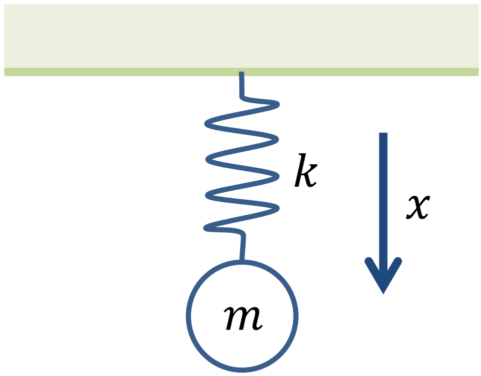
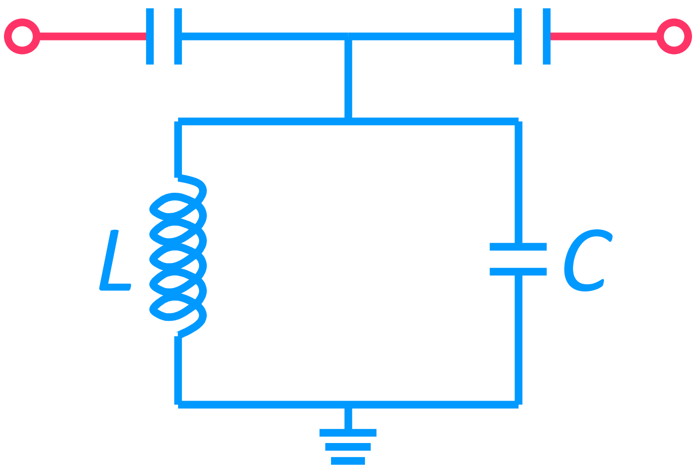
6.4
Physics
of superconducting
quantum circuits

General Hamiltonian

$$\hat{H}_{\mathrm{HO}} = E_{\mathrm{kin}} + E_{\mathrm{pot}} = \frac{\hat{p}^2}{2m} + \underbrace{\frac{1}{2} m\omega_{\mathrm{r}}^2 \hat{x}^2}_{\text{e.g., } \frac{1}{2}k\hat{x}^2 \text{ for mass on spring}}$$



LC resonant circuit

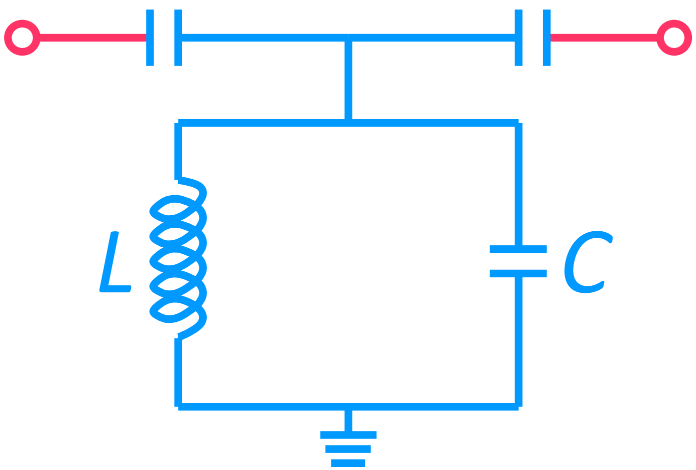


$$\hat{H}_{LC} = E_{\mathrm{kin}} + E_{\mathrm{pot}} = \frac{\hat{q}^2}{2C} + \frac{\hat{\Phi}^2}{2L} = \frac{\hat{q}^2}{2C} + \frac{1}{2} C \left(\frac{1}{LC}\right) \hat{\Phi}^2$$

Momentum $\hat{p}$	$\leftrightarrow$	Charge $\hat{q}$
Position $\hat{x}$	$\leftrightarrow$	Flux $\hat{\Phi}$
Mass m	$\leftrightarrow$	Capacitance C
Resonance frequency ω_{r}	$\leftrightarrow$	$\omega_{\mathrm{r}} = 1/\sqrt{LC}$

→ Completely analogous treatment

Continuous-variable operators



Momentum $\hat{p}$	$\leftrightarrow$ Charge $\hat{q}$
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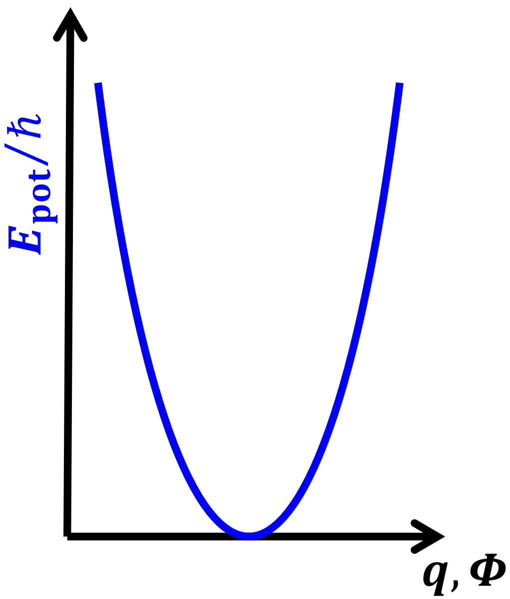


→ Parabolic potential

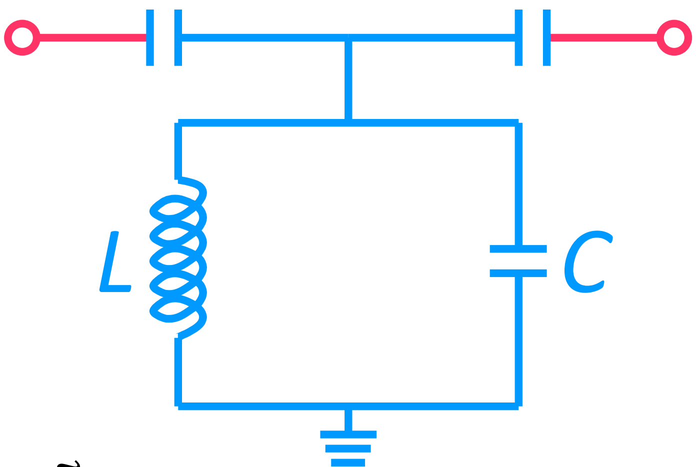
→ $\hat{q}$ and $\hat{\Phi}$ form a conjugate pair such as $\hat{x}$ and $\hat{p}$

→ Heisenberg uncertainty: $\Delta q \Delta \Phi \geq \frac{\hbar}{2}$

→ Commutation relation $[\hat{\Phi}, \hat{q}] = -i\hbar$

Discrete Basis (Fock states)

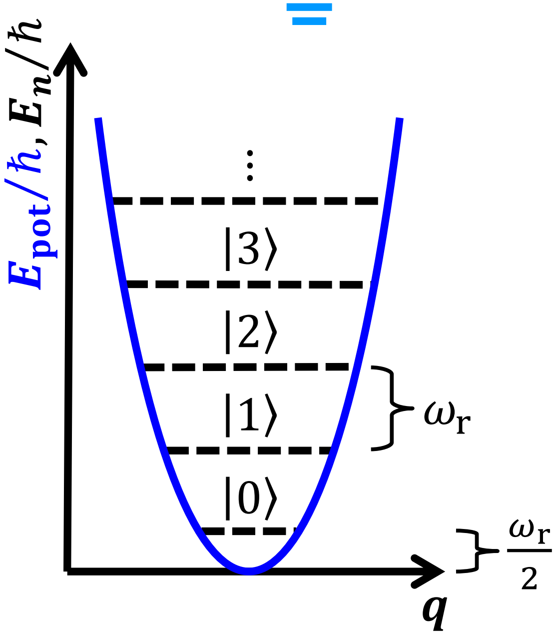
Momentum $\hat{p}$	$\leftrightarrow$ Charge $\hat{q}$
Position $\hat{x}$	$\leftrightarrow$ Flux $\hat{\Phi}$
Mass m	$\leftrightarrow$ Capacitance C
Resonance frequency ω_r	$\leftrightarrow$ $\omega_r = 1/\sqrt{LC}$



$$\hat{H}_{LC} = \hbar\omega_r\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

$\hat{a} \equiv \frac{\omega_r C\hat{\Phi} + i\hat{q}}{\sqrt{2\omega_r C\hbar}}$ is the annihilation operator

$\hat{a}^\dagger \equiv \frac{\omega_r C\hat{\Phi} - i\hat{q}}{\sqrt{2\omega_r C\hbar}}$ is the creation operator



→ **Photon number operator** $\hat{n} \equiv \hat{a}^\dagger\hat{a}$

→ Eigenstates are **Fock states** → $\hat{H}_{LC}|n\rangle = E_n|n\rangle$

→ Eigenvalues $E_n = \hbar\omega_r\left(n + \frac{1}{2}\right)$

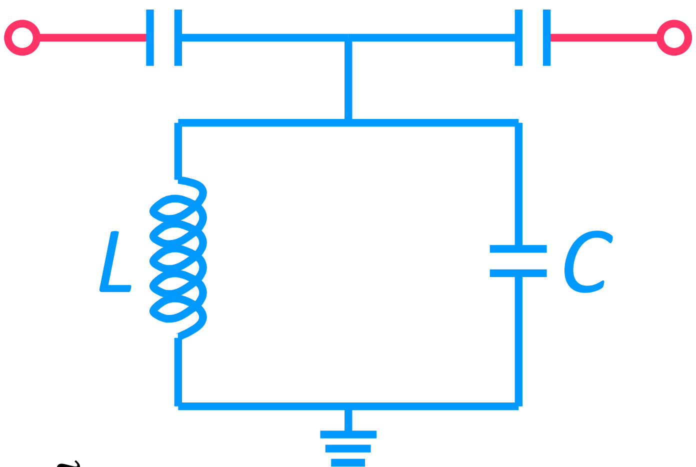
→ **Linear system** → Equidistant level spacing

→ $n = 0$ → Finite **vacuum energy** $E_0 = \frac{\hbar\omega_r}{2}$

→ $\{|n\rangle\}$ is the Fock or number basis

Annihilation and creation operator

Momentum $\hat{p}$	$\leftrightarrow$ Charge $\hat{q}$
Position $\hat{x}$	$\leftrightarrow$ Flux $\hat{\Phi}$
Mass m	$\leftrightarrow$ Capacitance C
Resonance frequency ω_r	$\leftrightarrow$ $\omega_r = 1/\sqrt{LC}$



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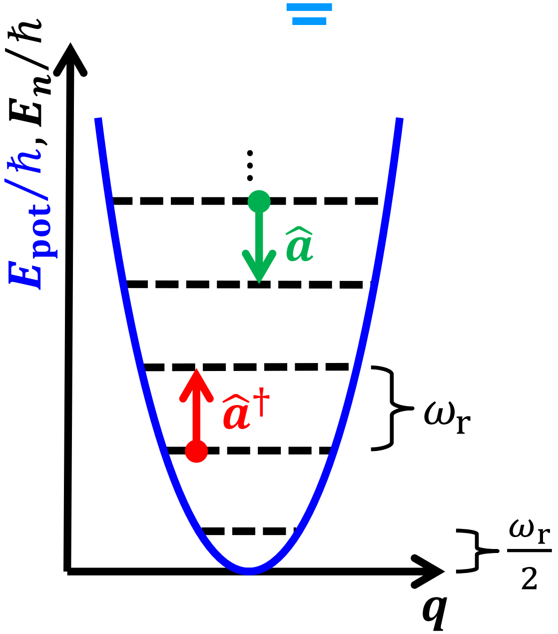


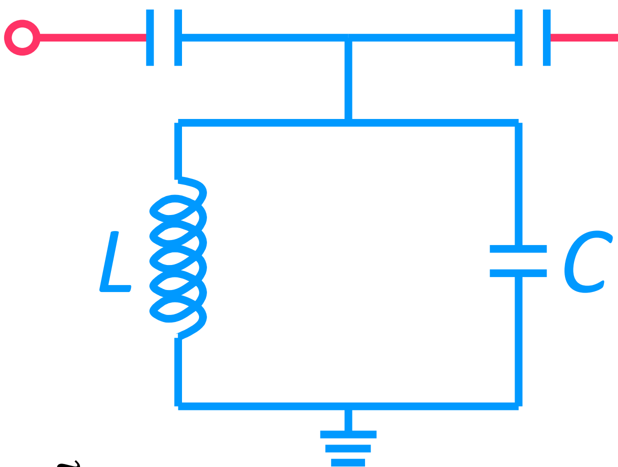
→ When applied to a Fock state, $\hat{a}$ annihilates a photon inside the oscillator

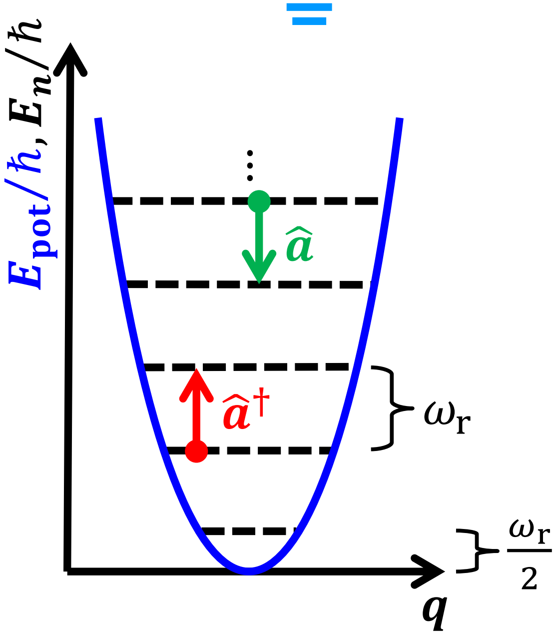
$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

→ When applied to a Fock state, $\hat{a}^\dagger$ creates a photon inside the oscillator

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

Coherent states





Momentum $\hat{p}$ $\leftrightarrow$ Charge $\hat{q}$
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$$\hat{H}_{LC} = \hbar\omega_{\mathrm{r}}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

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$\hat{a}^\dagger \equiv \frac{\omega_{\mathrm{r}}C\hat{\Phi}-i\hat{q}}{\sqrt{2\omega_{\mathrm{r}}C\hbar}}$ is the creation operator

$[\hat{a}, \hat{a}^\dagger] = 1$ → Bosonic commutation relation

Eigenstates of $\hat{a}$ → $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\alpha \in \mathbb{C}$

Coherent states $|\alpha\rangle$ → $|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$

$\{|\alpha\rangle\}$ normal but not orthogonal → Overcomplete

Practical importance of coherent states

Momentum $\hat{p}$	$\leftrightarrow$ Charge $\hat{q}$
Position $\hat{x}$	$\leftrightarrow$ Flux $\hat{\Phi}$
Mass m	$\leftrightarrow$ Capacitance C
Resonance frequency ω_r	$\leftrightarrow$ $\omega_\mathrm{r} = 1/\sqrt{LC}$

$$\hat{H}_{LC} = \hbar\omega_\mathrm{r}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

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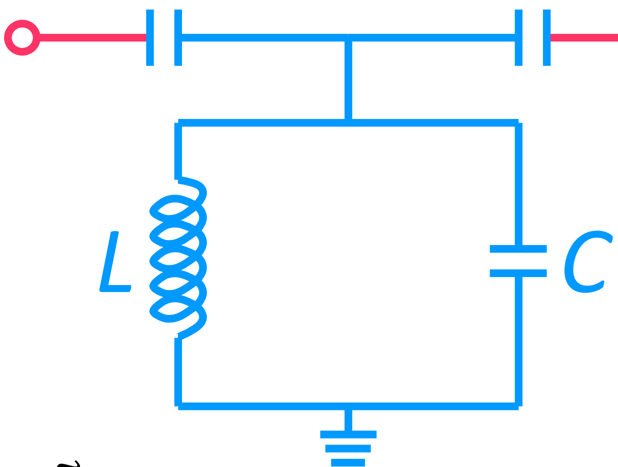
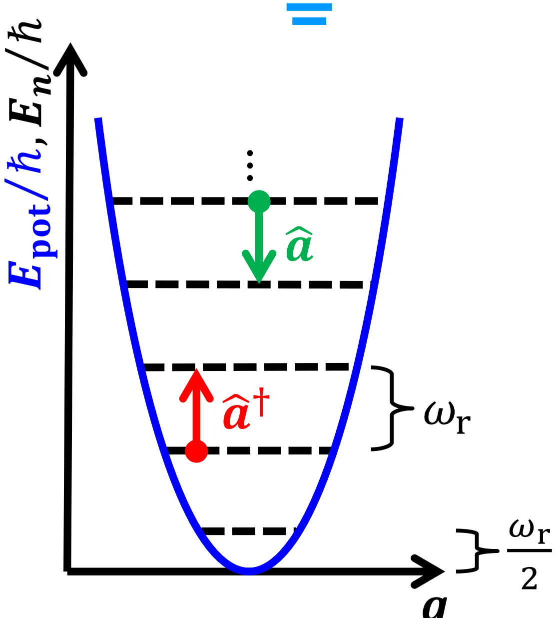




Bosonic field amplitude operator

$\rightarrow$ $\hat{A}(t) \equiv \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$

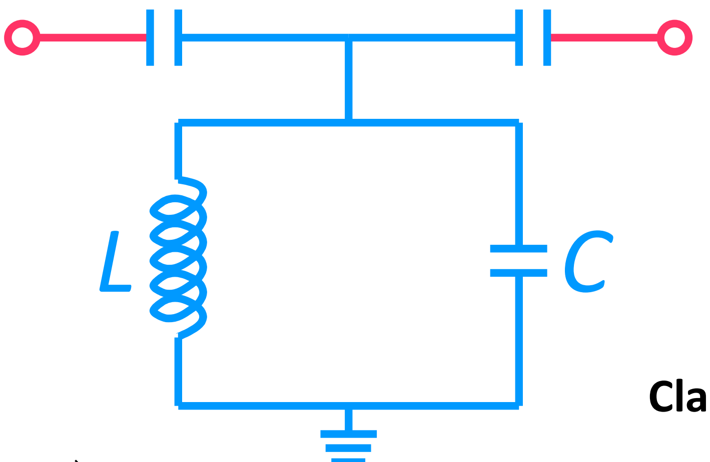
For intuitive understanding

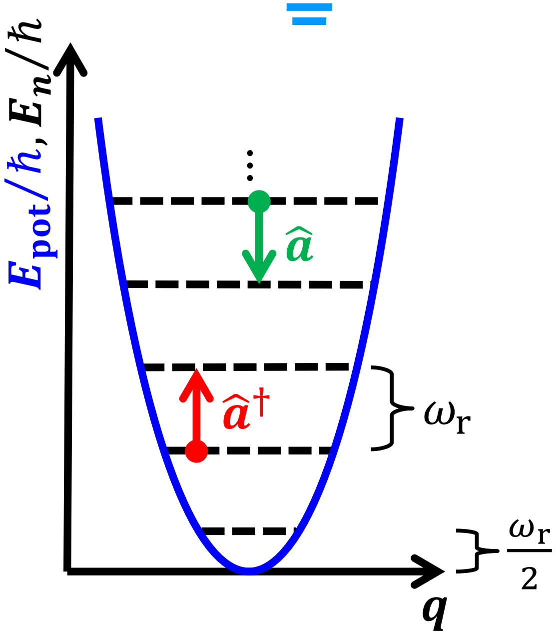
$\rightarrow$ Move to interaction picture

$\rightarrow$ $\hat{U}^{(\dagger)} = e^{(-)i\omega_\mathrm{r}t\hat{a}^\dagger\hat{a}}$

$\rightarrow$ $\hat{A}^I(t) \equiv \hat{U}\hat{A}\hat{U}^\dagger = \frac{1}{2}\left(\hat{a}e^{-i\omega_\mathrm{r}t} + \hat{a}^\dagger \mathrm{e}^{+i\omega_\mathrm{r}t}\right)$

Practical importance of coherent states





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Position $\hat{x}$	$\leftrightarrow$ Flux $\hat{\Phi}$
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Resonance frequency ω_r	$\leftrightarrow$ $\omega_\mathrm{r} = 1/\sqrt{LC}$

$$\hat{A}(t) \equiv \frac{1}{2}\left(\hat{a}e^{-i\omega_\mathrm{r}t} + \hat{a}^\dagger \mathrm{e}^{+i\omega_\mathrm{r}t}\right)$$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle,\ \alpha \in \mathbb{C}$$

Classical limit

$$\rightarrow \langle\alpha|\hat{A}(t)|\alpha\rangle = \frac{\alpha}{2}\underbrace{\langle\alpha|\alpha\rangle}_{=1}e^{-i\omega_\mathrm{r}t} + \frac{\alpha^\star}{2}\underbrace{\langle\alpha|\alpha\rangle}_{=1}\mathrm{e}^{+i\omega_\mathrm{r}t}$$

$$= \frac{|\alpha|}{2}\left(e^{-i(\omega_\mathrm{r}t+\phi)} + e^{+i(\omega_\mathrm{r}t+\phi)}\right)$$

$$= |\alpha|\cos(\omega_\mathrm{r}t + \phi)$$

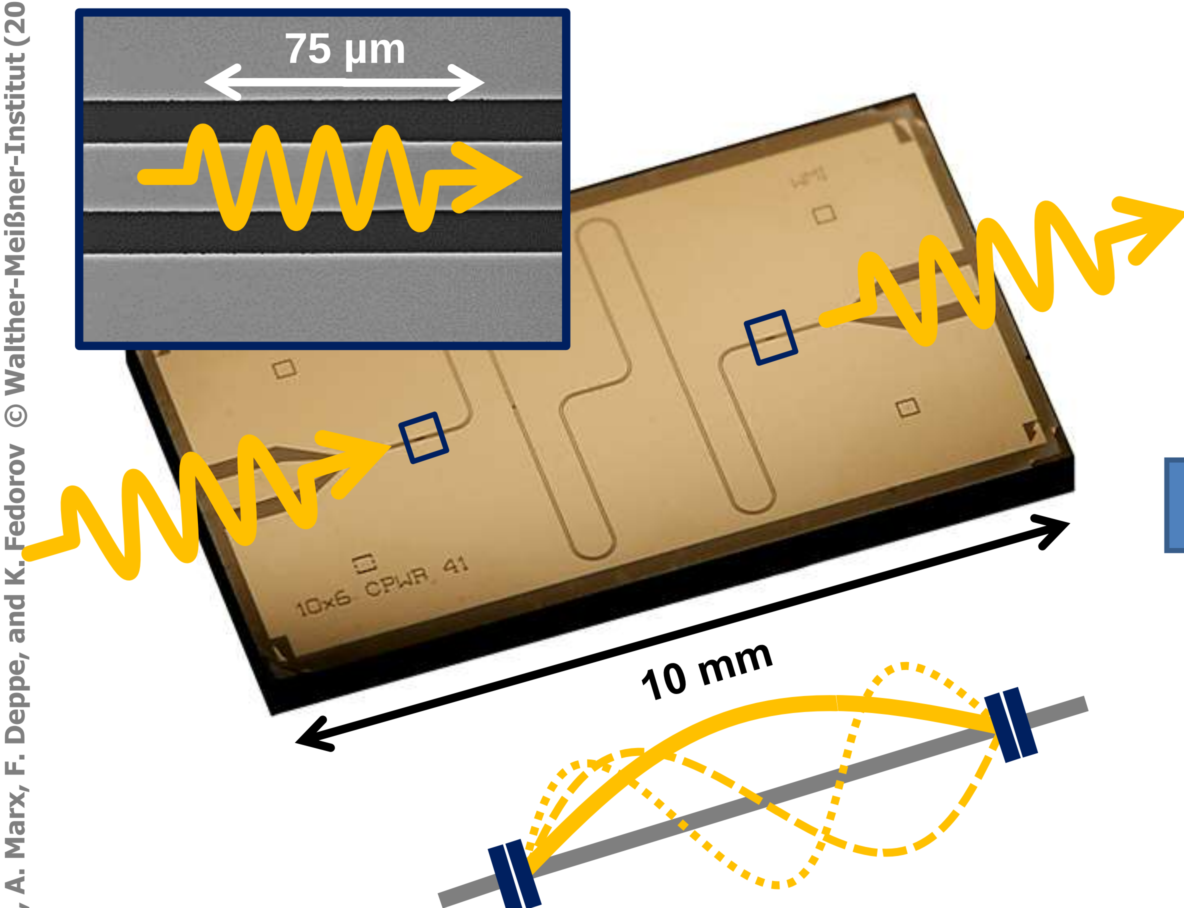
$\rightarrow$ **Oscillating field** with amplitude $|\alpha|$ and phase $\phi = \arg\alpha$

Coherent state $\rightarrow$ Most classical quantum state
(expectation values obey classical equations of motion)

6.4 Physics of superconducting quantum circuits

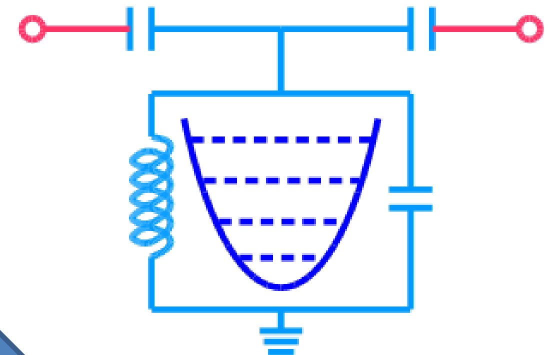
$\lambda/2$ coplanar waveguide resonator (quasi-1D)

$$\hat{H}_{LC} = \hbar\omega_{\mathrm{r}}\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right)$$



Standing waves ("modes")

Each mode n



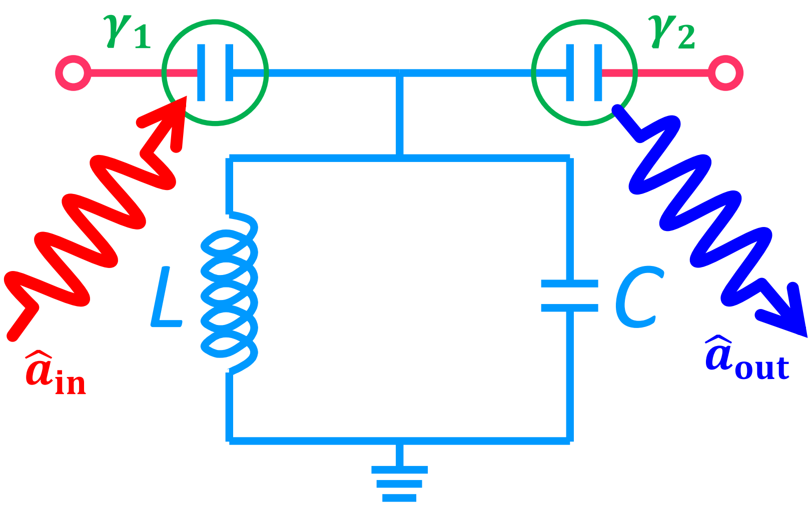
$k_{\mathrm{B}}T \ll \hbar\omega_n$

LC quantum harmonic oscillator

$$\hat{H}_{\mathrm{TL}} = \hbar\sum_n \omega_n \hat{a}_n^{\dagger}\hat{a}_n$$

Spectrum of $\widehat{H}_{LC}$

$$\widehat{H}_{LC} = \hbar\omega_{\rm r}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$



Must consider **coupling to external channel**
→ Loss rates γ_1 and γ_2
→ Simplification → $\gamma \equiv \gamma_1 = \gamma_2$

Challenge → Describe interaction between a single mode field and a continuum of modes!

