Chapter 1

Macroscopic Quantum Phenomena

1. Macroscopic Quantum Phenomena

1.1 The Macroscopic Quantum Model of Superconductivity

Macroscopic systems

Quantum mechanics → Physical quantities ($p, E, \ldots$) are quantized

Usually thermal motion masks quantum properties

→ No quantization effects on a macroscopic scale

Superconductivity:

→ Macroscopic quantum effects are observable

Example: quantization of flux through a loop

Why?

Electrons form highly correlated system

Milestones

1911 **Discovery of superconductivity**

1933 **Meißner-Ochsenfeld effect**

1935 **London-Laue-theory**
phenomenological model describing observations

1948 **London:** superelectron fluid as quantum mechanical entity
Superconductivity is an inherently quantum phenomenon manifested on a macroscopic scale

→ Macroscopic wave function $\Psi(\boldsymbol{r},t) = \Psi_0 \mathrm{e}^{i\theta(\boldsymbol{r},t)}$

→ London equations

1952 **Ginzburg-Landau theory**

→ Description by complex order parameter $\Psi(\boldsymbol{r})$

→ Treatment of spatially inhomogeneous situations near T_c

1957 **Microscopic BCS theory** (J. Bardeen, L.N. Cooper, J.R. Schrieffer)

→ BCS ground state Ψ_BCS (coherent many body state)

1962 Prediction of the **Josephson effect**

Macroscopic quantum model of superconductivity

Macroscopic wave function $\Psi(\boldsymbol{r}, t)$

→ Describes the behavior of the whole ensemble of superconducting electrons

→ Justified by microscopic BCS theory

- Small portion of electrons close to Fermi level are bound to Cooper pairs
- Center of mass motion of pairs is strongly correlated
- Example:

  Wave function $\Psi(\boldsymbol{r}, t) = \Psi_0 \mathrm{e}^{i\theta(\boldsymbol{r},t)} = \Psi_0 \mathrm{e}^{i(\boldsymbol{k}_\mathrm{s}\cdot\boldsymbol{r} - \omega t)}$

  Each pair has momentum $\hbar\boldsymbol{k}_\mathbf{s}$ or velocity $\boldsymbol{v}_\mathbf{s} = \frac{\hbar\boldsymbol{k}_\mathbf{s}}{m_\mathrm{s}}$

Basic quantum mechanics

Quantization of electromagnetic radiation (Planck, Einstein):
photons represent smallest amount of energy:

$$E = \hbar\omega \quad \text{with} \qquad \hbar = 1.054\,571\,596(82) \times 10^{-34}\ \text{Js}$$

Luis de Broglie describes classical particles as waves → wave particle duality
→ Particle – wave interrelations:

$$E = \hbar\omega \quad , \qquad\qquad p = \hbar\boldsymbol{k} = \frac{h}{\lambda}\widehat{\boldsymbol{k}}$$

Erwin Schrödinger developed a wave mechanics for particles
→ Complex wave function describes quantum particle

$$\Psi(\mathbf{r}, t) = A\ \exp[\imath(\mathbf{k}\cdot\mathbf{r} - \omega t)] \quad \Longleftrightarrow \quad \Psi(\mathbf{r}, t = A\ \exp[\frac{\imath}{\hbar}(\mathbf{p}\cdot\mathbf{r} - Et)]$$

$$\nabla^2\Psi = -k^2\Psi \qquad\qquad \nabla^2\Psi = -\frac{p^2}{\hbar^2}\Psi$$

$$\frac{\partial^2\Psi}{\partial t^2} = \omega^2\Psi \qquad\qquad \frac{\partial^2\Psi}{\partial t^2} = \frac{E^2}{\hbar^2}\Psi$$

Basic quantum mechanics

$$\Psi(\mathbf{r},t) = A\ \exp[\imath(\mathbf{k}\cdot\mathbf{r}-\omega t)] \quad \Longleftrightarrow \quad \Psi(\mathbf{r},t = A\ \exp[\frac{\imath}{\hbar}(\mathbf{p}\cdot\mathbf{r}-Et)]$$

$$\nabla^2\Psi - \frac{k^2}{\omega^2}\frac{\partial^2\Psi}{\partial t^2} = 0 \qquad\qquad \nabla^2\Psi - \frac{p^2}{E^2}\frac{\partial^2\Psi}{\partial t^2} = 0$$

with $\frac{p^2}{2m} = E - V \ \Rightarrow\ p^2 = 2m(E-V)$

$E = \hbar\omega$

$$\nabla^2\Psi - \frac{2m(E-V)}{\hbar^2\omega^2}\frac{\partial^2\Psi}{\partial t^2} = 0$$

multiply by $-\frac{\hbar^2}{2m}$

$$-\frac{\hbar^2}{2m}\nabla^2\Psi - \frac{V}{\omega^2}\frac{\partial^2\Psi}{\partial t^2} = -\frac{E}{\omega^2}\frac{\partial^2\Psi}{\partial t^2}$$

use $E = \hbar\omega$ $\quad \frac{\partial^2\Psi}{\partial t^2} = -\omega^2\Psi$

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = -\frac{\hbar}{\omega}\frac{\partial^2\Psi}{\partial t^2}$$

since $\frac{\partial\Psi}{\partial t} = -\imath\omega\Psi$

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = \imath\hbar\frac{\partial\Psi}{\partial t}$$

Basic quantum mechanics

Similar considerations → Schrödinger postulated a

General time dependent equation for massive quantum objects:

$$\boxed{\imath\hbar\frac{\partial}{\partial t}\,\Psi(\mathbf{r},t) = \underbrace{\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r},t)\right)}_{\text{Hamilton operator}}\;\Psi(\mathbf{r},t)}$$

Schrödinger equation
(differential equation)

We restrict ourselves to systems with constant total energy (conservative systems)

→ Due to $E = \hbar\omega$ also the frequency is constant

→ Prefactor of 2nd term on lhs of $\nabla^2\Psi - \frac{2m(E-V)}{\hbar^2\omega^2}\frac{\partial^2\Psi}{\partial t^2} = 0$ is constant

→ Solutions can be split in a two parts depending only on space and time

Basic quantum mechanics

$$\Psi(\mathbf{r},t) = \Psi(\mathbf{r},0)\ \exp(-\imath\omega t) \qquad \text{and} \qquad \nabla^2\Psi - \frac{2m(E-V)}{\hbar^2\omega^2}\frac{\partial^2\Psi}{\partial t^2} = 0$$

$$\nabla^2\Psi(\mathbf{r})\ \exp(-\imath\omega t) - \frac{2m(E-V)}{\hbar^2\omega^2}\cdot(-\omega^2)\ \Psi(\mathbf{r})\exp(-\imath\omega t)$$

$$\nabla^2\Psi(\mathbf{r}) + \frac{2m(E-V)}{\hbar^2}\ \Psi(\mathbf{r})$$

$$\boxed{\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\ \Psi(\mathbf{r}) = E\Psi(\mathbf{r})}$$

Stationary Schrödinger equation for Hamiltonians without explicit time dependence

Yields eigenenergies E_n and eigenstates $\Psi_n(\boldsymbol{r})$

Probability currents

Interpretation of complex wave function $\Psi(\mathbf{r},t) = \Psi_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

→ Note: EM fields represented as the real **or** imaginary part of a complex expression

→ Schrödinger equation suggests that phase has physical significance

Max Born: Interpretation of square magnitude as probability of a quantum object

$$\rho(\mathbf{r},t) = |\Psi(\mathbf{r},t)|^2 = \Psi^\star(\mathbf{r},t)\Psi(\mathbf{r},t)$$

$$\int \Psi^\star(\mathbf{r},t)\Psi(\mathbf{r},t)\, dV = 1$$

Conservation of probability density requires $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J}_\rho = 0$

→ Continuity equation describes evolution of probability in space and time

$\boldsymbol{J}_\rho$ describes probabilistic flow of a quantum object, not the motion of a charged particle in an electromagnetic field (forces depending on the motion of the particle itself)

Probability currents

→ Time evolution of $\rho(\mathbf{r}, t)$ defines probability current:

$$\mathbf{J}_\rho \equiv \frac{\hbar}{2m\imath}\left(\Psi^\star \boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^\star\right) = \Re\left\{\Psi^\star \frac{\hbar}{\imath m}\boldsymbol{\nabla}\Psi\right\} = \Re\left\{\Psi^\star \frac{\widehat{\mathbf{p}}}{m}\Psi\right\}$$

Example: J_ρ for a charged particle in an EM field

Start with classical equation of motion: $\frac{d}{dt}\,\mathbf{p} = -\boldsymbol{\nabla}V$

Canonical (kinetic and field) momentum

$$\boldsymbol{p} = m\boldsymbol{v} + q\boldsymbol{A}$$

Vector potential

Example: J_ρ for a charged particle in an EM field

Lorentz' law with $\boldsymbol{E}$ and $\boldsymbol{B}$ expressed in terms of potentials ϕ and $\boldsymbol{A}$

$$\frac{d\mathbf{p}}{dt} = -\boldsymbol{\nabla}\left\{q\phi - \frac{q}{m}(\mathbf{p}\cdot\mathbf{A}) + \frac{q^2}{2m}(\mathbf{A}\cdot\mathbf{A})\right\}$$

→ generalized potential:

$$V = q\phi - \frac{q}{m}(\mathbf{p}\cdot\mathbf{A}) + \frac{q^2}{2m}(\mathbf{A}\cdot\mathbf{A}) \qquad \Rightarrow \frac{d}{dt}\mathbf{p} = -\boldsymbol{\nabla}V$$

Lorentz's law:

$$m\frac{d\mathbf{v}}{dt} = q\left[\mathbf{E} + (\mathbf{v}\times\mathbf{B})\right]$$

with:

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$$

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \boldsymbol{\nabla}\phi$$

$$\boldsymbol{p} = m\boldsymbol{v} + q\boldsymbol{A}$$

$$\frac{d\boldsymbol{A}}{dt} = \frac{\partial\boldsymbol{A}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A}$$

Schrödinger equation:

$$\imath\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r},t)\right)\Psi(\mathbf{r},t)$$

$$E = E_{\text{kin}} + E_{\text{pot}}$$

$$E \Rightarrow \imath\hbar\frac{\partial}{\partial t}$$

insert expression for $V(\boldsymbol{r},t)$

$$\mathbf{p} \Rightarrow -\imath\hbar\boldsymbol{\nabla}$$

$$\imath\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + q\phi - \frac{q}{m}(\mathbf{p}\cdot\mathbf{A}) + \frac{q^2}{2m}(\mathbf{A}\cdot\mathbf{A})\right)\Psi(\mathbf{r},t)$$

Example: J_ρ for a charged particle in an EM field

Schrödinger equation

$$\imath\hbar\frac{\partial\Psi}{\partial t} = \frac{1}{2m}\left(\frac{\hbar}{\imath}\nabla - q\mathbf{A}\right)^2 \Psi + q\phi\,\Psi$$

