Chapter 3

Physics of Josephson Junctions:

The Voltage State

3. Physics of Josephson junctions: The voltage state

For bias currents $I > I_s^m$

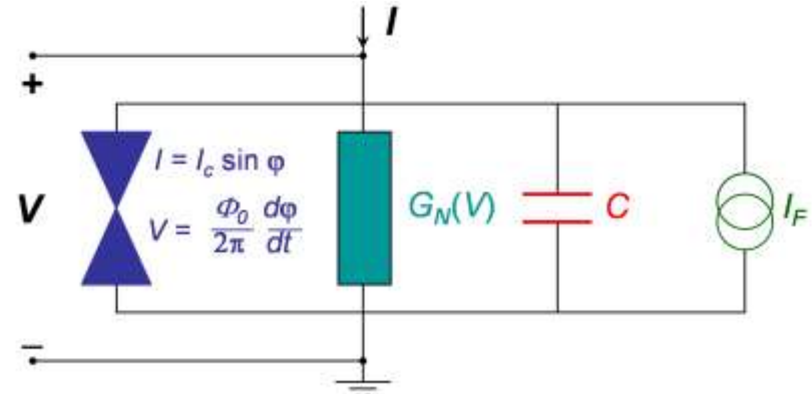
→ Finite junction voltage

→ Phase difference φ evolves in time: $\frac{d\varphi}{dt} \propto V$

→ Finite voltage state of the junction corresponds to a dynamic state

→ Only part of the total current is carried by the Josephson current

- → additional resistive channel
- → capacitive channel
- → noise channel



Key questions

→ How does the phase dynamics look like?

→ Current-voltage characteristics for $I > I_s^m$?

→ What is the influence of the resistive damping ?

At finite temperature $T > 0$

→ Finite density of “normal” electrons

→ Quasiparticles

→ Zero-voltage state: No quasiparticle current

→ For $V > 0$ → Quasiparticle current $=$ Normal current I_{N} → Resistive state

High temperatures close to T_{c}

→ For $T \lesssim T_c$ and $2\Delta(T) \ll k_B T$: (almost) all Cooper pairs are broken up

→ Ohmic current-voltage characteristic (IVC)

→ $I_{\mathrm{N}} = G_{\mathrm{N}} V$, where $G_{\mathrm{N}} \equiv \frac{1}{R_{\mathrm{N}}}$ is the normal conductance

Large voltage $V > V_{\mathrm{g}} = \frac{\Delta_1 + \Delta_2}{e}$

→ External circuit provides energy to break up Cooper pairs

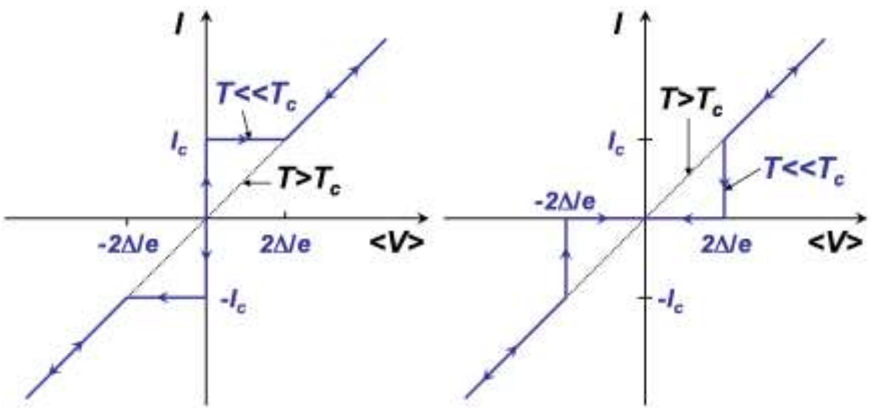
→ Ohmic IVC

For $T \ll T_{\mathrm{c}}$ and $|V| < V_g$

→ Vanishing quasiparticle density → No normal current

3.1.1 The normal current: Junction resistance

Current-voltage characteristic



Voltage state

→ Bias current I

→ $I_s(t) = I_c \sin\varphi(t)$ is time dependent

→ I_N is time dependent

→ Junction voltage $V = \frac{I_N}{G_N}$ is time dependent

→ IVC shows time-averaged voltage $\langle V \rangle$

For $T \ll T_c$ and $|V| < V_g$

→ IVC depends on sweep direction and on bias type (current/voltage)

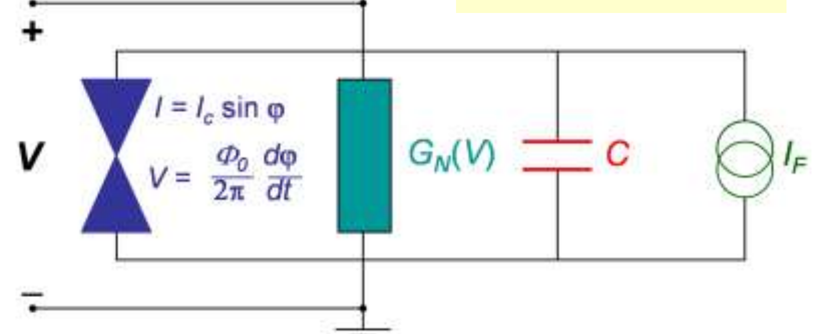
→ Hysteretic behavior

→ Current bias → $I = I_s + I_N = const.$

Equivalent conductance G_N at $T = 0$:

$$G_N(V) = \begin{cases} 0 & \text{for } |V| < 2\Delta/e \\ \frac{1}{R_N} & \text{for } |V| \geq 2\Delta/e \end{cases}$$

Circuit model



3.1.1 The normal current: Junction resistance

Finite temperature

→ Sub-gap resistance $R_{\mathrm{sg}}(T)$ for $|V| < V_{\mathrm{g}}$

→ $R_{\mathrm{sg}}(T)$ determined by amount of thermally excited quasiparticles

$$G_{sg}(T) = \frac{1}{R_{sg}(T)} = \frac{n(T)}{n_{\mathrm{tot}}} G_N$$

$n(T)$ → Density of excited quasiparticles

→ for $T > 0$ we get

$$G_N(V,T) = \begin{cases} \frac{1}{R_{sg}(T)} & \text{for } |V| < 2\Delta(T)/e \\ \frac{1}{R_N} & \text{for } |V| \geq 2\Delta(T)/e \end{cases}$$

→ Nonlinear conductance $G_{\mathrm{N}}(V,T)$

→ Characteristic voltage ($I_{\mathrm{c}} R_{\mathrm{N}}$-product)

$$V_c \equiv I_c R_N = \frac{I_c}{G_N}$$

Note: → There may be a frequency dependence of the normal channel

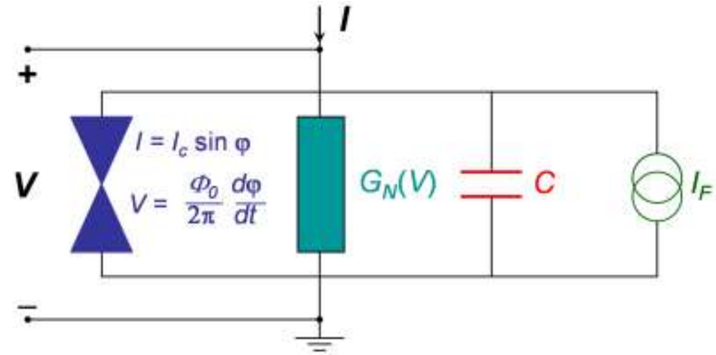
→ Normal channel depends on junction type

3.1.2 The displacement current: Junction capacitance

If $\frac{dV}{dt} \neq 0$ → Finite displacement current $I_D = C\frac{dV}{dt}$

→ C ↔ junction capacitance

→ For planar tunnel junction $C = \frac{\epsilon\epsilon_0 A_i}{d}$



→ Compare current values of different channels

With $V = L_c\frac{dI_s}{dt}$, $I_N = VG_N$, $I_D = C\frac{dV}{dt}$, $L_s = \frac{L_c}{\cos\varphi} \geq L_c$ and $G_N(V,T) = \frac{1}{R_N}$

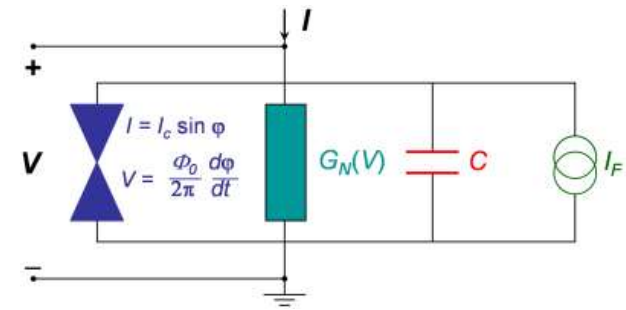
$L_c = \frac{\hbar}{2eI_c}$ Josephson inductance

$$I_s \leq \frac{V}{\omega L_c} \qquad I_N \leq \frac{V}{R_N} \qquad I_D \simeq \omega C V$$

3.1.3 Characteristic times and frequencies

Characteristic frequencies

Equivalent parallel LRC circuit
→ L_c, R_N, C
→ Three characteristic frequencies



Plasma frequency

$$\omega_p = \frac{1}{\tau_p} \equiv \frac{1}{\sqrt{L_c C}} = \sqrt{\frac{2eI_c}{\hbar C}}$$

→ $\omega_p \propto \sqrt{\frac{J_c}{C_A}}$, where $C_A \equiv \frac{C}{A}$ is the specific junction capacitance
→ $\omega < \omega_p$ → $I_D < I_s$

Inductive L_c/R_N time constant

$$\omega_c = \frac{1}{\tau_c} \equiv \frac{R_N}{L_c} = \frac{2e}{\hbar}V_c = \frac{2\pi}{\Phi_0}V_c$$

$$V_c = I_c R_N$$

→ Inverse relaxation time in the normal+supercurrent system
→ ω_c follows from V_c (2nd Josephson equation)
→ ω_c - characteristic frequency
→ $I_N < I_c$ for $V < V_c$ or $\omega < \omega_c$

Capacitive $R_N C$ time constant

$$\omega_{RC} = \frac{1}{\tau_{RC}} \equiv \frac{1}{R_N C} = \frac{\omega_p^2}{\omega_c}$$

→ $I_D < I_N$ for $\omega < \frac{1}{\tau_{RC}}$

3.1.3 Characteristic times and frequencies

Stewart-McCumber parameter and quality factor

→ Stewart-McCumber parameter $$\beta_C \equiv \frac{\omega_c^2}{\omega_p^2} = \frac{\omega_c}{\omega_{RC}} = \omega_c \tau_{RC} = \frac{2e}{\hbar} I_c R_N^2 C$$

→ Quality factor $$Q = \frac{RC}{\sqrt{LC}} = \frac{\omega_p}{\omega_{RC}} = \frac{\omega_c}{\omega_p} = \sqrt{\beta_C}$$

(Q compares the decay of oscillation amplitudes to the oscillation period)

Limiting cases

→ $\beta_C \ll 1$
- → Small capacitance and/or small resistance
- → Small $R_\mathrm{N}C$ time constants ($\tau_{RC}\omega_\mathrm{p} \ll 1$)
- → Highly damped (overdamped) junctions

→ $\beta_C \gg 1$
- → Large capacitance and/or large resistance
- → Large $R_\mathrm{N}C$ time constants ($\tau_{RC}\omega_\mathrm{p} \gg 1$)
- → Weakly damped (underdamped) junctions

3.1.4 The fluctuation current

Fluctuation/noise

→ Langevin method: include **random source** → fluctuating noise current

→ type of fluctuations: **thermal noise**, **shot noise**, **1/f noise**

Thermal Noise

Johnson-Nyquist formula for thermal noise ($k_\mathrm{B}T \gg eV, \hbar\omega$):

$$S_I(f) = \frac{4k_BT}{R_N} \qquad \text{(current noise power spectral density)}$$

$$S_V(f) = 4k_BTR_N \qquad \text{(voltage noise power spectral density)}$$

relative noise intensity (thermal energy/Josephson coupling energy):

$$\gamma \equiv \frac{k_BT}{E_J} = \frac{2e}{\hbar}\frac{k_BT}{I_c} \qquad \Rightarrow \gamma \equiv \frac{I_T}{I_C} \;\; \text{with} \;\; I_T = \frac{2e}{\hbar}k_BT$$

$I_T \equiv$ thermal noise current

$T = 4.2\ \mathrm{K} \rightarrow I_T \approx 0.15\ \mu\mathrm{A}$

Shot Noise

Schottky formula for shot noise ($eV \gg k_\mathrm{B}T$ → $V > 0.5$ mV @ 4.2 K): $S_I(f) = 2eI_N$

→ Random fluctuations due to the discreteness of charge carriers
→ Poisson process → Poissonian distribution
→ Strength of fluctuations → variance $\Delta I^2 \equiv \langle (I - \langle I \rangle)^2 \rangle$
→ Variance depends on frequency → Use noise power:

$$S(f) = \int_{-\infty}^{+\infty} \left(\langle I(t) I(0) \rangle - \langle I(0) \rangle^2 \right) dt$$

includes equilibrium fluctuations (white noise)

1/f noise

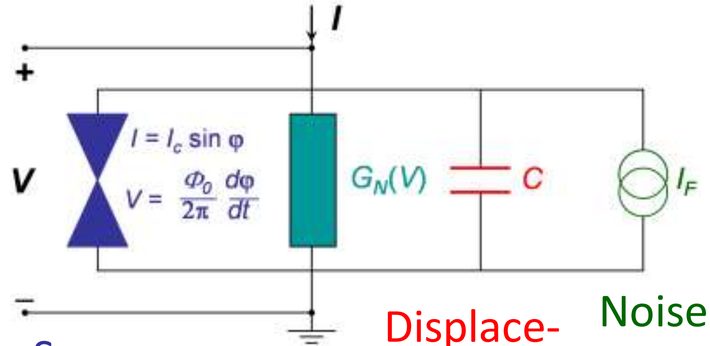
→ Dominant at low frequencies
→ Physical nature often unclear
→ Josephson junctions: dominant below about 1 Hz - 1 kHz → Not considered here

3.1.5 The basic junction equation

Kirchhoff's law: $I = I_s + I_N + I_D + I_F$

Voltage-phase relation: $\frac{d\varphi}{dt} = \frac{2eV}{\hbar}$

→ Basic equation of a Josephson junction



Super-current

Normal current

Displace-ment current

Noise current

$$\Rightarrow I = I_c \sin\varphi + G_N(V)V + C\frac{dV}{dt} + I_F$$

$$\Rightarrow I = I_c \sin\varphi + G_N(V)\frac{\Phi_0}{2\pi}\frac{d\varphi}{dt} + C\frac{\Phi_0}{2\pi}\frac{d^2\varphi}{dt^2} + I_F$$

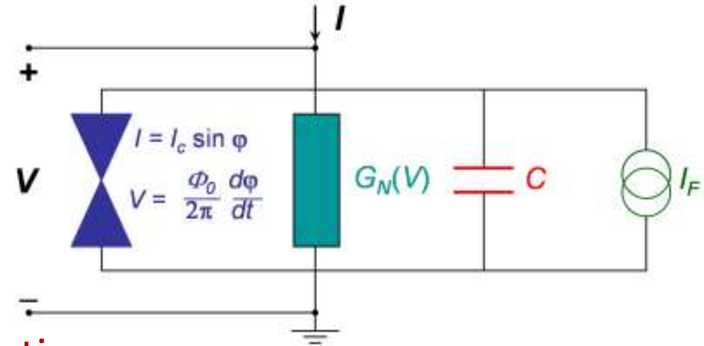
Nonlinear differential equation with nonlinear coefficients

→ Complex behavior, numerical solution

→ Use approximations (simple models)

3.2 The resistively and capacitively shunted junction (RCSJ) model

Resistively and Capacitively Shunted Junction (RCSJ) model



Approximation → $G_N(V) \equiv G = R^{-1} = \mathrm{const.}$

R = Junction normal resistance

Differential equation in dimesionless or energy formulation

$$\left(\frac{\hbar}{2e}\right) C \frac{d^2\varphi}{dt^2} + \left(\frac{\hbar}{2e}\right) \frac{1}{R}\frac{d\varphi}{dt} + I_c \left[\sin\varphi - \underbrace{\frac{I}{I_c}}_{\equiv\, i} + \underbrace{\frac{I_F(t)}{I_c}}_{\equiv\, i_F(t)}\right] = 0$$

$$\Rightarrow \left(\frac{\hbar}{2e}\right)^2 C \frac{d^2\varphi}{dt^2} + \left(\frac{\hbar}{2e}\right)^2 \frac{1}{R}\frac{d\varphi}{dt} + \frac{d}{d\varphi}\left\{E_{J0}\left[1 - \cos\varphi - i\varphi + i_F(t)\varphi\right]\right\} = 0$$

