Introduction to Chapter 6
(from Chapter 3 of the lecture notes)

Quantum treatment of JJ

Classical treatment of Josephson junctions (so far)

Phase φ and charge $Q = CV \propto \frac{d\varphi}{dt}$ → Purely **classical** variables

(Q, φ) are assumed to be measurable simultaneously

Dynamics → Tilted washboard potential, rotating pendulum

Classical energies:

→ Potential energy $U(\varphi)$
(Josephson coupling energy / Josephson inductance)

→ Kinetic energy $K(\dot{\varphi})$
(Charging energy via $\frac{1}{2}CV^2 = \frac{Q^2}{2C} \propto \left(\frac{d\varphi}{dt}\right)^2$ / junction capacitance)

Current-phase & voltage-phase relation from macroscopic quantum model

→ Quantum origin

→ Primary macroscopic quantum effects

Second quantization

→ Treat (Q, φ) as quantum variables (commutation relations, uncertainty)

→ Secondary macroscopic quantum effects

Validity of classical treatment

Consider an isolated, low-damping junction, $I = 0$

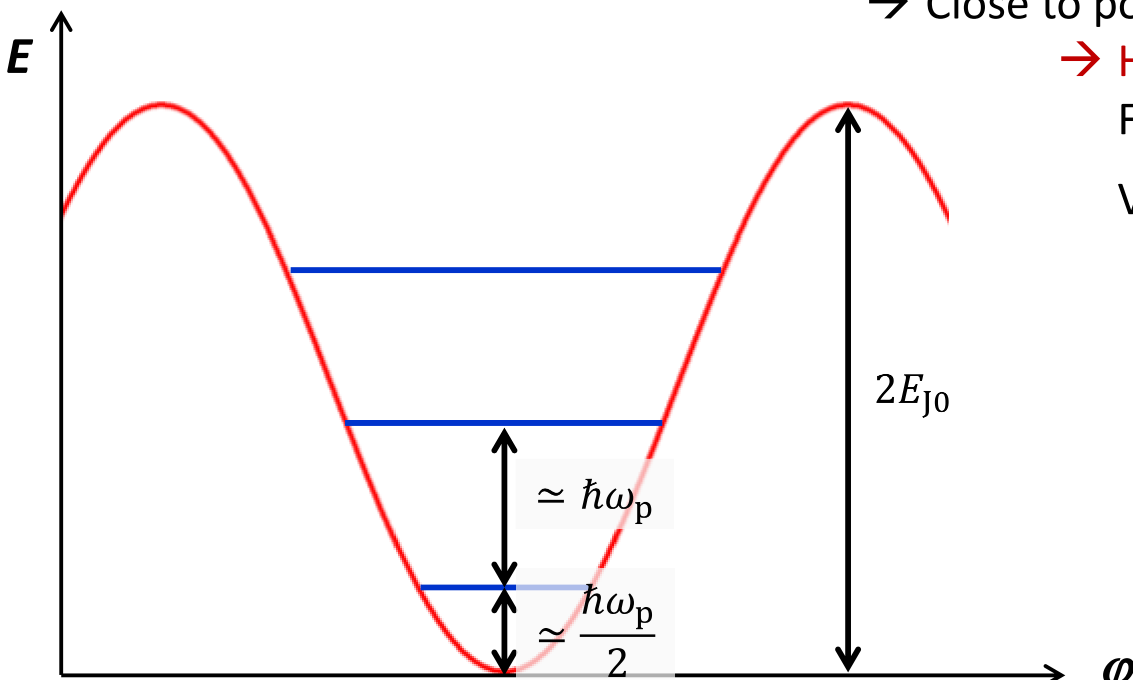
→ Cosine potential, depth $2E_{J0}$

→ Close to potential minimum

→ Harmonic oscillator

Frequency ω_p, level spacing $\hbar\omega_\mathrm{p}$

Vacuum energy $\frac{\hbar\omega_\mathrm{p}}{2}$



$$\hbar\omega_p = \sqrt{8E_{J0}E_C}$$
$$E_C = \frac{e^2}{2C}$$

→ Classical treatment valid for $\frac{E_{J0}}{\hbar\omega_p} \simeq \left(\frac{E_{J0}}{E_C}\right)^{1/2} \gg 1$ (Level spacing ≪ Potential depth)

$E_C \propto 1/C \propto 1/A$

$E_{J0} \propto I_c \propto A$

→ Enter quantum regime by decreasing junction area A

Parameters for the quantum regime

Example 1

Area $A = 10\ \mu\text{m}^2$, Tunnel barrier $d = 1\ \text{nm}, \varepsilon = 10$, $J_c = 100\,\frac{\text{A}}{\text{cm}^2}$

