Control of quantum two-level systems

6.3 Control of quantum two-level systems

General concept – How to control a qubit?

Does the answer depend on specific realization? → No, only its implementation

Qubit is **pseudo spin**

→ **General concept exists**

→ Independent of qubit realization

→ Methods from nuclear magnetic resonance (NMR)

Nuclear magnetic resonance

→ Method to explore the magnetism of nuclear spins

→ Important application → **Magnetic resonance imaging (MRI)** in medicine

→ MRI exploits the different nuclear magnetic signatures of different tissues

Brief MRI history

early suggestions → **H. Carr** (1950) and **V. Ivanov** (1960)

1972 → MRI imaging machine proposed by **R. Damadian** (SUNY)

1973 → 1st MRI image by **P. Lauterbur** (Urbana-Champaign)

Late 1970ies → Fast scanning technique proposed by **P. Mansfield** (Nottingham)

2003 Nobel Prize in Medicine → for P. Lauterbur and P. Mansfield

Overview: Important NMR techniques

Basic idea

→ Rotate spins by static or oscillating magnetic fields

→ Static fields parallel to quantization axis

  → Free precession

  → Changes φ on Bloch sphere

→ Oscillating fields perpendicular to quantization axis

  → Change population

  → Changes θ on Bloch sphere

Important protocols

→ Rabi → Population oscillations → Controlled excitations

→ Relaxation measurement → T_1

→ Ramsey fringes → T_2^*

→ Spin echo (Hahn echo) corrects T_2^* for reversible dephasing→ T_2

NMR reminder

→ Spin-lattice (longitudinal) relaxation time τ_1

→ Spin-spin (transversal) relaxation time $\tau_2 \leq 2\tau_1$

→ Free induction decay (FID) in presence of magnetic field inhomogeneities → $\tau_2^* \leq \tau_2$

→ In traditional NMR Hahn echo does not always yield τ_2 → Multiple echoes (CPMG)

The Hamiltonian of a quantum two-level system (TLS)

Arbitrary TLS → $\widehat{H} = \frac{1}{2}\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$ is Hermitian matrix

→ $H_{11}, H_{22} \in \mathbb{R}$ and $H_{21} = H_{12}^{\star}$

→ we choose $H_{11} > H_{22}$

Symmetrize → Subtract global energy offset $\frac{H_{11}+H_{22}}{4} \times \hat{1}$

→ $\widehat{H} = \frac{1}{2}\begin{pmatrix} \epsilon & \Delta - i\widetilde{\Delta} \\ \Delta + i\widetilde{\Delta} & -\epsilon \end{pmatrix} = \frac{\epsilon}{2}\hat{\sigma}_z + \frac{\widetilde{\Delta}}{2}\hat{\sigma}_y + \frac{\Delta}{2}\hat{\sigma}_x$

→ $\epsilon \equiv \frac{H_{11}-H_{22}}{2} > 0$ and $\Delta, \widetilde{\Delta} \in \mathbb{R}$

→ Without loss of generality, we usually assume $\tilde{\Delta} = 0$

Natural or physical basis $\{|\varphi_+\rangle, |\varphi_-\rangle\}$

→ $\widehat{H}_0 = \frac{1}{2}\begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix} = \frac{\epsilon}{2}\hat{\sigma}_z$

→ $\widehat{H}|\varphi_\pm\rangle = \pm\frac{\epsilon}{2}$

A quantum TLS is called „qubit“ in information processing!

C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics*, Volume One (Wiley –VCH)

6.3 Control of quantum two-level systems

$$\widehat{H} = \frac{1}{2}\begin{pmatrix} \epsilon & \Delta - i\widetilde{\Delta} \\ \Delta + i\widetilde{\Delta} & -\epsilon \end{pmatrix}$$

Diagonalization of $\widehat{H}$

Eigenvalues

$$\rightarrow \det\begin{pmatrix} \epsilon - 2E_\pm & \Delta - i\widetilde{\Delta} \\ \Delta + i\widetilde{\Delta} & -\epsilon - 2E_\pm \end{pmatrix} = 0$$

$$\rightarrow (\epsilon - 2E_\pm)(-\epsilon - 2E_\pm) - (\Delta - i\widetilde{\Delta})(\Delta + i\widetilde{\Delta}) = 0$$

$$\rightarrow E_\pm = \pm\frac{1}{2}\sqrt{\epsilon^2 + \Delta^2 + \widetilde{\Delta}^2} \equiv \pm\frac{1}{2}\hbar\omega_\mathrm{q}$$

Transition energy conveniently expressed in frequency units

Eigenvectors

$$\rightarrow \widehat{H}|\Psi_\pm\rangle = \mathrm{E}_\pm|\Psi_\pm\rangle$$

$$\rightarrow |\Psi_-\rangle = -e^{-i\varphi/2}\sin\frac{\theta}{2}|\varphi_+\rangle + e^{+i\varphi/2}\cos\frac{\theta}{2}|\varphi_-\rangle$$

$$\rightarrow |\Psi_+\rangle = +e^{-i\varphi/2}\cos\frac{\theta}{2}|\varphi_+\rangle + e^{+i\varphi/2}\sin\frac{\theta}{2}|\varphi_-\rangle$$

Recovers Bloch sphere picture!

$$\tan\theta \equiv \frac{\sqrt{\Delta^2 + \widetilde{\Delta}^2}}{\epsilon} \quad \text{with } 0 \leq \theta \leq \pi$$

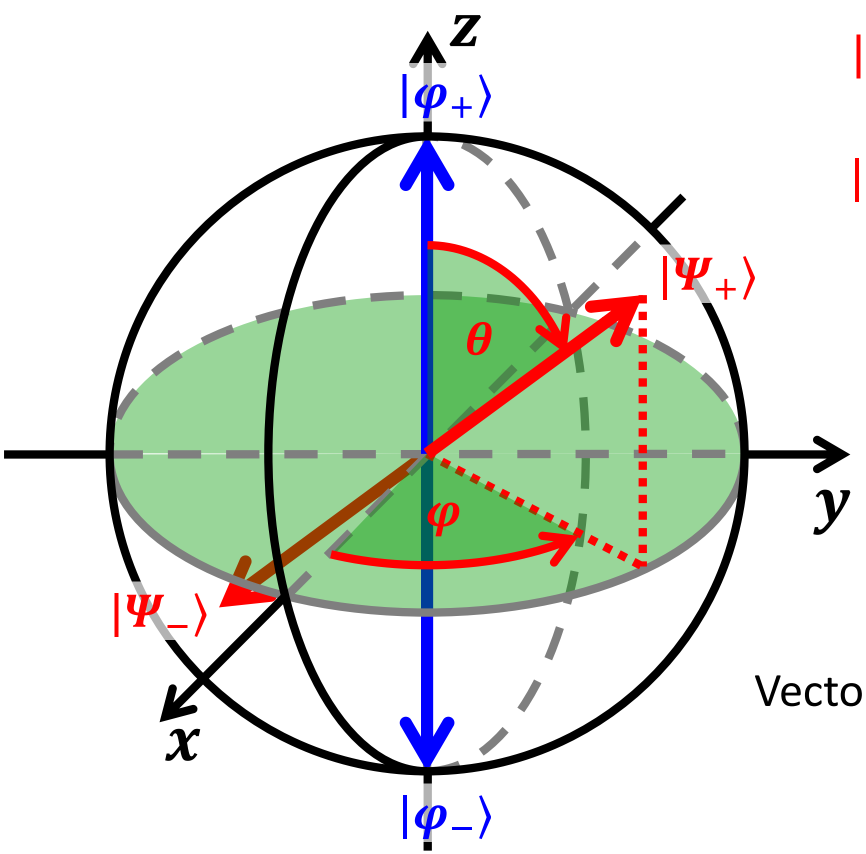
$$\tan\varphi \equiv \frac{\widetilde{\Delta}}{\Delta} \quad \text{with } 0 \leq \varphi \leq 2\pi$$

Bloch angles!

C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics*, Volume One (Wiley-VCH)

Basis $\{|\Psi_+\rangle, |\Psi_-\rangle\}$ → Energy eigenbasis (because $\widehat{H}$ has energy units)

Visualization on the Bloch sphere



$$|\Psi_+\rangle = +e^{-i\varphi/2}\cos\frac{\theta}{2}|\varphi_+\rangle + e^{i\varphi/2}\sin\frac{\theta}{2}|\varphi_-\rangle$$

$$|\Psi_-\rangle = -e^{-i\varphi/2}\sin\frac{\theta}{2}|\varphi_+\rangle + e^{i\varphi/2}\cos\frac{\theta}{2}|\varphi_-\rangle$$

$$\tan\theta \equiv \frac{\sqrt{\Delta^2+\tilde{\Delta}^2}}{\epsilon} \qquad \text{with } 0 \le \theta \le \pi$$

$$\tan\varphi \equiv \frac{\tilde{\Delta}}{\Delta} \qquad \text{with } 0 \le \varphi \le 2\pi$$

Basis rotation on Bloch sphere!

Vectors $|\Psi_-\rangle$ and $|\Psi_+\rangle$ define new quantization axis

→ $\hat{H} = \frac{\sqrt{\epsilon^2+\Delta^2+\tilde{\Delta}^2}}{2}\hat{\sigma}_z$ in basis $\{|\Psi_-\rangle, |\Psi_+\rangle\}$

→ $|\Psi_-\rangle$ and $|\Psi_+\rangle$ become new poles

Practical relevance

→ ε or Δ (and therefore the rotation angles) often depend on an external control parameter!

6.3 Control of quantum two-level systems

Analogy to spin ½ in static magnetic field

Fictitious spin ½ in fictitious magnetic field $\boldsymbol{B}$

$$\rightarrow \widehat{H}_{\uparrow} = -\gamma\hbar\, \boldsymbol{B}\cdot\boldsymbol{S} = -\frac{\gamma\hbar}{2}\begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$$

→ γ is the gyromagnetic ratio

→ $B = \left(B_x, B_y, B_z\right)^T$ is the magnetic field vector

Any quantum TLS has a **"built-in" static field**
↓
NMR situation!

Fictitous spin in fictitious $\boldsymbol{B}$-field	→	**quantum TLS**
$\vert\uparrow\rangle$	→	$\vert\varphi_+\rangle$
$\vert\downarrow\rangle$	→	$\vert\varphi_-\rangle$
$\vert\uparrow\rangle_{\boldsymbol{u}}$	→	$\vert\Psi_+\rangle$
$\vert\downarrow\rangle_{\boldsymbol{u}}$	→	$\vert\Psi_-\rangle$
($\boldsymbol{u}$ denotes the quantization axis along which $\widehat{H}_{\uparrow}$ is diagonal)		
$\hbar\vert\gamma\vert\vert\boldsymbol{B}\vert$	→	$\mathrm{E}_+ - \mathrm{E}_- = \hbar\omega_q$
Polar angles of $\boldsymbol{B}$	→	θ, φ
$-\gamma\hbar B_z$	→	ϵ
$-\gamma\hbar B_x$	→	Δ
$-\gamma\hbar B_y$	→	$\widetilde{\Delta}$

Field orientation with respect to the quantization axis depends on $\epsilon, \Delta, \widetilde{\Delta}$

6.3 Control of quantum two-level systems

Dynamics of the qubit state

Analogy to spin ½ in static magnetic field → Dynamics as known from NMR!