Mechanical analog

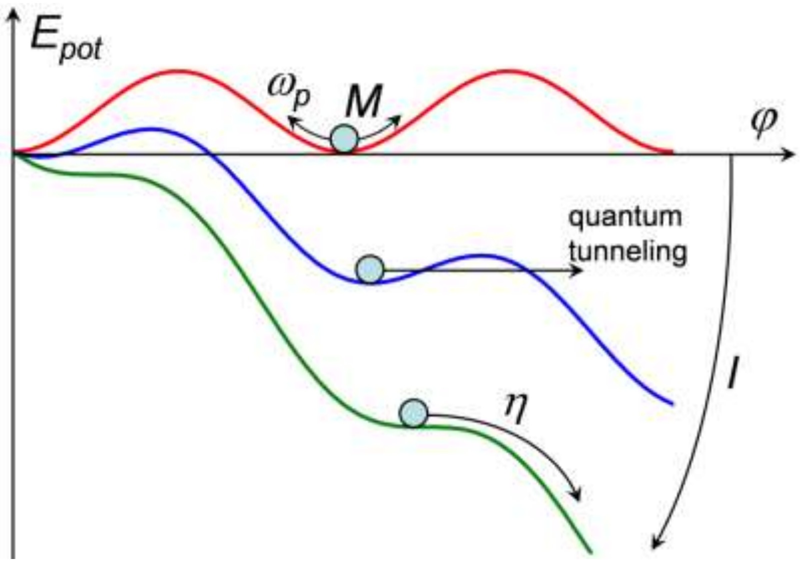
Gauge invariant phase difference ↔ Particle with mass M and damping η in potential U:

$$M\frac{d^2x}{dt^2} + \eta\frac{dx}{dt} + \nabla U = 0$$

with $M = \left(\frac{\hbar}{2e}\right)^2 C \qquad \eta = \left(\frac{\hbar}{2e}\right)^2 \frac{1}{R} \qquad U = E_{J0}\left[1 - \cos\varphi - i\varphi + i_F(t)\varphi\right]$

Tilted washboard potential

3.2 The resistively and capacitively shunted junction (RCSJ) model



Finite tunneling probability:
→ Macroscopic quantum tunneling (MQT)

Escape by thermal activation
→ Thermally activated phase slips

Normalized time: $\tau \equiv \frac{t}{\tau_c} = \frac{t}{2eI_cR/\hbar}$

Stewart-McCumber parameter: $\beta_C \equiv \frac{\omega_c^2}{\omega_p^2} = \frac{2e}{\hbar} I_c R_N^2 C$

$$\boxed{\beta_C \frac{d^2\varphi}{d\tau^2} + \frac{d\varphi}{d\tau} + \sin\varphi - i - i_F(\tau) = 0}$$

Motion of „phase particle" φ in the tilted washboard potential

Plasma frequency

→ Neglect damping, zero driving and small amplitudes ($\sin\varphi \approx \varphi$)

$$\beta_C \frac{d^2\varphi}{d\tau^2} + \varphi = 0$$

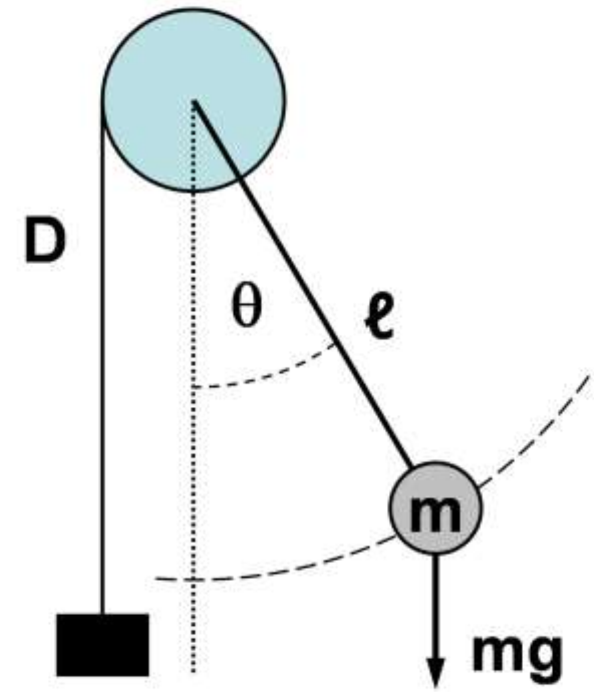
Solution: $\varphi = c \cdot \exp\left(\imath \frac{\tau}{\sqrt{\beta_C}}\right) = c \cdot \exp\left(\imath \frac{t}{\sqrt{\beta_C}\tau_c}\right) = c \cdot \exp\left(\imath\omega_p t\right)$

Plasma frequency = Oscillation frequency around potential minimum

3.2 The resistively and capacitively shunted junction (RCSJ) model

The pendulum analog

→ Plane mechanical pendulum in uniform gravitational field
→ Mass m, length ℓ, deflection angle θ
→ Torque D parallel to rotation axis
→ Restoring torque: $mg\ell\sin\theta$



Equation of motion → $D = \Theta\ddot{\theta} + \Gamma\dot{\theta} + mg\ell\sin\theta$

$\Theta = m\ell^2$ Moment of inertia
Γ Damping constant

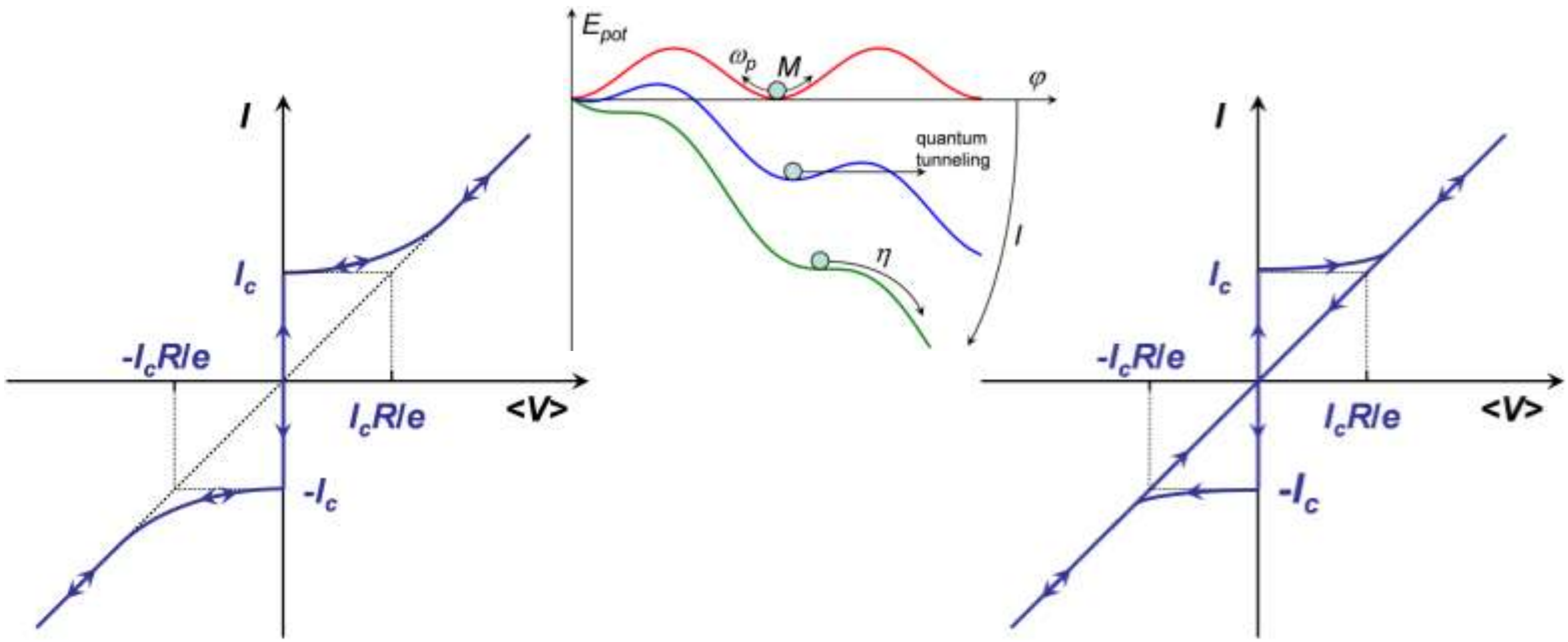
Analogies			
	I	$\leftrightarrow$	D
	I_c	$\leftrightarrow$	$mg\ell$
	$\frac{\Phi_0}{2\pi R}$	$\leftrightarrow$	Γ
	$\frac{C\Phi_0}{2\pi}$	$\leftrightarrow$	Θ
	φ	$\leftrightarrow$	θ

For $D = 0$ → Oscillations around equilibrium with

$$\omega = \sqrt{\frac{g}{\ell}} \leftrightarrow \text{Plasma frequency } \omega_p = \sqrt{\frac{2\pi I_c}{\Phi_0 C}}$$

Finite torque ($D > 0$) → Finite θ_0 → Finite, but constant φ_0 → Zero-voltage state
Large torque (deflection $> 90°$) → Rotation of the pendulum → Finite-voltage state
Voltage V $\leftrightarrow$ Angular velocity of the pendulum

3.2.1 Under- and overdamped Josephson junctions



Overdamped junction

$$\beta_C = \frac{2eI_cR^2C}{\hbar} \ll 1$$

Capacitance & resistance small

→ M small, η large

→ Non-hysteretic IVC

(Phase particle will retrap immediately at I_c because of large damping)

Underdamped junction

$$\beta_C = \frac{2eI_cR^2C}{\hbar} \gg 1$$

→ Capacitance & resistance large

→ M large, η small

→ Hysteretic IVC

(Once the phase is moving, the potential has to be tilt back almost into the horizontal position to stop ist motion)

Motivation

„Applied Superconductivity“ → One central question is

→ „How to extract information about the junction experimentally?“

Typical strategy

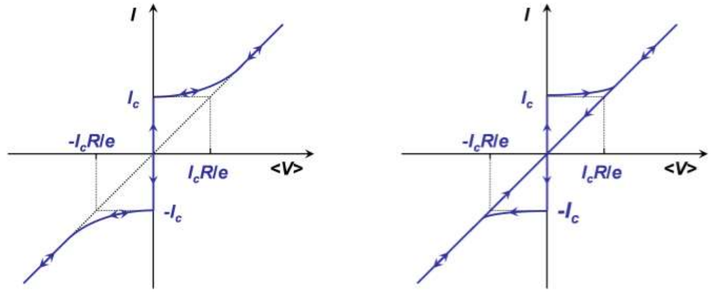
→ Drive junction with a probe signal and measure response

Examples for probe signals

→ Currents (magnetic fields)
→ Voltages (electric fields)
→ DC or AC
→ Josephson junctions → AC means microwaves!

Prototypical experiment

→ Measure junction IVC
→ Typically done with current bias



3.3.1 Response to a dc current source

T = Oscillation period

Time averaged voltage:

$$\langle V\rangle = \frac{1}{T}\int_0^T V(t)\,dt = \frac{1}{T}\int_0^T \frac{\hbar}{2e}\frac{d\varphi}{dt}dt = \frac{1}{T}\frac{\hbar}{2e}\underbrace{[\varphi(T)-\varphi(0)]}_{2\pi} = \frac{\Phi_0}{T}$$

Total current must be constant (neglecting the fluctuation source):

$$I = I_s(t) + I_N(t) + I_D(t) = I_c \sin\varphi(t) + \frac{V(t)}{R} + C\frac{dV(t)}{dt} = const$$

where: $\varphi(t) = \int_0^t \frac{2e}{\hbar} V(t)\,dt$

$I > I_c$ → Part of the current must flow as I_N or I_D

→ Finite junction voltage $|V| > 0$

→ Time varying I_s

→ $I_N + I_D$ varies in time

→ Time varying voltage, complicated non-sinusoidal oscillations of I_s,
Oscillating voltage has to be calculated self-consistently

→ Oscillation frequency $f = \langle V\rangle/\Phi_0$

3.3.1 Response to a dc current source

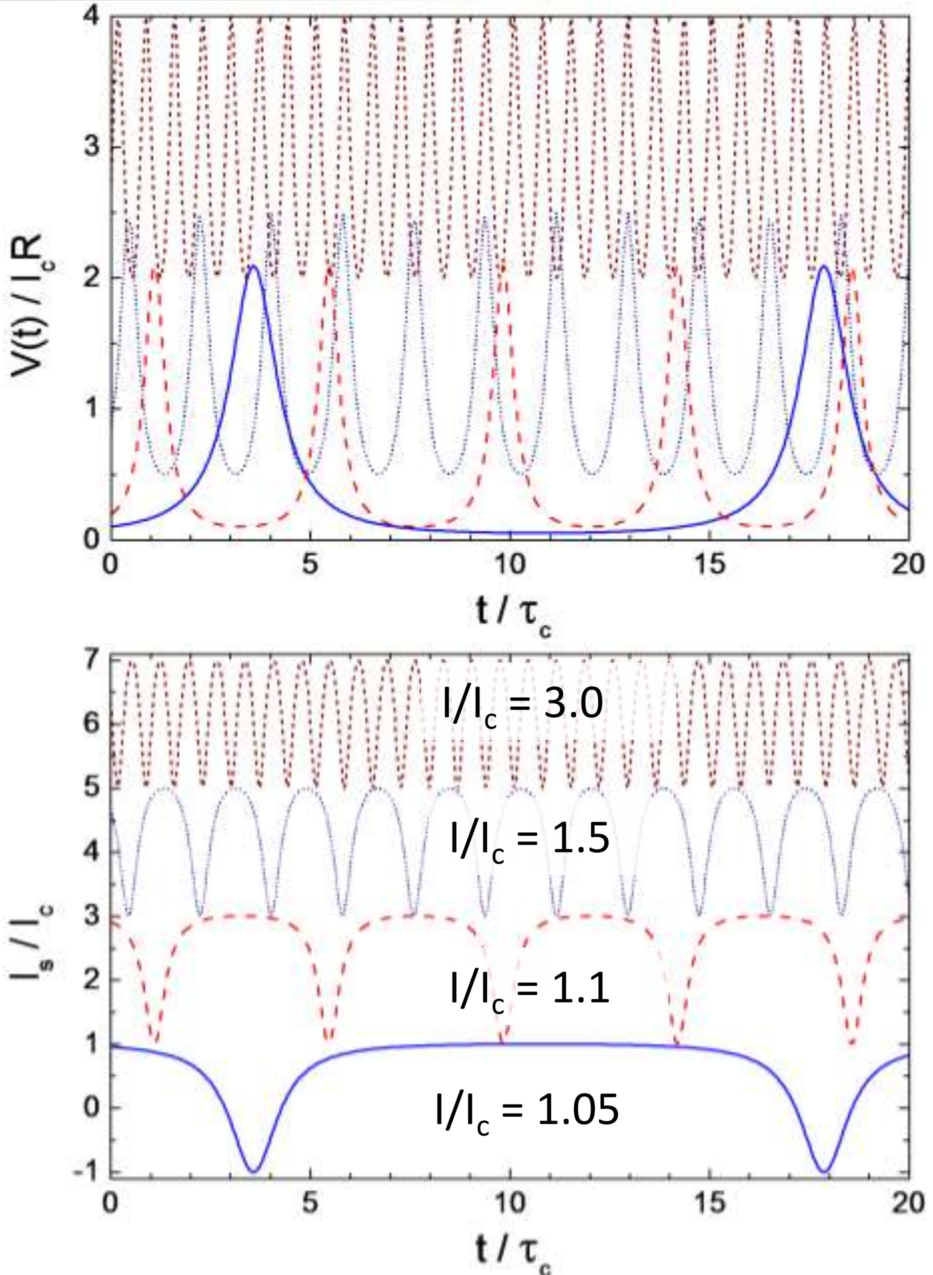
For $I \gtrsim I_c$

→ Highly non-sinusoidal oscillations
→ Long oscillation period
→ $\langle V \rangle \propto \frac{1}{T}$ is small

For $I \gg I_c$

→ Almost all current flows as normal current
→ Junction voltage is nearly constant
→ Almost sinusoidal Josephson current oscillations
→ Time averaged Josephson current almost zero
→ Linear/Ohmic IVC