Measurement requires two components
→ Input probe field $\hat{a}_{\rm in}$
→ Detected output field $\hat{a}_{\rm out}$

Properties of $\hat{a}_{\rm in}$ and $\hat{a}_{\rm out}$
→ free (propagating) multimode fields
→ Field $\hat{a}$ inside the resonator is a single mode-field
→ Transition mediated by coupling capacitors
→ Borrow input-output formalism from quantum optics!

D. F. Walls & G. Milburn, *Quantum Optics* (Springer, Berlin-Heidelberg, 2008)

Input-output fundamentals

Propagating on-chip microwave fields are quasi-1D → No polarization

Electric field operator of a free multi-mode field propagating along the $+x$-direction

$$\hat{E}^+(x,t) = i\sum_{n=0}^{\infty}\sqrt{\frac{\hbar\omega_n}{2\varepsilon_0 V}}\,\hat{b}_n e^{-i\omega_n\left(t-\frac{x}{c}\right)}$$

Propagation axis

Frequency

Mode sum

vacuum Amplitude

Dimensionless Fourier amplitudes of mode n

Mode volume

Mode spectrum centered on carrier frequency $\Omega \gg \omega$
& Continuum limit

→ Bosonic field operator of dimensions $\sqrt{s}$

$$\int_{-\infty}^{\infty} dt\, e^{-i(\omega_1-\omega_2)t} = 2\pi\,\delta(\omega_1-\omega_2)$$

$$\hat{b}(x,t) = e^{-i\Omega\left(t-\frac{x}{c}\right)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\,\hat{b}(\omega)e^{-i\omega\left(t-\frac{x}{c}\right)}$$

$$\left[\hat{b}(\omega_1),\hat{b}^\dagger(\omega_2)\right] = \delta(\omega_1-\omega_2)$$

Propagation phase often irrelevant
& frame rotating with Ω

$$\hat{b}(t) = \hat{b}(x>0,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\,\hat{b}(\omega)e^{-i\omega t}$$

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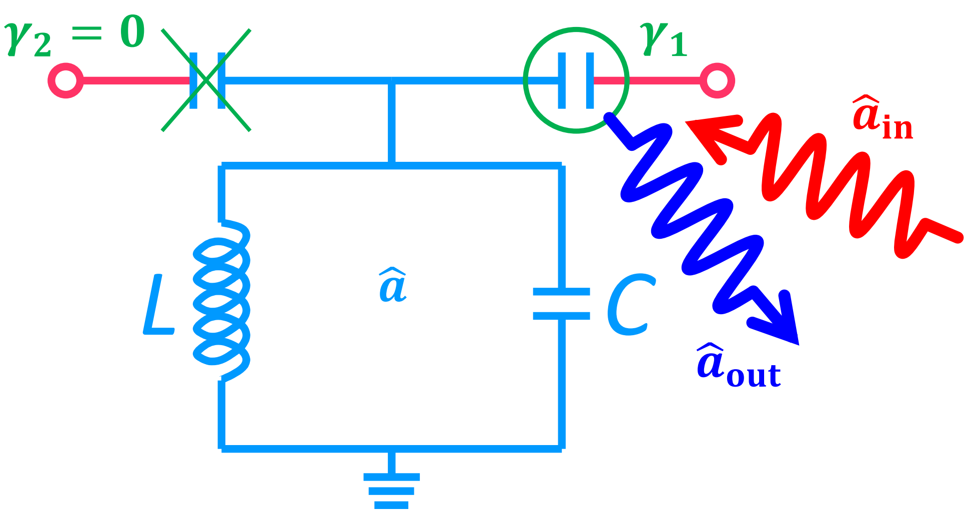
$\lambda/4$-resonator (single port)



$$\hat{H}_{LC} = \hbar\omega_{\text{r}}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

Frame rotating with Ω

Interaction energy

$$\text{V(t)} = i\hbar\int_{-\infty}^{\infty} d\omega g(\omega)\left[\hat{b}(\omega)\hat{a}^\dagger - \hat{b}^\dagger(\omega)\hat{a}\right]$$

Heisenberg picture → Wavefunction constant → Operators time-dependent

→ Explicit time dependence of frequency space operators $\hat{b}(\omega)$ → $\boldsymbol{\hat{b}(t,\omega)}$

$$\hat{H} = \hat{H}_{LC} + \hbar\omega\hat{b}^\dagger(t,\omega)\hat{b}(t,\omega) + i\hbar g(\omega)\left(\hat{b}(t,\omega)\hat{a}^\dagger - \hat{b}^\dagger(t,\omega)\hat{a}\right)$$

$$\frac{d}{dt}\hat{b}(t,\omega) = \frac{i}{\hbar}\left[\hat{H},\hat{b}(t,\omega)\right] + \frac{\partial}{\partial t}\hat{b}(t,\omega) \;\rightarrow\; \frac{d}{dt}\hat{b}(t,\omega) = -i\omega\hat{b}(t,\omega) + \text{g}(\omega)\hat{a}$$

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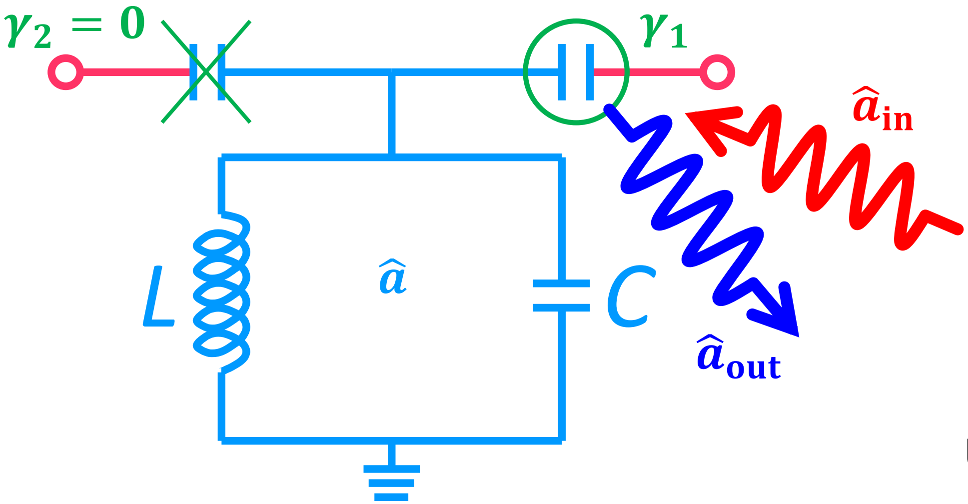
$\lambda/4$-resonator (single port)



$$\frac{d}{dt}\hat{b}(t,\omega) = -i\omega\hat{b}(t,\omega) + \mathrm{g}(\omega)\hat{a}$$

Initial condition $\hat{b}_0(\omega) \equiv \hat{b}(t_0,\omega)$

Final condition $\hat{b}_1(\omega) \equiv \hat{b}(t_1,\omega)$

Usually $t_0 = -\infty$ and $t_1 = \infty$ (free field)

Solutions

Input ($t_0 < t$) → $\hat{b}(t,\omega) = e^{-i\omega(t-t_0)}b_0(\omega) + g(\omega)\int_{t_0}^{t} dt'\, e^{-i\omega(t-t')}a(t')$

Output ($t < t_1$) → $\hat{b}(t,\omega) = e^{-i\omega(t-t_1)}b_1(\omega) - g(\omega)\int_{t}^{t_1} dt'\, e^{-i\omega(t-t')}a(t')$

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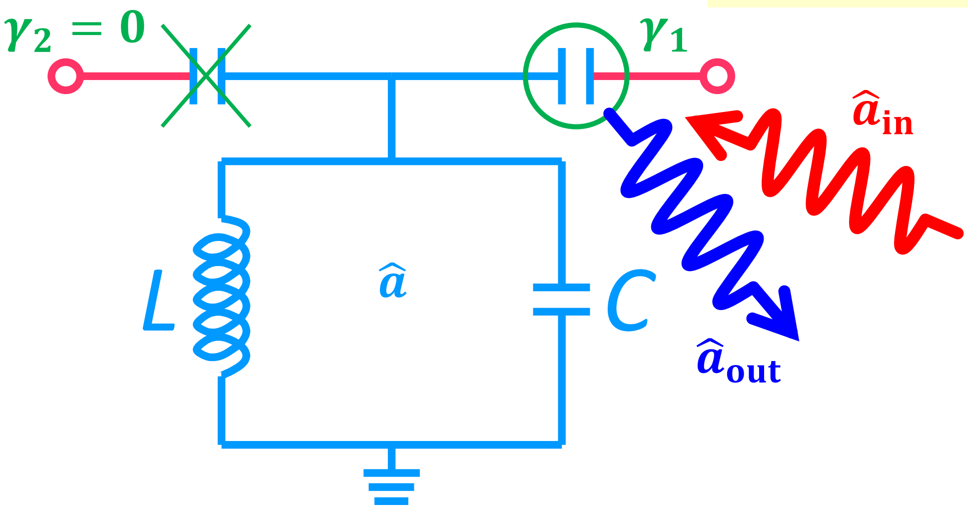
$\lambda/4$-resonator (single port)

$$\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} = 2\pi\delta(t-t')$$

$$\frac{d}{dt}\hat{b}(t,\omega) = -i\omega\hat{b}(t,\omega) + \mathrm{g}(\omega)\hat{a}$$



Resonator field – Equation of motion

$$\frac{d}{dt}\hat{a}(t) = \frac{i}{\hbar}[\hat{H}_{LC}, a] - \int_{-\infty}^{\infty} d\omega g(\omega)\hat{b}(t,\omega)$$

Coupling to multi-mode field!

$$\frac{d}{dt}\hat{a}(t) = \frac{i}{\hbar}[\hat{H}_{LC}, \hat{a}(t)] - \int_{-\infty}^{\infty} d\omega g(\omega) e^{-i\omega(t-t_0)}\hat{b}_0(\omega) - \int_{-\infty}^{\infty} d\omega g^2(\omega)\int_{t_0}^{t} dt' e^{-i\omega(t-t')}\hat{a}(t')$$

Input field $\quad \hat{a}_{\mathrm{in}}(t) \equiv -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t_0)}\hat{b}_0(\omega) \qquad [\hat{a}_{\mathrm{in}}(t), \hat{a}_{\mathrm{in}}^{\dagger}(t')] = \delta(t-t')$

$\gamma_1 \equiv 2\pi g^2$ Frequency-independent & well-behaved functions →

$$\frac{d}{dt}\hat{a}(t) = \frac{i}{\hbar}[\hat{H}_{LC}, \hat{a}(t)] + \frac{\gamma_1}{2}\hat{a}(t) + \sqrt{\gamma_1}\hat{a}_{\mathrm{in}}(t)$$

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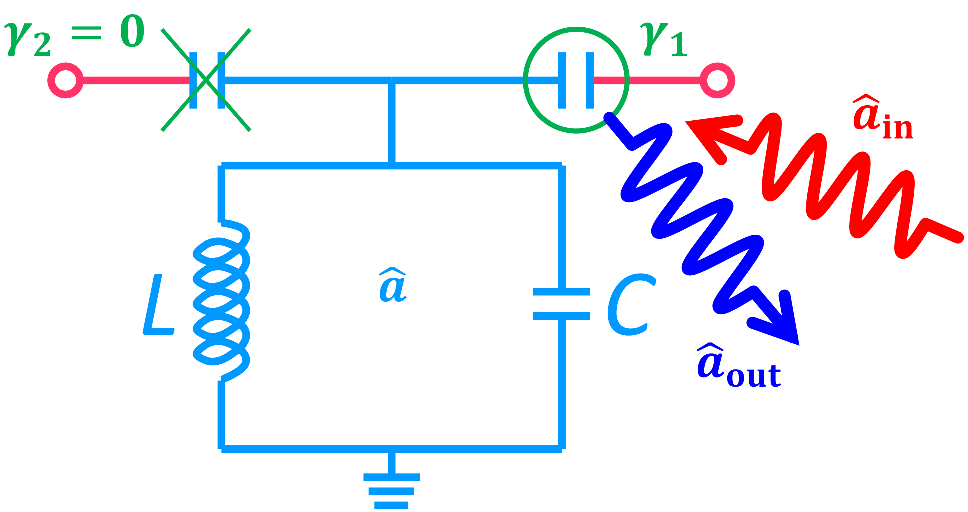
$\lambda/4$-resonator (single port)

$\gamma_1 \equiv 2\pi g^2(\omega)$

$\hat{a}_{\mathrm{in}}(t) \equiv -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t_0)}\hat{b}_0(\omega)$



Output field – Analogous result

$$\hat{a}_{\mathrm{out}}(t) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t_1)}\hat{b}_1(\omega)$$

$$\left[\hat{a}_{\mathrm{out}}(t), \hat{a}_{\mathrm{out}}^{\dagger}(t')\right] = \delta(t-t')$$

$$\frac{d}{dt}\hat{a}(t) = \frac{i}{\hbar}\left[\hat{H}_{LC}, \hat{a}(t)\right] - \frac{\gamma_1}{2}\hat{a}(t) + \sqrt{\gamma_1}\hat{a}_{\mathrm{in}}(t)$$

$$\frac{d}{dt}\hat{a}(t) = \frac{i}{\hbar}\left[\hat{H}_{LC}, \hat{a}(t)\right] + \frac{\gamma_1}{2}\hat{a}(t) - \sqrt{\gamma_1}\hat{a}_{\mathrm{out}}(t)$$

$$\hat{a}_{\mathrm{in}}(t) + \hat{a}_{\mathrm{out}}(t) = \sqrt{\gamma_1}\hat{a}(t)$$

Fourier integral

$$\hat{a}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t_0)}\hat{a}(\omega)$$

→ Solving differential equations simplifies to root finding!

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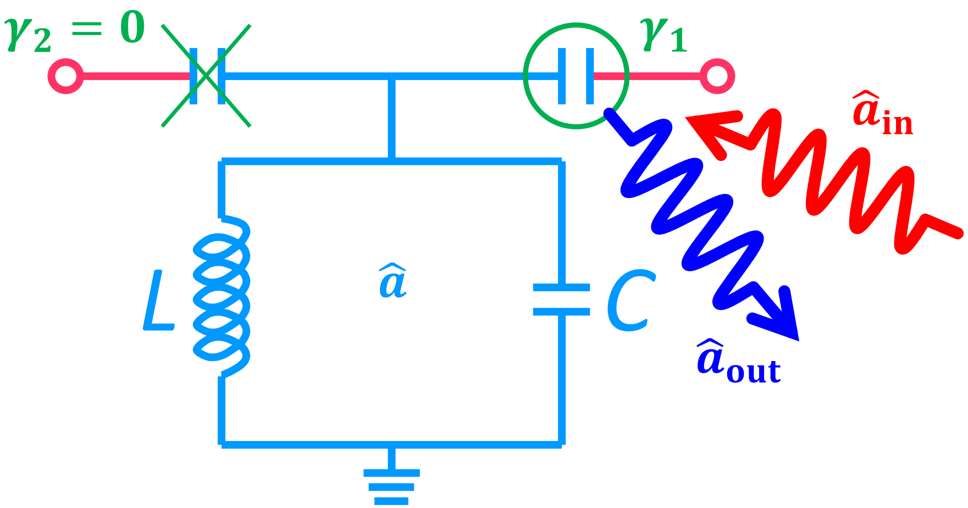
$\lambda/4$-resonator (single port)



$$\hat{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t_0)} \hat{a}(\omega)$$

$$\hat{H}_{LC} = \hbar\omega_{\mathrm{r}} \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$

$$\hat{a}_{\mathrm{in}}(t) + \hat{a}_{\mathrm{out}}(t) = \sqrt{\gamma_1}\hat{a}(t)$$

$$\frac{d}{dt}\hat{a}(t) = \frac{i}{\hbar}\left[\hat{H}_{LC}, \hat{a}(t)\right] - \frac{\gamma_1}{2}\hat{a}(t) + \sqrt{\gamma_1}\hat{a}_{\mathrm{in}}(t)$$

$$-i\omega\hat{a}(\omega) = -i\omega_{\mathrm{r}}\hat{a}(\omega) - \frac{\gamma_1}{2}\hat{a}(\omega) + \sqrt{\gamma_1}\hat{a}_{\mathrm{in}}(\omega)$$

after applying $\hat{H}_{LC}$ and Fourier transform

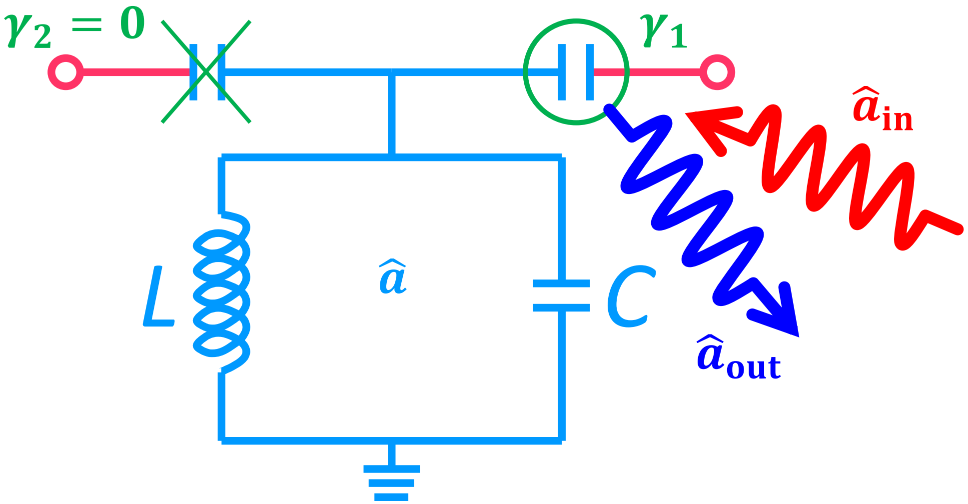
$$\hat{a}_{\mathrm{in}}(\omega) + \hat{a}_{\mathrm{out}}(\omega) = \sqrt{\gamma_1}\hat{a}(\omega)$$

D. F. Walls & G. Milburn, *Quantum Optics* (Springer, Berlin-Heidelberg, 2008)

$\lambda/4$-resonator (single port)

$$\hat{H}_{LC} = \hbar\omega_\mathrm{r}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

$$\hat{a}_\mathrm{in}(t) + \hat{a}_\mathrm{out}(t) = \sqrt{\gamma_1}\,\hat{a}(t)$$



Reflection coefficient $\Gamma \equiv \frac{\hat{a}_\mathrm{out}}{\hat{a}_\mathrm{in}} = \frac{\frac{\gamma_1}{2}+i(\omega-\omega_\mathrm{r})}{\frac{\gamma_1}{2}-i(\omega-\omega_\mathrm{r})} = \frac{\left(\frac{\gamma_1}{2}\right)^2+i(\omega-\omega_r)\gamma_1-(\omega-\omega_r)^2}{\left(\frac{\gamma_1}{2}\right)^2+(\omega-\omega_r)^2}$ → $|\Gamma| = 1$

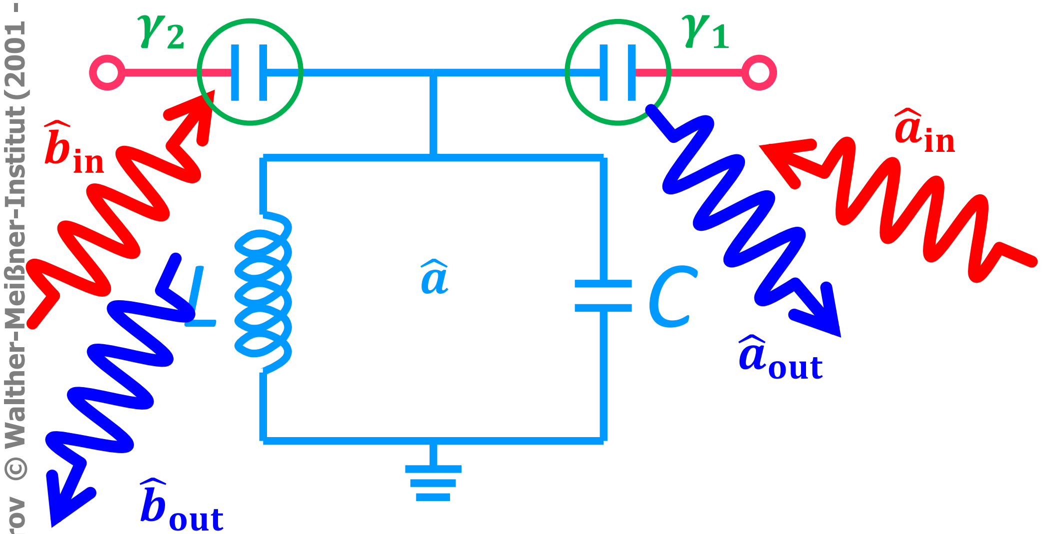
→ What goes in, must go out

Resonator field $\Gamma_\mathrm{r} \equiv \frac{\hat{a}}{\hat{a}_\mathrm{in}} = \frac{\sqrt{\gamma_1}}{\frac{\gamma_1}{2}-i(\omega-\omega_\mathrm{r})}$ → $|\Gamma_r|$ is Lorentzian

→ Field enhancement inside resonator

D. F. Walls & G. Milburn, *Quantum Optics* (Springer, Berlin-Heidelberg, 2008)

$\lambda/2$-resonator (two ports)



$$\hat{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t_0)} \hat{a}(\omega)$$

$$\hat{H}_{LC} = \hbar\omega_{\rm r}\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)$$

$$\hat{a}_{\rm in}(\omega) + \hat{a}_{\rm out}(\omega) = \sqrt{\gamma_1}\hat{a}(\omega)$$

$$\frac{d}{dt}\hat{a}(t) = \frac{i}{\hbar}\left[\hat{H}_{LC}, \hat{a}(t)\right] - \frac{(\gamma_1+\gamma_2)}{2}\hat{a}(t) + \sqrt{\gamma_1}\hat{a}_{\rm in}(t) + \sqrt{\gamma_2}\hat{b}_{\rm in}(t)$$

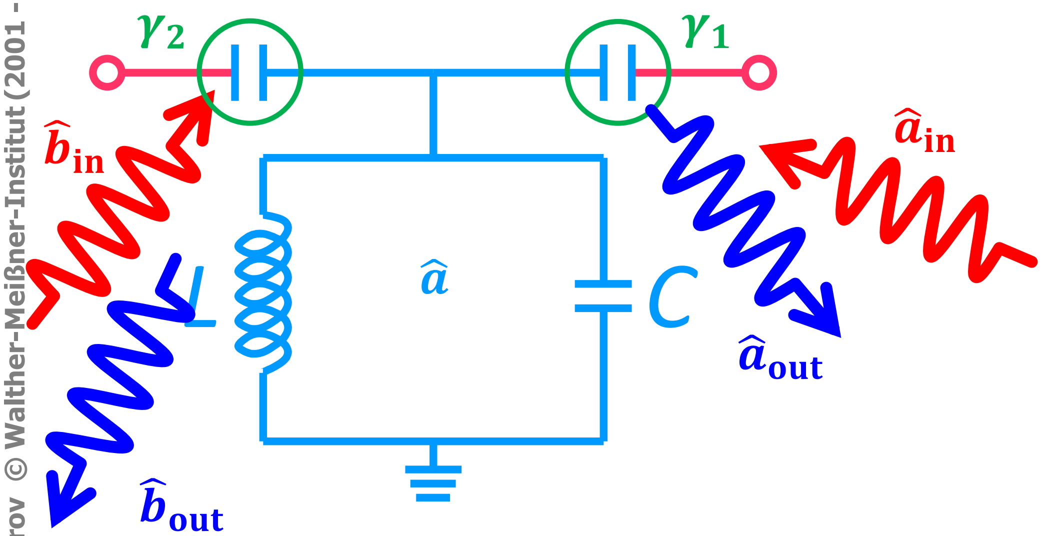
$$-{\rm i}\omega\hat{a}(t) = -i\omega_{\rm r}\hat{a}(t) - \frac{(\gamma_1+\gamma_2)}{2}\hat{a}(t) + \sqrt{\gamma_1}\hat{a}_{\rm in}(t) + \sqrt{\gamma_2}\hat{b}_{\rm in}(t)$$

after applying $\hat{H}_{LC}$ and Fourier transform

$$\hat{a}_{\rm in}(t) + \hat{a}_{\rm out}(t) = \sqrt{\gamma_1}\hat{a}(t) \quad \rightarrow \quad \hat{b}_{\rm in}(\omega) + \hat{b}_{\rm out}(\omega) = \sqrt{\gamma_2}a(\omega)$$