→ $E_{\text{J}0} = 3 \times 10^{-21}\ \text{J}$

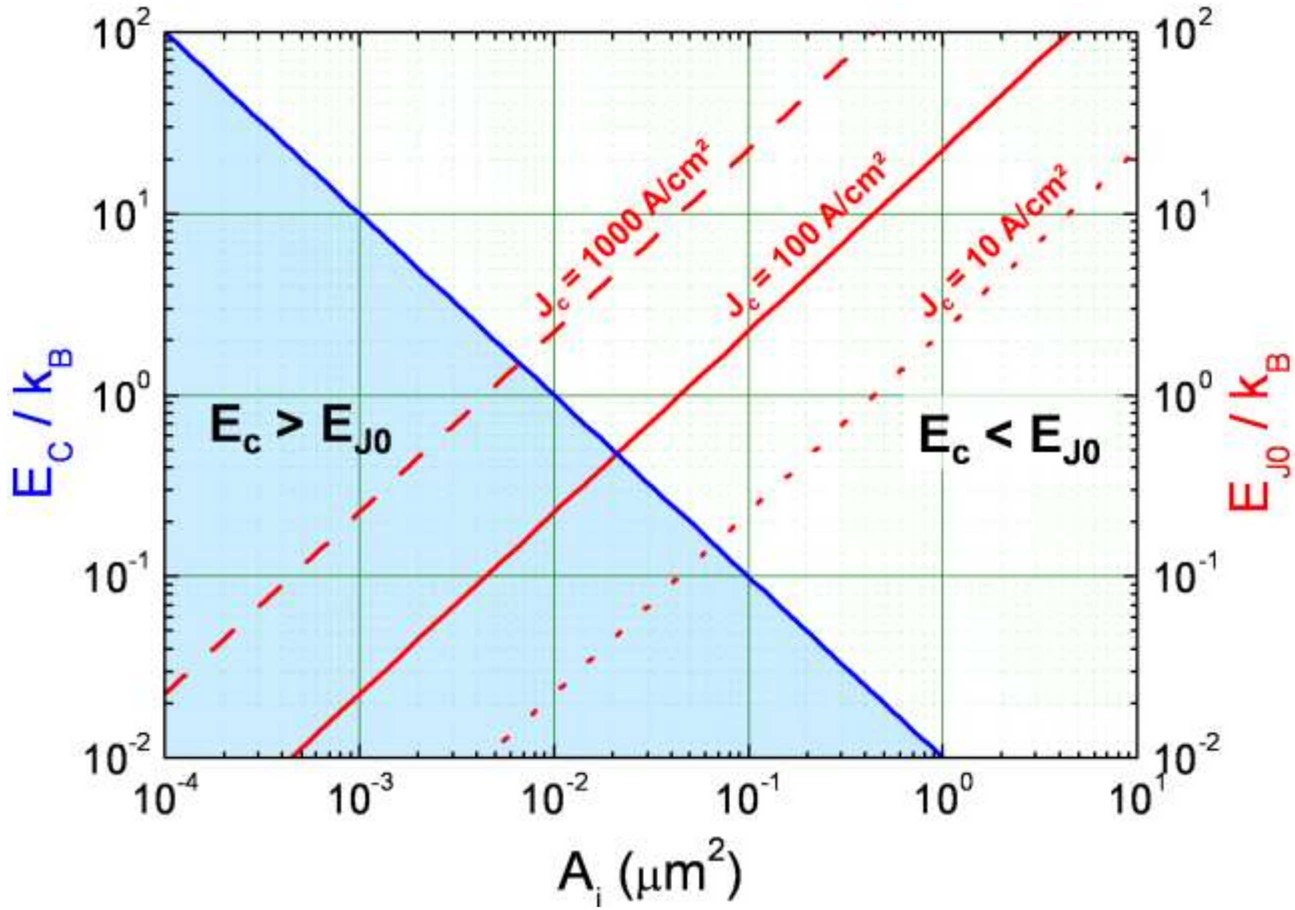
→ $E_{\text{J}0}/h = 4500\ \text{GHz}$

$C = \frac{\varepsilon\varepsilon_0 A}{d} = 0.9\ \text{pF}$

→ $E_C = 2 \times 10^{-26}\ \text{J}$

→ $\frac{E_C}{h} = 30\ \text{MHz}$

→ Classical junction



Example 2

Area $A = 0.02\ \mu\text{m}^2$

→ $C \simeq 1\ \text{fF}$ → $E_C \simeq E_{\text{J}0}$

→ Quantum junction

→ We also need $T \ll 500\ \text{mK}$ for $k_\text{B}T \ll E_{\text{J}0}, E_C$!

3.6.1 Quantum Consequences of the small Junction Capacitance

Hamiltonian of a strongly underdamped junction (with $\frac{d\varphi}{dt} \neq 0$)

Kinetic energy: $$K = \frac{1}{2}CV^2 = \frac{Q^2}{2C} = \frac{1}{2}C\left(\frac{\hbar}{2e}\right)^2 \dot{\varphi}^2 = \frac{1}{2}E_{J0}\frac{\dot{\varphi}^2}{\omega_p^2}$$

→ Energy due to extra charge Q on one junction electrode due to V

Total energy: $$E = K + U = E_{J0}\left(1 - \cos\varphi + \frac{1}{2}\frac{\dot{\varphi}^2}{\omega_p^2}\right)$$

$U(\varphi) \propto 1 - \cos\varphi$ → Potential energy

$K(\dot{\varphi}) \propto \dot{\varphi}^2$ → Kinetic energy

Consider $E(\varphi, \dot{\varphi})$ as junction Hamiltonian, rewrite kinetic energy

$$K = \frac{Q^2}{2C} = \frac{1}{2}\underbrace{\frac{1}{(\hbar/2e)^2 C}}_{1/\text{mass}}\underbrace{\left(\frac{\hbar}{2e}Q\right)^2}_{\text{momentum}}$$

$$K = p^2/2M$$

$$\rightarrow p = \left(\frac{\hbar}{2e}\right)Q$$

→ Position coordinate associated to phase φ, momentum associated to charge Q

Canonical quantization (operator replacement) $\frac{\hbar}{2e}Q \rightarrow -\imath\hbar\frac{\partial}{\partial\varphi}$

$p \rightarrow -\imath\hbar\frac{\partial}{\partial x}$

with $N = \frac{Q}{2e}$ → # of Cooper pairs $Q = -\imath 2e\frac{\partial}{\partial\varphi}$ $N = -\imath\frac{\partial}{\partial\varphi}$

we get the Hamiltonian

$$\mathcal{H} = \frac{Q^2}{2C} + E_{J0}(1-\cos\varphi) = -\frac{(2e)^2}{2C}\frac{\partial^2}{\partial\varphi^2} + E_{J0}(1-\cos\varphi)$$

$$\mathcal{H} = -4E_C\frac{\partial^2}{\partial\varphi^2} + E_{J0}(1-\cos\varphi)$$

→ Describes only Cooper pairs

$E_C = \frac{e^2}{2C}$ nevertheless defined as charging energy for a single electron charge

Commutation rules for the operators

$$[\varphi, Q] = \imath 2e \quad ; \quad [\varphi, N] = \imath \quad \text{or} \quad [\varphi, \frac{\hbar}{2e}Q] = \imath\hbar$$

$$\Delta A \Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$$

$N \equiv \frac{Q}{2e}$ → Deviation of # of CP in electrodes from equilibrium

Heisenberg uncertainty relation → $\Delta N \cdot \Delta\varphi \geq 1$

$$\frac{\hbar}{2e}Q \rightarrow -\imath\hbar\frac{\partial}{\partial\varphi}$$

Hamiltonian in the flux basis ($\phi = \frac{\hbar}{2e}\varphi = \frac{\Phi_0}{2\pi}\varphi$)

$$\mathcal{H} = \frac{Q^2}{2C} + E_{J0}\left(1 - \cos 2\pi\frac{\phi}{\Phi_0}\right) = -\frac{(2e)^2}{2C}\frac{\hbar^2}{(2e)^2}\frac{\partial^2}{\partial\phi^2} + E_{J0}\left(1 - \cos 2\pi\frac{\phi}{\Phi_0}\right)$$

$$= -\frac{\hbar^2}{2C}\frac{\partial^2}{\partial\phi^2} + E_{J0}\left(1 - \cos 2\pi\frac{\phi}{\Phi_0}\right)$$

Commutator $[\phi, Q] = i\hbar$

→ ϕ and Q are canonically conjugate (analogous to x and p)

→ Circuit variables are now quantized

→ **Superconducting quantum circuits**

The phase regime

$\hbar\omega_\mathrm{p} \ll E_\mathrm{J0}$, $E_C \ll E_\mathrm{J0}$ → Phase φ is a good quantum number!

Lowest energy levels localized near bottom of potential wells at $\varphi_n = 2\pi\, n$

Taylor series for $U(\varphi)$ → Harmonic oscillator,

Frequency ω_p, eigenenergies $E_n = \hbar\omega_\mathrm{p}\left(n + \frac{1}{2}\right)$

Ground state: narrowly peaked wave function at $\varphi = \varphi_n$

Large fluctuations of Q on electrodes since $\Delta Q \cdot \Delta\varphi \geq 2e$

(small E_C → pairs can easily fluctuate, large ΔQ)

Small phase fluctuations $\Delta\varphi$

Negligible $\Delta\varphi \Rightarrow$ classical treatment of phase dynamics is good approximation

The phase regime

Hamiltonian $$\mathcal{H} = -4E_C \frac{\partial^2}{\partial \varphi^2} + E_{J0}(1 - \cos\varphi)$$

define $a = (E - E_{J0})/E_C$, $b = E_{J0}/2E_C$ and $z = \varphi/2$

→ Mathieu equation $$\frac{\partial^2 \Psi}{\partial z^2} + (a + 2b\cos 2z)\,\Psi = 0$$

known from periodic potential problem in solid state physics
→ Energy bands

General solution $$\Psi(\varphi) = \sum_q c_q \psi_q$$

Bloch waves $$\psi_q(\varphi) = u_q(\varphi)\exp(\imath q\varphi) \quad \text{with} \quad u_q(\varphi) = u_q(\varphi + 2\pi)$$

Charge/pair number variable q is continuous (charge on capacitor!)

→ $\Psi(\varphi)$ is not 2π-periodic

The phase regime

1D problem → Numerical solution straightforward
Variational approach for **approximate** ground state

Trial function for $E_C \ll E_{J0}$ $\quad \Psi(\varphi) \propto \exp\left(-\frac{\varphi^2}{4\sigma^2}\right)$

Choose σ to find minimum energy:

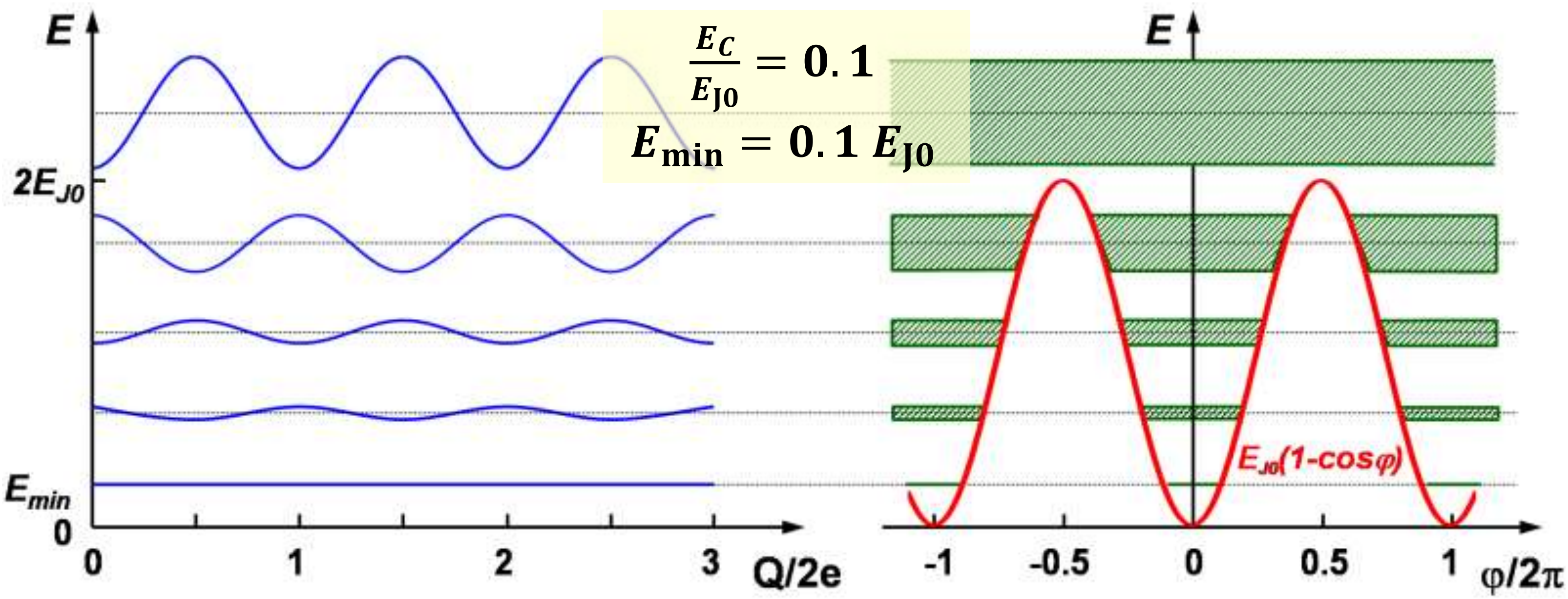
$$E_{\min} = E_{J0}\left(1 - \left[1 - \sqrt{\frac{2E_C}{E_{J0}}}\right]^2\right) = E_{J0}\left(1 - \left[1 - \frac{\hbar\omega_p}{2E_{J0}}\right]^2\right)$$

first order in E_{J0}
$E_{\min} \simeq 0$ for $E_C \ll E_{J0}$

$\hbar\omega_p = \sqrt{8E_{J0}E_C}$

$\frac{E_C}{E_{J0}} = 0.1$

$E_{\min} = 0.1\, E_{J0}$



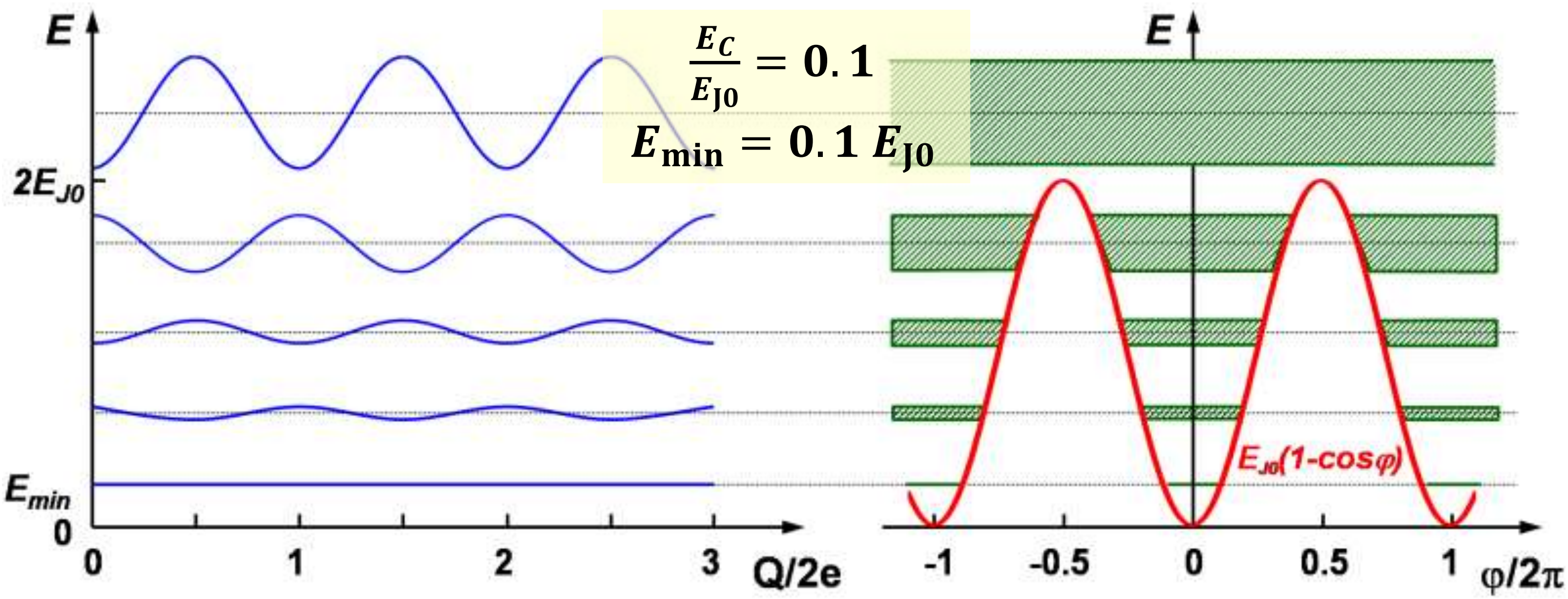
3.6.2 Limiting Cases: The Phase and Charge Regime

The phase regime

Tunneling coupling $\propto \exp\left(-\frac{2E_{J0}-E}{\hbar\omega_p}\right)$ → Very small since $\hbar\omega_p \ll E_{J0}$

→ Tunneling splitting of low lying states is exponentially small

$$E_{min} = E_{J0}\left(1-\left[1-\sqrt{\frac{2E_C}{E_{J0}}}\right]^2\right) = E_{J0}\left(1-\left[1-\frac{\hbar\omega_p}{2E_{J0}}\right]^2\right)$$



$$\frac{E_C}{E_{J0}} = 0.1$$

$$E_{min} = 0.1\,E_{J0}$$

The charge regime

$\hbar\omega_{\rm p} \gg E_{\rm J0}$, $E_C \gg E_{\rm J0}$ → Charge Q (momentum) is good quantum number

Kinetic energy $\propto E_c \left(\frac{d\varphi}{dt}\right)^2$ dominates

→ Complete delocalization of phase

→ Wave function should approach constant value, $\Psi(\varphi) \simeq \text{const.}$

→ Large phase fluctuations, small charge fluctuations $(\Delta Q \cdot \Delta\varphi \geq 2e)$

Hamiltonian

$$\mathcal{H} = -4E_C \frac{\partial^2}{\partial\varphi^2} + E_{J0}(1-\cos\varphi)$$

Appropriate trial function:

$$\Psi(\varphi) \propto (1-\alpha\cos\varphi) \qquad \alpha \ll 1$$

Approximate ground state energy

$$E_{\rm min} \simeq E_{J0}\left(1-\frac{E_{J0}}{8E_C}\right) = E_{J0}\left(1-\frac{E_{J0}^2}{(\hbar\omega_p)^2}\right)$$

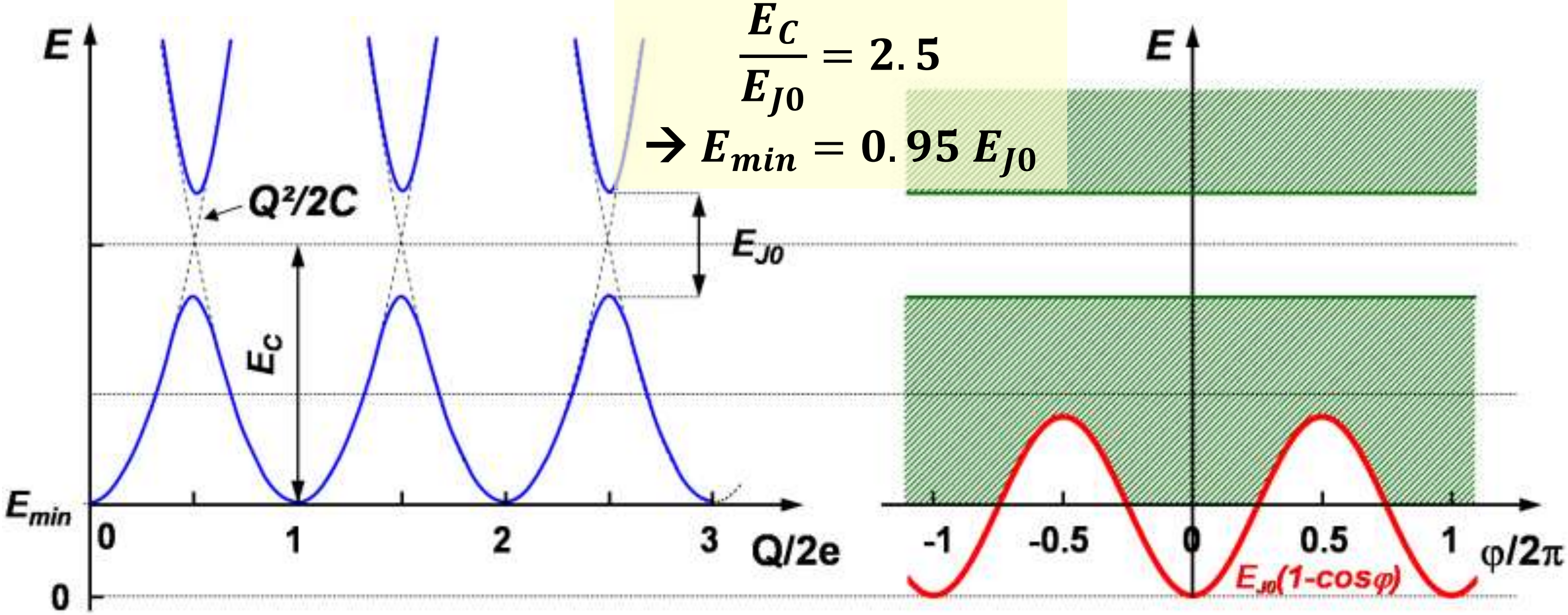
second order in E_{J0}

The charge regime

$$E_{\min} \simeq E_{J0}\left(1-\frac{E_{J0}}{8E_C}\right) = E_{J0}\left(1-\frac{E_{J0}^2}{(\hbar\omega_p)^2}\right)$$

$$\frac{E_C}{E_{J0}} = 2.5$$

$\rightarrow E_{min} = 0.95\, E_{J0}$

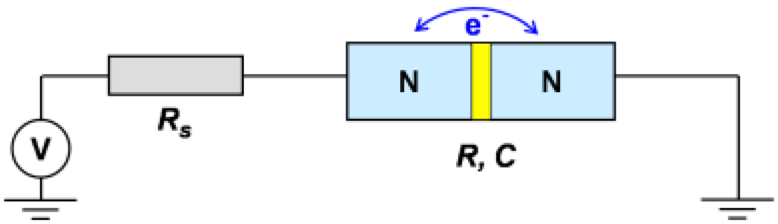


→ Periodic potential is weak

→ Strong coupling between neighboring phase states → Broad bands

→ Compare to electrons moving in strong (phase regime) or weak (charge regime) periodic potential of a crystal

Coulomb blockade in normal metal tunnel junctions



Voltage V → Charge $Q = CV$, energy $E = \frac{Q^2}{2C}$

Single electron tunneling

→ Charge on one electrode changes to $Q - e$

→ Electrostatic energy $E' = \frac{(Q-e)^2}{2C}$

→ Tunneling only allowed for $E' \leq E$

→ Coulomb blockade: Need $|Q| \geq e/2$ or $|V| \geq V_{\text{CB}} = V_{\text{c}} = e/2C$

Observation of CB requires small thermal fluctuations

→ $E_C = \frac{e^2}{2C} > k_B T \quad \Rightarrow \quad C < \frac{e^2}{2k_B T}$

→ $C \simeq 1$ fF at T $= 1$ K, $d = 1$ nm, and $\varepsilon = 5$ → $A \lesssim 0.02\ \mu\text{m}^2$ → Small junctions!

