Introduction to Chapter 6
(from Chapter 3 of the lecture notes)

Quantum treatment of JJ
Classical treatment of Josephson junctions (so far)

Phase $\varphi$ and charge $Q = CV \propto \frac{d\varphi}{dt}$ → Purely *classical* variables

$(Q, \varphi)$ are assumed to be measurable simultaneously

Dynamics → Tilted washboard potential, rotating pendulum

Classical energies:

→ Potential energy $U(\varphi)$
  
  (Josephson coupling energy / Josephson inductance)

→ Kinetic energy $K(\dot{\varphi})$
  
  (Charging energy via $\frac{1}{2} CV^2 = \frac{Q^2}{2C} \propto \left(\frac{d\varphi}{dt}\right)^2$ / junction capacitance)

Current-phase & voltage-phase relation from macroscopic quantum model

→ Quantum origin

→ Primary macroscopic quantum effects

Second quantization

→ Treat $(Q, \varphi)$ as quantum variables (commutation relations, uncertainty)

→ Secondary macroscopic quantum effects
3.6.1 Quantum Consequences of the small Junction Capacitance

Validity of classical treatment

Consider an isolated, low-damping junction, $I = 0$

→ Cosine potential, depth $2E_{J0}$

→ Close to potential minimum

→ Harmonic oscillator

Frequency $\omega_p$, level spacing $\hbar\omega_p$

Vacuum energy $\frac{\hbar\omega_p}{2}$

$\hbar\omega_p = \sqrt{8E_{J0}E_C}$

$E_C = \frac{e^2}{2C}$

→ Classical treatment valid for $\frac{E_{J0}}{\hbar\omega_p} \approx \left(\frac{E_{J0}}{E_C}\right)^{1/2} \gg 1$ (Level spacing $\ll$ Potential depth)

$E_C \propto \frac{1}{C} \propto \frac{1}{A}$

$E_{J0} \propto I_c \propto A$

→ Enter quantum regime by decreasing junction area $A$
### 3.6.1 Quantum Consequences of the small Junction Capacitance

**Parameters for the quantum regime**

**Example 1**
Area $A = 10 \, \mu m^2$, Tunnel barrier $d = 1 \, nm$, $\varepsilon = 10$, $J_c = 100 \frac{A}{cm^2}$

$\rightarrow E_{J0} = 3 \times 10^{-21} \, J$

$\rightarrow E_{J0}/h = 4500 \, GHz$

$C = \frac{\varepsilon \varepsilon_0 A}{d} = 0.9 \, pF$

$\rightarrow E_C = 2 \times 10^{-26} \, J$

$\rightarrow \frac{E_C}{h} = 30 \, MHz$

$\rightarrow$ Classical junction

**Example 2**
Area $A = 0.02 \, \mu m^2$

$\rightarrow C \approx 1 \, fF \rightarrow E_C \approx E_{J0}$

$\rightarrow$ Quantum junction

$\rightarrow$ We also need $T \ll 500 \, mK$ for $k_B T \ll E_{J0}, E_C$!
3.6.1 Quantum Consequences of the small Junction Capacitance

Hamiltonian of a strongly underdamped junction (with $\frac{d\varphi}{dt} \neq 0$)

Kinetic energy:

$$K = \frac{1}{2} CV^2 = \frac{Q^2}{2C} = \frac{1}{2} C \left( \frac{\hbar}{2e} \right)^2 \dot{\varphi}^2 = \frac{1}{2} E_{J0} \frac{\varphi^2}{\omega_p^2}$$

$\Rightarrow$ Energy due to extra charge $Q$ on one junction electrode due to $V$

Total energy:

$$E = K + U = E_{J0} \left( 1 - \cos \varphi + \frac{1}{2} \frac{\varphi^2}{\omega_p^2} \right)$$

$$U(\varphi) \propto 1 - \cos \varphi \quad \Rightarrow \quad \text{Potential energy}$$

$$K(\dot{\varphi}) \propto \dot{\varphi}^2 \quad \Rightarrow \quad \text{Kinetic energy}$$