→ Analogy to pendulum



3.3.1 Response to a dc current source

Strong damping

$\beta_C \ll 1$ & neglecting noise current

$$\cancel{\beta_C \frac{d^2\varphi}{d\tau^2}} + \frac{d\varphi}{d\tau} + \sin\varphi - i = 0$$

$i < 1$ → Only supercurrent, $\varphi = \sin^{-1} i$ is a solution, zero junction voltage
$i > 1$ → Finite voltage, temporal evolution of the phase

$$d\tau = \frac{d\varphi}{i - \sin\varphi}$$

Integration using

$$\int \frac{dx}{a - \sin x} = \frac{2}{\sqrt{a^2-1}} \tan^{-1}\left(\frac{-1 + a\tan(x/2)}{\sqrt{a^2-1}}\right)$$

gives

$$\tau - \tau_0 = \frac{2}{\sqrt{i^2-1}} \tan^{-1}\left(\frac{-1 + i\tan(\varphi/2)}{\sqrt{i^2-1}}\right)$$

Setting $\tau_0 = 0$ and using $\tau = \frac{t}{\tau_c}$

$$\Rightarrow \varphi(t) = 2\tan^{-1}\left\{\sqrt{1 - \frac{1}{i^2}}\tan\left(\frac{t\sqrt{i^2-1}}{2\tau_c}\right) + \frac{1}{i}\right\}$$

Periodic function with period $$T = \frac{2\pi\tau_c}{\sqrt{i^2-1}}$$

$\tan^{-1}(a\tan x + b)$
is π-periodic

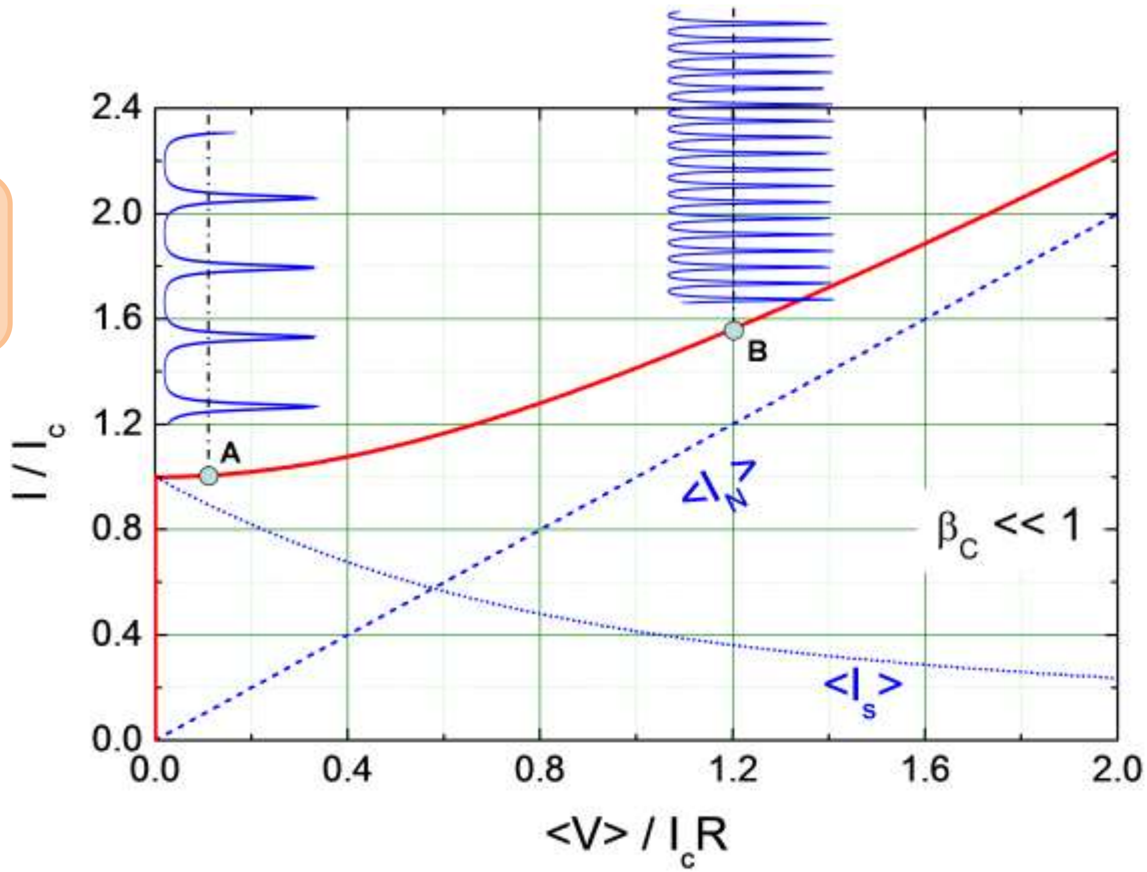
3.3.1 Response to a dc current source

with $\langle V(t) \rangle = \frac{1}{T} \int_0^T V(t)\, dt = \frac{\Phi_0}{T}$

and $\tau_c = \frac{\Phi_0}{2\pi} \frac{1}{I_c R}$ $\qquad T = \frac{2\pi\tau_c}{\sqrt{i^2 - 1}}$

We get for $i > 1$

$$\langle V(t) \rangle = I_c R \sqrt{\left(\frac{I}{I_c}\right)^2 - 1}$$



3.3.1 Response to a dc current source

Weak damping

$\beta_C \gg 1$ & neglecting noise current

$\omega_{RC} = \frac{1}{R_N C}$ is very small

→ Large C is effectively shunting oscillating part of junction voltage → $V(t) \simeq \bar{V}$

→ Time evolution of the phase

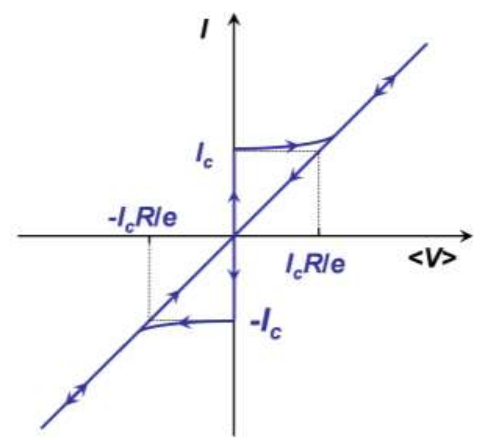
$$\varphi(t) = \frac{2e}{\hbar} \bar{V}\, t + \text{const}$$

→ Almost sinusoidal oscillation of Josephson current

$$\overline{I_s(t)} = \overline{I_c \sin\left(\frac{2e}{\hbar}\bar{V}t + \text{const}\right)} \simeq 0$$

→ Down to $\bar{V} \approx \frac{\hbar\omega_{RC}}{e} \ll (V_c = I_c R_N)$ $\Rightarrow \bar{I} = I_N(\bar{V}) = \frac{\bar{V}}{R}$

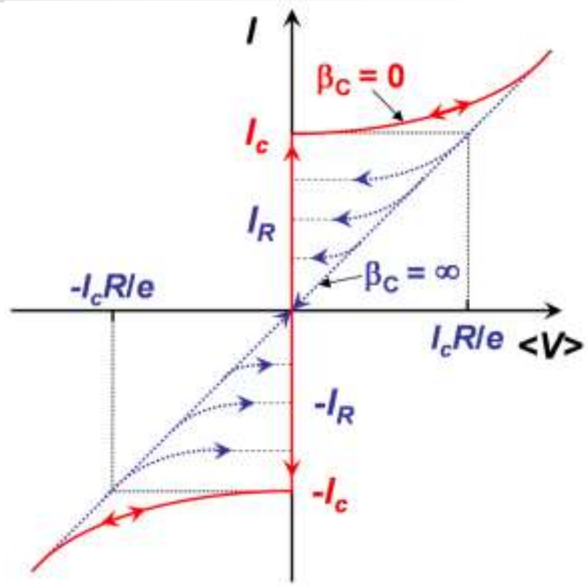
→ Corresponding current $\ll I_c$ → Hysteretic IVC



Ohmic result valid for $R_N = const.$

→ Real junction → IVC determined by voltage dependence of $R_N = R_N(V)$

3.3.1 Response to a dc current source



Intermediate damping

$\beta_C \simeq 1$

→ Numerically solve IVC

→ General trend

Increasing $\beta_C \leftrightarrow$ Increasing hysteresis

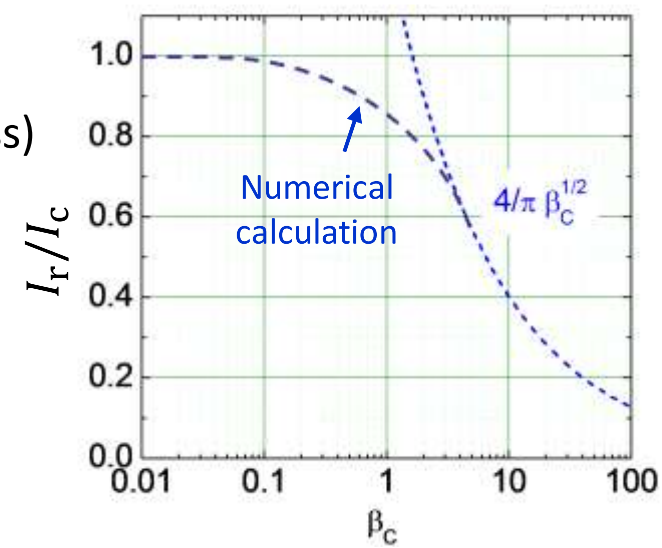
Hysteresis characterized by retrapping current I_r

→ $I_\mathrm{r} \propto$ washboard potential tilt where

Energy dissipated in advancing to next minimum = Work done by drive current

Analytical calculation possible for $\beta_C \gg 1$ (exercise class)

$$\frac{I_R}{I_c} = \frac{4}{\pi}\frac{1}{\sqrt{\beta_C}}$$



3.3.1 Response to a dc voltage source

Phase evolves linearly in time: $$\varphi(t) = \frac{2e}{\hbar} V_{dc} t + \text{const}$$

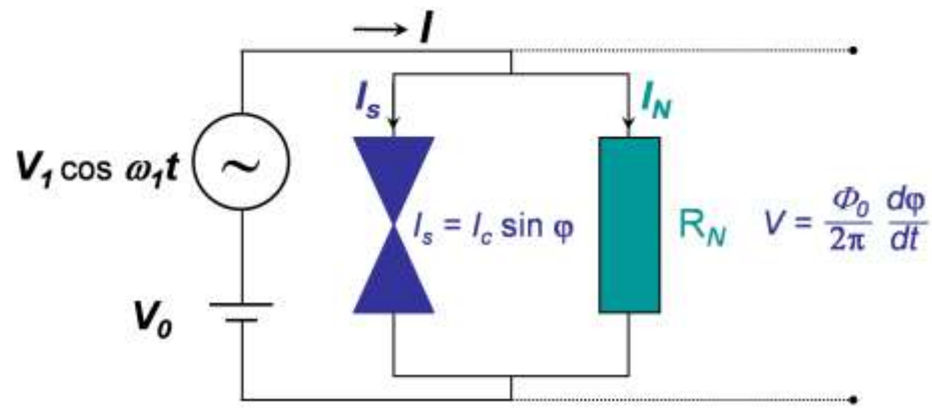
→ Josephson current I_s oscillates sinusoidally
→ Time average of I_s is zero
→ $I_\mathrm{D} = 0$ since $\frac{dV_\mathrm{dc}}{dt} = 0$
→ Total current carried by normal current → $I = \frac{V_\mathrm{dc}}{R_\mathrm{N}}$

RCSJ model → Ohmic IVC
General case $R = R_\mathrm{N}(V)$ → Nonlinear IVC

3.3.2 Response to ac driving sources

Response to an ac voltage source

Strong damping $\beta_C \ll 1$



$$V(t) = V_{\mathrm{dc}} + V_1 \cos \omega_1 t$$

Integrating the voltage-phase relation:

$$\varphi(t) = \varphi_0 + \frac{2\pi}{\Phi_0} V_{\mathrm{dc}} t + \frac{2\pi}{\Phi_0}\frac{V_1}{\omega_1} \sin \omega_1 t$$

Current-phase relation: $$I_s(t) = I_c \sin\left\{\varphi_0 + \frac{2\pi}{\Phi_0} V_{\mathrm{dc}} t + \frac{2\pi}{\Phi_0}\frac{V_1}{\omega_1} \sin \omega_1 t\right\}$$

Superposition of linearly increasing and sinusoidally varying phase

→ Supercurrent $I_s(t)$ and ac voltage V_1 have different frequencies

→ Origin → Nonlinear current-phase relation

Some maths for the analysis of the time-dependent Josephson current

Fourier-Bessel series identity: $$e^{\imath b \sin x} = \sum_{n=-\infty}^{+\infty} \mathcal{J}_n(b) e^{\imath n x}$$

$\mathcal{J}_n(b) = n^{\text{th}}$ order Bessel function of the first kind

and: $$\sin(a + b \sin x) = \Im\left\{e^{\imath(a+b\sin x)}\right\}$$

$\mathcal{J}_{-n}(b) = (-1)^n \mathcal{J}_n(b)$

$$\Rightarrow e^{\imath(a+b\sin x)} = \sum_{n=-\infty}^{+\infty} \mathcal{J}_n(b) e^{\imath(a+nx)} = \sum_{n=-\infty}^{+\infty} (-1)^n \mathcal{J}_n(b) e^{\imath(a-nx)}$$

$$\Rightarrow \sin(a + b\sin x) = \sum_{n=-\infty}^{+\infty} (-1)^n \mathcal{J}_n(b) \sin(a - nx)$$

Imaginary part

Ac driven junction → $x = \omega_1 t$, $b = \frac{2\pi}{\Phi_0 \omega_1}$ and $a = \varphi_0 + \omega_{\mathrm{dc}} t = \varphi_0 + \frac{2\pi}{\Phi_0} V_{\mathrm{dc}} t$

$$I_s(t) = I_c \sum_{n=-\infty}^{+\infty} (-1)^n \mathcal{J}_n\left(\frac{2\pi V_1}{\Phi_0 \omega_1}\right) \sin\left[(\omega_{\mathrm{dc}} - n\omega_1)t + \varphi_0\right]$$

→ Frequency ω_{dc} couples to multiples of the driving frequency

3.3.2 Response to ac driving sources

Shapiro steps

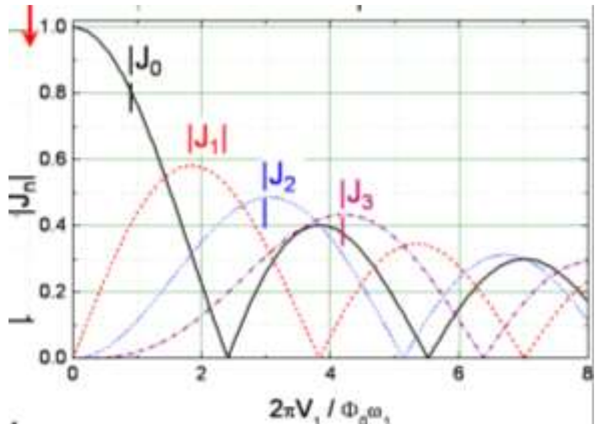
$$I_s(t) = I_c \sum_{n=-\infty}^{+\infty} (-1)^n \mathcal{J}_n\left(\frac{2\pi V_1}{\Phi_0\omega_1}\right) \sin\underbrace{[(\omega_{dc} - n\omega_1)t + \varphi_0]}$$

→ Ac voltage results in dc supercurrent if $[(\omega_{dc} - n\omega_1)t + \varphi_0]$ is time independent

$$\omega_{dc} = n\omega_1 \quad \text{or} \quad V_{dc} = V_n = n\frac{\Phi_0}{2\pi}\omega_1$$

→ Amplitude of average dc current for a specific step number n

$$|\langle I_s\rangle_n| = I_c \left|\mathcal{J}_n\left(\frac{2\pi V_1}{\Phi_0\omega_1}\right)\right|$$