Observation of CB requires small quantum fluctuations

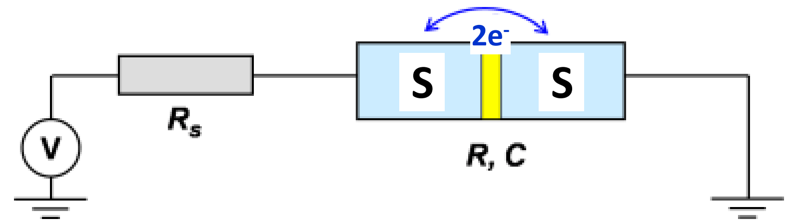
→ Quantum fluctuations due to Heisenberg principle → $\Delta E \cdot \Delta t \geq \hbar$

→ Finite tunnel resistance

→ $\tau_{RC} = RC$ (decay of charge fluctuations)

→ $\Delta t = 2\pi RC,\ \Delta E = \frac{e^2}{2C}$ → $R \geq \frac{h}{e^2} = R_K = 24.6\ \text{k}\Omega$ → Typically satisfied

Coulomb blockade in superconducting tunnel junctions

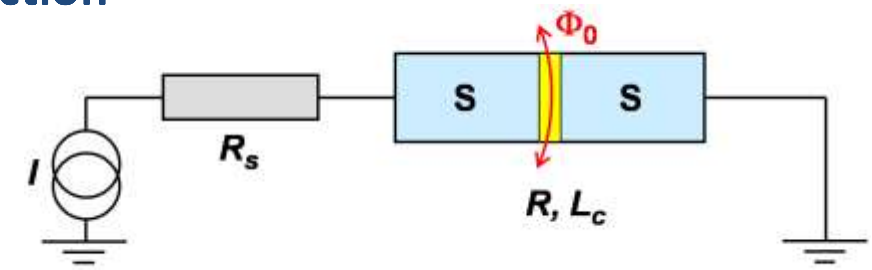


For $\frac{Q^2}{2C} > k_B T, eV \quad (Q = 2e)$ → No flow of Cooper pairs

Threshold voltage → $|V| \geq V_{CB} = V_c = \frac{2e}{2C} = \frac{e}{C}$

Coulomb blockade → Charge is fixed, phase is completely delocalized

Phase or flux blockade in a Josephson junction



Current I → Flux $\Phi = L\,I$, energy $E = \Phi^2/2L$

→ Phase is blocked due to large $E_{J0} = \Phi_0 I_c/2\pi$

→ I_c takes the role of V_{CB}

→ Phase change of 2π equivalent to flux change of Φ_0

→ Flux blockade $|I| \geq I_{FB} = I_c = \frac{(\Phi_0/2\pi)}{L_c}$

$L_c = \frac{\hbar}{2eI_c}$

→ Analogy to CB → $I \leftrightarrow V, \quad 2e \leftrightarrow \frac{\Phi_0}{2\pi}, \quad C \leftrightarrow L$

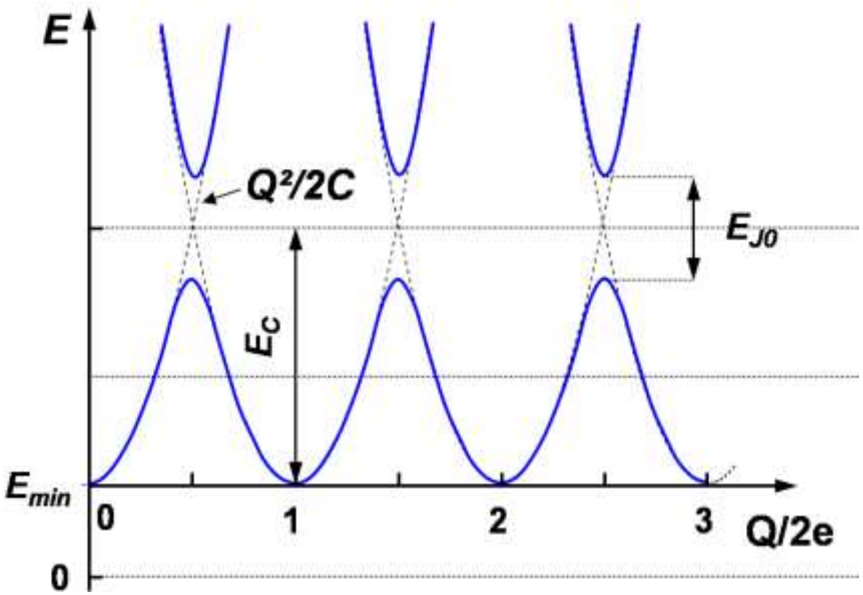
In presence of fluctuations we need

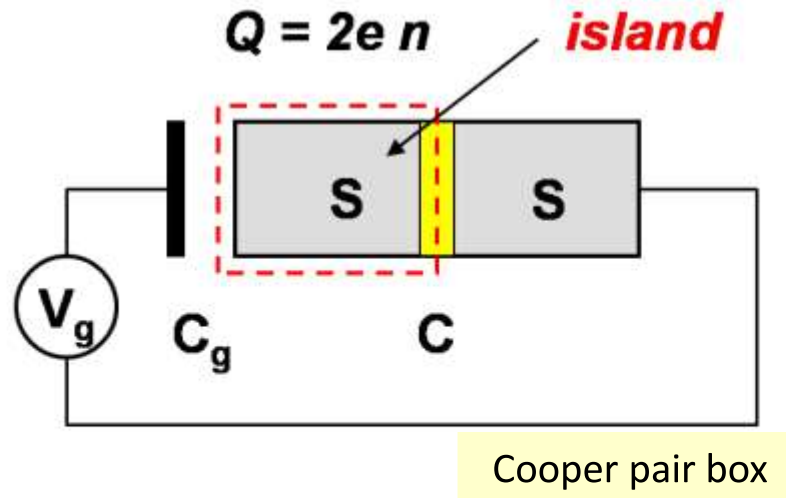
→ $E_{J0} \gg k_B T$ (large junction area)

→ And $\Delta E \cdot \Delta t \geq \hbar$ with $\Delta t = 2\pi \frac{L}{R}$ and $\Delta E = 2E_{J0}$ → $R \leq \frac{h}{(2e)^2} = \frac{1}{4} R_K$