Consider $E(\varphi, \dot{\varphi})$ as junction Hamiltonian, rewrite kinetic energy

$$K = \frac{Q^2}{2C} = \frac{1}{2} \left( \frac{\hbar/2e}{2e} \right)^2 C \left( \frac{\hbar Q}{2e} \right)^2$$

$$\Rightarrow \quad p = \left( \frac{\hbar}{2e} \right) Q$$

$\Rightarrow$ Position coordinate associated to phase $\varphi$, momentum associated to charge $Q$
3.6.1 Quantum Consequences of the small Junction Capacitance

**Canonical quantization** (operator replacement)

\[ \frac{\hbar}{2e} Q \rightarrow -i\hbar \frac{\partial}{\partial \varphi} \]

with \( N = \frac{Q}{2e} \rightarrow \# \text{ of Cooper pairs} \)

\( Q = -i2e \frac{\partial}{\partial \varphi} \quad N = -i \frac{\partial}{\partial \varphi} \)

we get the Hamiltonian

\[ \mathcal{H} = \frac{Q^2}{2C} + E_{J0}(1 - \cos \varphi) = -\left(\frac{2e}{2C}\right)^2 \frac{\partial^2}{\partial \varphi^2} + E_{J0}(1 - \cos \varphi) \]

\[ \mathcal{H} = -4E_C \frac{\partial^2}{\partial \varphi^2} + E_{J0}(1 - \cos \varphi) \]

→ Describes only Cooper pairs

\( E_C = \frac{e^2}{2C} \) nevertheless defined as charging energy for a single electron charge

Commutation rules for the operators

\[ [\varphi, Q] = i2e \quad ; \quad [\varphi, N] = i \quad \text{or} \quad [\varphi, \frac{\hbar}{2e} Q] = i\hbar \]

\( N \equiv \frac{Q}{2e} \rightarrow \text{Deviation of \# of CP in electrodes from equilibrium} \)

Heisenberg uncertainty relation → \( \Delta N \cdot \Delta \varphi \geq 1 \)
3.6.1 Quantum Consequences of the small Junction Capacitance

Hamiltonian in the flux basis \((\phi = \frac{\hbar}{2e} \varphi = \frac{\Phi_0}{2\pi} \varphi)\)

\[
\mathcal{H} = \frac{Q^2}{2C} + E_{J0} \left( 1 - \cos 2\pi \frac{\phi}{\Phi_0} \right) = -\frac{(2e)^2}{2C} \frac{\hbar^2}{(2e)^2} \frac{\partial^2}{\partial\phi^2} + E_{J0} \left( 1 - \cos 2\pi \frac{\phi}{\Phi_0} \right)
\]

Commutator \([\phi, Q] = i\hbar\)

→ \(\phi\) and \(Q\) are canonically conjugate (analogous to \(x\) and \(p\))

→ Circuit variables are now quantized

→ **Superconducting quantum circuits**
3.6.2 Limiting Cases: The Phase and Charge Regime

The phase regime

\[ \hbar \omega_p \ll E_J, \ E_C \ll E_J \rightarrow \text{Phase } \varphi \text{ is a good quantum number!} \]

Lowest energy levels localized near bottom of potential wells at \( \varphi_n = 2\pi n \)

Taylor series for \( U(\varphi) \rightarrow \) Harmonic oscillator,

\[ \text{Frequency } \omega_p, \text{ eigenenergies } E_n = \hbar \omega_p \left( n + \frac{1}{2} \right) \]

Ground state: narrowly peaked wave function at \( \varphi = \varphi_n \)

Large fluctuations of \( Q \) on electrodes since \( \Delta Q \cdot \Delta \varphi \geq 2e \)

(small \( E_C \rightarrow \) pairs can easily fluctuate, large \( \Delta Q \))

Small phase fluctuations \( \Delta \varphi \)

Negligible \( \Delta \varphi \Rightarrow \) classical treatment of phase dynamics is good approximation
3.6.2 Limiting Cases: The Phase and Charge Regime

The phase regime

Hamiltonian
\[ \mathcal{H} = -4E_C \frac{\partial^2}{\partial \varphi^2} + E_{J0} (1 - \cos \varphi) \]

define \( a = (E - E_{J0})/E_C \), \( b = E_{J0}/2E_C \) and \( z = \varphi/2 \)

→ Mathieu equation
\[ \frac{\partial^2 \psi}{\partial z^2} + (a + 2b \cos 2z) \psi = 0 \]

General solution
\[ \psi(\varphi) = \sum_q c_q \psi_q \]

Bloch waves
\[ \psi_q(\varphi) = u_q(\varphi) \exp(iq\varphi) \quad \text{with} \quad u_q(\varphi) = u_q(\varphi + 2\pi) \]

Charge/pair number variable \( q \) is continuous (charge on capacitor!)