$V_{dc} \neq V_n$

→ $[(\omega_{dc} - n\omega_1)t + \varphi_0]$ is time dependent

→ Sum of sinusoidally varying terms

→ Time average is zero

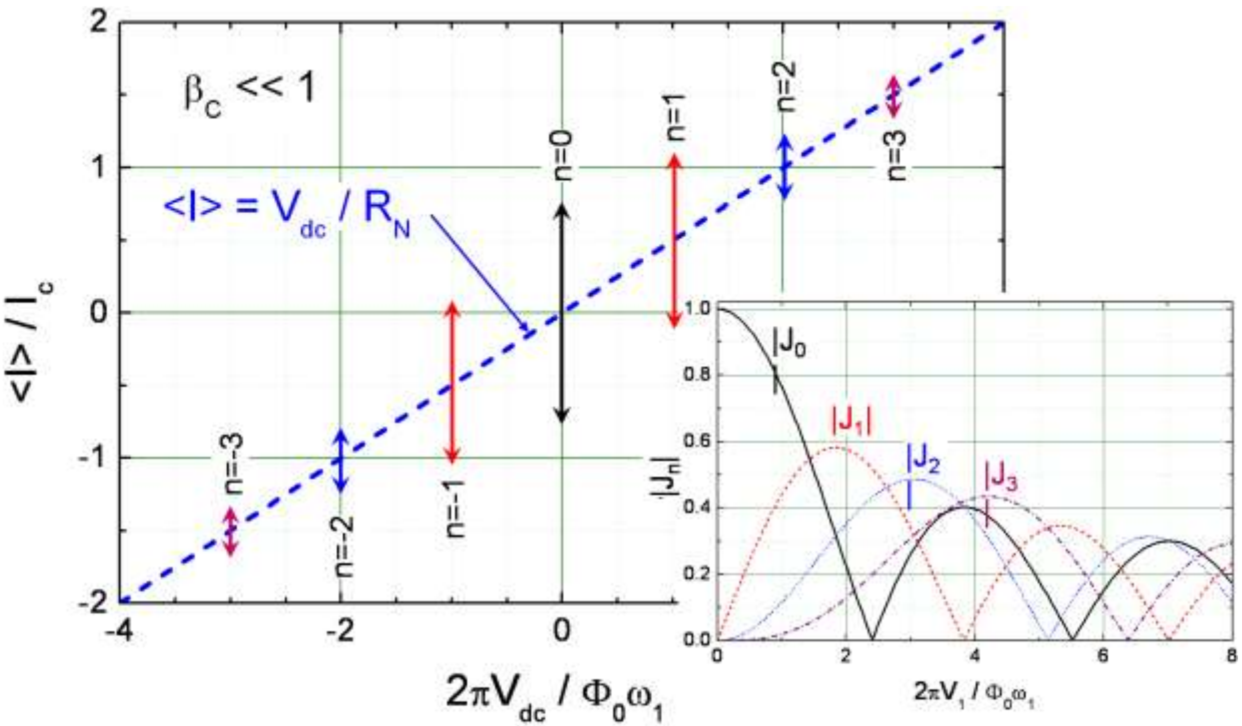
→ Vanishing dc component → $\langle I\rangle = \frac{V_{dc}}{R_N} + \langle\frac{V_1}{R_N}\cos\omega_1 t\rangle = \frac{V_{dc}}{R_N}$

3.3.2 Response to ac driving sources

→ Ohmic dependence with sharp current spikes at $V_{\mathrm{dc}} = V_n$
→ Current spike amplitude depends on ac voltage amplitude
→ n^{th} step → Phase locking of the junction to the n^{th} harmonic

$$V_n = n\frac{\Phi_0}{2\pi}\omega_1$$

$$|\langle I_s\rangle_n| = I_c\left|\mathcal{J}_n\left(\frac{2\pi V_1}{\Phi_0\omega_1}\right)\right|$$



Example: $\omega_1/2\pi = 10\ \mathrm{GHz}$

Constant dc current at $\mathrm{V_{dc}} = 0$ and $V_n = n\omega_1\frac{\Phi_0}{2\pi} \simeq n \times 20\ \mathrm{\mu V}$

3.3.2 Response to ac driving sources

Response to an ac current source

Strong damping $\beta_C \ll 1$ (experimentally relevant)

→ Kirchhoff's law (neglecting I_D) → $I_c \sin\varphi + \frac{1}{R_N}\frac{\Phi_0}{2\pi}\frac{d\varphi}{dt} = I_{dc} + I_1 \sin\omega_1 t$

Difficult to solve → Qualitative discussion with washboard potential

→ Increase I_{dc} at constant I_1

→ Zero-voltage state for $I_{dc} + I_1 \leq I_c$, finite voltage state for $I_{dc} + I_1 > I_c$

→ Complicated dynamics!

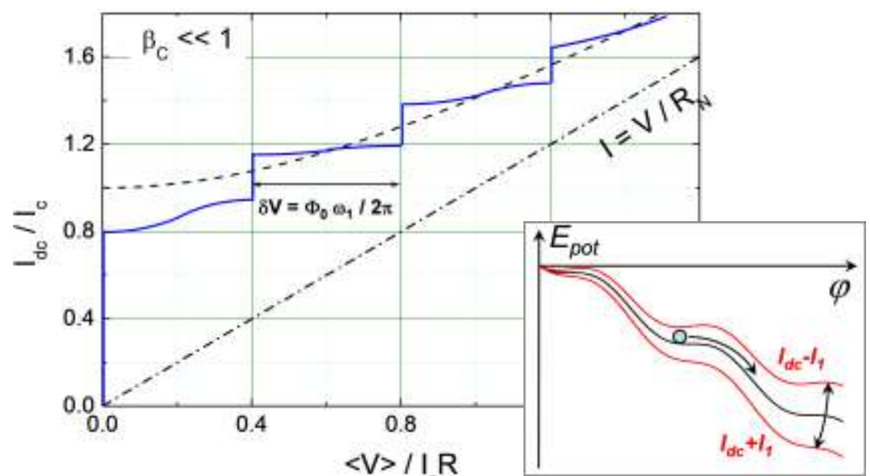
→ $V_n = n\omega_1 \frac{\Phi_0}{2\pi}$ → Motion of phase particle synchronized by ac driving

Simplifying assumption

→ During each ac cycle the phase particle moves down n minima

→ Resulting phase change $\dot{\varphi} = n\frac{2\pi}{T} = n\omega_1$

→ Average dc voltage $\langle V\rangle = n\frac{\Phi_0}{2\pi}\omega_1 \equiv V_n$

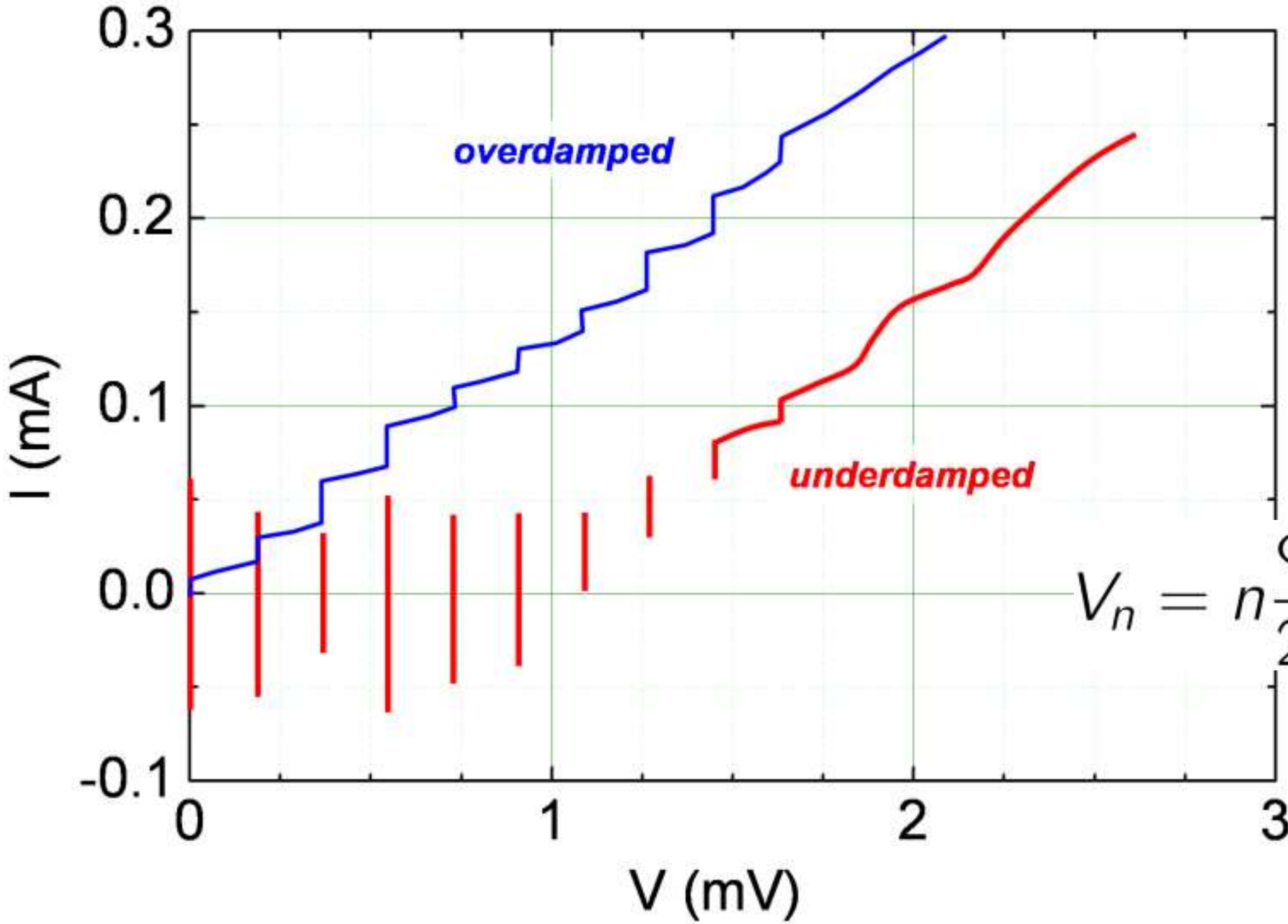


Exact analysis

→ Synchronization of phase dynamics with external ac source for a certain bias current interval → Steps

3.3.2 Response to ac driving sources

Experimental IVCs obtained for an underdamped and overdamped Niobium Josephson junction under microwave irradiation



$$V_n = n\frac{\Phi_0}{2\pi}\omega_1$$

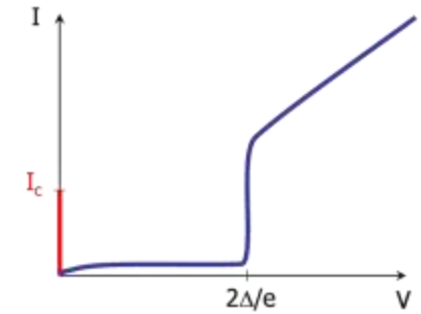
3.3.4 Photon-assisted tunneling

Superconducting tunnel junction → Highly nonlinear $R(V)$

→ Sharp step at $V_g = \frac{2\Delta}{e}$

→ Use quasiparticle (QP) tunneling current $I_{qp}(V)$

→ Include effect of ac source on QP tunneling

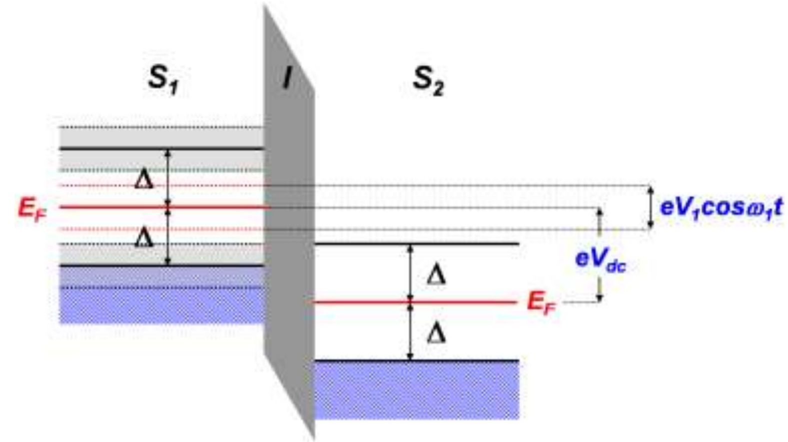


Model of Tien and Gordon:

→ Ac driving shifts levels in electrode up and down
QP energy: $E_{qp} + eV_1 \cos\omega_1 t$

→ QM phase factor

$$\exp\left(-\frac{\imath}{\hbar}\int (E_{qp} + eV_1\cos\omega_1)dt\right)$$
$$= \exp\left(-\frac{\imath}{\hbar}E_{qp}t\right)\cdot\exp\left(-\imath\frac{eV_1}{\hbar\omega_1}\sin\omega_1 t\right)$$



Bessel function identity for V_1-term → Sum of terms $\mathcal{J}_n(eV_1/\hbar\omega_1)e^{-\imath n\omega_1 t}$

→ Splitting of qp-levels into many levels $E_{qp} \pm n\hbar\omega_1$ → Modified density of states!

→ Tunneling current $I_{qp}(V) = \sum_{n=-\infty}^{+\infty} \mathcal{J}_n^2\left(\frac{eV_1}{\hbar\omega_1}\right) I_{qp}^0(V + n\hbar\omega_1/e)$

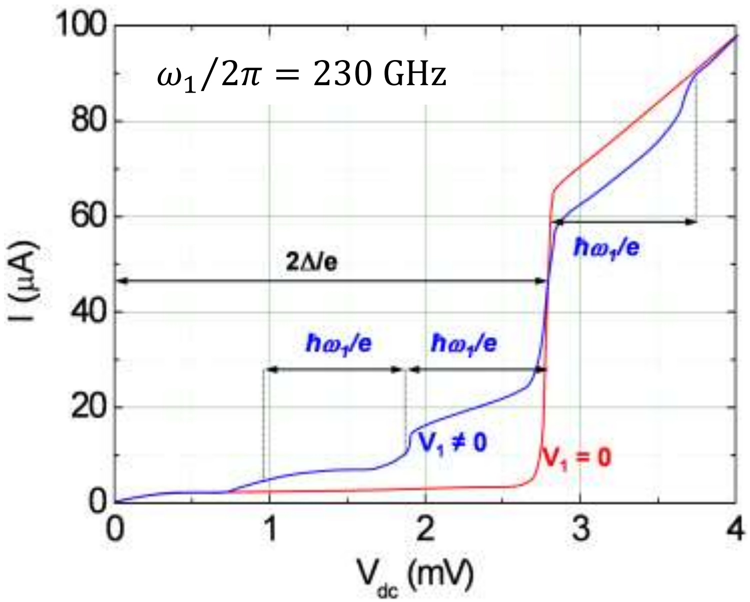
→ Sharp increase of the $I_{qp}(V)$ at $V = V_g$ is broken up into many steps of smaller current amplitude at $V_n = V_g \pm \frac{n\hbar\omega_1}{e}$

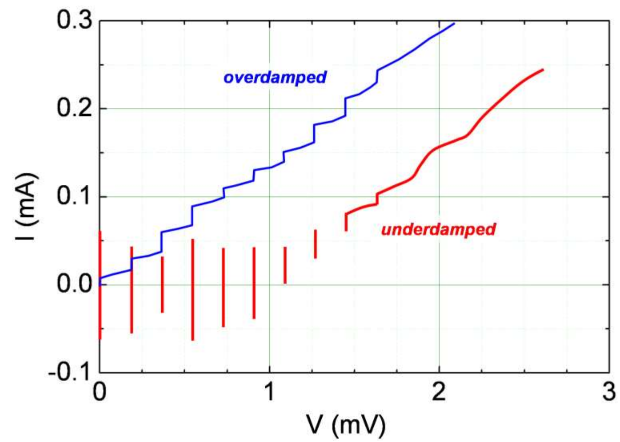
3.3.4 Photon-assisted tunneling

Example

→ QP IVC of a Nb SIS Josephson junction without & with microwave irradiation

→ Frequency $\omega_1/2\pi = 230\ \mathrm{GHz}$ corresponding to $\hbar\omega_1/e \simeq 950\ \mu\mathrm{V}$





QP steps

→ Appear at $V_n = n\frac{\hbar}{e}\omega_1$

→ Amplitude $J_n\left(\frac{eV_1}{\hbar\omega_1}\right)$

→ Broadended steps (depending on $I_{\mathrm{qp}}(V)$)

Shapiro steps

→ Appear at $V_n = n\frac{\hbar}{2e}\omega_1$

→ Amplitude $J_n\left(\frac{2eV_1}{\hbar\omega_1}\right)$

→ Sharp steps

3.4 Effect of thermal fluctuations

Thermal fluctuations with correlation function: $\langle I_F(t) I_F(t+\tau)\rangle = \frac{2k_BT}{R_N}\delta(\tau)$

