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Chapter E

Foundations of Quantum Bits and Gates

I What is a quantum bit ?

Classical computing is based on classical (c-) bits that are usually represented by "0" and "1". Mathematically we have to deal with a binary variable

$$x \in \{0,1\} \tag{I.1}$$

with the property $x^2 = x$. Physically, these two states can be represented in different ways as for example by "charge on a capacitor" and "no charge on a capacitor" , by "magnetization direction to the left" and "magnetization direction to the right" or by "hole in the punch card" and "no hole in the punch card". The bits are manipulated by classical single (e.g. NOT) or multiple bit gates (e.g. AND, NAND, OR, NOR, ...) as discussed in more detail in section IV. For example, a two bit gate is transferring the two bits x and y with $(x,y) \in \{0,1\}$ to $f(x,y)$ with $f(x,y) \in \{0,1\}$.

I.1 Single-Qubit Systems

Whereas classical computers operate with ***classical (c-) bits***, quantum computers operate with ***quantum (qu-) bits*** usually denoted as ***qubits***. Physically, a qubit can be represented by every two level quantum system. With the basis states of a two level quantum system (e.g. a spin-1/2 system)

$$|\phi_1\rangle = |0\rangle = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{I.2}$$

$$|\phi_2\rangle = |1\rangle = |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{I.3}$$

we can define a one qubit state in the following way:

A qubit $|\Psi\rangle$ is the superposition of two computational basis states

$$|\Psi(t)\rangle = a(t)|0\rangle + b(t)|1\rangle = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} , \tag{I.4}$$

where $a(t)$ and $b(t)$ are complex amplitudes.

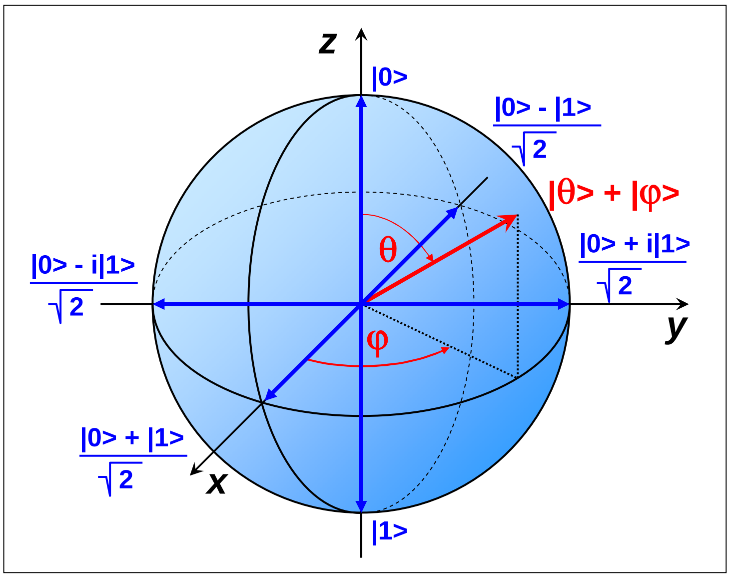

Figure E.1: Geometrical representation of a qubit state as a vector on the Bloch sphere S^2.

It is important to note that a and b are continuous analogue variables. If we are measuring the quantum state of a qubit, we obtain the result $|0\rangle$ with probability $|a(t)|^2$ and the result $|1\rangle$ with probability $|b(t)|^2$. Since the total probability must be unity we have the normalization condition

$$\langle\Psi(t)|\Psi(t)\rangle \;=\; |a(t)|^2+|b(t)|^2 \;=\; 1 \;. \tag{I.5}$$

We see that the qubit exists in a continuum of states. It is a superposition of two basis states and therefore can be represented as a unit vector in a two-dimensional Hilbert space $\mathscr{H}_2$.