D. F. Walls & G. Milburn, *Quantum Optics* (Springer, Berlin-Heidelberg, 2008)

$\lambda/2$-resonator (two ports)



$$\hat{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t_0)} \hat{a}(\omega)$$

$$\hat{H}_{LC} = \hbar\omega_{\mathrm{r}} \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$

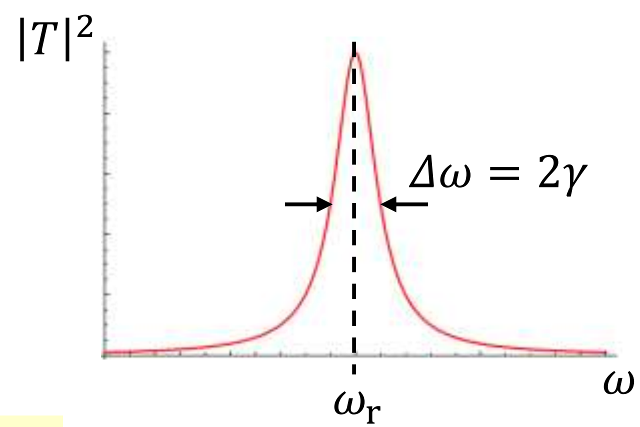
$$\hat{a}_{\mathrm{in}}(\omega) + \hat{a}_{\mathrm{out}}(\omega) = \sqrt{\gamma_1} \hat{a}(\omega)$$

Transmission coefficient
$(\hat{b}_{\mathrm{in}} = 0, \gamma_1 = \gamma_2 \equiv \gamma)$

→ Now reflection and transmission (two ports) → $\Gamma + T = 1$

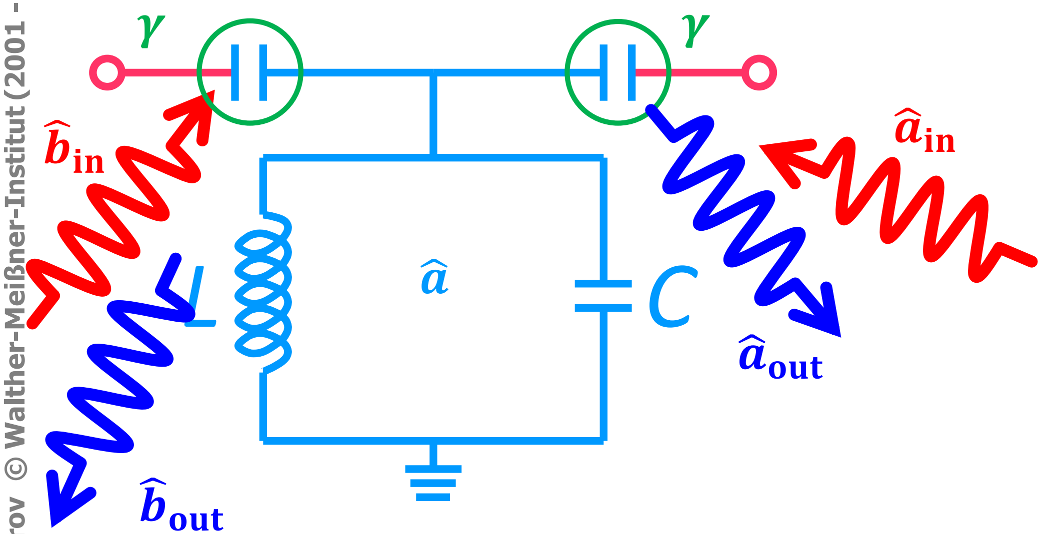
→ $T \equiv \frac{\hat{b}_{\mathrm{out}}}{\hat{a}_{\mathrm{in}}} = \frac{\gamma}{\gamma - i(\omega - \omega_{\mathrm{r}})}$

→ Transmitted power $|T|^2$ is Lorentzian!
(equals the classical result)

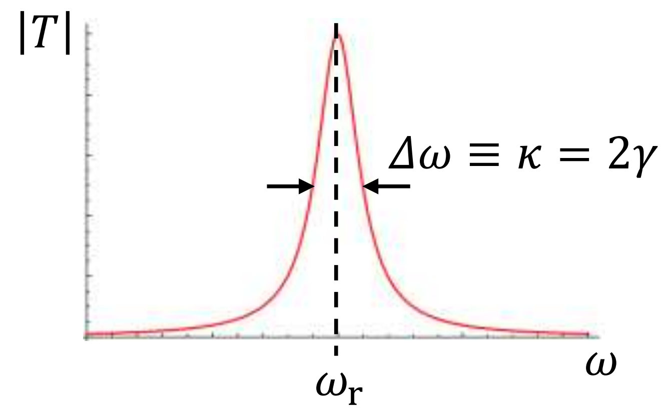


D. F. Walls & G. Milburn, *Quantum Optics* (Springer, Berlin-Heidelberg, 2008)

$\lambda/2$-resonator (two ports)



$\rightarrow T = \frac{\gamma}{\gamma - i(\omega - \omega_r)}$



Storage time of HO characterized by its quality factor $Q \equiv \frac{f_r}{\Delta f} = \frac{\omega_r}{\Delta\omega} = \frac{\omega_r}{\kappa}$, where $\omega_r \equiv 2\pi f_r$

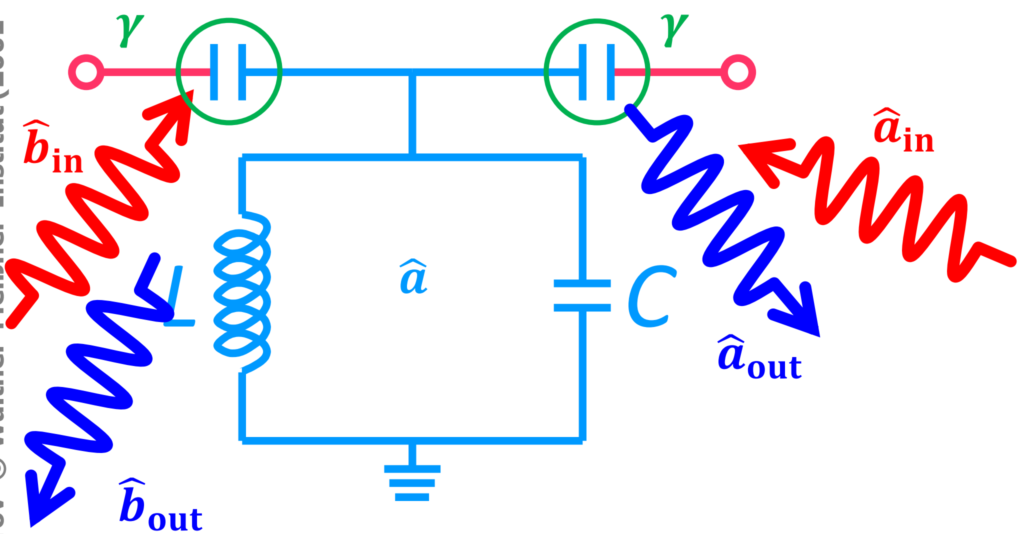
Energy-time uncertainty $\rightarrow \Delta E \Delta t \simeq \hbar \rightarrow \Delta\omega\Delta t \simeq 1$

$E = \hbar\omega$

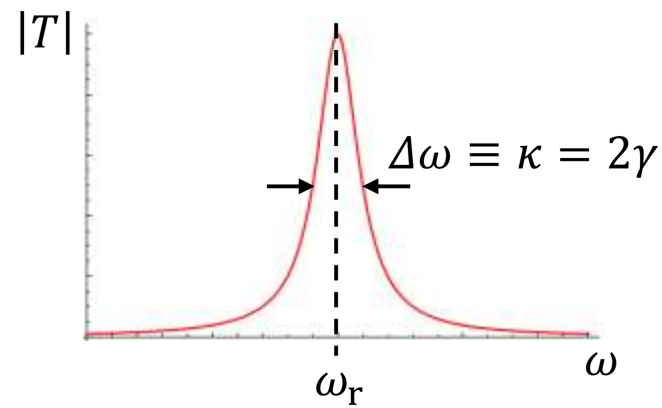
Identify Δt with relaxation time $T_1 = \frac{1}{2\pi\Delta f}$ and dephasing time $T_2 = \frac{1}{\pi\Delta f}$

Typically, ω, γ, κ are angular frequencies, but $f, T_1, T_2, T_2^*, T_\varphi$ are normal frequencies

$\lambda/2$-resonator (two ports)



$\rightarrow T = \frac{\gamma}{\gamma - i(\omega - \omega_{\mathrm{r}})}$



$Q \propto \gamma^{-1}$ → Quality factor determined by loss ports

Many types of loss ports
→ Coupling capacitors Q_{c}
→ Internal dissipative/dielectric losses → Q_{i}
→ Radiation losses → Q_{rad}
→ …

Loaded quality factor $\frac{1}{Q} = \frac{1}{Q_{\mathrm{c}}} + \frac{1}{Q_{\mathrm{i}}} + \frac{1}{Q_{\mathrm{rad}}} + \cdots$

$$T_2 = \frac{1}{\pi \Delta f}$$

2D resonators

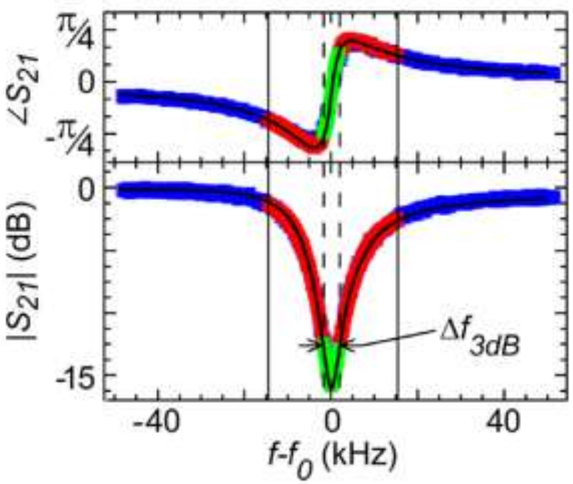
Niobium on SiO_2-coated high-resistivity Si substrate @mK temperatures

→ $f_0 = \text{few GHz}$
→ $Q_\mathrm{i} \approx 10^5$
→ $T_2 \lesssim 10\ \mu\mathrm{s}$

MBE grown (epitaxially) Al on sapphire substrate @ mK temperatures

→ $f_0 = 6.121\ \mathrm{GHz}$
→ $Q_\mathrm{i} = 1.7 \times 10^6$, $Q_\mathrm{c} = 4 \times 10^5$
→ $T_2 \lesssim 0.1\ \mathrm{ms}$

Megrant *et al.*, APL 100, 113510 (2012)



3D resonators

$$T_2 = \frac{1}{\pi \Delta f}$$

Loss sources for planar superconducting resonators

→ Resistive or QP losses → Superconductivity & low temperatures

→ Radiation losses → Clever design

→ Problem: Dielectric losses from material defects (spurious TLS)

  → TLS in bulk substrate → Use clean single crystal (sapphire, intrinsic Si)

  → TLS at substrate-metal interface

  → Need clean materials & growth processes!

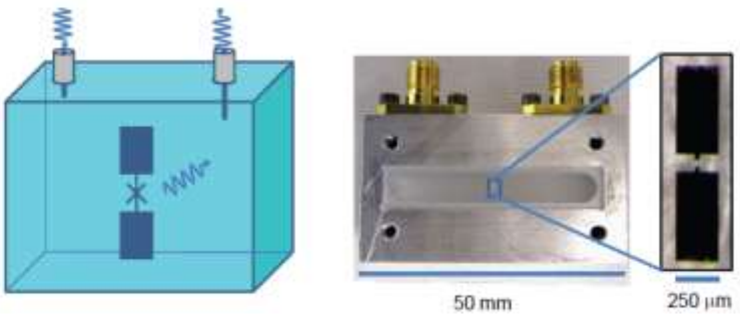
→ Alternative: 3D cavity resonators

  → No more dielectric → No TLS in cavity → $Q_\mathrm{i} \approx 10^7 - 10^8$

  → Reduce participation ratio of interace losses for embedded circuits

  → 3D transmon qubit → $T_1 \lesssim 0.15\ \mathrm{ms}$

Paik *et al.*, PRL 107, 240501 (2011)



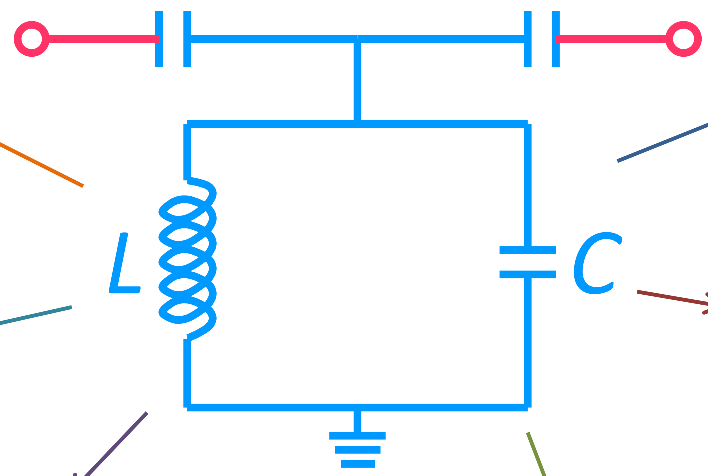
Applications of quantum harmonic oscillators

Quantum HO is linear → Not a qubit → Not directly useful for quantum computation!

Nevertheless indirect use:

Quantum simulation of many-body Hamiltonians

Mediate coupling between qubits → Quantum bus



Qubit readout („dispersive readout")

Typically long coherence times → Quantum memory

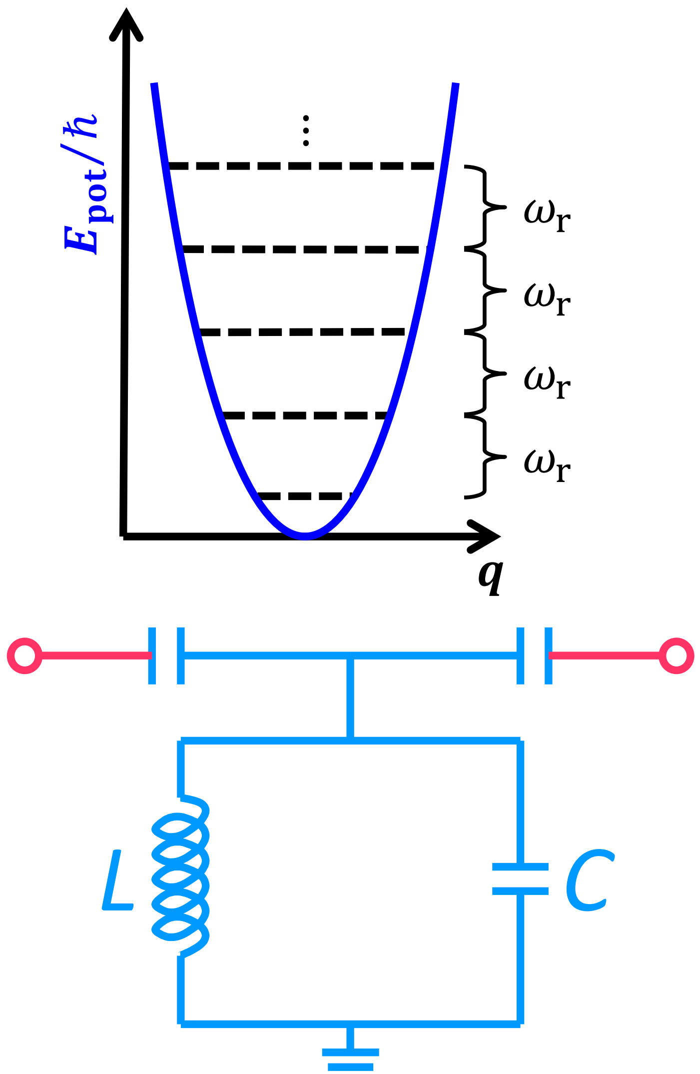
Ancilla qubit/nonlinearity → Explore quantum physics (Fock states, squeezing etc.)

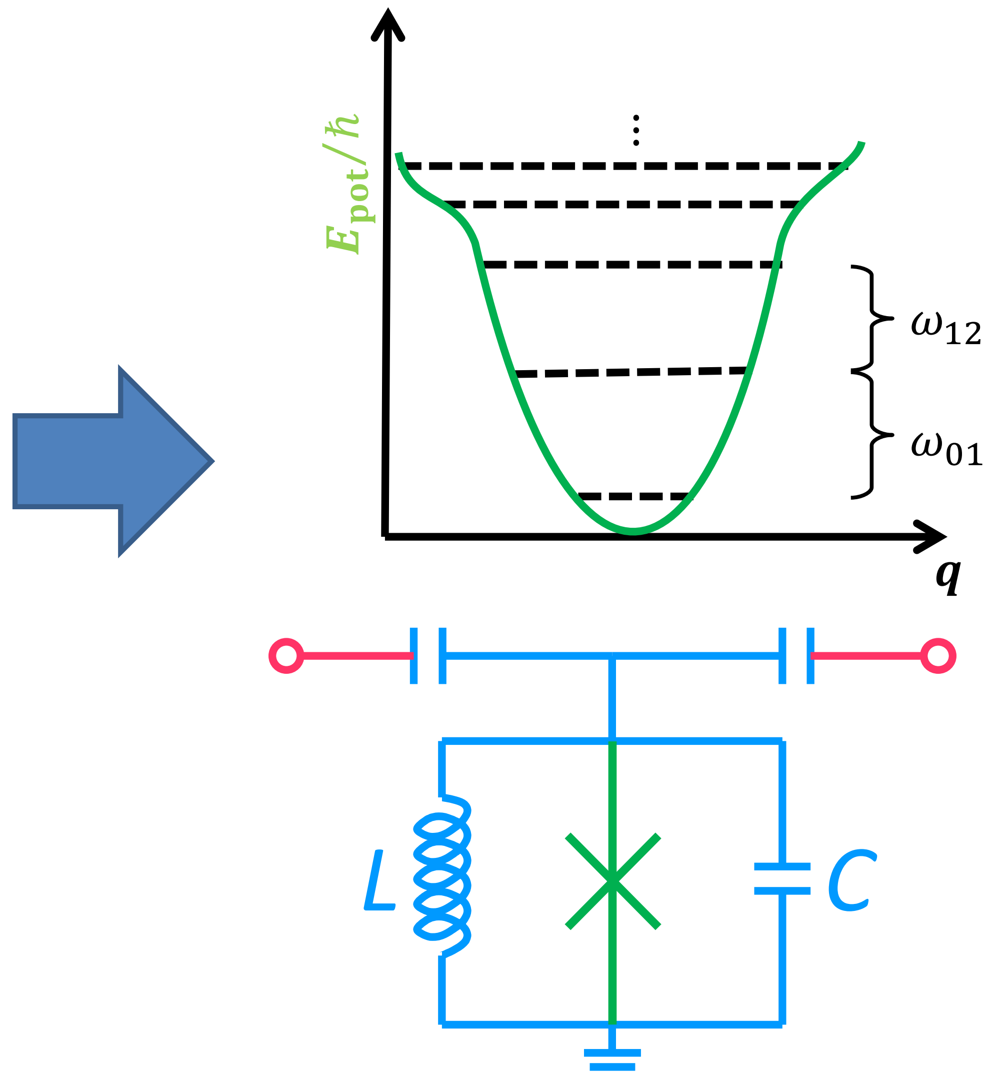
Identify decoherence sources in superconducting quantum circuits

Josephson nonlinearity

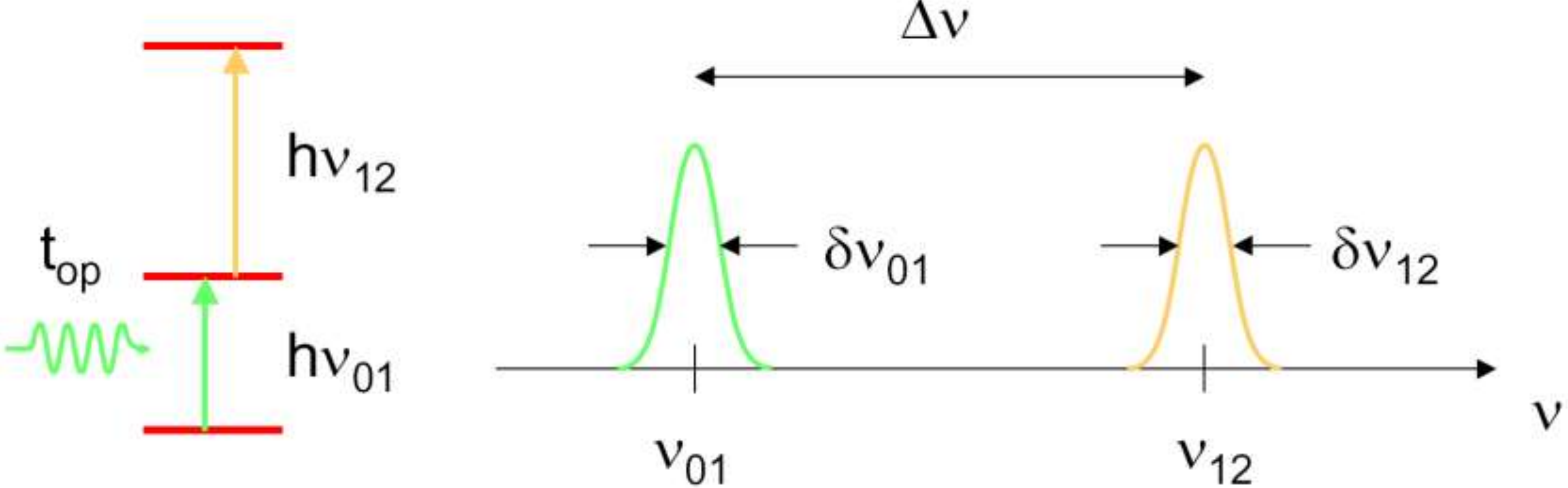
Linear system (parabolic potential)

Nonlinear system (general potential)





How much anharmonicity is required ?

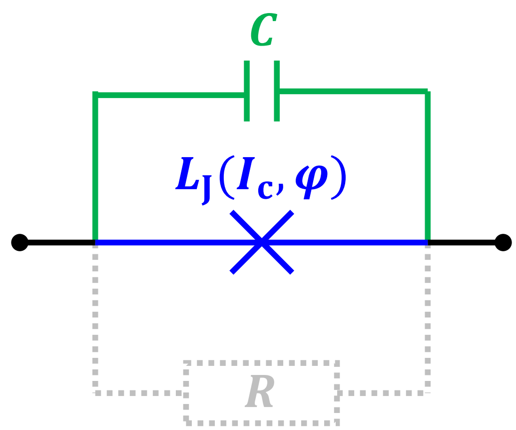


Coherence time: $T_{\mathrm{dec}} \approx \frac{1}{\delta\nu_{01}} = \frac{\nu_{01}}{\delta\nu_{01}}\frac{1}{\nu_{01}} = Q\frac{1}{\nu_{01}}$

1 bit operation time: $t_{\mathrm{op}} > \frac{1}{\Delta\nu}$ (otherwise $|1\rangle \rightarrow |2\rangle$-transitions are induced!)