Small fluctuations → Phase fluctuations around equilibrium

$S(f) = 4k_BT/R_N$

Larger fluctuations

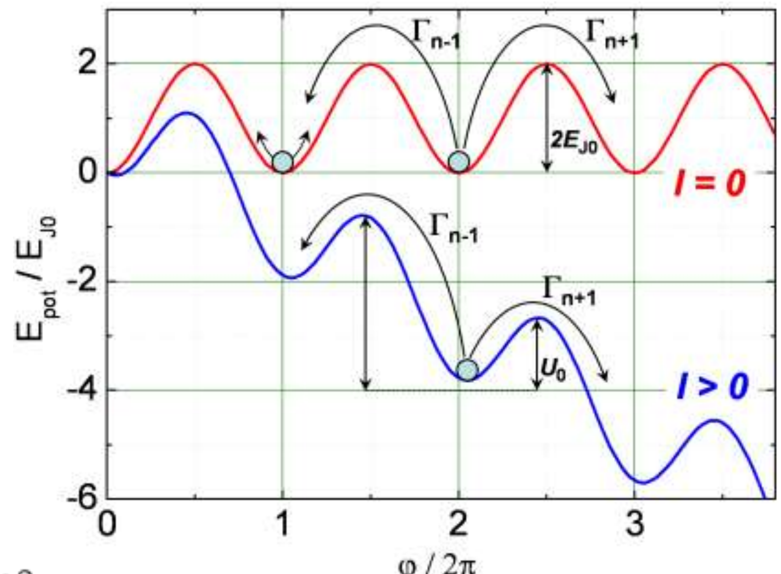
→ Increase probability for escape out of potential well

→ Escape at rates $\Gamma_{n\pm1}$

→ Escape to next minimum

→ Phase change of 2π

→ $I > 0$ → $\Gamma_{n+1} > \Gamma_{n-1}$ → $\left\langle \frac{d\varphi}{dt} \right\rangle > 0$



Langevin equation for RCSJ model

$$I = I_c \sin\varphi + \frac{1}{R_N}\frac{\Phi_0}{2\pi}\frac{d\varphi}{dt} + C\frac{\Phi_0}{2\pi}\frac{d^2\varphi}{dt^2} + I_F$$

→ Equivalent to Fokker-Planck equation:

$$\frac{1}{\omega_c}\frac{\partial\sigma}{\partial t} + \frac{\partial}{\partial\varphi}(\sigma v) + \frac{1}{\beta_C}\frac{\partial}{\partial v}(\sigma[f(\varphi) - v]) = \frac{\gamma}{\beta_C^2}\frac{\partial^2\sigma}{\partial v^2}$$

Normalized force $\quad f(\varphi) = -\frac{1}{E_{J0}}\frac{\partial U(\varphi)}{\partial\varphi} = \frac{I}{I_c} - \sin\varphi$

Normalized momentum $\quad v = \frac{d\varphi/dt}{\omega_c} = \frac{V}{I_cR_N}$

3.4 Effect of thermal fluctuations

$\sigma(v, \varphi, t)$ → Probability density of finding system at (v, φ) at time t

$$\langle X \rangle(t) = \iint\limits_{-\infty}^{+\infty} \sigma(\varphi, v, t) X(\varphi, v, t) d\varphi dv$$

statistical average of variable X

Small fluctuations

→ Static solution ($\frac{d\sigma}{dt} = 0$) $\quad \sigma(v, t) = \mathcal{F}^{-1} \exp\left(-\frac{G(\varphi, \sigma)}{k_B T}\right)$

with: $\mathcal{F} = \iint\limits_{-\infty}^{+\infty} \exp\left(-\frac{G(\varphi, \sigma)}{k_B T}\right) d\varphi dv$

→ Boltzmann distribution ($G = E - Fx$ is total energy, E is free energy)

→ Constant probability to find system in n^{th} metastable state $\quad p = \int\limits_{-\infty}^{+\infty} dv \int\limits_{\varphi \approx \varphi_n} \sigma(\varphi, v) d\varphi$

3.4 Effect of thermal fluctuations

Large fluctuations

p can change in time → $\frac{dp}{dt} = (\Gamma_{n+1} - \Gamma_{n-1})p$ Amount of phase slippage

for $\Gamma_{n+1} \gg \Gamma_{n-1}$ and $\frac{\omega_A}{\Gamma_{n+1}} \gg 1$ → $\boxed{\Gamma_{n+1} = \frac{\omega_A}{2\pi} \exp\left(-\frac{U_0}{k_B T}\right)}$

Kramers approximation
ω_A = Attempt frequency

Attempt frequency ω_A

$$\omega_A = \omega_0 = \omega_p(1 - i^2)^{1/4} \quad \text{for} \quad \omega_c\tau \gg 1, \qquad \text{(underdamped junction)}$$

$$\omega_A = \tau^{-1} = \omega_c(1 - i^2)^{1/2} \quad \text{for} \quad \omega_c\tau \ll 1 \qquad \text{(overdamped junction)}$$

Weak damping ($\beta_C = \omega_c \tau_{RC} \gg 1$)
→ $I = 0$ → $\omega_A = \omega_p$ (Oscillation frequency in the potential well)
→ $I \ll I_c$ → $\omega_A \gg \omega_p$

Strong damping ($\beta_C = \omega_c \tau_{RC} \ll 1$)
→ $\omega_p \to \omega_c$ (Frequency of an overdamped oscillator)

3.4.1 Underdamped junctions: Critical current reduction by premature switching

For $E_{J0} \gg k_B T$ → Small escape probability $\propto \exp\left(-\frac{U_0(I)}{k_B T}\right)$ at each attempt

Barrier height: $$U_0(I) \simeq 2E_{J0}\left(1 - \frac{I}{I_c}\right)^{3/2}$$

→ $2E_{J0}$ for $I = 0$

→ 0 for $I \to I_c$

Escape probability → $\omega_A/2\pi$ for $I \to I_c$

After escape → Junction switches to IR_N

Experiment

→ Measure distribution of escape current I_M

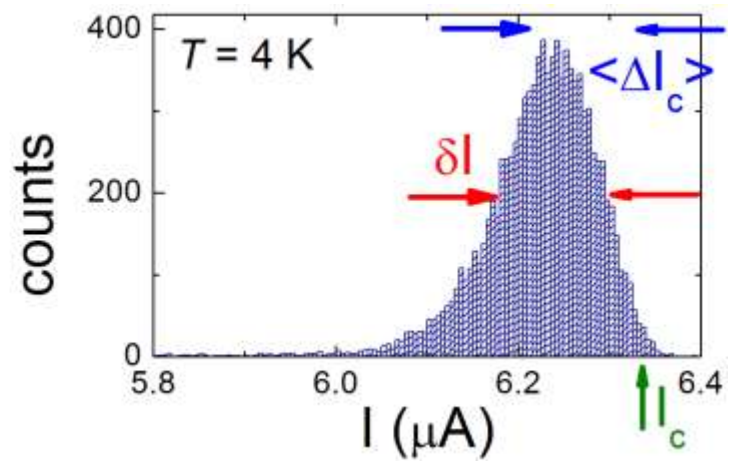
→ Width δI and mean reduction $\langle \Delta I_c \rangle = I_c - \langle I_M \rangle$

→ Use approximation for $U_0(I)$ and escape rate

$$\omega_A/2\pi \exp\left(-\frac{U_0(I)}{k_B T}\right)$$

$$\langle \Delta I_c \rangle = I_c - \langle I_M \rangle \simeq I_c \left[\frac{k_B T}{2E_{J0}} \ln\left(\frac{\omega_p \Delta t}{2\pi}\right)\right]^{2/3}$$

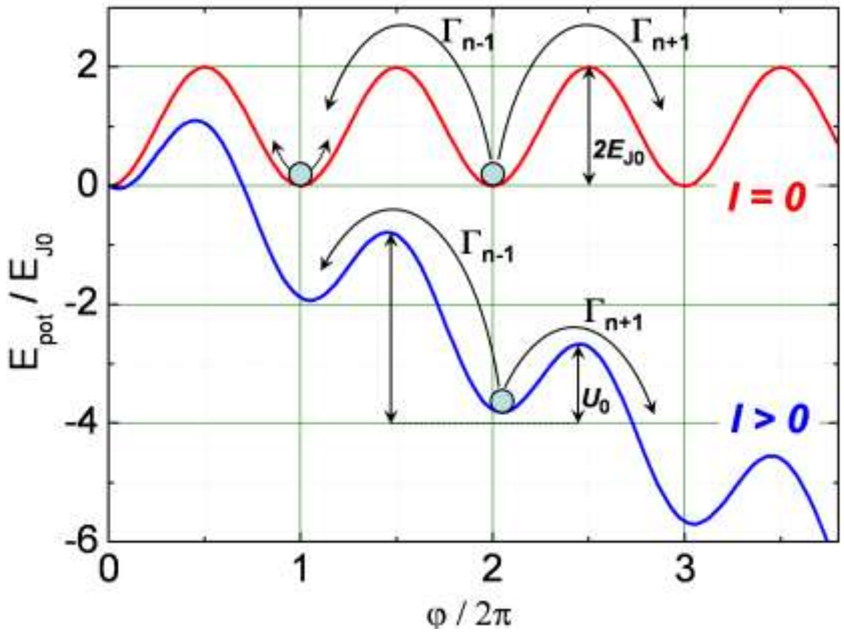
→ Considerable reduction of I_c when $k_B T > 0.05\, E_{J0}$

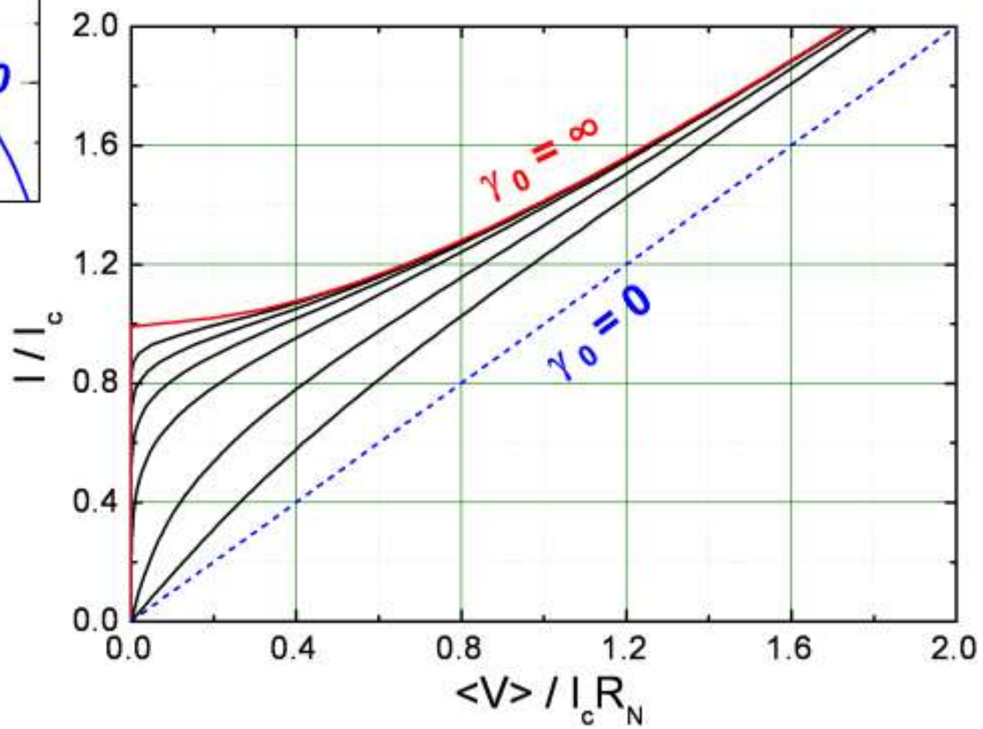


→ Provides experimental information on real or effective temperature!

3.4.2 Additional topic: Overdamped junctions - The Ambegaokar-Halperin theory

Calculate voltage $\langle V \rangle$ induced by thermally activated phase slips as a function of current





Important parameter:

$$\gamma_0(T) = \frac{2E_{J0}(T)}{k_B T} = \frac{\Phi_0 I_c(T)}{\pi k_B T}$$

3.4.2 Additional topic:
Overdamped junctions - The Ambegaokar-Halperin theory

Amgegaokar-Halperin theory

Finite amount of phase slippage

→ Nonvanishing voltage for $I \to 0$

→ Phase slip resistance for strong damping ($\beta_C \ll 1$), for $U_0 = 2E_{J0}$:

$$R_p = \lim_{I\to 0} \frac{\langle V \rangle}{I} = R_N \left\{ \mathcal{I}_0 \left[\frac{\gamma_0(T)}{2} \right] \right\}^{-2} \qquad \gamma_0(T) = \frac{2E_{J0}(T)}{k_B T} = \frac{\Phi_0 I_c(T)}{\pi k_B T}$$

Modified Bessel function

$\frac{E_{J0}}{k_B T} \gg 1$ → Approximate Bessel function → $\mathcal{I}_0(x) = e^x/2\pi\sqrt{x}$

$$\frac{R_p(T)}{R_N} \propto E_{J0} \, \exp\left(-\frac{2E_{J0}}{k_B T}\right)$$

attempt frequency

or $$\langle \dot{\varphi} \rangle \propto \frac{2eI_c R_N}{\hbar} \exp\left(-\frac{2E_{J0}}{k_B T}\right) = \omega_c \exp\left(-\frac{2E_{J0}}{k_B T}\right)$$

Attempt frequency is characteristic frequency ω_c

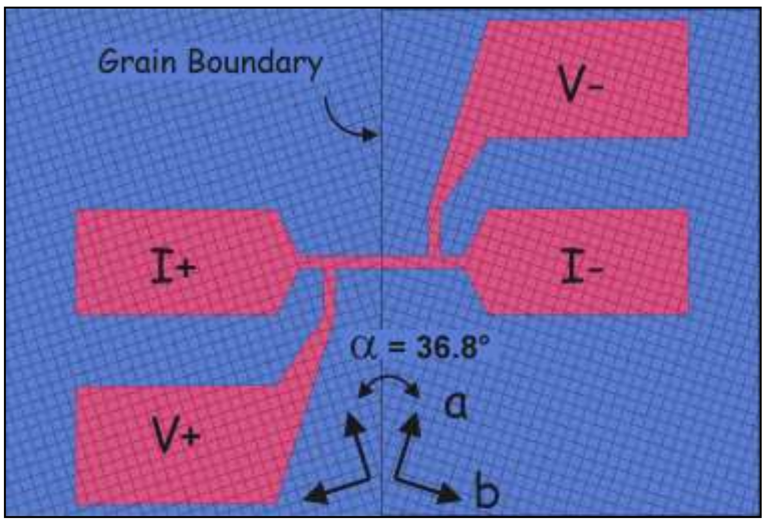
Plasma frequency has to be replaced by frequency of overdamped oscillator:

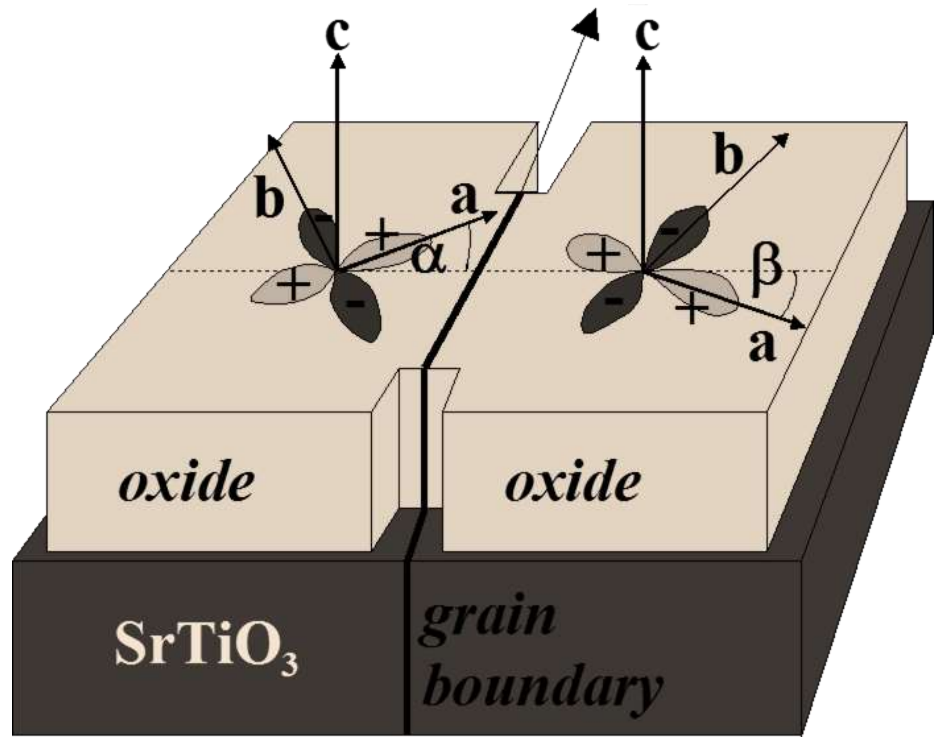
$$\omega_A = \omega_p \sqrt{\beta_C} = \omega_p \sqrt{\omega_c R_N C} = \omega_c$$