Probability current

$$\mathbf{J}_\rho = \Re\left\{\Psi^\star\left(\frac{\hbar}{\imath m}\nabla - \frac{q}{m}\mathbf{A}\right)\Psi\right\} = \Re\left\{\Psi^\star\,\frac{\widehat{\mathbf{p}}}{m}\,\Psi\right\}$$

$$\widehat{\mathbf{p}} = \frac{\hbar}{\imath}\nabla - q\mathbf{A}$$

→ Central expression in the quantum description of superconductivity

→ Wave function of a single charged particle will be replaced by the macroscopic wave function describing all superelectrons

Normal metals

Electrons as weakly/non-interacting particles
→ Ordinary Schrödinger equation: $$\imath\hbar\frac{\partial\Psi}{\partial t} = \widehat{H}\,\Psi$$

where $\Psi(\mathbf{r},t) = \Psi_0(\mathbf{r},t)\,\mathrm{e}^{\imath\theta(\mathbf{r},t)}$ is the complex wave function of a particle

Stationary case: $$\hbar\frac{\partial\theta}{\partial t} = -E$$

→ Quantum behavior reduced to that of wave function phase
→ Fermi statistics → different time evolution of phase for different energies
→ Phases are uniformly distributed, phase drops out for macroscopic quantities

Central hypothesis of macroscopic quantum model:

There exists a macroscopic wave function $\psi(\mathbf{r},t) = \psi_0(\mathbf{r},t)\, e^{i\theta(\mathbf{r},t)}$

describing the behavior of all superelectrons in a superconductor

(motivation: superconductivity is a coherent phenomenon of all sc electrons)

Normalization condition:

$$\int \psi^\star(\mathbf{r},t)\psi(\mathbf{r},t)\, dV = N_s$$

$$|\psi(\mathbf{r},t)|^2 = \psi^\star(\mathbf{r},t)\psi(\mathbf{r},t) = n_s(\mathbf{r},t)$$

N_s and $n_s(\mathbf{r},t)$ are the total and local density of superconducting electrons

→ Charged superfluid (analogy to fluid mechanics)

→ Similarities in the description of superconductivity and superfluids

→ No explanation of microscopic origin of superconductivity

→ Relevant issue: describe superelectron fluid as quantum mechanical entity

Some basic relations:

- General relations in electrodynamics:

$$\text{Electric field} \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi$$

$$\text{Flux density} \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$\boldsymbol{A}$ = Vector potential
ϕ = Scalar potential

- Electrical current is driven by gradient of electrochemical potential:

$$\phi(\mathbf{r},t) \rightarrow \phi(\mathbf{r},t) + \mu(\mathbf{r},t)/q$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\left(\phi + \frac{\mu}{q}\right)$$

- Canonical momentum: $\mathbf{p} = m\mathbf{v} + q\mathbf{A}$

$$p_x = \partial\mathcal{L}/\partial\dot{x},$$
$$\mathcal{L} = \text{Lagrange function}$$

- Kinematic momentum: $m\mathbf{v} = \frac{\hbar}{i}\nabla - q\mathbf{A}$

1.1.2 Macroscopic Quantum Currents in Superconductors

- Schrödinger equation for charged particle:

$$\frac{1}{2m}\left(\frac{\hbar}{\imath}\nabla - q\mathbf{A}(\mathbf{r},t)\right)^2 \Psi(\mathbf{r},t) + \underbrace{[q\phi(\mathbf{r},t) + \mu(\mathbf{r},t)]}_{\text{electro-chemical potential}} \Psi(\mathbf{r},t) = \imath\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}$$

- Insert macroscopic wave-function $\psi(\mathbf{r},t) = \psi_0(\mathbf{r},t)\, e^{\imath\theta(\mathbf{r},t)}$

$\Psi \rightarrow \psi$, $\quad q \rightarrow q_s$, $\quad m \rightarrow m_s$

- Split up into real and imaginary part and assume $|\psi_0(\mathbf{r},t)|^2 = n_s(\mathbf{r},t)| = const.$

Real part:

$$\boxed{-\hbar\frac{\partial\theta}{\partial t} = \frac{1}{2n_s}\Lambda\mathbf{J}_s^2 + \underbrace{q_s\phi + \mu}}$$

Energy-phase relation

Kinetic energy Potential energy

Imaginary part:

$$\boxed{\mathbf{J}_s = q_s n_s \left\{\frac{\hbar}{m_s}\nabla\theta(\mathbf{r},t) - \frac{q_s}{m_s}\mathbf{A}(\mathbf{r},t)\right\}}$$

Current-phase relation

Superelectron velocity $v_s \rightarrow J_s = n_s q_s v_s$

1.1.2 Macroscopic Quantum Currents in Superconductors

- We start from the Schrödinger equation:

$$\underbrace{\frac{1}{2m_s}\left(\frac{\hbar}{\imath}\nabla - q_s\mathbf{A}(\mathbf{r},t)\right)^2 \Psi(\mathbf{r},t)}_{II} + \underbrace{[q_s\phi(\mathbf{r},t) + \mu(\mathbf{r},t)]}_{\textit{electro-chemical potential}}\,\Psi(\mathbf{r},t) \;=\; \underbrace{\imath\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}}_{I}$$

- We use the definition $S \equiv \hbar\theta$ and obtain with $\psi(\mathbf{r},t) \;=\; \psi_0(\mathbf{r},t)\,\mathrm{e}^{\imath\theta(\mathrm{r},t)}$

$$I = \imath\hbar\frac{\partial\Psi}{\partial t} = \left[\imath\hbar\frac{\partial\Psi_0}{\partial t} - \Psi_0\frac{\partial S}{\partial t}\right]\,\mathrm{e}^{\imath S/\hbar}$$

$$II = \frac{1}{2m_s}\left(\frac{\hbar}{\imath}\nabla - q_s\mathbf{A}\right)^2\Psi = \frac{1}{2m_s}\left[\underbrace{-\hbar^2\nabla^2}_{1} + \underbrace{\imath\hbar q_s\nabla\cdot\mathbf{A}}_{2} + \underbrace{\imath\hbar q_s\mathbf{A}\cdot\nabla}_{3} + \underbrace{q_s^2\mathbf{A}^2}_{4}\right]\Psi_0\mathrm{e}^{\imath S/\hbar}$$

$$II = \frac{1}{2m_s}\left(\frac{\hbar}{\imath}\nabla - q_s\mathbf{A}\right)^2\Psi = \frac{1}{2m_s}\left[\underbrace{-\hbar^2\nabla^2}_{1} + \underbrace{\imath\hbar q_s\nabla\cdot\mathbf{A}}_{2} + \underbrace{\imath\hbar q_s\mathbf{A}\cdot\nabla}_{3} + \underbrace{q_s^2\mathbf{A}^2}_{4}\right]\Psi_0 e^{\imath S/\hbar}$$

$$1 = -\frac{\hbar^2\nabla^2}{2m_s}\Psi_0 e^{\imath S/\hbar} = \frac{1}{2m_s}\left[-\hbar^2\nabla^2\Psi_0 + \Psi_0(\nabla S)^2 - 2\imath\hbar\nabla\Psi_0(\nabla S) - \imath\hbar\Psi_0\nabla^2 S\right] e^{\imath S/\hbar}$$

$$2 = \frac{1}{2m_s}\imath\hbar q_s\Psi_0(\nabla\cdot\mathbf{A})e^{\imath S/\hbar} + \text{term } 3$$

$$3 = \frac{1}{2m_s}\left[\imath\hbar q_s\mathbf{A}\cdot(\nabla\Psi_0) - q_s\Psi_0\mathbf{A}(\nabla S)\right] e^{\imath S/\hbar}$$

$$2+3 = \frac{1}{2m_s}\left[\imath\hbar q_s\Psi_0(\nabla\cdot\mathbf{A}) + 2\imath\hbar q_s\mathbf{A}\cdot(\nabla\Psi_0) - 2q_s\Psi_0\mathbf{A}(\nabla S)\right] e^{\imath S/\hbar}$$

$$4 = \frac{1}{2m_s}q_s\Psi_0\mathbf{A}^2 e^{\imath S/\hbar}$$

$$II = \left[\Psi_0\frac{(\nabla S - q_s\mathbf{A})^2}{2m_s} - \frac{\hbar^2\nabla^2}{2m_s}\Psi_0 - \frac{\imath}{2m_s}\underbrace{(2\hbar\nabla\Psi_0 + \hbar\Psi_0\nabla)(\nabla S - q_s\mathbf{A})}_{=\frac{\hbar}{\Psi_0}\nabla\Psi_0^2(\nabla S - q_s\mathrm{A})}\right] e^{\imath S/\hbar}$$

$$= \left[\Psi_0\frac{(\nabla S - q_s\mathbf{A})^2}{2m_s} - \frac{\hbar^2\nabla^2}{2m_s}\Psi_0 - \imath\frac{\hbar}{2\Psi_0}\nabla\left(\frac{\Psi_0^2}{m_s}(\nabla S - q_s\mathbf{A})\right)\right] e^{\imath S/\hbar}$$

$$I = \imath\hbar\frac{\partial\Psi}{\partial t} = \left[\imath\hbar\frac{\partial\Psi_0}{\partial t} - \Psi_0\frac{\partial S}{\partial t}\right] e^{\imath S/\hbar}$$

$$II = \left[\Psi_0\frac{(\nabla S - q_s\mathbf{A})^2}{2m_s} - \frac{\hbar^2\nabla^2}{2m_s}\Psi_0 - \imath\frac{\hbar}{2\Psi_0}\nabla\left(\frac{\Psi_0^2}{m_s}(\nabla S - q_s\mathbf{A})\right)\right] e^{\imath S/\hbar}$$

- equation for real part:

$$\left[\Psi_0\left(\frac{(\nabla S - q_s\mathbf{A})^2}{2m_s} + q_s\phi\right) - \frac{\hbar^2\nabla^2}{2m_s}\Psi_0\right] e^{\imath S/\hbar} = -\Psi_0\frac{\partial S}{\partial t}e^{\imath S/\hbar}$$

$$\mathbf{J}_s = \frac{q_s n_s}{m_s}(\hbar\nabla\theta - q_s\mathbf{A})$$

$$\Rightarrow \quad \frac{\partial S}{\partial t} + \underbrace{\frac{(\nabla S - q_s\mathbf{A})^2}{2m_s}}_{=\frac{1}{2}m_s v_s^2 = \frac{1}{2n_s}\Lambda J_s^2} + q_s\phi = \frac{\hbar^2\nabla^2\Psi_0}{2m_s\Psi_0}$$

$$\Lambda = \frac{m_s}{n_s q_s^2}$$

$$\Rightarrow \quad \boxed{\hbar\frac{\partial\theta}{\partial t} + \frac{1}{2n_s}\Lambda J_s^2 + q_s\phi = \cancel{\frac{\hbar^2\nabla^2\Psi_0}{2m_s\Psi_0}}}$$

$$S \equiv \hbar\theta$$

energy-phase relation (term of order $\nabla^2 n_s$ is usually neglected)

