Applied Superconductivity:

Josephson Effect and Superconducting Electronics

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The Josephson Equations

The Josephson equations can be derived by considering two quantum systems that are weakly coupled. One starts with the completely uncoupled systems that are described by the macroscopic wave functions ψ_1 and ψ_2. The time dependence of these wave functions is given by

$$\frac{\partial \psi_1}{\partial t} = -\frac{\imath}{\hbar} E_1 \psi_1 \quad \text{(A..1)}$$

$$\frac{\partial \psi_2}{\partial t} = -\frac{\imath}{\hbar} E_2 \psi_2 \ . \quad \text{(A..2)}$$

We now switch on a weak symmetric coupling of the two systems. In this case the time evolution of system 1 will also be determined by system 2 and vice versa. We account for this situation by introducing the coupling constant K in (A..1) and (A..1):

$$\frac{\partial \psi_1}{\partial t} = -\frac{\imath}{\hbar} \left[E_1 \psi + K \psi_2 \right] \quad \text{(A..3)}$$

$$\frac{\partial \psi_2}{\partial t} = -\frac{\imath}{\hbar} \left[E_2 \psi_2 + K \psi_1 \right] \ . \quad \text{(A..4)}$$

The coupling of the two systems means that Cooper pairs can be exchanged between the two systems, or equivalently, that the macroscopic wave functions are weakly overlapping (e.g. within the insulating barrier). The magnitude of the coupling constant is determined by the strength of the overlap.

Note that the coupling of two wave functions is very well known from molecules, e.g. the H_2^+-molecule. However, in the case of two coupled superconductors the wavefunctions represent the whole ensemble of the superelectrons, whereas in a molecule the wavefunctions represent single electrons. Therefore, the amplitude of the superconducting wavefunction is proportional to the density of the superelectrons and the wavefunctions can be expressed as (compare (1.1.46) in section 1.1)

$$\psi_1 = \sqrt{n_1^\star}\, e^{\imath\theta_1} \quad \text{(A..5)}$$

$$\psi_2 = \sqrt{n_2^\star}\, e^{\imath\theta_2} \ . \quad \text{(A..6)}$$

Using these wavefunctions in (A..3) and (A..4) we obtain

$$\frac{\dot{n_1^\star} e^{\imath\theta_1}}{2\sqrt{n_1^\star}} + \imath\sqrt{n_1^\star} e^{\imath\theta_1}\dot{\theta}_1 = -\frac{\imath}{\hbar}\left[E_1\sqrt{n_1^\star} e^{\imath\theta_1} + K\sqrt{n_2^\star} e^{\imath\theta_2}\right] \quad \text{(A..7)}$$

$$\frac{\dot{n_2^\star} e^{\imath\theta_2}}{2\sqrt{n_2^\star}} + \imath\sqrt{n_2^\star} e^{\imath\theta_2}\dot{\theta}_2 = -\frac{\imath}{\hbar}\left[E_2\sqrt{n_2^\star} e^{\imath\theta_2} + K\sqrt{n_1^\star} e^{\imath\theta_1}\right] . \quad \text{(A..8)}$$

Separation into real and imaginary part yields

$$\frac{1}{2}\frac{\dot{n_1^\star}}{\sqrt{n_1^\star}} = \frac{K}{\hbar}\sqrt{n_2^\star}\sin(\theta_2 - \theta_1) \quad \text{(A..9)}$$

$$\frac{1}{2}\frac{\dot{n_2^\star}}{\sqrt{n_2^\star}} = \frac{K}{\hbar}\sqrt{n_1^\star}\sin(\theta_1 - \theta_2) \quad \text{(A..10)}$$

and

$$\imath\sqrt{n_1^\star}\dot{\theta}_1 = -\frac{\imath}{\hbar}\left[E_1\sqrt{n_1^\star} + K\sqrt{n_2^\star}\cos(\theta_2 - \theta_1)\right] \quad \text{(A..11)}$$

$$\imath\sqrt{n_2^\star}\dot{\theta}_2 = -\frac{\imath}{\hbar}\left[E_2\sqrt{n_2^\star} + K\sqrt{n_1^\star}\cos(\theta_1 - \theta_2)\right] . \quad \text{(A..12)}$$

In order to have current conservation we have to demand $\dot{n_1^\star} = -\dot{n_2^\star}$. For simplicity, we also assume two identical superconductors, that is $n_1^\star = n_2^\star = n^\star$. Then, we obtain from (A..9) and (A..10)

$$\dot{n_1^\star} = \frac{2K}{\hbar} n^\star \sin(\theta_2 - \theta_1) = -\dot{n_2^\star} . \quad \text{(A..13)}$$

We have to recall that the change of the particle density in superconductor 1 multiplied by the volume of 1 just corresponds to the change of the particle number in superconductor 1 and, hence, the particle flow through the junction. In order to obtain the supercurrent I_s flowing from superconductor 1 to 2 we only have to multiply by the charge $q^\star$ of the superelectrons and obtain

$$I_s = I_c \sin(\theta_2 - \theta_1) \quad \text{(A..14)}$$

with the maximum or critical Josephson current[1]

[1]Note that in order to keep the density of superelectrons in the superconducting electrodes constant, the junction has to be attached to a current source. The current source supplies and removes charge from the two electrodes thereby keeping $n^\star$ constant.

$$I_c = \frac{2K \cdot q^\star}{\hbar} V \cdot n^\star \ . \tag{A..15}$$

From (A..11) and (A..12) we obtain a differential equation for the temporal change of the phase difference $\theta_2 - \theta_1$. With $n_1^\star = n_2^\star = n^\star$ we obtain

$$\frac{\partial}{\partial t}(\theta_2 - \theta_1) = \frac{1}{\hbar}(E_1 - E_2) \ . \tag{A..16}$$

We see that for $E_1 = E_2$ the phase difference between the two superconductors stays constant. However, if there is a constant electric field $\mathbf{E}$ across the junction, the energy difference can be expressed as

$$E_1 - E_2 = q^\star \int_1^2 \mathbf{E}(\mathbf{r},t) \cdot d\ell \tag{A..17}$$

and we recover the second Josephson equation (1.3.12).