Washboard potential → Phase diffuses over barrier → Activated nonlinear resistance

Example: $YBa_2Cu_3O_7$ grain boundary Josephson junctions

→ Strong effect of thermal fluctuations due to high operation temperature



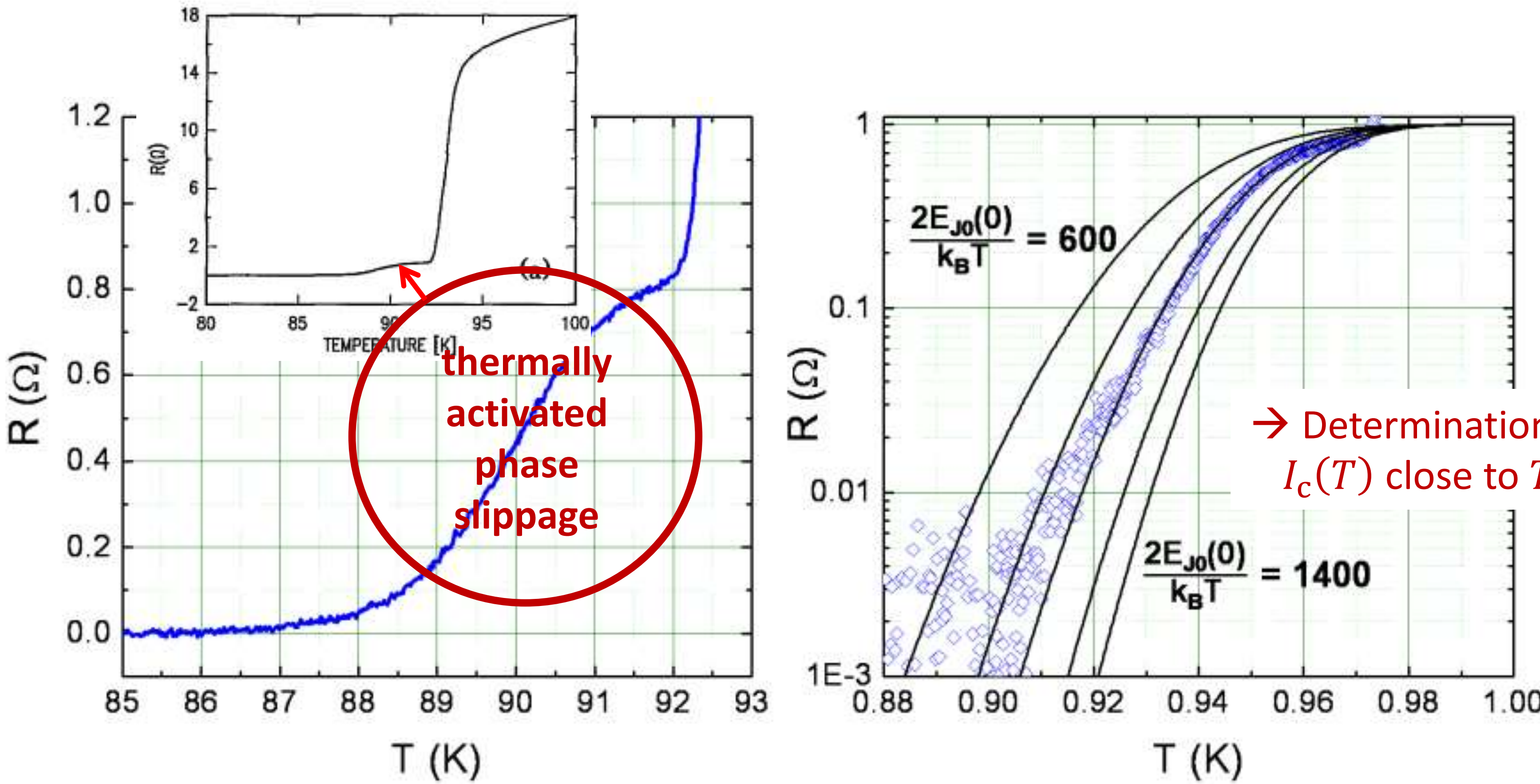


epitaxial *$YBa_2Cu_3O_7$* film on
$SrTiO_3$ bicrystalline substrate

R. Gross et al., Phys. Rev. Lett. 64, 228 (1990)
Nature 322, 818 (1988)

3.4.2 Additional topic: Overdamped junctions - The Ambegaokar-Halperin theory

Overdamped $YBa_2Cu_3O_7$ grain boundary Josephson junction



→ Determination of $I_c(T)$ close to T_c

R. Gross et al., Phys. Rev. Lett. 64, 228 (1990)

3.5 Voltage state of extended Josephson junctions

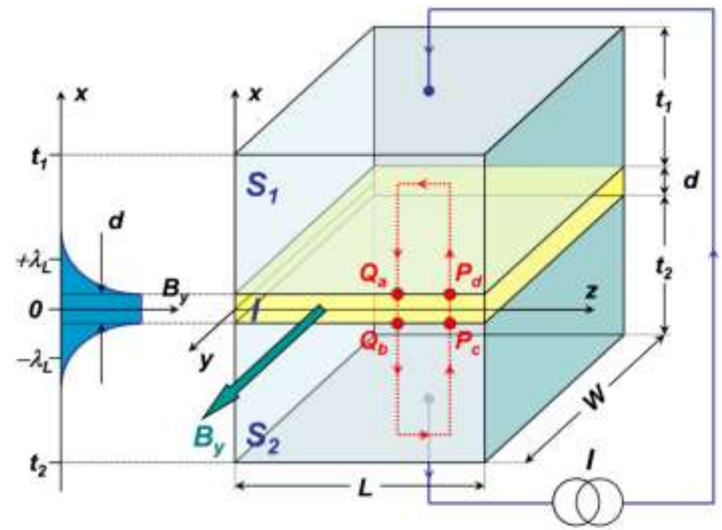
So far

→ Junction treated as lumped element circuit element
→ Spatial extension neglected

Spatially extended junctions

→ Specific geometry as as in Chapter 2
- → Insulating barrier in yz-plane
- → In-plane B field in y-direction
- → Thick electrodes $\gg \lambda_{\mathrm{L}1,2}$
- → Magnetic thickness $t_B = d + \lambda_{L,1} + \lambda_{L,2}$
- → Bias current in x-direction

→ Phase gradient along z-direction
- → $\frac{\partial\varphi(z,t)}{\partial z} = \frac{2\pi}{\Phi_0} t_B B_y(z,t)$



Expected effects

→ Voltage state → E-field and time-dependence become important
→ Short junction and long junction case

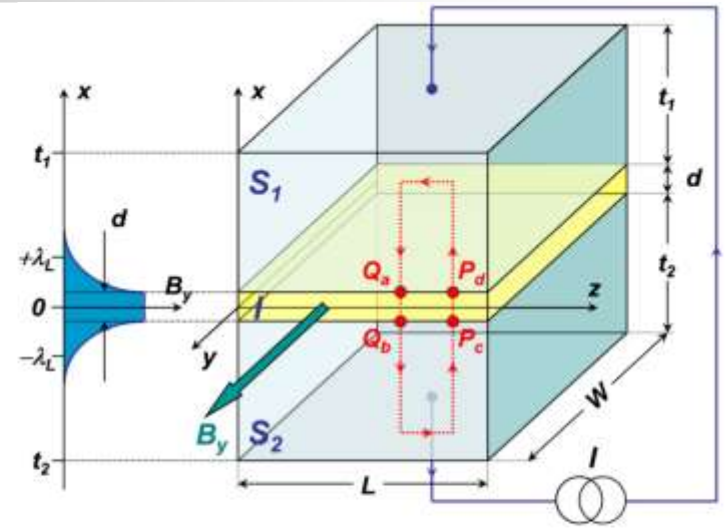
3.5.1 Negligible Screening Effects

Short junctions ($L \ll \lambda_J$) - neglecting self-fields

→ $\boldsymbol{B} = \boldsymbol{B}^{\mathbf{ex}}$

→ Junction voltage $V =$ Applied voltage V_0

→ Gauge invariant phase difference:



$$\frac{\partial \varphi}{\partial t} = \frac{2e}{\hbar} V_0 = \omega_0$$

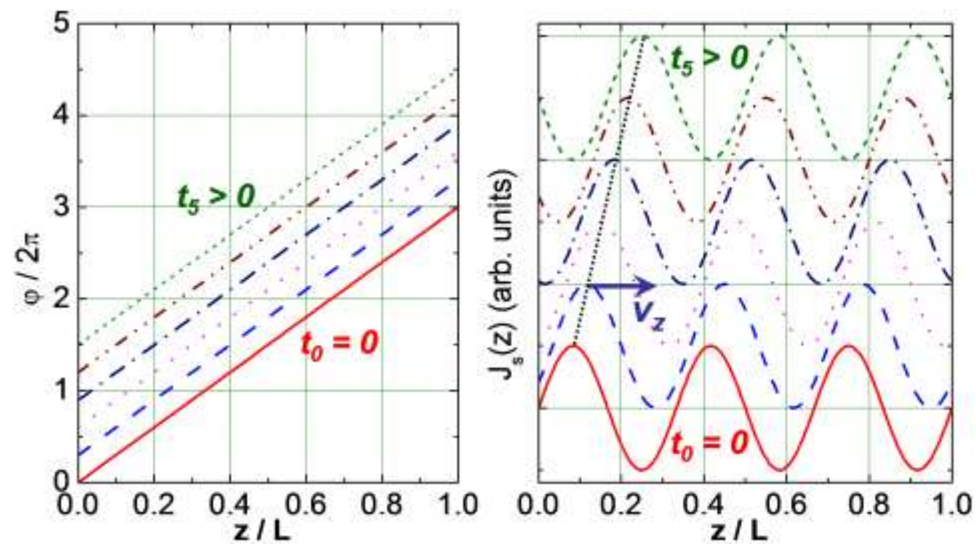
$$\frac{\partial \varphi(z,t)}{\partial z} = \frac{2\pi}{\Phi_0} t_B B_y(z,t)$$

$$\Rightarrow \varphi(z,t) = \varphi_0 + \omega_0 t + \frac{2\pi}{\Phi_0} B_y t_B \cdot z = \varphi_0 + \omega_0 t + k \cdot z$$

$$\Rightarrow J_s(z,t) = J_c \sin\left(\omega_0 t + k \cdot z + \varphi_0\right)$$

→ Josephson vortices moving in z-direction with velocity

$$v_z = \frac{\omega_0}{k} = \frac{V_0}{B_y t_B}$$



3.5.2 The time dependent Sine-Gordon equation

Long junctions ($L \gg \lambda_J$)

→ Effect of Josephson currents has to be taken into account

→ Magnetic flux density = External + Self-generated field

with **B** = μ_0**H** and **D** = ε_0**E**:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon\epsilon_0\mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

in contrast to static case, now $\partial \mathbf{E}/\partial t \neq 0$

consider 1D junction extending in z-direction, $B = B_y$, current flow in x-direction

$$\frac{\partial B_y(z,t)}{\partial z} = -\mu_0 J_x(z,t) - \epsilon\epsilon_0\mu_0 \frac{\partial E_x(z,t)}{\partial t}$$

$$\Rightarrow \frac{\partial^2\varphi(z,t)}{\partial z^2} = -\frac{2\pi}{\Phi_0} t_B \left\{ \mu_0 J_x(z,t) + \epsilon\epsilon_0\mu_0 \frac{\partial E_x(z,t)}{\partial t} \right\}$$

$$\frac{\partial\varphi(z,t)}{\partial z} = \frac{2\pi}{\Phi_0} t_B B_y(z,t)$$

with $E_x = -V/d$, $J_x = -J_c \sin\varphi$ and $\partial\varphi/\partial t = 2\pi V/\Phi_0$:

$$\frac{\partial E_x}{\partial t} = \frac{\partial^2\varphi}{\partial t^2}\frac{\Phi_0}{2\pi d}(z,t)$$

$$\frac{\partial^2\varphi(z,t)}{\partial z^2} = \underbrace{\frac{2\pi t_B \mu_0 J_c}{\Phi_0}} \sin\varphi(z,t) + \underbrace{\frac{\epsilon\epsilon_0\mu_0 t_B}{d}} \frac{\partial^2\varphi(z,t)}{\partial t^2}$$

$$\lambda_J \equiv \sqrt{\frac{\Phi_0}{2\pi\mu_0 t_B J_c}}$$

(Josephson penetration depth)

$$\overline{c} = \sqrt{\frac{d}{\epsilon\epsilon_0\mu_0 t_B}}$$

(propagation velocity)

3.5.2 The time dependent Sine-Gordon equation

$$\Rightarrow \frac{\partial^2\varphi(z,t)}{\partial z^2} - \frac{1}{\bar{c}^2}\frac{\partial^2\varphi(z,t)}{\partial t^2} - \frac{1}{\lambda_J^2}\sin\varphi(z,t) = 0$$

Time dependent
Sine-Gordon equation

with the Swihart velocity

$$\bar{c} = \sqrt{\frac{d}{\epsilon\epsilon_0\mu_0 t_B}} = \frac{1}{\sqrt{\epsilon_0\mu_0}}\sqrt{\frac{d}{\epsilon(2\lambda_L + d)}} = c\sqrt{\frac{1}{\epsilon(1+2\lambda_L/d)}}$$

$$\lambda_J \equiv \sqrt{\frac{\Phi_0}{2\pi\mu_0 t_B J_c}}$$

$\bar{c}$ = velocity of TEM mode in the junction transmission line

Example: $\varepsilon \simeq 5 - 10, \frac{2\lambda_L}{d} \simeq 50 - 100 \rightarrow \bar{c} \simeq 0.1c$

→ Reduced wavelength

→ For f = 10 GHz → Free space: 3 cm, in junction: 1 mm

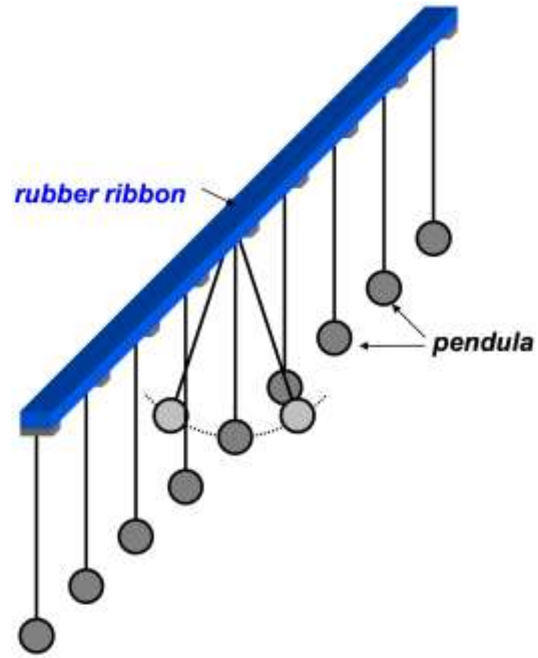
Other form of time-dependent Sine-Gordon equation

$$\lambda_J^2\frac{\partial^2\varphi(z,t)}{\partial z^2} - \frac{4\pi^2}{\omega_p^2}\frac{\partial^2\varphi(z,t)}{\partial t^2} - \sin\varphi(z,t) = 0$$

$$\omega_p^2 = 2eI_c/\hbar C \qquad C/A_i = \epsilon\epsilon_0/d \qquad I_c/A_i = J_c \qquad c^2 = 1/\epsilon_0\mu_0 \qquad \Rightarrow \omega_p/2\pi = \bar{c}/\lambda_J$$

Time-dependent Sine-Gordon equation:

$$\lambda_J^2 \frac{\partial^2 \varphi(z,t)}{\partial z^2} - \frac{4\pi^2}{\omega_p^2}\frac{\partial^2 \varphi(z,t)}{\partial t^2} - \sin\varphi(z,t) = 0$$