- Interpretation of energy-phase relation

$$\boxed{\hbar\frac{\partial\theta}{\partial t} + \frac{1}{2n_s}\Lambda J_s^2 + q_s\phi = 0}$$

$S(\mathbf{r}, t) \equiv \hbar\theta(\mathbf{r}, t)$ corresponds to action

→ In the quasi-classical limit $\hbar \to 0$, the energy-phase-relation becomes the Hamilton-Jacobi equation

$$\frac{\partial S(\mathbf{r}, t)}{\partial t} = -\mathcal{H}(\mathbf{r}, t)$$

1.1.2 Macroscopic Quantum Currents in Superconductors

$$I = \imath\hbar\frac{\partial\Psi}{\partial t} = \left[\imath\hbar\frac{\partial\Psi_0}{\partial t} - \Psi_0\frac{\partial S}{\partial t}\right] e^{\imath S/\hbar}$$

$$II = \left[\Psi_0\frac{(\nabla S - q_s\mathbf{A})^2}{2m_s} - \frac{\hbar^2\nabla^2}{2m_s}\Psi_0 - \imath\frac{\hbar}{2\Psi_0}\nabla\left(\frac{\Psi_0^2}{m_s}(\nabla S - q_s\mathbf{A})\right)\right] e^{\imath S/\hbar}$$

- equation for imaginary part:

$$\imath\hbar\frac{\partial\Psi_0}{\partial t}e^{\imath S/\hbar} = -\imath\frac{\hbar}{2\Psi_0}\nabla\left(\frac{\Psi_0^2}{m_s}(\nabla S - q_s\mathbf{A})\right)e^{\imath S/\hbar}$$

$$\Rightarrow \quad 2\Psi_0\frac{\partial\Psi_0}{\partial t} = -\nabla\left(\frac{\Psi_0^2}{m_s}(\nabla S - q_s\mathbf{A})\right)$$

$$\Rightarrow \quad \underbrace{\frac{\partial\Psi_0^2}{\partial t}}_{=\partial\rho_s/\partial t} = -\nabla\underbrace{\left(\frac{\Psi_0^2}{m_s}(\nabla S - q_s\mathbf{A})\right)}_{=\rho_s v_s = J_\rho}$$

continuity equation for probability density $\rho = \Psi_0^2 = n_s$ and probability current density $\boldsymbol{J}_\rho$

→ Conservation law for probability density

1.1.2 Macroscopic Quantum Currents in Superconductors

- *energy-phase relation*

$$\boxed{-\hbar\frac{\partial\theta}{\partial t} = \frac{1}{2n_s}\Lambda\mathbf{J}_s^2 + q_s\phi + \mu} \quad (1)$$

- *supercurrent density-phase relation*

$$\Lambda = \frac{m_s}{n_s {q_s}^2} \quad \textit{(London parameter)}$$

$$\boxed{\mathbf{J}_s = q_s n_s \left\{\frac{\hbar}{m_s}\nabla\theta(\mathbf{r},t) - \frac{q_s}{m_s}\mathbf{A}(\mathbf{r},t)\right\}} \quad (2)$$

$$\boxed{\Lambda\mathbf{J}_s = -\left\{\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q_s}\nabla\theta(\mathbf{r},t)\right\}}$$

➔ ***equations (1) and (2) have general validity for charged and uncharged superfluids***

$$q_s = k\cdot q, \qquad m_s = k\cdot m, \qquad n_s = n/k$$

(i) $q = -e, \quad k = 2$ — *superconductor with Cooper pairs of charge q_s = -2e*

(ii) $q = 0, \quad k = 1$ — *neutral Bose superfluid, e.g. ^{4}He*

(iii) $q = 0, \quad k = 2$ — *neutral Fermi superfluid, e.g. ^{3}He*

➔ ***we use equations (1) and (2) to derive London equations***

- note: $\Lambda = \frac{km}{(n/k)(kq)^2} = \frac{m}{nq^2}$ independent of k!

Additional topic: Gauge invariance

→ expression for the supercurrent density must be gauge invariant

$$\mathbf{J}_s = q_s n_s \left\{ \frac{\hbar}{m_s} \nabla\theta(\mathbf{r},t) - \frac{q_s}{m_s}\mathbf{A}(\mathbf{r},t) \right\}$$

with the ***gauge invariant phase gradient***:

$$\gamma = \nabla\theta - \frac{q_s}{\hbar}\mathbf{A} = \nabla\theta - \frac{2\pi}{\Phi_0}\mathbf{A}$$

the supercurrent is

$$\mathbf{J}_s = \frac{q_s n_s \hbar}{m_s}\gamma = \frac{\hbar}{q_s \Lambda}\gamma$$

London coefficient

$$\Lambda \equiv \frac{m_s}{n_s q_s^2}$$

London penetration depth

$$\lambda_L \equiv \sqrt{\frac{\Lambda}{\mu_0}} = \sqrt{\frac{m_s}{\mu_0 n_s q_s^2}}$$

Summary:

The macroscopic wave function $\psi(\mathbf{r},t) = \sqrt{n_s(\mathbf{r},t)}\, \mathrm{e}^{\imath\theta(\mathbf{r},t)}$

describes the whole ensemble of superelectrons with

$$\int \psi^\star(\mathbf{r},t)\psi(\mathbf{r},t)\, dV = N_s \qquad |\psi(\mathbf{r},t)|^2 = \psi^\star(\mathbf{r},t)\psi(\mathbf{r},t) = n_s(\mathbf{r},t)$$

The current-phase relation (supercurrent equation) is $(n_s(\boldsymbol{r},t) = const)$

$$\mathbf{J}_s = q_s n_s \left\{ \frac{\hbar}{m_s}\boldsymbol{\nabla}\theta(\mathbf{r},t) - \frac{q_s}{m_s}\mathbf{A}(\mathbf{r},t) \right\} = q_s n_s \mathbf{v}_s$$

The gauge invariant phase gradient is

$$\gamma = \boldsymbol{\nabla}\theta - \frac{2\pi}{\Phi_0}\mathbf{A} \qquad \Rightarrow \mathbf{J}_s = \frac{q_s n_s \hbar}{m_s}\gamma = \frac{\hbar}{q_s\Lambda}\gamma$$

The energy-phase relation is $(n_s(\boldsymbol{r},t) = const)$

$$-\hbar\frac{\partial\theta}{\partial t} = \frac{1}{2n_s}\Lambda\mathbf{J}_s^2 + q_s\phi + \mu$$

Importance of current-phase and energy-phase relation

We can derive

→ 1. and 2. London equation

→ Flux(oid) quantization

→ Josephson equations

1.1.3 The London Equations

Fritz London (1900 – 1954)

London equations are purely phenomenological

→ Describe the behavior of superconductors

→ Starting point: (super)current-phase relation (CPR)

$$\boxed{\Lambda \mathbf{J}_s = -\left\{\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q_s}\nabla\theta(\mathbf{r},t)\right\}} \qquad \Lambda = \frac{m_s}{n_s q_s^2} \quad \text{(London parameter)}$$

2nd London equation – Meißner-Ochsenfeld effect:

Take the curl of CPR

→ second London equation

$$\nabla \times (\Lambda \mathbf{J}_s) = -\nabla \times \mathbf{A} = -\mathbf{B}$$

$$\nabla^2 \mathbf{B} = \frac{\mu_0}{\Lambda}\mathbf{B} = \frac{1}{\lambda_L^2}\mathbf{B}$$

Maxwell:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_s$$
$$\nabla \times \nabla \times \mathbf{B} = \nabla \times \mu_0 \mathbf{J}_s$$
$$\nabla \times \nabla \times \mathbf{B} = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mu_0 \mathbf{J}_s = -\nabla^2 B$$

describes Meißner-Ochsenfeld effect

applied field decays exponentially inside superconductor,

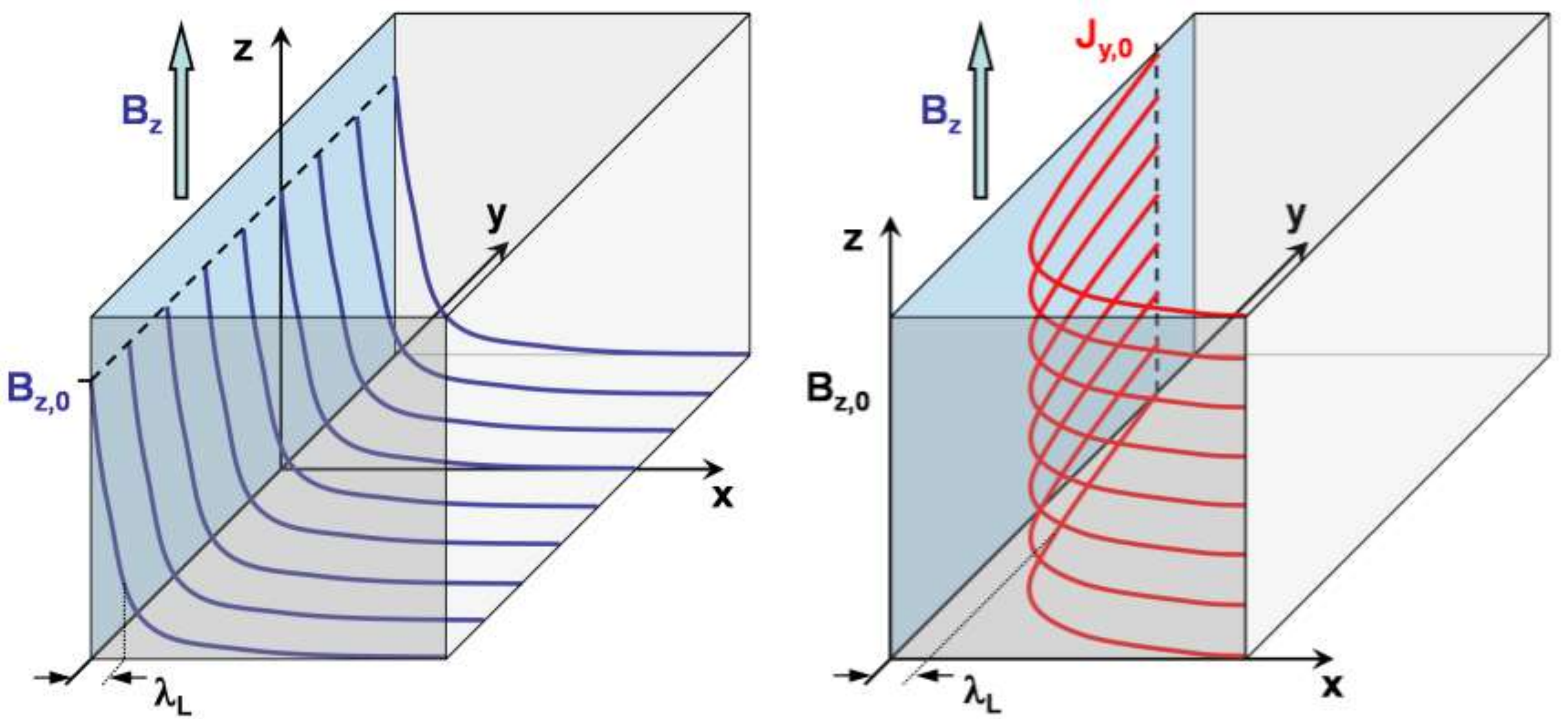
→ decay length $\lambda_L \equiv \sqrt{\dfrac{m_s}{\mu_0 n_s q_s^2}}$ (London penetration depth)