→ \( \Psi(\varphi) \) is not 2\( \pi \)-periodic

known from periodic potential problem in solid state physics

→ Energy bands
3.6.2 Limiting Cases: The Phase and Charge Regime

The phase regime

1D problem → Numerical solution straightforward
Variational approach for approximate ground state

Trial function for \( E_C \ll E_{J0} \)
\[ \Psi(\varphi) \propto \exp\left(-\frac{\varphi^2}{4\sigma^2}\right) \]

Choose \( \sigma \) to find minimum energy:

\[ E_{\text{min}} = E_{J0} \left(1 - \left[1 - \sqrt{\frac{2E_C}{E_{J0}}}\right]^2\right) = E_{J0} \left(1 - \left[1 - \frac{\hbar\omega_p}{2E_{J0}}\right]^2\right) \]

First order in \( E_{J0} \)
\( E_{\text{min}} \approx 0 \) for \( E_C \ll E_{J0} \)

\[ \hbar\omega_p = \sqrt{8E_{J0}E_C} \]

\[ \frac{E_C}{E_{J0}} = 0.1 \]
\[ E_{\text{min}} = 0.1E_{J0} \]
3.6.2 Limiting Cases: The Phase and Charge Regime

The phase regime

Tunneling coupling $\propto \exp\left(-\frac{2E_J-E}{\hbar\omega_p}\right) \rightarrow$ Very small since $\hbar\omega_p \ll E_J$

$\rightarrow$ Tunneling splitting of low lying states is exponentially small

$$E_{\min} = E_J \left(1 - \left[1 - \sqrt{\frac{2E_C}{E_J}}\right]^2\right) = E_J \left(1 - \left[1 - \frac{\hbar\omega_p}{2E_J}\right]^2\right)$$

$\frac{E_C}{E_J} = 0.1$

$E_{\min} = 0.1 E_J$
3.6.2 Limiting Cases: The Phase and Charge Regime

The charge regime

\[ \hbar \omega_p \gg E_{J0}, \ E_C \gg E_{J0} \rightarrow \text{Charge } Q \text{ (momentum) is good quantum number} \]

Kinetic energy \( \propto E_C \left(\frac{d\varphi}{dt}\right)^2 \) dominates

\[ \rightarrow \text{Complete delocalization of phase} \]
\[ \rightarrow \text{Wave function should approach constant value, } \Psi(\varphi) \approx \text{const.} \]
\[ \rightarrow \text{Large phase fluctuations, small charge fluctuations } (\Delta Q \cdot \Delta \varphi \geq 2e) \]

Hamiltonian

\[ \mathcal{H} = -4E_C \frac{\partial^2}{\partial \varphi^2} + E_{J0}(1 - \cos \varphi) \]

Appropriate trial function:

\[ \Psi(\varphi) \propto (1 - \alpha \cos \varphi) \quad \alpha \ll 1 \]

Approximate ground state energy

\[ E_{\min} \approx E_{J0} \left(1 - \frac{E_{J0}}{8E_C}\right) = E_{J0} \left(1 - \frac{E_{J0}^2}{(\hbar \omega_p)^2}\right) \]

second order in \( E_{J0} \)
3.6.2 Limiting Cases: The Phase and Charge Regime

The charge regime

\[ E_{\text{min}} \approx E_{J0} \left( 1 - \frac{E_{J0}}{8E_C} \right) = E_{J0} \left( 1 - \frac{E_{J0}^2}{(\hbar \omega_p)^2} \right) \]

\[ \frac{E_C}{E_{J0}} = 2.5 \]

\[ E_{\text{min}} = 0.95 E_{J0} \]