Mechanical analogue

→ Chain of mechanical pendula attached to a twistable rubber ribbon

→ Restoring torque $\lambda_J^2 \frac{\partial^2 \varphi}{\partial z^2}$

→ Short junction w/o magnetic field

→ $\partial^2\varphi/\partial z^2 = 0$

→ Rigid connection of pendula

→ Corresponds to single pendulum

3.5.3 Solutions of the time dependent SG equation

Simple case

→ 1D junction ($W \ll \lambda_J$), short and long junctions

$$\lambda_J^2 \frac{\partial^2 \varphi(z,t)}{\partial z^2} - \frac{4\pi^2}{\omega_p^2}\frac{\partial^2 \varphi(z,t)}{\partial t^2} - \sin\varphi(z,t) = 0$$

Short junctions ($L \ll \lambda_J$) @ low damping

→ Neglect z-variation of φ

$$\frac{\partial^2 \varphi(z,t)}{\partial t^2} + \frac{\omega_p^2}{4\pi^2}\sin\varphi(z,t) = 0$$

→ Equivalent to RCSJ model for $G_N = 0, I = 0$

Small amplitudes → Plasma oscillations

(Oscillation of φ around minimum of washboard potential)

$$C/A_i = \epsilon\epsilon_0/d$$

$$I_c/A_i = J_c$$

Long junctions ($L \gg \lambda_J$)

→ Solution for infinitely long junction → Soliton or fluxon

$$\varphi(z,t) = 4\arctan\left\{\exp\left(\pm\frac{\frac{z-z_0}{\lambda_J} - \frac{v_z}{c}t}{\sqrt{1-\left(\frac{v_z}{c}\right)^2}}\right)\right\}$$

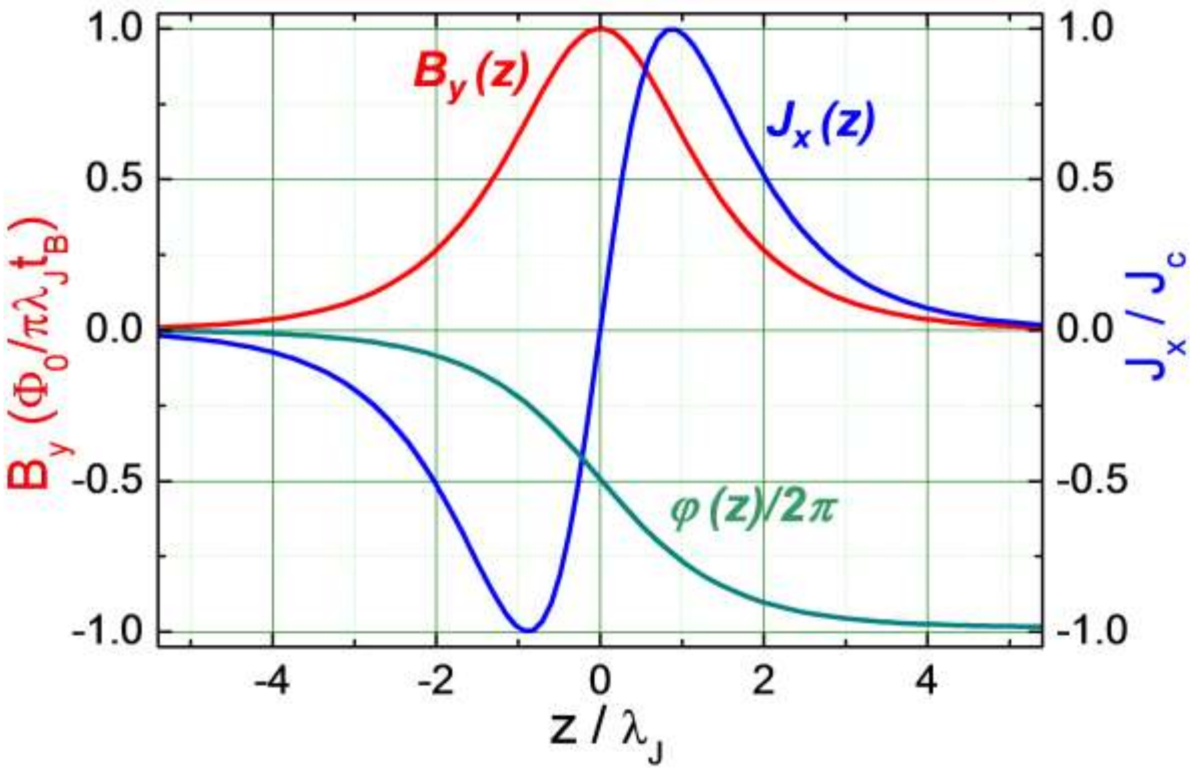
$\varphi = \pi$ at $z = z_0 + v_z t$

goes from 0 to 2π for $-\infty \to z \to \infty$

→ Fluxon (antifluxon: $\infty \to z \to -\infty$)

3.5.3 Solutions of the time dependent SG equation

$\varphi = \pi$ at $z = z_0 + v_z t$
goes from 0 to 2π for $-\infty \to z \to \infty$
→ Fluxon (antifluxon: $\infty \to z \to -\infty$)



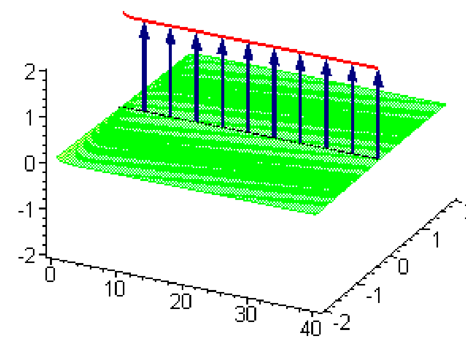
Pendulum analog
→ Local 360° twist of
rubber ribbon

Applied current
→ Lorentz force → Motion of phase twist (fluxon)

Fluxon as particle → Lorentz contraction for $v_z \to \bar{c}$

Local change of phase difference → Voltage

→ Moving fluxon = Voltage pulse



Other solutions: Fluxon-fluxon collisions, breathers, bound states,…

Josephson plasma waves

Linearized Sine-Gordon equation

$$\varphi(z,t) = \varphi_0(z) + \varphi_1(z,t)$$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

φ_1 = Small deviation

→ Approximation $\sin\varphi \simeq \sin\varphi_0 + \varphi_1\cos\varphi_0$

Substitution (keeping only linear terms):

$$\frac{\partial^2\varphi_0}{\partial z^2} + \frac{\partial^2\varphi_1(z,t)}{\partial z^2} - \frac{1}{\bar{c}^2}\frac{\partial^2\varphi_1(z,t)}{\partial t^2} - \frac{1}{\lambda_J^2}\sin\varphi_0 - \frac{1}{\lambda_J^2}\cos\varphi_0\,\varphi_1(z,t) = 0$$

φ_0 solves time independent SGE $\frac{\partial^2\varphi_0}{\partial z^2} = \lambda_\mathrm{J}^{-2}\sin\varphi_0$

$$\frac{\partial^2\varphi_1(z,t)}{\partial z^2} - \frac{1}{\bar{c}^2}\frac{\partial^2\varphi_1(z,t)}{\partial t^2} - \frac{1}{\lambda_J^2}\cos\varphi_0\varphi_1(z,t) = 0$$

φ_0 slowly varying
→ $\varphi_0 \approx const.$

3.5.3 Solutions of the time dependent SG equation

Solution: $\varphi_1(z,t) = \exp\left(-\imath[kz - \omega t]\right)$ (small amplitude plasma waves)

Dispersion relation ω(k) : $\omega^2 = \overline{c}^2 k^2 + \omega_{p,J}^2$

Josephson plasma frequency $\frac{\omega_{p,J}^2}{4\pi^2} = \frac{\overline{c}^2}{\lambda_J^2}\cos\varphi_0 = \frac{\omega_\mathrm{p}^2}{4\pi^2}\cos\varphi_0$

$\omega < \omega_\mathrm{p,J}$

→ Wave vector k imaginary → No propagating solution

$\omega > \omega_\mathrm{p,J}$

→ Mode propagation

→ Pendulum analogue → Deflect one pendulum → Relax → Wave like excitation

$\omega = \omega_\mathrm{p,J}$

→ Infinite wavelength Josephson plasma wave

→ Analogy to plasma frequency in a metal

→ Typically junctions $\omega_\mathrm{p,J} \simeq 10\ \mathrm{GHz}$

Plane waves

$$\Rightarrow \frac{\partial^2\varphi(z,t)}{\partial z^2} - \frac{1}{\overline{c}^2}\frac{\partial^2\varphi(z,t)}{\partial t^2} - \cancel{\frac{1}{\lambda_J^2}\sin\varphi(z,t)} = 0$$

For very large λ_J or very small *I*

→ Neglect $\frac{\sin\varphi}{\lambda_\mathrm{J}^2}$ term → Linear wave equation → Plane waves with velocity $\bar{c}$

3.5.4 Resonance phenomena

Interaction of fluxons or plasma waves with oscillating Josephson current

→ Rich variety of interesting resonance phenomena
→ Require presence of $\boldsymbol{B}^{\mathbf{ex}}$
→ Steps in IVC (junction upconverts dc drive)

Flux-flow steps and Eck peak

For $B^{\mathrm{ex}} > 0$

→ Spatially modulated Josephson currrent density moves at $v_z = V/B_y t_B$
→ Josephson current can excite Josephson plasma waves
→ On resonance, em waves couple strongly to Josephson current if $\boldsymbol{\bar{c} = v_z}$

Corresponding junction voltage:

$$V_{\mathrm{Eck}} = \bar{c} B_y t_B = \sqrt{\frac{d}{\epsilon\epsilon_0\mu_0 t_B}} B_y t_B = \frac{\omega_p}{2\pi}\frac{\lambda_J}{L} B_y t_B L = \frac{\omega_p}{2\pi}\frac{\lambda_J}{L}\Phi_0\frac{\Phi}{\Phi_0}$$

→ **Eck peak** at frequency: $\omega_{\mathrm{Eck}} = \frac{2e}{\hbar} V_{\mathrm{Eck}} = \omega_p \frac{\lambda_J}{L}\frac{\Phi}{\Phi_0}$

$$\bar{c} = \frac{\omega_p}{2\pi}\lambda_J$$
$$\Phi = B_y t_B L$$

3.5.4 Resonance phenomena

→ Traveling current wave only excites traveling em wave of same direction
  → Low damping, short junctions → Em wave is reflected at open end
  → Eck peak only observed in long junctions at medium damping when the backward wave is damped

Alternative point of view

→ Lorentz force → Josephson vortices move at $v_z = \frac{V}{B_y t_B}$

→ Increase driving force → Increase v_z

→ Maximum possible speed is $v_z = \bar{c}$

→ Further increase of I does not increase V)

→ Flux-flow step in IVC

→ $V_{\mathrm{ffs}} = \bar{c} B_y t_B = \bar{c} \frac{\Phi}{L} = \frac{\omega_{\mathrm{p}}}{2\pi} \frac{\lambda_{\mathrm{J}}}{L} \Phi_0 \frac{\Phi}{\Phi_0}$

→ Corresponds to Eck voltage

Fiske steps

$\lambda/2$-cavity
→ $\omega_n = (n\pi v_{\mathrm{ph}})/L$
v_{ph} = Phase velocity

Standing em waves in junction "cavity" at $\omega_n = 2\pi f_n = 2\pi \frac{\bar{c}}{2L} n = \frac{\pi \bar{c}}{L} n$

→ Fiske steps at voltages

$$V_n = \frac{\hbar}{2e}\omega_n = \Phi_0 \frac{\bar{c}}{2L} n = \frac{\omega_p}{2\pi}\frac{\lambda_J}{L}\Phi_0 \frac{n}{2}$$

for L » 100 µm
first Fiske step » 10 GHz
(few 10s of µV)

Interpretation

→ Wave length of Josephson current density is $\frac{2\pi}{k}$

→ Resonance condition $L = \frac{\bar{c}}{2 f_n} n = \frac{\lambda}{2} n \Rightarrow kL = n\pi$ or $\Phi = n \frac{\Phi_0}{2}$
where maximum Josephson current of short junction vanishes

→ Standing wave pattern of em wave and Josephson current match

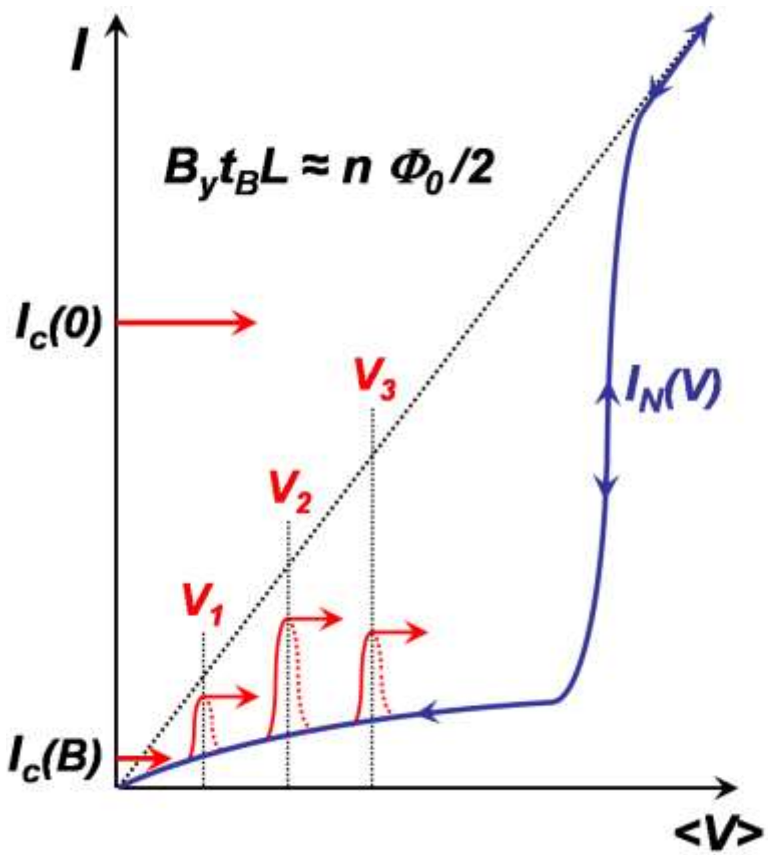
→ Steps in IVC

Influence of dissipation

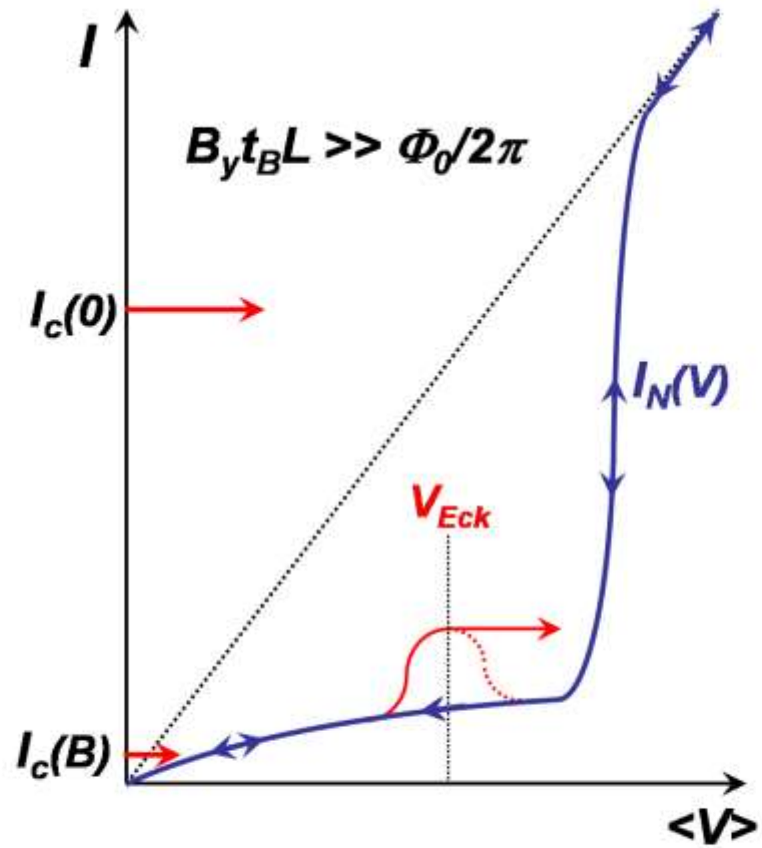
→ Damping of standing wave pattern by dissipative effects
  → Broadening of Fiske steps
  → Observation only for small and medium damping

