6.2
Introduction to quantum information processing
6.2 Introduction to information processing

Information

→ General concept (similar to energy)
→ Many forms: Mechanical, thermal, electric, ...
→ Can be packed into equivalent forms:

0 1 ↓ ↑ ○ ●

→ Information is physical (Landauer 1991)
  - Ink on paper
  - Charge on capacitor
  - Currents in leads
  - Spins
  - ...

Information has uncertainty!
→ Shannon entropy
→ Measure of this uncertainty

Can be quantified: The random variable $X$ distributed according to $p(x)$ contains the information

$$ S[p(x)] = - \sum_x p(x) \log_2 p(x) $$

E.g., in the process of throwing a dice one may gain the information

$$ S = - \log_2 (1/6) $$
6.2 Introduction to information processing

Konrad Zuse (1945)

→ Built the first binary digital computer („Z1“) in 1938.
→ First programmable electromechanical computer („Z3“) completed in 1941.
→ Developed the first algorithmic programming language („Plankalkül“).
6.2 Introduction to information processing

John von Neumann (1945)

→ Proposed the EDVAC computer in 1945.
→ Introduced the concept of a computer that is controlled by a stored program.
6.2 Introduction to information processing

The „electronic numerical integrator and computer“ (ENIAC, 1946)

→ Built at the University of Pennsylvania
→ 18,000 vacuum tubes
→ Weight of 30 tons
→ Space of 160 m²
→ Required 6 operators

Left to right:
6.2 Introduction to information processing

Moore’s law

In 1965, Moore observed an exponential growth in the number of transistors per integrated circuit → Prediction: trend continues

Moore’s Law – The number of transistors on integrated circuit chips (1971-2018)

Moore's law describes the empirical regularity that the number of transistors on integrated circuits doubles approximately every two years. This advancement is important as other aspects of technological progress – such as processing speed or the price of electronic products – are linked to Moore’s law.


The data visualization is available at OurWorldInData.org. There you find more visualizations and research on this topic.
6.2.1 Computational complexity

Complexity of a „yes/no“-problem

Example: Is \( m \) a prime number?

→ Expressed in terms of required resources (memory space & computing time) as a function of the problem size \( N \)

→ Prime number factorization \( \rightarrow N = \log_2 m \) (number of bits)

<table>
<thead>
<tr>
<th>Class</th>
<th>Description</th>
<th>Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L )</td>
<td>Logarithmic memory space</td>
<td>( \mathcal{O}(\ln N) )</td>
</tr>
<tr>
<td>( \text{PSPACE} )</td>
<td>Polynomial memory space</td>
<td>( \mathcal{O}(\ln N^k) )</td>
</tr>
<tr>
<td>( \text{P, QP} )</td>
<td>Polynomial execution time</td>
<td>( \mathcal{O}(N^k) )</td>
</tr>
<tr>
<td>( \text{NP} )</td>
<td>Not solvable in polynomial time</td>
<td>( \mathcal{O}(\ln N) )</td>
</tr>
<tr>
<td>( \text{EXP} )</td>
<td>Exponential execution time</td>
<td>( \mathcal{O}(k^N) )</td>
</tr>
</tbody>
</table>

Known relations between the complexity classes \( \rightarrow L \subseteq P \subseteq NP \subseteq \text{PSPACE} \subseteq \text{EXP} \)
6.2.1 Computational complexity: Factorization

**Integer factorization** → NP hard on classical computer!

- Algorithm is known, but it takes too long
- Exponential speedup by quantum algorithm (Shor)
6.2.1 Computational complexity: Factorization

**Integer factorization**

- **Best known classical algorithm** (number field sieve)
  - Required time $\propto \exp \left[ 2 \left( \frac{1}{3} \ln m \right) \left( \ln \ln m \right)^{\frac{2}{3}} \right]$
  - Exponential in bit number $N \equiv \log_2 m$
  - Factorization of 400-digit decimal takes $\approx 10^{10}$ years

- **Shor quantum algorithm** (Peter Shor, 1994)
  - Required time $\propto (\ln m)^3$
  - Factorization of 400-digit decimal takes a few years

**Largest known prime number** (2017):

$$2^{74207281} - 1 \rightarrow 22\ 338\ 618 - \text{digit decimal}$$

- Quantum algorithms can reduce the execution time to human timescales, where classical algorithms would take „forever“ in terms of these timescales!
6.2.1 Computational complexity: Quantum simulation

Example: N interacting spins \((S = \frac{1}{2})\)

\[ N = 1000 \rightarrow \text{Dimension of Hilbert space} = 2^{1000} > \text{number of atoms in universe} \]

**Richard Feynman (1981):**

“... nature isn’t classical, dammit, and if you want to make a simulation of nature, you'd better make it quantum mechanical, and by golly it's a wonderful problem because it doesn't look so easy.”