# of 1 bit operations: $\frac{T_{\mathrm{dec}}}{t_{\mathrm{op}}} = \frac{Q}{\nu_{01}\, t_{\mathrm{op}}} < Q\,\underbrace{\frac{\Delta\nu}{\nu_{01}}}_{\text{Anharmonicity}}$

Repetition -- Quantum description of JJ



Charging energy
``*kinetic*''

$$E_{\mathrm{kin}} = \frac{(2eN)^2}{2C} = 4E_C N^2$$

Josephson energy
``*potential*''

$$E_{\mathrm{pot}} = \frac{I_c \Phi_0}{2\pi} = E_J\,(1 - \cos\varphi)$$

Quantization

$$N \to \widehat{N} = -i\frac{\partial}{\partial\varphi} \qquad \varphi \to \hat{\varphi} \qquad (\Delta N)(\Delta\varphi) \geq 1$$

$$\hat{q} = 2e\widehat{N}$$

$$\widehat{\Phi} = \frac{\Phi_0}{2\pi}\hat{\varphi}$$

Hamiltonian

$$\widehat{H}_J = -4E_C\left(\frac{\partial}{\partial\varphi}\right)^2 + E_J\,(1 - \cos\hat{\varphi})$$

Characteristic energies – Typical parameters

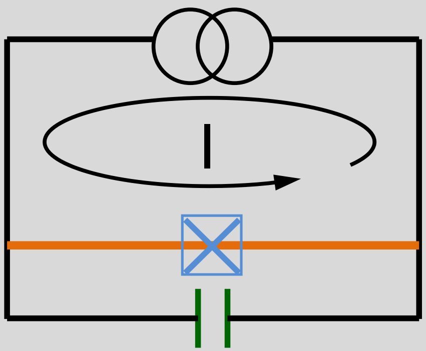
Al/AlO$_x$/Al junction with $A = 100 \times 100\ \mathrm{nm}^2$ → $C \approx 1\ \mathrm{fF}$ and $I_c \approx 300\ \mathrm{nA}$

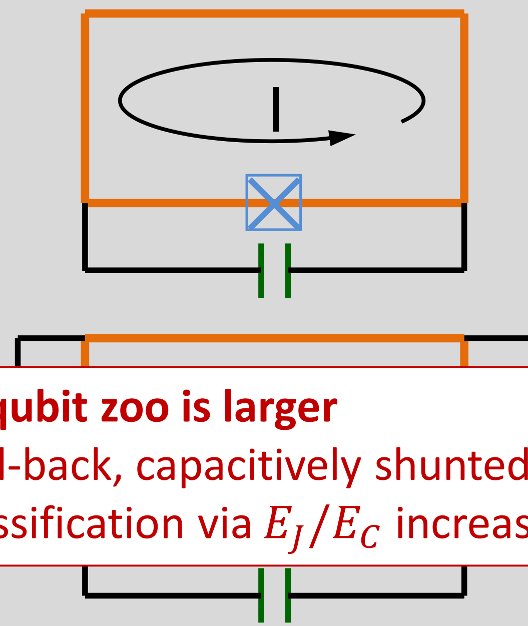
→ $E_C \approx 60\ \mu\mathrm{eV}$ and $E_J \approx 600\ \mu\mathrm{eV}$

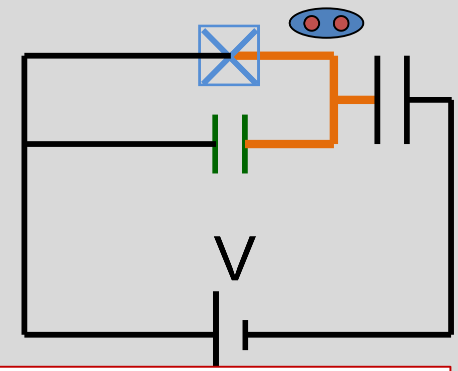
→ Quantum effects observable only at $T \ll 0.5\ \mathrm{K}$ $\quad (100\ \mu\mathrm{eV} \approx 1\ K)$

6.4 Physics of superconducting quantum circuits

phase qubit	**flux qubit**	**charge qubit**
($E_J >> E_C$)	($E_J > E_C$)	($E_J < E_C$)
current biased JJ	fluxon boxes	Cooper pair boxes



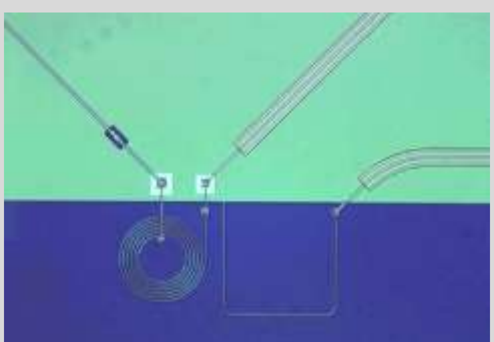


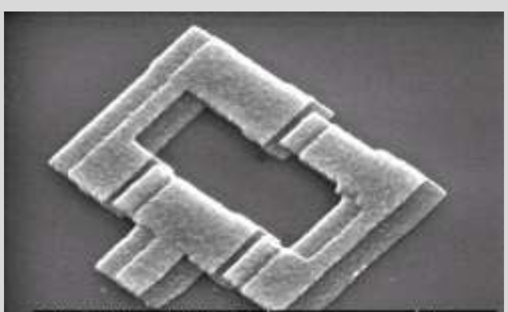


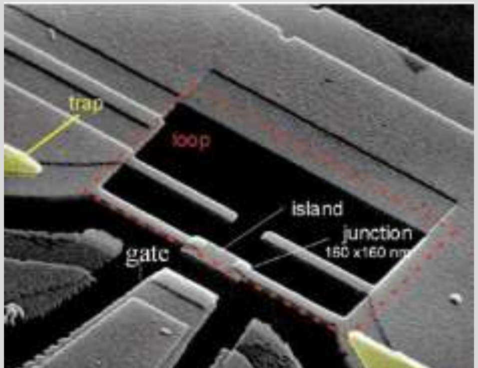
Nowadays superconducting qubit zoo is larger

→ Transmon, camel-back, capacitively shunted 3JJ-FQB, fluxonium…

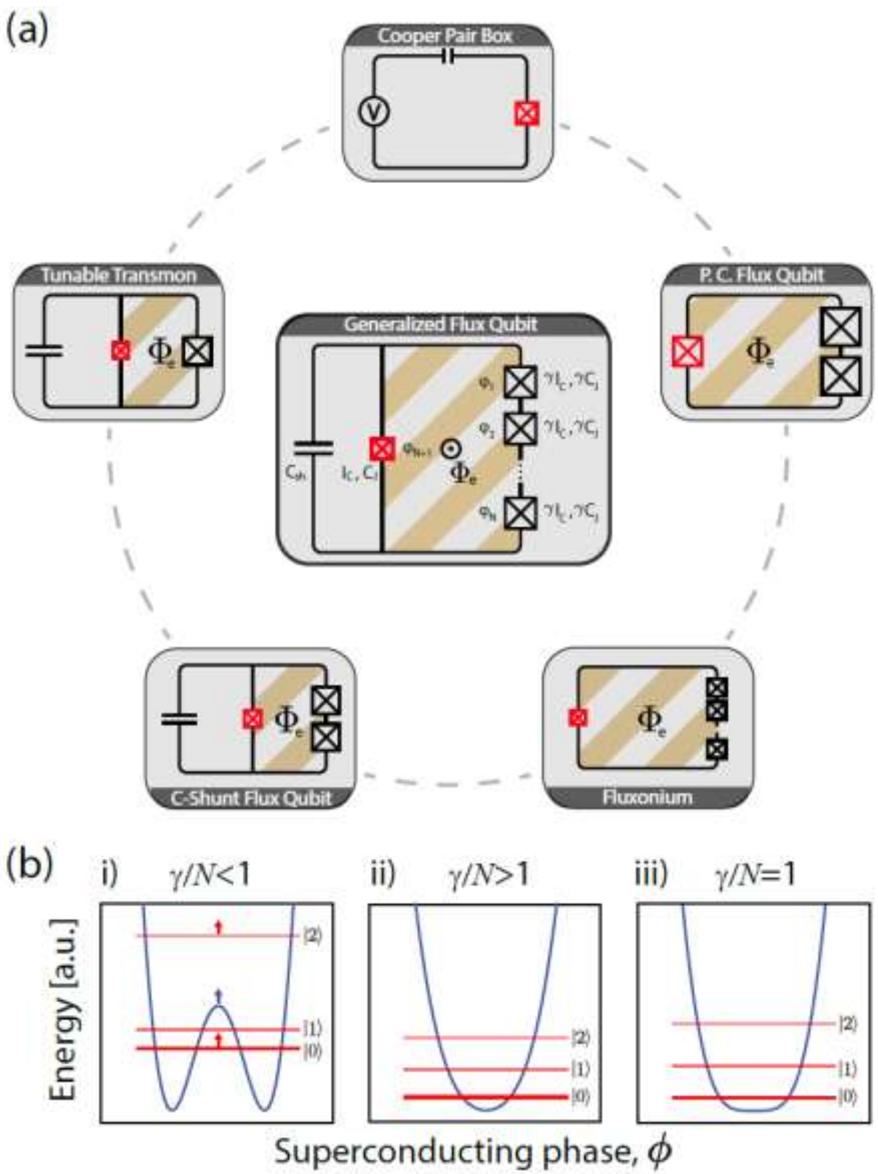
→ "Traditional" classification via E_J/E_C increasingly difficult

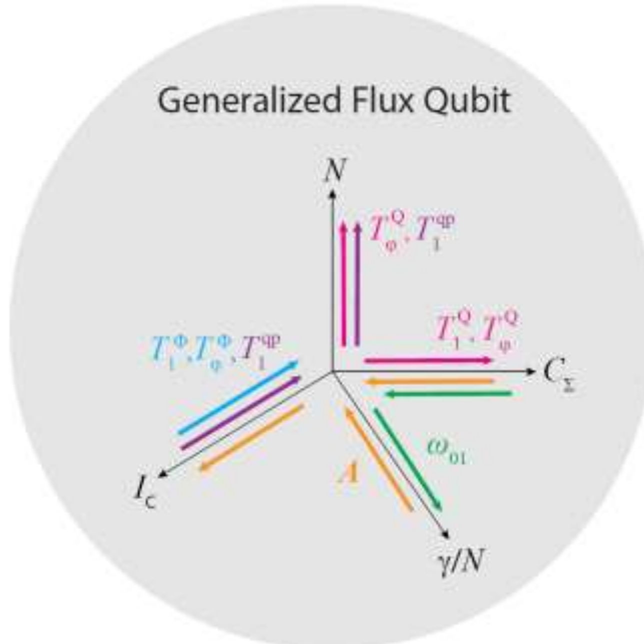

J. Martinis (NIST)


H. Mooij (Delft)



V. Bouchiat (Quantronics)



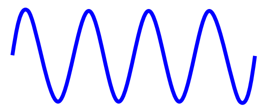


F Yan et al.,
Engineering Framework for Optimizing Superconducting Qubit Designs,
arXiv:2006.04130v1(2020).

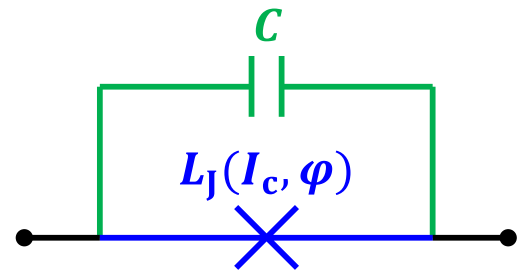
6.4 Physics of superconducting quantum circuits

Engineer potential

$$\hat{H}_J = -4E_C\left(\frac{\partial}{\partial\varphi}\right)^2 + E_J\,(1-\cos\hat{\varphi})$$

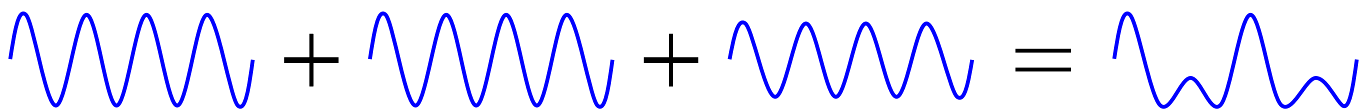


Unsuitable for TLS!

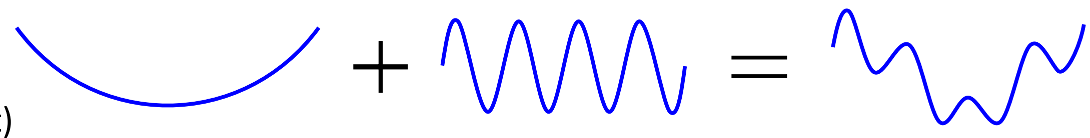


Flux/phase engineering

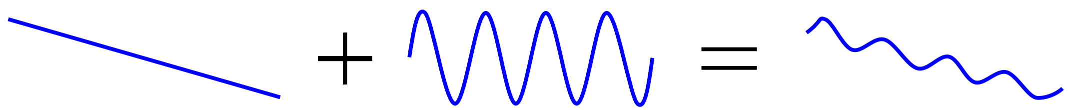
→ Add junctions
(3JJ flux qubit)



→ Add inductance
(rf SQUID &phase qubit)

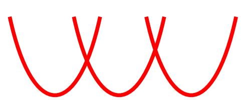


→ Add bias current
(phase qubit)



Charge engineering

→ Add gate capacitor
(charge qubit)

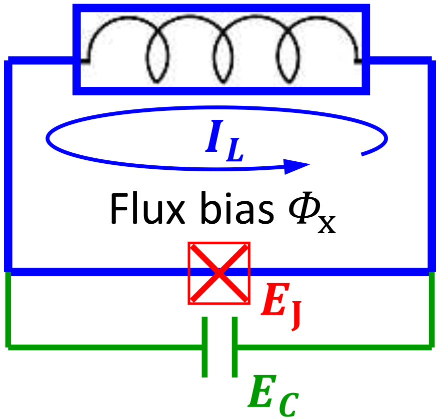


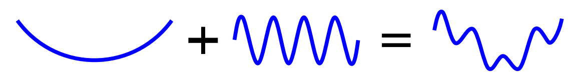
E_J naturally induces anticrossings

→ Add shunt capacitor
(transmon qubit)

→ Slightly delocalized charge states with residual anharmonicity

6.4 Physics of superconducting quantum circuits



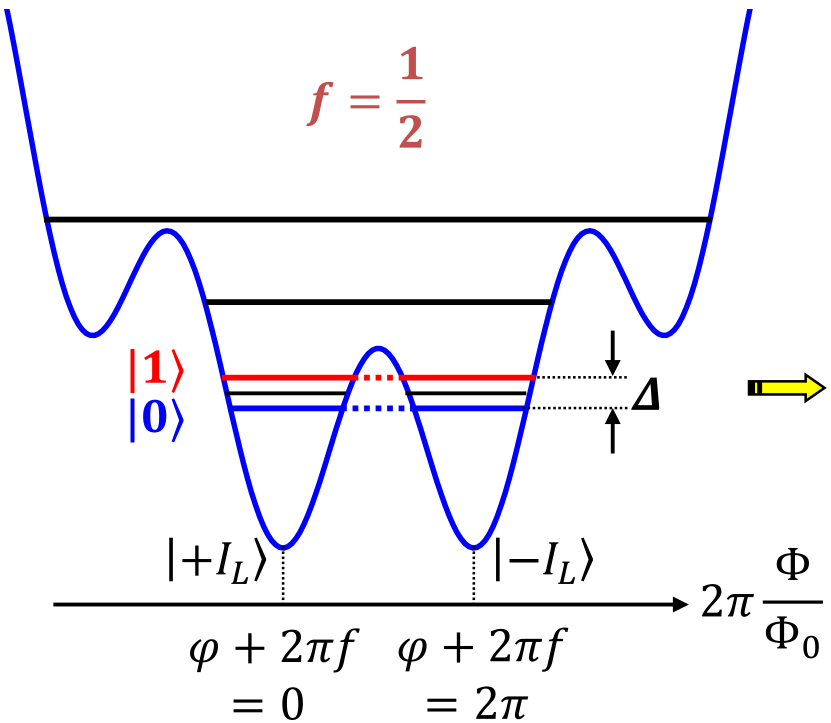


$$\hat{N} = -i\frac{\partial}{\partial\varphi}$$

$$\hat{H} = 4E_C\hat{N}^2 + E_J(1-\cos\varphi) + E_L\frac{(\varphi - 2\pi f)^2}{2}$$

Inductive energy $E_L \equiv \frac{\Phi_0^2}{2L}$ Frustration $f \equiv \frac{\Phi_x}{\Phi_0}$

Circulating current due to external flux → $I_L = \frac{\Phi_x}{L} = \frac{f\Phi_0}{L}$



$E_J \gg E_C$ (phase/flux regime)

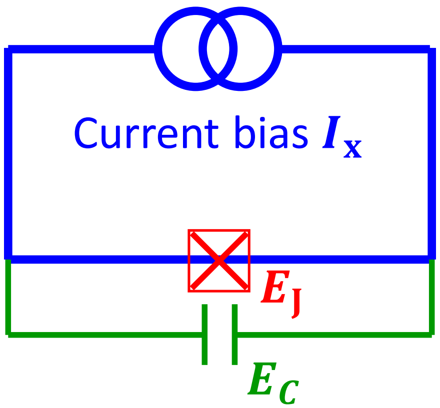
$\hbar\omega_p \gg k_B T$ $(\omega_p = \sqrt{8E_J E_C})$

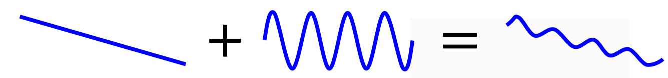
$I_C L \approx \Phi_0$ → Requires large L
(otherwise flux quantization kills our quantum variable)

MQT causes level splitting Δ
→ $|0\rangle, |1\rangle$ are symmetric and antisymmetric superpositions of $|+I_L\rangle, |-I_L\rangle$

Theoretical prediction:	Leggett (1984)
Experimental realization:	Friedman et al. (2000)

6.4 Physics of superconducting quantum circuits

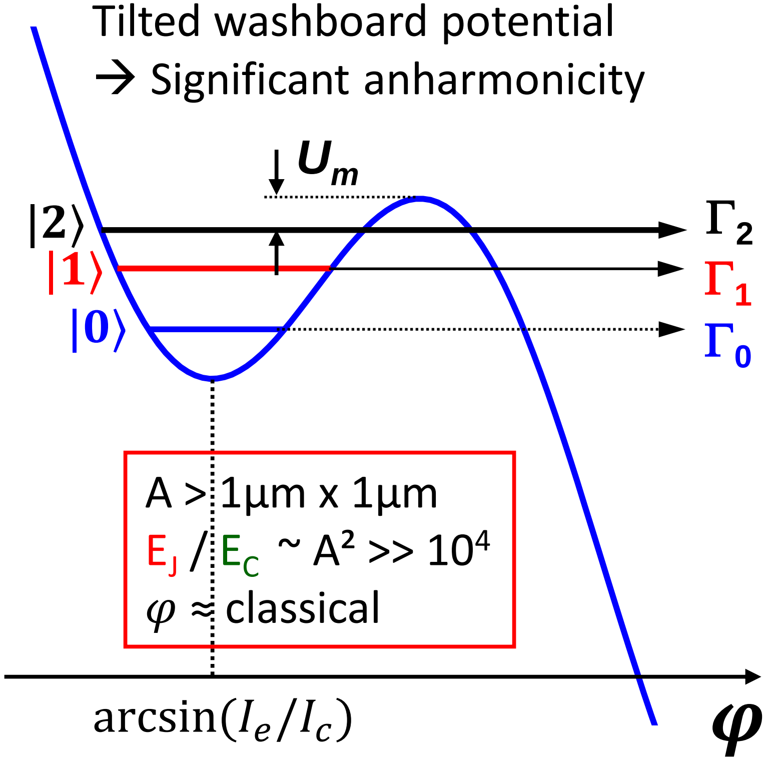




$$\hat{H} = 4E_C\hat{N}^2 + E_J(1-\cos\varphi) + \frac{\hbar}{2e}I_x\varphi$$

Additional „force term“ due to current source

Tilted washboard potential
→ Significant anharmonicity

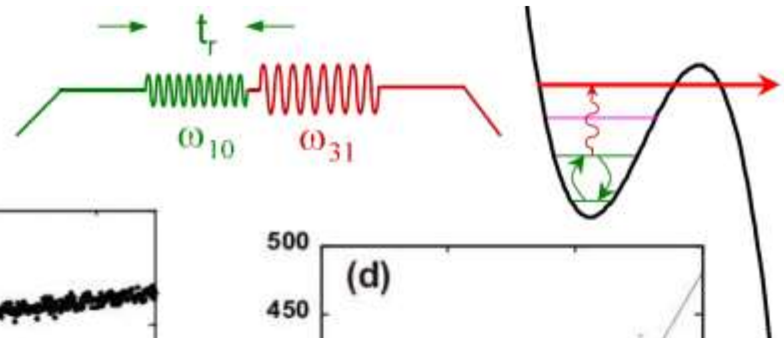


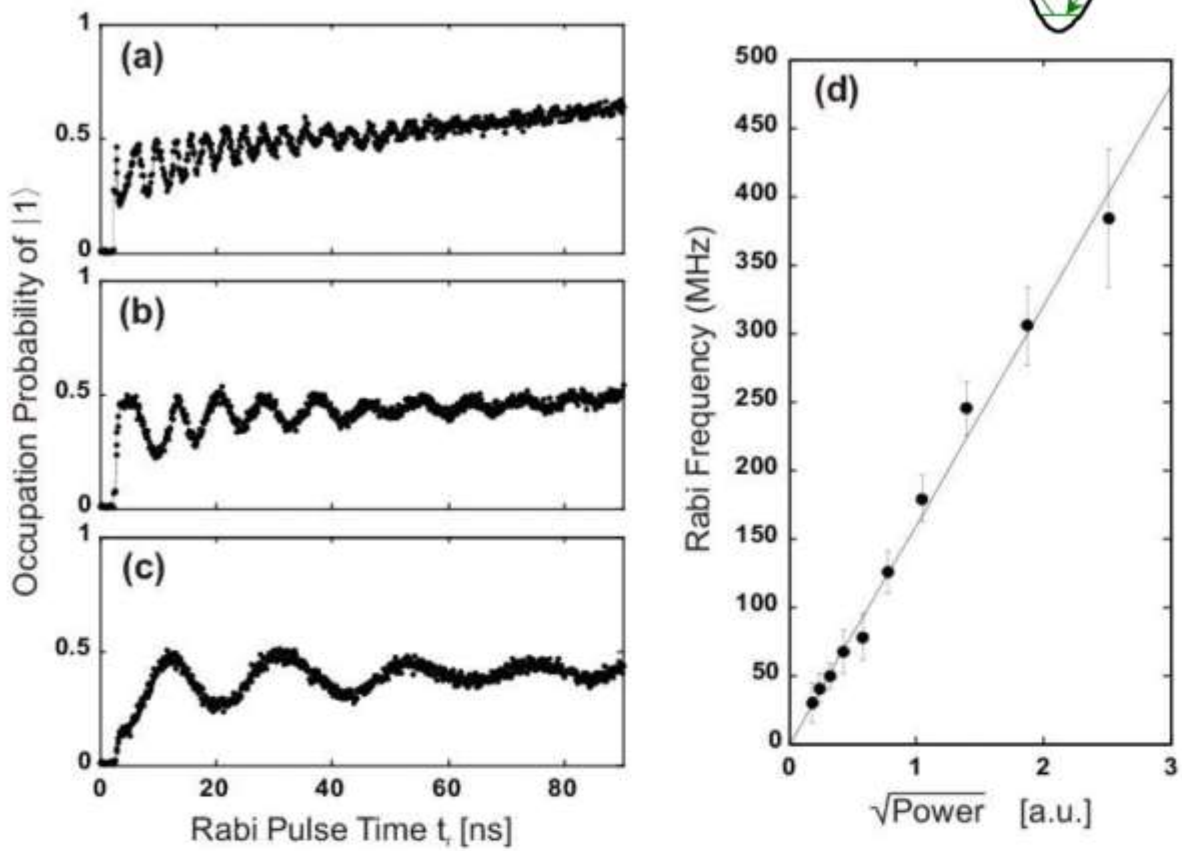
$$\Rightarrow \Gamma_{\mathrm{MQT}} \propto \omega_p\sqrt{\frac{U_M}{\hbar\omega_p}}\exp\left(-\frac{7.2U_m}{\hbar\omega_p}\right)$$

→ levels $|0\rangle$, $|1\rangle$ form the qubit
→ oscillator states differ in phase
→ ***phase qubit***
→ $\Gamma_2 \gg \Gamma_1, \Gamma_0$ → pump ω_{12} for readout
→ readout detects running phase (voltage)

First Rabi oscillations in phase qubit

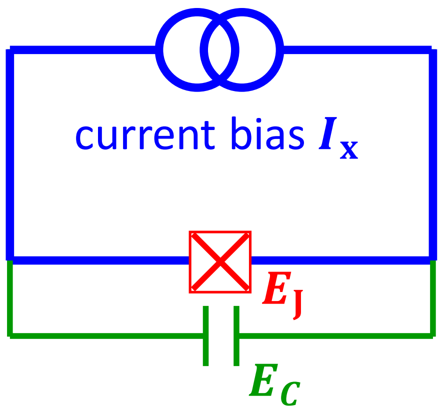
Martinis et al (2002)

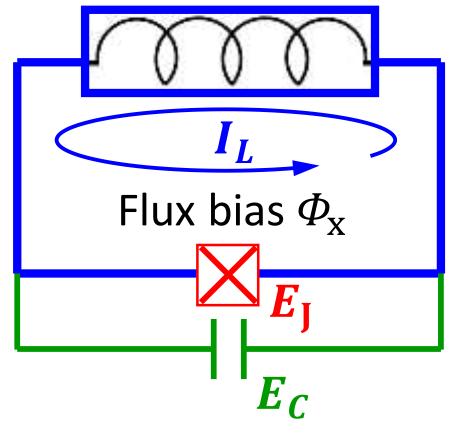




NIST National Institute of Standards and Technology • Technology Administration • U.S. Department of Commerce

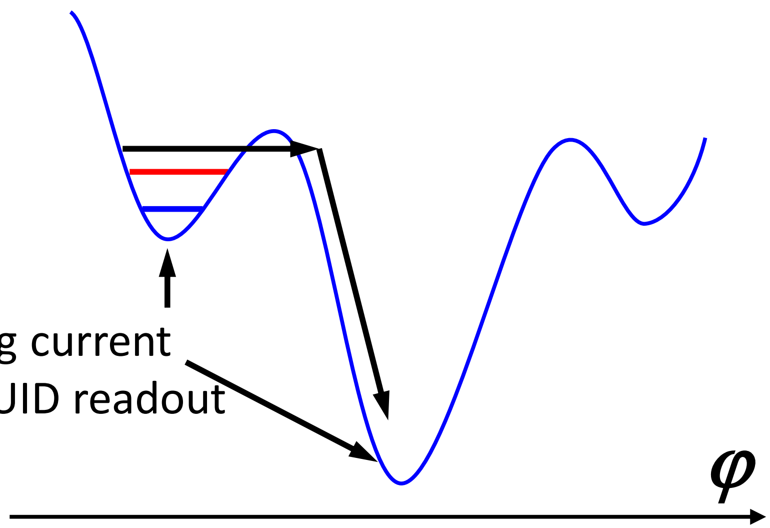
Construction principle





parameters similar to RF SQUID qubit
→ $E_J >> E_C$
→ $\hbar\omega_p > k_BT$
→ $I_cL \approx \Phi_0$

opposite circulating current
→ dc SQUID readout



→ Better decoupling from readout electronics → significantly longer decoherence times
→ Preferred over current-biased version