3.5.4 Resonance phenomena

Fiske steps at small damping and/or small magnetic field



Eck peak at medium damping and/or medium magnetic field



For $V \neq V_{\mathrm{Eck}}$ and $V \neq V_n \rightarrow \langle I_s \rangle = \langle I_c \sin(\omega_0 t + kz + \varphi_0) \rangle \simeq 0 \rightarrow I = I_{\mathrm{N}}(V) = V/R_{\mathrm{N}}(V)$

3.5.4 Resonance phenomena

Zero field steps

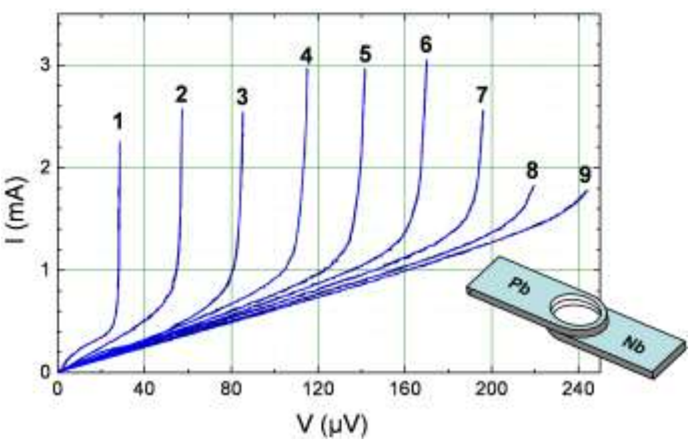
→ Motion of trapped flux due to Lorentz force (w/o magnetic field)
→ Junction of length L, moving back and forth
→ Phase change of 4π in period $T = \frac{2L}{v_z}$
→ At large bias currents ($v_z \to \bar{c}$)

$$V_{\text{zfs}} = \dot{\varphi}\frac{\hbar}{2e} = \frac{4\pi}{T}\frac{\hbar}{2e} = \frac{4\pi}{2L/\bar{c}}\frac{\hbar}{2e} = \frac{h}{2e}\frac{\bar{c}}{L} = \frac{\omega_{\text{p}}}{\pi}\frac{\lambda_{\text{J}}}{L}\Phi_0$$

For n fluxons

→ $V_{n,\text{zfs}} = nV_{zfs}$
→ $V_{n,\text{zfs}} = 2 \times$ Fiske voltage V_n (fluxon has to move back and forth)
→ $V_{\text{ffs}} = V_{n,\text{zfs}}$ for $\Phi = n\Phi_0$ (introduce n fluxons = generate n flux quanta)

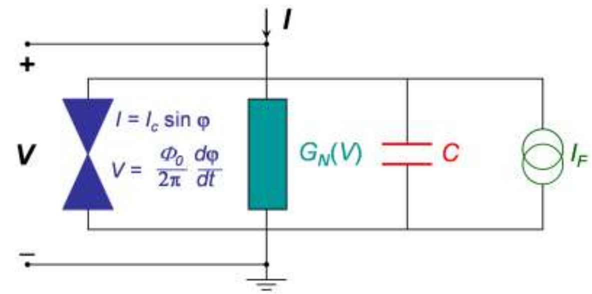
Example:
IVCs of annular Nb/insulator/Pb Josephson junction containing a different number of trapped fluxons



Summary (Voltage state of short junctions)

Voltage state: (Josephson + normal + displacement + fluctuation) current = total current

$$\Rightarrow I = I_c \sin\varphi + G_N(V)V + C\frac{dV}{dt} + I_F$$



$$\frac{d\varphi}{dt} = \frac{2eV}{\hbar}$$

$$\Rightarrow I = I_c \sin\varphi + G_N(V)\frac{\Phi_0}{2\pi}\frac{d\varphi}{dt} + C\frac{\Phi_0}{2\pi}\frac{d^2\varphi}{dt^2} + I_F$$

Equation of motion for phase difference φ:

$$\tau \equiv \frac{t}{\tau_c} = \frac{t}{2eI_cR/\hbar}$$

RCSJ-model ($G_N(V) = const.$)

$$\beta_C\frac{d^2\varphi}{d\tau^2} + \frac{d\varphi}{d\tau} + \sin\varphi - i - i_F(\tau) = 0$$

Motion of phase particle in the tilted washboard potential $U = E_{J0}[1 - \cos\varphi - (I/I_c)\varphi]$

Equivalent LCR resonator, characteristic frequencies:

$$\omega_p = \sqrt{\frac{1}{L_cC}} = \sqrt{\frac{2eI_c}{\hbar C}} \qquad \omega_c = \frac{R}{L_c} = \frac{2eI_cR}{\hbar} \qquad \omega_{RC} = \frac{1}{RC}$$

Quality factor: $Q^2 = \beta_C \equiv \frac{2e}{\hbar}I_cR^2C$ $\qquad \beta_C$ = Stewart-McCumber parameter

Summary (Voltage state of short junctions)

IVC for strong damping and $\beta_C \ll 1$ $\quad \langle V(t)\rangle = I_c R\sqrt{\left(\frac{I}{I_c}\right)^2 - 1}$ for $\frac{I}{I_c} > 1$

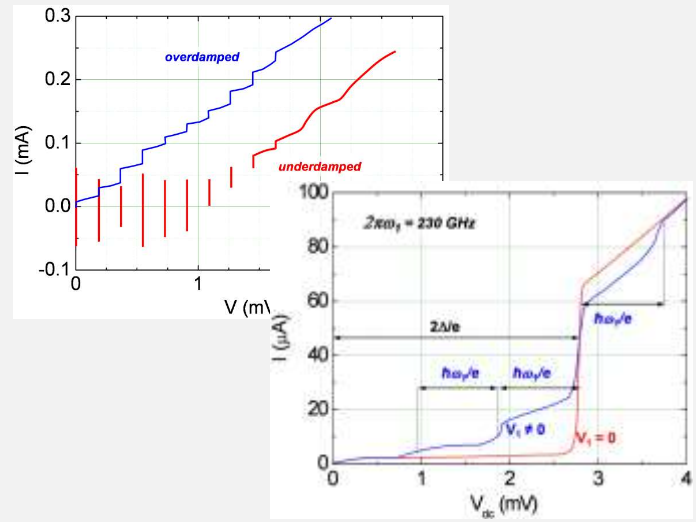
Driving with $V(t) = V_{\mathrm{dc}} + V_1 \cos\omega_1 t$

→ Shapiro steps at $V_n = n\frac{\Phi_0}{2\pi}\omega_1$

with amplitudes $|\langle I_s\rangle_n| = I_c\left|\mathcal{J}_n\left(\frac{2\pi V_1}{\Phi_0\omega_1}\right)\right|$

→ Photon assisted tunneling

Voltage steps at $V_n = n\frac{\Phi_0}{\pi}\omega_1$ due to nonlinear QP resistance

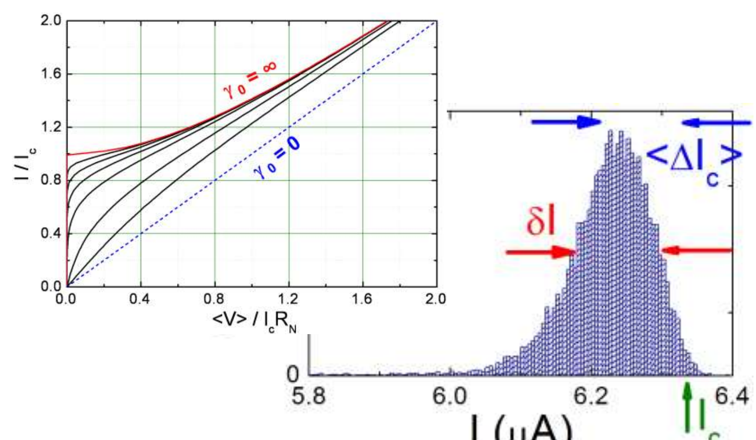


Effect of thermal fluctuations

→ Phase-slips at rate $\Gamma_{n+1} = \frac{\omega_{\mathrm{A}}}{2\pi}\exp\left(-\frac{U_0}{\mathrm{k_B}T}\right)$

→ Finite phase-slip resistance R_{p} even below I_{c}

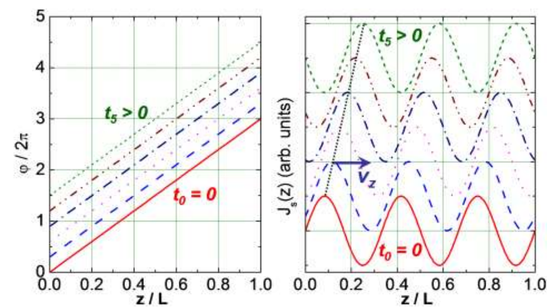
→ Premature switching



Summary

- voltage state of **extended** junctions w/o self-field:

$$\Rightarrow J_s(z,t) = J_c \sin\left(\omega_0 t + k\cdot z + \varphi_0\right)$$



- with self-field: time dependent Sine-Gordon equation

$$\Rightarrow \frac{\partial^2\varphi(z,t)}{\partial z^2} - \frac{1}{\overline{c}^2}\frac{\partial^2\varphi(z,t)}{\partial t^2} - \frac{1}{\lambda_J^2}\sin\varphi(z,t) = 0$$

$$\overline{c} = \sqrt{\frac{d}{\epsilon\epsilon_0\mu_0 t_B}} = c\sqrt{\frac{1}{\epsilon(1+2\lambda_L/d)}} \qquad \lambda_J \equiv \sqrt{\frac{\Phi_0}{2\pi\mu_0 t_B J_c}}$$

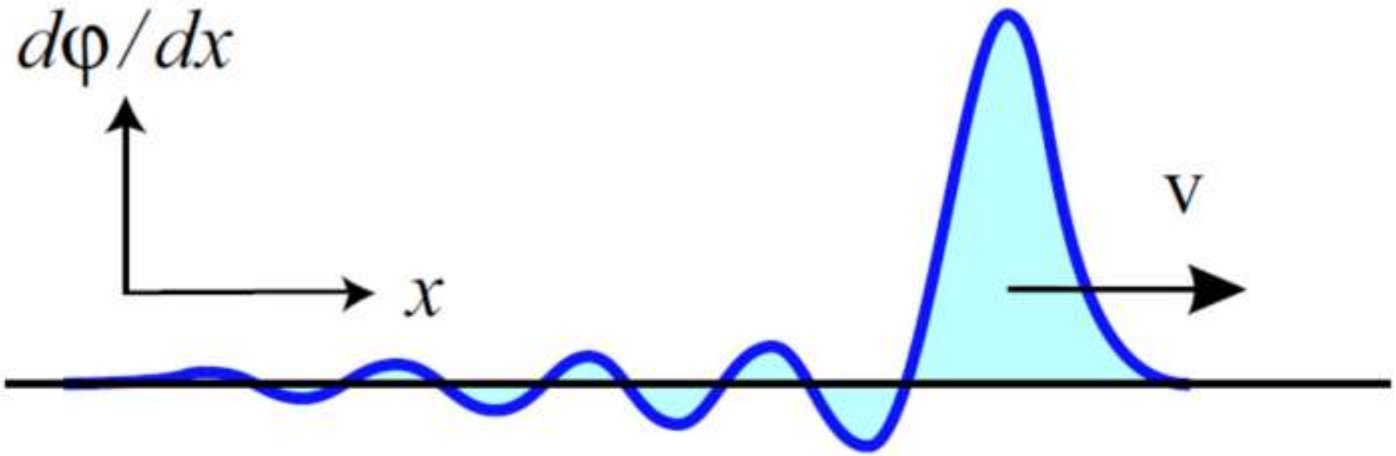
characteristic **velocity** of TEM mode in the **junction transmission line**

characteristic **screening length**

Prominent solutions: plasma oscillations and solitons

nonlinear interactions of these excitations with Josephson current:
→ ***flux-flow steps, Fiske steps, zero-field steps***

Solitons



A **soliton** is a self-reinforcing solitary wave (a wave packet or pulse) that maintains its shape while it travels at constant speed.

Solitons are caused by a cancellation of nonlinear and dispersive effects in the medium.

Solitons

1-d classic soliton in the water canal, discovered in 1834 by J.S.Russel :

Described by Kortweg de Friz (KdF) equation – similar to SG equation.

Solitons

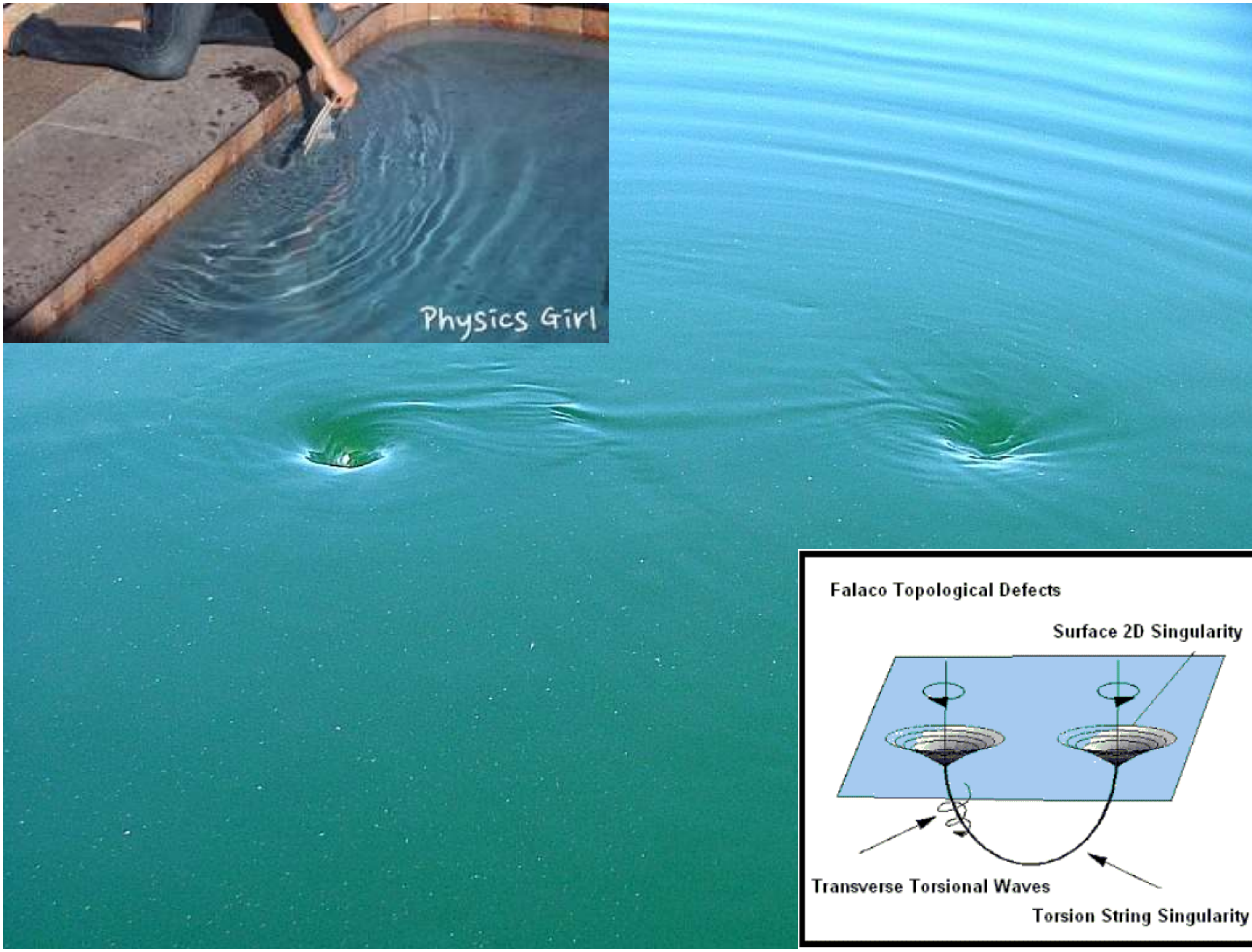
2-d solitons
on the shallow
water surface:

Usually
described
by cnoidal wave
solution of KdF
equation.

Solitons

3-d Falaco soliton in the water pool:

Two vortices are linked together with a turbulent channel deep in the water and moving as the whole.



Solitons

“Morning Glory Clouds”, Australia, coastline: