partial t} = \frac{2\pi}{\Phi_0} \ V
\]

\[
\varphi(t) = \varphi_0 + \frac{2\pi}{\Phi_0} \ V \cdot t
\]

• \(I_s\) is oscillating at the Josephson frequency \(f = V/\Phi_0\):

\[
I_s(t) = I_c \sin \varphi(t)
\]

\[
= I_c \sin \left( \frac{2\pi}{\Phi_0} \ V \cdot t \right)
\]

\[
\frac{f}{V} = \frac{\omega}{2\pi V} = \frac{1}{\Phi_0} \approx 483.597898(19) \ \frac{MHz}{\mu V}
\]

\(\Rightarrow\) voltage controlled oscillator

• applications:
  - Josephson voltage standard
  - microwave sources

• derivation of Josephson equations for SIS junction from time-dependent Schrödinger equation:
  - see exercise sheet
5.1.1 Special Topic: Superconducting Tunnel Junctions

- **Josephson effect in superconducting tunnel junctions**
  - *insulating* tunneling barrier, thickness $d$

- what determines the **maximum Josephson current density $J_c$**?
  - calculation by **wave matching method**

- **energy-phase relation**:

  $$-\hbar \frac{\partial \theta}{\partial t} = \frac{1}{2n_s} \mathbf{J}_s^2 + q_s \phi + \mu$$

  $$-\hbar \frac{\partial \theta}{\partial t} = \frac{1}{2n_s} \mathbf{J}_s^2$$

  $\Rightarrow$ time dependent macroscopic wave function:

  $$\psi(\mathbf{r}, t) = \psi(\mathbf{r}) e^{-i(E_0/\hbar)t}$$

- **wave function within** barrier with height $V_0 > E_0$:
  - only *elastic* processes:
    - time evolution is the **same** outside and inside barrier
    - consider only **time independent** part
    - time independent Schrödinger(-like) equation for region of constant potential
5.1.1 Josephson Effect

\[ -\frac{\hbar^2}{2m^*} \nabla^2 \psi(r) = (E_0 - V_0)\psi(r) \]

- **assumption:**
  homogeneous barrier and supercurrent flow \( \rightarrow 1D \) problem

- **solutions:**
  - in superconductors
    \[ \psi_{1,2} = \sqrt{n_{1,2}} e^{i\theta_{1,2}} \]
  - in insulator: sum of decaying and growing exponentials
    \[ \psi(x) = A \cosh(\kappa x) + B \sinh(\kappa x) \]

  - characteristic decay constant:
    \[ \kappa = \sqrt{\frac{2m_s(V_0 - E_0)}{\hbar^2}} \]

- coefficients A and B are determined by the boundary conditions at \( x = \pm d/2 \):
  \[
  \begin{align*}
  \psi(-d/2) &= \sqrt{n_1} e^{i\theta_1} \\
  \psi(+d/2) &= \sqrt{n_2} e^{i\theta_2}
  \end{align*}
  \]

\( n_{1,2}, \theta_{1,2} \): Cooper pair density and wave function phase at the boundaries \( x = \pm d/2 \)

\[
\begin{align*}
\sqrt{n_1} e^{i\theta_1} &= A \cosh(\kappa d/2) - B \sinh(\kappa d/2) \\
\sqrt{n_2} e^{i\theta_2} &= A \cosh(\kappa d/2) + B \sinh(\kappa d/2)
\end{align*}
\]
5.1.1 Josephson Effect

• solving for A and B:

\[ A = \frac{\sqrt{n_1} e^{i\theta_1} + \sqrt{n_2} e^{i\theta_2}}{2 \cosh(\kappa d/2)} \]

\[ B = -\frac{\sqrt{n_1} e^{i\theta_1} - \sqrt{n_2} e^{i\theta_2}}{2 \sinh(\kappa d/2)} \]

• supercurrent density:

\[ J_s = \frac{q_s}{m_s} \Re \left\{ \psi^* \left( \frac{\hbar}{i} \nabla \right) \psi \right\} = \frac{\hbar q_s}{m_s} \Im \left\{ \psi^* \nabla \psi \right\} \]

\[ J_s = \frac{q_s}{m_s} \kappa \hbar \Im \{ A^* B \} \]

• substituting the coefficients A and B:

\[ J_s = J_c \sin(\theta_2 - \theta_1) \]

⇒ maximum Josephson current density \( J_c \):

\[ J_c = -\frac{q_s}{m_s} \kappa \hbar \frac{\sqrt{n_1 n_2}}{2 \sinh(\kappa d/2) \cosh(\kappa d/2)} = -\frac{q_s \hbar \kappa}{m_s} \frac{\sqrt{n_1 n_2}}{\sinh(2\kappa d)} \]

• real junctions:

\[ V_0 \approx \text{few eV} \Rightarrow 1/\kappa < 1 \text{ nm}, \quad d \text{ few nm} \Rightarrow \kappa d \ll 1, \text{ then:} \quad \sinh(2\kappa d) \approx \frac{1}{2} \exp(2\kappa d) \]

• maximum Josephson current decays exponentially with increasing barrier thickness \( d \):

\[ J_c = -\frac{q_s \hbar \kappa}{m_s} 2\sqrt{n_1 n_2} \exp(-2\kappa d) \]
5.1.2 Ambegaokar-Baratoff Relation

- quasiparticle tunneling:
  - current-voltage characteristics

at $eV \gg 2\Delta(T)$

\[ J_{qp} \propto \frac{1}{R_nA} \cdot \exp(-2\kappa d) \]

$R_n$ = normal resistance resistance of NIN tunnel junctions

- Cooper pair tunneling:
  \[ J_c = \frac{e\hbar \kappa}{m} 2\sqrt{n_1n_2} \exp(-2\kappa d) \]

$V > 0$: time average of supercurrent vanishes:

\[ \langle J_c \sin \frac{2eV}{\hbar} t \rangle = 0 \]
5.1.2 Ambegaokar-Baratoff Relation

- ratio of $J_c$ and $J_{qp}(eV \gg 2\Delta) = const \Rightarrow J_c R_n A = I_c R_n A = I_c R_n = const$

- exact calculation yields Ambegaokar-Baratoff relation:

$$I_c R_n = \frac{\pi}{2e} \frac{\Delta(T)}{2k_B T} \cdot \tanh \left( \frac{\Delta(T)}{2k_B T} \right)$$


M.D. Fiske, Rev. Mod. Phys. **36**, 221–222
Temperature and Magnetic Field Dependences of the Josephson Tunneling Current
5.1 Summary

Macroscopic wave function $\Psi$:
- describes ensemble of macroscopic number of superconducting pairs
- $|\Psi|^2$ describes density of superconducting pairs

Current density in a superconductor:

$$J_s = \frac{\hbar n_s q_s}{m_s} \left\{ \nabla \theta(r, t) - \frac{q_s}{\hbar} A(r, t) \right\} = \frac{\hbar n_s q_s}{m_s} \left\{ \nabla \theta(r, t) - \frac{2\pi}{\Phi_0} A(r, t) \right\}$$

Gauge invariant phase gradient:

$$\gamma(r, t) = \nabla \theta(r, t) - \frac{q_s}{\hbar} A(r, t) = \nabla \theta(r, t) - \frac{2\pi}{\Phi_0} A(r, t)$$

Phenomenological London equations:

$$\frac{\partial}{\partial t} (\Lambda J_s) = E \quad \nabla \times (\Lambda J_s) = -B \quad (\Lambda = m_s/n_s q_s^2 = \mu_0 \lambda_L^2)$$

Flux/fluxoid quantization:

$$\oint_C (\Lambda J_s) \cdot dl + \int_S B \cdot ds = n \Phi_0$$
5.1 Summary

Josephson equations:

\[ J_s(r, t) = J_c(r, t) \sin \varphi(r, t) \]

\[ \frac{\partial \varphi}{\partial t} = \frac{2eV}{\hbar} = \frac{2\pi}{\Phi_0} V \]

\((\omega/2\pi = 483.6 \text{ GHz/mV})\)

**maximum** Josephson current density \( J_c \):

**wave matching method**

\[ J_s = J_c \sin(\theta_2 - \theta_1) \]

\[ J_c = -\frac{q_s \hbar \kappa}{m_s} 2\sqrt{n_1 n_2} \exp(-2\kappa d) \]

Tunneling current of unpaired electrons (quasiparticles, cf. chapter 4.4.2):

\[ J_q = f(V) \cdot \exp(-2\kappa d) \]
5.2 Josephson Coupling Energy

• the two weakly coupled superconductors form “molecule” analogous to H₂ molecule → what is the binding energy of this molecule?

• consider a JJ with initial current & phase difference equal to zero then: *increase junction current from zero to finite value*
  - phase difference has to change
  - voltage-phase relation: finite junction voltage
  - external source has to supply energy (to accelerate the superelectrons)
  - stored in kinetic energy of moving superelectrons
  - integral of the supplied power $I \cdot V$ to increase current to $I(\varphi) = I_c \sin \varphi$

\[
E_J = \int_{0}^{t_0} I_s V \, dt = \int_{0}^{t_0} (I_c \sin \varphi) \left( \frac{\Phi_0}{2\pi} \frac{d\varphi}{dt} \right) \, dt
\]
5.2 Josephson Coupling Energy

with $\varphi(0) = 0$ and $\varphi(t_0) = \varphi$:

$$E_J = \frac{\Phi_0 I_c}{2\pi} \int_0^\varphi \sin \tilde{\varphi} \, d\tilde{\varphi}$$

integration:

$$E_J = \frac{\Phi_0 I_c}{2\pi} (1 - \cos \varphi) = E_{J0} (1 - \cos \varphi)$$

- order of magnitude:
  - typically: $I_c \sim 1$ mA $\Rightarrow$ $E_{J0} \approx 3 \times 10^{-19}$ J
  - corresponds to thermal energy $k_B T$ for $T \approx 20 000$ K
  - junction with very small critical current: $I_c \approx 1 \mu A$ $\Rightarrow$ thermal energy $\approx k_B \times 20$ K
5.2.1 Josephson Junction with Applied Current

- analysis of **stability** of (junction + current source) – system:

  potential energy $E_{\text{pot}}$ of the system under action of external force: $E_J - F \cdot x$

  $E_J$: intrinsic free energy of the subsystem junction
  $F$: generalized force ($F = I$)
  $x$: generalized coordinate $\Rightarrow F \cdot \partial x / \partial t = \text{power flowing into subsystem} (I \cdot V)$:

  $$x = \int \mathcal{V} \, dt = \frac{\hbar}{2e} \varphi + c = \frac{\Phi_0}{2\pi} \varphi + c$$

  $\rightarrow$ potential energy:

  $$E_{\text{pot}}(\varphi) = E_J(\varphi) - I \left( \frac{\Phi_0}{2\pi} \varphi + c \right)$$

  $$= E_{J0} \left[ 1 - \cos \varphi - \frac{I}{I_c} \varphi \right] + \tilde{c}$$

  **tilted washboard potential**
  stable minima $\varphi_n$, unstable maxima $\tilde{\varphi}_n$,
  states for different $n$: equivalent

- junction dynamics: motion of $\varphi$ in tilted washboard potential (not discussed here)
5.2.1 Josephson Junction with Applied Current

\[-I_c < I < I_c \Rightarrow \text{constant phase difference:}\]

\[\varphi = \varphi_n = \arcsin \left( \frac{l}{I_c} \right) + 2\pi n\]

\[\varphi = \tilde{\varphi}_n = \pi - \arcsin \left( \frac{l}{I_c} \right) + 2\pi n\]

→ zero junction voltage:
zero voltage state / ordinary (S) state
5.3 Applications of the Josephson Effect

large number of applications in analog and digital electronics
⇒ detailed discussion in lecture „Applied Superconductivity“

- $I_s^m = I_s^m(B)$:
  ⇒ magnetic field sensors (SQUIDs)

- $\beta_c \gg 1$
  ⇒ bistability: zero/voltage state
  ⇒ switching devices, Josephson computer

- 2nd Josephson equation
  ⇒ VCO, voltage standard

- nonlinear IVC
  ⇒ mixers up to THz, oscillators

- macroscopic quantum behavior
  ⇒ superconducting qubits
5.3 Applications of the Josephson Effect

- $V = 0$: Josephson current
- $V \neq 0$: quasiparticle current

hysteresis:
- fast switching device
- very low power consumption
- $\Rightarrow$ Josephson digital electronics

slope $= 1/R$

switching time $\approx 1$ ps

$I_{\text{max}}$
5.3 Applications of the Josephson Effect

**principle of switching element:**

- magnetic field dependence of the maximum Josephson current

\[
I_{\text{max}}(0) \quad \text{stable for } B = 0 \\
I_{\text{max}}(B) \quad \text{unstable for } B > 0
\]
5.3 Applications of the Josephson Effect

World's fastest digital IC - operates to 750 GHz

http://insti.physics.sunysb.edu/physics/news_fast_ic.htm

Dividers

<table>
<thead>
<tr>
<th></th>
<th>RSFQ</th>
<th>Semiconductor</th>
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</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>750 GHz</td>
<td>60 GHz</td>
</tr>
<tr>
<td>Power Dissipation</td>
<td>1.5 μW</td>
<td>0.5 W</td>
</tr>
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</table>

\[ f_{\text{max}} = 750 \text{ GHz} \]

for details see: http://gamayun.physics.sunysb.edu/RSFQ/

- problem: integration of large number of JJs (> 10⁵ ) with high yield and small parameter spread
5.3 Applications of the Josephson Effect

http://pavel.physics.sunysb.edu/RSFQ/
5.3 Applications of the Josephson Effect

superconducting quantum bits

F. Deppe et al., PRB 76, 214503 (2007)
T. Niemczyk et al., SUST 22, 034009 (2009)

superconducting flux quantum bits fabricated at WMI
5.3 Applications of the Josephson Effect

superconducting quantum bits

superconducting quantum circuits fabricated at WMI
5.3 Applications of the Josephson Effect: metrology

precise definition of the electrical voltage, current and the resistance by using fundamental quantum effects:

- **Josephson effect:** \[ V = \frac{h}{2e} \cdot f = \Phi_0 \cdot f \] (relation between voltage and time/frequency by flux quantum)

- **Single electron pump:** \[ I = e \cdot f \] (relation between current and time by charge quantum)

- **Quantum Hall effect:** \[ V = \frac{h}{e^2} \cdot I = R_K \cdot I \] (relation between voltage and current by quantum resistance, unit = 1 Klitzing)

allows the reproduction of the physical units Volt, Ampère and Ohm with a very high precision and largely uninfluenced by environmental parameters at any place in the world

realization of the Ampère by single electron pump not realized so far at sufficient precision ➔ would allow an important experimental test of the consistency of the relations between the fundamental constants illustrated in the “electrical triangle”