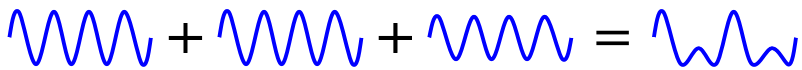
3-Josephson-junction persistent current flux qubit

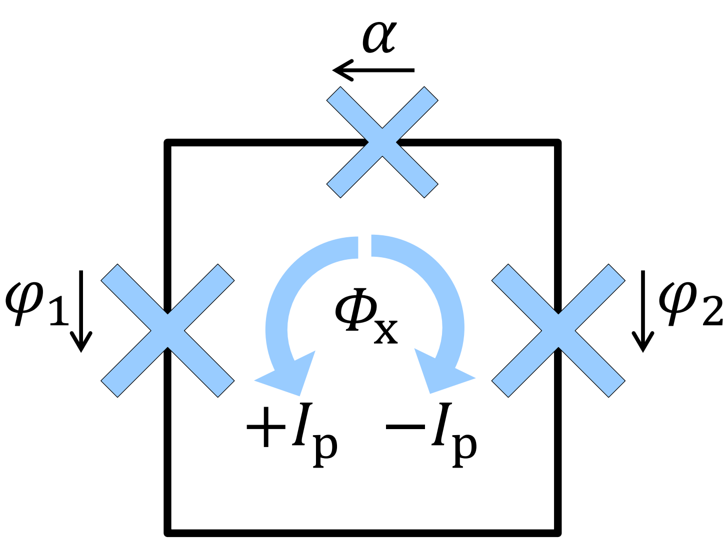
single JJ in a loop (RF-SQUID flux & flux biased phase qubit)

→ Single quantum degree of freedom

→ Flux quantization fixes JJ phase for $\beta_L \ll 1$ → Requires $\beta_L \simeq 1$

→ Alternative: More than one JJ in the loop





Persistent current flux qubit

→ 3 JJ in superconducting loop 1 ($L_\mathrm{loop} \ll L_\mathrm{J} \rightarrow \beta_L \ll 1$)

→ Two junctions have identical size

→ Third junction smaller by factor α

→ $E_\mathrm{J} \equiv E_{\mathrm{J}1} = E_{\mathrm{J}2}$ and $E_C \equiv E_{C1} = E_{C2}$

→ $E_{\mathrm{J}3} = \alpha E_\mathrm{J}$ and $E_{C3} = \frac{E_C}{\alpha}$

→ $E_\mathrm{J} > E_C$ (phase regime) & $\hbar\omega_\mathrm{p} \gg k_\mathrm{B}T$

Control knob

→ External flux Φ_x applied to loop

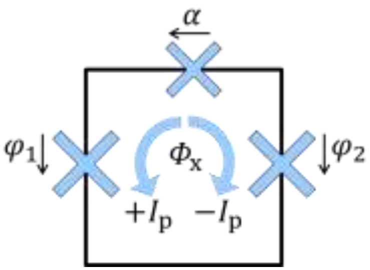
→ Still 2 quantum degrees of freedom left after flux quantization

J.E. Mooij et al., Science 285, 1036 (1999)

T. P. Orlando et al., PRB 60, 15399-15413 (1999)

Double-well potential

$$U_J = E_J(1 - \cos\varphi)$$



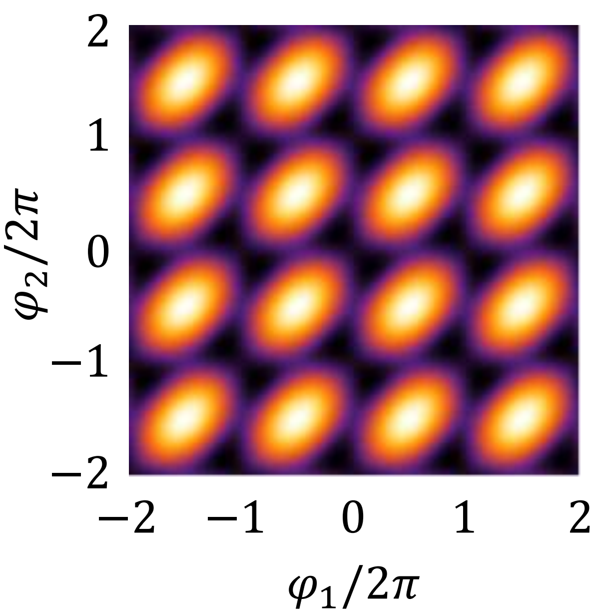
$$U(\varphi_1, \varphi_2, \varphi_3) = E_J\,[2 - \cos\varphi_1 - \cos\varphi_2 + \alpha(1 - \cos\varphi_3)]$$

Flux quantization

→ $\varphi_1 - \varphi_2 + \varphi_3 = -2\pi f$ with frustration $f \equiv \frac{\Phi_x}{\Phi_0}$

→ Signs are mere convention!

$$U(\varphi_1, \varphi_2) = E_J\,[2 + \alpha - \cos\varphi_1 - \cos\varphi_2 - \alpha\cos(2\pi f + \varphi_1 - \varphi_2)]$$

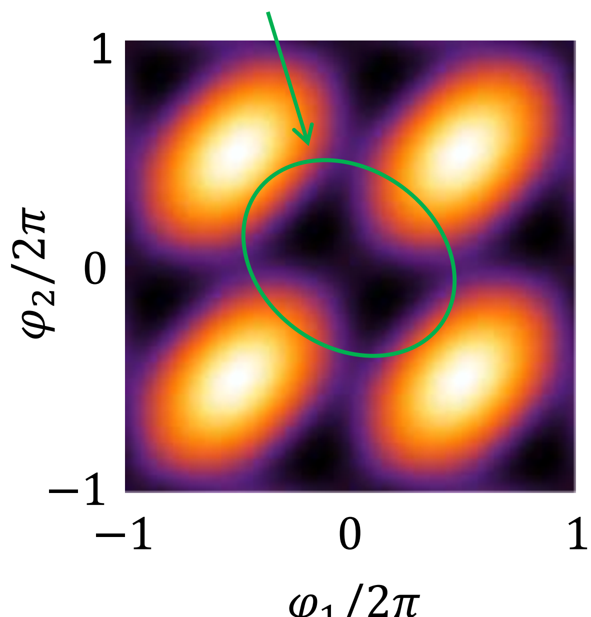


Color code: $\frac{U(\varphi_1,\varphi_2)}{E_J}$

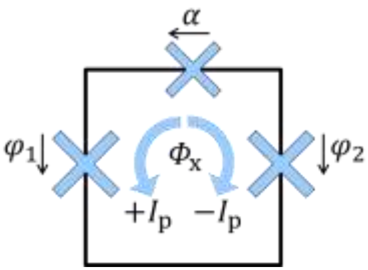
$\Phi_x = \frac{\Phi_0}{2}$

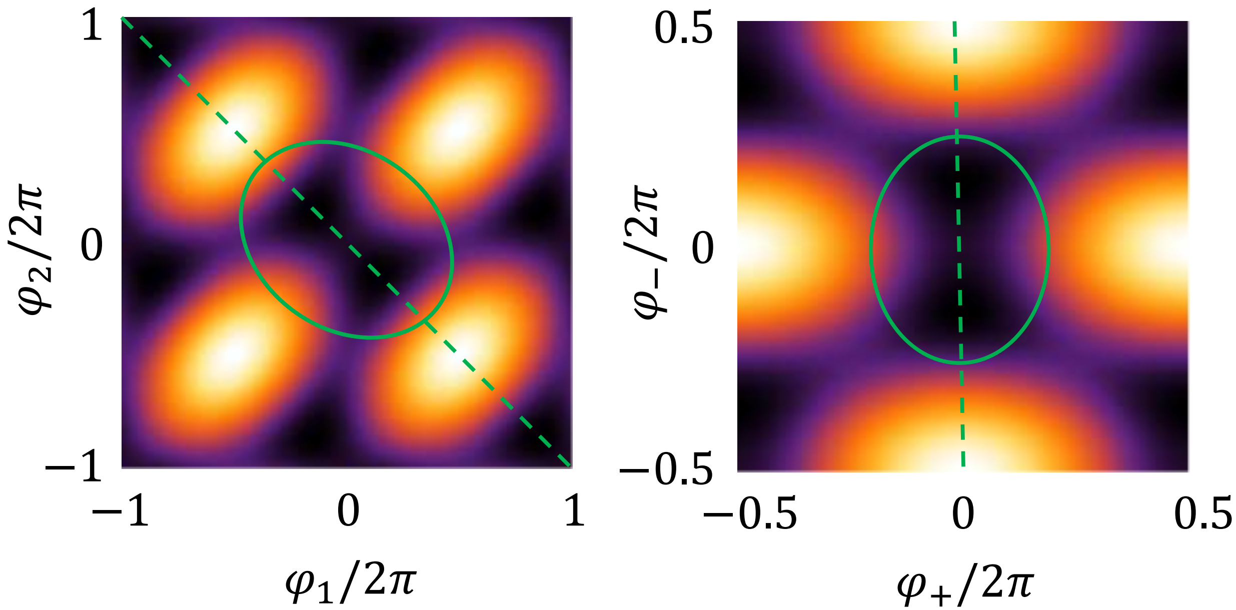
$\alpha = 0.8$

Double-well potential in each unit cell!



Double-well potential





Two stable minima at $(\varphi^*, -\varphi^*)$ and $(-\varphi^*, \varphi^*)$, where $\cos\varphi^* \equiv \frac{1}{2\alpha}$

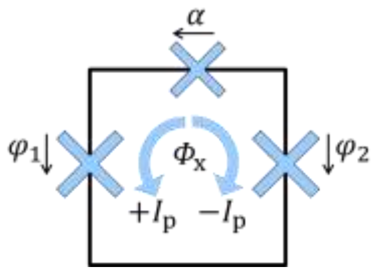
Double well rotated by 45° in the $\varphi_1\varphi_2$-plane → Variable transformation

$$\left.\begin{aligned}\varphi_+ &\equiv \frac{1}{2}(\varphi_1 + \varphi_2)\\ \varphi_- &\equiv \frac{1}{2}(\varphi_1 - \varphi_2)\end{aligned}\right\} \quad U(\varphi_+, \varphi_-) = E_\mathrm{J}\,[2 + \alpha - 2\cos\varphi_+ \cos\varphi_- - \alpha\cos(2\pi f + 2\varphi_-)$$

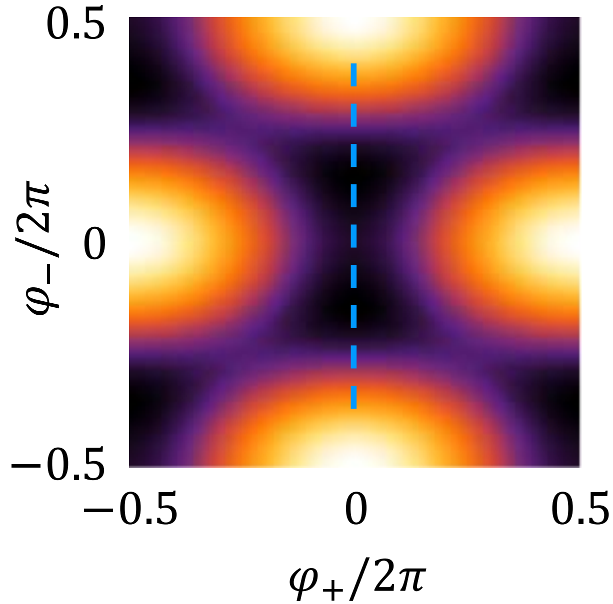
Variable relevant for qubit dynamics → φ_-

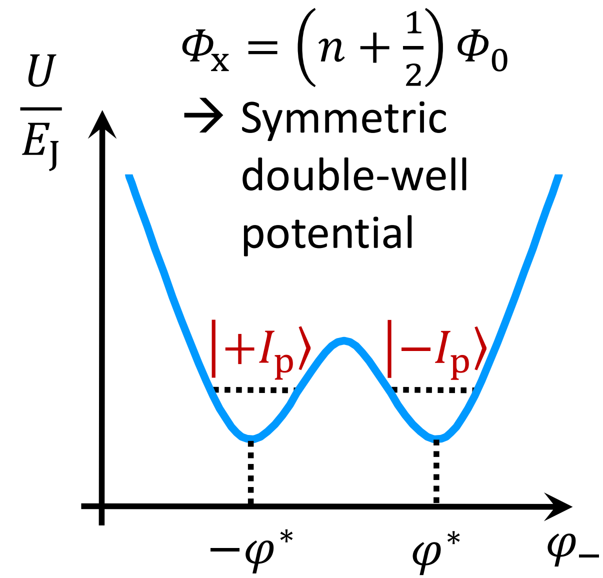
Double-well potential

$$U(\varphi_+,\varphi_-) = E_\mathrm{J}\,[2+\alpha-2\cos\varphi_+\cos\varphi_- - \alpha\cos(2\pi f+2\varphi_-)$$



Thermodynamics: $I = -\frac{\partial U}{\partial \Phi}$





→ No tunneling → Degenerate ground state

→ Left/right well correspond to clockwise/anticlockwise circulating current

→ Persistent current $\pm I_\mathrm{p}$ $= -\left.\frac{\partial U(\varphi_- = -\varphi^*)}{\partial \Phi_\mathrm{x}}\right|_{\Phi_\mathrm{x}=\Phi_0/2} = -E_\mathrm{J}\alpha[-\sin(2\pi f - 2\varphi^*)]\left.\frac{2\pi}{\Phi_0}\right|_{\Phi_\mathrm{x}=\Phi_0/2}$

$$= I_c\alpha\sin(\pi-2\varphi^*) = -2I_c\alpha\sin 2\varphi^*\cos\pi = 2I_c\alpha\sin\varphi^*\cos\varphi^* = I_c\sqrt{1-\left(\frac{1}{2\alpha}\right)^2}$$

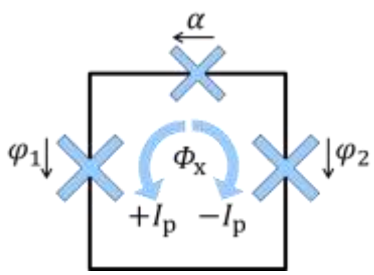
$\sin(x+y) = \sin x\cos y + \cos x \sin y$

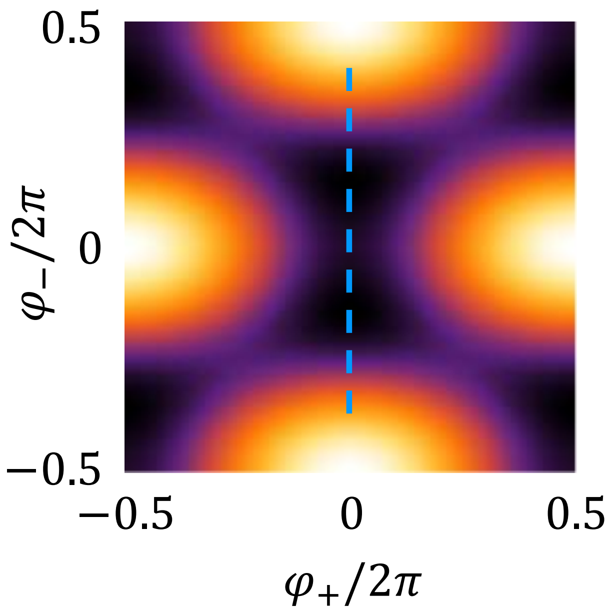
$\sin 2x = 2\sin x \cos y$

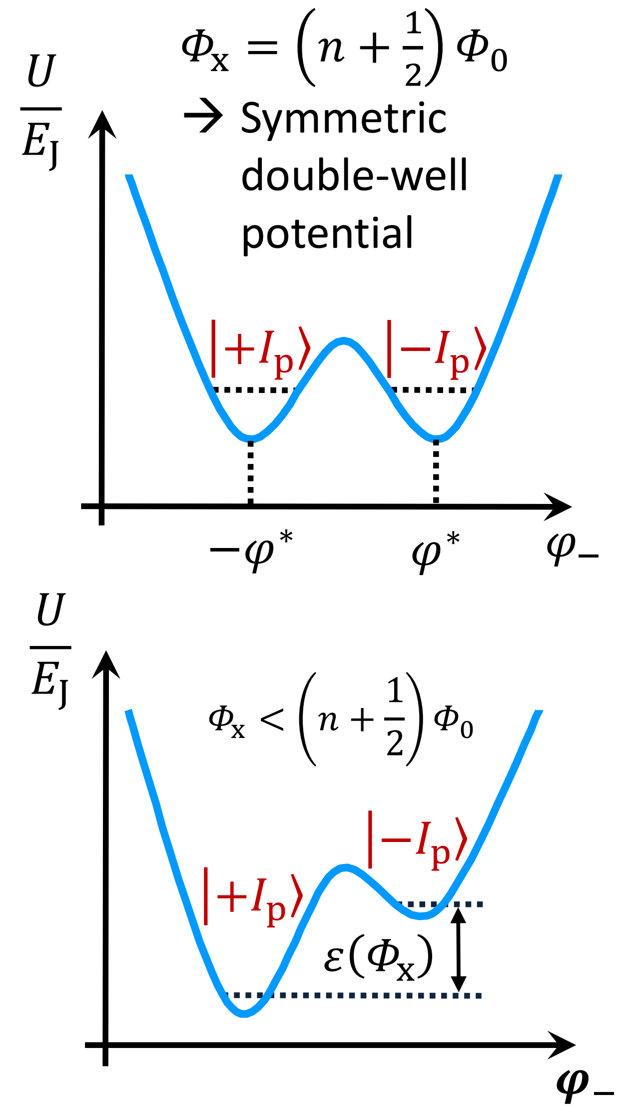
$\cos\varphi^* \equiv \frac{1}{2\alpha}$

Double-well potential

$$U(\varphi_+, \varphi_-) = E_J\,[2 + \alpha - 2\cos\varphi_+ \cos\varphi_- - \alpha\cos(2\pi f + 2\varphi_-)]$$







→ No tunneling → Degenerate ground state
→ Left/right well correspond to clockwise/anticlockwise circulating current
→ Persistent current $\pm I_p$

$\Phi_x \neq \left(n + \frac{1}{2}\right)\Phi_0$ → Tilted double-well potential
→ Flux bias induces energy bias $\varepsilon(\Phi_x)$
→ Near $\Phi_x = \left(n + \frac{1}{2}\right)\Phi_0$ → $\varepsilon(\Phi_x) = 2 I_p \Phi_0 \left(f - \mathrm{n} - \frac{1}{2}\right)$

Quantum treatment

$\hat{N}_{1,2} \equiv -i \frac{\partial}{\partial \varphi_{1,2}}$ $E_C \equiv \frac{e^2}{2C}$ $\varphi_+ \equiv \frac{1}{2}(\varphi_1 + \varphi_2)$ $\varphi_- \equiv \frac{1}{2}(\varphi_1 - \varphi_2)$

$$\hat{H} = 4E_C\left[\hat{N}_1^2 + \hat{N}_2^2\right] + E_J\left[2 + \alpha - 2\cos\hat{\varphi}_+ \cos\hat{\varphi}_- - \alpha\cos(2\pi f + 2\hat{\varphi}_-)\right.$$

Task → convert $\hat{N}_1$ and $\hat{N}_2$ into $\hat{N}_+$ and $\hat{N}_-$

$$\frac{\hat{N}_1^2 + \hat{N}_2^2}{-i} = \left(\frac{\partial}{\partial\varphi_1}\right)^2 + \left(\frac{\partial}{\partial\varphi_2}\right)^2 = \left(\frac{\partial}{\partial\varphi_+}\frac{d\varphi_+}{d\varphi_1} + \frac{\partial}{\partial\varphi_-}\frac{d\varphi_-}{d\varphi_1}\right)^2 + \left(\frac{\partial}{\partial\varphi_+}\frac{d\varphi_+}{d\varphi_2} + \frac{\partial}{\partial\varphi_-}\frac{d\varphi_-}{d\varphi_2}\right)^2$$

$$= \frac{1}{4}\left(\frac{\partial}{\partial\varphi_+} + \frac{\partial}{\partial\varphi_-}\right)^2 + \frac{1}{4}\left(\frac{\partial}{\partial\varphi_+} - \frac{\partial}{\partial\varphi_-}\right)^2 = \frac{1}{2}\left[\left(\frac{\partial}{\partial\varphi_+}\right)^2 + \left(\frac{\partial}{\partial\varphi_-}\right)^2\right] = \frac{\frac{1}{2}\left[\hat{N}_+^2 + \hat{N}_-^2\right]}{-i}$$

Flux qubit Hamiltonian

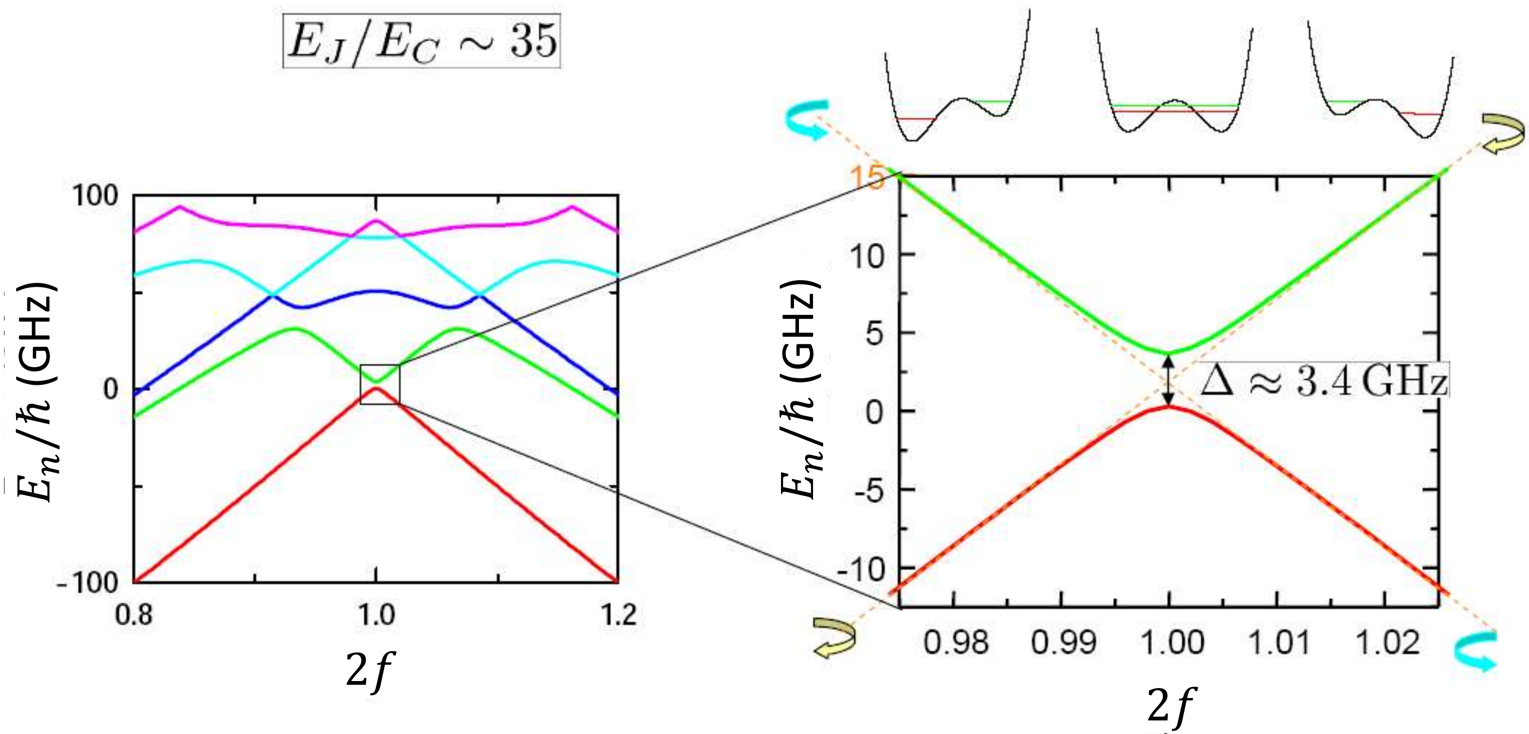
$$\hat{H} = 2E_C\left[\hat{N}_+^2 + \hat{N}_-^2\right] + E_J\left[2 + \alpha - 2\cos\hat{\varphi}_+ \cos\hat{\varphi}_- - \alpha\cos(2\pi f + 2\hat{\varphi}_-)\right.$$

Quantum treatment

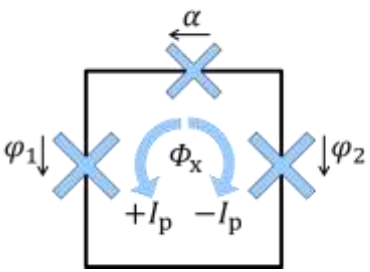
$f = \Phi_x/\Phi_0$

$$\hat{H} = 2E_C\left[\hat{N}_+ + \hat{N}_-\right] + E_J\left[2 + \alpha - 2\cos\hat{\varphi}_+\cos\hat{\varphi}_- - \alpha\cos(2\pi f + 2\hat{\varphi}_-)\right.$$

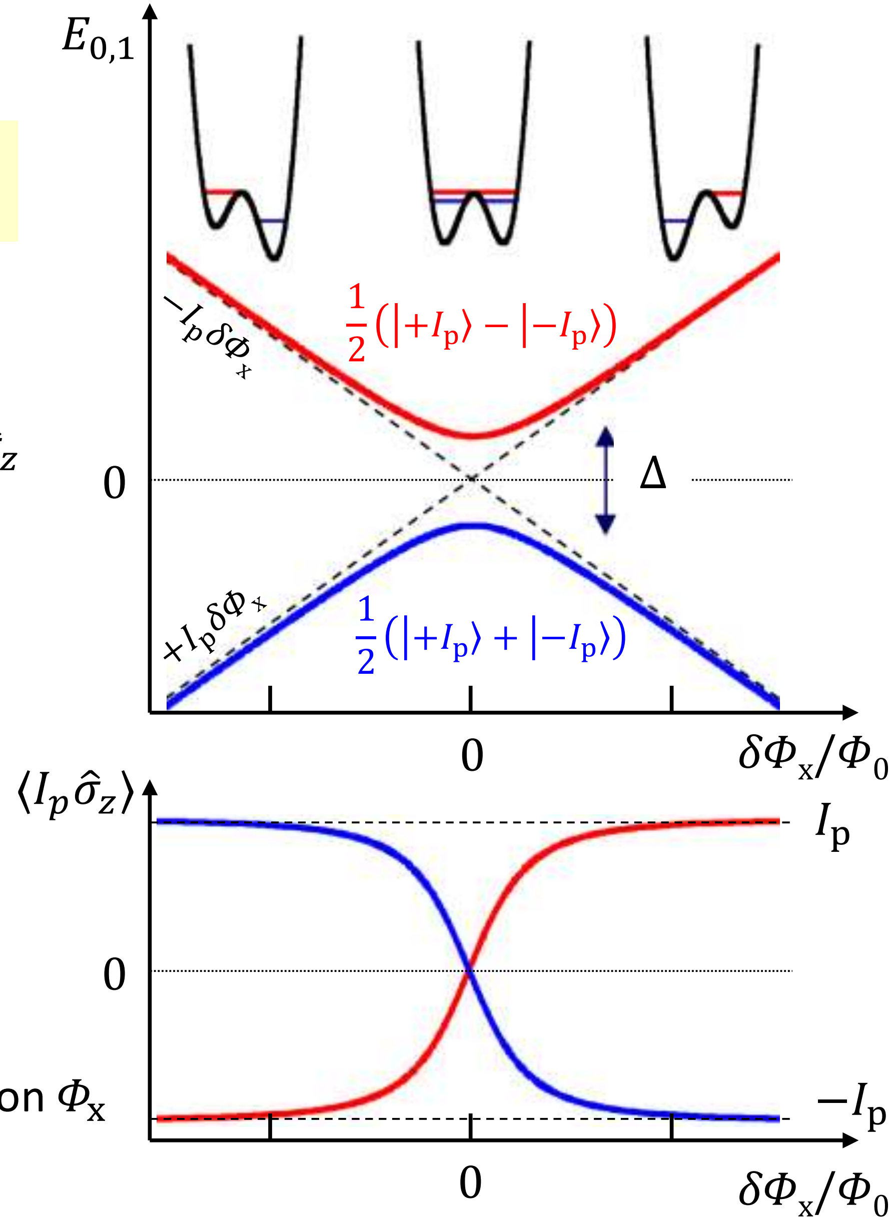
Numerical diagonalization → Eigenenergies E_n



Near $\Phi_x = \left(n + \frac{1}{2}\right)\Phi_0$ → Approximate as two-level system with linear energy bias!



$$\delta\Phi_x \equiv \Phi_0\left(f - n - \frac{1}{2}\right)$$



Energy bias $\varepsilon(\Phi_x) = 2I_p\delta\Phi_x$

→ $|+I_p\rangle$ and $|-I_p\rangle$ are eigenstates of $\varepsilon(\Phi_x)\hat{\sigma}_z$

Tunneling rate Δ/h over potential barrier

→ Tunnel splitting $\Delta \propto \exp\left(-\sqrt{E_J/E_C}\right)$

$$\hat{H} = \varepsilon(\Phi_x)\hat{\sigma}_z + \Delta\hat{\sigma}_x$$

$$E_1 - E_0 \equiv \hbar\omega_q(\Phi_x) = \sqrt{\varepsilon^2(\Phi_x) + \Delta^2}$$

Bloch angle θ denotes operation point Φ_x

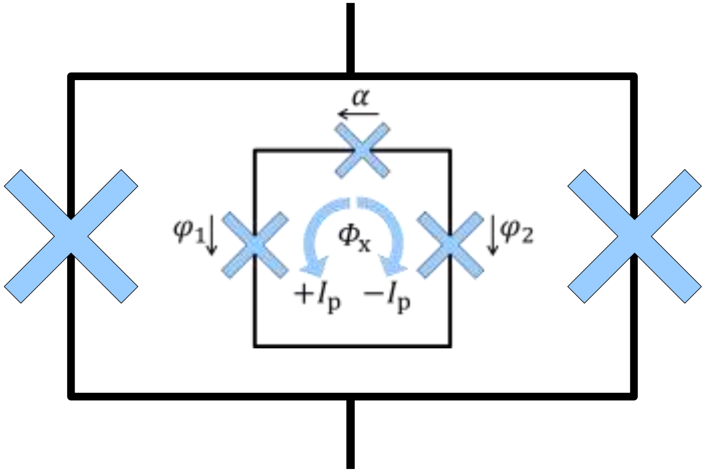
→ $\sin\theta \equiv \frac{\Delta}{\hbar\omega_q(\Phi_x)}$ and $\cos\theta \equiv \frac{\varepsilon(\Phi_x)}{\hbar\omega_q(\Phi_x)}$

Persistent circulating current $\langle I_p\hat{\sigma}_z\rangle$ depends on Φ_x

→ $\langle I_p\hat{\sigma}_z\rangle = I_p\cos\theta$

Readout requirements

$|L_\mathrm{J}| \geq \frac{\Phi_0}{2\pi I_\mathrm{c}}$



Inductive coupling to readout circuit

→ Goal: $\frac{E_\mathrm{J}}{E_C} \simeq 50$

→ Junction $I_\mathrm{c} \simeq 750$ nA, $C \simeq 3$ fF, $\alpha \simeq 0.7$

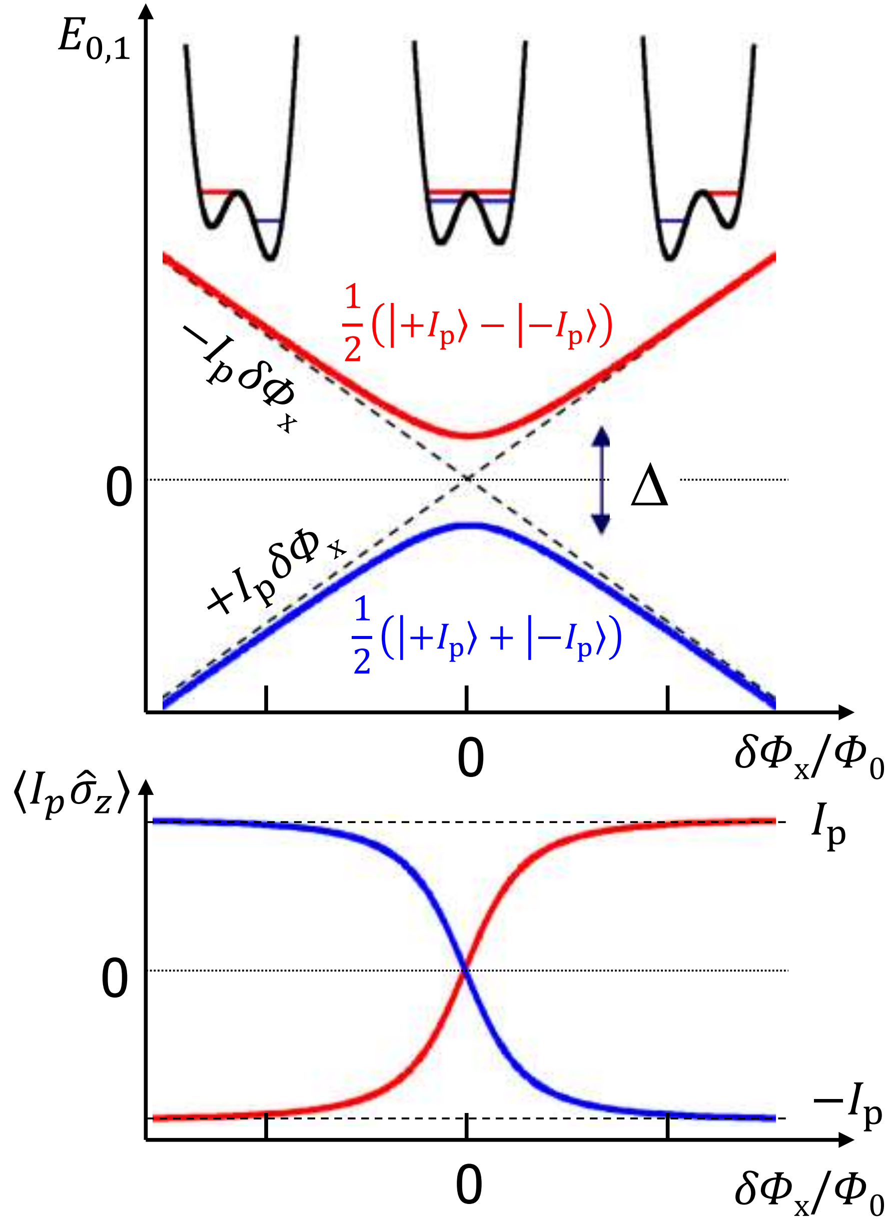
→ Loop: $d \simeq 10$ µm → $L \approx \mu_0 d \simeq 10$ pH

→ $L \ll |L_\mathrm{J}| \simeq 400$ pH

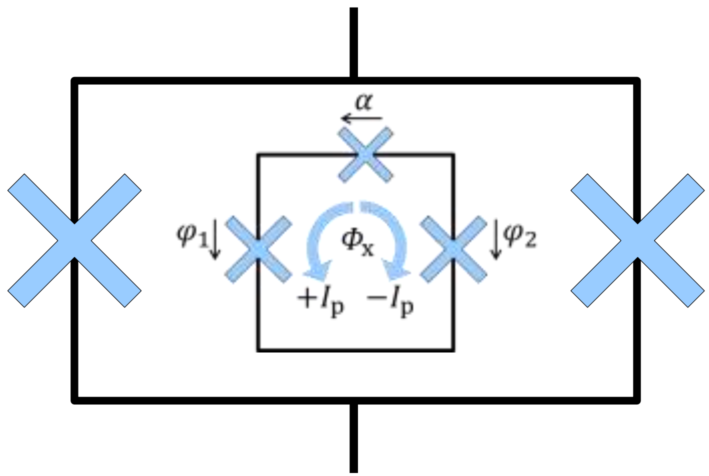
→ Self-induced flux negligible

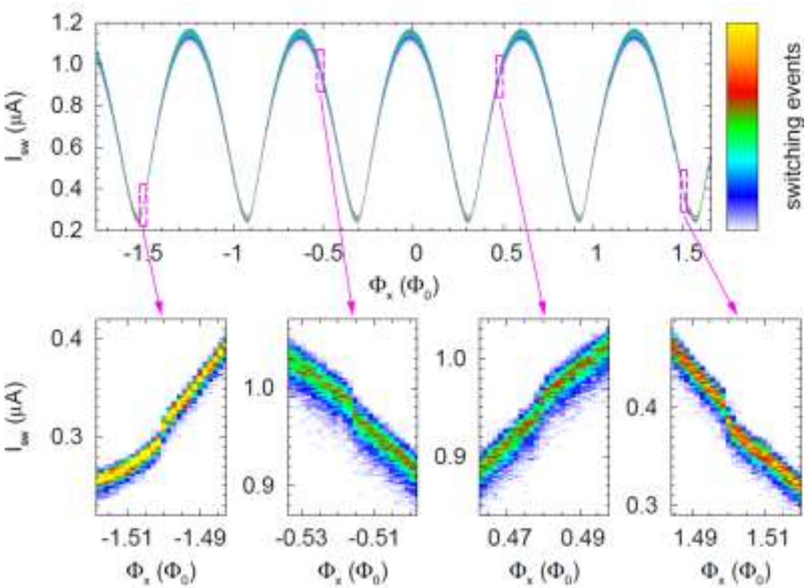
→ Flux signal $L I_\mathrm{p} \approx L\alpha I_\mathrm{c} \simeq 3$ mΦ_0

→ $|\pm I_\mathrm{p}\rangle$ can be distinguished by an on-chip dc SQUID

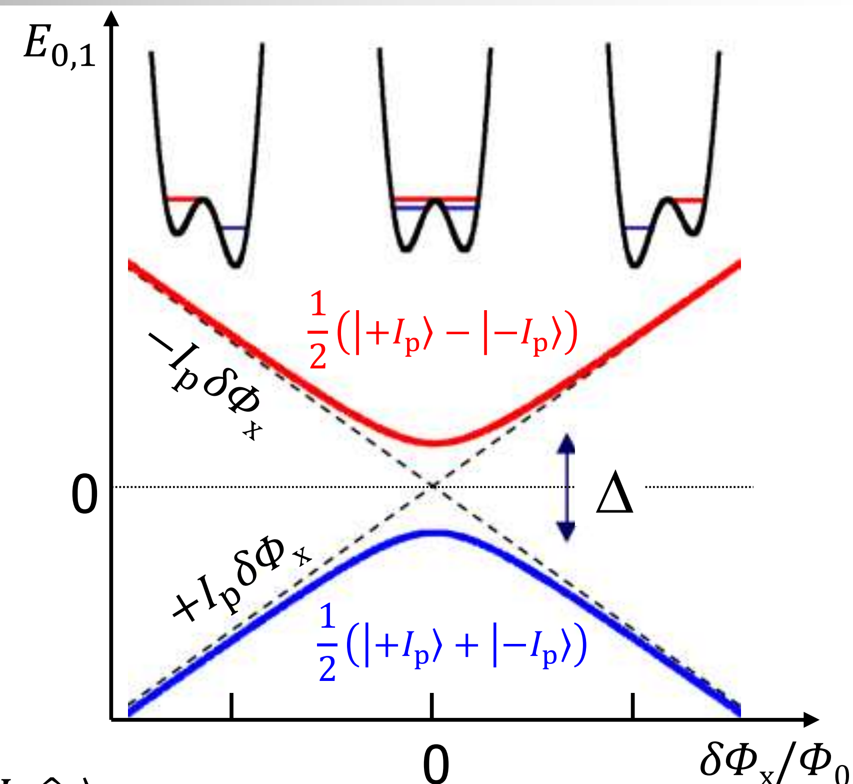


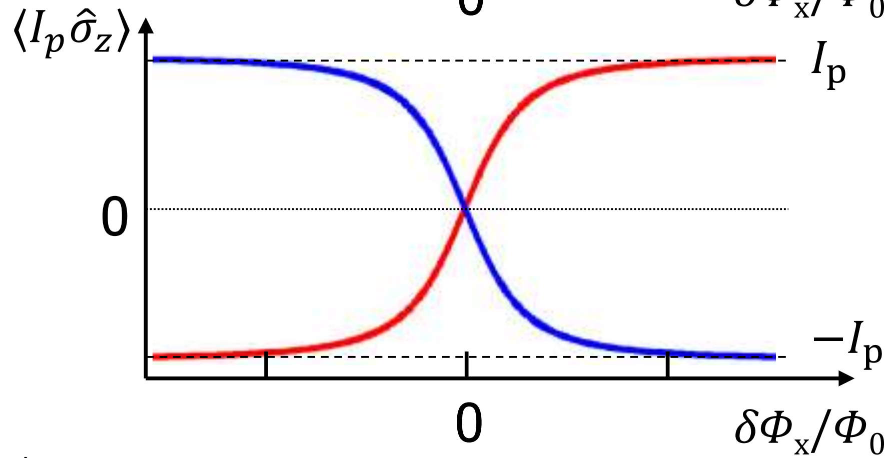
Continuous-wave readout with dc SQUID



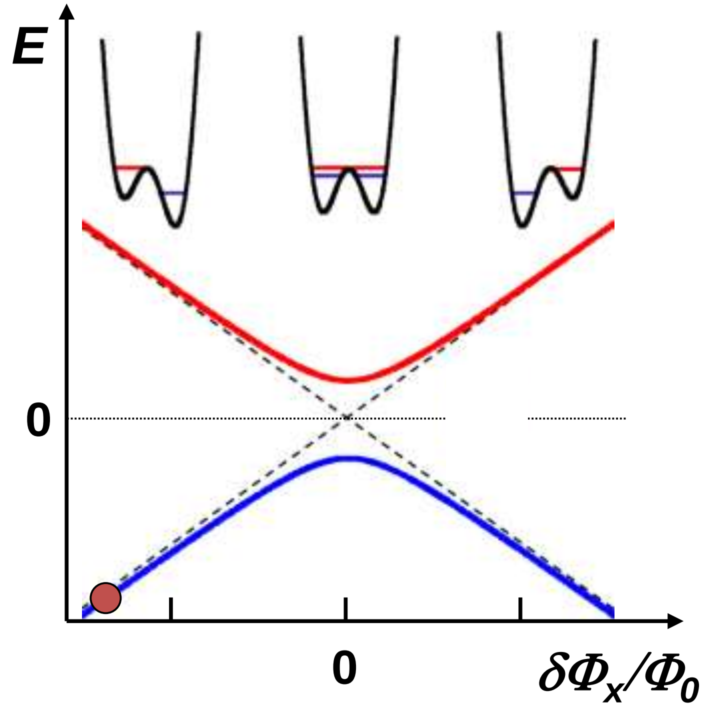


F. Deppe, Ph.D. thesis (Technische Universität München, 2009)



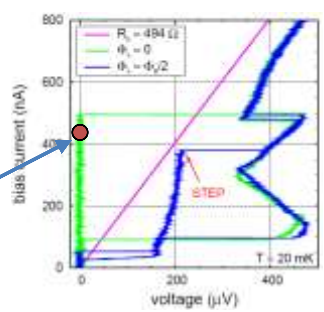


Pulsed dc SQUID readout away from the degeneracy point



time

Bias here



Principle

→ Away from $\Phi_\mathrm{x} = \Phi_0/2$, qubit eigenstates coincide with (anti-)clockwise circulating current states

→ Add/remove flux from SQUID loop

→ Increase/decrease $I_\mathrm{c,SQUID}(\Phi_\mathrm{x,SQUID})$

Protocol

→ Bias SQUID just below switching point (short switching pulse)

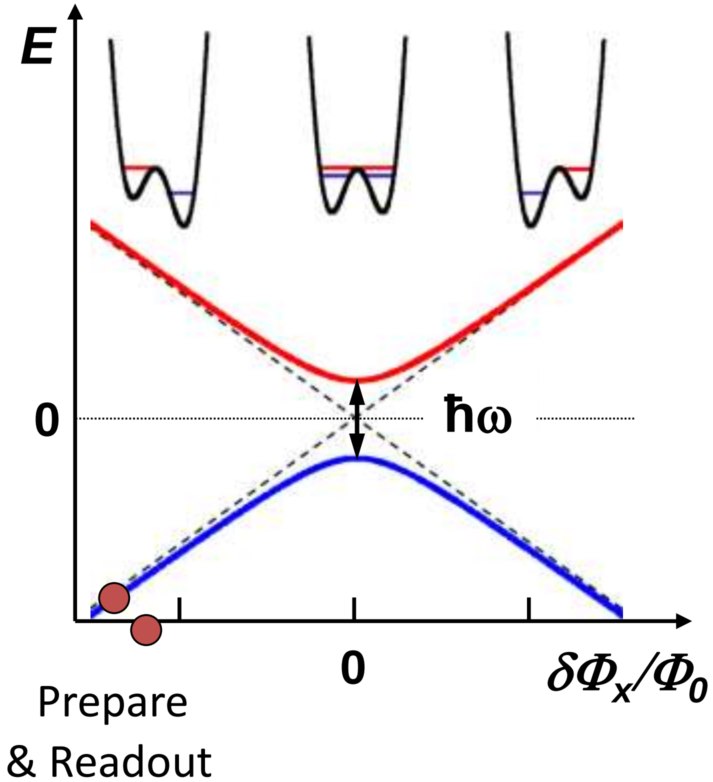
→ Depending on qubit state switching or not

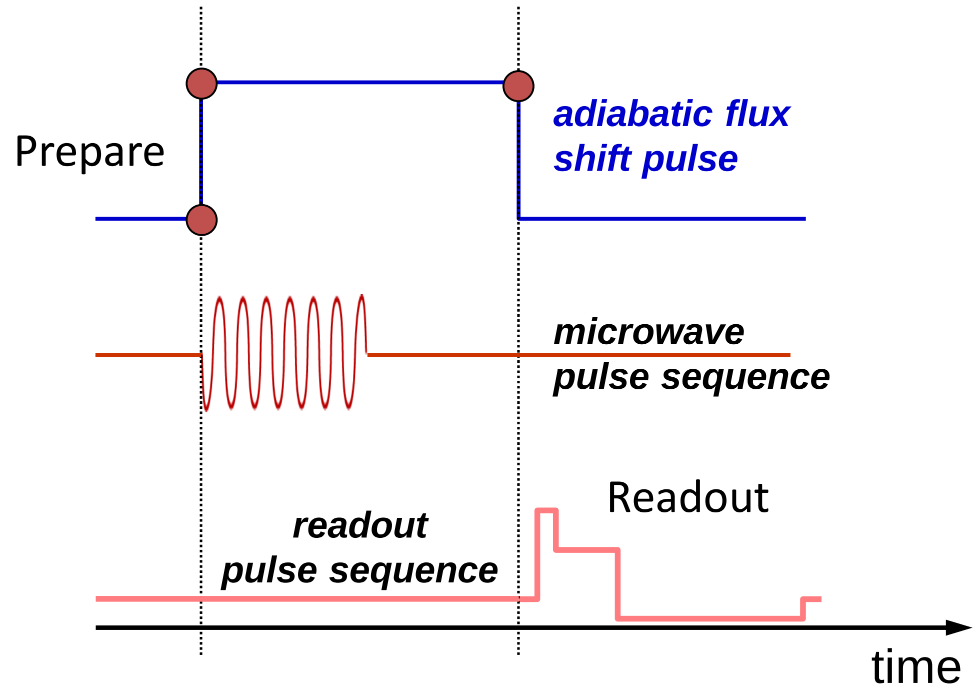
→ Measure voltage or not

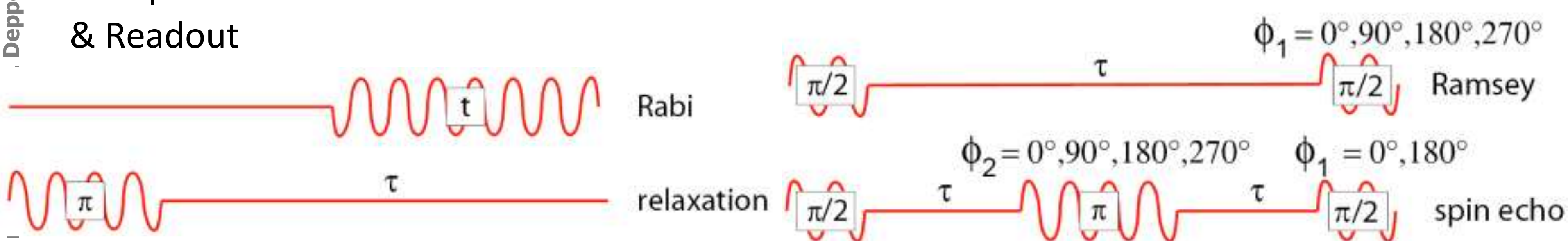
→ Hold pulse avoids retrapping → More signal

K. Kakuyanagi et al., Phys. Rev. Lett. 98, 047004 (2007)
F. Deppe et al., Phys. Rev. B 76, 214503 (2007)

Pulsed dc SQUID readout at the degeneracy point







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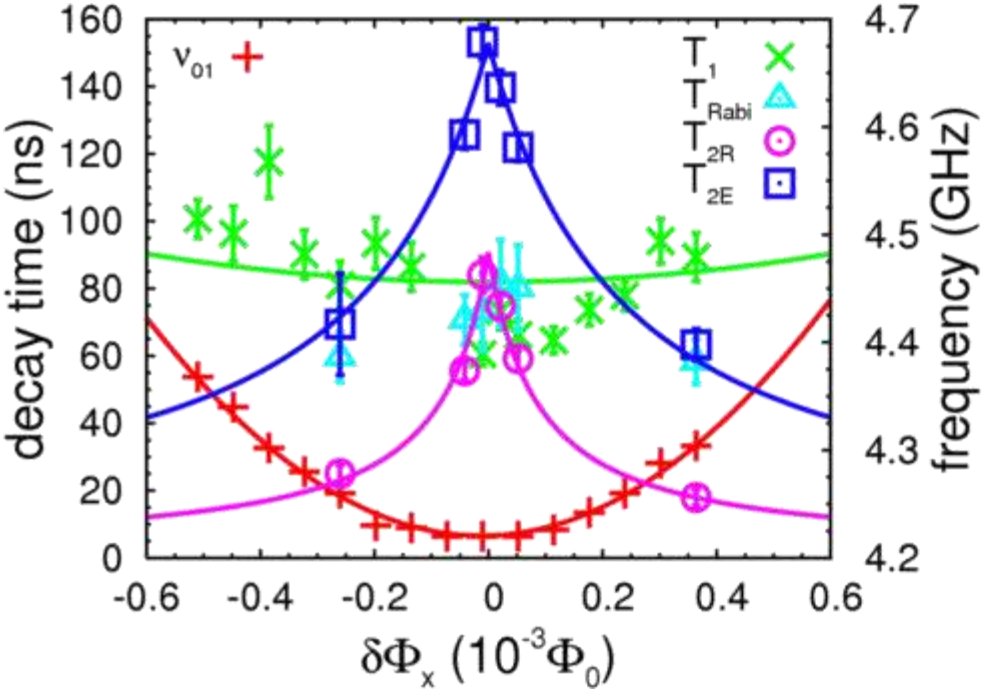