Example Meißner-Ochsenfeld effect

Plane surface extending in yz-plane, magnetic field B_z parallel to z-axis:

→ exponential decay $$B_z(x) = B_{z,0}\, \mathrm{e}^{-x/\lambda_L}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_s \quad \rightarrow \quad J_{s,y}(x) = \frac{B_{z,0}}{\mu_0 \lambda_L}\, \mathrm{e}^{-x/\lambda_L} = \frac{H_{z,0}}{\lambda_L}\, \mathrm{e}^{-x/\lambda_L} = J_{y,0}\, \mathrm{e}^{-x/\lambda_L}$$



1st London equation – perfect conductivity

Time derivative of CPR →

$$\frac{\partial}{\partial t}(\Lambda \mathbf{J}_s) = -\left\{\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} - \frac{\hbar}{q_s}\boldsymbol{\nabla}\left(\frac{\partial\theta(\mathbf{r},t)}{\partial t}\right)\right\}$$

Use energy-phase relation

$$-\hbar\frac{\partial\theta}{\partial t} = \frac{1}{2n_s}\Lambda \mathbf{J}_s^2 + q_s\phi + \mu$$

and

$$\mathbf{E} = -\partial \mathbf{A}/\partial t - \boldsymbol{\nabla}\phi$$

First London equation

$$\frac{\partial}{\partial t}(\Lambda \mathbf{J}_s) = \mathbf{E} - \frac{1}{n_s q_s}\boldsymbol{\nabla}\left(\frac{1}{2}\Lambda \mathbf{J}_s^2\right)$$

Linearized form

$$\frac{\partial}{\partial t}(\Lambda \mathbf{J}_s) = \mathbf{E}$$

Time-independent supercurrent → electric field inside the superconductor vanishes
→ dissipationless supercurrent

Processes that could cause a decay of J_s

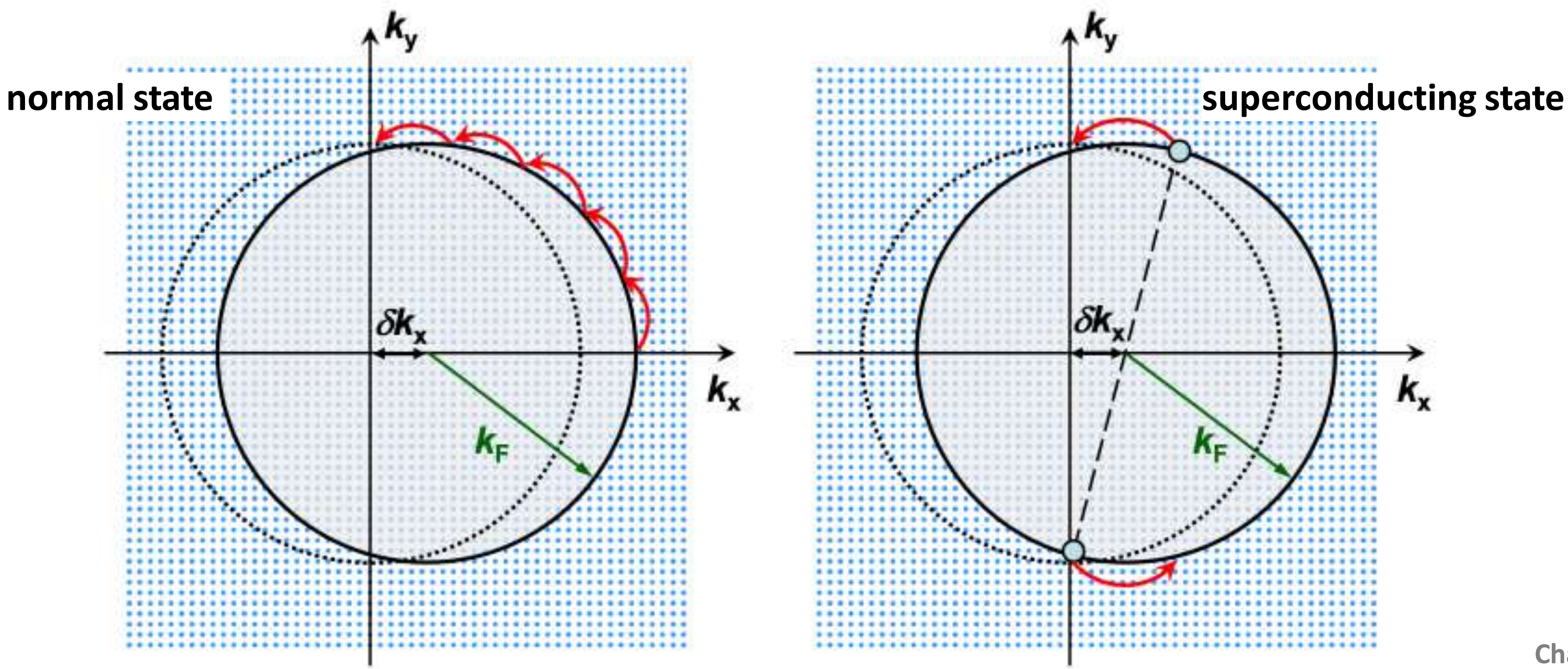
Example: Fermi sphere in two dimensions in the $k_x k_y$ – plane

→ T = 0: all states inside the Fermi circle are occupied

→ Current in x-direction → shift of Fermi circle along k_x by $\pm\delta k_x$

normal state: Relaxation into states with lower energy (obeying Pauli principle)
→ centered Fermi sphere → current relaxes

supercond. state: All Cooper pairs must have the same center of mass moment
→ only scattering around the sphere → no decay of supercurrent



Additional topic: Linearized 1. London Equation

Usually, 1. London equation is linearized: $$\frac{\partial}{\partial t}(\Lambda \mathbf{J}_s) = \mathbf{E} - \cancel{\frac{1}{n_s q_s} \nabla \underbrace{\left(\frac{1}{2}\Lambda \mathbf{J}_s^2\right)}_{\text{kinetic energy of superelectrons}}}$$

→ Allowed if |E| >> $|v_s|$ |B|
Condition is satisfied in most cases
Equivalent to neglecting magnetic contribution in Lorentz‘ law

The nonlinear first London equation results from the Lorentz's law and the second London equation
→ Exact form describes the zero dc resistance in superconductors

The first London equation is derived using the second London equation
→ Meißner-Ochsenfeld effect is more fundamental than vanishing dc resistance

Additional topic: The London Gauge (see lecture notes)

rigid phase: $\nabla\phi = 0 \;\Rightarrow\; \mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla\phi = \partial\mathbf{A}/\partial t \;\Rightarrow\; \Lambda\mathbf{J}_s = -\mathbf{A}$

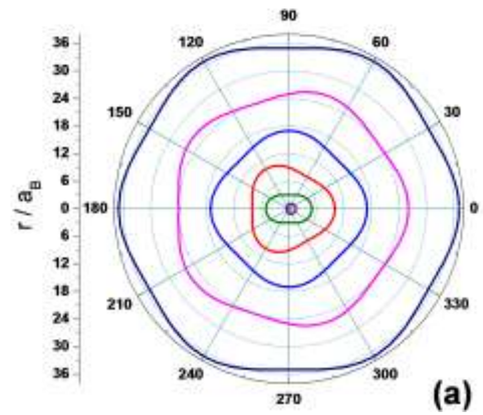
no conversion of J_s in J_n: $\nabla\cdot\mathbf{J}_s = 0 \;\Rightarrow\; \nabla\cdot\mathbf{A} = 0$

1.2. Fluxoid Quantization

Gedanken-experiment

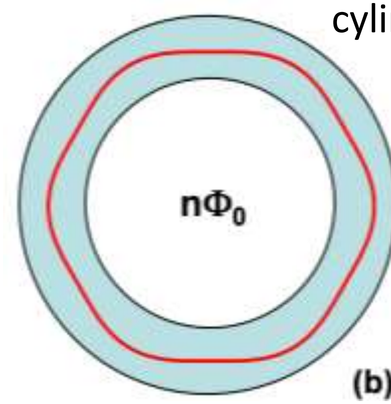
Generate supercurrent in a ring → Zero dc-resistance
→ Stationary state → Determined by quantum conditions

Bohr‘s model for atoms



Stationary supercurrent:

Superconducting cylinder



→ Angular momentum quantization
→ No destructive interference of electron wave
→ Stationary state

→ Macroscopic wave function is not allowed to interfere destructively
→ quantization condition

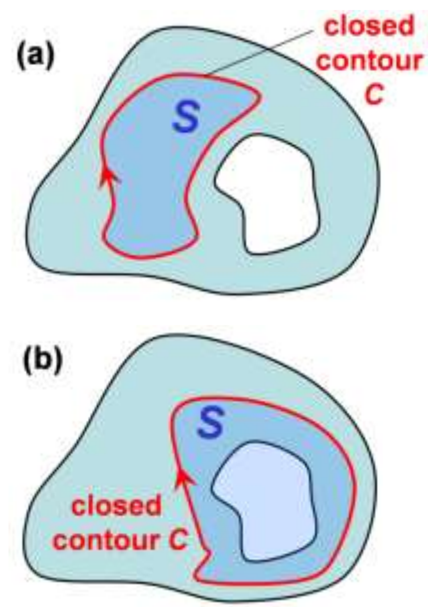
Derivation of the quantization condition

(based on macroscopic quantum model of superconductivity)

Start with supercurrent density: $\Lambda \mathbf{J}_s = -\left\{ \mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q_s}\nabla\theta(\mathbf{r},t) \right\}$

Integration of expression for supercurrent density around a closed contour

$$\Lambda \mathbf{J}_s = -\left\{ \mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q_s} \nabla\theta(\mathbf{r},t) \right\}$$



Stoke's theorem (path C in simply or multiply connected region):

$$\oint_C \mathbf{A}\cdot d\mathbf{l} = \int_S (\nabla\times\mathbf{A})\cdot d\mathbf{s} = \int_S \mathbf{B}\cdot d\mathbf{s}$$

applied to supercurrent:

$$\oint_C (\Lambda\mathbf{J}_s)\cdot d\mathbf{l} + \int_S \mathbf{B}\cdot d\mathbf{s} = \frac{\hbar}{q_s}\oint_C \nabla\theta\cdot d\mathbf{l}$$

Integral of phase gradient: $$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \nabla\theta\cdot d\mathbf{l} = \theta(\mathbf{r}_2,t) - \theta(\mathbf{r}_1,t)$$

