Chapter III
Quantum Transport in Nanostructures
Chapter III: Quantum Transport in Nanostructures

Contents:

III.1 Introduction
   III.1.1 General Remarks
   III.1.2 Mesoscopic Systems
   III.1.3 Characteristic Length Scales
   III.1.4 Characteristic Energy Scales
   III.1.5 Transport Regimes

III.2 Description of Electron Transport by Scattering of Waves
   III.2.1 Electron Waves and Waveguides
   III.2.2 Landauer Formalism
   III.2.3 Multi-terminal Conductors

III.3 Quantum Interference Effects
   III.3.1 Double Slit Experiment
   III.3.2 Two Barriers – Resonant Tunneling
   III.3.3 Aharonov-Bohm Effect
   III.3.4 Weak Localization
   III.3.5 Universal Conductance Fluctuations

III.4 From Quantum Mechanics to Ohm’s Law

III.5 Coulomb Blockade
Chapter III: Quantum Transport in Nanostructures

Literature:

1. Introduction to Mesoscopic Physics
   Yoseph Imry

2. Electronic Transport in Mesoscopic Systems
   Supriyoto Datta

3. Mesoscopic Electronic in Solid State Nanostructures
   Thomas Heinzel

4. Quantum Transport
   Yuli V. Nazarov, Yaroslav M. Blanter
III.1 Introduction

III.1.1 General Remarks

• **macroscopic solid state systems**
  - *usually consideration of thermodynamic limit*
    \[ N \to \infty, \Omega \to \infty, N/\Omega = \text{const}. \]

• **what happens if system size becomes small?**
  - *discrete spectrum of electronic levels*
  - *coherent motion of electrons*
    \[ \to \text{phase memory due to lack of inelastic scattering within system size:} \]
    \[ \text{system size } L \text{ smaller than } \text{phase coherence length } L_{\phi} \]
    \[ \to \text{new interference phenomena} \]
  - *validity of Boltzmann theory of electronic transport and concept of resistivity?*
    \[ \to \text{system size } L \text{ smaller than } \text{mean free path } \ell: \text{ballistic transport} \]
  - *discreteness of electric charge and magnetic flux becomes important*
    \[ \to \text{single electron and single flux effects} \]
  - *concept of impurity ensemble breaks down*
    \[ \to \text{sample properties show „fingerprint“ of detailed arrangement of impurities} \]
III.1 Introduction

III.1.2 Mesoscopic Systems

- **mesoscopic systems** (coined by Van Kampen in 1981):

  - system size is between microscopic (e.g. atom, molecule) and macroscopic system (e.g. bulk solid)
  - system size $L$ is smaller than phase coherence length $L_\phi$ (typically in nm - µm regime)
    - quantum coherent phenomena become important
    - statistical concepts no longer applicable due to smallness of system size
    - still coupling to environment/reservoir present
      (in contrast to microscopic objects such as atoms)

- **properties of mesoscopic systems are usually studied at low temperatures**

  - phase coherence length $L_\phi$ decreases rapidly with increasing $T$
  - $L < L_\phi$ can usually be satisfied only at low $T$

  - observation of level quantization effects require $k_B T < \Delta E \approx 1/L^2$

  **study of nanostructures at low temperature**
III.1 Introduction

III.1.2 Mesoscopic Systems – The World of Solid State Nanostructures
Die folgende graphische Animation zeigt den Anflug auf eine Einzelelektronen-Schaltung.

Sie beginnt mit der Ansicht des gesamten Wafers und endet mit der elektronenmikroskopischen Aufnahme einer realen Struktur.
superconducting flux quantum circuit
III.1 Introduction

III.1.2 Mesoscopic Systems – Miniaturization of Electronic Devices

Gate length of transistors

- 65 nm process 2005
- 45 nm process 2007
- 32 nm 2009
- 22 nm 2011

(Source: Intel Inc.)
III.1 Introduction
III.1.3 Characteristic Length Scales

*from microscopic to macroscopic systems*

<table>
<thead>
<tr>
<th>microscopic ↔ mesoscopic ↔ macroscopic</th>
</tr>
</thead>
</table>

Fermi wave length: $\lambda_F < 1 \text{ nm}$ (for metals)

⇒ "size" of charge carrier

electron mean free path: $\ell \approx 10 - 100 \text{ nm}$

⇒ distance between (elastic) scattering events

phase coherence length: $L_\phi \approx 1 \text{ \mu m}$

⇒ loss of phase memory

sample size: $L, W \approx 0.01 - 1 \text{ \mu m}$

mesoscopic regime: $L < L_\phi (T)$
### III.1 Introduction

#### III.1.3 Characteristic Length Scales

<table>
<thead>
<tr>
<th>Length</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 mm</td>
<td>mean free path in the Quantum Hall regime</td>
</tr>
<tr>
<td>100 µm</td>
<td>mean free path / phase coherence length in high mobility semiconductors at ( T &lt; 4 \text{ K} )</td>
</tr>
<tr>
<td>10 µm</td>
<td>phase coherence length in clean metal films</td>
</tr>
<tr>
<td>1 µm</td>
<td>size of commercial semiconductor devices</td>
</tr>
<tr>
<td>100 nm</td>
<td>Fermi wave length in semiconductors</td>
</tr>
<tr>
<td>10 nm</td>
<td>mean free path in polycrystalline metal films</td>
</tr>
<tr>
<td>1 nm</td>
<td>Fermi wave length in metals</td>
</tr>
<tr>
<td>0.1 nm</td>
<td>distance between atoms</td>
</tr>
</tbody>
</table>
III.1 Introduction

III.1.3 Characteristic Length Scales

- **electron wavelength:**
  \[ \lambda_F = \frac{\hbar}{\sqrt{2m^\star E_F}} = \frac{2\pi}{(3\pi^2 n)^{1/3}} \]  
  (Fermi wavelength)

- **mean free path:**
  \[ \ell = v_F \cdot \tau_m \]
  \[ \tau_m^{-1} = \tau_c^{-1} \cdot \alpha_m \]
  \[ \uparrow \text{effectiveness of collision: } 0 < \alpha_m < 1 \]

- **phase relaxation length:**
  \[ L_\phi = v_F \tau_\phi \]
  \[ \tau_\phi^{-1} = \tau_c^{-1} \cdot \alpha_\phi \]
  \[ \downarrow \text{effectiveness of collision in destroying phase coherence: } 0 < \alpha_\phi < 1 \]

  \[ \rightarrow \text{elastic impurity scattering: } \tau_\phi \to 0 \text{ or } \alpha_\phi \to 0 \]

  \[ \rightarrow \text{electron-phonon scattering: } \tau_\phi \approx \tau_{e-ph} ?? \]

  \[ \rightarrow \text{electron-electron scattering: } \tau_\phi \approx \tau_{e-e} ?? \]

  \[ \rightarrow \text{electron-impurity scattering} \quad (\text{with internal degree of freedom, e.g. spin}) \]
• **question:** what is the effectiveness of an *inelastic scattering process* regarding destruction of phase coherence?

• *Altshuler, Aronov, Khmelnitsky (1982):*

If $\hbar \omega$ is characteristic energy of an *inelastic process* (e.g. phonon energy), then the mean-squared energy spread of electron after collision is

\[
\langle \Delta E \rangle^2 = (\hbar \omega)^2 \frac{\tau \varphi}{\tau_c}
\]

$\langle \Delta E \rangle^2$ is square of energy change, \( \tau_c \) is number of scattering events.

\( \tau \varphi \) is time required to acquire a phase change of \( \approx 2 \pi \)

\[
\Delta \varphi \approx \frac{\Delta E}{\hbar} \tau \varphi \approx 2 \pi \quad \Rightarrow \quad \tau \varphi \approx \left( \frac{\tau_c}{\omega^2} \right)^{1/3}
\]

- low-frequency excitations are less effective in destroying phase coherence!!

• at low $T$: e-e scattering is dominating
III.1 Introduction

III.1.4 Characteristic Energy Scales - Size Quantization

- electron in a box:

![Graph showing level spacing](image)

- level spacing:

\[ \Delta E = \frac{\hbar^2}{2m^*} \left( \frac{1}{L} \right)^2 \]

1 nm \( \leftrightarrow \) 10.000 K \( \leftrightarrow \) 800 meV
10 nm \( \leftrightarrow \) 100 K \( \leftrightarrow \) 8 meV
100 nm \( \leftrightarrow \) 1 K \( \leftrightarrow \) 0.08 meV

- Fermi wavelength:

\[ \lambda_F = \frac{\hbar}{\sqrt{2m^*E_F}} = \frac{2\pi}{(3\pi^2n)^{1/3}} \]

- if \( \lambda_F > L_x, L_y, L_z \) \( \rightarrow \) reduction of dimension by size quantization

3D \( \rightarrow \) 2D \( \rightarrow \) 1D \( \rightarrow \) 0D

- for metals: \( n \approx 10^{22} - 10^{23} \text{ cm}^{-3} \rightarrow \lambda_F \approx 1 \text{ nm} \)

- for semiconductors: \( n \approx 10^{16} - 10^{19} \text{ cm}^{-3} \rightarrow \lambda_F \approx 10 - 100 \text{ nm} \)

- single charge/flux effects:

\[ \frac{e^2}{2C} > k_B T, \quad \frac{\Phi_0^2}{2L} > k_B T \]
III.1 Introduction

III.1.4 Size Quantization – DOS in 3D, 2D, 1D, and 0D

- **bulk**: $D(E) \propto \sqrt{E}$
- **superlattice**: $D(E) = \text{const}$
- **quantum well**: $D(E) \propto 1/\sqrt{E}$
- **quantum wire**: $D(E) = \delta(E - E_i)$
- **quantum dot**: $D(E)$