Flux qubit – Relaxation rate

fit by Bloch-Redfield model
(or Golden rule argument):

$$T_1^{-1} = \frac{\pi}{\hbar}\left(\frac{\partial\varepsilon}{\partial\Phi_x}\sin\theta\right)^2 S_\Phi(\omega_q)$$

Transfer function — Actual flux noise

High frequency noise causing resonant transitions

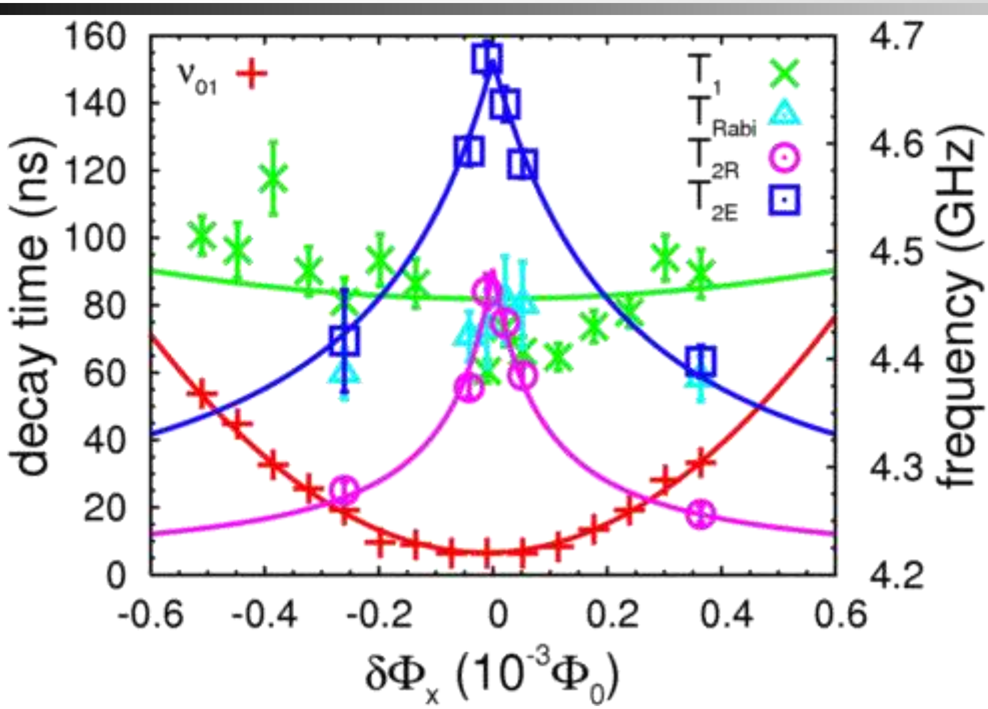
→ Physical coupling via σ_z (flux through loop)
   → $\sigma_z \rightarrow \cos\theta\,\sigma_z - \sin\theta\,\sigma_x$
   → $\langle 0|\sigma_z|1\rangle = \sin\theta$
   → Stronger near magic point ($\sin\theta \simeq 1$)

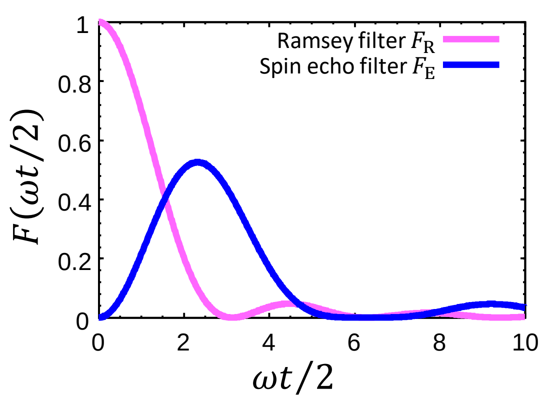
→ Inductive coupling MI_pI_{noise}
   → Stronger for large $\frac{\partial\varepsilon}{\partial\Phi_x} = 2I_p$

This data

→ $S_\Phi\left(\frac{\Delta}{\hbar}\right) \simeq 2\times 10^{-20}\,\frac{\Phi_0^2}{\text{Hz}}$

F. Deppe et al., Phys. Rev. B 76, 214503 (2007)





Flux qubit – Dephasing rates

Depend on type of low frequency noise!

→ Rates depend on the measurement protocol (filters!)

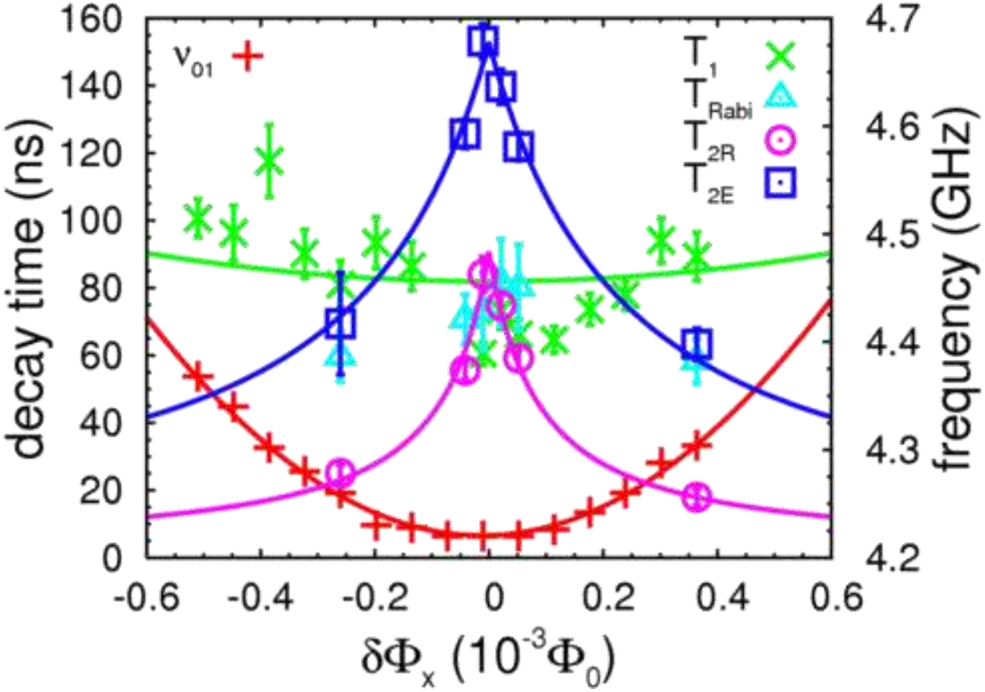
→ Ramsey

- → Probes low-frequency noise
- → $T_{2R}^{-1} = (2T_1^{-1}) + T_{\varphi R}^{-1}$

→ Spin echo

- → $T_{2E}^{-1} = (2T_1^{-1}) + T_{\varphi E}^{-1}$
- → Filters low-frequency noise

G. Ithier *et al.*, Phys. Rev. B **72**, 134519 (2005).
A. Shnirman *et al.*, Phys. Rev. Lett. **94**, 127002 (2005).
K. Kakuyanagi, F. Deppe *et al.*, PRL 98, 047004 (2007)
F. Deppe et al., Phys. Rev. B 76, 214503 (2007)



Flux qubit – dephasing rates

Predictions

→ White noise (Bloch-Redfield)

→ $T_\varphi^{-1} = \frac{\pi}{\hbar}\left(\frac{\partial\varepsilon}{\partial\Phi_\mathrm{x}}\cos\theta\right)^2 \times S_\Phi^{\mathrm{BR}}(\omega=0)$

→ Low-frequency spectrum important

→ $\langle 1|\sigma_z|1\rangle = \cos\theta$

→ $T(\Phi_\mathrm{x})$ is smooth function

→ 1/f-noise

→ $S_\Phi(\omega) = \frac{A_{1/f}}{\omega}$

→ $T_{\varphi E}^{-1} = \frac{1}{\hbar}\left|\frac{\partial\varepsilon}{\partial\Phi_\mathrm{x}}\cos\theta\right|\sqrt{A\ln 2}$

→ Φ_x-dependence cusped

→ This experiment

→ Clearly cusped

→ 1/f-noise dominates

→ Additional small white noise contribution

G. Ithier *et al.*, Phys. Rev. B **72**, 134519 (2005).
A. Shnirman *et al.*, Phys. Rev. Lett. **94**, 127002 (2005).
K. Kakuyanagi, F. Deppe *et al.*, PRL 98, 047004 (2007)
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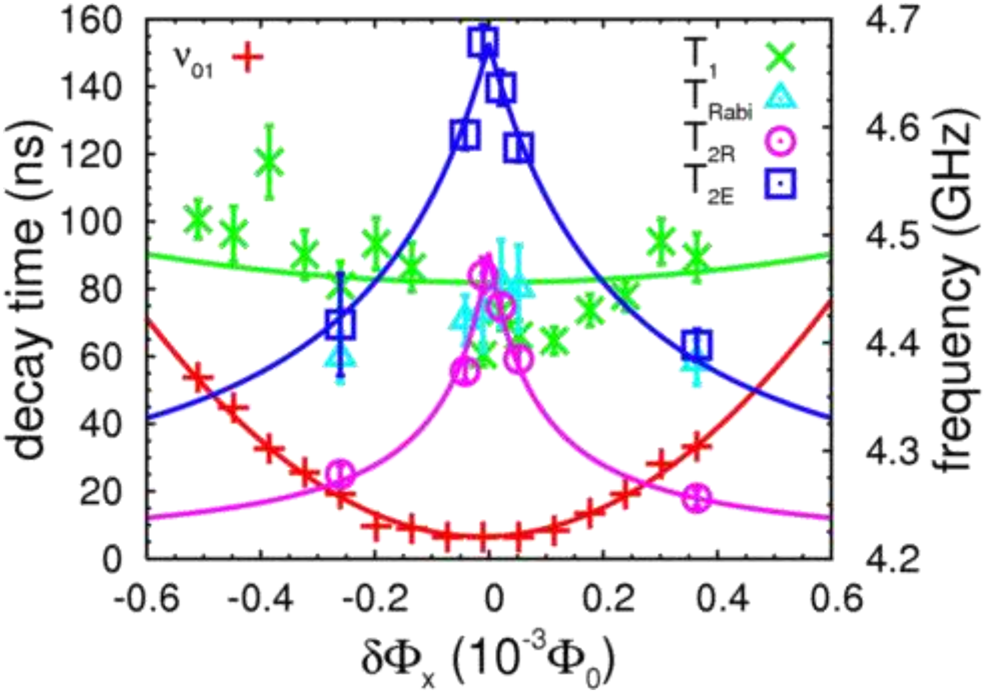


Flux qubit – dephasing rates

Summary & numbers

→ $T_{2E}^{-1} \approx (2T_1)^{-1}$ at magic point
- → Relaxation-limited
- → $T_1 \ll T_{\varphi E} \simeq 2\ \mu s$

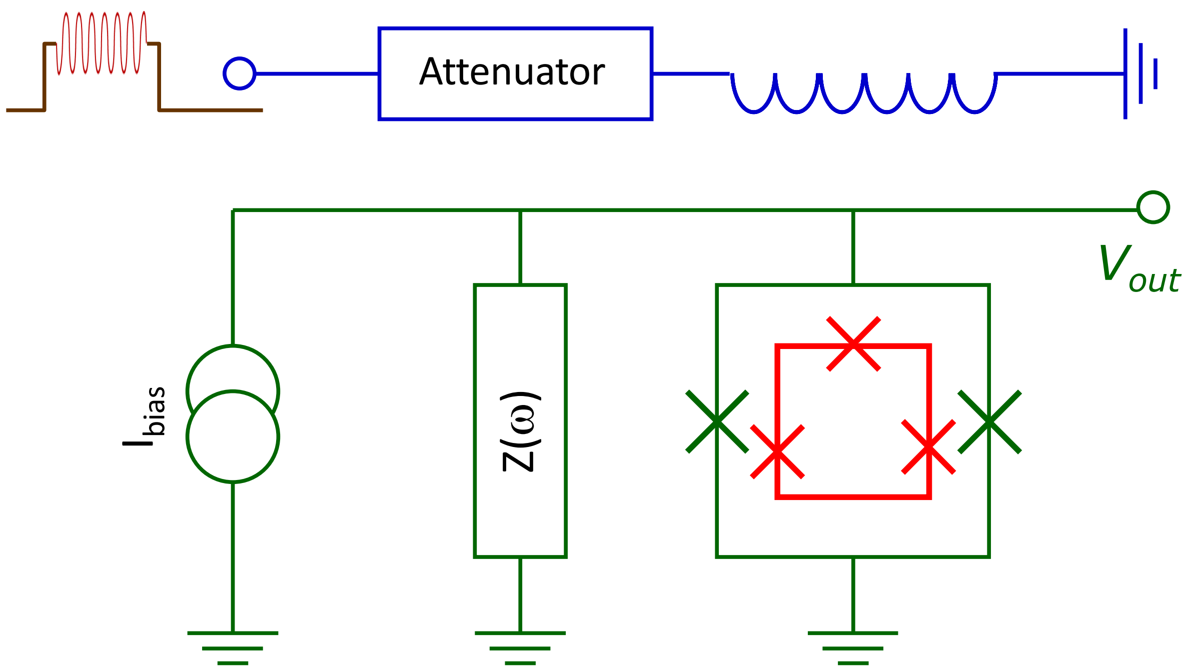
→ Strong 1/f-contribution
- → $A_{1/f} \simeq (10^{-6}\Phi_0)^2$
- → Typical value (known from SQUIDs)

→ Additional white-noise contribution
- → $S_\Phi^{BR}(\omega \to 0) = 4 \times 10^{-20} \frac{\Phi_0^2}{Hz}$
- → $S_\Phi^{BR}(\omega \to 0) \approx S_\Phi\left(\frac{\Delta}{\hbar}\right)$
- → Consistent (white noise should be frequency-independent)

G. Ithier *et al.*, Phys. Rev. B **72**, 134519 (2005).
A. Shnirman *et al.*, Phys. Rev. Lett. **94**, 127002 (2005).
K. Kakuyanagi, F. Deppe *et al.*, PRL 98, 047004 (2007)
F. Deppe et al., Phys. Rev. B 76, 214503 (2007)

Flux qubit – noise sources

White noise



White noise from control line

→ Cold attenuators

White noise from readout circuit

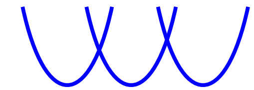
→ Engineer $Z(\omega)$

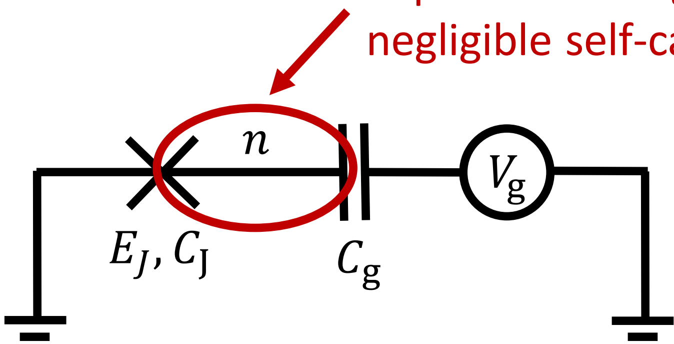
1/f noise

1/f noise independent of area of qubit loop
local noise sources in close qubit environment

→ Two-level fluctuators → Improve fabrication technology

Charge engineering – The Cooper pair box (CPB)





Charge regime $E_C \gtrsim E_J$
→ Charge is good quantum number

Hamiltonian (for any E_J, $E_C \equiv \frac{e^2}{2(C_g + C_J)}$, n_g)

$$\hat{H}_{CPB} = 4E_C(\hat{n} - n_g)^2 + E_J(1 - \cos\hat{\varphi})$$

Gate charge $n_g \equiv \frac{C_g V_g}{2e}$ induced by source V_g

→ Adds/removes excess CP to/from island (always many equilibrium CP)

→ Classical quantity

→ May assume fractional values!

Schrödinger equation $\hat{H}_{CPB}|\Psi_k\rangle = E_k|\Psi_k\rangle$

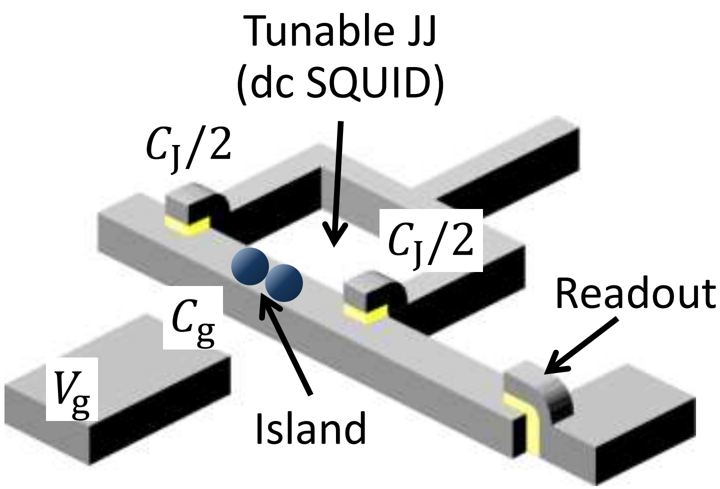
→ Mathieu equation for $|\tilde{\Psi}_k\rangle \equiv |\Psi_k\rangle e^{-in_g\varphi}$

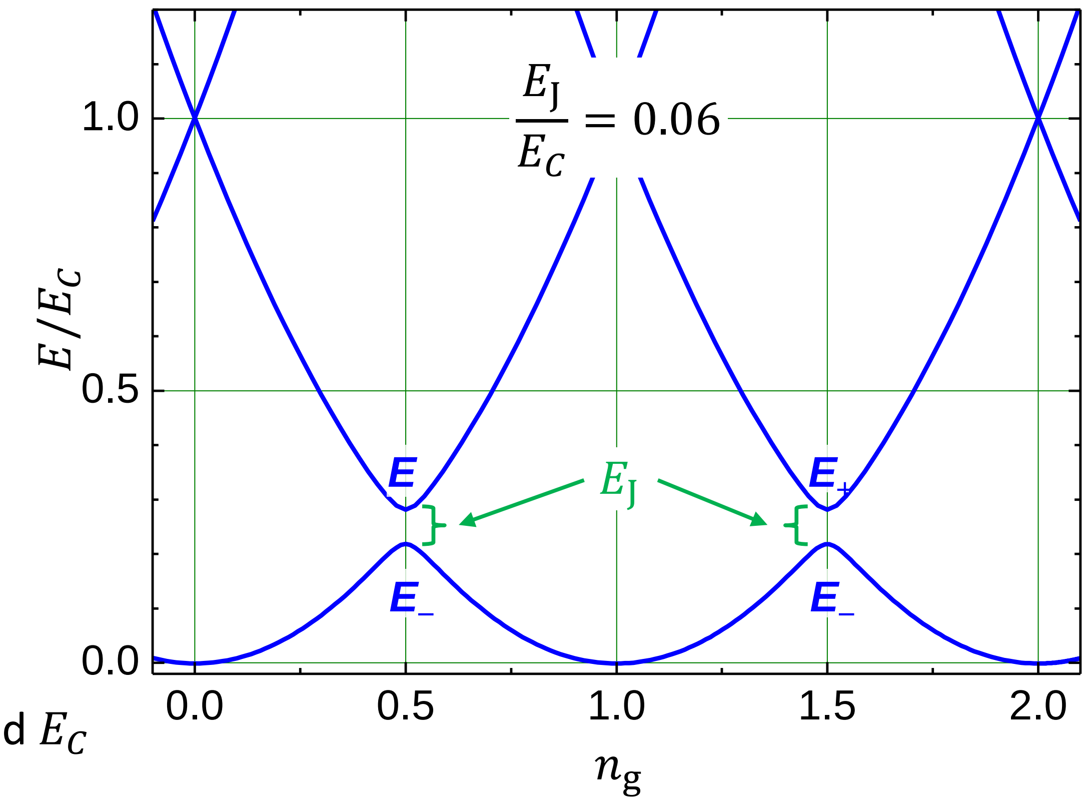
$$\frac{\partial^2|\tilde{\Psi}_k\rangle}{\partial\alpha^2} - 2\left(\frac{2E_J}{E_C}\right)\cos\alpha\,|\tilde{\Psi}_k\rangle = \frac{4E_k}{E_C}|\tilde{\Psi}_k\rangle$$

$$\alpha \equiv \varphi/2$$

Numerical solution → Eigenenergies E_k

The split Cooper pair box





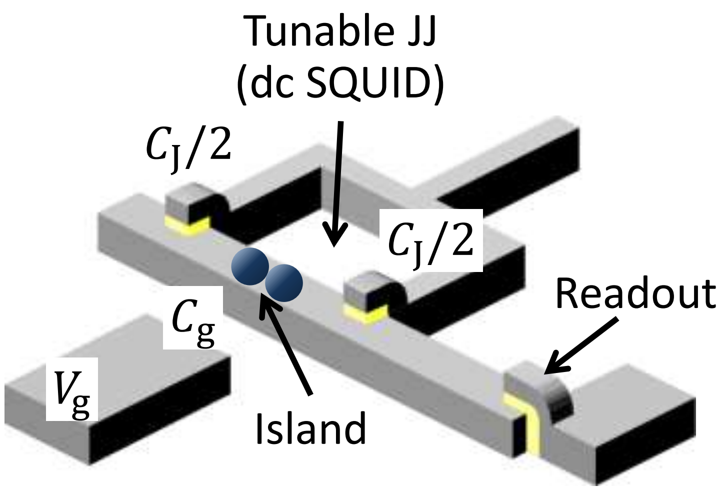
- → Dc SQUID with $\beta_L \ll 1$
  - → Tunable JJ with effective E_J and E_C
- → Gate voltage V_g is control knob
- → Readout with additional JJ
  - → Detect number of excess Cooper pairs on island
- → Josephson energy $E_J \cos\hat{\varphi}$
  - → Couples charge states/parabolas
  - → Avoided crossings

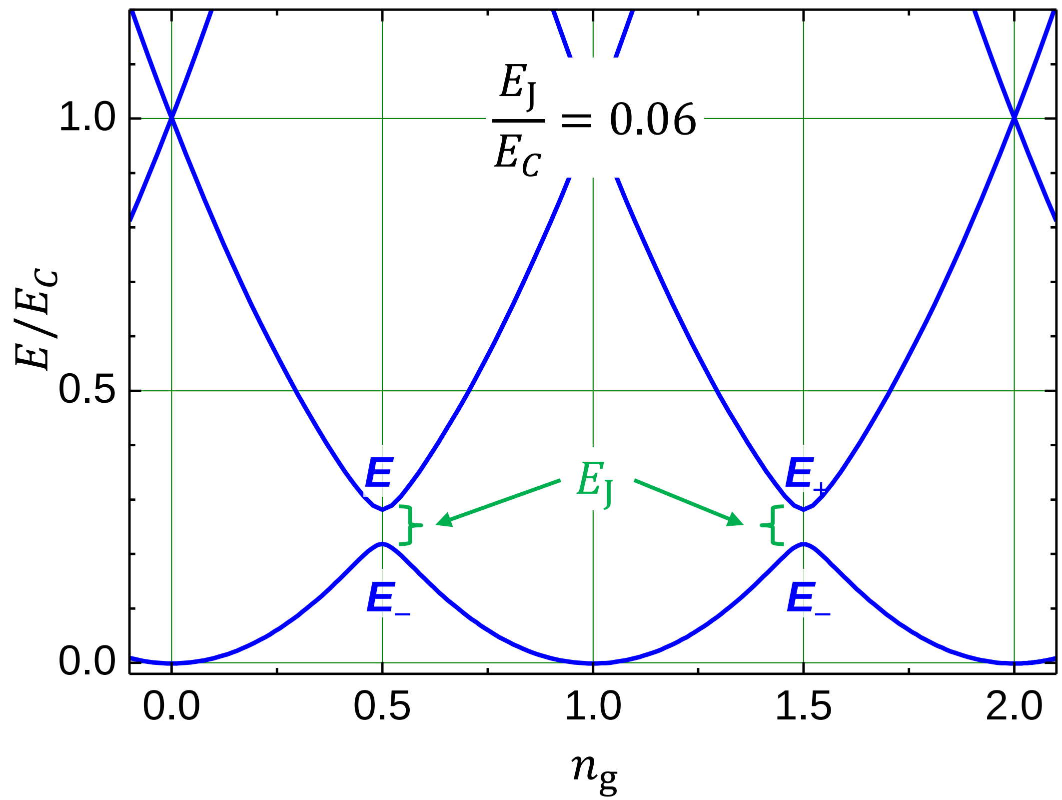
$$\hat{H}_{CPB} = 4E_C(\hat{n} - n_g)^2 + E_J \cos\hat{\varphi}$$

More typical parameters

→ $E_C \simeq 5\text{ GHz}$, $E_J \simeq 5\text{ GHz}$

The split Cooper pair box





Theory questions

→ Why is coupling exactly E_J?

→ Near $n_g = \frac{1}{2}$, energy levels look like hyperbola. Correct?

$$\hat{H}_{CPB} = 4E_C(\hat{n} - n_g)^2 + E_J \cos\hat{\varphi}$$

More typical parameters

→ $E_C \simeq 5$ GHz, $E_J \simeq 5$ GHz

Two-level-representation of the CPB

Goal → Express $\widehat{H}_{\mathrm{CPB}} = 4E_C\left(\hat{n} - n_{\mathrm{g}}\right)^2 + E_{\mathrm{J}} \cos\hat{\varphi}$ as TLS near $n_{\mathrm{g}} = \frac{1}{2}$

Charge states $|n\rangle$ → $\hat{n}|n\rangle = n|n\rangle$

$$\rightarrow \left(\hat{n} - n_{\mathrm{g}}\right)^2 = \left(\hat{n} - n_{\mathrm{g}}\right)\left(\sum_n |n\rangle\langle n|\right)\left(\hat{n} - n_{\mathrm{g}}\right) = \sum_n \left(n - n_{\mathrm{g}}\right)^2 |n\rangle\langle n|$$

Commutation relations $[\hat{\mathrm{n}}, \hat{\varphi}] = 1$

$$\rightarrow |n\rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\varphi \exp(-in\hat{\varphi})\,|\varphi\rangle$$

$$\rightarrow \exp(ip\hat{\varphi})\,|n\rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\varphi \exp[-i(n+p)\hat{\varphi}]\,|\varphi\rangle = |n+p\rangle$$

$$\rightarrow \exp(\pm i\hat{\varphi})\,|n\rangle = |n \pm 1\rangle \rightarrow \cos\hat{\varphi} = \frac{1}{2}\left(\exp(i\hat{\varphi}) + \exp(-i\hat{\varphi})\right) =$$

$$= \frac{1}{2}\left(\sum_n |n\rangle\langle n|\right)\left(\exp(i\hat{\varphi}) + \exp(-i\hat{\varphi})\right)\left(\sum_n |n\rangle\langle n|\right) = \frac{1}{2}\sum_n \left(|n\rangle\langle n+1| + |n+1\rangle\langle n|\right)$$

Two-level-representation of the CPB

$|n\rangle \rightarrow \hat{n}|n\rangle = \mathrm{n}|n\rangle$

$\exp(\pm i\hat{\varphi})\,|n\rangle = |n \pm 1\rangle$

TLS → $n \in \{0,1\}$

$$\rightarrow \sum_n (n - n_\mathrm{g})^2 |n\rangle\langle n| \;\rightarrow \left(\frac{1}{2} - n_\mathrm{g}\right)\hat{\sigma}_z$$

$$\widehat{H}_\mathrm{CPB} = 4E_C(\hat{n} - n_\mathrm{g})^2 + E_\mathrm{J}\cos\hat{\varphi}$$

↓

$$\rightarrow \frac{1}{2}\left(\sum_n |n\rangle\langle n|\right)(\exp(i\hat{\varphi}) + \exp(-i\hat{\varphi}))\left(\sum_n |n\rangle\langle n|\right)$$

$$= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)(\exp(i\hat{\varphi}) + \exp(-i\hat{\varphi}))(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

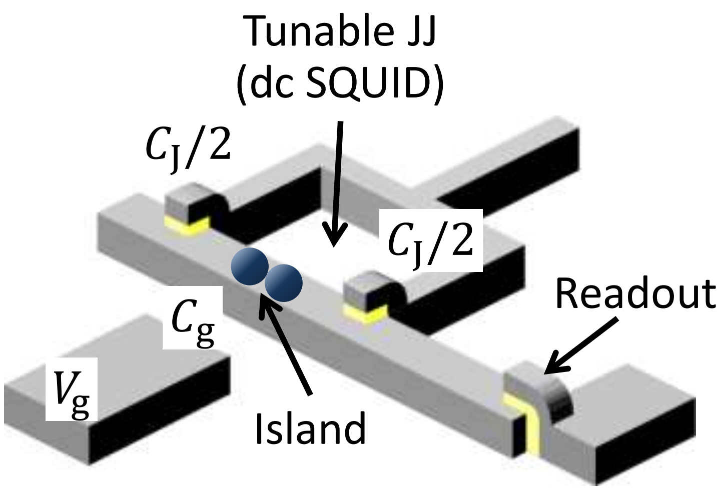
$$= \frac{1}{2}(|0\rangle\langle 0|\exp(i\hat{\varphi})\,|0\rangle\langle 0| + |0\rangle\langle 0|\exp(i\hat{\varphi})\,|1\rangle\langle 1|$$
$$+ |1\rangle\langle 1|\exp(i\hat{\varphi})\,|0\rangle\langle 0| + |1\rangle\langle 1|\exp(i\hat{\varphi})\,|1\rangle\langle 1|$$
$$+ |0\rangle\langle 0|\exp(-i\hat{\varphi})\,|0\rangle\langle 0| + |0\rangle\langle 0|\exp(-i\hat{\varphi})\,|1\rangle\langle 1|$$

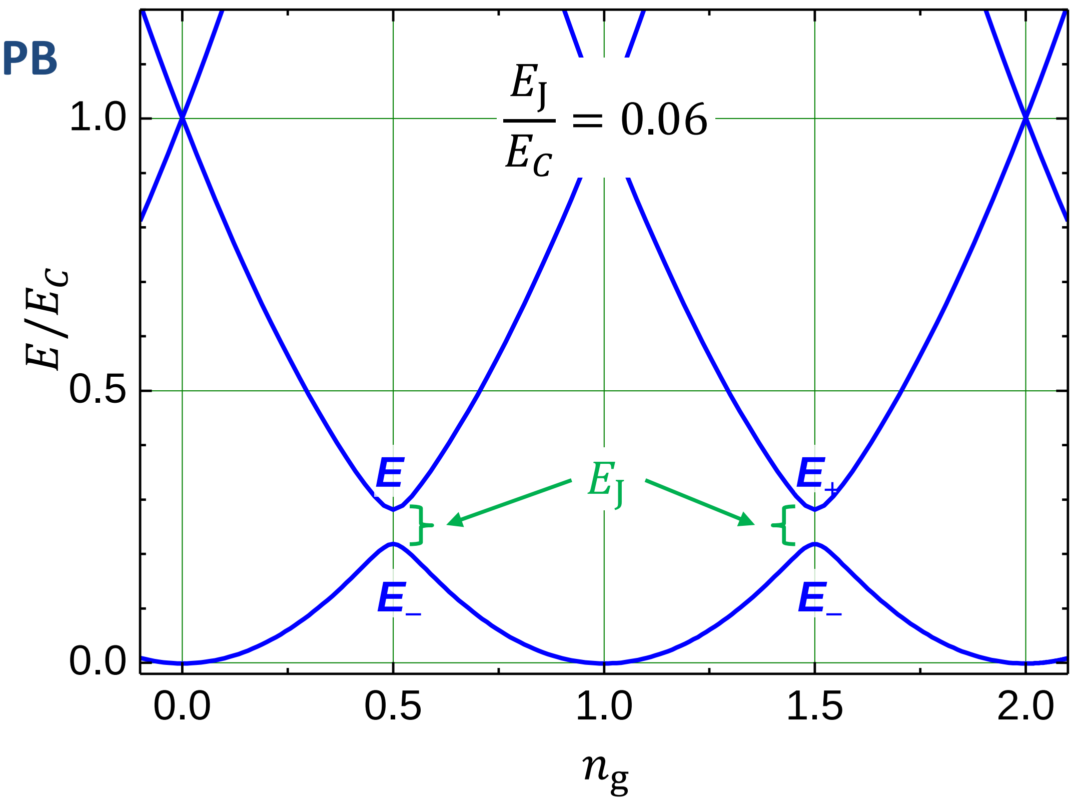
$$\widehat{H}_\mathrm{CPB} = \frac{E_\mathrm{el}}{2}\hat{\sigma}_z + \frac{E_\mathrm{J}}{2}\hat{\sigma}_x$$

$$E_\mathrm{el} \equiv 4E_C\left(\frac{1}{2} - n_\mathrm{g}\right)$$

$$= \frac{1}{2}(|0\rangle\langle 0|1\rangle\langle 0| + |0\rangle\langle 0|2\rangle\langle 1| + |1\rangle\langle 1|1\rangle\langle 0| + |1\rangle\langle 1|2\rangle\langle 1|$$
$$+ 0 + |0\rangle\langle 0|0\rangle\langle 1| + 0 + |1\rangle\langle 1|0\rangle\langle 1|) \;= \frac{1}{2}\hat{\sigma}_x$$

Two-level-representation of the CPB





→ Why is coupling exactly E_J?

→ Because of the two-level representation

→ Near $n_g = \frac{1}{2}$, energy levels look like hyperbola. Correct?

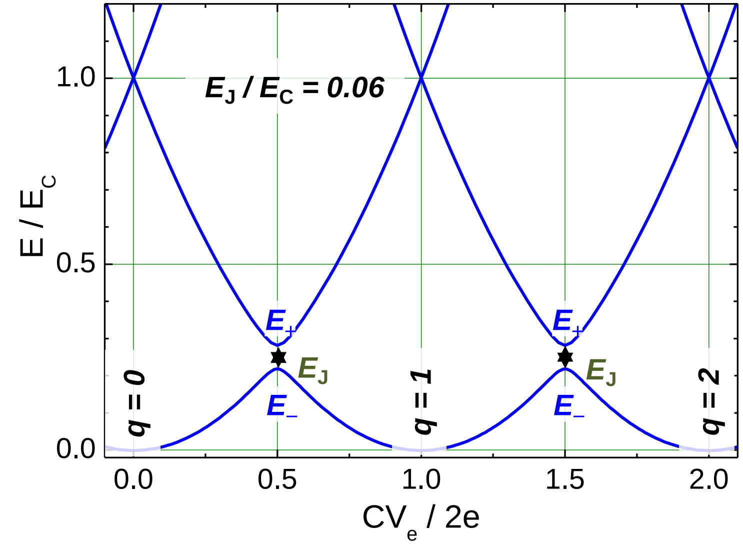
→ Yes, because $E_{el}(V_g)$ can be linearized for $(n - n_g) \ll 1$

$$\hat{H}_{CPB} = \frac{E_{el}}{2}\hat{\sigma}_z + \frac{E_J}{2}\hat{\sigma}_x \qquad E_{el} \equiv 4E_C\left(\frac{1}{2} - n_g\right)$$

Practical devices parameters

→ $E_C \simeq 5$ GHz, $E_J \simeq 5$ GHz

From the Cooper pair box to the transmon qubit



Advantages of the CPB

→ Simple design (2JJ, $\beta_L \ll 1$)

→ Coupling element $E_\mathrm{J} \propto I_\mathrm{c}$
(Flux qubit: $\Delta \propto \exp\left(-\sqrt{E_J/E_C}\right)$)

→ Voltages convenient for
  → Coupling to other qubits
  → Coupling to readout circuitry
  → Coupling to control signals

→ Large anharmonicity (few GHz)

→ In first order insensitive to charge fluctuations at „sweet spot" $n_\mathrm{g} = n + \frac{1}{2}$

Big disadvantage

→ Coherence times short due to susceptibility to $1/f$ charge noise

→ In practice: Coherence times of a few tens of nanoseconds even at the sweet spot!
(Typical charge noise magnitude ≫ Typical flux noise magnitude)

→ Idea → Flatten energy dispersion

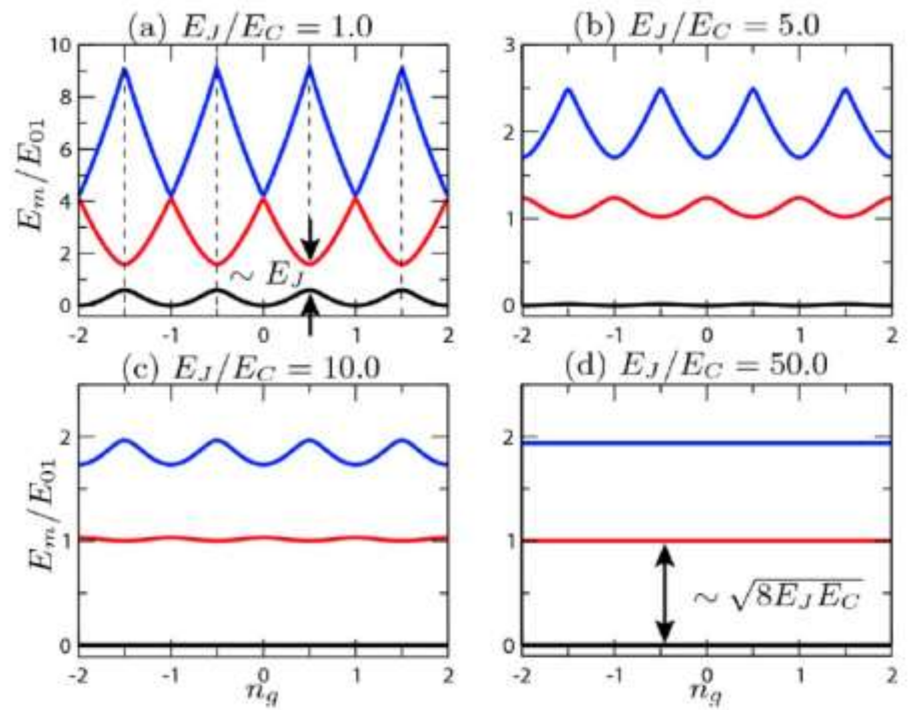
The transmon qubit

(Transmission line shunted plasma oscillation qubit)

Take a CPB geometry and increase E_J/E_C

$$E_m(n_g) \approx E_m\left(n_g = \frac{1}{4}\right) - \frac{\epsilon_m}{2}\cos 2\pi n_g$$

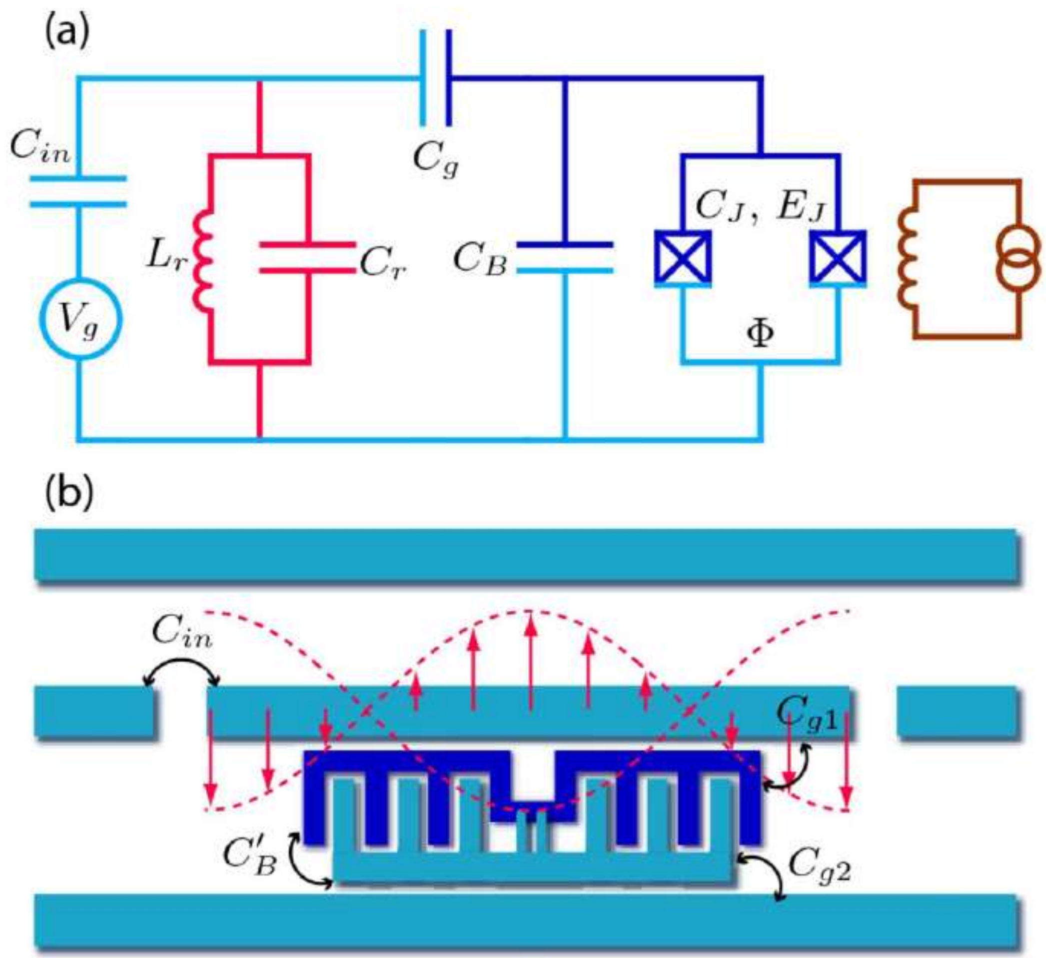
$$\epsilon_m \approx (-1)^m E_C \frac{2^{4m+5}}{m}\sqrt{\frac{2}{\pi}}\left(\frac{E_J}{2E_C}\right)^{\frac{m}{2}+\frac{3}{4}} e^{-\sqrt{\frac{8E_J}{E_C}}}$$



→ Charge dispersion decreases exponentially with E_J/E_C
→ Anharmonicity decreases only polynomially with E_J/E_C
→ Optimum trade-off for $E_\mathrm{J}/E_C \approx 50$
→ Few hundreds of MHz anharmonicity left
→ Charge no longer good quantum number
→ Not tunable via gate voltage anymore → Tune via flux (dc SQUID)

J. Koch *et al.*, Phys. Rev. A **76**, 042319 (2007).

The transmon qubit



Embed into a resonator for
→ Readout
→ Filtering
→ Control

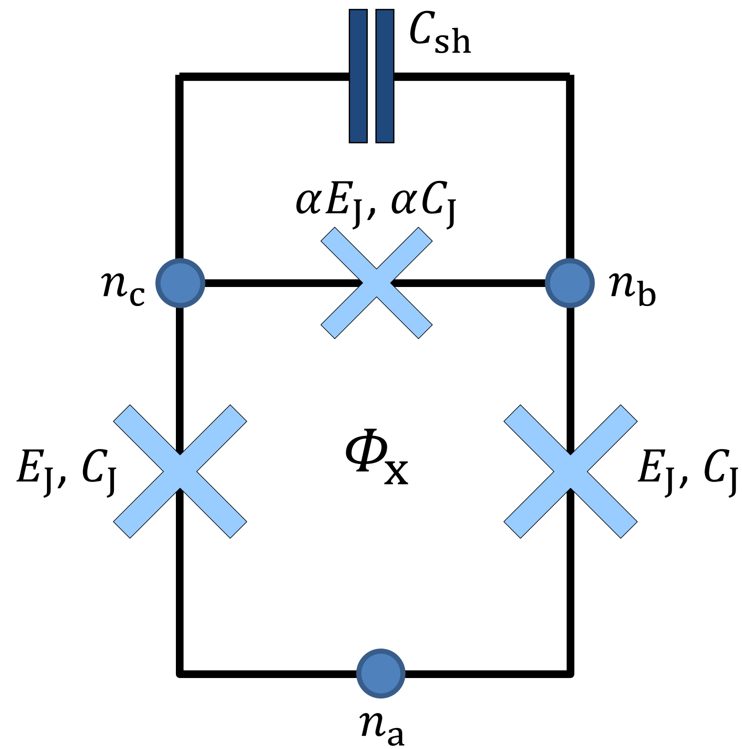
The transmon is currently most successful qubit with respect to coherence times

Coherence of transmons mostly limited by spurious TLS (defects) in substrate and metal-substrate interface

→ 2D geometries → 10 – 40 μs
→ 3D geometries → up to 200 μs

J. Koch *et al.*, Phys. Rev. A **76**, 042319 (2007).

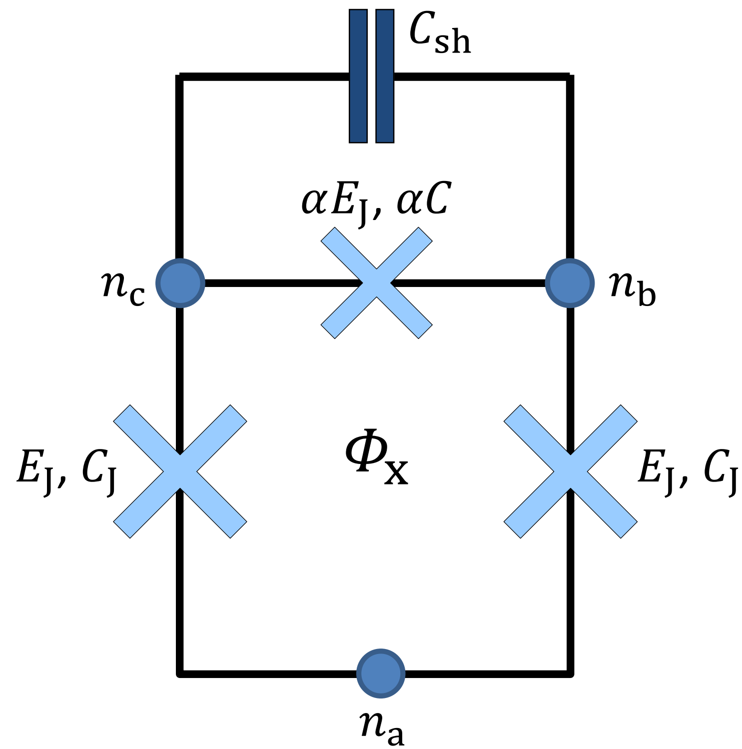
The C-shunted flux qubit



→ Decreasing $E_{\rm J}/E_C$ and/or α reduces influence of flux noise by level flattening
→ However, sensitivity to charge noise on islands a,b,c is increased
→ Suppress charge noise by shunt capacitance $C_{\rm sh} = (\beta - \alpha)C_{\rm J}$
→ Typically, $C_{\rm sh} \simeq 100\ {\rm fF} \gg C_{\rm J} \simeq 5\ {\rm fF}$
→ First promising results $T_2^* \approx T_1 \simeq 1.5\ \mu{\rm s}$

J. Q. You *et al.*, Phys. Rev. B **75**, 140515(R) (2007).
M. Steffen et al., Phys. Rev. Lett 105, 100502 (2010).

The C-shunted flux qubit



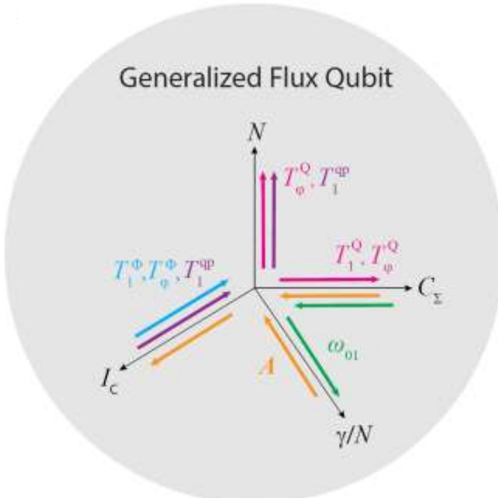
→ Optimize thin film fabrication as in transmon
- → Anharmonicity 800 MHz slightly larger than for typical transmon qubits
- → Neverthless, transmon-like design

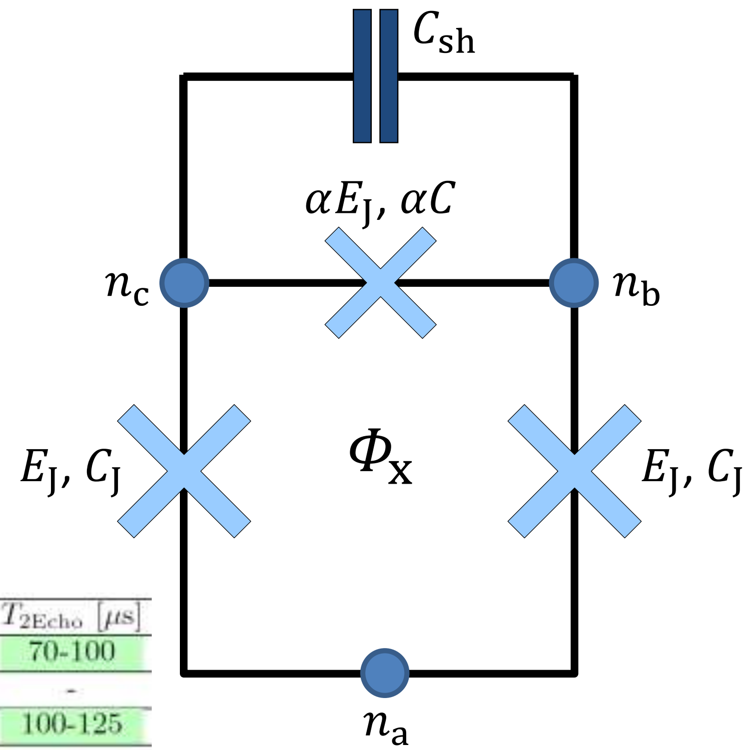
→ Noise sources limiting $T_1 \lesssim 55$ µs @$\Phi_0/2$
- → Resonator loss
- → Ohmic charge noise
- → $1/f$ flux noise
- → Temporal variations attributed to quasiparticles

→ Noise sources limiting $T_2 \simeq 85$ µs @$\Phi_0/2$
- → Photon shot noise from residual thermal photons in the readout resonator

F. Yan *et al.*, Nat. Comm. **7**, 12964 (2016).

The C-shunted flux qubit





Device	N	I_c [nA]	C_{sh} [fF]	γ/N	ω_{01} [GHz]	$\mathcal{A}$ [GHz]	$\mathcal{A}/\omega_{01}$	T_1 [μs]	T_{2Echo} [μs]
A	8	21	20	0.92	3.6	1.0	0.28	43.1 ± 7.5	70-100
B	8	21	30	0.95	2.8	0.8	0.29	23	-
C	8	21	30	0.92	2.6	1.0	0.38	82.9 ± 7.9	100-125
D	8	18	30	0.92	2.4	0.9	0.38	20	-
E	8	18	30	0.93	2.0	1.0	0.50	50	12
F	8	40	20	0.93	3.7	1.6	0.43	-	-
G	8	40	20	0.98	4.7	1.2	0.26	10	7
H	8	40	20	1.0	3.4	1.9	0.56	23	15
I	16	14	20	0.84	3.0	0.9	0.30	50.6 ± 9.2	110-140
J	16	14	20	1.09	3.8	0.6	0.16	30	-
K	16	27	20	0.88	2.6	1.0	0.38	30	20
CSFQ	2	60	50	1.2	4.7	0.5	0.11	35-55	70-90

F. Yan et al., arXiv:2006.04130v1(2020).