→ Periodic potential is weak
→ Strong coupling between neighboring phase states → Broad bands
→ Compare to electrons moving in strong (phase regime) or weak (charge regime) periodic potential of a crystal
3.6.3 Coulomb and Flux Blockade

Coulomb blockade in normal metal tunnel junctions

Voltage $V \rightarrow$ Charge $Q = CV$, energy $E = \frac{Q^2}{2C}$

Single electron tunneling

$\rightarrow$ Charge on one electrode changes to $Q - e$
$\rightarrow$ Electrostatic energy $E' = \frac{(Q-e)^2}{2C}$
$\rightarrow$ Tunneling only allowed for $E' \leq E$

$\rightarrow$ Coulomb blockade: Need $|Q| \geq e/2$ or $|V| \geq V_{\text{CB}} = V_c = e/2C$

Observation of CB requires small thermal fluctuations

$\rightarrow$ $E_C = \frac{e^2}{2C} > k_B T \Rightarrow C < \frac{e^2}{2k_B T}$
$\rightarrow$ $C \approx 1 \text{ fF at } T = 1 \text{ K, } d = 1 \text{ nm, and } \varepsilon = 5 \Rightarrow A \lesssim 0.02 \mu \text{m}^2 \rightarrow \text{Small junctions!}$

Observation of CB requires small quantum fluctuations

$\rightarrow$ Quantum fluctuations due to Heisenberg principle $\rightarrow \Delta E \cdot \Delta t \geq \hbar$ 
$\rightarrow$ Finite tunnel resistance
$\rightarrow$ $\tau_{RC} = RC$ (decay of charge fluctuations)

$\rightarrow$ $\Delta t = 2\pi RC$, $\Delta E = \frac{e^2}{2C} \rightarrow R \geq \frac{\hbar}{e^2} = R_K = 24.6 \text{ k}\Omega \rightarrow \text{Typically satisfied}$
3.6.3 Coulomb and Flux Blockade

Coulomb blockade in superconducting tunnel junctions

For $\frac{Q^2}{2C} > k_B T, eV$ ($Q = 2e$) $\Rightarrow$ No flow of Cooper pairs

Threshold voltage $\Rightarrow |V| \geq V_{CB} = V_c = \frac{2e}{2C} = \frac{e}{C}$

Coulomb blockade $\Rightarrow$ Charge is fixed, phase is completely delocalized
3.6.3 Coulomb and Flux Blockade

Phase or flux blockade in a Josephson junction

Current $I \rightarrow$ Flux $\Phi = LI$, energy $E = \Phi^2/2L$

$\rightarrow$ Phase is blocked due to large $E_{J0} = \Phi_0 I_c / 2\pi$
  $\rightarrow I_c$ takes the role of $V_{CB}$
  $\rightarrow$ Phase change of $2\pi$ equivalent to flux change of $\Phi_0$

$\rightarrow$ Flux blockade $|I| \geq I_{FB} = I_c = \frac{(\Phi_0/2\pi)}{L_c}$

$\rightarrow$ Analogy to CB $\rightarrow I \leftrightarrow V$, $2e \leftrightarrow \frac{\Phi_0}{2\pi}$, $C \leftrightarrow L$

In presence of fluctuations we need

$\rightarrow E_{J0} \gg k_B T$ (large junction area)

$\rightarrow$ And $\Delta E \cdot \Delta t \geq \hbar$ with $\Delta t = 2\pi \frac{L}{R}$ and $\Delta E = 2E_{J0}$ $\rightarrow R \leq \frac{\hbar}{(2e)^2} = \frac{1}{4} R_K$
3.6.4 Coherent Charge and Phase States

Coherent charge states
Island charge continuously changed by gate

\[ E_{J0} > 0 \]

\( \rightarrow \) Interaction of \( |n\rangle \) and \( |n + 1\rangle \) at the level crossing points \( Q = \left( n + \frac{1}{2} \right) \cdot 2e \)

\( \rightarrow \) Avoided level crossing (anti-crossing)

\( \rightarrow \) Coherent superposition states \( |\Psi_\pm\rangle = \alpha |n\rangle \pm \beta |n + 1\rangle \)