3-dim | 2-dim | 1-dim | 0-dim
III.1 Introduction
III.1.4 Characteristic Energy Scales – Thouless Energy

• how long does it take for an electron to diffuse through a sample of length $L$

\[ L = \sqrt{Dt} \quad \Rightarrow \quad t = \frac{L^2}{D} \]

• mean diffusion time is related to the characteristic energy (uncertainty relation)

\[ E_{Th} = \frac{\hbar}{t} = \frac{\hbar D}{L^2} \]  (Thouless energy)

• ballistic transport regime (see below):

\[ t = \frac{L}{v_F} \]
\[ E_{Th} = \frac{\hbar}{t} = \frac{\hbar v_F}{L} \]  ($v_F$: Fermi velocity)

• macroscopic samples:

\[ E_{Th} \ll k_B T \]

• mesoscopic samples:

\[ E_{Th} > k_B T \]

- low $T$
- small $L$
- clean samples (large $D$)

$L < \sqrt{\frac{\hbar D}{k_B T}}$
III.1 Introduction

III.1.4 Characteristic Energy Scales – Thouless Energy

- meaning of Thouless energy \( E_{Th} = \frac{\hbar}{t} = \frac{\hbar D}{L^2} \)

→ electrons in energy interval \( \Delta E = E_{Th} \) stay phase coherent in sample of length \( L \)

\[
E_{Th} = \frac{\hbar}{t} = \frac{\hbar D}{L^2}
\]

\[
\Delta \varphi = 2\pi
\]

after length \( L \), if \( \Delta E = E_{Th} \)

\[
\text{if } \Delta E \leq E_{Th}, \text{ acquired phase shift is less than } 2\pi
\]

example: \( D = 10^3 \text{ cm}^2/\text{s}, \ L = 1 \mu\text{m} \rightarrow E_{Th}/k_B \approx 1 \text{ K} \)
III.1 Introduction

III.1.5 Transport Regimes

<table>
<thead>
<tr>
<th>macroscopic sample</th>
<th>mesoscopic sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>diffusive: $L,W &gt;&gt; \ell$</td>
<td>ballistic: $L,W &lt; \ell$</td>
</tr>
<tr>
<td>incoherent: $L &gt;&gt; L_\phi$</td>
<td>quasi-ballistic: $W &lt; \ell$</td>
</tr>
<tr>
<td></td>
<td>coherent: $L &lt; L_\phi$</td>
</tr>
</tbody>
</table>

- @ 300 K: $\ell \sim 10$ nm due to e-ph scattering
- @ at low $T$: $\ell$ is limited by impurity and e-e scattering $\rightarrow$ sample quality matters
- $L_\phi$ is limited by inelastic processes: e-ph and e-e scattering:
  - strong $T$ dependence: $L_\phi$ increases with decreasing $T$
  - $L_\phi \approx 1$ μm @ 1K
III.2 Quantum Transport in Nanostructures

Contents:

III.1 Introduction
   III.1.1 General Remarks
   III.1.2 Mesoscopic Systems
   III.1.3 Characteristic Length Scales
   III.1.4 Characteristic Energy Scales
   III.1.5 Transport Regimes

III.2 Description of Electron Transport by Scattering of Waves
   III.2.1 Electron Waves and Waveguides
   III.2.2 Landauer Formalism
   III.2.3 Multi-terminal Conductors

III.3 Quantum Interference Effects
   III.3.1 Double Slit Experiment
   III.3.2 Two Barriers – Resonant Tunneling
   III.3.3 Aharonov-Bohm Effect
   III.3.4 Weak Localization
   III.3.5 Universal Conductance Fluctuations

III.4 From Quantum Mechanics to Ohm’s Law

III.5 Coulomb Blockade
III.2 Description of Electron Transport by Scattering of Waves

III.2.1 Electron Waves and Waveguides

- **electrons as plane waves** (true only in vacuum)

\[
\Psi(r, t) = \frac{1}{\sqrt{V}} \exp \left( ik \cdot r - \frac{i}{\hbar} E(k) t \right)
\]

- \(\Psi(r, t)\) wave function
- \(|\Psi(r, t)|^2\) probability to find electron at position \(r\) at time \(t\)
- \(V\) normalization volume
- \(k\) wave vector
- \(p = \hbar k\) momentum
- \(E = \frac{\hbar^2 k^2}{2m}\) energy
III.2 Description of Electron Transport by Scattering of Waves

III.2.1 Electron Waves and Waveguides

- **electrons as Fermions:**

  → Pauli principle (state either occupied by single electron or empty)

  → density of states in k-space: \( \frac{V}{(2\pi)^3} \) (factor 2 due to spin)

  → fraction of filled states: \( f \)

- important quantities:

  \[
  \begin{align*}
  \text{density} & : \rho \\
  \text{energy density} & : E \\
  \text{current density} & : J
  \end{align*}
  \]

  \[
  J = \int 2 \frac{d^3 k}{(2\pi)^3} \left( \begin{array}{c} 1 \\ E(k) \\ e v(k) \end{array} \right) f(k)
  \]

- \( f \) determined by statistics:

  \[
  f(k) = \frac{1}{\exp\left(\frac{E(k) - \mu}{k_B T}\right) + 1}
  \]

  Fermi statistics for electrons

not by qm !!
**III.2 Description of Electron Transport by Scattering of Waves**

**III.2.1 Electron Waves and Waveguides**

- **ballistic conductor as waveguide**: 1D free motion of charge carriers, e.g. in x-direction confinement in y,z-direction

\[
\Psi_{k_x,n}(r, t) = \phi_n(y, z) \exp[i(k_x x - \omega t)]
\]

- **mode index** \( n \)
- **standing wave**
- **plane wave**

\[
E_n(k_x) = \frac{\hbar^2 k_x^2}{2m^*} + E_n; \quad E_n = \frac{\pi^2 \hbar^2}{2m^*} \left( \frac{n_y^2}{a^2} + \frac{n_z^2}{b^2} \right)
\]

Source: Handouts Nazarov, TU Delft
III.2 Description of Electron Transport by Scattering of Waves

III.2.1 Electron Waves and Waveguides

- wave guide with potential barrier

\[ E = \frac{\hbar^2 k_x^2}{2m^*} \]

\[ E - U_0 = \frac{\hbar^2 \kappa^2}{2m^*} \]

**4 unknown variables:**

- \( A, B, r, t \)

- \( t \): transmission amplitude

- \( r \): reflection amplitude

**4 equations**

(wave function matching at interfaces)
• wave guide with potential barrier → example: *rectangular barrier*

\[
T(E) \equiv |t|^2 = \frac{1}{1 + \left(\frac{k_x^2 - \kappa^2}{2k_x\kappa}\right)^2} \sinh^2 \kappa d
\]

for \(\kappa d \gg 1\):

\[
\sinh^2 (\kappa d) = [\exp(\kappa d) - \exp(-\kappa d)]^2 \approx \exp(2\kappa d)
\]
III.2 Description of Electron Transport by Scattering of Waves

III.2.1 Electron Waves and Waveguides

• modeling of nanostructures as complex waveguides:

→ transport channels + potential barrier

• description of transport by a set of transmission coefficients $T_n$

sufficient to describe transport!!

examples:
(i) adiabatic quantum transport
(ii) quantum point contact

ideal waveguides

reservoir

$T_n$

reservoir

scattering region
III.2 Description of Electron Transport by Scattering of Waves

III.2.1 Electron Waves and Waveguides

- example: *adiabatic quantum transport* → *constriction as a potential barrier*

\[ E_n(k_x, x) = \frac{\hbar^2 k_x^2}{2m^*} + \frac{\pi^2 \hbar^2}{2m^*} \left( \frac{n_y^2}{a^2(x)} + \frac{n_z^2}{b^2(x)} \right) \]

adiabatic waveguide: variation of dimensions occurs on length scale large compared to width → waveguide walls can be assumed parallel locally
III.2 Description of Electron Transport by Scattering of Waves

III.2.1 Electron Waves and Waveguides

- example: quantum point contact

![Diagram of quantum point contact]

Net current:
\[ I = I_l - I_r \]

\[ I_l = T \frac{2}{2\pi} \int dk_x e v_x \cdot f_l(k_x) \]

\[ I_r = T \frac{2}{2\pi} \int dk_x e v_x \cdot f_r(k_x) \]

\[ v_x = \frac{1}{\hbar} \frac{\partial E}{\partial k_x} \]

Quantized conductance:
\[ G_Q = \frac{e^2}{h} N_{\text{open}} V \]

\[ I = \frac{2e}{2\pi\hbar} \sum_{\text{open ch}} \int dE \cdot [f_l(E) - f_r(E)] = \frac{2e}{2\pi\hbar} N_{\text{open}} (\mu_l - \mu_r) = 2\frac{e^2}{h} N_{\text{open}} V \]
III.2 Description of Electron Transport by Scattering of Waves
Electron Waves and Waveguides

- what is the meaning of the quantity

\[ G = \frac{l}{V} = 2 \frac{e^2}{h} N_{\text{open}} = 2G_Q N_{\text{open}} \]

- for ballistic transport and reflectionless contacts there should not be any resistance!