Intrinsic dynamics → Free evolution → Precession about z-axis

Driven evolution → Population (Rabi) oscillations

Pulsed driving schemes
→ Ramsey fringes
→ Hahn echo

NMR-type control and pulsed control of $\hbar\omega_{\mathrm{q}}$ → Single qubit gates

6.3 Control of quantum two-level systems

$$|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|\mathrm{e}\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|\mathrm{g}\rangle$$

Free precession

In the energy eigenbasis $\{|\mathrm{g}\rangle, |\mathrm{e}\rangle\}$, the qubit state vector $|\Psi_0\rangle$ is

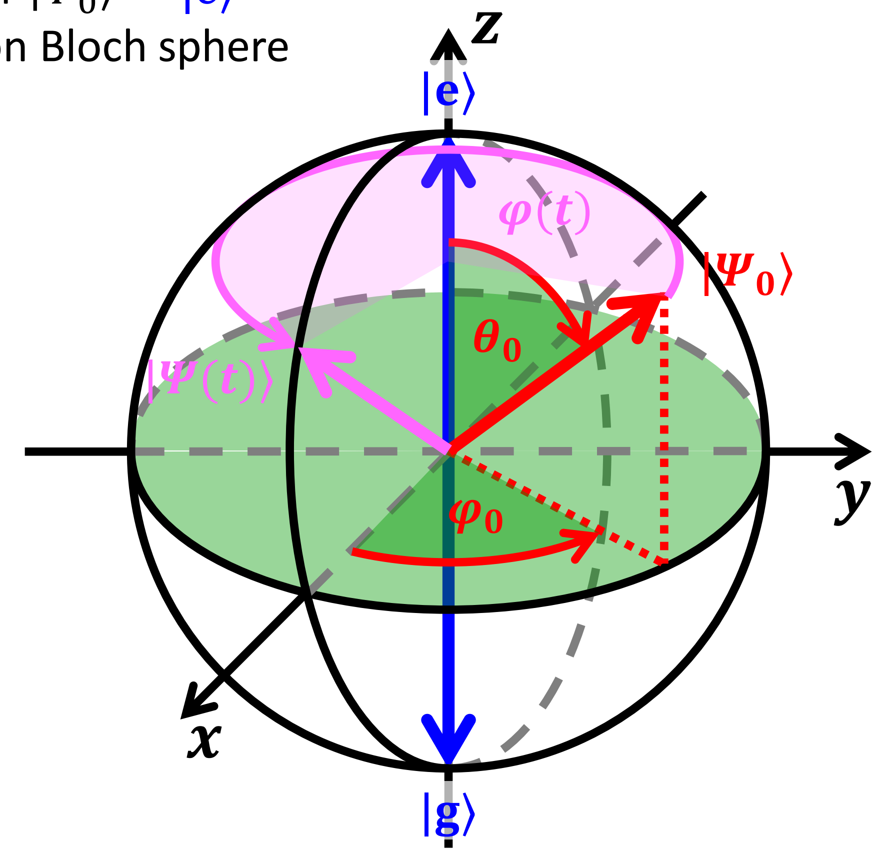
→ parallel to built-in field for $|\Psi_0\rangle = |\mathrm{g}\rangle$
→ antiparallel to the built-in field for $|\Psi_0\rangle = |\mathrm{e}\rangle$
→ Built-in field points along z-axis on Bloch sphere

When qubit state vector $|\Psi_0\rangle$ is not parallel or antiparallel to built-in field

→ Free evolution corresponds to free precession about the z-axis

→ Also called Larmor precession

→ In absence of decoherence, only $\varphi(t)$ evolves linearly with time t



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$$|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|\mathrm{e}\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|\mathrm{g}\rangle$$

Larmor precession – formal calculation

Energy eigenbasis → $\widehat{H} = \frac{\hbar\omega_q}{2}\hat{\sigma}_z$ → fictitious field aligned along quantization axis

Time-independent $\widehat{H}$ → Develop into stationary states $|\Psi(t)\rangle = e^{-i\frac{E}{\hbar}t}|\Psi_0\rangle$

$$|\Psi(t)\rangle = \langle e|\Psi_0\rangle e^{-i\frac{\omega_q}{2}t}|\mathrm{e}\rangle + \langle g|\Psi_0\rangle e^{i\frac{\omega_q}{2}t}|\mathrm{g}\rangle$$

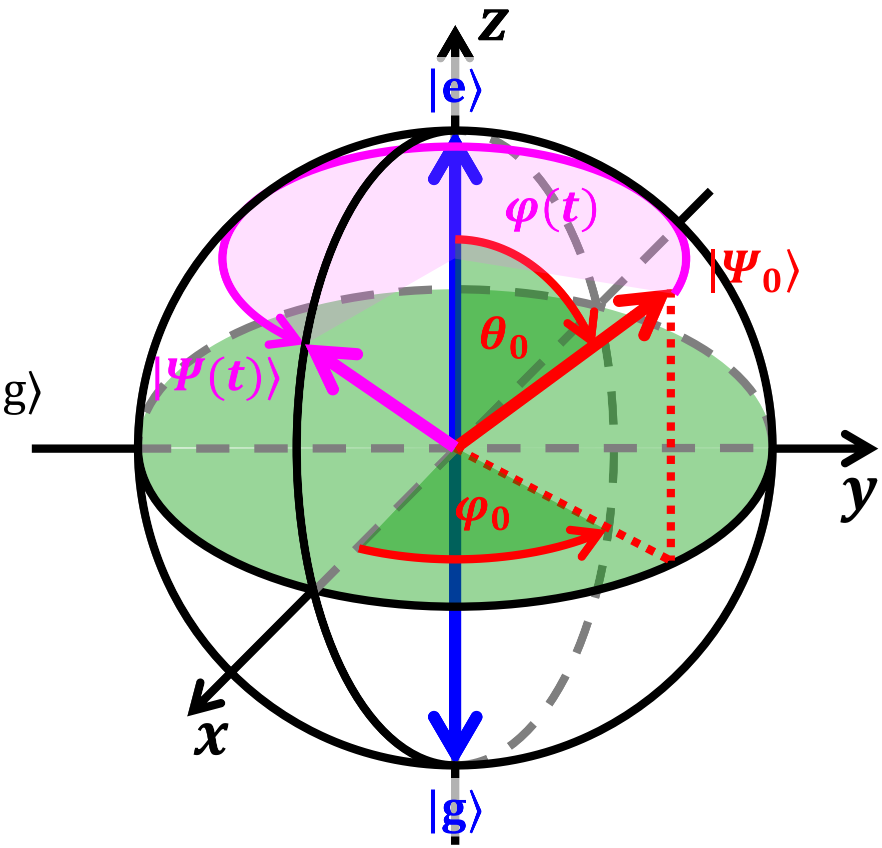
$$|\Psi_0\rangle = \cos\left(\frac{\theta_0}{2}\right)|\mathrm{e}\rangle + e^{i\varphi_0}\sin\left(\frac{\theta_0}{2}\right)|\mathrm{g}\rangle$$

$$|\Psi(t)\rangle = e^{-i\frac{\omega_\mathrm{q}}{2}t}\cos\frac{\theta_0}{2}|\mathrm{e}\rangle + e^{i\varphi_0}e^{i\frac{\omega_\mathrm{q}}{2}t}\sin\frac{\theta_0}{2}|\mathrm{g}\rangle$$

Global phase arbitrary

$$|\Psi(t)\rangle = \cos\frac{\theta_0}{2}|\mathrm{e}\rangle + e^{i(\varphi_0+\omega_\mathrm{q}t)}\sin\frac{\theta_0}{2}|\mathrm{g}\rangle$$

$\varphi(t)$



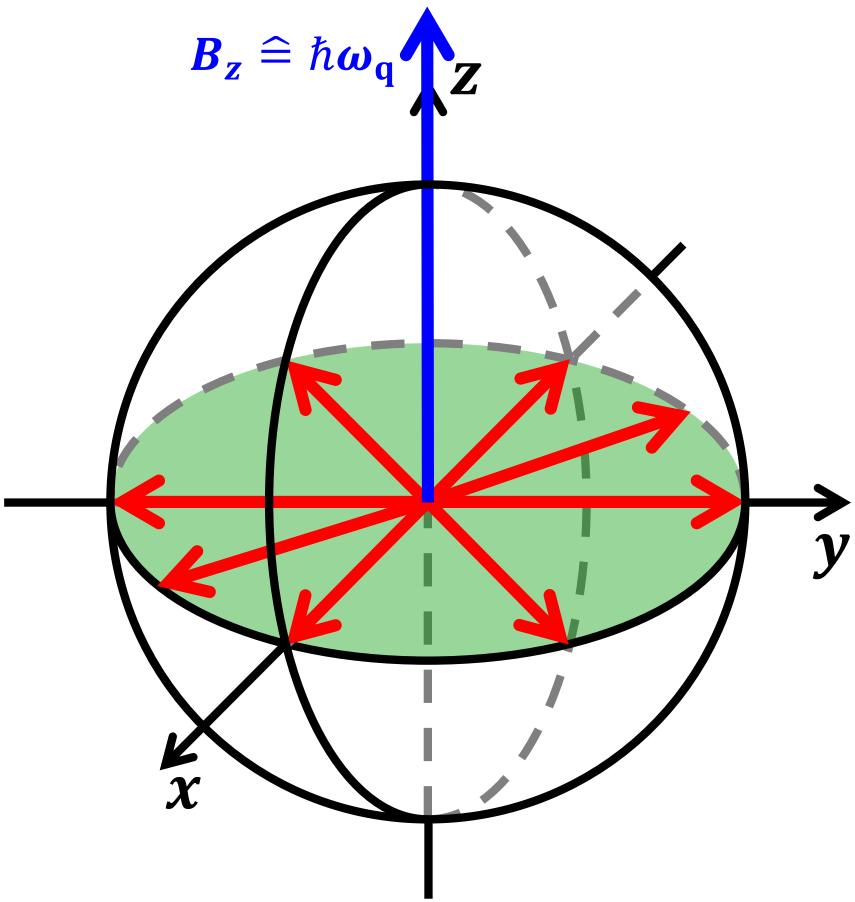
6.3 Control of quantum two-level systems

$$\widehat{H}_\uparrow = -\frac{\gamma\hbar}{2}\begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$$

Rotating drive field

Consider the qubit state vector $|\Psi\rangle$ in energy eigenbasis $\{|\mathrm{g}\rangle, |\mathrm{e}\rangle\}$ of the undriven system

Apply a drive field with power $\hbar\omega_\mathrm{d}$ rotating around the z-axis at frequency ω



$$\rightarrow \quad \widehat{H}(t) = \frac{\hbar}{2}\begin{pmatrix} \omega_\mathrm{q} & \omega_\mathrm{d} e^{-i\omega t} \\ \omega_\mathrm{d} e^{i\omega t} & -\omega_\mathrm{q} \end{pmatrix}$$

ωt	B_x	B_y
0	ω_d	0
$\pi/4$	$\omega_\mathrm{d}/\sqrt{2}$	$\omega_\mathrm{d}/\sqrt{2}$
$\pi/2$	0	ω_d
$3\pi/4$	$-\omega_\mathrm{d}/\sqrt{2}$	$\omega_\mathrm{d}/\sqrt{2}$
π	$-\omega_\mathrm{d}$	0
$5\pi/4$	$-\omega_\mathrm{d}/\sqrt{2}$	$-\omega_\mathrm{d}/\sqrt{2}$
$3\pi/2$	0	$-\omega_\mathrm{d}$
$7\pi/4$	$\omega_\mathrm{d}/\sqrt{2}$	$-\omega_\mathrm{d}/\sqrt{2}$

6.3 Control of quantum two-level systems

Driven quantum TLS

$$\hat{H}(t) = \frac{\hbar}{2}\begin{pmatrix} \omega_\mathrm{q} & \omega_\mathrm{d} e^{-i\omega t} \\ \omega_\mathrm{d} e^{i\omega t} & -\omega_\mathrm{q} \end{pmatrix} = \underbrace{\frac{\hbar\omega_\mathrm{q}}{2}\hat{\sigma}_z}_{\equiv \hat{H}_0} + \underbrace{\frac{\hbar\omega_\mathrm{d}}{2}\left(\hat{\sigma}_- e^{+i\omega t} + \hat{\sigma}_+ e^{-i\omega t}\right)}_{\equiv \hat{H}_\mathrm{d}}$$

Operators $\hat{\sigma}_+ \equiv |\mathrm{e}\rangle\langle\mathrm{g}|$ and $\hat{\sigma}_- \equiv |\mathrm{g}\rangle\langle\mathrm{e}|$ create or annihilate an excitation in the TLS

→ Note: $\hat{\sigma}_+$ and $\hat{\sigma}_-$ are unintuitive in the IT convention, because of the unintuitive assosiation of a negative eigenvalue of $\hat{\sigma}_z$ with state $|1\rangle$!

Matrix representation → $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\hat{\sigma}_+ = |\mathrm{e}\rangle\langle\mathrm{g}|$
$\hat{\sigma}_- = |\mathrm{g}\rangle\langle\mathrm{e}|$

6.3 Control of quantum two-level systems

Time evolution of driven quantum TLS

$$\widehat{H}(t) = \frac{\hbar}{2}\begin{pmatrix} \omega_{\mathrm{q}} & \omega_{\mathrm{d}} e^{-i\omega t} \\ \omega_{\mathrm{d}} e^{i\omega t} & -\omega_{\mathrm{q}} \end{pmatrix}$$

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = H|\Psi(t)\rangle$$

Qubit state $|\Psi(t)\rangle = a_{\mathrm{e}}(t)|\mathrm{e}\rangle + a_{\mathrm{g}}(t)|\mathrm{g}\rangle$
obeys **Schrödinger equation**

$$i\frac{\mathrm{d}}{\mathrm{d}t}a_{\mathrm{e}}(t) = \frac{\omega_{\mathrm{q}}}{2}a_{\mathrm{e}}(t) + \frac{\omega_{\mathrm{d}}}{2}e^{-i\omega t}a_{\mathrm{g}}(t)$$

$$i\frac{\mathrm{d}}{\mathrm{d}t}a_{\mathrm{g}}(t) = \frac{\omega_{\mathrm{d}}}{2}e^{i\omega t}a_{\mathrm{e}}(t) - \frac{\omega_{\mathrm{q}}}{2}a_{\mathrm{g}}(t)$$

Time-dependent coupled differential equations → Difficult to solve

→ **Move to rotating frame**

$$b_{\mathrm{e}}(t) \equiv e^{i\omega t/2}a_{\mathrm{e}}(t)$$

$$b_{\mathrm{g}}(t) \equiv e^{-i\omega t/2}a_{\mathrm{g}}(t)$$

→ Schrödinger equation **looses explicit time dependence**

$$i\frac{\mathrm{d}}{\mathrm{d}t}b_{\mathrm{e}}(t) = \frac{\omega_{\mathrm{q}} - \omega}{2}b_{\mathrm{e}}(t) + \frac{\omega_{\mathrm{d}}}{2}b_{\mathrm{g}}(t)$$

$$i\frac{\mathrm{d}}{\mathrm{d}t}b_{\mathrm{g}}(t) = \frac{\omega_{\mathrm{d}}}{2}b_{\mathrm{e}}(t) - \frac{\omega_{\mathrm{q}} - \omega}{2}b_{\mathrm{g}}(t)$$

6.3 Control of quantum two-level systems

$$\hat{H}(t) = \frac{\hbar}{2}\begin{pmatrix} \omega_\mathrm{q} & \omega_\mathrm{d} e^{-i\omega t} \\ \omega_\mathrm{d} e^{i\omega t} & -\omega_\mathrm{q} \end{pmatrix}$$

Interpretation of the rotating frame

$$i\frac{\mathrm{d}}{\mathrm{d}t} b_\mathrm{e}(t) = \frac{\omega_\mathrm{q} - \omega}{2} b_\mathrm{e}(t) + \frac{\omega_\mathrm{d}}{2} b_\mathrm{g}(t)$$

$$i\frac{\mathrm{d}}{\mathrm{d}t} b_\mathrm{g}(t) = \frac{\omega_\mathrm{d}}{2} b_\mathrm{e}(t) - \frac{\omega_\mathrm{q} - \omega}{2} b_\mathrm{g}(t)$$

The **frame rotates at the angular speed $\boldsymbol{\omega}$** of the drive

→ Driving field appears at rest

→ Drive can be in resonance with Larmor precession frequency ω_q

→ Away from resonance $|\omega - \omega_\mathrm{q}| \gg 0$

→ red terms dominate → no $|\mathrm{g}\rangle \leftrightarrow |\mathrm{e}\rangle$ transitions induced by drive

→ Near resonance $\omega \approx \omega_\mathrm{q}$

→ blue terms dominate → $|\mathrm{g}\rangle \leftrightarrow |\mathrm{e}\rangle$ transitions induced by drive

Formal treatment → Effective Hamiltonian $\widetilde{H}$ describing the same dynamics as $\hat{H}(t)$

$$\widetilde{\mathrm{H}} = \frac{\hbar}{2}\begin{pmatrix} -\Delta\omega & \omega_\mathrm{d} \\ \omega_\mathrm{d} & \Delta\omega \end{pmatrix} \qquad i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\widetilde{\Psi}(t)\right\rangle = \widetilde{H}\left|\widetilde{\Psi}(t)\right\rangle$$

with $\Delta\omega \equiv \omega - \omega_\mathrm{q}$ $\qquad \left|\widetilde{\Psi}(t)\right\rangle \equiv b_\mathrm{e}(t)|\mathrm{e}\rangle + b_\mathrm{g}(t)|\mathrm{g}\rangle$

6.3 Control of quantum two-level systems

Dynamics of the effective Hamiltonian

$$|\Psi_+\rangle = +e^{-i\varphi/2}\cos\frac{\theta}{2}|\varphi_+\rangle + e^{+i\varphi/2}\sin\frac{\theta}{2}|\varphi_-\rangle$$
$$|\Psi_-\rangle = -e^{-i\varphi/2}\sin\frac{\theta}{2}|\varphi_+\rangle + e^{+i\varphi/2}\cos\frac{\theta}{2}|\varphi_-\rangle$$

$$\widetilde{\mathrm{H}} = \frac{\hbar}{2}\begin{pmatrix} -\Delta\omega & \omega_\mathrm{d} \\ \omega_\mathrm{d} & \Delta\omega \end{pmatrix}$$

$$\Delta\omega \equiv \omega - \omega_\mathrm{q}$$

→ **Diagonalize** → $\widetilde{\mathrm{H}} = \dfrac{-\hbar\sqrt{\Delta\omega^2 + \omega_\mathrm{d}^2}}{2}\hat{\sigma}_z$

$$\tan\theta = -\frac{\omega_\mathrm{d}}{\Delta\omega}$$

$$\tan\varphi = 0$$

→ New eigenstates →

$$|\widetilde{\Psi}_+\rangle = +\cos\frac{\theta}{2}|\mathrm{e}\rangle + \sin\frac{\theta}{2}|\mathrm{g}\rangle$$
$$|\widetilde{\Psi}_-\rangle = -\sin\frac{\theta}{2}|\mathrm{e}\rangle + \cos\frac{\theta}{2}|\mathrm{g}\rangle$$

Expand into stationary states

$$|\widetilde{\Psi}(t)\rangle = \langle\widetilde{\Psi}_-|\widetilde{\Psi}_0\rangle e^{+\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}|\widetilde{\Psi}_-\rangle + \langle\widetilde{\Psi}_+|\widetilde{\Psi}_0\rangle e^{-\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}|\widetilde{\Psi}_+\rangle$$

Initial state $|\boldsymbol{\Psi}_0\rangle = |\mathrm{g}\rangle$ (energy ground state)

$$|\widetilde{\Psi}(t)\rangle = \cos\frac{\theta}{2}e^{+\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}|\widetilde{\Psi}_-\rangle + \sin\frac{\theta}{2}e^{-\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}|\widetilde{\Psi}_+\rangle$$

6.3 Control of quantum two-level systems

Probability P_e to find TLS in $|\mathrm{e}\rangle$

$$|\widetilde{\Psi}(t)\rangle = \cos\frac{\theta}{2}\, e^{+\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}|\widetilde{\Psi}_-\rangle + \sin\frac{\theta}{2}\, e^{-\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}|\widetilde{\Psi}_+\rangle$$

$$|\widetilde{\Psi}_+\rangle = +\cos\frac{\theta}{2}|\mathrm{e}\rangle + \sin\frac{\theta}{2}|\mathrm{g}\rangle$$

$$|\widetilde{\Psi}_-\rangle = -\sin\frac{\theta}{2}|\mathrm{e}\rangle + \cos\frac{\theta}{2}|\mathrm{g}\rangle$$

$$\Delta\omega \equiv \omega - \omega_\mathrm{q}$$

$$\tan\theta = -\frac{\omega_\mathrm{d}}{\Delta\omega}$$

$$P_\mathrm{e} \equiv |\langle \mathrm{e}|\Psi(t)\rangle|^2 = \left|\langle \mathrm{e}|\widetilde{\Psi}(t)\rangle\right|^2 =$$

$$= \left|\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(e^{-\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}} - e^{+\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}\right)\right|^2$$

$$= \left|\sin\theta\ \sin\left(\frac{t\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}{2}\right)\right|^2$$

$$\boxed{P_\mathrm{e} = \frac{\omega_\mathrm{d}^2}{\Delta\omega^2+\omega_\mathrm{d}^2}\ \sin^2\left(\frac{t\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}{2}\right)}$$

Driven Rabi oscillations

TLS population oscillates with Rabi frequency $\omega_\mathrm{R} \equiv \sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}$ under transversal drive

Rabi Oscillations on the Bloch sphere

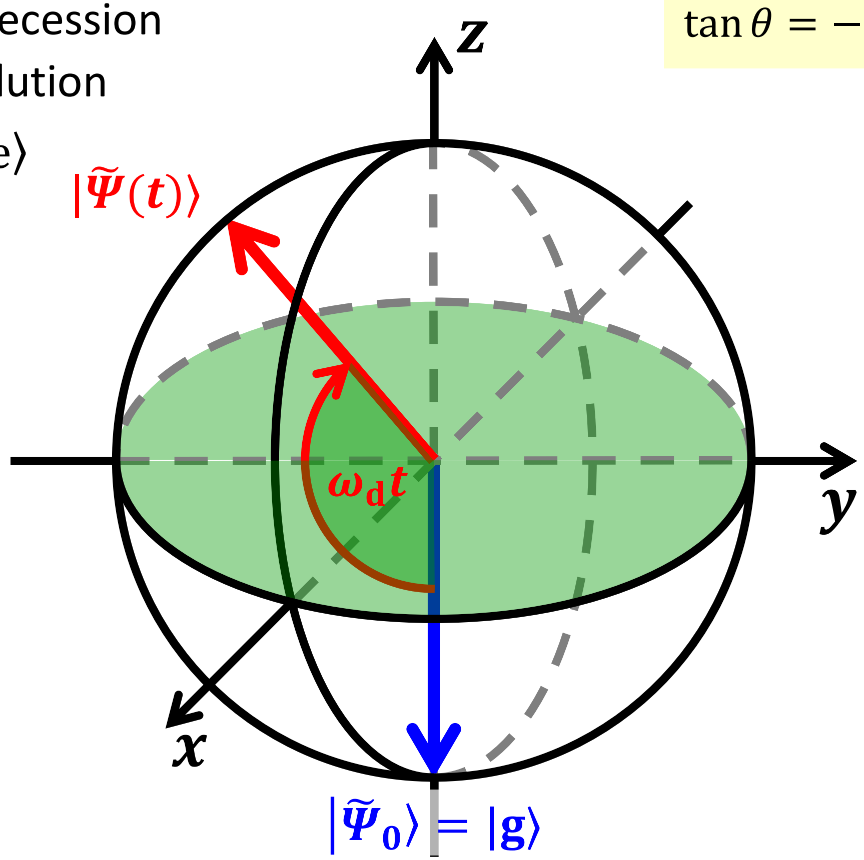
$$|\widetilde{\Psi}(t)\rangle = \cos\frac{\theta}{2}\, e^{+\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}|\widetilde{\Psi}_-\rangle + \sin\frac{\theta}{2}\, e^{-\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}|\widetilde{\Psi}_+\rangle$$

$$\Delta\omega \equiv \omega - \omega_\mathrm{q}$$

$$\tan\theta = -\frac{\omega_\mathrm{d}}{\Delta\omega}$$

On resonance $\omega = \omega_\mathrm{q}$

→ Rotating frame cancels Larmor precession
→ State vector $|\widetilde{\Psi}(t)\rangle$ has no φ-evolution
→ $|\widetilde{\Psi}(t)\rangle = \cos\frac{\omega_\mathrm{d}t}{2}|\mathrm{g}\rangle + i\sin\frac{\omega_\mathrm{d}t}{2}|\mathrm{e}\rangle$
→ Rotation about x-axis



Finite detuning $|\Delta\omega| > 0$

→ Additional precession of at $\Delta\omega$
→ Population oscillates faster
→ Reduced oscillation amplitude

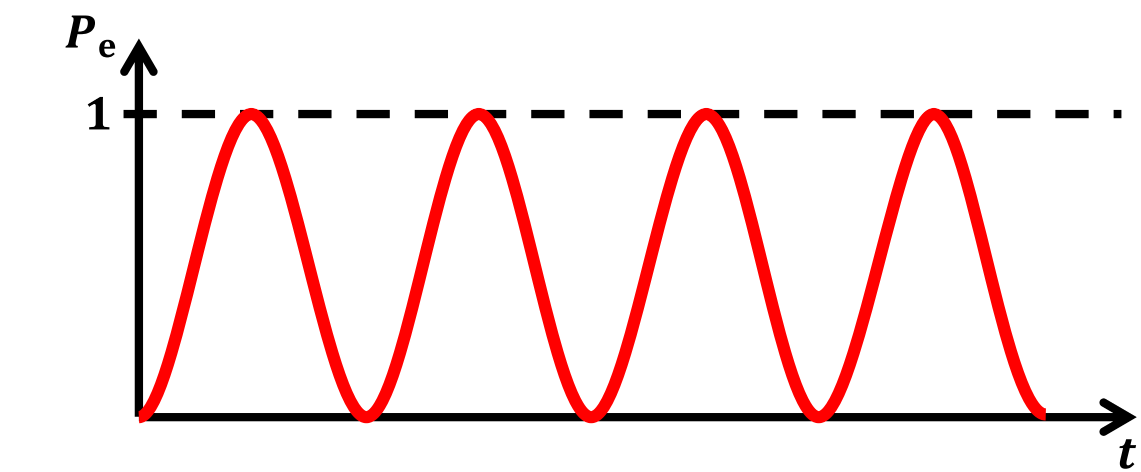
$$\boxed{P_\mathrm{e} = \frac{\omega_\mathrm{d}^2}{\Delta\omega^2+\omega_\mathrm{d}^2}\sin^2\left(\frac{t\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}{2}\right)}$$

Arbitrary equatorial axis → $\widehat{H}_\mathrm{d} = \frac{\hbar\omega_\mathrm{d}}{2}\left(\hat{\sigma}_- e^{+i(\omega t+\varphi)} + \hat{\sigma}_+ e^{-i(\omega t+\varphi)}\right)$

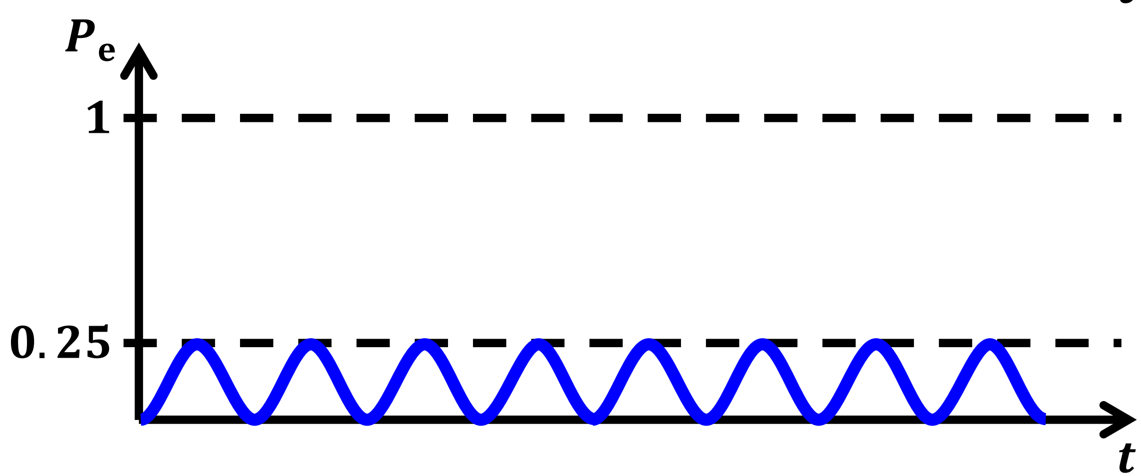
Rabi Oscillations – Graphical representation

$$P_\mathrm{e} = \frac{\omega_\mathrm{d}^2}{\Delta\omega^2 + \omega_\mathrm{d}^2} \sin^2\left(\frac{t\sqrt{\Delta\omega^2 + \omega_\mathrm{d}^2}}{2}\right)$$

On resonance $\omega = \omega_\mathrm{q}$



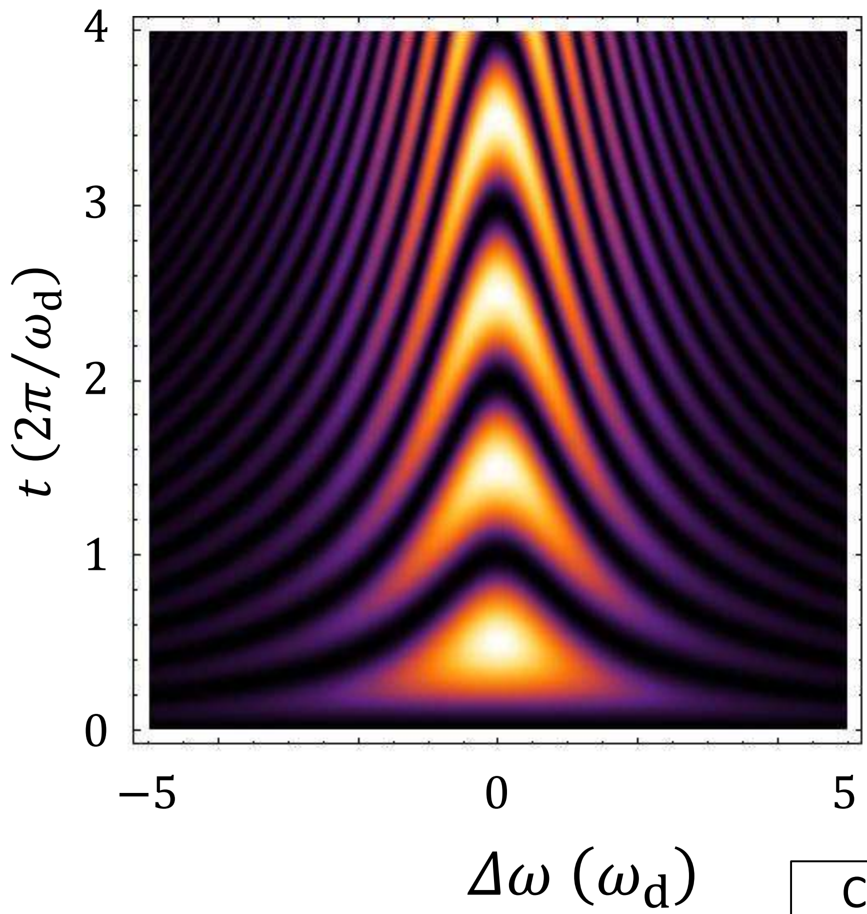
Detuning $|\Delta\omega| = \sqrt{3}\,\omega_\mathrm{d}$

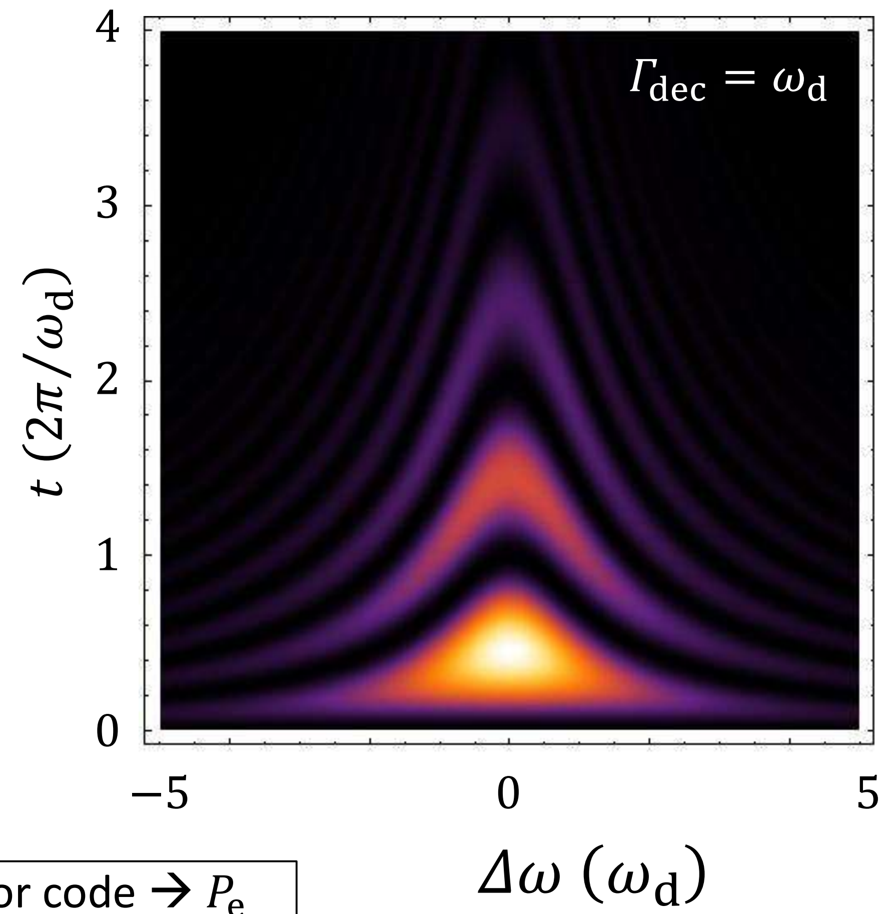


6.3 Control of quantum two-level systems

Rabi Oscillations – Graphical representation

$$P_{\mathrm{e}} = \frac{\omega_{\mathrm{d}}^2}{\Delta\omega^2 + \omega_{\mathrm{d}}^2}\,\sin^2\left(\frac{t\sqrt{\Delta\omega^2 + \omega_{\mathrm{d}}^2}}{2}\right)$$





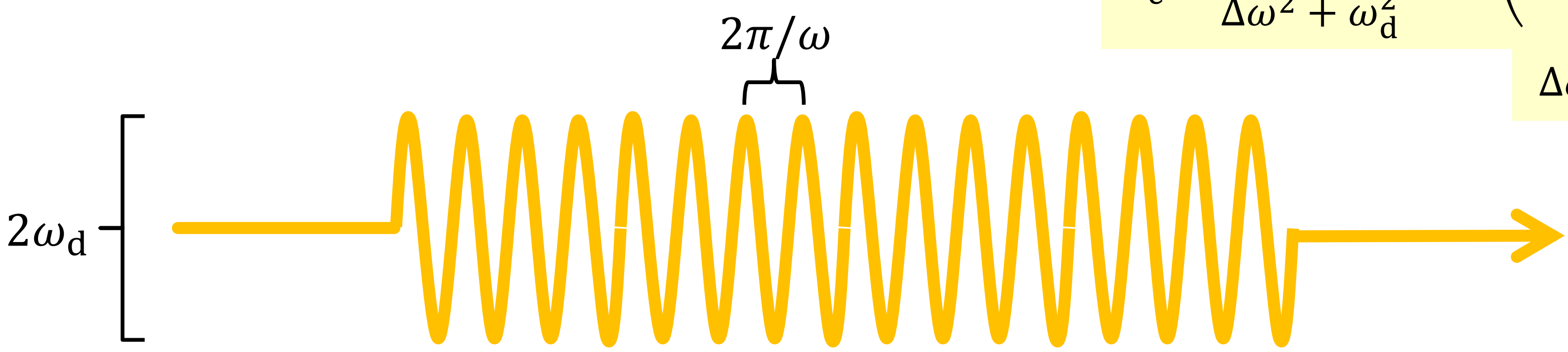
Color code → P_{e}
(0 =black, 1 =white)

6.3 Control of quantum two-level systems

Oscillating vs. rotating drive – Microwave pulses

$$P_{\mathrm{e}} = \frac{\omega_{\mathrm{d}}^2}{\Delta\omega^2 + \omega_{\mathrm{d}}^2} \sin^2\left(\frac{t\sqrt{\Delta\omega^2 + \omega_{\mathrm{d}}^2}}{2}\right)$$

$$\Delta\omega \equiv \omega - \omega_{\mathrm{q}}$$



Oscillating drive $2\hbar\omega_{\mathrm{d}} \cos\omega t = \hbar\omega_{\mathrm{d}}\left(e^{+i\omega t} + e^{-i\omega t}\right)$

→ In frame rotating with $+\omega$ the $e^{-\mathrm{i}\omega t}$ -component rotates fast with -2ω

→ For $\omega_{\mathrm{d}} \ll \omega$, this fast contribution averages out on the timescale of the slowly rotating component → Rotating wave approximation

→ Interaction picture:

→ removing eigenenergies by going into rotating frame with ω_{q} → qubit-drive interaction

→ $\hat{\sigma}_{\pm} \to \hat{\sigma}_{\pm} e^{\pm i\omega_{\mathrm{q}} t}$ → $\cos\omega t\, \hat{\sigma}_x \to \cos\omega t \left(\hat{\sigma}_+ e^{+i\omega_{\mathrm{q}} t} + \hat{\sigma}_- e^{-i\omega_{\mathrm{q}} t}\right)$

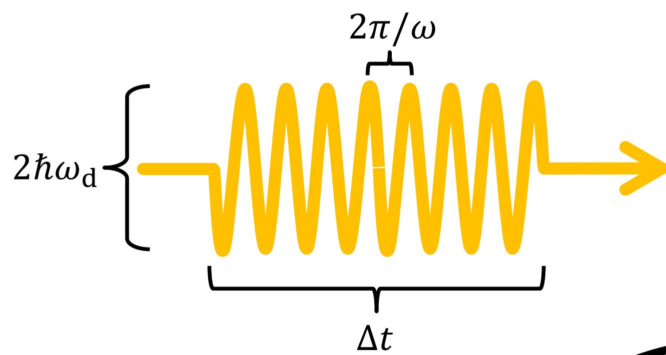
→ $2\hbar\omega_{\mathrm{d}}\left(e^{+i\omega t} + e^{-i\omega t}\right)\left(\hat{\sigma}_+ e^{+i\omega_{\mathrm{q}} t} + \hat{\sigma}_- e^{-i\omega_{\mathrm{q}} t}\right) =$

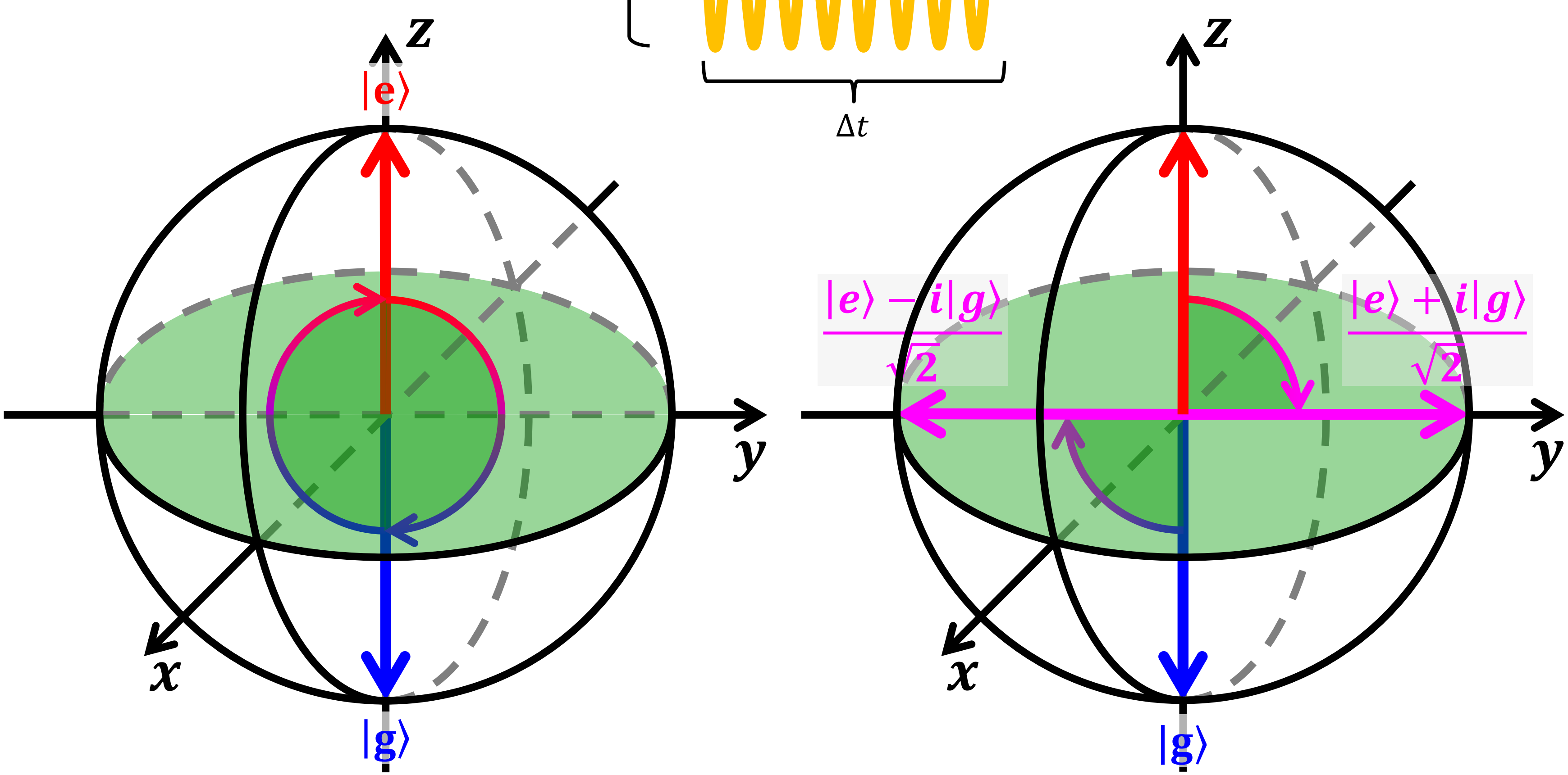
$$= 2\hbar\omega_{\mathrm{d}}\left(\cancel{\hat{\sigma}_+ e^{+i(\omega+\omega_{\mathrm{q}})t} + \hat{\sigma}_- e^{-i(\omega+\omega_{\mathrm{q}})t}} + \hat{\sigma}_+ e^{-i\Delta\omega\, t} + \hat{\sigma}_- e^{+i\Delta\omega\, t}\right)$$

$\approx 2\hbar\omega_{\mathrm{d}}\left(\hat{\sigma}_+ e^{-i\Delta\omega\, t} + \hat{\sigma}_- e^{+i\Delta\omega\, t}\right)$ (if $2\omega_d \ll \omega, \omega_{\mathrm{q}}$)

→ Rabi oscillations just as for rotating drive

Important drive pulses on the Bloch sphere





π-pulse $(\omega_\mathrm{d}\Delta t = \pi)$
$|\mathrm{g}\rangle \leftrightarrow |\mathrm{e}\rangle$ flips, refocus phase evolution

$\pi/2$-pulse $(\omega_\mathrm{d}\Delta t = \pi/2)$
Rotates into equatorial plane and back

6.3 Control of quantum two-level systems

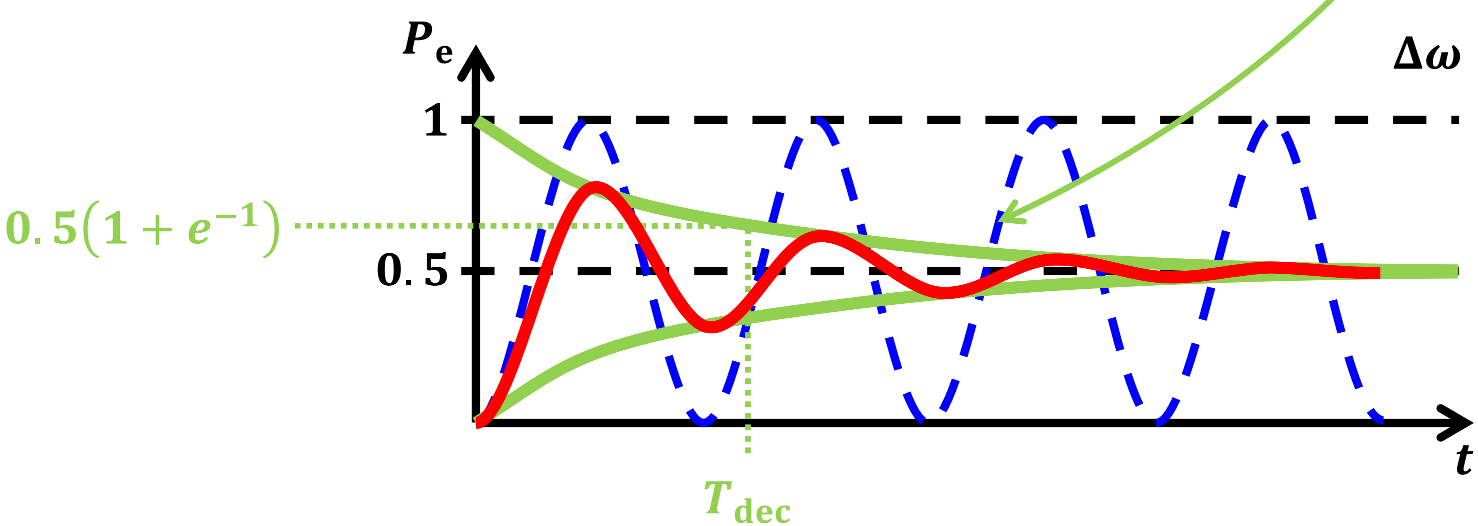
Rabi Oscillations in presence of decoherence

$$P_e = \frac{\omega_d^2}{\Delta\omega^2 + \omega_d^2} \sin^2\left(\frac{t\sqrt{\Delta\omega^2 + \omega_d^2}}{2}\right)$$

Effect of decoherence (qualitative approach)

→ Loss of coherent properties to environment at a constant rate $\Gamma_{\mathrm{dec}} = 2\pi/T_{\mathrm{dec}}$
→ "Change of coherence" (time derivative) proportional to "amount of coherence"

→ Exponential decay of coherent property with factor $e^{-\frac{\Gamma_{\mathrm{dec}} t}{2\pi}} = e^{-\frac{t}{T_{\mathrm{dec}}}}$
→ Argument holds well for population decay (energy relaxation, T_1)
→ Loss of phase coherence more diverse depending on environment (exponential, Gaussian, or power law)
→ Experimental timescales range from few ns to 100 µs



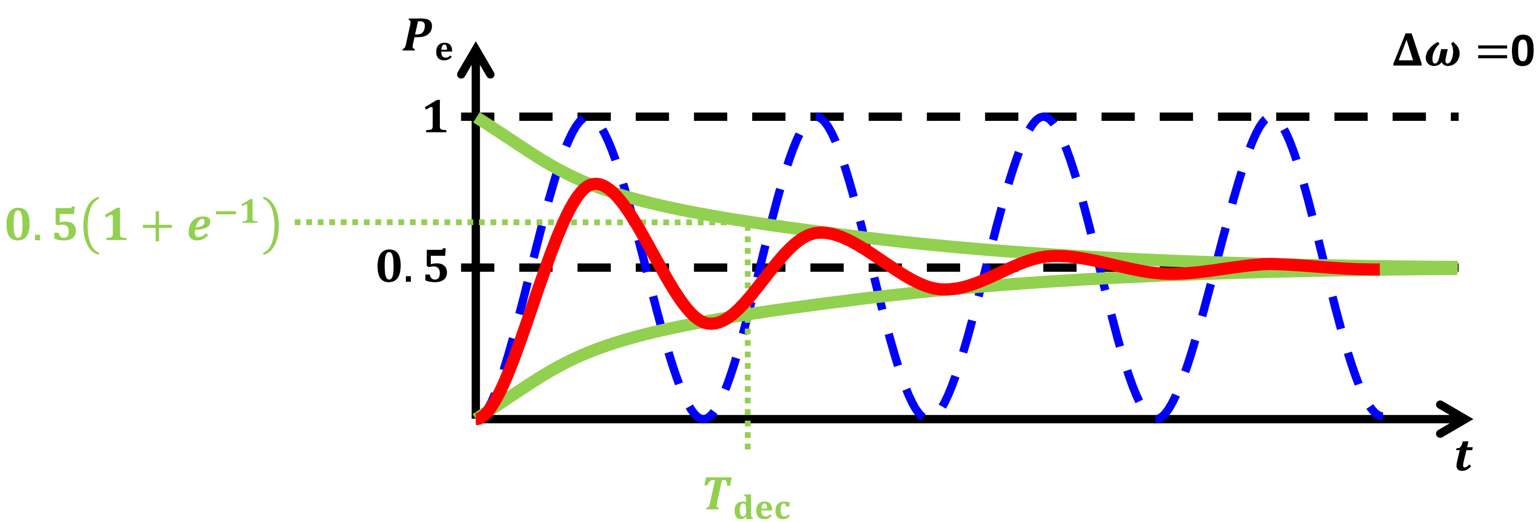
Rabi decay time

$$P_\mathrm{e} = \frac{\omega_\mathrm{d}^2}{\Delta\omega^2 + \omega_\mathrm{d}^2} \sin^2\left(\frac{t\sqrt{\Delta\omega^2 + \omega_\mathrm{d}^2}}{2}\right)$$

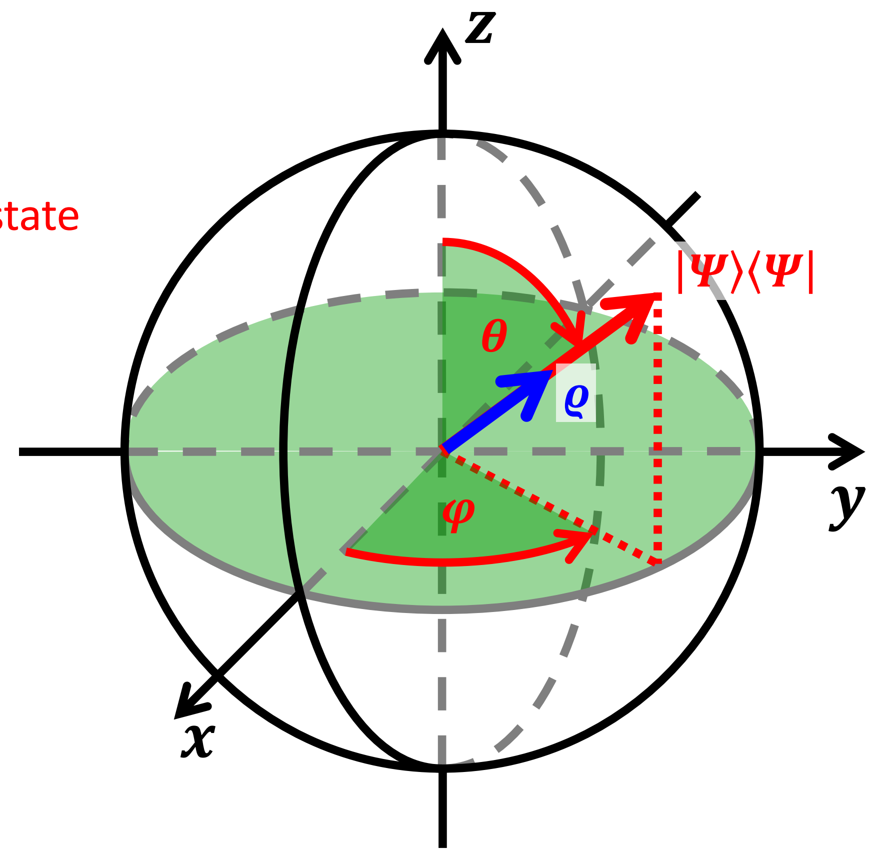
Complicated interplay between T_1, T_2^*, and the drive

→ At long times, small oscillations persist for oscillatory drive
→ Nevertheless useful **order-of-magnitude check** for T_dec
→ Important tool for single-qubit gates

→ To determine T_1, T_2^*, T_2 correctly, more sophisticated protocols are required
→ Energy relaxation measurements, Ramsey fringes, spin echo



Pure and mixed states on the Bloch sphere



No decoherence → Unitary evolution

→ State vector $|\Psi\rangle$ always well defined → Pure state

→ Describe via density matrix $\varrho \equiv |\Psi\rangle\langle\Psi|$

→ Hermitian → $\varrho^\dagger = \varrho$

→ Expectation value $\langle\Psi|\hat{A}|\Psi\rangle = \mathrm{Tr}[\varrho\hat{A}]$

→ $\mathrm{Tr}\,\varrho = 1$ → Bloch vector length 1

Decoherence → Loss of unitary evolution

→ State vector $|\Psi\rangle$ only known probabilistically

→ Mixed state $\varrho \equiv \sum_{i=1}^{n} p_i |\Psi_i\rangle\langle\Psi_i|$

→ $\mathrm{Tr}\,\varrho^2 < 1$ for $n > 1$

→ Bloch vector length < 1

→ p_i are classical probabilities!

→ Center of Bloch sphere → Completely depolarized state $\varrho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

→ General state → $\varrho = \frac{1}{2}(1 + \boldsymbol{a}\cdot\boldsymbol{\sigma})$ with $\boldsymbol{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)^T$

→ Bloch vector $\boldsymbol{a} \equiv (a_x, a_y, a_z)^T$ with $|\boldsymbol{a}| \leq 1$

→ Eigenvalues of ϱ are $\frac{1}{2}(1 \pm |\boldsymbol{a}|)$

Example: Pure state density matrix on the Bloch sphere

Goal → Get intuition for the Bloch vector $\boldsymbol{a}$

$$|\Psi\rangle = \cos\frac{\theta}{2}|\mathrm{e}\rangle + e^{i\varphi}\sin\frac{\theta}{2}|\mathrm{g}\rangle$$

$$\varrho = |\Psi\rangle\langle\Psi| = \left(\cos\frac{\theta}{2}|\mathrm{e}\rangle + e^{i\varphi}\sin\frac{\theta}{2}|\mathrm{g}\rangle\right)\left(\cos\frac{\theta}{2}\langle\mathrm{e}| + e^{-i\varphi}\sin\frac{\theta}{2}\langle\mathrm{g}|\right) =$$

$$= \cos^2\frac{\theta}{2}|\mathrm{e}\rangle\langle\mathrm{e}| + \sin^2\frac{\theta}{2}|\mathrm{g}\rangle\langle\mathrm{g}| + e^{i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}|\mathrm{e}\rangle\langle\mathrm{g}| + e^{-i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}|\mathrm{g}\rangle\langle\mathrm{e}|$$

$$\hat{\sigma}_+ = |\mathrm{e}\rangle\langle\mathrm{g}| \qquad \hat{\sigma}_- = |\mathrm{g}\rangle\langle\mathrm{e}|$$

$$= \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & 1-\cos\theta \end{pmatrix}$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$
$$1+\cos\theta = \cos^2\frac{\theta}{2}$$
$$1-\cos\theta = \sin^2\frac{\theta}{2}$$

Bloch vector → $\varrho = \frac{1}{2}(1 + \boldsymbol{a}\cdot\boldsymbol{\sigma}) = \frac{1}{2}\begin{pmatrix} 1+a_z & a_x - ia_y \\ a_x + ia_y & 1-a_z \end{pmatrix}$

Compare coefficients → $\boldsymbol{a} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$ → Vector on suface of a sphere
Unit length & polar angles (θ, φ)

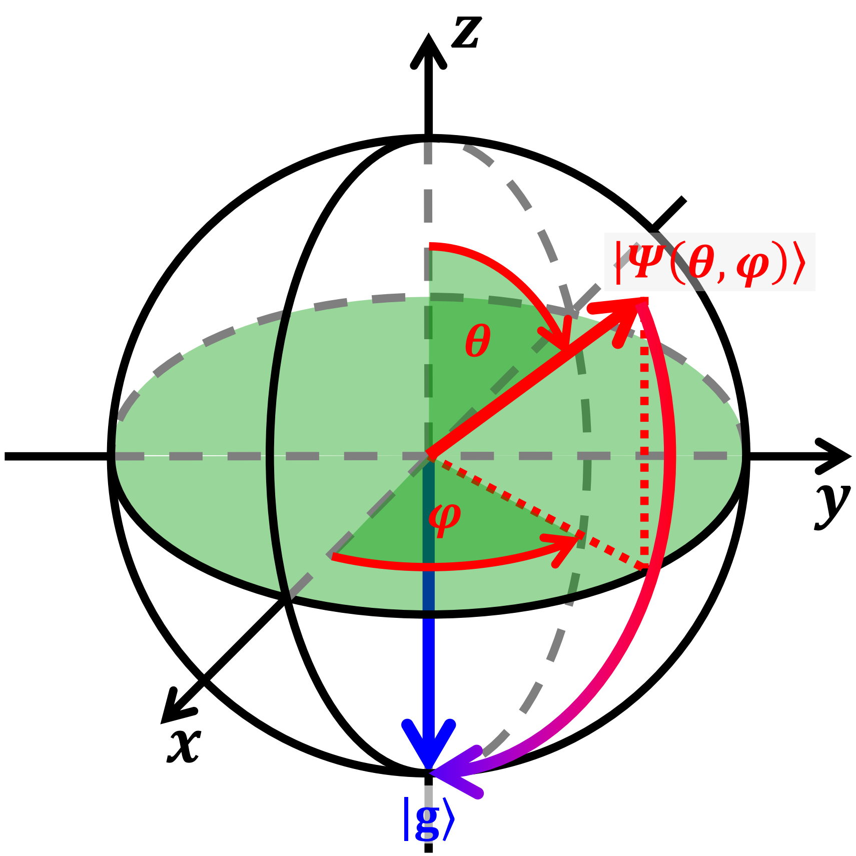
Dynamics of the state vector on the Bloch sphere with decoherence

Equations of motion for spin ½ under influence of the control fields $\boldsymbol{B}(t)$

$$\frac{d\boldsymbol{r}(t)}{dt} = -\boldsymbol{B}(t) \times \boldsymbol{r}(t) - \frac{1}{T_1}[r_z(t) - r_z(0)]\,\hat{z} - \frac{1}{T_2}\left[r_x(t)\,\hat{x} + r_y(t)\hat{y}\right]$$

→ $r(t)$ is state vector on the Bloch sphere
→ Introduced in 1946 by Bloch for NMR systems
→ Describe simple exponential decays with time constants T_1 and T_2
→ In simple scenarios confirmed by quantum master equations

Energy relaxation on the Bloch sphere



Environment induces energy loss

→ State vector collapses to $|\mathrm{g}\rangle$
→ Implies also loss of phase information
→ Intrinsically irreversible
→ T_1-time → rate $\Gamma_1 = \frac{2\pi}{T_1}$

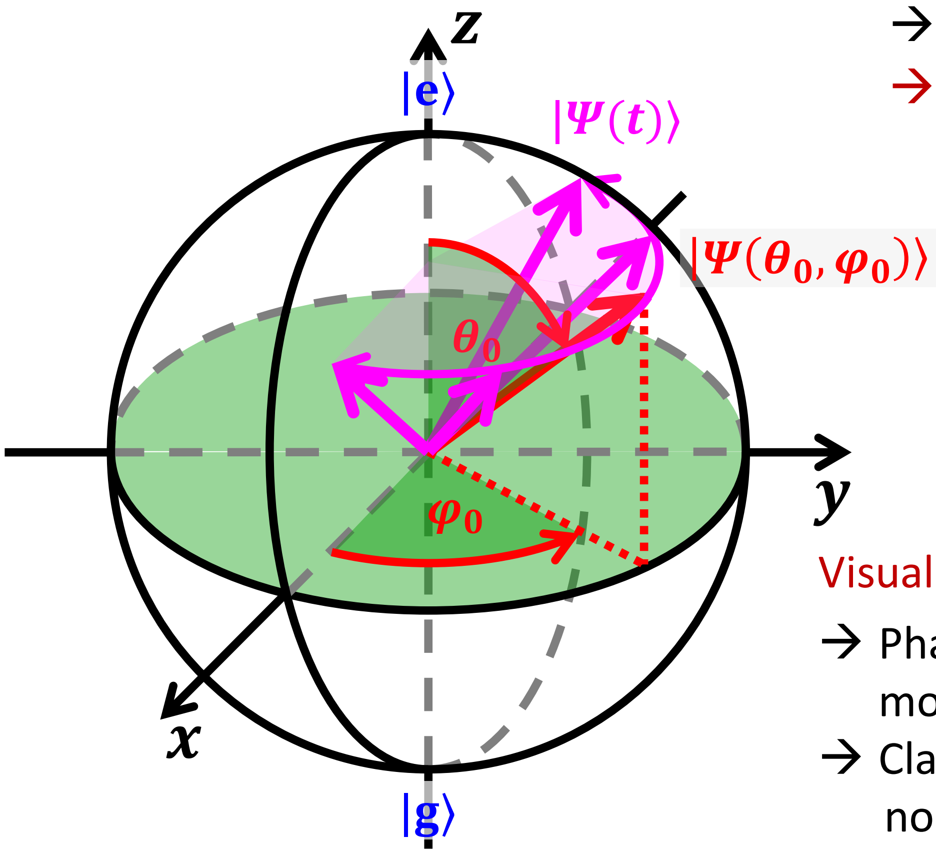
Golden Rule argument

→ $\Gamma_1 \propto S(\omega_\mathrm{q})$
→ $S(\omega)$ is noise spectral density
→ High frequency noise
→ Intuition: Noise induces transitions

Quantum jumps

→ Single-shot, quantum nondemolition measurement yields a discrete jump to $|\mathrm{g}\rangle$ at a random time
→ Probability is equal for each point of time
→ Exponential decay with $e^{-\frac{t}{T_1}}$

Dephasing on the Bloch sphere



Environment induces **random phase changes**

→ $\Gamma_\varphi \propto S(\omega \to 0)$, $T_\varphi = \frac{2\pi}{\Gamma_\varphi}$, (for Markovian bath)

→ Low-frequency noise is detuned from ω_q

→ No energy transfer

→ $1/f$-noise → $S(\omega) \propto \frac{1}{\omega}$

→ Example: two-level fluctuator bath

→ To some extent reversible

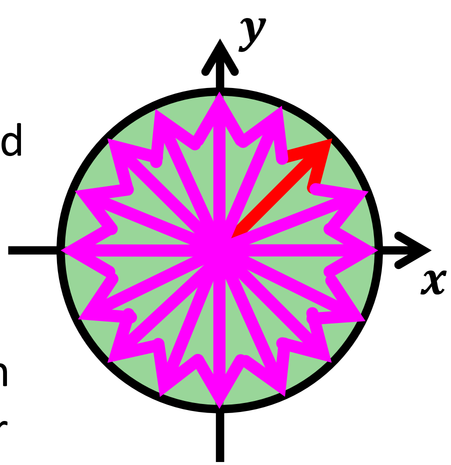
→ Decay laws $e^{-\frac{t}{T_\varphi}}$, $e^{-\left(\frac{t}{T_\varphi}\right)^2}$, $\left(\frac{t}{T_\varphi}\right)^\beta$

Visualization

→ Phase φ becomes more and more unknown with time

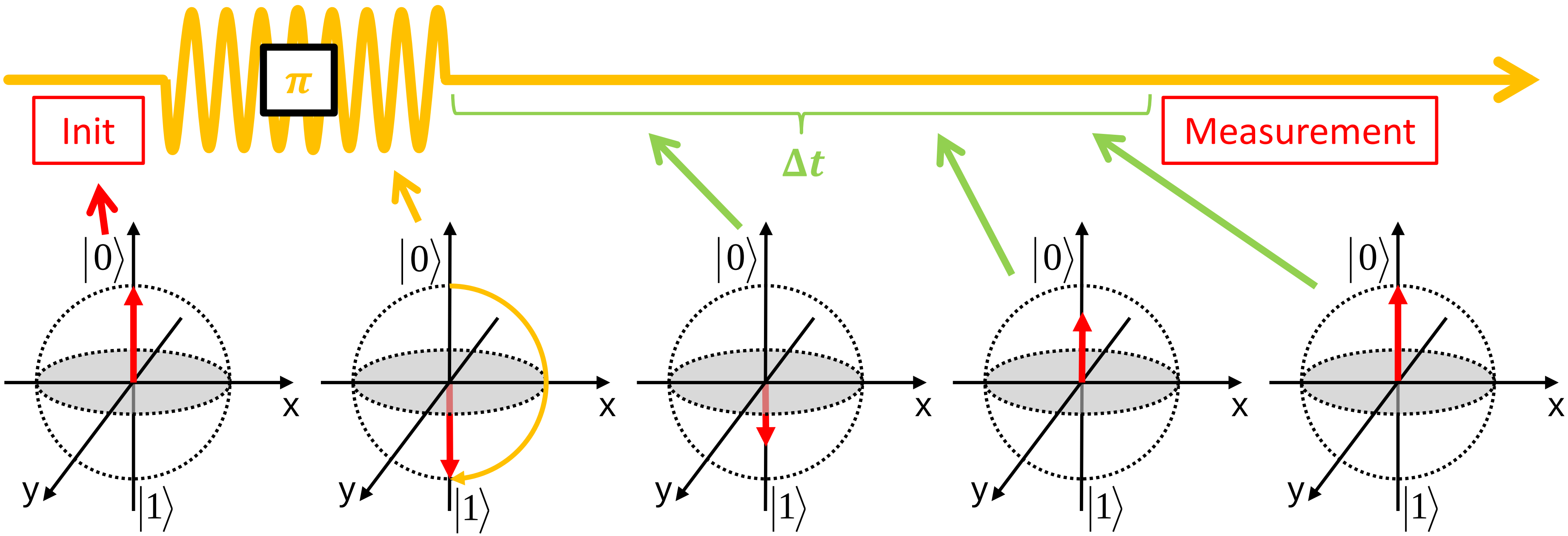
→ Classical probability, no superposition!

→ Phase coherence lost when arrows are distributed over whole equatorial plane

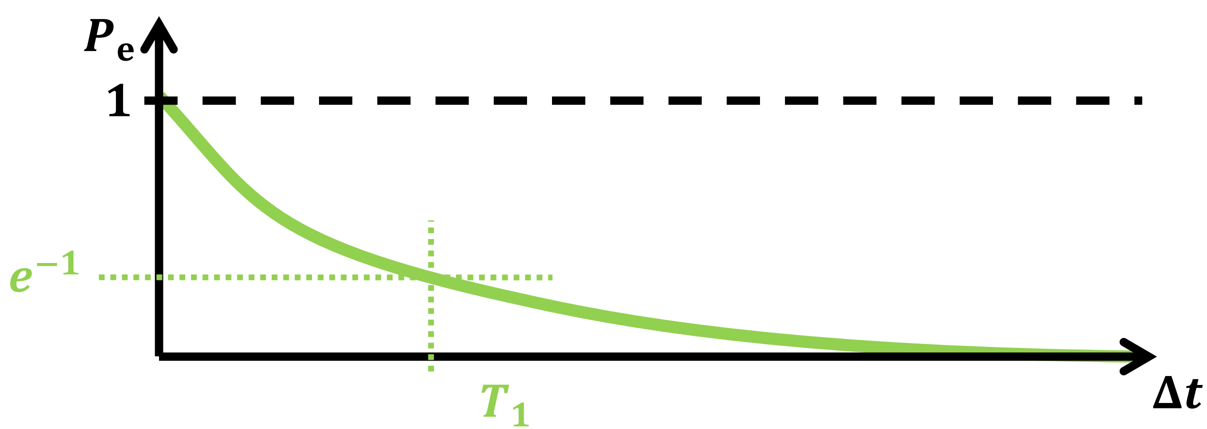


6.3 Control of quantum two-level systems

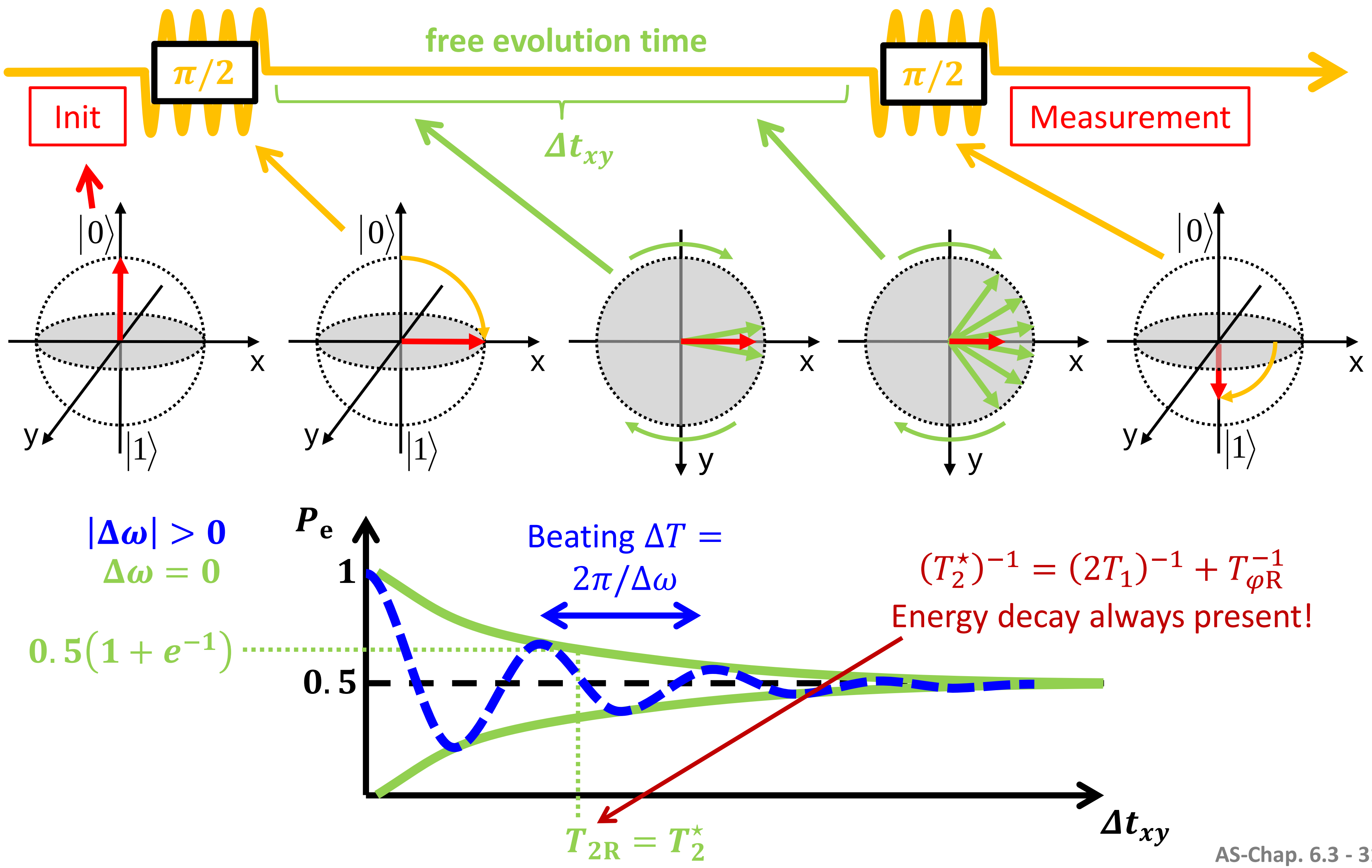
Qubit dynamics – Relaxation (T_1)



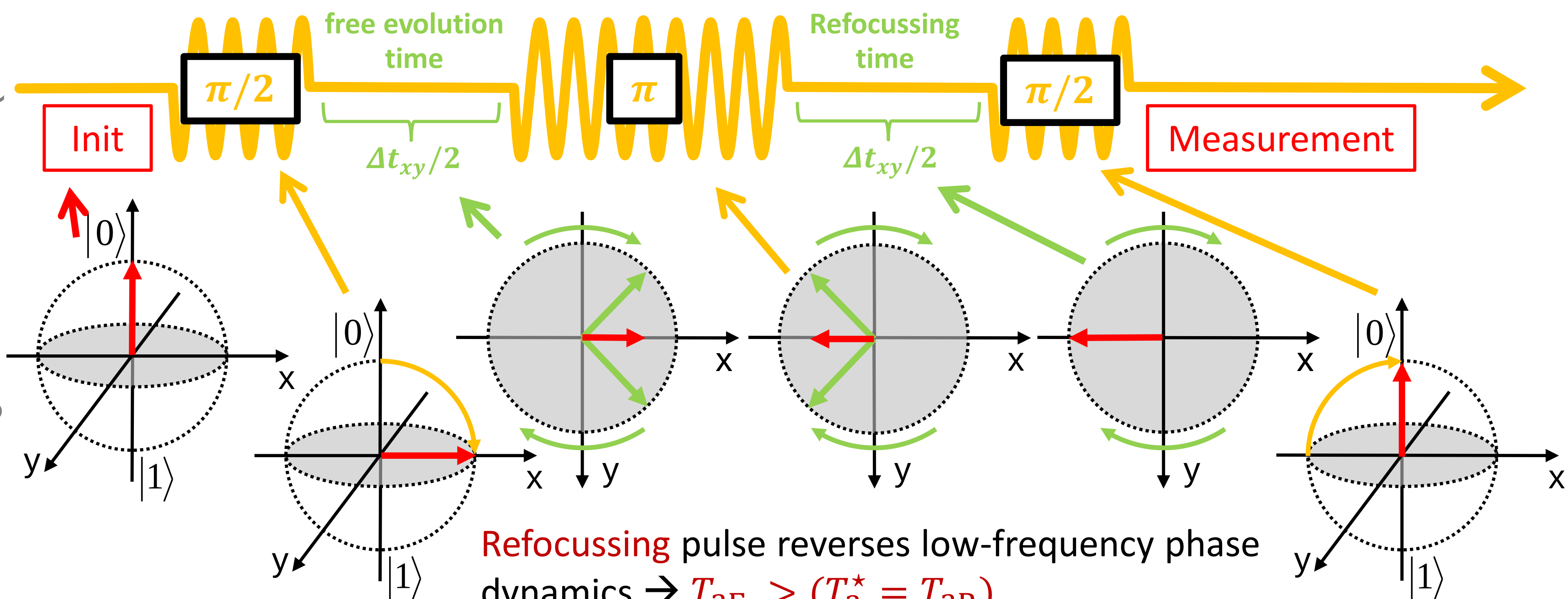
Rotating frame & no detuning ($\Delta\omega = \omega - \omega_{\mathrm{q}} = 0$) → no xy-evolution



Qubit dynamics – Ramsey fringes ($T_2^\star$)



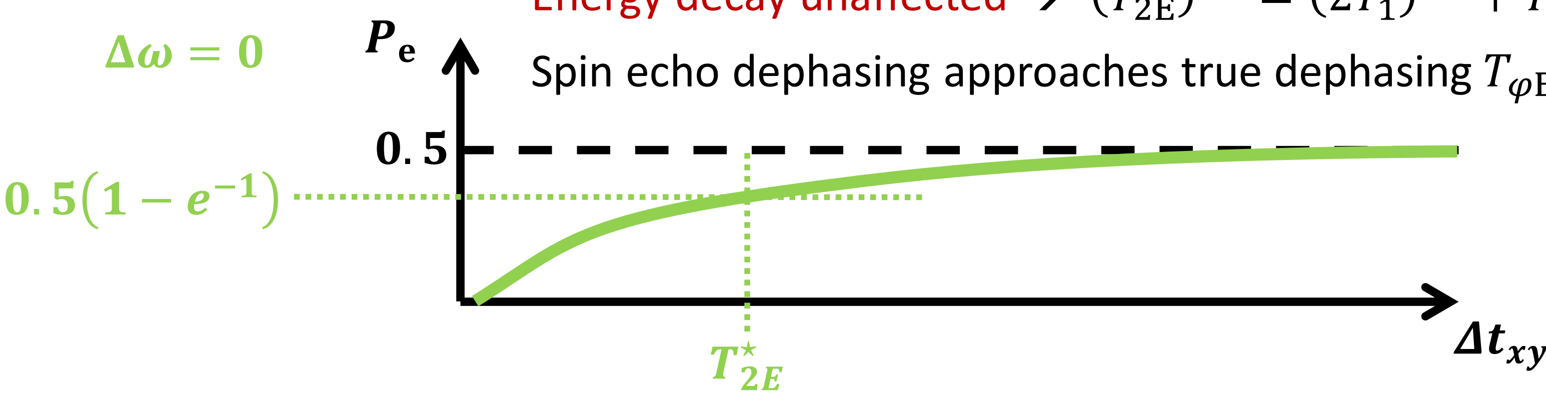
Qubit dynamics – Spin echo (T_{2E}^{*})



Refocussing pulse reverses low-frequency phase dynamics → $T_{2\mathrm{E}} > (T_2^{\star} = T_{2\mathrm{R}})$

Energy decay unaffected → $(T_{2\mathrm{E}})^{-1} = (2T_1)^{-1} + T_{\varphi\mathrm{E}}^{-1}$

Spin echo dephasing approaches true dephasing $T_{\varphi\mathrm{E}} \approx T_{\varphi}$



Ramsey vs. spin echo sequence

Simple picture → $T_{2\mathrm{E}} \simeq T_2$

More precise → Spin echo cancels the effect of low-frequency noise

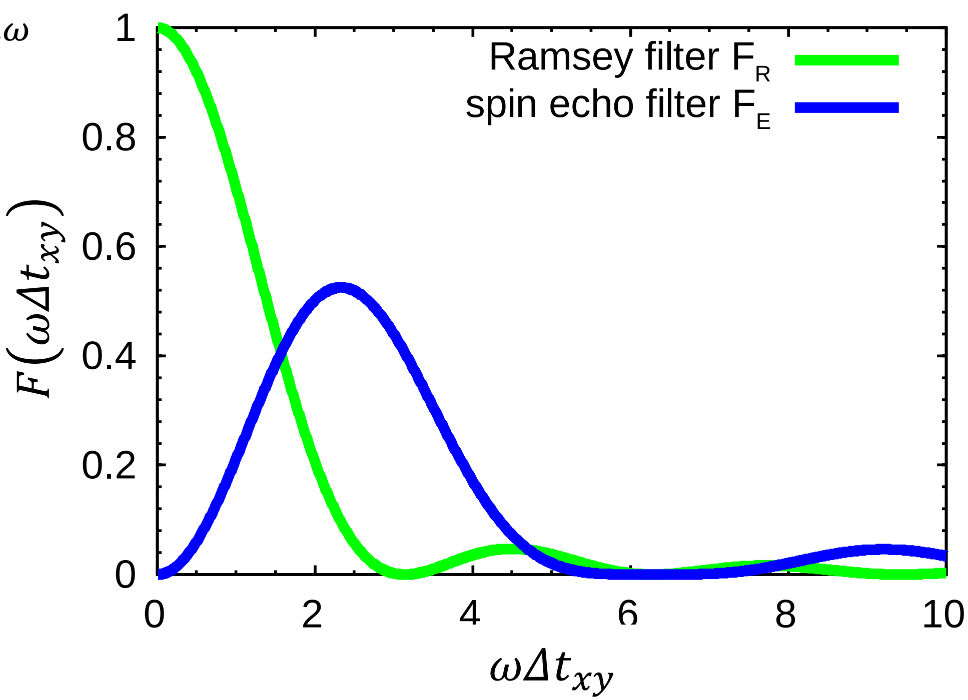
→ Pulse sequences act as filters!

→ Environment described by noise spectral density $S(\omega)$

Decay envelope ∝ $e^{-t^2 \int_{-\infty}^{+\infty} S(\omega) F_{\mathrm{R,E}}\left(\frac{\omega \Delta t_{xy}}{2}\right) d\omega}$

$$F_{\mathrm{R}}(\omega \Delta t_{xy}) = \frac{\sin^2 \frac{\omega \Delta t_{xy}}{2}}{\left(\frac{\omega \Delta t_{xy}}{2}\right)^2}$$

$$F_{\mathrm{E}}(\omega \Delta t_{xy}) = \frac{\sin^4 \frac{\omega \Delta t_{xy}}{4}}{\left(\frac{\omega \Delta t_{xy}}{4}\right)^2}$$



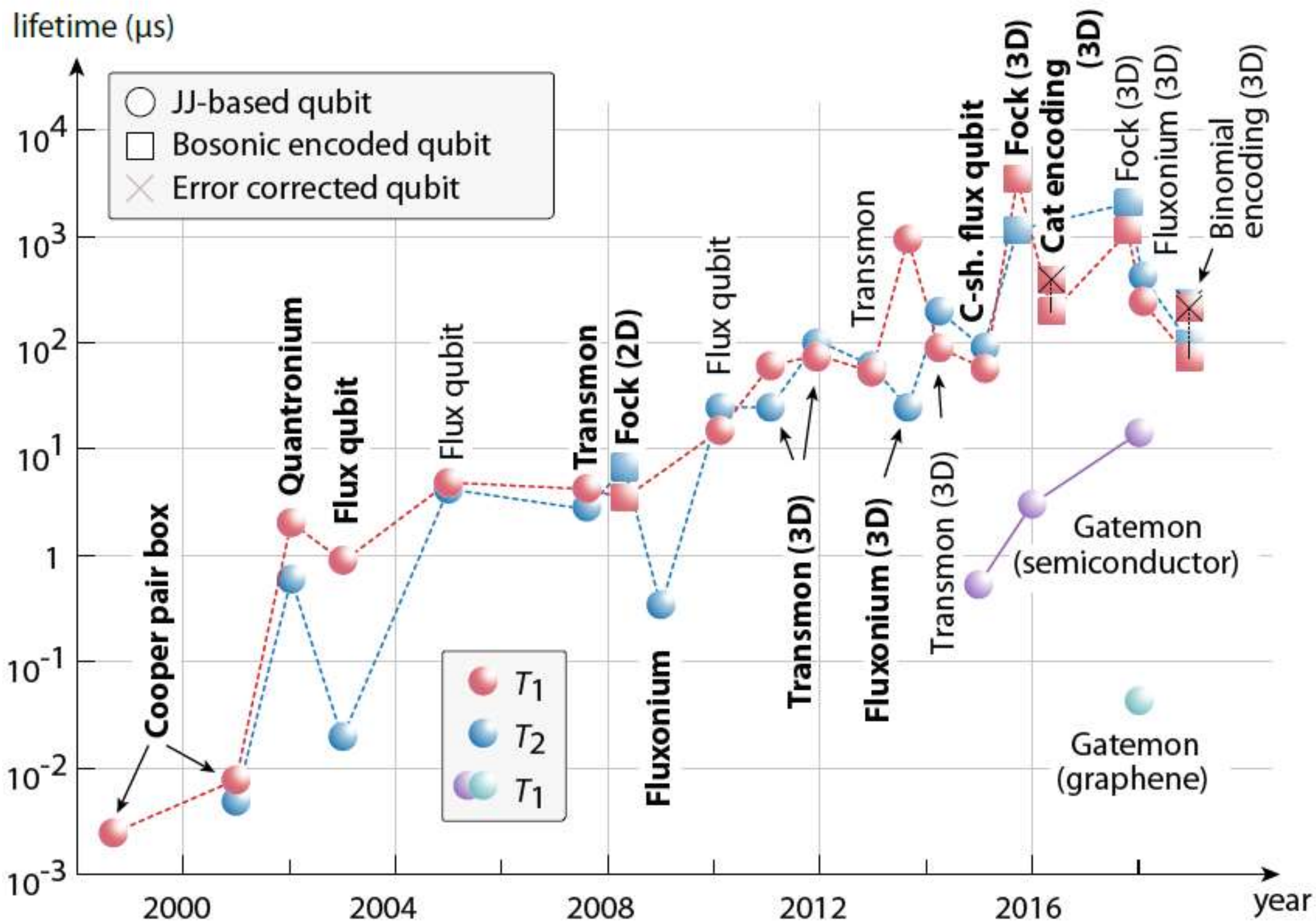
Sequence length Δt_{xy} is important!

Spin echo sequence

→ Filters low-frequency noise for$\omega \Delta t_{xy} \to 0$

→ $\omega \Delta t_{xy} \approx 2$ → Noise field fluctuates synchronously with π-pulse → No effect

Superconducting qubits – Improvement of decoherence times



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