# Applied Superconductivity:

## Josephson Effect and Superconducting Electronics

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## I Power of Blackbody Radiation

Bolometers used for the detection of thermal radiation are influenced by thermal backgrounds. Therefore, we briefly summarize some aspects of thermal radiation. It is usual to work with an optical system which limits the beam to an area $A$ and a solid angle $\Omega$, and has filters with transmittance $\tau(f)$, where $f$ is the optical frequency. The power transmitted through such a system from a blackbody source with Planck spectral brightness $B(f,T)$ can be expressed as

$$P = \int_0^\infty P_f df = \int_0^\infty A\Omega T(f)B(f,T)\,df \ . \tag{I.1}$$

The throughput $A\Omega$ in sr m$^2$ is an invariant in an optical system, as can easily be proved from geometrical optics.

When diffraction is important, the throughput depends on the wavelength $\lambda$. The antenna theorem states that for a single spatial mode, or diffraction limited beam, the throughput is exactly $A\Omega = \lambda^2$. This principle is easily illustrated by using Fraunhoffer diffraction theory to compute the solid angle of divergence $\Omega$ of a plane wave after passing through a circular aperture of area $A$. When the solid angle or the area is not uniformly illuminated, an equivalent solid angle (or area) must be used for exact results. Infrared telescopes often use throughputs somewhat larger than the diffraction limit. For a uniformly illuminated circular aperture, 84% of the energy from a point source appears in a throughput $A\Omega = 3.7\lambda^2$. Spatially incoherent light can be thought of as being made up from many modes. The number of modes is $N = A\Omega/\lambda^2$ for a single polarization. This picture can also be applied for the intermediate case of partially coherent light, although a rigorous treatment is complicated.[1]

A well-known result of thermal physics is that for a blackbody source the power per mode is

$$P_f df = \frac{hf\,df}{\exp(hf/k_BT)-1} \ . \tag{I.2}$$

This expression approaches $k_BTdf$ for $hf \ll k_BT$. The power $P(f,T)df$ in a multimode source is just the number of modes times the power per mode, i.e. $P(f,T)df = \frac{c}{2}u(f,T)df = \frac{c}{2}hfnD(f)$. Here, $u(f,T)$

[1]J. M. Lamarre, Appl. Opt. **25**, 870 (1986).

is the energy spectral density, $n$ the occupation number and $D(f) = 4f^2/c^3$ the spectral density of the modes (for two polarizations). This gives the Planck result for the spectral brightness of a blackbody to

$$B(f,T)\,df \quad = \quad \frac{2hf^3\,df}{c^2[\exp(hf/k_BT)-1]} \; . \tag{I.3}$$

This expression implicitly identifies the thermal equilibrium number $n = [\exp(hf/k_BT)-1]^{-1}$ of photons per standing wave mode in a box at temperature $T$ with the number of photons per s per Hz of infrared bandwidth in a spatial mode propagating in free space. With this identification we can use the textbook expression for the thermal average variance in the number of photons per mode inside the box, $\langle(\Delta n)^2\rangle = n+n^2$ to compute the fluctuations in the number of photons arriving per second from the free space beam. Note that for $hf \ll k_BT$ we have $n \ll 1$ and the fluctuations obey Poisson statistics, $\langle(\Delta n)^2\rangle = n$. In this case, the photon arrival is random. When there are many photons per mode, $n \gg 1$, the photons arrive in bunches and we have $\langle(\Delta n)^2\rangle = n^2$.

Since a bolometer detects the incoming power, we are interested in the mean square energy fluctuation, which can be written as $h^2f^2\langle(\Delta n)^2\rangle$. If we make the simple assumption that fluctuations in energy in different modes and in different infrared bandwidths are uncorrelated, then their mean square fluctuations are additive. Then the mean square fluctuation in the energy arriving within the time interval of 1 s is $\int h^2f^2 2N(n+n^2)df$, where $N$ is the number of contributing modes. Since the bandwidth associated with a 1 s unweighted average is 1/2 Hz, the mean square noise power per unit post-detection bandwidth $B$ referred to the absorbed power at the input is

$$\begin{aligned}\frac{P_N^2}{B} &= 2\int 2Nn\,(hf)^2\,df + 2\int (2Nnhf)^2\,\frac{hf}{2N}\,df \\ &= 2\int P_f\,hf\,df + \int P_f^2\,\frac{c^2}{A\Omega^2 f^2}\,df \;,\end{aligned} \tag{I.4}$$

where we have used $P_f = 2Nnhf$ for the spectral power absorbed in the bolometer and $N = A\Omega/\lambda^2 = a\Omega f^2/c^2$ for the number of modes. Note that Planck's constant $h$ does not appear in the second term, which is a property of classical waves.

The first term in (D.I.4) can be obtained more directly. For Poisson statistics, the mean square fluctuation in the number of photons arriving in a time interval of 1 s is just equal to the number of photons arriving, i.e. $\langle(\Delta n)^2\rangle = P_f/hf$. If we multiply by $h^2f^2$ to obtain fluctuations in power and by $2B$ to convert a 1 s average to a bandwidth of $B$ Hz, we obtain the first term in (D.I.4). This term has been verified experimentally in many experiments. The second term, by contrast, has not been measured unambiguously.