- where does the resistance come from?

  \[ \text{contact resistance from the interface between the ballistic conductor and the contact pads} \]

  \[ \text{resistance is denoted as } \textit{contact resistance} \]

\[ G_c^{-1} = \frac{h}{e^2} \frac{1}{2N_{\text{open}}} = G_Q^{-1} \frac{1}{2N_{\text{open}}} \]

\[ \text{quantum resistance} \]

\[ 25 \ 812.807 \ \Omega = 1 \text{ Klitzing} \]

- \( G_Q \) determined by fundamental constants, does not depend on materials properties, geometry or size of nanostructure
III.2 Description of Electron Transport by Scattering of Waves

III.2.1 Electron Waves and Waveguides

- voltage drop at interfaces !!
III.2 Description of Electron Transport by Scattering of Waves

Electron Waves and Waveguides

- **quantum point contact**: experimental results

Thomas et al., PRB 58, 4846 (1998)


Gate voltage narrows channel $\rightarrow$ reduction of $N_{\text{open}}$

$T = 0.6$ K
III.2 Description of Electron Transport by Scattering of Waves

III.2.1 Electron Waves and Waveguides

conduction through a single atom!

(Elke Scheer, Univ. Konstanz)
• considered examples have been too simple: $T$ only 1 (open) or 0 (closed)

• more complicated situation: *ideal sample + scattering sites*

transmission probability of the different modes will no longer be only 0 or 1

$0 \leq T \leq 1$

• $T$ represents the **average probability** that an electron injected at one end will be **transmitted** to the other end

• treatment of the situation by a **scattering matrix**
III.2 Description of Electron Transport by Scattering of Waves

III.2.2 Landauer Formalism: scattering matrix

\[
\begin{align*}
N_l + N_r \text{ incoming amplitudes } & a_l, a_r \\
N_l + N_r \text{ outgoing amplitudes } & b_l, b_r \\
\end{align*}
\]

\[
b = \hat{s} a
\]

\[
\begin{bmatrix}
 b_l \\
 b_r 
\end{bmatrix} = 
\begin{bmatrix}
 \hat{s}_{ll} & \hat{s}_{lr} \\
 \hat{s}_{rl} & \hat{s}_{rr} 
\end{bmatrix}
\begin{bmatrix}
 a_l \\
 a_r 
\end{bmatrix} = 
\begin{bmatrix}
 \hat{r} & \hat{t}' \\
 \hat{t} & \hat{r}' 
\end{bmatrix}
\begin{bmatrix}
 a_l \\
 a_r 
\end{bmatrix}
\]

scattering matrix

scattering matrix
III.2 Description of Electron Transport by Scattering of Waves

III.2.2 Landauer Formalism: scattering matrix

- time reversal symmetry:
  \[ \hat{t}^T = \hat{t}' \quad \Rightarrow \quad \hat{S}^T = \hat{S} \]
  (sym. matrix)

- electrons do not disappear:
  \[ \sum_{n'} |r_{nn'}|^2 + \sum_m |t_{mn}|^2 = (\hat{S}^\dagger \hat{S})_{nn} = 1 \]

  \[ R_n = (\hat{r}^\dagger \hat{r})_{nn} \]
  \[ T_n = (\hat{t}^\dagger \hat{t})_{nn} \]

  conjugate transpose of \( \hat{S} \)

\[ \hat{S}^\dagger \hat{S} = \hat{1} \]

unitary matrix
III.2 Description of Electron Transport by Scattering of Waves

III.2.2 Landauer Formalism: scattering matrix

- example:

one channel scatterer

\[
\begin{bmatrix}
    b_l \\
    b_r
\end{bmatrix}
= \begin{bmatrix}
    r & t' \\
    t & r'
\end{bmatrix}
\begin{bmatrix}
    a_l \\
    a_r
\end{bmatrix}
\]

\( r, t, r', t' \) are complex numbers

condition of unitarity → only three independent parameters

\[
\begin{align*}
    r &= \sqrt{R} e^{i\theta} \\
    t' &= \sqrt{T} e^{i\eta} \\
    t &= \sqrt{T} e^{i\eta} \\
    r' &= -\sqrt{R} e^{i(2\eta - \theta)}
\end{align*}
\]

\( R = 1 - T \)

follows from condition of unitarity
III.2 Description of Electron Transport by Scattering of Waves

III.2.2 Landauer Formalism: scattering matrix

- condition of unitarity: \( \hat{S}^\dagger \hat{S} = \mathbb{1} \)

\[
\begin{bmatrix}
\hat{r}^* & \hat{t'}^*
\end{bmatrix}
\begin{bmatrix}
\hat{r} & \hat{t}
\end{bmatrix}
= \begin{bmatrix}
|r|^2 + |t'|^2 & r^*t + t'^*r' \\
|t|^2 + |r'|^2 & t^*r + r'^*t'
\end{bmatrix} = \mathbb{1}
\]

\[= 0\]

(i) \( r^*t + t'^*r' = 0 \)
\[
\sqrt{T} e^{-i\eta} - \sqrt{T} e^{-i\eta} \sqrt{R} e^{i(2\eta - \theta)} =
\]
\[
\sqrt{T} R e^{-i(\theta - \eta)} - \sqrt{T} R e^{-i(\theta - \eta)} = 0 \quad !!
\]

(ii) \( t^*r + r'^*t' = 0 \)
\[
\sqrt{T} e^{-i\eta} \cdot \sqrt{R} e^{i\theta} - \sqrt{R} e^{-i(2\eta - \theta)} \cdot \sqrt{T} e^{i\eta} =
\]
\[
\sqrt{T} R e^{i(\theta - \eta)} - \sqrt{T} R e^{i(\theta - \eta)} = 0 \quad !!
\]
III.2 Description of Electron Transport by Scattering of Waves

III.2.2 Landauer Formalism: scattering matrix

- the Hermitian matrix $\hat{t}^\dagger \hat{t}$ has a set of eigenvalues $T_p$ (for each energy $E$)

- expression for the current:

$$I = \frac{2e}{2\pi \hbar} \sum_p \int dE \ T_p(E) \cdot [f_l(E) - f_r(E)] = 2G_Q \sum_p T_p \cdot V$$

Landauer formula

this gives just the number of open channels, if $T_p$ is either 0 or 1

Einstein relation $\iff$ Landauer formula

$$\sigma = 2e^2 N(E_F) D \iff G = 2 \frac{e^2}{\hbar} N T$$

Landauer formula $\rightarrow$ ‘mesoscopic version’ of Einstein relation
• consider a conductor with a single conduction channel

• reservoir biased at $V$ sends out the following number of electrons:

$$N(t) = Z(k) \Delta k \cdot \frac{1}{\hbar} \frac{\Delta E}{\Delta k} \cdot t = \frac{2}{2\pi} \Delta k \cdot \frac{eV}{\hbar \Delta k} \cdot t = \frac{2eV}{\hbar} t$$

• the chance to pass is $T_0$, then the passed charge is just

$$Q(t) = eT_0 N(t)$$

• the average current is charge per time:

$$I = \frac{Q}{t} = 2 \frac{e^2}{\hbar} T_0 V$$

• many channels: just sum up to obtain

$$I = 2G_Q \sum_p T_p V$$
• **restrictions:**
  
  → only *elastic* scattering (electrons pass the conductor at constant energy)  
  → *no interactions* between electrons

• **limitations:**

  → low temperatures and low voltages  
  → short conductors (shorter than inelastic scattering length)
so far discussion of two-terminal systems, extension to multi-terminal conductors?

how to express currents in terms of voltages using the Landauer formalism?
• conduction matrix $G_{kl}$

$$I_k = \sum_l G_{kl} V_l$$

• properties of conduction matrix:

→ current conservation:

$$\sum_k I_k = 0 \implies \sum_k G_{kl} = 0$$

→ no current, if potential is shifted by the same amount in all leads

$$\sum_l G_{kl} = 0$$
III.2 Description of Electron Transport by Scattering of Waves

III.2.2 Additional Topic: Multi-terminal conductors

• simplest case: two-terminal conductor

\[
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix} = \begin{bmatrix}
-G & G \\
G & -G
\end{bmatrix} \begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
\]

• the conduction matrix only has a *single independent element*:
III.2 Description of Electron Transport by Scattering of Waves

III.2.2 Additional Topic: Multi-terminal conductors

- scattering matrix for *multi-terminal* conductors

- number of modes: $N = N_1 + N_2 + N_3 + \ldots$
  $\rightarrow$ scattering matrix is $N \times N$ matrix

$N_1 \begin{cases} 
S_{11,12} \\
S_{12,12} \\
S_{13,12}
\end{cases}$

- meaning of $S_{\beta m,\alpha n}$:
  $\rightarrow$ propagation amplitude
  from terminal $\alpha$, transport channel $n$,
  to the terminal $\beta$, transport channel $m$

- transmission probability:
  \[
  \overline{T_{\alpha\beta}} = \sum_{m=1}^{N_\beta} \sum_{n=1}^{N_\alpha} T_{nm} = \sum_{m=1}^{N_\beta} \sum_{n=1}^{N_\alpha} |S_{\alpha n,\beta m}|^2 
  \]
III.2 Description of Electron Transport by Scattering of Waves

III.2.2 Additional Topic: Multi-terminal conductors

• properties of scattering matrix:

→ reflection back from $\alpha$: $S_{\alpha n,\alpha m}$

→ transmission from $\beta$ to $\alpha$: $S_{\alpha n,\beta m}$

• current conservation requires

$$\hat{S}^\dagger \hat{S} = \hat{1}$$

(unitary matrix)

$$\sum_{\alpha n} s^*_{\alpha n,\gamma l} s_{\alpha n,\beta m} = \delta_{\gamma \beta} \delta_{lm}$$

• time reversal symmetry:

$$s_{\alpha n,\beta m}(B) = s_{\beta m,\alpha n}(-B)$$
### III.2 Description of Electron Transport by Scattering of Waves
#### III.2.2 Additional Topic: Multi-terminal conductors

- **sum rules:**
  \[
  \sum_{\alpha} \overline{T}_{\alpha\beta} = N_{\beta} \quad \text{number of modes in lead } \beta \\
  \sum_{\alpha} \overline{T}_{\beta\alpha} = N_{\beta} 
  \]