Quantum simulation \(\rightarrow\) Encode the dynamics of a difficult-to-access quantum system in another quantum system, which can be more easily accessed

Example \(\rightarrow\) Dirac dynamics of a relativistic electron in (nonrelativistic) trapped ion system (Nature 463, 68-71, 2010)

Advantage \(\rightarrow\) Less demanding than a universal quantum computer
### 6.2.2 Improvement of IP systems

#### Exploitation of new functionalities

<table>
<thead>
<tr>
<th>field</th>
<th>electronics</th>
<th>opto electronics</th>
<th>fluxonics</th>
<th>mechatronics</th>
<th>spintronics</th>
</tr>
</thead>
<tbody>
<tr>
<td>degree of freedom</td>
<td>charge</td>
<td>charge + optical</td>
<td>charge + fluxonic</td>
<td>charge + mechanical</td>
<td>charge + spin degree</td>
</tr>
<tr>
<td></td>
<td></td>
<td>degree of freedom</td>
<td>degree of freedom</td>
<td>degree of freedom</td>
<td>degree of freedom</td>
</tr>
</tbody>
</table>

- RSFQ-Logic
- ultrafast AD-converters
  
- MRAM
- GMR read heads
  - spin transistor
### 6.2.2 Improvement of IP systems

- **Today**: Multi-electron, spin, fluxon, photon devices, classical description.
- **Near future**: Single-electron, spin, fluxon, photon devices, quantifiable, but not quantum description.
- **Far future**: Quantum-electron, spin, fluxon, photon devices, quantum description.

- **65 nm process 2005**
- **Single electron transistor**
- **Superconducting qubit**

**Notes:**
- Intel 30nm process
- Source, tunnel junctions, drain
- Gate, 10 μm scale
6.2.2 Improvement of IP systems
6.2 Introduction to QIP

**Classical**

Point on a classical trajectory $x(t)$ and $p(t)$ can be measured simultaneously at arbitrary precisions.

- No noise $\Rightarrow (\Delta x) = (\Delta p) = 0$
- Classical noise $\Rightarrow$ Probability distribution $\Rightarrow$ Nonnegative

**Quantum mechanical**

Quantum parallelism

$\Rightarrow$ Superposition of states

$\Rightarrow$ Entanglement of states

Vacuum noise (quantum)

$\Rightarrow (\Delta x)(\Delta p) \geq \hbar/2$

$\Rightarrow$ Quasiprobability distribution

$\Rightarrow$ Can become negative (e.g., Fock states)
6.2 Introduction to QIP

Superposition of (basis) states
A quantum degree of freedom is described through a wave function.

Entanglement between states
Two quantum degrees of freedom can exhibit stronger correlations than any classical system. (“Superposition between Hilbert spaces”)
6.2 Introduction to QIP

Not entangled, separable (product) states: \( |\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle \)

(Quantum) Entanglement \( |\Psi\rangle \neq |\Psi_1\rangle \otimes |\Psi_2\rangle \)

Uniquely quantum mechanical property of a composite system

No/restricted knowledge on subsystems despite perfect knowledge on combined system

- Quantum correlations

Entanglement between spatially separate subsystems

- Quantum teleportation
- Quantum communication
- Quantum illumination (radar)
- Quantum cryptography
6.2 Introduction to QIP

Realization of a quantum two-level system

Natural systems:
- Natural spins $|\downarrow\rangle$, $|\uparrow\rangle$
- Orthogonally polarized light $|\downarrow\rangle$, $|\leftrightarrow\rangle$

Effective systems:
- Isolate two levels from a manifold structure
- Mechanisms: Energy separation, selection rules, ...
- Requires nonlinearity!

\[ |e\rangle \quad \begin{array}{c}
\vdots \\
|2\rangle \\
|3\rangle \\
\end{array} \quad \text{Other system states} \]

\[ |g\rangle \quad \{ \text{Two-level system} \} \]
6.2 Introduction to QIP

microscopic

quantum optics
trapped ions

NIST, Innsbruck, Munich

ENS Paris

easily quantum,
but difficult to scale

meso- and macroscopic

solid state devices

Oxford, Stanford,
IBM, MIT...

contact

scalable,
but not easily quantum

Oxford, Stanford,
IBM, MIT...
6.2 Introduction to QIP

**single particle states**

- nuclear spins of P in Si (Kane)
- electron spins in quantum dots
- ......

**global states**

- superconducting flux, charge, phase, charge-phase qubits
### 6.2 Introduction to QIP

#### Classical bits

- **0, 1**
- E.g., implementation by charge states
- 0: \( Q = 0 \)
- 1: \( Q = Q_0 \)

#### Basic gates:
- Single-bit-gate: **NOT**
- Two-bit-gates: **AND**
  **OR**

#### Quantum bits

\[
|\Psi(t)\rangle = \cos\left(\frac{\theta}{2}\right)|e\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|g\rangle
\]

Implementation by any quantum two-level system

- Spin
- Polarization
- Persistent current

#### Basic gates:
- Single-qubit-gate: \( \{ \text{rotations} \)
- Hadamard
- Two-qubit-gates: **C-NOT**
6.2 Introduction to QIP

Classical computer: Bits & gates

Bits $\rightarrow$ Capacitors

$V = 0 \rightarrow 0$

$V > 0 \rightarrow 1$

Gates $\rightarrow$ transistors

$V_g = 0 \rightarrow$ closed

$V_g > 0 \rightarrow$ open

Example for a single-bit gate $\rightarrow$ NOT

Example for a two-bit gate $\rightarrow$ AND

Classical computer: Bits & gates
6.2 Introduction to QIP

Universality theorem
A small number of Single-bit gates (e.g., NOT) and two-bit gates allows for any manipulation on classical bits (universal set)

Truth tables for various 2-bit gates

<table>
<thead>
<tr>
<th>(a,b)</th>
<th>AND</th>
<th>NAND</th>
<th>OR</th>
<th>NOR</th>
<th>EQUIV</th>
<th>XOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0 1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1 0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1 1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Examples

\[
a \text{ OR } b = (a \text{ NAND } a) \text{ NAND } (b \text{ NAND } b)
\]

\[
a \text{ AND } b = (a \text{ NAND } b) \text{ NAND } (a \text{ NAND } b)
\]

\[
\text{NOT } a = a \text{ NAND } a
\]

OR, AND, NOT also form universal set
6.2 Introduction to QIP

Irreversible classical logic (1940 - current):

Logical irreversibility
→ No ability to reconstruct the input from the output

Physical irreversibility
→ Heat dissipated during a gate operation

Intimate connection
→ Increase of entropy due to loss of 1 bit of information, $\Delta S = k_B \ln 2$
→ Dissipative heat → Irreversibility

Si-based computers
→ Way above the ideal limit by a factor
  → $10^{10}$ (transistor)
  → 100 (DNA copying mechanism in human cell)
→ Discussion somewhat academic
6.2 Introduction to QIP

Reversible classical logic (Bennett, 1973)

Conditions

\[ \rightarrow \text{Absence of physical dissipation} \]
\[ \rightarrow \text{Number of output bits equals to number of inputs} \]
\[ \rightarrow \text{Gate produces all input combinations at output} \]

Intrinsically irreversible | Logically reversible
---|---
(a,b) | (a, a NAND b) | (a, a XOR b)
(0,0) | (0,1) | (0,0)
(0,1) | (0,1) | (0,1)
(1,0) | (1,1) | (1,1)
(1,1) | (1,0) | (1,0)

CNOT gate (reversible XOR)
6.2 Introduction to QIP

Three-bit gates (Bennett, 1973)

**Universality**
→ Reversible logic requires three-bit gates

**Toffoli gate**
→ Controlled CNOT gate (CCNOT)
→ Applies a NOT to bit c when both bit a and bit b are “1”

```
    a
   ___________

    b
   ___________

    c
   ___________

```

→ Some two-bit gates are intrinsically irreversible (e.g., NAND)
→ Third bit required to store input safely
6.2 Introduction to QIP

Definition of a quantum bit

Classical bit → Deterministic, either in ground state "g" or in excited state "e"

Quantum bit (qubit) → Superposition of two computational basis states

\[ |\Psi(t)\rangle = a(t)|g\rangle + b(t)|e\rangle \]

\[ a(t), b(t) \in \mathbb{C} \text{ with } |a(t)|^2 + |b(t)|^2 = 1 \]

→ All states can be visualized on the surface of a sphere

Global phase unobservable
→ Bloch sphere representation

\[ |\Psi(t)\rangle = \cos\left(\frac{\theta(t)}{2}\right)|e\rangle + e^{i \varphi(t)} \sin\left(\frac{\theta(t)}{2}\right)|g\rangle \]

Bloch angles

\[ \theta(t) \rightarrow \text{Amplitude} \rightarrow \text{Energy, population} \]
\[ \varphi(t) \rightarrow \text{Phase} \rightarrow \text{Coherence} \]
6.2 Introduction to QIP

Linear algebra notation of operators and state vectors

Qubit states can be written as vectors

\[ a|e\rangle + b|g\rangle \quad \Rightarrow \quad a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \]

Qubit operators (gates) can be written as matrices

\[ a|e\rangle\langle e| + b|g\rangle\langle g| + c|e\rangle\langle g| + d|g\rangle\langle e| \]

\[ \Rightarrow a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} + a \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} a & c \\ d & b \end{pmatrix} \]
6.2 Introduction to QIP

Pseudo spin and Pauli matrices

\[ |\Psi(t)\rangle = \cos\left(\frac{\theta(t)}{2}\right) |e\rangle + e^{i\varphi(t)} \sin\left(\frac{\theta(t)}{2}\right) |g\rangle \]

Pseudo spin

→ \(|\Psi\rangle\) equivalent to spin wavefunction in external magnetic field

Unitary operations

→ \(\hat{U}|\Psi\rangle\) expressed via the Hermitian Pauli spin matrices \(\hat{1}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\)

\[
\begin{align*}
\hat{1} &\equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\hat{\sigma}_x &\equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\hat{\sigma}_y &\equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
\hat{\sigma}_z &\equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]

\(|g\rangle\) and \(|e\rangle\) are the eigenvectors of \(\hat{\sigma}_z\)
6.2 Introduction to QIP

Conventions: Pauli matrices and Bloch sphere

\[
\hat{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \hat{\sigma}_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\sigma}_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

These definitions contain several conventions, such as:

→ The global scaling factor

→ The position of the minus sign in \( \hat{\sigma}_z \)

→ Here, we show two examples with fixed \( \hat{\sigma}_z \)

→ Physics convention \( \rightarrow |g\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ |e\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \)

→ Ground state energy negative (more „physical“)

Information theory (IT) convention

→ M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press

→ \( |0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)

→ Ground state energy positive („unphysical“)

→ Easily generalized, more „logical“

→ Unless otherwise mentioned \( \rightarrow \) Physics convention!

→ Formal resolution \( \rightarrow \) Equate \( |g\rangle \) to \( |1\rangle \) and \( |e\rangle \) to \( |0\rangle \) \( \rightarrow \) Used in this lecture!
6.2 Introduction to QIP

Important states on the Bloch sphere

Physics

\[ |\Psi(t)\rangle = \cos\left(\frac{\theta}{2}\right) |e\rangle + e^{i\phi} \sin\left(\frac{\theta}{2}\right) |g\rangle \]

IT

\[ |\Psi(t)\rangle = \cos\left(\frac{\theta}{2}\right) |0\rangle + e^{i\phi} \sin\left(\frac{\theta}{2}\right) |1\rangle \]
### 6.2 Introduction to QIP

**Interpretation of the Pauli matrices**

\[
\hat{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\sigma}_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

→ The Pauli matrices can expressed in terms of projection operators

\[\hat{\sigma}_+ = |e\rangle\langle g| \quad \rightarrow \quad \text{Puts an excitation into the qubit}\]

\[\hat{\sigma}_- = |g\rangle\langle e| \quad \rightarrow \quad \text{Removes an excitation from the qubit}\]

\[\hat{\sigma}_x = \hat{\sigma}_- + \hat{\sigma}_+ \quad \rightarrow \quad \text{Induce transitions between } |g\rangle \text{ and } |e\rangle\]

\[\hat{\sigma}_y = i(\hat{\sigma}_- - \hat{\sigma}_+) \quad \rightarrow \quad \langle \hat{\sigma}_y \rangle \text{ gives the qubit population}\]

\[\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g| \quad \rightarrow \quad \langle \hat{\sigma}_z \rangle \text{ gives the qubit population}\]

\[\hat{1} = |g\rangle\langle g| + |e\rangle\langle e| \quad \rightarrow \quad \text{Reflects normalization}\]

→ Combination of basis definition and operator description in terms of projection operators → Matrix from of operators

→ In this lecture, we fix the matrix definitions of the Pauli matrices

→ “Physical” intuition in \{|g\rangle, |e\rangle\}-notation

→ Notation consistent with Nielsen & Chuang and most physics papers!
6.2 Introduction to QIP

**Definition of a single qubit gate**

Single qubit gate

→ Unitary operation $\hat{U}$ on state $|\Psi\rangle$
→ Described by rotations on Bloch sphere + global phase

Rotation matrices

→ Around x-axis  $\rightarrow \hat{R}_x(\alpha) \equiv e^{-i\alpha\hat{\sigma}_x/2} = \begin{pmatrix} \cos \frac{\alpha}{2} & -i \sin \frac{\alpha}{2} \\ -i \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{pmatrix}$
→ Around y-axis  $\rightarrow \hat{R}_y(\alpha) \equiv e^{-i\alpha\hat{\sigma}_y/2} = \begin{pmatrix} \cos \frac{\alpha}{2} & - \sin \frac{\alpha}{2} \\ \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{pmatrix}$
→ Around z-axis  $\rightarrow \hat{R}_z(\alpha) \equiv e^{-i\alpha\hat{\sigma}_z/2} = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$

Why? General unitary expressed by rotations

→ $\hat{U} = e^{i\varepsilon}\hat{R}_z(\beta)\hat{R}_y(\gamma)\hat{R}_z(\delta)$ with $\varepsilon, \beta, \gamma, \delta \in \mathbb{R}$
→ Z-Y decomposition (others possible)
→ $\varepsilon$ is a global phase (unobservable)
### 6.2 Introduction to QIP

**Examples for 1-qubit gates**

<table>
<thead>
<tr>
<th>Gate</th>
<th>Matrix Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identität</td>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix}$</td>
</tr>
<tr>
<td>NOT</td>
<td>$\begin{pmatrix} 0 &amp; 1 \ 1 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>Pauli-Y</td>
<td>$\begin{pmatrix} 0 &amp; -i \ i &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>Pauli-Z</td>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; -1 \end{pmatrix}$</td>
</tr>
<tr>
<td>$\sqrt{\text{NOT}}$</td>
<td>$\frac{1}{2} \begin{pmatrix} 1+i &amp; 1-i \ 1-i &amp; 1+i \end{pmatrix}$</td>
</tr>
<tr>
<td>$\sqrt{X}$</td>
<td>$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 &amp; 1 \ 1 &amp; -1 \end{pmatrix}$</td>
</tr>
<tr>
<td>$\text{Phase} (\sqrt{Z})$</td>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; i \end{pmatrix}$</td>
</tr>
<tr>
<td>$\frac{\pi}{8} (\sqrt{S})$</td>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; e^{i\pi/4} \end{pmatrix}$</td>
</tr>
</tbody>
</table>

**Graphical representation example**

```
NOT a  ⊕  ¬ a
```

Matrix representation (taken from QI theory books) typically follow IT convention!
6.2 Introduction to QIP

Hadamard gate $\hat{H}$ is of particular importance

→ Applied to one of the basis states $|g\rangle$ or $|e\rangle$, it results in a superposition state of the basis states

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} (\hat{\sigma}_x + \hat{\sigma}_z)$$

→ Physics convention

$$\hat{H} = \frac{1}{\sqrt{2}} (|e\rangle\langle e| - |g\rangle\langle g| + |e\rangle\langle g| + |g\rangle\langle e|)$$

$$\hat{H} |g\rangle = \frac{1}{\sqrt{2}} (|e\rangle - |g\rangle)$$

$$\hat{H} |e\rangle = \frac{1}{\sqrt{2}} (|e\rangle + |g\rangle)$$
6.2 Introduction to QIP

Quantum coherence

$$|\Psi(t)\rangle = \cos\left(\frac{\theta(t)}{2}\right)|e\rangle + e^{i\varphi(t)}\sin\left(\frac{\theta(t)}{2}\right)|g\rangle$$

- \(\theta(t)\) → Amplitude → Energy, population
- \(\varphi(t)\) → Phase → Coherence

Ideal quantum system

- Completely isolated
- In reality, however, ...

Environment must interact with \(|\Psi(t)\rangle\) for control

- Uncontrolled interactions (noise) also exist
- Quantum effects (population oscillations, quantum interference, superpositions, entanglement) unobservable after characteristic time

Decoherence time \(T_{\text{dec}}\)

- After \(T_{\text{dec}}\), quantum effects have decayed to \(1/e\) of their original level
- \(T_{\text{dec}}\) is a **time scale** rather than a strict time
- Term “decoherence” originally only referred to phase
- Nowadays sloppily comprises both phase and amplitude effects
6.2 Introduction to QIP

Energy and phase relaxation

$$|\Psi(t)\rangle = \cos\left(\frac{\theta(t)}{2}\right)|e\rangle + e^{i\varphi(t)} \sin\left(\frac{\theta(t)}{2}\right)|g\rangle$$

- $\theta(t) \rightarrow$ Amplitude $\rightarrow$ Energy, population
- $\varphi(t) \rightarrow$ Phase $\rightarrow$ Coherence

Population

- $\rightarrow$ Energy relaxation time $T_1$ or $T_1$
- $k_B T \ll \hbar \omega_{ge} \rightarrow$ decay from $|e\rangle$ to $|g\rangle$
- Nonadiabatic (irreversible) processes
- Induced by high-frequency fluctuations ($\omega \approx \omega_{ge}$)

Phase

- $\rightarrow$ Pure dephasing time $T_\varphi$
- Adiabatic (reversible) processes
- Induced by low-frequency fluctuations ($\omega \rightarrow 0$)
- Often encountered: 1/f-noise
- Real measurements always contain $T_1$-effects

$$T_2^{-1} = (2T_1)^{-1} + T_\varphi^{-1}$$

Nomenclature is not very consistent in literature!
6.2 Introduction to QIP

From single to multi-qubit systems

Single qubit (IT) →

\[ 0 \equiv |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]
\[ 1 \equiv |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ |\Psi\rangle = c_1 |0\rangle + c_2 |1\rangle \]

Two qubits (IT) →

\[ |\Psi\rangle = c_1 |00\rangle + c_2 |10\rangle + c_3 |01\rangle + c_4 |11\rangle \]

\[ |00\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]
\[ |01\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \]
\[ |10\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]
\[ |11\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]
Two-qubit operators

\[ A \otimes B = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \otimes \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11} \begin{pmatrix} B_{11} & B_{12} \end{pmatrix} & A_{12} \begin{pmatrix} B_{11} & B_{12} \end{pmatrix} \\ A_{21} \begin{pmatrix} B_{11} & B_{12} \end{pmatrix} & A_{22} \begin{pmatrix} B_{11} & B_{12} \end{pmatrix} \end{pmatrix} \]

→ Tensor product (blockwise product) of single qubit operators
6.2 Introduction to QIP

The Bell states

→ The Bell states are of particular importance in many QIP protocols
→ Created via a Hadamard and a CNOT gate

\[
\begin{align*}
|x\rangle &\quad \hat{H} \quad |\beta_{xy}\rangle \\
|y\rangle &\quad \quad \quad \quad \quad \quad \quad \\
\end{align*}
\]

\[
|00\rangle \quad |\beta_{00}\rangle \equiv \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \\
|01\rangle \quad |\beta_{01}\rangle \equiv \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \\
|10\rangle \quad |\beta_{10}\rangle \equiv \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) \\
|11\rangle \quad |\beta_{11}\rangle \equiv \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)
\]

\[
\hat{H} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\]
Example: Two spins

Uncorrelated: 4 product states

Correlated: Linear combination of product states

→ Entanglement

\[ |\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle) \]

→ Alice and Bob perform measurements in z-direction on the entangled spins
→ Suppose: Alice first measures her spin and finds \(|\uparrow\rangle\) (50% chance)
→ Then: Bob always measures his spin \(|\downarrow\rangle\) (100%), although he may be far away from Alice → Quantum mechanics is nonlocal!

→ Repetition of the experiment
   → Always the same result → The two entangled spins are fully correlated
→ Heisenberg relation violated if conjugate quantities measured by Alice and Bob („EPR paradox“)?
   → No, Bob’s measurements in x- or y-direction yield equal probabilities
→ Superluminal information exchange?
   → Only if quantum copying („cloning“) was allowed
6.2 Introduction to QIP

No-cloning theorem

Classical bits can be copied easily: C → CC

Quantum bits (quantum states) cannot be copied → No-cloning theorem

→ Proof: Assume that there is a unitary transformation $\hat{U}$ producing copies of $|\alpha\rangle$ and $|\beta\rangle$

\[
\hat{U}|\alpha0\rangle = |\alpha\alpha\rangle \text{ and } \hat{U}|\beta0\rangle = |\beta\beta\rangle
\]

→ However, the quantum copying machine fails in copying state

\[
|\gamma\rangle = \frac{1}{\sqrt{2}} (|\alpha\rangle + |\beta\rangle)
\]

\[
\hat{U}|\gamma0\rangle = \frac{1}{\sqrt{2}} (|\alpha\alpha\rangle + |\beta\beta\rangle) \neq |\gamma\gamma\rangle
\]

Combination of the EPR paradox and the no-cloning theorem

→ Rescues the consistency between quantum mechanics and special relativity

→ No superliminal communication!
Quantum teleportation

- No-cloning theorem forbids copying state $|\phi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$
- However, vanishing at one place and reappearing at another is allowed
- Teleportation
- Teleporting a quantum state (qubit) requires that Alice and Bob share an entangled state $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ ("EPR pair")

Teleportation protocol

1. Alice entangles her spin $|\uparrow\rangle$ with the unknown state $|\phi\rangle$
2. Alice measures what state her two spins are and tells Bob, which of the four possible results she has found
   - Classical communication
3. Bob carries out the appropriate rotation of his spin $|\uparrow\rangle$ by $\pi$
4. As a result, Bob ends up with his spin in the state $|\phi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$
6.2 Introduction to QIP

Quantum gates

Intrinsically reversible

→ Any irreversible manipulation would be associated with heat dissipation
  → Destruction of quantum coherence

→ Universal set of gates → E.g., single qubit rotations and CNOT

→ Three-qubit Toffoli is required (also important for quantum error correction)

Represented by unitary transformations

→ Normalization → length of state vector (on Bloch sphere) stays constant

→ Reversibility requires that matrix can be inverted

→ Complex matrix elements, since components of spinor are complex
6.2 Introduction to QIP

SWAP gate

\[
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\quad \begin{array}{cc}
\circ & \circ \\
\circ & \circ
\end{array}
\quad \begin{array}{c}
\text{b} \\
\text{a}
\end{array}
\]

resp.

\[
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\quad \begin{array}{cc}
\times & \times \\
\times & \times
\end{array}
\quad \begin{array}{c}
\text{b} \\
\text{a}
\end{array}
\]

Interchange of a and b

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

CNOT gate

\[
\begin{array}{c|c|c|c|c}
\text{a} & \text{b} & \text{a}' & \text{b}' \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0
\end{array}
\]
6.2 Introduction to QIP

Universal quantum processor: Required elements

- **qubit** (two-level quantum system)
- **2 qubit gates** (e.g., C-NOT) (controlled interactions)
- **single qubit gate**
- **read out**
# 6.2 Introduction to QIP

| Initialization | $|0> \otimes |0> \otimes .... \otimes |0>$ |
|----------------|---------------------------------------------|
| Preparation of superposition states | $|0> + |1> \otimes |0> + |1> \otimes .... \otimes |0> + |1>$ |
| Example: 3 bit system | $|Y> = a|000> + b|001> + c|010> + d|100> + e|011> + f|101> + g|110> + h|111>$ |

## Computational steps

- Unitary transformations
  - Single-qubit gates
  - Two-qubit gates
- Program
- Parameters

## Quantum error correction

- E.g., Shor, Steane, surface code, cat code

## Quantum algorithm

- Factorization (Shor)
- Database search (Grover)
- ...
6.2 Introduction to QIP

DiVincenzo criteria for scalable QIP

- **Qubits:**
  > The system has to provide a well defined two-level quantum system

- **Preparation of the initial state:**
  > It must be possible to prepare the initial state with sufficient accuracy

- **Decoherence:**
  > The phase coherence time must be long enough to allow for a sufficiently large number (typically >10⁴) of coherent manipulations

- **Quantum gates:**
  > There must be sufficient control over the qubit Hamiltonian to perform the necessary unitary transformations, i.e., single- and two-qubit operations

- **Quantum measurement:**
  > For read-out of the quantum information a quantum measurement is needed

- **Scalability:**
  > There should be the possibility to increase to number of qubits

6.2 Introduction to QIP