Coherent charge states

Island charge continuously changed by gate





Independent charge states ($E_{J0} = 0$)

→ Parabola $E(Q) = (Q - n \cdot 2e)^2 / 2C_\Sigma$

$E_{J0} > 0$

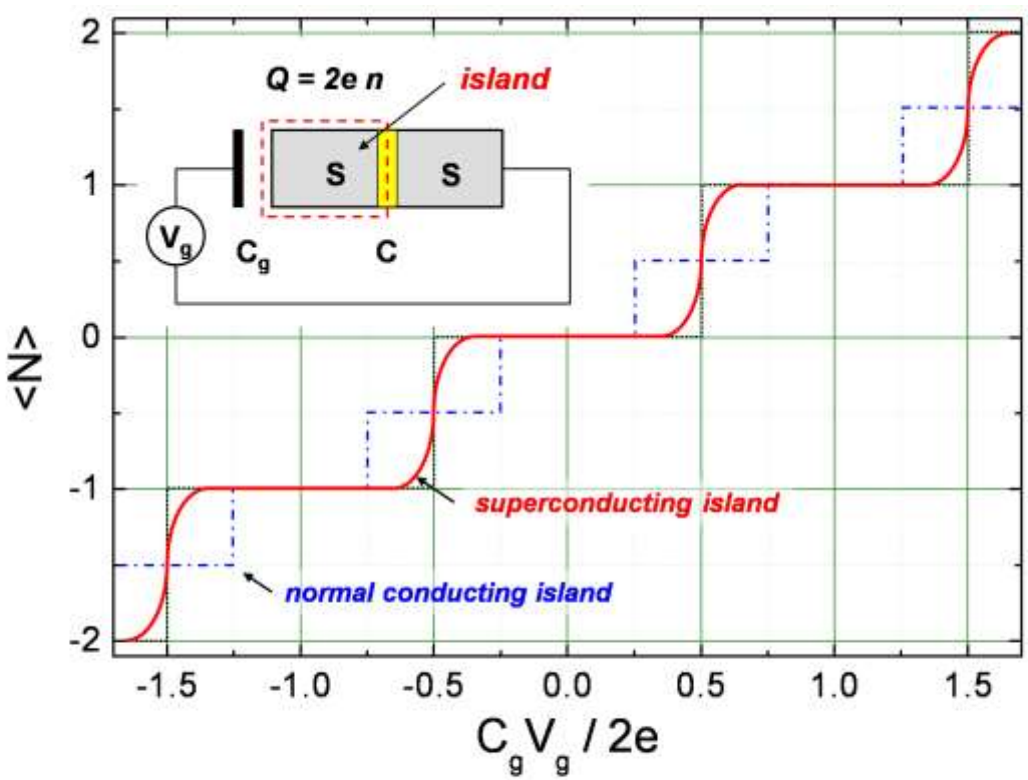
→ Interaction of $|n\rangle$ and $|n+1\rangle$ at the level crossing points $Q = \left(n + \frac{1}{2}\right) \cdot 2e$

→ Avoided level crossing (anti-crossing)

→ Coherent superposition states $|\Psi_\pm\rangle = \alpha|n\rangle \pm \beta|n+1\rangle$

Coherent charge states

Average charge on the island as a function of the applied gate voltage
→ Quantized in units of $2e$ (no coherence yet)



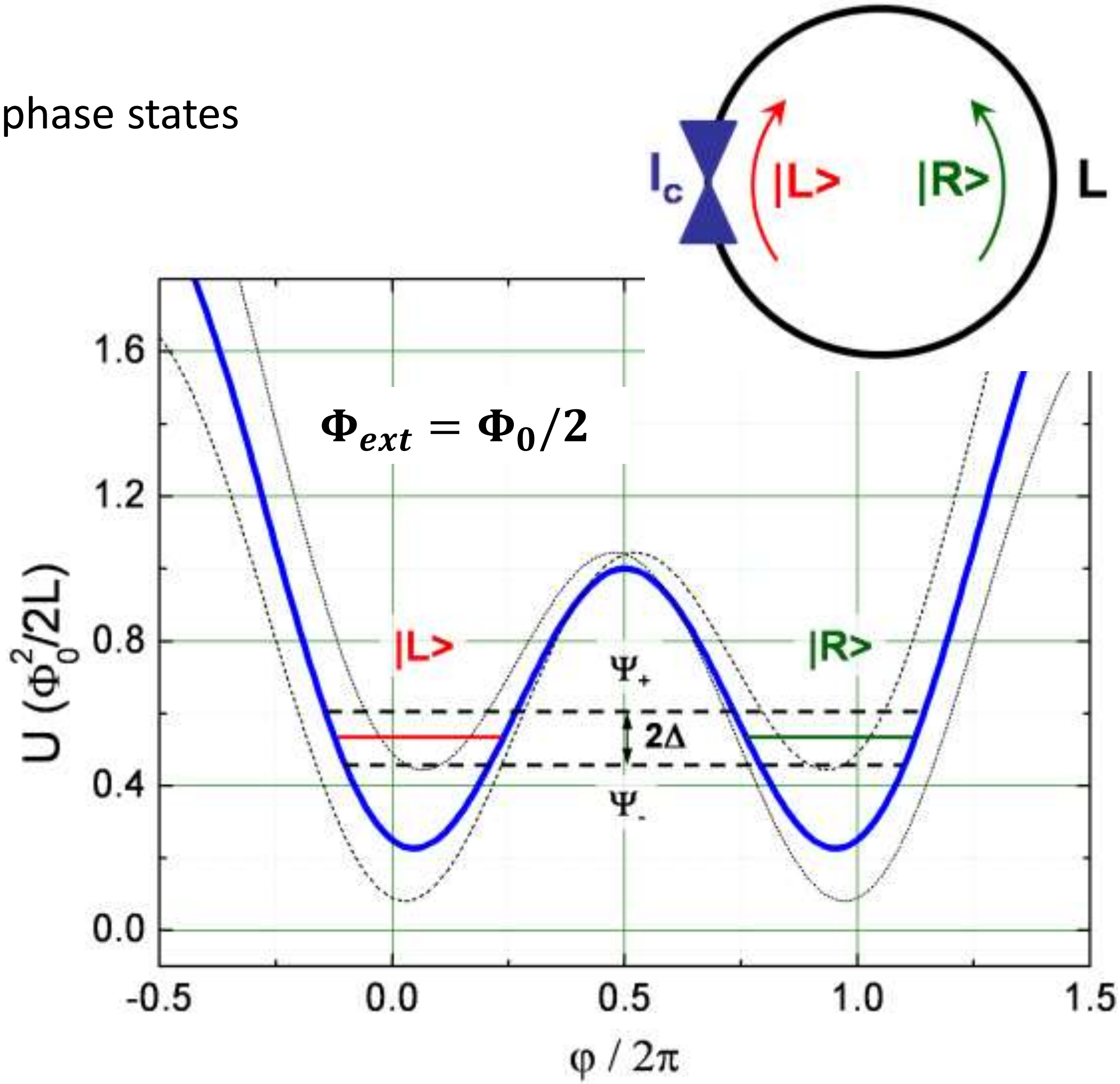
First experimental demonstration of coherent superposition charge states by Nakamura, Pashkin, Tsai (1999, not this picture)