- **example: two-terminal conductor:**

  \[
  \overline{T}_{\beta\alpha}(E) \quad \begin{array}{c|cc}
  \alpha = 1 & \alpha = 2 & \text{sum =} \\
  \hline
  \beta = 1 & \overline{T}_{11} & \overline{T}_{12} & N_1 \\
  \beta = 2 & \overline{T}_{21} & \overline{T}_{22} & N_2 \\
  \hline
  \text{sum =} & N_1 & N_2 
  \end{array}
  \]

- since \( \overline{T}_{11} + \overline{T}_{12} = \overline{T}_{11} + \overline{T}_{21} = N_1 \) \( \Rightarrow \overline{T}_{12} = \overline{T}_{21} \)

  *transmission function is reciprocal!*

  \( \Rightarrow \) *time reversal symmetry*
III.2 Description of Electron Transport by Scattering of Waves
III.2.2 Additional Topic: Multi-terminal conductors

• multi-terminal expression of Landauer formula relates currents to voltages via a scattering matrix

\[ I_\alpha = -\frac{G}{e} \int dE \sum_\beta \bar{T}_{\alpha\beta} f_\beta (E) \]

• probability for transmission from \( \alpha \) to \( \beta \):

\[ \bar{T}_{\alpha\beta} = \text{Tr} \left[ \delta_{\alpha\beta} - \hat{s}^+ \hat{s} \right] \]

trace includes all possible transport channels

• plausibility check:

\( \rightarrow \) current conservation is satisfied \((\text{follows from unitarity})\)
\( \rightarrow \) no current is flowing in equilibrium, same voltage at all terminals \((\text{follows also from unitarity})\)
III.2 Description of Electron Transport by Scattering of Waves

III.2.2 Additional Topic: Multi-terminal conductors

- linear transport regime: 
  \[ G_{\alpha\beta} = -G_Q \text{Tr} \left[ \delta_{\alpha\beta} - \hat{s}_{\alpha\beta}^+ \hat{s}_{\alpha\beta} \right] \]

- relation to two-terminal expression: \( \alpha, \beta = l, r \)
  \[ G_{lr} = G_Q \text{Tr} \left[ \hat{s}_{lr}^+ \hat{s}_{lr} \right] = G_Q \text{Tr} \left[ t^+ t \right] \]

- time reversal symmetry: 
  \[ G_{\alpha\beta} (B) = G_{\beta\alpha} (-B) \]

*this is in agreement with Onsager symmetry relations!*
III.2 Description of Electron Transport by Scattering of Waves

III.2.2 Additional Topic: Multi-terminal conductors

• three-terminal scattering element:

\[
\hat{S}_{BS} = \begin{pmatrix}
-1/2 & 1/2 & 1/\sqrt{2} \\
1/2 & -1/2 & 1/\sqrt{2} \\
1/\sqrt{2} & 1/\sqrt{2} & 0
\end{pmatrix}
\]

• scattering matrix:

\[
\hat{S}_{BS} = \begin{pmatrix}
-1/2 & 1/2 & 1/\sqrt{2} \\
1/2 & -1/2 & 1/\sqrt{2} \\
1/\sqrt{2} & 1/\sqrt{2} & 0
\end{pmatrix}
\]

• conductance matrix:

\[
G_{\alpha\beta} = G_Q \begin{pmatrix}
-3/4 & 1/4 & 1/2 \\
1/4 & -3/4 & 1/2 \\
1/2 & 1/2 & -1
\end{pmatrix}
\]

• example: \( V_1 = V_2 = V; V_3 = 0 \)
III.3 Quantum Interference Effects

Contents:

III.1 Introduction
   III.1.1 General Remarks
   III.1.2 Mesoscopic Systems
   III.1.3 Characteristic Length Scales
   III.1.4 Characteristic Energy Scales
   III.1.5 Transport Regimes

III.2 Description of Electron Transport by Scattering of Waves
   III.2.1 Electron Waves and Waveguides
   III.2.2 Landauer Formalism
   III.2.3 Multi-terminal Conductors

III.3 Quantum Interference Effects
   III.3.1 Double Slit Experiment
   III.3.2 Two Barriers – Resonant Tunneling
   III.3.3 Aharonov-Bohm Effect
   III.3.4 Weak Localization
   III.3.5 Universal Conductance Fluctuations

III.4 From Quantum Mechanics to Ohm’s Law

III.5 Coulomb Blockade
III.3 Quantum Interference Effects

III.3.1 Double Slit Experiment

**coherent charge carriers** $L_\phi > L$

low temperatures ($\rightarrow L_\phi$ gets large), nanoscale samples ($L$ gets small)

- interference of multiply scattered charge carriers
- **corrections to the classical conductance**

- macroscopic and mesoscopic samples:
  - **weak localization (WL)**

- mesoscopic samples:
  - **Aharonov-Bohm (AB) oscillations**
  - **Universal Conductance Fluctuations (UCF)**
III.3 Quantum Interference Effects

III.3.1 Revision: Characteristic Length Scales (see III.1.3)

- **electron wavelength:**
  \[ \lambda_F = \frac{h}{\sqrt{2m^*E_F}} = \frac{2\pi}{(3\pi^2n)^{1/3}} \]  
  (Fermi wavelength)

- **mean free path:**
  \[ \ell = v_F \cdot \tau_m \quad \tau_m^{-1} = \tau_c^{-1} \cdot \alpha_m \]
  effectiveness of collision: \( 0 < \alpha_m < 1 \)

- **phase relaxation length:**
  \[ L_\phi = v_F \tau_\phi \quad \tau_\phi^{-1} = \tau_c^{-1} \cdot \alpha_\phi \]
  effectiveness of collision in destroying phase coherence: \( 0 < \alpha_\phi < 1 \)

  \[ L_\phi = \sqrt{D \tau_\phi} = \frac{1}{\sqrt{3}} v_F^2 \tau_m \tau_\phi \]

  \[ \rightarrow \text{elastic impurity scattering:} \quad \tau_\phi \rightarrow 0 \quad \text{or} \quad \alpha_\phi \rightarrow 0 \]

  \[ \rightarrow \text{electron-phonon scattering:} \quad \tau_\phi \approx \tau_{e-ph} \quad ?? \]

  \[ \rightarrow \text{electron-electron scattering:} \quad \tau_\phi \approx \tau_{e-e} \quad ?? \]

  \[ \rightarrow \text{electron-impurity scattering} \quad (\text{with internal degree of freedom, e.g. spin}) \]
III.3 Quantum Interference Effects

III.3.1 Double Slit Experiment

- basic quantum mechanics: *double slit experiment*
- probability of propagation from point A to point B:

\[ P_{AB} = |A_1 + A_2|^2 = |A_1|^2 + |A_2|^2 + A_1A_2^* + A_1^*A_2 + 2\text{Re}[A_1A_2^*] \]

- classical term
- interference term: quantum mechanical
III.3 Quantum Interference Effects
III.3.1 Double Slit Experiment

interference terms may be *destructive* or *constructive*

\[ P_{AB} = P_{\text{classical}} + 2 \sqrt{P_1 P_2} \cos \phi \]

\( \rightarrow \) depends on phase shift \( \phi \)

**problem:**
calculate phase shift \( \phi \) as a function of geometry, electric potential, magnetic field, ...
III.3 Quantum Interference Effects

III.3.1 Double Slit Experiment

- **Phase shifts:**

  \[ \psi(x) = \exp[i\phi(x)] = \exp[ik(x)x] \]

  \[ \frac{d\phi}{dx} = k(x) = \sqrt{2m[E - V(x)]/\hbar} \]

  - if \( V(x) = \text{const.} \), then \( k = \text{const.} \) and hence \( \phi = kL \) (**geometric phase**)
  - usually, absolute value of phase is not interesting but the relative phase shift between different paths

- **Energy dependence:**

  \[ \frac{d\phi}{dE} = \frac{d\phi}{dk} \frac{dk}{dE} = \frac{d\phi}{dk} \frac{1}{\hbar \nu(x)} = \int_{x_1}^{x_2} \frac{dx}{\hbar \nu(x)} = \int_{t_1}^{t_2} \frac{dt}{\hbar} = \frac{\tau}{\hbar} \]

  \[ \nu = \frac{1}{\hbar} \frac{dE}{dk} \]

  \[ \Delta\phi = \frac{d\phi}{dE} \Delta E = \int_{x_1}^{x_2} eV(x) \frac{dx}{\hbar \nu(x)} = \int_{t_1}^{t_2} eV(x) \frac{dt}{\hbar} \approx \frac{eV \tau}{\hbar} \]

  dynamical phase shift
  e.g. by potential \( V \) along path
  \( \to \) same for time-reversed path

local wave vector at position \( x \)

\[ \frac{d}{dk} \int \frac{d\phi}{dx} \ dx = \frac{d}{dk} \int k \ dx = \int dx \]

time of flight between points \( x_1 \) and \( x_2 \) at energy \( E \)
III.3 Quantum Interference Effects

III.3.1 Double Slit Experiment

- magnetic field dependence:

  canonical momentum:  \( p = mv + qA \)

  \[
  k(x) \rightarrow k(x) - \frac{q}{\hbar} A(x)
  \]

  results in phase shift \( \phi_{mag} \)

  phase shift accumulated along the trajectory due to magnetic field:

  \[
  \phi_{mag} = \frac{e}{\hbar} \int_{x_1}^{x_2} A \cdot dx \quad (q = -e)
  \]

  phase shift along closed path (electron returns to the same point):

  \[
  \phi_{mag} = \frac{e}{\hbar} \oint A \cdot d\mathbf{x} = \frac{e}{\hbar} \int B \cdot d\mathbf{F} = 2\pi \frac{\Phi}{\Phi_0}
  \]

  Stokes theorem

  \[
  \Phi_0 = \frac{\hbar}{e} \quad \text{("normal" flux quantum)}
  \]

  \( \text{in superconductors we have } q_s = -2e \) and therefore \( \Phi_0 = \hbar/2e \)
III.3 Quantum Interference Effects

III.3.2 Two Barriers – Resonant Tunneling

- we consider only a single conductance channel
- no magnetic field

```
\[ \frac{1}{r_{\text{tot}}} \quad 1 \quad t_{\text{tot}} \]
```

• „classical“ expectation:
  (tunneling) resistances are added
  → product of transmission probabilities \( T_L \cdot T_R \)

• what is the role of quantum interference?