If $r_1 \to r_2$ (closed path), then integral $\to 0$

But → Phase only specified within modulo 2π of principal value $[-\pi,\pi]$: $q_n = q_0 + 2\pi n$

$$\oint_C \nabla\theta\cdot d\mathbf{l} = \lim_{\mathbf{r}_2\to\mathbf{r}_1}[\theta(\mathbf{r}_2,t) - \theta(\mathbf{r}_1,t)] = 2\pi n$$

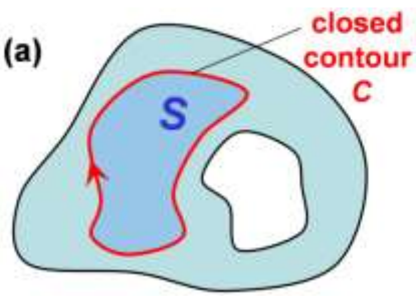
then:

$$\underbrace{\oint_C (\Lambda \mathbf{J}_s)\cdot d\mathbf{l} + \int_S \mathbf{B}\cdot d\mathbf{s}} = n\,\frac{h}{q_s} = n\,\Phi_0$$

Fluxoid is quantized

Flux quantum: $\Phi_0 \equiv \dfrac{h}{|q_s|} = \dfrac{h}{2e} = 2.067\ 833\ 636(81) \times 10^{-15}\,\mathrm{Vs}$

(a)

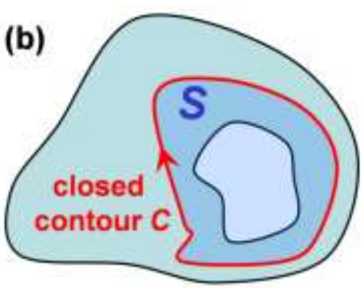


Simply connected superconductor

Quantization condition holds for all contour lines including contour that has shrunk to single point

→ $r_1 = r_2$ in limit $r_1 \rightarrow r_2$ → $n = 0$

(b)



Multiply connected superconductor

Contour line can no longer shrink to single point

→ Inclusion of non-superconducting region in contour

→ $r_1 \neq r_2$ in limit $r_1 \rightarrow r_2$ → $n \neq 0$ possible

$$\oint_C (\Lambda \mathbf{J}_s) \cdot d\mathbf{l} + \int_S \mathbf{B} \cdot d\mathbf{s} = n\,\Phi_0$$

Fluxoid quantization

Total flux = externally applied flux + flux generated by induced supercurrent
Must have discrete values

Flux Quantization (without „oid")

Superconducting cylinder, wall much thicker than λ_{L}
Application of small magnetic field at $T < T_{\mathrm{c}}$
→ Screening currents, **no** flux inside

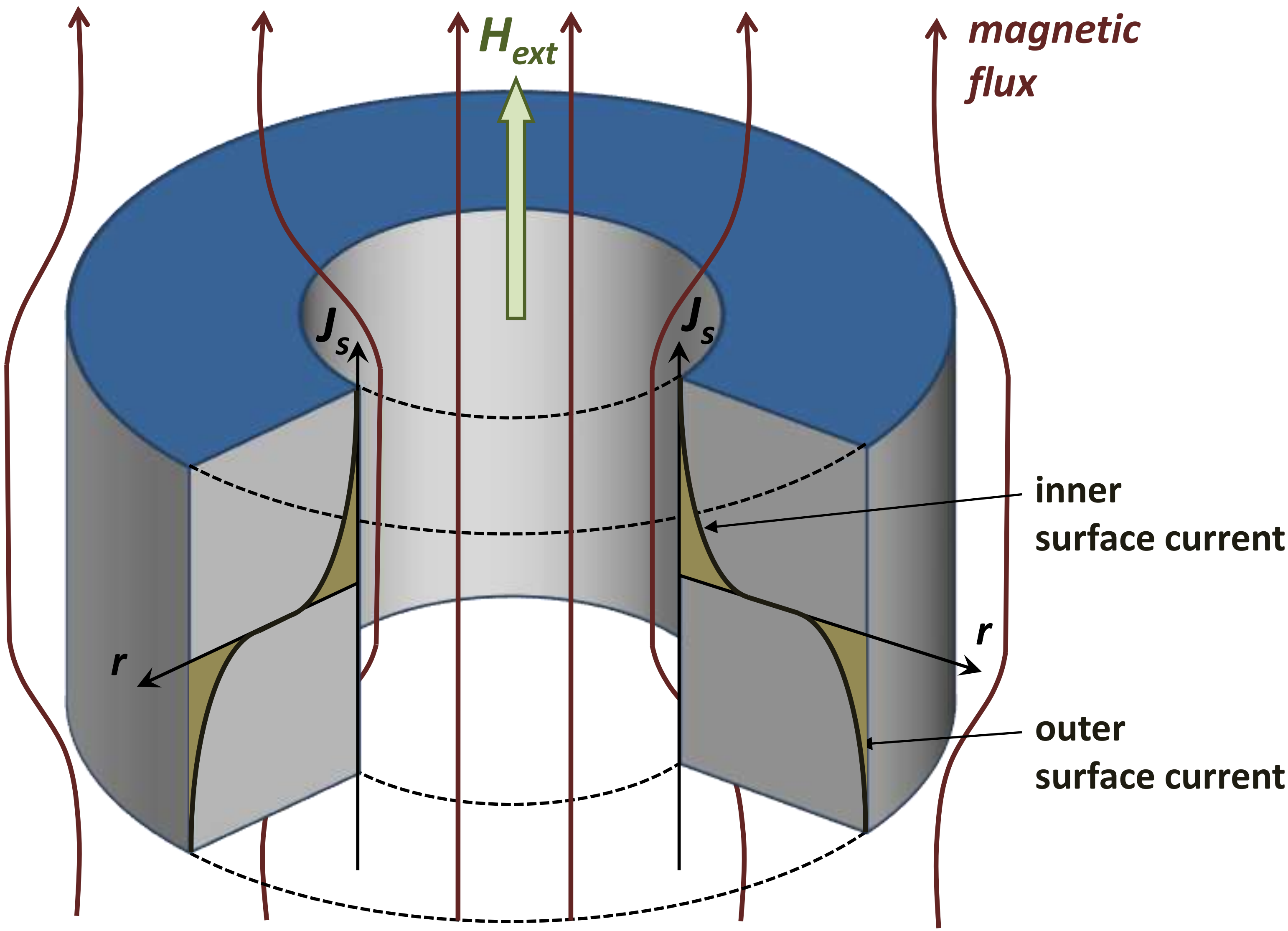
Application of H_{ext} **during cool down** → Screening current on outer **and** inner wall
Amount of flux trapped in cylinder satisfies fluxoid quantization condition
Wall thickness $\gg \lambda_{\mathrm{L}}$ → closed contour **deep inside** with $J_{\mathrm{s}} = 0$

$$\int_S \mathbf{B} \cdot d\mathbf{s} = n\,\Phi_0$$

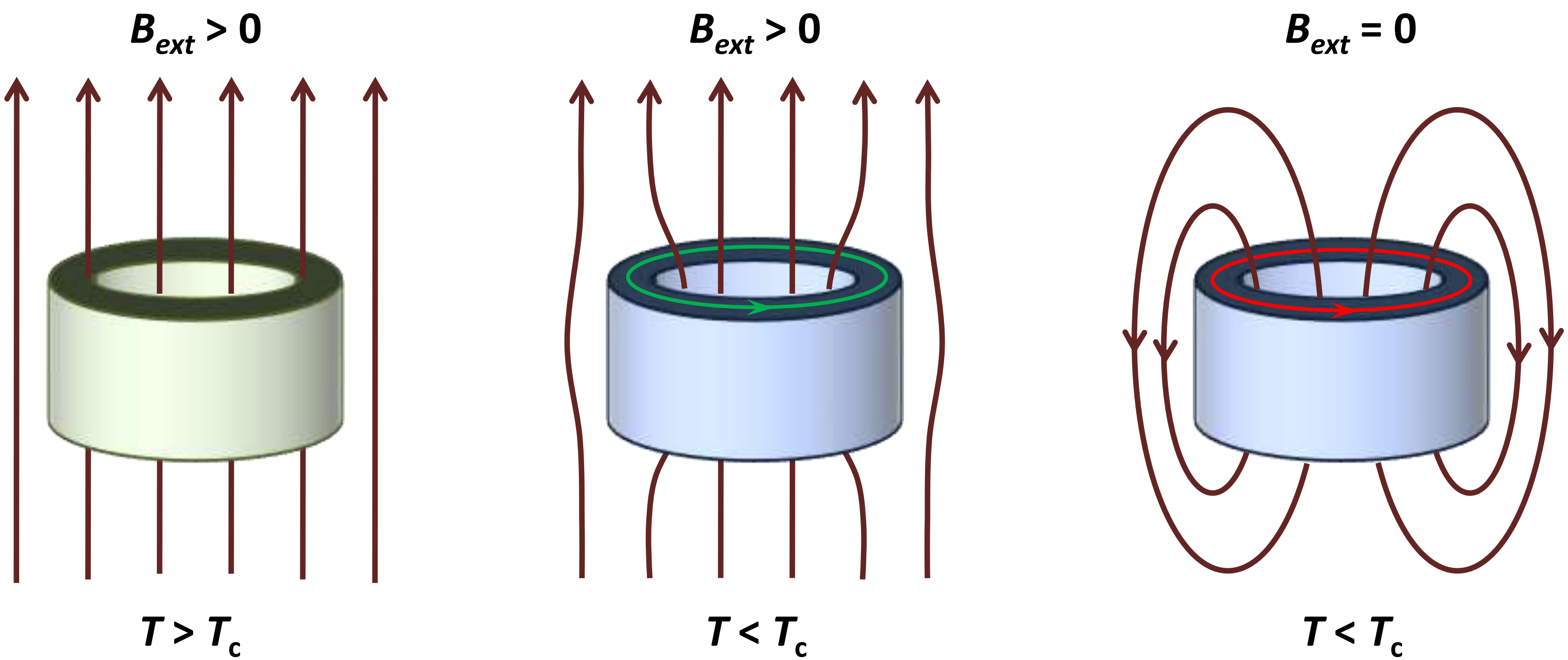
→ Flux quantization

Remove field after cooling down → Trapped flux is integer multiple of Φ_0

1.2.1 Fluxoid and Flux Quantization



1.2.1 Fluxoid and Flux Quantization



Flux Trapping: why is flux not expelled after switching off external field

– $\partial J_s/\partial t = 0$ according to 1st London equation: ***E*** = 0 deep inside (supercurrent only on surface within λ_L)

$$\frac{\partial}{\partial t}(\Lambda \mathbf{J}_s) = \mathbf{E}$$

with $\mathbf{E} = -\partial \mathbf{A}/\partial t - \nabla\phi$ and $\nabla\phi = 0$ we get:

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{\partial}{\partial t}\oint \mathbf{A}\cdot d\mathbf{l} = -\frac{\partial}{\partial t}\int_S \mathbf{B}\cdot d\mathbf{s} = -\frac{\partial\Phi}{\partial t}$$

Φ: magnetic flux enclosed in loop

contour deep inside the superconductor: ***E*** = 0 and therefore $\frac{\partial\Phi}{\partial t} = 0$

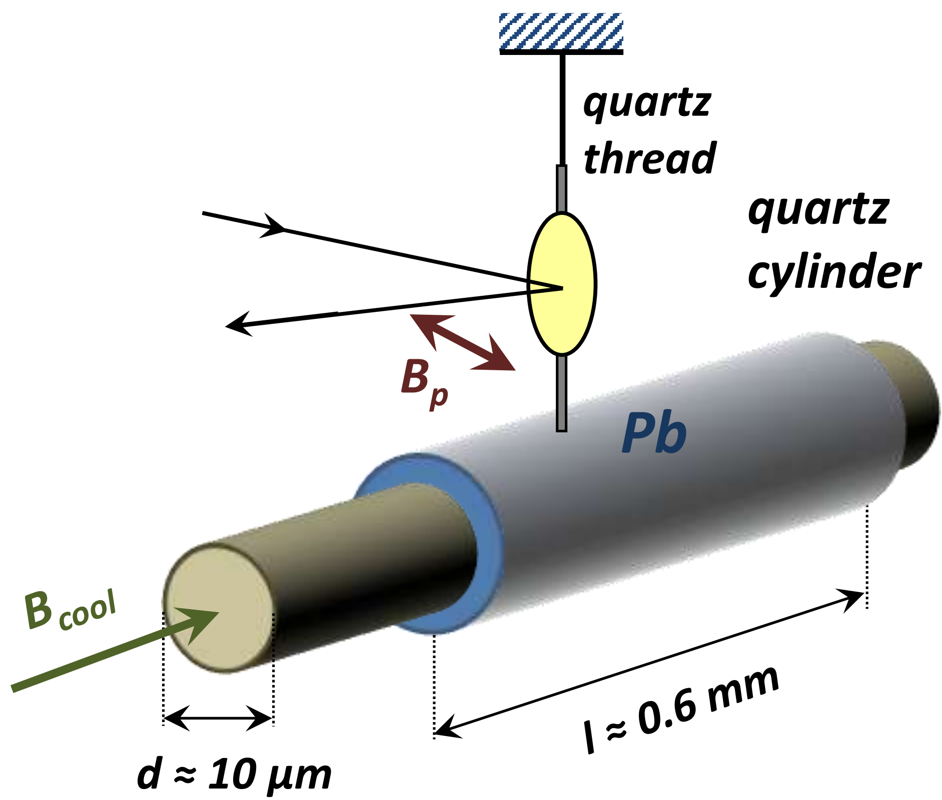
→ **Flux enclosed in cylinder stays constant**

1.2.2 Experimental Proof of Flux Quantization

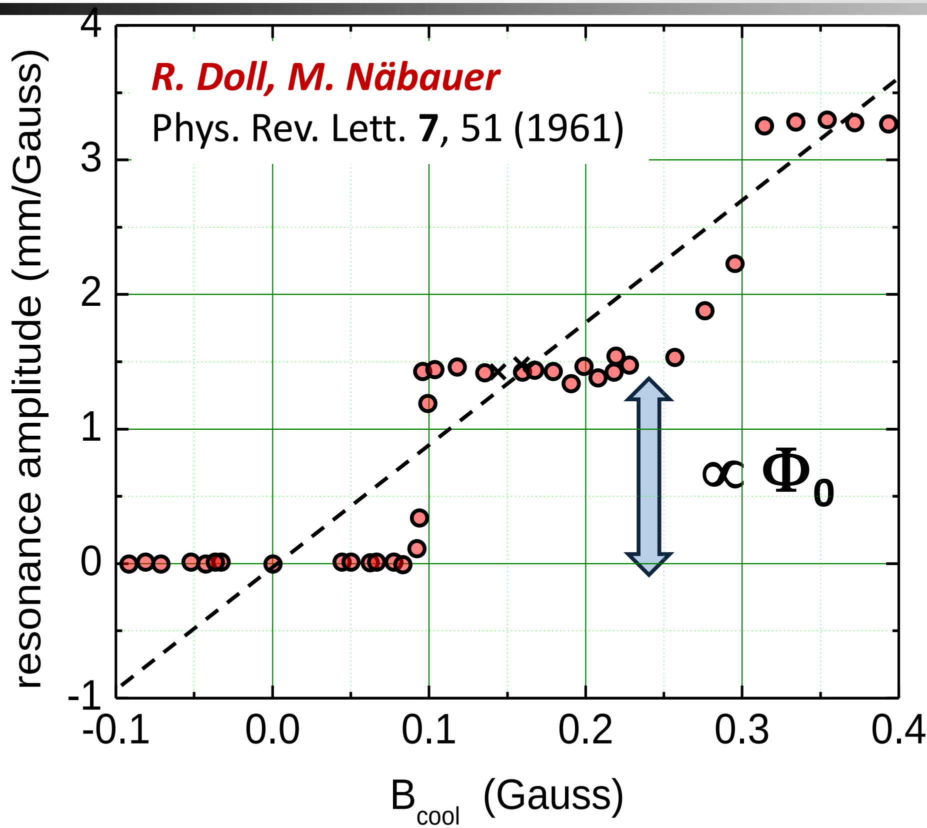
- 1961 by **Doll/Näbauer** at Munich, **Deaver/Fairbanks** at Stanford

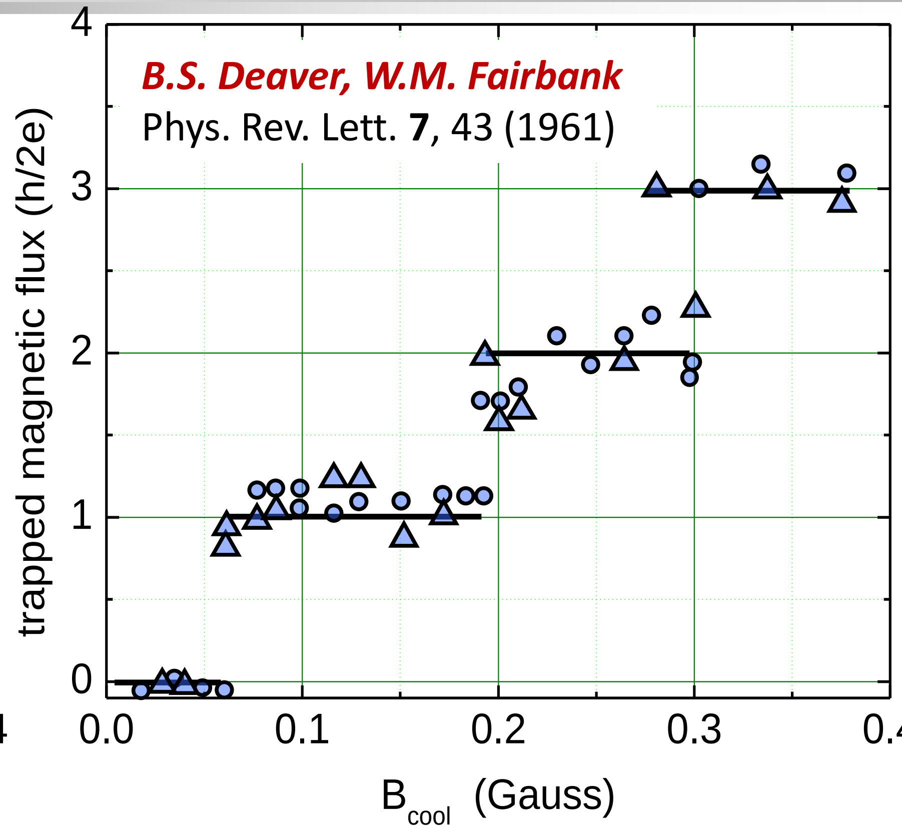
  → ***quantization of magnetic flux*** in superconducting cylinder
  → ***Cooper pairs*** with $\boldsymbol{q_s = -2e}$

→ Cylinder with wall thickness $\gg \lambda_{\mathrm{L}}$
→ Different amounts of flux are frozen in during cooling down in B_{cool}
→ Measure amount of trapped flux
→ Demanding! Required:
Large relative changes of magnetic flux
Small fields
Small diameter d
→ For $d = 10\ \mu\mathrm{m}$ we need:
2×10^{-5} T for one flux quantum
Measurement of (very small) torque
$\boldsymbol{D} = \boldsymbol{\mu} \times \boldsymbol{B}_{\mathrm{p}}$ due to probe field $\boldsymbol{B}_{\mathrm{p}}$
→ Resonance method:
Amplitude of rotary
oscillation ∝ exciting torque



1.2.2 Experimental Proof of Flux Quantization





$$\Phi_0 = \frac{h}{2e}$$

Prediction by Fritz London: h/e

→ First experimental evidence for the existence of Cooper pairs

Paarweise im Fluss, D. Einzel and R. Gross, Physik Journal **10**, No. 6, 45-48 (2011)

1.3 Josephson Effect

Brian David Josephson (born 04. 01. 1940)

Nobel Prize in Physics 1973

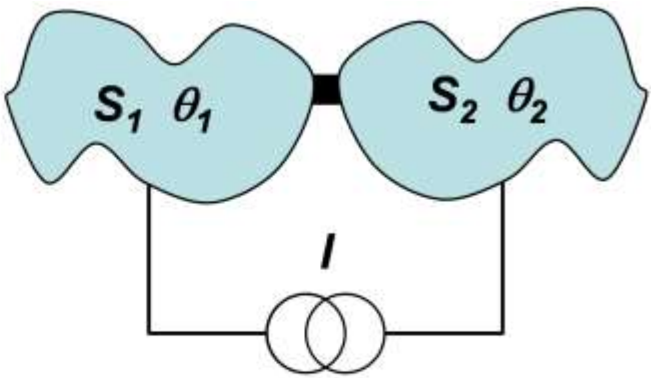
"for his theoretical predictions of the properties of a supercurrent through a tunnel barrier, in particular those phenomena which are generally known as the Josephson effects"

(together with Leo Esaki and Ivar Giaever)

1.3.1 Josephson Equations

What happens if we weakly couple two superconductors?

Mechanisms → Tunneling barriers, point contacts, normal conductors, etc.
Do they form a bound state such as a molecule?
If so, what is the binding energy?



Cooper pairs can tunnel through thin insulating barrier!