Coherent phase states

→ Interaction of two adjacent phase states
→ Example is rf SQUID

magnetic energy
of flux $\phi = \left(\frac{\Phi_0}{2\pi}\right)\ \varphi$ in the ring

$$U(\phi) = \frac{(\phi - \phi_{\text{ext}})^2}{2L} + E_{J0}\left(1 - \cos 2\pi \frac{\phi}{\Phi_0}\right)$$



Tunnel coupling → $\Psi_{\pm} = a|L\rangle \pm b|R\rangle$
Experimental evidence for quantum coherent superposition (Mooij et al., 1999)

3.6.5 Quantum Fluctuations

Violation of conservation of energy on small time scales, obey $\Delta E \cdot \Delta t \geq \hbar$

→ Creation of **virtual excitations**

→ Include Langevin force I_F with adequate statistical properties

→ Fluctuation-dissipation theorem

$$S_I(f) = 2\pi S_I(\omega) = 4\frac{E(\omega,T)}{R_N}$$

$E(\omega,T)$ = energy of a quantum oscillator

$$E(\omega,T) = \underbrace{\frac{\hbar\omega}{2}}_{\text{vacuum fluctuations}} + \hbar\omega \underbrace{\frac{1}{\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1}}_{\text{occupation probability of oscillator (Planck distribution)}} = \frac{\hbar\omega}{2}\coth\left(\frac{\hbar\omega}{2k_B T}\right)$$

→ Transition from “thermal” Johnson-Nyquist noise to quantum noise:

classical limit ($\hbar\omega, eV \ll k_B T$): $$S_I(\omega) = \frac{1}{2\pi}\frac{4k_B T}{R_N}$$

quantum limit ($\hbar\omega, eV \gg k_B T$): $$S_I(\omega) = \frac{1}{2\pi}\frac{2\hbar\omega}{R_N} = \frac{1}{2\pi}\frac{2eV}{R_N}$$

Escape of the "phase particle" from minimum of washboard potential by tunneling

→ Macroscopic, i.e., phase difference is tunneling (collective state)
→ States easily distinguishable

Competing process

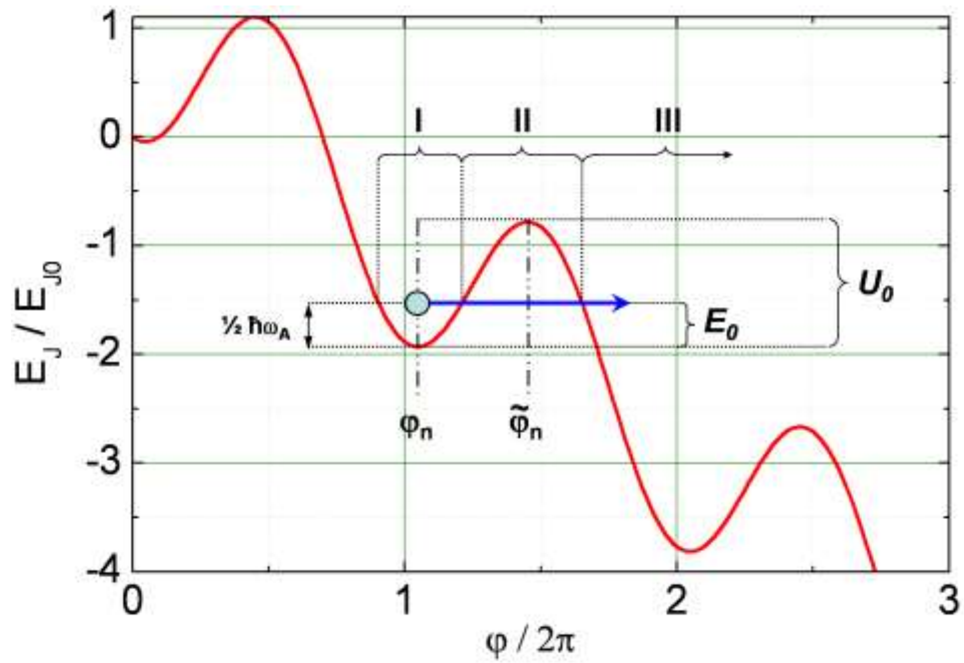
→ Thermal activation
→ Low temperatures

Neglect damping

Dc-bias → Term $-\frac{\hbar I \varphi}{2e}$ in Hamiltonian

Curvature at potential minimum:

$$\frac{\partial^2 U}{\partial \varphi^2} = E_{J0}\sqrt{1-i^2} \qquad i = I/I_c$$



(Classical) small oscillation frequency:

$$\omega_A = \omega_p(1-i^2)^{1/4} \qquad \text{(attempt frequency)}$$

3.6.6 Macroscopic Quantum Tunneling

Quantum mechanical treatment

→ Tunnel coupling of bound states to outgoing waves → Continuum of states
→ But only states corresponding to quasi-bound states have high amplitude
→ In-well states of width $\Gamma = \hbar/\tau$ (τ = lifetime for escape)

Determination of wave functions

→ Wave matching method
→ Exponential prefactor within WKB approximation
→ Decay in barrier:

$$|\Psi(x)|^2 \propto \exp\left\{-\frac{2}{\hbar}\int_{\text{II}} \sqrt{2M[V(x)-E]}\,dx\right\}$$

decay of wave function of particle with mass M and energy E

for $U(\varphi) \gg E_0 = \hbar\omega_A/2$

$$|\Psi(\varphi)|^2 \propto \exp\left\{-\frac{2}{\hbar}\int_{\text{II}} \sqrt{2\underbrace{\left(\frac{\hbar}{2e}\right)^2 C}_{\text{mass}}\underbrace{\left[U(\varphi)-\frac{\hbar\omega_A}{2}\right]}_{\text{effective barrier height}}}\,d\varphi\right\}$$

Constant barrier height → Escape rate

$$|\Psi(\varphi)|^2 \propto \exp\left\{-\frac{2}{\hbar}\int\sqrt{2\left(\frac{\hbar}{2e}\right)^2 C\left[U(\varphi)-\frac{\hbar\omega_A}{2}\right]}\,d\varphi\right\}$$

$$|\Psi(\varphi)|^2 \propto \exp\left\{-\sqrt{\frac{U_0}{E_C}}\,\Delta\varphi\right\} \quad \Rightarrow \Gamma = \frac{\omega_A}{2\pi}\exp\left\{-\sqrt{\frac{U_0}{E_C}}\,\Delta\varphi\right\}$$

very small for $i \simeq 0$

Increasing bias current →

$$U_0 \simeq 2E_{J0}(1-i^2)^{3/2} \quad \text{and} \quad \Delta\varphi \simeq \pi\sqrt{1-i^2}$$