• how do individual \textit{scattering matrices} have to be \textit{combined}?

Ohm’s law:

\[
\begin{align*}
R &= R_1 + R_2 \\
G &= \frac{G_1 G_2}{G_1 + G_2}
\end{align*}
\]
III.3 Quantum Interference Effects

III.3.2 Two Barriers – Resonant Tunneling

\[ \varphi = k \cdot s \]

acquired phase during propagation between barriers

scattering matrix of left barrier

propagation between barriers

scattering matrix of right barrier
III.3 Quantum Interference Effects

III.3.2 Two Barriers – Resonant Tunneling

\[
\begin{align*}
[r_{\text{tot}} & \ a] = \begin{bmatrix} \hat{r}_L & \hat{t}_L' \end{bmatrix} \begin{bmatrix} 1 \\ de^{i\phi} \end{bmatrix} \\
[1 & \ a] = \begin{bmatrix} \hat{r}_L & \hat{t}_L' \end{bmatrix} \begin{bmatrix} 1 \\ de^{i\phi} \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix} d & t_{\text{tot}} \end{bmatrix} = \begin{bmatrix} \hat{r}_R & \hat{t}_R' \end{bmatrix} \begin{bmatrix} ae^{i\phi} \\ 0 \end{bmatrix}
\end{align*}
\]

outgoing modes  incoming modes  outgoing modes  incoming modes
III.3 Quantum Interference Effects

III.3.2 Two Barriers – Resonant Tunneling

process

amplitude

| \begin{align*}
t_L t_R & e^{i\phi} \\
t_L t_R r'_L r'_R & e^{3i\phi}
\end{align*} |

probability

| \begin{align*}
T_L T_R \\
T_L T_R R_L R_R
\end{align*} |

... sum of all amplitudes: \( t_{tot} = \frac{t_L t_R}{1 - r'_L r'_R e^{2i\phi}} \)

coherent

| \begin{align*}
T_{tot} = |t_{tot}|^2
\end{align*} |

... sum of all probabilities: \( T_{cl} = \frac{T_L T_R}{1 - R_L R_R} \) incoherent

path can be viewed as Feynman path

\[ |t_{tot}|^2 = \frac{T_L T_R}{1 + R_L R_R - 2\sqrt{R_L R_R} \cos \chi} \]
III.3 Quantum Interference Effects

III.3.2 Two Barriers – Resonant Tunneling

The figure shows a system with two barriers, labeled L and R, connected by a scattering region S. The transmission coefficient $T_{tot}$ relates to the total transmission probability and is given by:

\[ T_{tot} = \frac{t_L t_R}{1 - r'_L r_R e^{2i\varphi}} \]

where $r_{tot}$ is the reflection coefficient, $a$ and $d$ are the transmission coefficients, $ae^{i\varphi}$ and $de^{i\varphi}$ are the phase shifts, and $T_L$ and $T_R$ are transmission coefficients.

The total transmission probability $T(E)$ is:

\[ T(E) = \left| T_{tot} \right|^2 = \frac{T_L T_R}{1 + R_L R_R - 2 \sqrt{R_L R_R} \cos \chi} \]

where $\chi = 2\varphi = 2ks$ is the phase accumulated during the round trip.

The figure also includes a question: **What is the relation to the double slit experiment?**
### III.3 Quantum Interference Effects

#### III.3.2 Two Barriers – Resonant Tunneling

\[ T(E) = \left| t_{\text{tot}} \right|^2 = \frac{T_L T_R}{1 + R_L R_R - 2 \sqrt{R_L R_R} \cos \chi} \]

\[ \hbar \frac{d\phi}{dt} = E \quad \Rightarrow \quad \Delta t = \frac{\hbar}{E} \Delta \phi \]

- **quantum result:** transmission coefficient *depends on energy*
  (not the case for classical result!)

\[ \chi = 2\varphi = 2ks \]

- assume \( T_L = T_R = T \ll 1 \), \( R_L = R_R = R \approx 1 \)

- between peaks: \( T(E) \approx T^2 \)

- peak values: \( T_{\text{max}} = 1 \ (@ \chi = 2\pi \cdot n) \)

**resonant tunneling**

(or Fabry-Perot resonances)

\[ \Rightarrow \text{double barrier structure behaves as an optical interferometer} \]

\[ \Rightarrow \text{resonant tunneling is quantum interference effect} \]
III.3 Quantum Interference Effects

III.3.2 Two Barriers – Resonant Tunneling

\[ T(E) = |t_{\text{tot}}|^2 = \frac{T_L T_R}{1 + R_L R_R - 2\sqrt{R_L R_R} \cos \chi} \]

(i) \( \chi = 0 \):

\[ T_{\text{max}} = \frac{T_L T_R}{(1 - \sqrt{R_L R_R})^2} \]

(ii) \( \chi = \pi \):

\[ T_{\text{min}} = \frac{T_L T_R}{(1 + \sqrt{R_L R_R})^2} \]

\( T_{L,R} \ll 1, \ R_{L,R} \to 1 \):

\[ T_{\text{min}} \sim T_L T_R \ll 1 \]

\[ T_{\text{max}} \sim \frac{4T_L T_R}{(T_L + T_R)^2} \sim 1 \]

(expanding the denominator up to linear term in \( T_{L,R} \))
how does the transmission $T(E)$ look like close to the transmission resonances?

\[
\cos \chi = \cos(2ks) \approx 1 - \frac{1}{2} (2ks)^2 \quad \text{for } \chi \ll 1
\]

\[
\cos \chi \approx 1 - \frac{E - E_{\text{res}}}{2D}
\]

$D$ = level spacing in potential well of width $s$

• after some math:

\[
T(E) = \frac{T_L T_R}{(\frac{T_L + T_R}{2})^2 + (\frac{E - E_{\text{res}}}{D})^2}
\]

transmission assumes Lorentzian shape

\[
T(E) = \frac{D^2 T_L T_R}{(\frac{D(T_L + T_R)}{2})^2 + (E - E_{\text{res}})^2}
\]

energy width of transmission resonance:

\[
d = D(T_L + T_R)
\]

→ interpretation in terms of a particle that moves back and forth between the two potential wells and escapes at a certain tunneling rates $\Gamma_L$ and $\Gamma_R$

→ with $d = \hbar (\Gamma_L + \Gamma_R)$ according to uncertainty relation we obtain well-known Breit-Wigner formula
III.3 Quantum Interference Effects

III.3.3 Aharonov-Bohm Effect

- **quantum interference effects in multiply connected conductors, e.g. rings**
- **phase shift due to magnetic field**

Two trajectories enclosing magnetic flux

Phase: \( ikx \)

With vector potential: \( \vec{k} \to \vec{k} - \frac{q}{\hbar} \vec{A}(x) \)

\[ \theta_{1,2} = k L_{1,2} + \frac{e}{\hbar} \int_{1,2} \vec{A} \cdot d\vec{l} \]

\[ \theta_2 - \theta_1 = k (L_2 - L_1) + \frac{e}{\hbar} \int \vec{A} \cdot d\vec{l} \]

- **all quantities are periodic in \( \Phi/\Phi_0 \), even if there is NO magnetic field at the trajectories!**

Flux quantum: \( \Phi_0 = \frac{\hbar}{e} \)

(in superconductors we have \( q_s = -2e \) and therefore \( \Phi_0 = \hbar/2e \))
III.3 Quantum Interference Effects

III.3.3 Aharonov-Bohm Effect

- description of Aharonov-Bohm ring by two beam splitters

\[
\begin{pmatrix}
  r \\
  b_1 \\
  d_1
\end{pmatrix}
= \begin{pmatrix}
  0 & 1/\sqrt{2} & 1/\sqrt{2} \\
  1/\sqrt{2} & -1/2 & 1/2 \\
  1/\sqrt{2} & 1/2 & -1/2
\end{pmatrix}
\begin{pmatrix}
  1 \\
  a_1 \\
  c_1
\end{pmatrix}
\]
III.3 Quantum Interference Effects

III.3.3 Aharonov-Bohm Effect

- description of Aharonov-Bohm ring by two beam splitters

\[
\begin{pmatrix}
0 & 1/\sqrt{2} & 1/\sqrt{2} \\
1/\sqrt{2} & -1/2 & 1/2 \\
1/\sqrt{2} & 1/2 & -1/2 \\
\end{pmatrix}
\begin{pmatrix}
0 & e^{i\varphi_1 + i\phi_1/2} \\
e^{i\varphi_1 - i\phi_1/2} & 0 \\
\end{pmatrix}
\begin{pmatrix}
-1/2 & 1/2 & 1/\sqrt{2} \\
1/2 & -1/2 & 1/\sqrt{2} \\
1/\sqrt{2} & 1/\sqrt{2} & 0 \\
\end{pmatrix}
\]

\[\varphi = kL / 2\]
\[\phi = 2\pi\Phi / \Phi_0\]
### III.3 Quantum Interference Effects

#### III.3.3 Aharonov-Bohm Effect

\[ \phi_{AB} = 2\pi \Phi / \Phi_0 \]

\[ \varphi = kL / 2 \]