Although the form given in (D.I.4) appears frequently in the literature,[2,3,4] there are theoretical arguments and indirect experimental data, which show that it is not correct. The argument can be understood from the central limit theorem of probability theory. When the fluctuations from enough modes are combined, the resulting distribution should be Gaussian (of which Poisson statistics is a special case). It has been argued that the second term in (D.I.4) should have a factor $q = 2A\Omega\Delta f\Delta T/\lambda^2$ in the denominator.[5,6] Here, $q$ is the number of modes of one polarization detected in the frequency band $\Delta f$ during the time $\Delta T$, which is a very large number for most bolometric systems. Indeed, experimental evidence for this averaging effect was found in the scattering of visible light from dielectric spheres.[7] Although we

[2] J. C Mather, Appl. Opt. **23**, 3181 (1984); J. C. Mather, Appl. Opt. **23**, 584 (1984).
[3] K. M. van Vliet, Appl. Opt. **6**, 1145 (1967).
[4] J. C. Mather, Appl. Opt. **21**, 1125 (1982).
[5] E. Jakeman and E. R. Pike, J. Phys. A (Proc. Phys. Soc.) **1**, 128 (1968).
[6] J. M. Lamarre, Appl. Opt. **25**, 870 (1986).
[7] E. Jakeman, C. J. Oliver, and E. R. Pike, J. Phys. A (Proc. Phys. Soc.) **1**, 406 (1968).

will use the full expression (D.I.4) with the factor $q$ in the following, it may prove that the second term can be neglected in almost all practical situations.

We have to generalize our discussion now to treat real systems, which have sources with emissivity $\varepsilon$, cold filters with transmissivity $\tau(f)$, and bolometers with absorptivity $\eta$. We will assume that the bolometer is cold enough that fluctuations in the power emitted by the bolometer can be neglected. The number of photons per s per mode and per Hz, which is $n = [\exp(hf/k_BT) - 1]^{-1}$ for a blackbody becomes $n = \varepsilon\tau\eta[\exp(hf/k_BT) - 1]^{-1}$ inside the bolometer, where the nonlinear processing takes place. For T=300 K and $\varepsilon\tau\eta = 1$ we have $n = 1$ at $\lambda \simeq 100\,\mu$m. For the more realistic case, $\varepsilon\tau\eta = 0.1$, we have $n = 1$ at $\lambda \simeq 1$ mm. Consequently, the first term in (D.I.4) typically dominates for infrared systems and the second term with $q = 1$ would be important for millimeter wave systems. Although the two limits of photon noise are analogous to the Rayleigh-Jeans and Wien limits of the Planck theory, it is clear that the fundamental variable is $n$, the number of photons per second in one mode in one Hz of infrared bandwidth, and not $hf/k_BT$.

# II Noise Equivalent Power

A frequently used figure of merit is the noise equivalent power (NEP), which is defined as the incident signal power required to obtain a signal equal to the noise in a one Hz bandwidth. That is, the NEP is a measure of signal to noise ratio (SNR) and not just noise. If we refer the NEP to the inside of the detector, the signal power absorbed in the detector that is required is just $\mathrm{NEP}_{\mathrm{abs}} = P_N B^{-1/2}$ according to (D.I.4). Usually, one refers however the NEP to the detector input. The signal power incident on the detector required to produce SNR=1 is then

$$\mathrm{NEP}^2 = \frac{2}{\eta^2}\int P_f\, hf\, df + \frac{1}{q\eta^2}\int P_f^2\,\frac{c^2}{A\Omega^2 f^2}\,df \;, \tag{II.5}$$

where $P_f$ is the power absorbed in the detector. Expression (D.II.5) is used to calculate the photon noise contribution to the detector noise for an existing system when the throughput $A\Omega$ is known from the optical geometry and the absorbed power spectral density can be estimated from the filter bands, the bolometer output and the absorbed power responsivity $S$. Although originally introduced to describe photoconductors, the term BLIP (Background Limited Infrared Photodetector) is often used to describe any detector whose noise in a given application comes only from photon fluctuations in the infrared background.

Calculations of the background power and photon noise expected from thermal sources are important in the design of bolometric detector systems. From (D.I.1) the absorbed power in the frequency band from $f_1$ to $f_2$ is

$$P = \frac{2k_B^4}{c^2h^3}\,T^4 A\Omega\,\varepsilon\tau\eta \int_{x_1}^{x_2} \frac{x^3}{\exp(x)-1}\,dx \;, \tag{II.6}$$

where we have used the substitution $x = hf/k_BT$. An analogous expression for the photon noise limited NEP referred to the detector input is obtained by writing the absorbed power spectral density in (D.II.5) as $P_f = A\Omega\,\varepsilon T\eta B(f,T)$. We obtain

$$\mathrm{NEP}^2 = \frac{4k_B^5}{c^2h^3}\,\frac{T^5 A\Omega\,\varepsilon\tau}{\eta}\int_{x_1}^{x_2}\frac{x^4}{\exp(x)-1}\,dx + \frac{\varepsilon\tau\eta}{q}\int_{x_1}^{x_2}\frac{x^4}{[\exp(x)-1]^2}\,dx \;. \tag{II.7}$$

We see that $\text{NEP}^2 \propto AT^5$, i.e. the photon noise increases strongly with increasing temperature and increasing detector area. Expressions related to (D.II.7) are seen in the literature for the NEP or the noise equivalent photon rate (NEN) of photon detectors such as photoconductors and photovoltaic diodes. The derivation differs from that given above in that the mean square fluctuation in the photon rate $\langle(\Delta n)^2\rangle$ for different infrared bandwidths are added directly and not multiplied by $(hf)^2$ to obtain energy fluctuations before adding as was done above.