→ Expectation → Extremely small

Tunneling probability for pairs $\approx (|T|^2)^2 \simeq (10^{-4})^2$

→ B. D. Josephson (1962):

Tunneling probability for pairs $\approx |T|^2$

Coherent tunneling of pairs

„Tunneling of macroscopic wave function“

→ Finite supercurrent at zero applied voltage

→ Oscillating supercurrent at constant applied voltage

} **Josephson effects**

→ Finite binding energy (Josephson coupling energy)

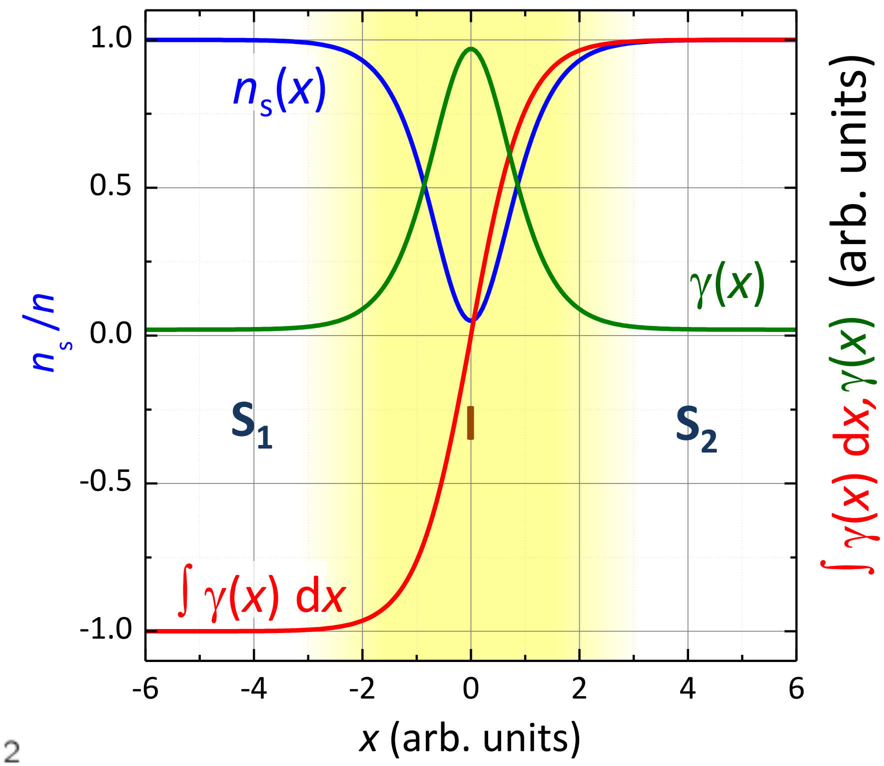
1.3.1 Josephson Equations

Coupling is weak → Supercurrent density is small → $|\Psi|^2 = n_s$ is not changed
Supercurrent density depends on gauge invariant phase gradient:

$$J_s(\mathbf{r},t) = \frac{q_s n_s \hbar}{m_s}\left[\nabla\theta(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\mathbf{A}(\mathbf{r},t)\right] = \frac{q_s n_s \hbar}{m_s}\gamma(\mathbf{r},t)$$

→ Simplifying assumptions:
Current density is homogeneous
γ varies negligibly in electrodes
$\boldsymbol{J_s}$ same in electrodes and junction area

→ γ varies in superconducting electrodes much smaller than in the tunnel barrier



→ Replace gauge invariant phase gradient γ by gauge invariant phase difference

$$\varphi(\mathbf{r},t) = \int_1^2 \gamma(\mathbf{r},t) = \int_1^2 \left(\nabla\theta - \frac{2\pi}{\Phi_0}\mathbf{A}\right)\cdot d\mathbf{l}$$

$$= \theta_2(\mathbf{r},t) - \theta_1(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot d\mathbf{l}$$

First Josephson equation:

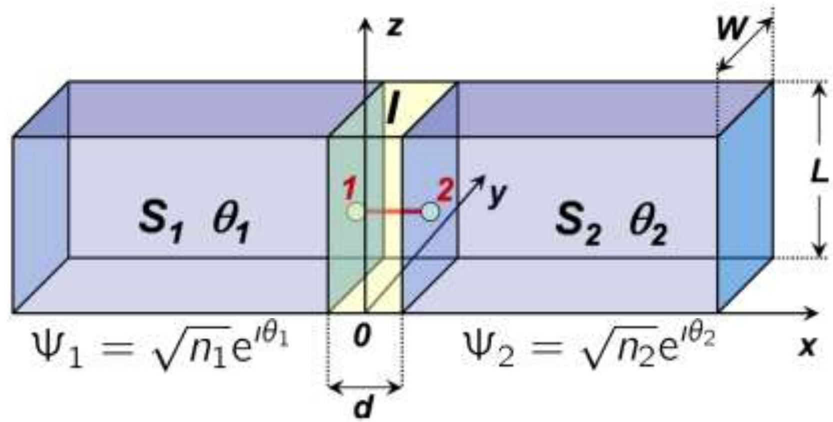
Expectation:

$$J_s = J_s(\varphi)$$

$$J_s(\varphi) = J_s(\varphi + n \cdot 2\pi)$$

$J_s = 0 \rightarrow$ Phase difference must be zero:

$$J_s(0) = J_s(n \cdot 2\pi) = 0$$



$$\boxed{J_s(\varphi) = J_c \sin\varphi + \sum_{m=2}^{\infty} J_m \sin(m\varphi)}$$

J_c: critical current density
(maximum Josephson current density)

(General formulation of 1st Josephson equation)

Weak coupling → Keep only 1st term

derived by Josephson for SIS junctions

1. Josephson equation: $J_s(\varphi) = J_c \sin\varphi$

Current – phase relation

supercurrent density varies sinusoidally with $\varphi = \theta_2 - \theta_1$ w/o external potentials

Spatially inhomogeneous supercurrent density:

$$J_s(y,z,t) = J_c(y,z) \; \sin\varphi(y,z,t)$$

1.3.1 Josephson Equations

Other argument why there are only sin contributions to Josephson current

$$\boxed{J_s(\varphi) = J_c \sin\varphi + \sum_{m=2}^{\infty} J_m \sin(m\varphi)}$$

→ Time reversal symmetry

If we reverse time, the Josephson current should flow in opposite direction
→ $t \to -t, \;\; J_s \to -J_s$

The time evolution of the macroscopic wave functions is $\propto e^{i\theta(t)} = e^{i\omega t}$
→ If we reverse time, we have

$$\varphi(t) = \theta_2(t) - \theta_1(t) \xrightarrow{t \to -t} \begin{aligned} \varphi(-t) &= \theta_2(-t) - \theta_1(-t) \\ &= -[\theta_2(t) - \theta_1(t)] \\ &= -\varphi(t) \end{aligned}$$

If the Josephson effect stays unchanged under time reversal, we have to demand

$$J_s(\varphi) = -J_s(-\varphi)$$

→ Satisfied only by sin-terms

Second Josephson equation:

Time derivative of the gauge invariant phase difference:

$$\varphi(\mathbf{r},t) = \theta_2(\mathbf{r},t) - \theta_1(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot d\mathbf{l}$$

$$\frac{\partial\varphi}{\partial t} = \frac{\partial\theta_2}{\partial t} - \frac{\partial\theta_1}{\partial t} - \frac{2\pi}{\Phi_0}\frac{\partial}{\partial t}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot d\mathbf{l}$$

Substitution of the energy-phase relation

$$-\hbar\frac{\partial\theta}{\partial t} = \frac{1}{2n_s}\Lambda\mathbf{J}_s^2 + q_s\phi$$

$$\frac{\partial\varphi}{\partial t} = -\frac{1}{\hbar}\left(\frac{\Lambda}{2n_s}\left[\mathbf{J}_s^2(2) - \mathbf{J}_s^2(1)\right] + q_s\left[\phi(2) - \phi(1)\right]\right) - \frac{2\pi}{\Phi_0}\frac{\partial}{\partial t}\int_1^2 \mathbf{A}\cdot d\mathbf{l}$$

Supercurrent density across the junction is continuous ($\boldsymbol{J}_\mathbf{s}(1) = \boldsymbol{J}_\mathbf{s}(2)$):

$$\frac{\partial\varphi}{\partial t} = \frac{2\pi}{\Phi_0}\int_1^2 \left(-\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}\right)\cdot d\mathbf{l}$$

$$E = -\nabla\phi - \frac{\partial A}{\partial t}$$

2. Josephson equation: $$\frac{\partial\varphi}{\partial t} = \frac{2\pi}{\Phi_0}\underbrace{\int_1^2 \mathbf{E}(\mathbf{r},t)\cdot d\mathbf{l}}_{\text{Voltage drop across barrier}}$$

Voltage – phase relation

Second Josephson equation:

For a constant voltage across the junction:
$$\frac{\partial \varphi}{\partial t} = \frac{2\pi}{\Phi_0} V$$
$$\varphi(t) = \varphi_0 + \frac{2\pi}{\Phi_0} V \cdot t$$

I_s oscillates at the Josephson frequency $\nu = V/\Phi_0$

$$I_s(t) = I_c \sin \varphi(t) = I_c \sin\left(\frac{2\pi}{\Phi_0} V \cdot t\right)$$

$$\frac{\nu}{V} = \frac{\omega}{2\pi V} = \frac{1}{\Phi_0} \simeq 483.597898(19)\ \frac{\mathrm{MHz}}{\mu\mathrm{V}}$$

→ Voltage controlled oscillator

Applications: Josephson voltage standard
Microwave sources

Maximum Josephson current density

Insulating tunneling barrier of thickness d

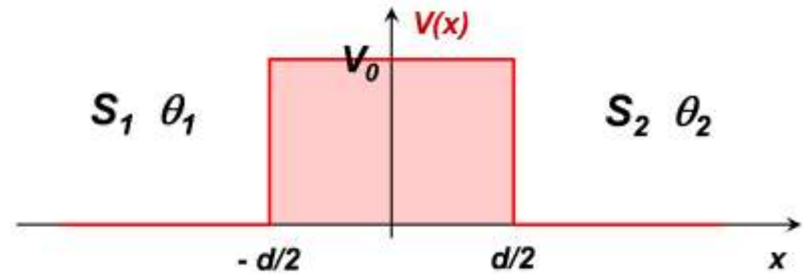
→ Calculation by wave matching method

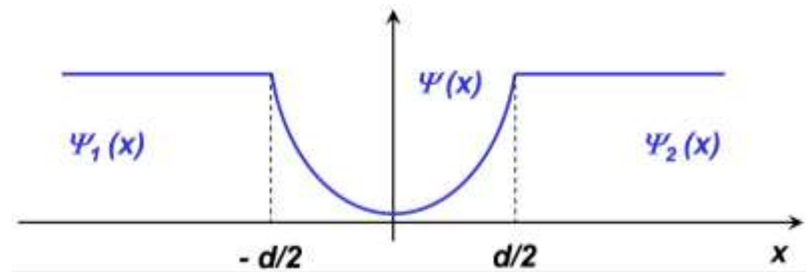
Energy-phase relation:

$$-\hbar\frac{\partial\theta}{\partial t} = \frac{1}{2n_s}\Lambda\mathbf{J}_s^2$$

$$\frac{\partial\theta}{\partial t} = -\frac{E_0}{\hbar}$$

E_0 = kinetic energy





→ Time-dependent macroscopic wave function $\psi(\mathbf{r},t) = \psi(\mathbf{r})\,\mathrm{e}^{-\imath(E_0/\hbar)t}$