The figure illustrates the classical and interference contributions to the conductance. The expression for the transmission probability is given by:

\[ t = \alpha_0 + \alpha_1 e^{i\varphi_1 + i\varphi_1/2} + \alpha_{-1} e^{i\varphi_2 - i\varphi_2/2} + \alpha_2 e^{i[2(\varphi_1 + \varphi_1/2) + (\varphi_2 + \varphi_2/2)]} + \alpha_{-2} e^{i[2\varphi_1 + (\varphi_2 - \varphi_2/2)]} + \ldots \]

- **Clockwise**:\[ P_{AB} = 2 \text{Re} t_1^* t_{-1} \]
- **Counter Clockwise**:\[ P_{AB} \propto \cos(\varphi_1 - \varphi_2 + \phi_{AB}) \]

**universal conductance fluctuations**

**extra turn**

**weak localization**

**twice shorter period**

Chapt. III - 70
III.3 Quantum Interference Effects

III.3.3 Aharonov-Bohm Effect

**Aharonov-Bohm effect: flux dependent transmission**
III.3 Quantum Interference Effects

III.3.3 Aharonov-Bohm Effect

Aharonov-Bohm (AB) oscillations:

- period: $h/e$
- amplitude: $2e^2/h$
- one channel in Landauer model

Fourier analysis shows that there are also weak oscillations with half period

$\rightarrow$ higher order interferences: Altshuler-Aronov-Spivak (AAS) oscillations

- period: $h/2e$
- exactly same traces
- constructive interference for $B = 0$
- coherent backscattering

R. Webb et al, PRL 54, 2696 (1985)
III.3 Quantum Interference Effects

III.3.3 Aharonov-Bohm Effect

- AB oscillations vanish in an ensemble of small ring (phases $2\pi\Phi/\Phi_0$ are random)
- AAS oscillations survive ensemble averaging

Test of ensemble averaging:
- Ag loops
- Area 940 x 940 nm$^2$
- Width of wires 80 nm

Umbach et al, PRL 56, 386 (1986)
III.3 Quantum Interference Effects

III.3.3 Aharonov-Bohm Effect

- Conductance of a Cu ring in units of $e^2/h$, as a function of magnetic field at $T = 100$ mK.
- Narrow AB oscillations $\Delta B \approx 2.5$ mT are superimposed on larger and broader universal conductance fluctuations.

Cu ring on Si, width 80 nm

F. Pierre et al., PRL 89, 206804 (2002)
III.3 Quantum Interference Effects

III.3.3 Aharonov-Bohm Effect

**Benzene ring:**

- Dimensions: 0.5 nm

**Ring accelerator:**

Large Electron Positron Collider at CERN (Geneva)

- Diameter: 8.6 km

**AB effect:**

- One flux quantum \( (h/e) \) through ring area:

\[
\frac{h/e}{\pi r^2} = 5000 \ \text{T}
\]

\[
\frac{h/e}{\pi r^2} = 7 \times 10^{-23} \ \text{T}
\]
magnetoresistance of a Mg film $(d = 8.4 \text{ nm})$ as a function of the magnetic field $H$. [Physics Reports, 107, 1 (1984), G. Bergmann]

- classically: resistance would be completely field independent because $\omega_c \tau \ll 1$
- magnetoresistance would increase with the magnetic field, relative increase of order $(\omega_c \tau)^2$
- classical theory could not explain the observed behavior
Weak localization:
interference of time reversed
electron paths
III.3 Quantum Interference Effects

III.3.4 Weak Localization
III.3 Quantum Interference Effects

III.3.4 Weak Localization

\[ P_{AB} = |A_1 + A_2|^2 = |A_1|^2 + |A_2|^2 + 2 \text{Re}[A_1^* A_2] \]

2 |A_1 A_2| \cos \varphi
\langle \cos \varphi \rangle = 0

does averaging over many paths destroy interference effects in diffusive conductor?

**special trajectories:**
consider now a closed loop with 1 = 2
then the amplitude A_2 is just
a time reversal of A_1. Hence

\[ |A_1 + A_2|^2 = |A_1 + A_1^*|^2 = 4|A_1|^2 \]

- the backscattering probability is enhanced by factor 2 !!!
- this is a predecessor of localization.
III.3 Quantum Interference Effects

III.3.4 Weak Localization

- **magnetic field dependence of WL:**

  calculate phase difference of time reversed paths:

  \[ \phi_{A_2} - \phi_{A_1} = \frac{2e}{\hbar} \oint \vec{A} \cdot ds \]

  loss of constructive interference for:

  \[ \phi_{A_2} - \phi_{A_1} = \frac{2e}{\hbar} \oint \vec{A} \cdot ds = 4\pi \frac{B \cdot F}{\Phi_0} \]

  characteristic field:

  \[ B^* = \frac{\hbar}{2eL^2_\phi} \]

  \[ F = \text{area of the enclosed loop} \]
  \[ B \cdot F = \text{flux enclosed in the loop} \]

  \[ \phi_{A_2} - \phi_{A_1} = 2\pi \]

  \[ F \approx L^2_\phi \]
III.3 Quantum Interference Effects

III.3.4 Weak Localization

- coherent backscattering: called the **weak localization**
  (the relative number of contributing closed loops is small)

- effect is important, since it is **sensitive to weak magnetic fields**:
  - **small fields**: contributions of large rings oscillate rapidly,
    phase difference in small rings almost unchanged
  - the larger the **field**, the fewer loops/rings contribute to constructive
    backscattering
  - resistance drops to classical value for large fields, if phase shift in
    smallest rings is about $2\pi$

- WL has to be distinguished from strong localization
  (due to strong disorder)
III.3 Quantum Interference Effects

III.3.4 Weak Localization

- **requirement**: sample larger than elastic scattering length: \( L > \ell \)
  conductivity reduced by \( \approx 2e^2/h \) for \( B = 0 \)
- large \( B \): Shubnikov de-Haas oscillations

weak localization in SiGe 2-dimensional quantum well with hole gas

dependence of magnitude of WL on the coherence time $\tau_\phi \sim L_\phi^2/D$ is known:

weak localization experiments can be used to determine $\tau_\phi$
III.3 Quantum Interference Effects

III.3.5 Universal Conductance Fluctuations

influence of magnetic field on conductance of simply connected conductor

\[ T_i = \left| \sum_p A_p e^{i\phi_p} \right|^2 = \sum_p A_p^2 + \sum_{p \neq p'} A_p A_{p'} e^{i(\phi_p - \phi_{p'})} \]

\[ \Delta G(e^2/h) \]

\[ B(T) \]

random phase shifts

position of scatters becomes important
Experimental observations:

- irregular conductance variations as a function of magnetic field ($B$), carrier density ($n$), and voltage ($V$)
- conductance variations are symmetric with respect to $B$ (2 probe setup)
- different in each individual sample (”magnetic fingerprint”)
- caused by irregular quantum interference
- fluctuations characterize impurity configuration
- no sample size dependence
- (border & impurity scattering)
- amplitude of conductance variations is of the order $e^2/h$
- not noise
- theory based on ergodicity theorem
phase shift of individual electron trajectories depends on

- magnetic field $B$
- voltage $V$
- Fermi energy $E_F$ (carrier density)
- impurity (scatterer) configuration

consider an ensemble of macroscopically identical but microscopically different samples (different configurations of scattering centers)

$\Rightarrow$ variance of ensemble conductance:

$$\langle (G - \langle G \rangle)^2 \rangle = \frac{e^4}{h^2} \left\langle \left( \sum_{mn} T_{mn} - \sum_{mn} \langle T_{mn} \rangle \right)^2 \right\rangle$$

$$T_{mn} = |t_{mn}|^2 \quad \Rightarrow \text{complicated calculation}$$
III.3 Quantum Interference Effects

III.3.5 Universal Conductance Fluctuations

![Image of nanoscale structure with gold layers and annotations](image)

- $L_{\phi}$
- $\ell \gg \lambda_F$

Graph showing conductance fluctuations $G - \langle G \rangle$ with $e^2/h$ versus magnetic field $B$ (T), with red and blue curves taken at different days without warming up the sample.

$T = 20$ mK

Walther-Meißner-Institut
III.3 Quantum Interference Effects

III.3.5 Universal Conductance Fluctuations

UCF in gold nanowire

\( L = 600 \text{ nm} \)

\( W = 60 \text{ nm} \)

III.3 Quantum Interference Effects

III.3.5 Universal Conductance Fluctuations

data from Heinzel (2003)
Contents:

III.1 Introduction
   III.1.1 General Remarks
   III.1.2 Mesoscopic Systems
   III.1.3 Characteristic Length Scales
   III.1.4 Characteristic Energy Scales
   III.1.5 Transport Regimes

III.2 Description of Electron Transport by Scattering of Waves
   III.2.1 Electron Waves and Waveguides
   III.2.2 Landauer Formalism
   III.2.3 Multi-terminal Conductors

III.3 Quantum Interference Effects
   III.3.1 Double Slit Experiment
   III.3.2 Two Barriers – Resonant Tunneling
   III.3.3 Aharonov-Bohm Effect
   III.3.4 Weak Localization
   III.3.5 Universal Conductance Fluctuations

III.4 From Quantum Mechanics to Ohm’s Law

III.5 Coulomb Blockade
• two different points of view:

⇒ quantum transport
  (electron waves, scattering matrix)

⇒ classical transport
  (electric currents, charged particles, friction due to scattering, Ohm’s law)

What is the bridge between these limiting cases??
III.4 From Quantum Mechanics to Ohm’s Law

- consider two conductors with transmission probabilities $T_1$ and $T_2$ connected in series

- what is the transmission probability $T_{12}$?