Independent charge states \( (E_{J0} = 0) \)

\( \rightarrow \) Parabola \( E(Q) = (Q - n \cdot 2e)^2 / 2C_{\Sigma} \)
3.6.4 Coherent Charge and Phase States

Coherent charge states

Average charge on the island as a function of the applied gate voltage → Quantized in units of $2e$ (no coherence yet)

First experimental demonstration of coherent superposition charge states by Nakamura, Pashkin, Tsai (1999, not this picture)
Coherent phase states

→ Interaction of two adjacent phase states
→ Example is rf SQUID

Magnetic energy of flux \( \phi = \left( \frac{\Phi_0}{2\pi} \right) \varphi \) in the ring

\[
U(\phi) = \frac{(\phi - \phi_{\text{ext}})^2}{2L} + E_{J0} \left( 1 - \cos 2\pi \frac{\phi}{\Phi_0} \right)
\]

\( \Phi_{\text{ext}} = \Phi_0/2 \)

Tunnel coupling \( \rightarrow \) \( \Psi_{\pm} = a|L\rangle \pm b|R\rangle \)

Experimental evidence for quantum coherent superposition (Mooij et al., 1999)
3.6.5 Quantum Fluctuations

Violation of conservation of energy on small time scales, obey $\Delta E \cdot \Delta t \geq \hbar$

→ Creation of virtual excitations
→ Include Langevin force $I_F$ with adequate statistical properties
→ Fluctuation-dissipation theorem

$$S_1(f) = 2\pi S_1(\omega) = 4 \frac{E(\omega, T)}{R_N}$$

$$E(\omega, T) = \frac{\hbar \omega}{2} + \hbar \omega \frac{1}{\exp \left( \frac{\hbar \omega}{k_B T} \right) - 1} = \frac{\hbar \omega}{2} \coth \left( \frac{\hbar \omega}{2k_B T} \right)$$

vacuum fluctuations
occupation probability of oscillator (Planck distribution)

→ Transition from “thermal” Johnson-Nyquist noise to quantum noise:

classical limit ($\hbar \omega, eV \ll k_B T$): $$S_1(\omega) = \frac{1}{2\pi} \frac{4k_B T}{R_N}$$

quantum limit ($\hbar \omega, eV \gg k_B T$): $$S_1(\omega) = \frac{1}{2\pi} \frac{2\hbar \omega}{R_N} = \frac{1}{2\pi} \frac{2eV}{R_N}$$
Escape of the “phase particle” from minimum of washboard potential by tunneling
→ Macroscopic, i.e., phase difference is tunneling (collective state)
→ States easily distinguishable

Competing process
→ Thermal activation
→ Low temperatures

Neglect damping
Dc-bias → Term $-\frac{\hbar I \varphi}{2e}$ in Hamiltonian

Curvature at potential minimum:
$$\frac{\partial^2 U}{\partial \varphi^2} = E_0 \sqrt{1 - i^2} \quad i = I/I_c$$

(Classical) small oscillation frequency:
$$\omega_A = \omega_p (1 - i^2)^{1/4} \quad \text{(attempt frequency)}$$
3.6.6 Macroscopic Quantum Tunneling

Quantum mechanical treatment

→ Tunnel coupling of bound states to outgoing waves → Continuum of states
→ But only states corresponding to quasi-bound states have high amplitude
→ In-well states of width $\Gamma = \hbar/\tau$ ($\tau =$ lifetime for escape)

Determination of wave functions

→ Wave matching method
→ Exponential prefactor within WKB approximation
→ Decay in barrier:

$$|\psi(x)|^2 \propto \exp \left\{ -\frac{2}{\hbar} \int_{\text{II}} \sqrt{2M[V(x) - E]} \, dx \right\}$$

for $U(\varphi) \gg E_0 = \hbar \omega_A / 2$

$$|\psi(\varphi)|^2 \propto \exp \left\{ -\frac{2}{\hbar} \int_{\text{II}} \sqrt{2 \left( \frac{\hbar}{2e} \right)^2 C \left[ U(\varphi) - \frac{\hbar \omega_A}{2} \right]} \, d\varphi \right\}$$

mass effective barrier height
3.6.6 Macroscopic Quantum Tunneling

Constant barrier height $\Rightarrow$ Escape rate

$$|\psi(\varphi)|^2 \propto \exp\left\{-\sqrt{\frac{U_0}{E_C}} \Delta \varphi\right\} \Rightarrow \Gamma = \frac{\omega_A}{2\pi} \exp\left\{-\sqrt{\frac{U_0}{E_C}} \Delta \varphi\right\}$$