Within barrier (height $V_0 > E_0$) → Only elastic processes

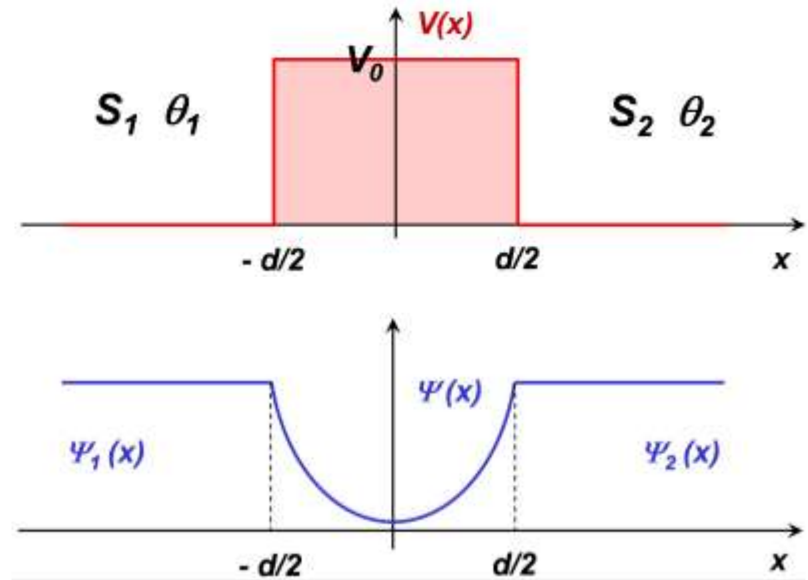
→ Time evolution is the same outside and inside barrier

→ Consider only time-independent part

→ Time-independent Schrödinger(-like) equation for region of constant potential

Maximum Josephson current density

$$-\frac{\hbar^2}{2m^\star}\nabla^2\psi(\mathbf{r}) = (E_0 - V_0)\psi(\mathbf{r})$$



barrier properties

Homogeneous barrier and supercurrent flow

→ 1D problem

Solutions:

→ Superconductor: $\psi_{1,2} = \sqrt{n_{1,2}}\,\mathrm{e}^{\imath\theta_{1,2}}$

→ Insulator: Decaying + growing exponential

$$\psi(x) = A\cosh(\kappa x) + B\sinh(\kappa x)$$

→ Characteristic decay constant: $\kappa = \sqrt{\frac{2m_s(V_0 - E_0)}{\hbar^2}}$

Coefficients A and B are determined by the boundary conditions at $x = \pm\, d/2$:

$$\psi(-d/2) = \sqrt{n_1}\,\mathrm{e}^{\imath\theta_1}$$
$$\psi(+d/2) = \sqrt{n_2}\,\mathrm{e}^{\imath\theta_2}$$

$n_{1,2}$, $\theta_{1,2}$: Cooper pair density and wave function phase at the boundaries $x = \pm\, d/2$

$$\sqrt{n_1}\,\mathrm{e}^{\imath\theta_1} = A\cosh(\kappa d/2) - B\sinh(\kappa d/2)$$
$$\sqrt{n_2}\,\mathrm{e}^{\imath\theta_2} = A\cosh(\kappa d/2) + B\sinh(\kappa d/2)$$

Maximum Josephson current density

Solving for A and B:
$$A = \frac{\sqrt{n_1}\, e^{\imath\theta_1} + \sqrt{n_2}\, e^{\imath\theta_2}}{2\cosh(\kappa d/2)} \qquad B = -\frac{\sqrt{n_1}\, e^{\imath\theta_1} - \sqrt{n_2}\, e^{\imath\theta_2}}{2\sinh(\kappa d/2)}$$

Supercurrent density
$$\mathbf{J}_s = \frac{q_s}{m_s}\Re\left\{\psi^\star\left(\frac{\hbar}{\imath}\nabla\right)\psi\right\} \rightarrow \mathbf{J}_s = \frac{q_s}{m_s}\kappa\hbar\ \Im\{A^\star B\}$$

Substituting the coefficients A and B $\mathbf{J}_s = \mathbf{J}_c\ \sin(\theta_2 - \theta_1)$

Current-phase relation

$$\mathbf{J}_c = -\frac{q_s}{m_s}\kappa\hbar\frac{\sqrt{n_1 n_2}}{2\sinh(\kappa d/2)\cosh(\kappa d/2)} = -\frac{q_s\hbar\kappa}{m_s}\frac{\sqrt{n_1 n_2}}{\sinh(2\kappa d)}$$

Real junctions:

$V_0 \approx$ few meV → $1/\kappa < 1$ nm, d ≈ few nm → $\kappa d \gg 1$ → $\sinh(2\kappa d) \simeq \frac{1}{2}\exp(2\kappa d)$

Maximum Josephson current decays exponentially with increasing thickness

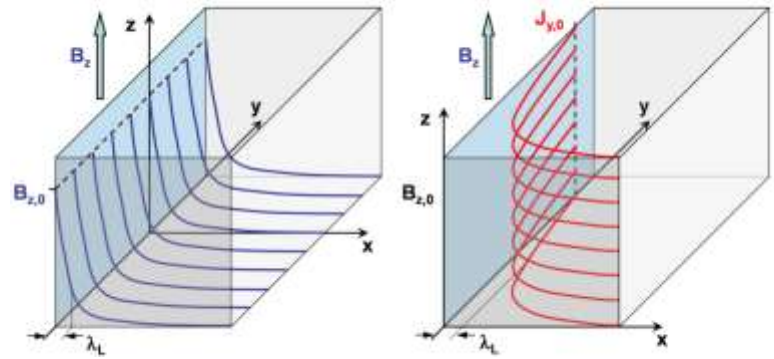
$$\mathbf{J}_c = \frac{e\hbar\kappa}{m}2\sqrt{n_1 n_2}\ \exp(-2\kappa d)$$

Summary

The supercurrent equation is $$\mathbf{J}_s = q^\star n_s^\star(\mathbf{r},t)\left\{\frac{\hbar}{m^\star}\nabla\theta(\mathbf{r},t) - \frac{q^\star}{m^\star}\mathbf{A}(\mathbf{r},t)\right\}$$

2nd London equation: $$\nabla \times (\Lambda \mathbf{J}_s) = -\nabla \times \mathbf{A} = -\mathbf{B}$$

which leads to: $$\nabla^2 \mathbf{B} = \frac{\mu_0}{\Lambda}\,\mathbf{B} = \frac{1}{\lambda_L^2}\,\mathbf{B}$$

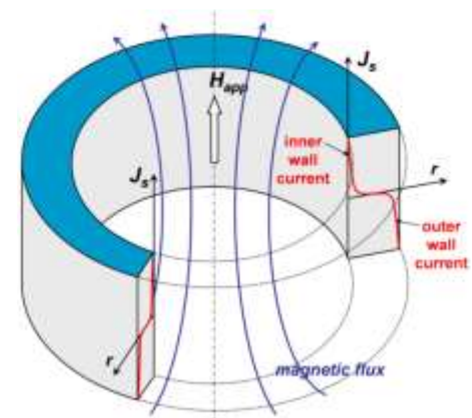


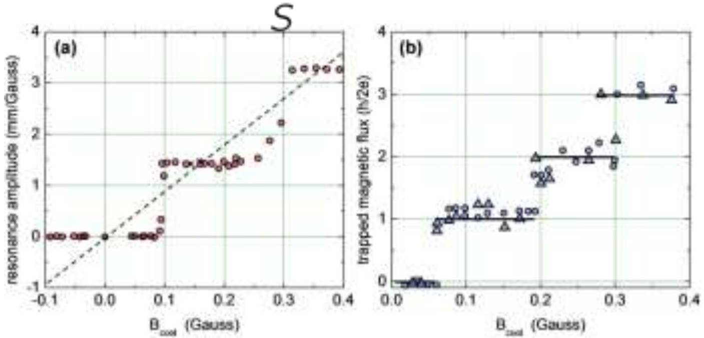
1st London equation:

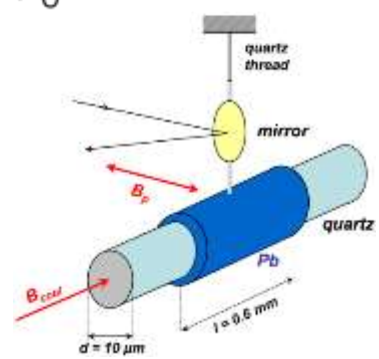
$$\frac{\partial}{\partial t}(\Lambda \mathbf{J}_s) = \mathbf{E} - \frac{1}{n_s^\star q^\star}\nabla\left(\frac{1}{2}\Lambda \mathbf{J}_s^2\right)$$

Fluxiod quantization

$$\oint_C (\Lambda \mathbf{J}_s)\cdot d\mathbf{l} + \int_S \mathbf{B}\cdot d\mathbf{s} = n\,\frac{h}{q^\star} = n\,\Phi_0$$







Summary

Macroscopic wave function $|\Psi\rangle$

describes ensemble of macroscopic number of superconducting pairs

$|\Psi|^2$ describes density of superconducting pairs

Current density in a superconductor:

$$\mathbf{J}_s = \frac{\hbar n_s q_s}{m_s}\left\{\nabla\theta(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t)\right\} \qquad \Lambda\mathbf{J}_s = -\left\{\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q_s}\nabla\theta(\mathbf{r},t)\right\}$$

Gauge invariant phase gradient:

$$\gamma(\mathbf{r},t) = \nabla\theta(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t) = \nabla\theta(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\mathbf{A}(\mathbf{r},t)$$

Phenomenological London equations:

$$\frac{\partial}{\partial t}(\Lambda\mathbf{J}_s) = \mathbf{E} \qquad \nabla\times(\Lambda\mathbf{J}_s) = -\mathbf{B} \qquad (\Lambda = m_s/n_s q_s^2 = \mu_0\lambda_L^2)$$

Flux(oid) quantization:

$$\oint_C (\Lambda\mathbf{J}_s)\cdot d\mathbf{l} + \int_S \mathbf{B}\cdot d\mathbf{s} = n\,\Phi_0$$

Summary

Josephson equations: $$\mathbf{J}_s(\mathbf{r},t) = \mathbf{J}_c(\mathbf{r},t)\sin\varphi(\mathbf{r},t)$$

$$\frac{\partial\varphi}{\partial t} = \frac{2eV}{\hbar} = \frac{2\pi}{\Phi_0}V \qquad (\omega/2\pi = 483.6\,\mathrm{GHz/mV})$$

Maximum Josephson current density

$$\mathbf{J}_s = \mathbf{J}_c\ \sin(\theta_2 - \theta_1)$$

$$\mathbf{J}_c = \frac{e\hbar\kappa}{m}\,2\sqrt{n_1 n_2}\,\exp(-2\kappa d)$$