- If $T_{12} = T_1 T_2$, then for a chain of scatterers we would expect the transmission probability to drop exponentially with the length of the chain:

  $$T(L) = \exp\left(-\frac{L}{L_0}\right)$$

  $\to$ no Ohm’s law

- **problem**: if we assume $T_{12} = T_1 T_2$, then we do not take into account multiple reflections

  $\to$ to obtain the correct result we have to add the probabilities of multiply reflected paths
III.4 From Quantum Mechanics to Ohm’s Law

**two scatterers in series**

\[
\begin{align*}
T_1 T_2 \\
+ \\
T_1 T_2 R_1 R_2 \\
+ \\
T_1 T_2 R_1^2 R_2^2 \\
+ \\
\ldots
\end{align*}
\]

\[
T_{12} = \frac{T_1 T_2}{1 - R_1 R_2}
\]

(incoherent processes)

\[
\frac{1 - T_{12}}{T_{12}} = \frac{1 - T_1}{T_1} + \frac{1 - T_2}{T_2}
\]

additive property

with \( T_1 = 1 - R_1 \) and \( T_2 = 1 - R_2 \)
III.4 From Quantum Mechanics to Ohm’s Law

**N scatterers in series:**

\[
\frac{1 - T(N)}{T(N)} = N \cdot \frac{1 - T}{T}
\]

\[
T(N) = \frac{T}{N(1 - T) + T}
\]

- The number of scatterers in a conductor of length \(L\) can be written as \(N = \nu L\), where \(\nu\) is the linear density.

\[
T(L) = \frac{L_0}{L + L_0}
\]

with

\[
L_0 = \frac{T}{\nu(1 - T)}
\]

- \(L_0\) is of the order of the mean free path \(\ell\)

\[
\ell \approx \frac{1}{\nu(1 - T)}
\]

\[
\ell \approx \frac{1}{\nu(1 - T)} \approx L_0 \quad \text{(for } T \text{ close to 1)}
\]
quantum conductance for N channels:

• wide conductor with \( M \approx k_F W / \pi \) modes:

\[
G \approx 2 \frac{e^2 M}{h} T \approx \frac{e^2 W}{\pi} T \frac{2k_F}{h}
\]

• 2D density of transverse modes:

\[
n = \frac{1}{2\pi} \frac{2m}{\hbar^2} \rightarrow n v_F = \frac{1}{2\pi} \frac{2m}{\hbar^2} m = \frac{2k_F}{h}
\]

\[
G \approx \frac{e^2 W}{\pi} T n v_F
\]

• using \( T(L) = \frac{L_0}{L + L_0} \) yields:

\[
G \approx \frac{W}{L + L_0} e^2 n v_F L_0 \pi
\]

\[
\approx \text{diffusion constant}
\]

\[
\approx \sigma (\text{Einstein relation})
\]

\[
R = \frac{1}{G} \approx \frac{L + L_0}{W} \frac{1}{\sigma} = \frac{L}{\sigma W} + \frac{L_0}{\sigma W}
\]

resistance obeying Ohm’s law

length independent interface resistance
conclusions:

• Ohm’s law is obtained from the expression for the quantum conductance
  
  → by summing up *probabilities of multiply reflected paths*

  → note that by summing up probabilities *coherence effects are neglected*
    *(of course these are not contained in Ohm’s law, incoherent transport)*

• sample size $L \gg$ phase coherence length $L_\phi$: large phase shifts
  
  (also affected by disorder)

  formally identical samples: - very different phase shifts,
    - but same ohmic resistance, since interference
      effects average out for $L \gg L_\phi$

• $L < L_\phi$: interference effects play important role
  
  → deviation from Ohm’s law
  → different resistance for formally identical samples due to different
    impurity configurations
III.4 From Quantum Mechanics to Ohm’s Law

**Where is the resistance??**

- expression for quantum conductance: \[ G = 2 \frac{e^2}{M T} \]
  
  \( \rightarrow \) scatterers give rise to resistance by reducing \( T \)

- example: waveguide with \( M \) modes and a single scatterer

\[
\frac{1}{G} = \frac{h}{2e^2 M} + \frac{h}{2e^2 M} \frac{1 - T}{T}
\]

"interface" resistance \quad "scatterer" resistance

\( \rightarrow \) scatterer resistance determined by properties of scatterer via its transmissivity

- remaining questions:
  
  \( \rightarrow \) can we associate a resistance with the scatterer ?
  \( \rightarrow \) what about the potential drop ? Does it occur across the scatterer ?
  \( \rightarrow \) what about Joule heating ? Dissipation at the scatterer ?
Contents:

III.1 Introduction
   III.1.1 General Remarks
   III.1.2 Mesoscopic Systems
   III.1.3 Characteristic Length Scales
   III.1.4 Characteristic Energy Scales
   III.1.5 Transport Regimes

III.2 Description of Electron Transport by Scattering of Waves
   III.2.1 Electron Waves and Waveguides
   III.2.2 Landauer Formalism
   III.2.3 Multi-terminal Conductors

III.3 Quantum Interference Effects
   III.3.1 Double Slit Experiment
   III.3.2 Two Barriers – Resonant Tunneling
   III.3.3 Aharonov-Bohm Effect
   III.3.4 Weak Localization
   III.3.5 Universal Conductance Fluctuations

III.4 From Quantum Mechanics to Ohm’s Law

III.5 Coulomb Blockade
III.5 Coulomb Blockade

**Charge quantization and charging energy:**

- *electric charge* is quantized for an isolated island

- charging energy:

\[
E = \frac{Q^2}{2C} = \frac{n^2 e^2}{2C} = n^2 E_c \quad \quad E_c = \frac{e^2}{2C}
\]

- how large is \( E_c \) for island of size \( L \) (*bring charge* \( e \) *from* \( \infty \) *to island*)

\[
E_c \approx \frac{e^2}{\varepsilon_0 L} \approx \frac{10 \, \text{eV}}{L \, \text{[nm]}}
\]

- level splitting in nm-sized island:

\[
\delta \approx \frac{E_F}{N_{\text{atom}}} \approx \frac{1 \, \text{eV}}{L^3 \, \text{[nm]}}
\]

typically in *meV regime* for 100 nm-sized samples

typically in *μeV regime* for 100 nm-sized samples
### III.5 Coulomb Blockade

**Single Electron Box:**

- **Electrostatic energy:**
  
  $$E_{el} = \frac{Q_1^2}{2C} + \frac{Q_2^2}{2C_g} - Q_2V_g = \frac{1}{2}CV_1^2 + \frac{1}{2}C_gV_2^2 - C_gV_2V_g$$

  - Work done by the voltage source

- **Boundary conditions:**
  
  $$V_g = V_1 + V_2 = \frac{Q_1}{C} + \frac{Q_2}{C_g}$$

  - Voltage drops over two capacitors

  $$ne = -Q_2 + Q_1$$

  - Charge quantization on island

- **With induced charge** $Q = C_gV_g$:
  
  $$Q_1 = \frac{(ne - Q)}{1 + C_g/C}$$

  $$Q_2 = -\frac{(ne - Q)}{1 + C_g/C} - Q$$

- **Gate: Induces charge** $C_gV_g$

- **Tunneling barrier**
III.5 Coulomb Blockade

- electrostatic energy:

\[ E_{\text{el}} = \frac{e^2}{2(C + C_g)} \left( n - \frac{Q}{e} \right)^2 - \frac{Q^2}{2C_g} \]

constant term (independent of N) is omitted

\[ Q = C_g V_g \]
### III.5 Coulomb Blockade

**Islands and Barriers:**

- **metal** | **island** | **metal**

  *tunneling barriers (characterized by tunneling resistance $R$)*

- weak coupling of island to metallic leads (reservoirs)

  - too weak: no electron transfer
  - too strong: strong leakage, no conservation of charge number

**too little** | **just right** | **too much**
Requirements for the Observation of the Coulomb Blockade:

- thermal fluctuations must be small enough:

\[ E_c > k_B T \implies C < \frac{e^2}{2k_B T} \approx 1 \text{ fF @ 1 K} \]

- quantum fluctuations must be small enough:

\[ E_c > \frac{\hat{h}}{\tau} \approx \frac{\hat{h}}{RC} \implies R > \frac{\hbar}{e^2} = R_Q = 25 \text{ k}\Omega \]

- requirement for voltage:

\[ E_c > eV \implies V < \frac{e}{2C} \approx 80 \mu\text{V at 1 fF} \]
III.5 Coulomb Blockade

Single Electron Transistor (SET):

\[ E_{el}(n, Q_g) = E_C \left( n - \frac{Q_g}{e} \right)^2 \]

\[ E_C = \frac{e^2}{2C_\Sigma} \]

\[ C_\Sigma = C_1 + C_2 + C_g \]

\[ Q_g = V_1 C_1 + V_2 C_2 + V_g C_g \]

- energy change by adding a single electron:

\[ E_{el}(n \pm 1) - E_{el}(n) = 2 \left( \pm n + \frac{1}{2} \mp \frac{Q_g}{e} \right) E_C \]
III.5 Coulomb Blockade

Electron Transfer Processes:

1. from the left: \( n \rightarrow n+1: \) \[ \Delta E_{FL}(n) = E(n+1) - E(n) - eV_1 \]
2. to the left: \( n \rightarrow n-1: \) \[ \Delta E_{TL}(n) = E(n-1) - E(n) + eV_1 \]
3. from the right: \( n \rightarrow n+1: \) \[ \Delta E_{FR}(n) = E(n+1) - E(n) - eV_2 \]
4. to the right: \( n \rightarrow n-1: \) \[ \Delta E_{TR}(n) = E(n-1) - E(n) + eV_2 \]
III.5 Coulomb Blockade