Increasing bias current $\Rightarrow$

$$U_0 \approx 2E_{J0}(1-i^2)^{3/2} \quad \text{and} \quad \Delta \varphi \approx \pi \sqrt{1-i^2}$$

$U_0$ decreases with $i \Rightarrow \Gamma$ becomes measurable

Temperature $T^*$ where $\Gamma_{\text{tunnel}} = \Gamma_{\text{TA}} \approx \exp\left(-\frac{U_0}{k_BT}\right)$

for $I \approx 0$: 

$$U_0 \approx 2E_{J0} \quad \hbar \omega_p = \sqrt{8E_{J0}E_C} \approx 2\sqrt{U_0E_C} \quad \Delta \varphi \approx \pi$$

$$\Rightarrow \Gamma = \frac{\omega_p}{2\pi} \exp\left\{-2\pi \frac{U_0}{\hbar \omega_p}\right\} \Rightarrow k_B T^* \approx \frac{\hbar \omega_p}{2\pi}$$

for $I > 0$:

$$\sqrt{U_0} \propto (1-i^2)^{3/4} \quad \Delta \varphi \propto (1-i^2)^{1/2}$$

$$\Rightarrow k_B T^* \approx \frac{\hbar \omega_A}{2\pi} = \frac{\hbar \omega_p}{2\pi} (1-i^2)^{1/4}$$

For $\omega_p \approx 10^{11} \, \text{s}^{-1}$

$\Rightarrow T^* \approx 100 \, \text{mK}$
3.6.6 Macroscopic Quantum Tunneling

Additional topic: Effect of damping

Junction couples to the environment (e.g., Caldeira-Leggett-type heat bath)

→ Crossover temperature \( k_B T^* \approx \frac{\hbar \omega_R}{2\pi} \) with \( \omega_R = \omega_A (\sqrt{1 + \alpha^2} - \alpha) \) and \( \alpha \equiv \frac{1}{2R_N C \omega_A} \)

→ Strong damping \( \alpha \gg 1 \rightarrow \omega_R \ll \omega_A \rightarrow \text{Lower } T^* \)

→ Damping suppresses MQT

Phase diffusion by MQT
See lecture notes

**Summary (secondary quantum macroscopic effects)**

**Classical description** only in the phase regime (large junctions): $E_C \ll E_{J0}$

For $E_C \gg E_{J0}$: **quantum description** (negligible damping):

$$\mathcal{H} = 4E_C \frac{\partial^2}{\partial \varphi^2} + E_{J0}(1 - \cos \varphi)$$

Phase difference $\varphi$ and Cooper pair number $N = \frac{Q}{2e}$ are canonically conjugate variables

$$[\varphi, \frac{\hbar}{2e} Q] = i\hbar \quad \Rightarrow \Delta N \cdot \Delta \varphi \geq 1$$

phase regime: $\Delta \varphi \to 0$ and $\Delta N \to \infty$
charge regime: $\Delta N \to 0$ and $\Delta \varphi \to \infty$

Charge regime at $T = 0$

→ Coulomb blockade → Tunneling only for $V_{CB} \geq \frac{e}{C}$

Flux regime at $T = 0$

→ Flux blockade → Flux motion only for $I_{FB} \geq \frac{\Phi_0}{2\pi L_c}$

At $I < I_c$

→ Escape out of the washboard by **thermal activation** or **macroscopic quantum tunneling**

**TA-MQT crossover temperature** $T^*$

$$k_B T^* \simeq \frac{\hbar \omega_A}{2\pi} = \frac{\hbar \omega_p}{2\pi} \left[1 - \left(\frac{l}{l_c}\right)^2\right]^{1/4}$$