Appendix 1: Additional Information to Section I.3.2 of Chapter I

- ***microscopic description of superfluids is very difficult***
  → phenomenological treatment using hydrodynamics

- anomalous properties of superfluid ^{4}He can be well described by two-fluid model
  (L. Tisza, J. de Phys. et Radium 1, 164 (1938))

- formal description of superfluid ^{4}He as the sum of a normal and a superfluid component

$$\rho = \rho_s + \rho_n$$

- ***T* = 0:** $\rho_s = \rho$ and $\rho_n = 0$

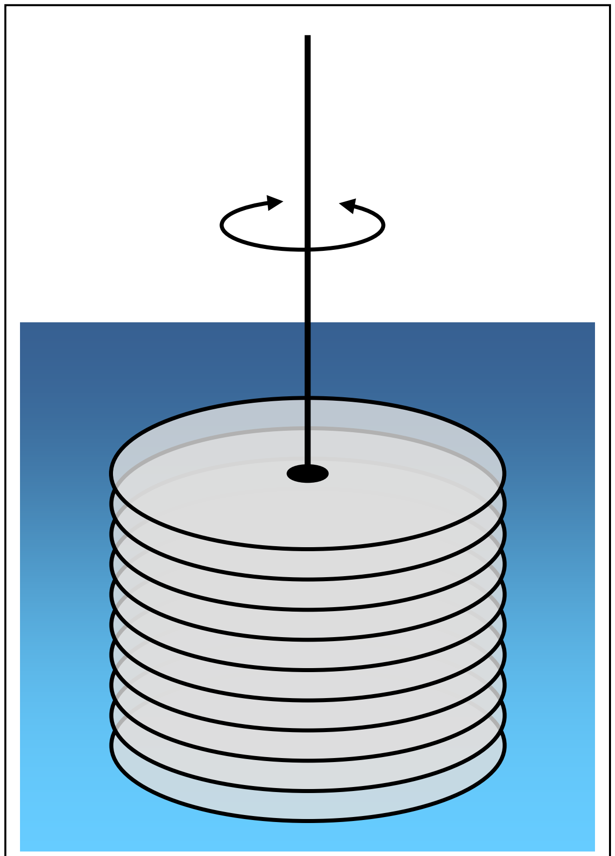
- ***T* = *T*$_\lambda$:** $\rho_s = 0$ and $\rho_n = \rho$

• superfluid component:	***no entropy: $S_s = 0$,***	***zero viscosity: $\eta_s = 0$***
• normal component:	***carries total entropy: $S_n = S$,***	***finite viscosity: $\eta_n = \eta$***

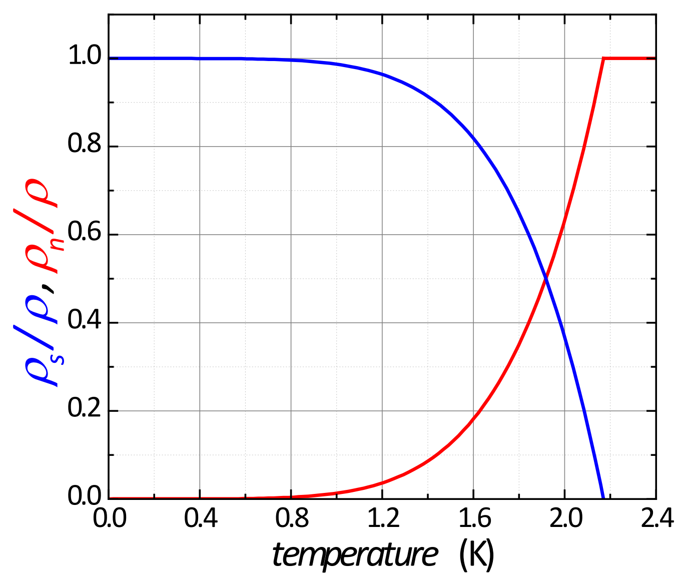
I.3.2 Two-Fluid Model:

Experiment of Andronikashvili

- determination of ρ_n *(E.L. Andronikashvili, Zh. Eksperim. i. Teor. Fiz. 18, 424 (1948))*
- torsional pendulum experiment: 50 Al disks, thickness: 13 µm, diameter: 3.5 cm, disk separation: 210 µm



- *only normal component is moved during rotation*
- *skin depth of viscous wave:* $(2\eta_n/\rho_n\omega)^{1/2}$
  → *large compared to disk separation below* T_λ
  → *complete normal component is moved*
- *moved mass of normal component results in change of moment of inertia*
  → *change of oscillation frequency:* $\omega \sim 1/\text{mass}$



empirical relation:

$$\frac{\rho_n}{\rho(T_\lambda)} = \left(\frac{T}{T_\lambda}\right)^{5.6}$$

cf. normal fluid density of ideal BEC:

$$\frac{\rho_n(T)}{\rho} = \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2}$$

for details of hydrodynamic description see appendix 1

- ***starting point for hydrodynamic description are Navier-Stokes equations:***

  - ***normal fluid***:

entropy per mass

$$\underbrace{\rho_n \frac{D\mathbf{v}_n}{Dt}}_{\text{inertia}} = \underbrace{-\frac{\rho_n}{\rho}\nabla\mathbf{p}}_{\text{pressure gradient}} - \underbrace{\rho_s \sigma \nabla T}_{\text{temperature gradient}} - \underbrace{\frac{\rho_s \rho_n}{2\rho}\nabla(\mathbf{v}_n - \mathbf{v}_s)^2}_{\text{additional term due to compressibility}} + \underbrace{\eta_n \nabla^2 \mathbf{v}_n}_{\text{viscosity}}$$

  - with the definition of the substantive derivative *Dv/Dt*:

$$\frac{D\mathbf{v}}{Dt} \equiv \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} \qquad \text{and} \qquad \mathbf{v}\cdot\nabla\mathbf{v} = \nabla\left(\frac{\|\mathbf{v}\|^2}{2}\right) + \underbrace{(\nabla\times\mathbf{v})}_{\substack{\text{curl of the velocity (called vorticity)}\\ \text{dissappears for irrotational flow}\\ \text{(no vortices)}}} \times \mathbf{v}$$

the tensor derivative of the velocity vector

I.3.2 Two-Fluid Model: Hydrodynamics

- ***superfluid fluid***:

➔ **no viscous friction**

$$\underbrace{\rho_s \frac{D\mathbf{v}_s}{Dt}}_{\text{inertia}} = -\underbrace{\frac{\rho_s}{\rho}\boldsymbol{\nabla}\mathbf{p}}_{\text{pressure gradient}} + \underbrace{\rho_s \sigma \boldsymbol{\nabla} T}_{\text{temperature gradient}} - \underbrace{\frac{\rho_s \rho_n}{2\rho}\boldsymbol{\nabla}(\mathbf{v}_n - \mathbf{v}_s)^2}_{\text{additional term due to compressibility}} + \underbrace{\cancel{\eta_s \boldsymbol{\nabla}^2 \mathbf{v}_s}}_{\text{viscosity: } \eta_s = 0}$$

➔ **Euler type equation for superfluid** (if there are no vortices)

➔ detailed treatment including vortices: **Gross-Pitaevskii equation**

- ***first sound:***

  normal and superfluid component move ***in phase:*** $\boxed{v_n = v_s}$

  $v_1 = 238$ m/s for $T \to 0$

- ***second sound:***

  normal and superfluid component move ***out of phase:***

  (Peshkov 1944)

  $$\boxed{\rho_n v_n + \rho_s v_s = 0}$$

  $$v_2 = \frac{v_1}{\sqrt{3}} \approx 137 \text{ m/s} \quad \text{for } T \to 0$$

- ***third sound:*** ***waves propagating in thin He films***

- ***fourth sound:*** ***compression wave propagating in a super leak***

  (super leak: very small opening or capillary, in which normal fluid cannot move because of its finite viscosity)

I.3.2 Two-Fluid Model: Hydrodynamics

- *simple derivation of equations of motion*

- mass flow density: $\mathbf{J} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s$

  normal *superfluid component*

- conservation of mass: $$\frac{\partial \rho}{\partial t} = -\mathrm{div}\mathbf{J}$$

- viscosity of normal component is very small and can be neglected in most situations

  → superfluid He is ideal liquid

  → description by ***Euler equation*** (*motion of a compressible, inviscid fluid*)

$$\frac{\partial \mathbf{J}}{\partial t} + \underbrace{\rho \mathbf{v} \cdot \mathrm{grad}\, \mathbf{v}}_{\approx 0} = -\mathrm{grad}\, \mathbf{p}$$

(*p* = pressure)

  → if **v** is small, quadratic terms in **v** can be neglected

- since we are neglecting dissipative effects (no viscous friction), entropy is conserved:

$$\frac{\partial (\rho\sigma)}{\partial t} = -\mathrm{div}\, (\rho\sigma\mathbf{v}_n)$$

entropy per mass

I.3.2 Two-Fluid Model: Hydrodynamics

- exact derivation of equation of motion of normal and superfluid phase is difficult
  → motivation of equation of motion by „Gedanken“ experiment (after Landau)

- suppose, we are adding superfluid component at constant volume
  → resulting change of inner energy:

$$dU = \underbrace{TdS}_{=0} - \underbrace{pdV}_{=0} + Gdm$$

(G = free entalpy per mass)

(the resulting operation only changes the mass of the superfluid component)

$$dU = Gdm$$

(G can be considered as the potential energy/mass of the superfluid phase,
→ - grad G is corresponding force)

→ we obtain: $$\frac{\partial \mathbf{v}_s}{\partial t} = -\mathrm{grad}\ G$$

- we use the thermodynamic relation $dG = -SdT + dp/\rho$ and can substitute grad G:
  → equation of motion of superfluid component

$$\boxed{\frac{\partial \mathbf{v}_s}{\partial t} = -\sigma\ \mathrm{grad}\ T - \frac{1}{\rho}\mathrm{grad}\ p}$$

(σ: entropy per mass)

I.3.2 Two-Fluid Model: Hydrodynamics

- equation of motion of normal component:

  → substitute $\partial v_s/\partial t$ by using $\mathbf{J} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s$ and $\frac{\partial \mathbf{J}}{\partial t} + \rho \mathbf{v} \cdot \mathrm{grad}\, \mathbf{v} = -\mathrm{grad}\, \mathbf{p}$

$$\frac{\partial \mathbf{v}_n}{\partial t} = -\frac{\rho_s}{\rho_n}\sigma\, \mathrm{grad}\, T - \frac{1}{\rho}\mathrm{grad}\, \mathbf{p}$$

- note that equations of motion are valid only for the linear regime !

I.3.2 Two-Fluid Model: Sound Propagation

- insert time derivative of $\dfrac{\partial \rho}{\partial t} = -\mathrm{div}\mathbf{J}$ into $\dfrac{\partial \mathbf{J}}{\partial t} + \rho\mathbf{v} \cdot \mathrm{grad}\ \mathbf{v} = -\mathrm{grad}\ \mathbf{p}$

$$\rightarrow \quad \frac{\partial^2 \rho}{\partial t^2} = -\nabla^2\ p$$

- elimination of terms with $\mathbf{v}_n$ and $\mathbf{v}_s$ (cannot be observed experimentally)
  → after some steps one obtains

$$\frac{\partial^2 \sigma}{\partial t^2} = -\frac{\rho_s}{\rho_n}\sigma^2 - \nabla^2\ T$$

- in the two equation we have four variables: ρ, σ, p and T
  → only two of them are independent, we choose ρ and σ

- dependence of p and T on ρ and σ:

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_\sigma \delta\rho + \left(\frac{\partial p}{\partial \sigma}\right)_\rho \delta\sigma$$

$$\delta T = \left(\frac{\partial T}{\partial \rho}\right)_\sigma \delta\rho + \left(\frac{\partial T}{\partial \sigma}\right)_\rho \delta\sigma$$

I.3.2 Two-Fluid Model: Sound Propagation

- insertion of these relations yields

$$\frac{\partial^2 \rho}{\partial t^2} = \left(\frac{\partial p}{\partial \rho}\right)_\sigma \nabla^2 \rho + \left(\frac{\partial p}{\partial \sigma}\right)_\rho \nabla^2 \sigma$$

$$\frac{\partial^2 \sigma}{\partial t^2} = \frac{\rho_s}{\rho_n}\sigma^2 \left[\left(\frac{\partial T}{\partial \rho}\right)_\sigma \nabla^2 \rho + \left(\frac{\partial T}{\partial \sigma}\right)_\rho \nabla^2 \sigma\right]$$

- solutions are plane waves of the form

$$\rho = \rho_0 + \rho_1 \exp\left[\imath\omega(t - x/v)\right]$$

$$\sigma = \sigma_0 + \sigma_1 \exp\left[\imath\omega(t - x/v)\right]$$

- insertion yields

$$\left[\left(\frac{v}{v_1}\right)^2 - 1\right]\rho_1 + \left(\frac{\partial p}{\partial \sigma}\right)_\rho \left(\frac{\partial \rho}{\partial p}\right)_\sigma \sigma_1 = 0$$

$$\left[\left(\frac{v}{v_2}\right)^2 - 1\right]\sigma_1 + \left(\frac{\partial T}{\partial \rho}\right)_\sigma \left(\frac{\partial \sigma}{\partial T}\right)_\rho \sigma_1 = 0$$

with

$$v_1^2 = \left(\frac{\partial p}{\partial \rho}\right)_\sigma$$

$$v_2^2 = \frac{\rho_s}{\rho_n}\sigma^2 \left(\frac{\partial T}{\partial \sigma}\right)_\rho$$

I.3.2 Two-Fluid Model: Sound Propagation

- system of linear equations → equation for coefficients

$$\left[\left(\frac{v}{v_1}\right)^2 - 1\right] \cdot \left[\left(\frac{v}{v_2}\right)^2 - 1\right] = \left(\frac{\partial p}{\partial \sigma}\right)_\rho \left(\frac{\partial \rho}{\partial p}\right)_\sigma \left(\frac{\partial T}{\partial \rho}\right)_\sigma \left(\frac{\partial \sigma}{\partial T}\right)_\rho$$

- use of thermodynamic relations:

$$\left[\left(\frac{v}{v_1}\right)^2 - 1\right] \cdot \left[\left(\frac{v}{v_2}\right)^2 - 1\right] = \frac{C_p - C_V}{C_p}$$

→ each expression in the brackets represents the dispersion relation of a special type of wave, coupling via $(C_p\text{-}C_V)/C_P$

- since for superfluid He $(C_p\text{-}C_V)/C_P$ is very small, we can use the approximation

$$\left[\left(\frac{v}{v_1}\right)^2 - 1\right] \cdot \left[\left(\frac{v}{v_2}\right)^2 - 1\right] \simeq 0$$

I.3.2 Two-Fluid Model: First Sound

- propagation of usual sound waves with velocity v_1 given by

$$\left[\left(\frac{v}{v_1}\right)^2 - 1\right]\rho_1 + \left(\frac{\partial p}{\partial \sigma}\right)_\rho \left(\frac{\partial \rho}{\partial p}\right)_\sigma \sigma_1 = 0$$

if we set $\rho_1 \neq 0$ and $\sigma_1 = 0$, we have grad $T \approx 0$ as for a usual sound wave (***first sound***)

- from $\frac{\partial \mathbf{J}}{\partial t} + \rho \mathbf{v} \cdot \mathrm{grad}\, \mathbf{v} = -\mathrm{grad}\, \mathbf{p}$ and $\frac{\partial \mathbf{v}_s}{\partial t} = -\sigma\, \mathrm{grad}\, T - \frac{1}{\rho}\mathrm{grad}\, \mathbf{p}$ we obtain

1

$$v_s = v_n$$

➔ ***normal and superfluid component move in phase (first sound)***
➔ density changes are adiabatic
➔ superfluid He behaves as usual liquid
➔ sound velocity ***v_1 = 238 m/s*** for ***$T \rightarrow 0$***
➔ $v_1(T)$ shows anomaly at T_λ

I.3.2 Two-Fluid Model: Second Sound

- in case of $\rho_1 = 0$ and $\sigma_1 \neq 0$, expression

$$\left[\left(\frac{v}{v_2}\right)^2 - 1\right]\sigma_1 + \left(\frac{\partial T}{\partial \rho}\right)_\sigma \left(\frac{\partial \sigma}{\partial T}\right)_\rho \sigma_1 = 0$$

describes temperature waves moving at velocity v_2

- from $\frac{\partial^2 \rho}{\partial t^2} = -\nabla^2 p$ we obtain grad $p = 0$.

Then, we obtain the following relation from Euler's equations

$$\frac{\partial(\rho_n v_n)}{\partial t} + \frac{\partial(\rho_s v_s)}{\partial t} = 0$$

that is, the mass momentum density must be either constant or zero $\mathbf{J} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s$

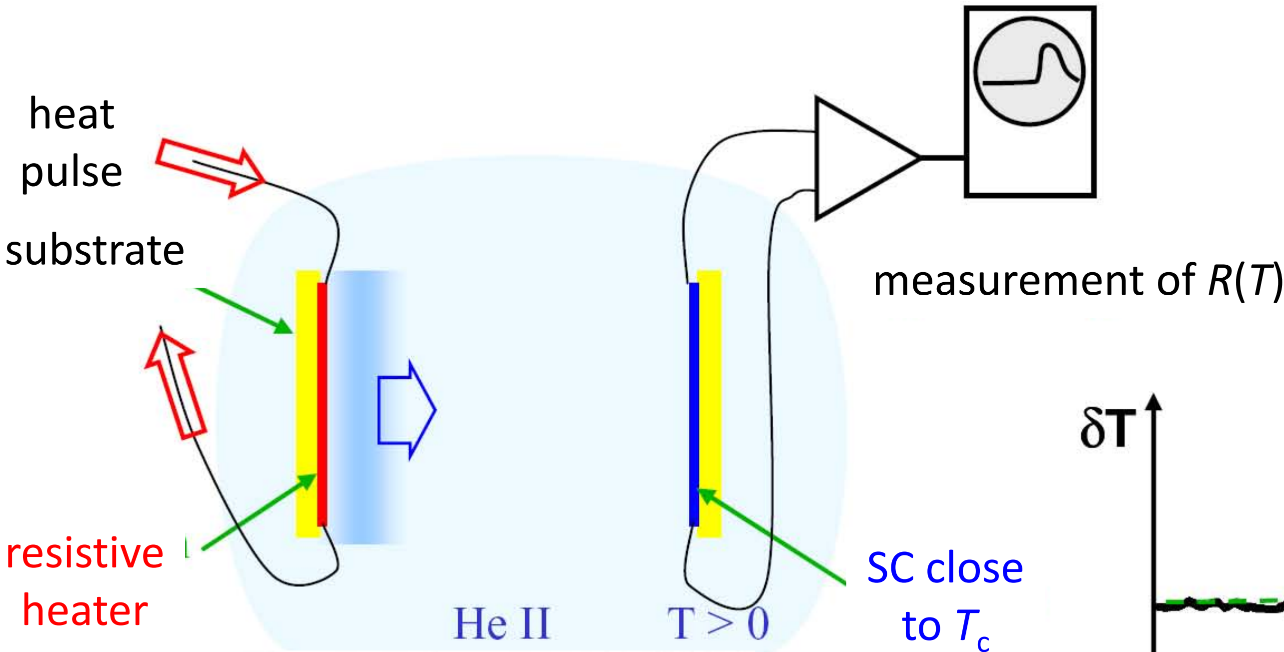
- since in a closed container there cannot be a constant mass flow, we have

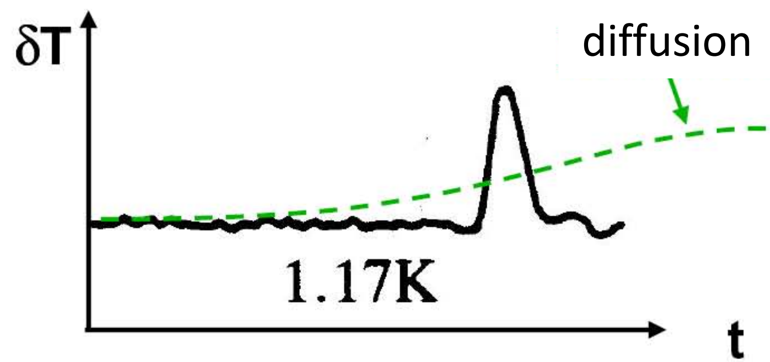
$$\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s = 0$$

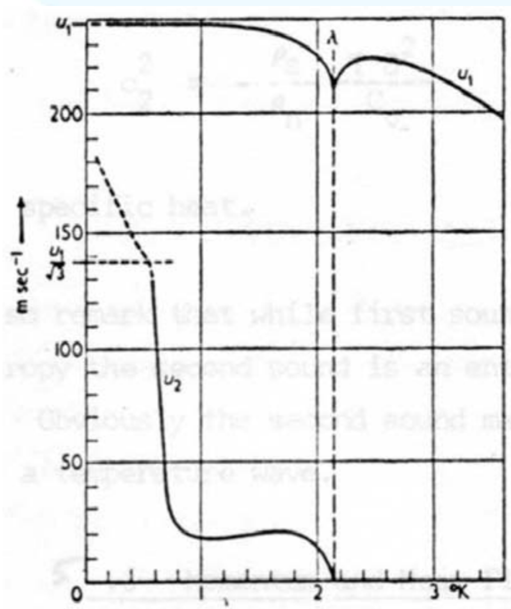
➔ ***normal and superfluid component move out of phase*** (***second sound***)

➔ $v_2 = \sqrt{\frac{\rho_s}{\rho_n}\sigma^2 \left(\frac{\partial T}{\partial \sigma}\right)_\rho} = \sqrt{\frac{\rho_s}{\rho_n}\frac{T\sigma^2}{C_p}}$ for $T \to 0$: $v_2 = \frac{v_1}{\sqrt{3}} \simeq 137\ \mathrm{m/s}$

I.3.2 Two-Fluid Model: Second Sound - Experiment







why are there temperature waves in superfluid helium ??

- starting point is usual expression for thermal conductance: $J_Q = -\kappa \frac{\partial T}{\partial x}$

  → velocity proportional to driving force (viscous motion)

- superfluid He: → no Umklapp processes
  → no impurity scattering
  → no friction !!!

- heat transported by excitations: phonons, rotons
  → they have momentum
  → requires force for momentum changes

  → add force term to $J_Q = -\kappa \frac{\partial T}{\partial x}$

➔ $$\frac{\partial J_Q}{\partial t} + \frac{1}{\tau} J_Q = -\frac{\kappa}{\tau} \frac{\partial T}{\partial x} \qquad (1)$$

τ is characteristic relaxation time of momentum change due to friction

I.3.2 Two-Fluid Model: Second Sound

- energy conservation: $$\boxed{\frac{\partial Q}{\partial t} = C\frac{\partial T}{\partial t} + \frac{\partial J_Q}{\partial x} = 0} \qquad (2)$$

- differentiate (1) and (2) with respect to *x* and *t*

  → eliminate J_Q *and neglect viscous term (damping)*

$$C\frac{\partial^2 T}{\partial t^2} = \frac{\kappa}{\tau}\frac{\partial^2 T}{\partial x^2}$$

- use $\kappa = \frac{1}{3}C v^2 \tau$ from kinetic gas theory ➔

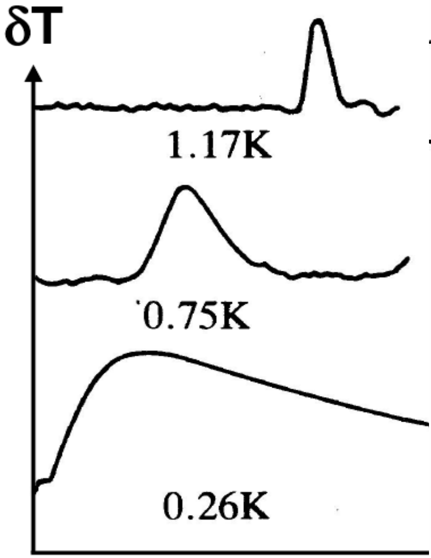
$$\boxed{\frac{\partial^2 T}{\partial t^2} = \frac{1}{3}v^2\frac{\partial^2 T}{\partial x^2}}$$

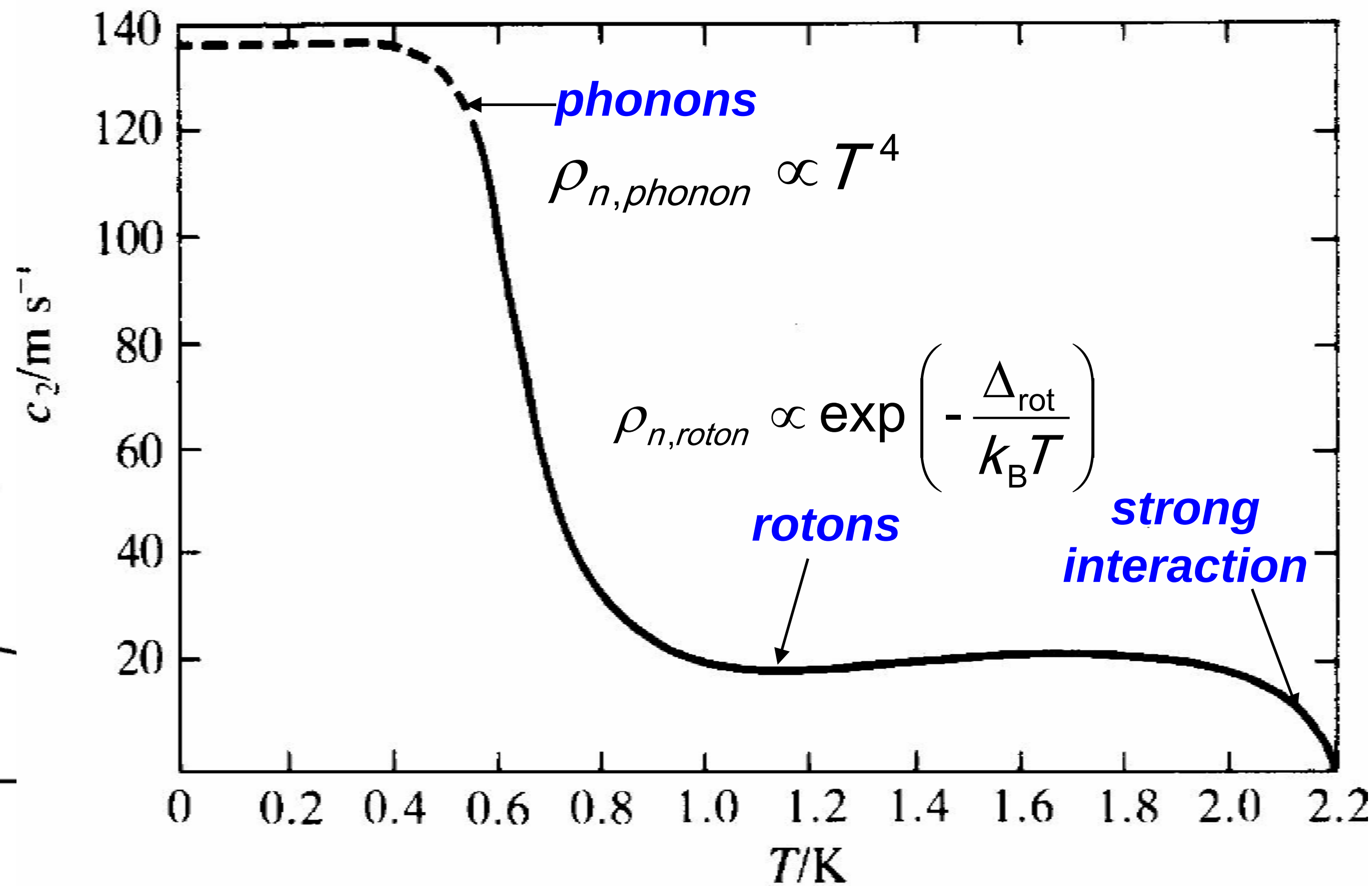
- Ansatz: $T = T_0 + T_1 \exp\left[\imath(kx - \omega t)\right]$ ➔ $\omega^2 = \frac{1}{3}v^2 k^2$

- same linear dispersion as for sound waves: $\omega = c_2 k$

I.3.2 Two-Fluid Model: Second Sound

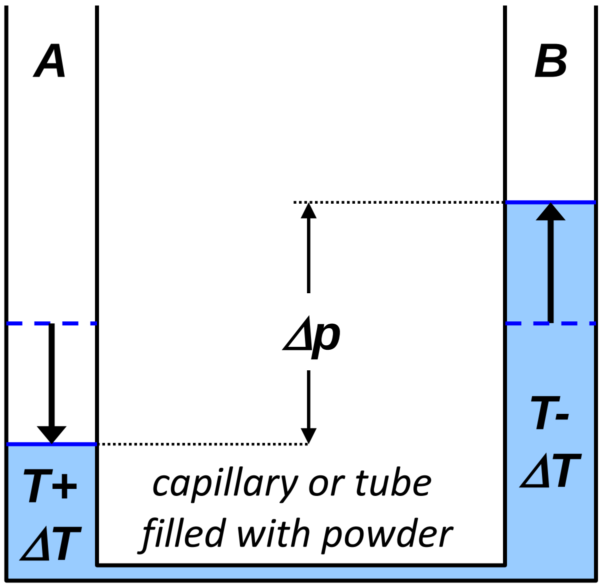
pulse experiment





$T \to T_\lambda$:

→ collision probability increases
→ picture of elementary excitations no longer valid
→ vortex rings



- work required for mass current from A to B:

$$\Delta W = \Delta V \cdot \Delta p = \Delta p \frac{\Delta m}{\rho}$$

- energy gained by heat flow from B to A:

$$\Delta Q = \Delta m \cdot \sigma \cdot \Delta T \qquad \sigma = S/m$$

- energy balance yields:

$$\Delta p \frac{\Delta m}{\rho} = \Delta m \cdot \sigma \cdot \Delta T$$

$$\Delta p = \rho \cdot \sigma \cdot \Delta T = \rho_n \cdot \sigma_n \cdot \Delta T$$

since σ = 0 for superfluid and $\rho = \rho_n + \rho_s$

➔ Δp causes ΔT and vice versa

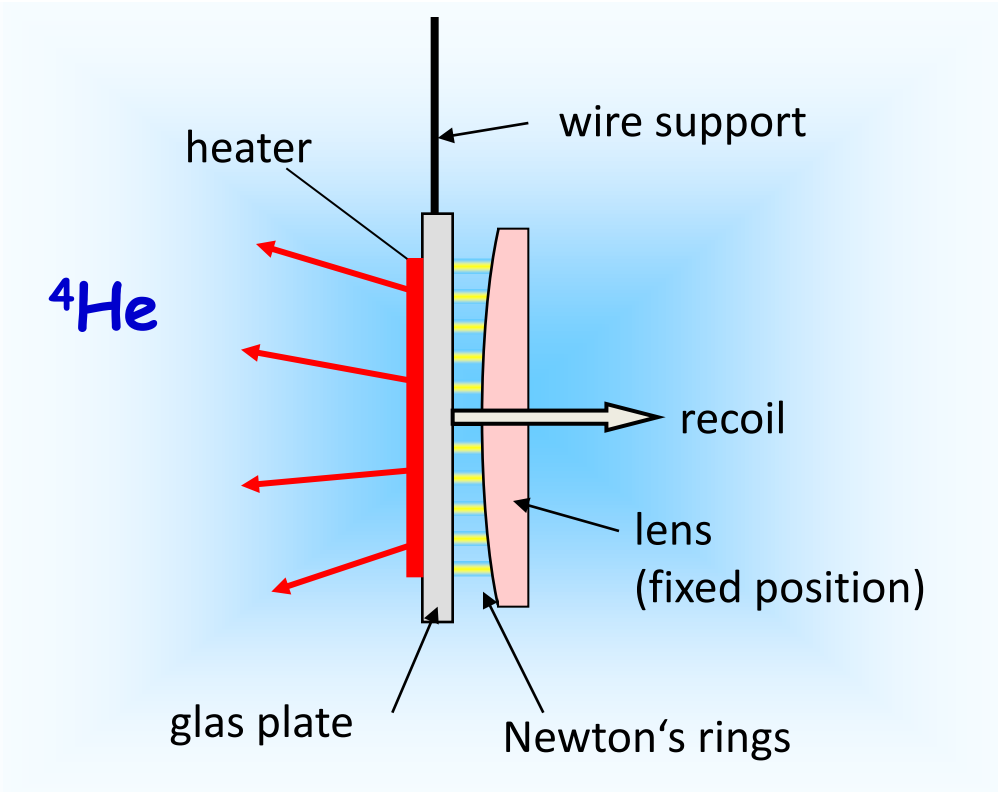
➔ $\Delta p \rightarrow 0$ *for* $\rho_n \rightarrow 0$

same result follows from

$$\frac{\partial \mathbf{v}_s}{\partial t} = -\sigma \,\mathrm{grad}\, T - \frac{1}{\rho}\mathrm{grad}\, p$$

with dv_s/dt = 0 in equilibrium

I.3.2 Two-Fluid Model
– momentum transfer due to heat flow



- momentum flow per volume: given by $\rho\,\mathbf{v}\cdot\mathbf{v}$
- resulting pressure on heat source:

$$p = \rho_n v_n^2 + \rho_s v_s^2$$

- with heat flow per area:

$$J_Q = \rho\sigma T v_n$$

- we obtain with $\rho_n v_n + \rho_s v_s = 0$ (no mass transport) pressure

$$p = \frac{\rho_n J_Q^2}{\rho_s \rho T^2 \sigma^2} = \frac{J_Q^2}{v_2^2 \rho C_p T}$$

- *heater generates normal fluid: phonons and rotons*
- *radiation to the left*
- *recoil to the right*
- *measurement of displacement of the glas plate via Newton's rings between glas plate and lens*