Electron Transfer Processes:

- $T > 0$: all transfer processes are allowed (by thermal activation)
- $T = 0$: only transfer processes with $\Delta E < 0$ are allowed

Coulomb blockade

$\Delta E_{FL, TL, FR, TR} (n) > 0$

single electron tunneling

$\Delta E_{FL} (n) < 0$  $\Delta E_{TR} (n) < 0$

$\Delta E_{FL} (n+1) > 0$  $\Delta E_{TR} (n-1) > 0$

no second additional or missing electron on island!!
electron transfer processes at varying gate voltage:

- in which range of the gate voltage is the electron transport blocked?
- assumptions: \( C_1 = C_2 \), \( V_1 = -V_2 = V/2 \)
- we use \( E_{el}(n \pm 1) - E_{el}(n) = \Delta E(n, Q_g) = 2 \left( \pm n + \frac{1}{2} \mp \frac{Q_g}{e} \right) E_C \)

1. from the left: \( \Delta E_{FL}(0) = E(1) - E(0) - eV_1 = 2E_C \left( 1/2 + \frac{Q_g}{e} \right) - eV/2 \)
2. to the left: \( \Delta E_{TL}(0) = E(-1) - E(0) + eV_1 = 2E_C \left( -1/2 + \frac{Q_g}{e} \right) + eV/2 \)
3. from the right: \( \Delta E_{FR}(0) = E(1) - E(0) - eV_2 = 2E_C \left( 1/2 + \frac{Q_g}{e} \right) + eV/2 \)
4. to the right: \( \Delta E_{TR}(0) = E(-1) - E(0) + eV_2 = 2E_C \left( -1/2 + \frac{Q_g}{e} \right) - eV/2 \)

- @ \( T = 0 \): \( I = 0 \) for

\[
\left| \frac{Q_g}{e} - n - 1/2 \right| > \frac{e|V|}{4E_C}
\]

blockade regimes:

- „Coulomb diamonds“
III.5 Coulomb Blockade

*Single Electron Transistor – Coulomb Diamonds:*

blue regions of vanishing conductance correspond to the Coulomb blockade regime (no current flow)

Quelle: ETH Zürich
III.5  Coulomb Blockade  
(additional topic)

Current-Voltage Characteristics (IVCs):

- facts:  (i) charging state is determined by \( n \)  
- (ii) no quantum coherence between different states

- probability \( p_n(t) \) to find system in state \( n \) at time \( t \):

\[ \frac{d}{dt} p_n(t) = -\left[ \Gamma_F(n) + \Gamma_T(n) \right] p_n(t) + \Gamma_T(n-1) p_{n-1}(t) + \Gamma_F(n+1) p_{n+1}(t) \]

\( \Gamma_F = \Gamma_{FL} + \Gamma_{FR} \)
\( \Gamma_T = \Gamma_{TL} + \Gamma_{TR} \)

- if we know \( p_n \) for stationary state, we get currents

\[ \begin{align*}
    I_L &= e \sum_n \left[ \Gamma_{FL}(n) - \Gamma_{TL}(n) \right] p_n \\
    I_R &= e \sum_n \left[ \Gamma_{TR}(n) - \Gamma_{FR}(n) \right] p_n \\
    I &= I_L - I_R
\end{align*} \]

\( \Gamma \): tunneling rates
### III.5 Coulomb Blockade

**Tunneling Rates for Single Tunnel Junction:**

**tunneling without CB**

\[ I = G_T V \]

\[ \Gamma = \frac{I}{e} = \frac{G_T}{e^2} eV \]

**tunneling with CB**

energy interval for available final states

\[ eV - E_C \]

\[ eV > E_C : \quad \Gamma = \frac{G_T}{e^2} (eV - E_C) \]

\[ eV < E_C : \quad \Gamma = 0 \]  

blockade regime
III.5 Coulomb Blockade

IVC for tunneling with Coulomb Blockade:

\[ 2 R_T C_J I / e \]

- Large quantum fluctuations
- \( R = 0 \)
- \( R \) sufficiently large

no quantum fluctuations
### III.5 Coulomb Blockade

**Tunneling Rates and IVC:**

- electrostatic energy changes as electron tunnels
  - determine tunneling rate at electron energy change of $\Delta E$:
  - **Fermi’s Golden Rule**
    
    $$
    \Gamma_{i \rightarrow f} = \frac{2\pi}{\hbar} \left| \langle i \mid H_t \mid f \rangle \right|^2 \delta(E_f - E_i - \Delta E)
    $$

- total transition rate from conductor 1 (source) to 2 (island):
  - tunneling rate proportional to density of states $D$
  - occupation probability given by Fermi functions $f(E)$
  - integration over all energies

$$
\Gamma_{1 \rightarrow 2}(\Delta E) = \frac{2\pi}{\hbar} \int_{-\infty}^{\infty} dE \left| \langle i \mid H_t \mid f \rangle \right|^2 D_i(E)f(E)D_f(E - \Delta E)[1 - f(E - \Delta E)]
$$
### III.5 Coulomb Blockade

**Tunneling Rates and IVC:**

- **simplifying assumptions:**
  1. $H_t$ is energy independent
  2. $D$ is energy independent

- $f(E)[1 - f(E - \Delta E)] = \frac{f(E) - f(E - \Delta E)}{1 - \exp \left( \frac{\Delta E}{k_B T} \right)}$

  \[
  \Gamma_{1\rightarrow2}(\Delta E) = \frac{1}{e^2 R} \frac{\Delta E}{e^{\Delta E/k_B T} - 1}
  \]

- **net current**

  \[
  I = e \left[ \Gamma_{1\rightarrow2}(\Delta E_{1\rightarrow2}) - \Gamma_{2\rightarrow1}(\Delta E_{2\rightarrow1}) \right]
  \]

- **current from current source (1) to island (2) in steady state**

  \[
  I = e \sum_n p(n) \left\{ \Gamma_{1\rightarrow2}(\Delta E_{1\rightarrow2}(n)) - \Gamma_{2\rightarrow1}(\Delta E_{2\rightarrow1}(n)) \right\}
  \]

  (equivalent expression for current from island to drain)

**Fermi functions \approx step functions**

\[
R = \frac{\hbar}{2\pi e^2} \left| \langle i | H_t | f \rangle \right|^2 D^2
\]
III.5  Coulomb Blockade

\[ \Gamma e^2 R \text{ vs. } \Delta E / k_B T \]

- **High T**
- **Low T**

The graphs show the variation of \( \Gamma e^2 R \) with \( \Delta E / k_B T \) for different temperatures.
III.5 Coulomb Blockade

Tunneling Rates and IVC:

Quelle: ETH Zürich
### III.5 Coulomb Blockade

**Tunneling Rates and IVC – Coulomb Staircase:**

- **1st step in IVC**
  - $V_g \downarrow \frac{V}{2}$
  - $V_g \uparrow \frac{V}{2}$

- **2nd step in IVC**
  - $V_g \downarrow \frac{V}{2}$
  - $V_g \uparrow \frac{V}{2}$

The movie shows variation of IVC with varying gate voltage.

Quelle: lt.px.tsukuba.ac.jp
III.5 Coulomb Blockade

Variation of the Gate Voltage – Coulomb Oscillations:

- gate voltage shifts up and down the energy levels of the island
- at small voltages: conductance can be varied considerably by gate voltage

⇒ Coulomb Oscillations
III.5 Coulomb Blockade

*Coulomb Oscillations – Variation of the Gate Voltage*

\[ \frac{eV}{2E_C} = 1.1, 1.0, 0.9, 0.75, 0.5, 0.25 \]

- large \( \frac{dl}{dV_G} \)

\[ \rightarrow \text{use as ultra-sensitive electrometer} \]
III.5 Coulomb Blockade

Coulomb Oscillations – Variation of the Gate Voltage

experimental data on Al/AlO$_x$/Al/AlO$_x$/Al - SET

V (source-drain) varied for different curves

J. Schuler, Ph.D. (WMI 2005)
III.5 Coulomb Blockade

Coulomb Oscillations – Effect of Single Fluctuating Background Charges

$\mathrm{Al/AlO_x/Al/AlO_x/Al}$ - SET

J. Schuler, Ph.D. (WMI 2005)

shift of $I(V_g)$ curve due to fluctuating background charge
### III.5 Coulomb Blockade

**SET fabrication:**

Fabrication of sub-μm Josephson Junctions by shadow evaporation technique

---

III.5 Coulomb Blockade

SET fabrication – Optical Lithography

(a) (b) (c)

SET fabrication – Electron Beam Lithography

two-layer e-beam resist

Si substrate
after liftoff
III.5 Coulomb Blockade

SET Fabrication
III.5 Coulomb Blockade

**Applications:**

- sensitive electrometers: $\delta Q/Q \approx 10^{-5} \, e$

- electron pumps
  
  $\rightarrow$ transporting electrons one by one: counting of electrons
  $\rightarrow$ current standard: $I = e \, f$
  $\rightarrow$ application of oscillating gate voltage

- charge Qubits
  
  $\rightarrow$ basic element for quantum information systems

---

Quantum oscillations in two coupled charge qubits
Nature 421, 823-826 (20 February 2003)
III.5 Coulomb Blockade

quantum metrological triangle

$$V = \Phi_0 \cdot f$$

$$I = e \cdot f$$

$$V = R_K \cdot I$$

1990:

$$K_{J-90} \equiv \frac{2e}{h} = \frac{1}{\Phi_0} = 483\,597\,891(12) \text{ Hz mV}$$

$$R_{K-90} \equiv \frac{h}{e^2} = 25\,812.807\,557(18) \Omega$$

$$K_{I-90} = \frac{1}{e}$$

not yet available