U_0 decreases with i → Γ becomes measurable

Temperature $T^\star$ where $\Gamma_{\text{tunnel}} = \Gamma_{\text{TA}} \approx \exp\left(-\frac{U_0}{k_B T}\right)$

for $I \simeq 0$: $U_0 \simeq 2E_{J0}$ $\quad \hbar\omega_p = \sqrt{8E_{J0}E_C} \simeq 2\sqrt{U_0 E_C}$ $\quad \Delta\varphi \simeq \pi$

$$\Rightarrow \Gamma = \frac{\omega_p}{2\pi}\exp\left\{-2\pi\frac{U_0}{\hbar\omega_p}\right\} \qquad \Rightarrow k_B T^\star \simeq \frac{\hbar\omega_p}{2\pi}$$

for $I > 0$: $\sqrt{U_0} \propto (1-i^2)^{3/4}$ $\quad \Delta\varphi \propto (1-i^2)^{1/2}$

$$\Rightarrow k_B T^\star \simeq \frac{\hbar\omega_A}{2\pi} = \frac{\hbar\omega_p}{2\pi}(1-i^2)^{1/4}$$

For $\omega_p \approx 10^{11}\ s^{-1}$
→ $T^* \approx 100$ mK

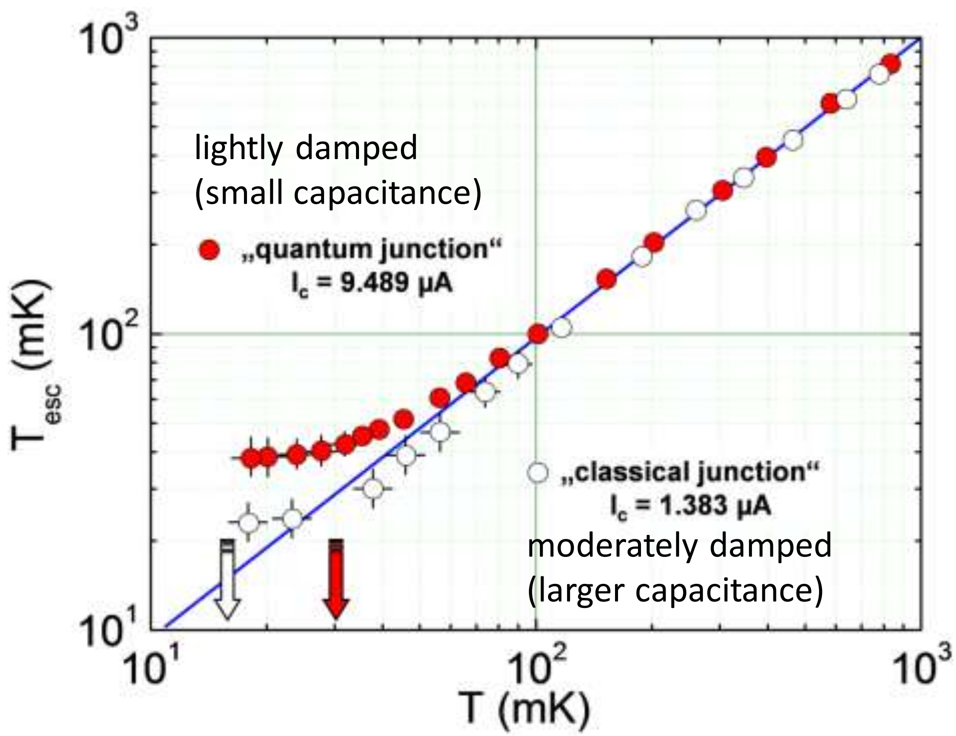
Additional topic: Effect of damping

Junction couples to the environment (e.g., Caldeira-Leggett-type heat bath)

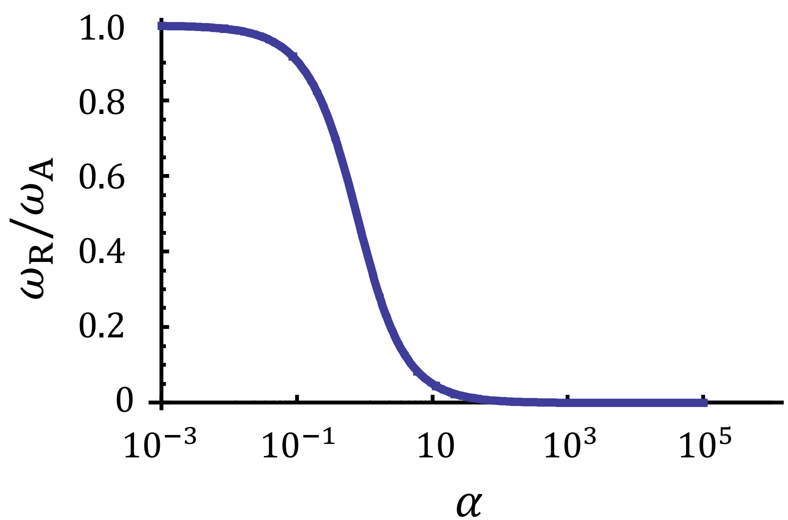
→ Crossover temperature $k_B T^\star \approx \frac{\hbar\omega_\mathrm{R}}{2\pi}$ with $\omega_\mathrm{R} = \omega_A\left(\sqrt{1+\alpha^2} - \alpha\right)$ and $\alpha \equiv \frac{1}{2R_\mathrm{N} C\omega_\mathrm{A}}$

→ Strong damping $\alpha \gg 1$ → $\omega_\mathrm{R} \ll \omega_\mathrm{A}$ → Lower $T^\star$

→ Damping suppresses MQT



After: Martinis *et al.*, Phys. Rev. B. **35**, 4682 (1987).



Phase diffusion by MQT

See lecture notes

Summary (secondary quantum macroscopic effects)

Classical description only in the phase regime (large junctions): $E_C \ll E_{J0}$

For $E_c \gg E_{J0}$: quantum description (negligible damping):

$$\mathcal{H} = 4E_C \frac{\partial^2}{\partial\varphi^2} + E_{J0}(1 - \cos\varphi)$$

Phase difference φ and Cooper pair number $N = \frac{Q}{2e}$ are canonically conjugate variables

$$[\varphi, \frac{\hbar}{2e}Q] = \imath\hbar \qquad \Rightarrow \Delta N \cdot \Delta\varphi \geq 1$$

phase regime: $\Delta\varphi \to 0$ and $\Delta N \to \infty$
charge regime: $\Delta N \to 0$ and $\Delta\varphi \to \infty$

Charge regime at $T = 0$
→ Coulomb blockade → Tunneling only for $V_{\mathrm{CB}} \geq \frac{e}{C}$
Flux regime at $T = 0$
→ Flux blockade → Flux motion only for $I_{\mathrm{FB}} \geq \frac{\Phi_0}{2\pi L_c}$

At $I < I_c$
→ Escape out of the washboard by thermal activation or macroscopic quantum tunneling

TA-MQT crossover temperature $T^\star$

$$k_B T^\star \simeq \frac{\hbar\omega_A}{2\pi} = \frac{\hbar\omega_p}{2\pi}\left[1 - \left(\frac{I}{I_c}\right)^2\right]^{1/4}$$