A general form of the one-qubit state satisfying (E.I.5) is given by

$$|\Psi(t)\rangle \;\equiv\; |\theta,\varphi\rangle \;=\; \cos\frac{\theta}{2}e^{-\imath\varphi/2}|0\rangle+\sin\frac{\theta}{2}e^{+\imath\varphi/2}|1\rangle \;=\; \begin{pmatrix} \cos\frac{\theta}{2}e^{-\imath\varphi/2} \\ \sin\frac{\theta}{2}e^{+\imath\varphi/2} \end{pmatrix} \;. \tag{I.6}$$

The geometrical representation of the qubit state can hence be given by a point on the Bloch sphere S^2 as shown in Fig. E.1.

We immediately can write down some special case for the qubit state $|\theta,\varphi\rangle$:

$$|\theta,\varphi\rangle \;=\; |0,\varphi\rangle \;=\; |0\rangle \tag{I.7}$$

$$|\theta,\varphi\rangle \;=\; |\pi,\varphi\rangle \;=\; |1\rangle \tag{I.8}$$

$$|\theta,\varphi\rangle \;=\; |\frac{\pi}{2},0\rangle \;=\; \frac{|0\rangle+|1\rangle}{\sqrt{2}} \tag{I.9}$$

$$|\theta,\varphi\rangle \;=\; |\frac{\pi}{2},\pi\rangle \;=\; \frac{|0\rangle-|1\rangle}{\sqrt{2}} \;. \tag{I.10}$$

These states are also indicated in Fig. E.1.

Note that there is an infinite number of possible qubit states. However, any measurement on the qubit state results in a collapse of the state and a reduction of the state to one of its basis states. Information on

a and b is only obtained by performing measurements on an ensemble of identical qubits and a statistical analysis. This is a specific advantage of the use of quantum bits in quantum information processing: As long as the quantum system is not perturbed, i.e. as long as we do not perform any measurement, the state keeps all continuous variables for the description of the state. That is, the quantum system keep all possible options until the state is destroyed by a measuring process. This results in a massive quantum parallelism that can speed up computing processes.

I.2 The spin-1/2 system

Since spin systems have been widely studied and today's magnetic resonance techniques are capable to prepare a spin system in any state and let it evolve in time, it is quite common to adopt the language of spin-1/2 systems to describe the preparation and manipulation of qubits (see Fig. E.2). We also will often do so in the following. The spin state again can be considered as a vector on the Bloch sphere shown in Fig. E.1. The controlled evolution of the spin state corresponding to the motion of the end point of the vector on the Bloch sphere can be obtained by applying control fields B_z and B_x or resonant microwave pulses to the system as discussed in more detail in Appendix G.

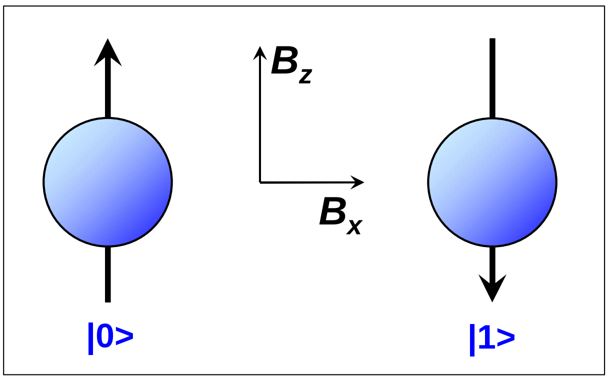

Figure E.2: The spin-1/2 system as an example for a two-level quantum system. The two basis state $|0\rangle$ and $|1\rangle$ correspond to the two possible spin orientations $|\uparrow\rangle$ and $|\downarrow\rangle$ with respect to the quantization axis given by the magnetic field B_z. A perpendicular magnetic field B_x results in the mixing of the two basis states.

I.3 Two-Qubit Systems

It is instructive to consider first two classical bits. The four possible states of a ***classical two-bit system*** are

$$|\phi_1\rangle \;=\; |00\rangle = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \tag{I.11}$$

$$|\phi_2\rangle \;=\; |01\rangle = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \tag{I.12}$$

$$|\phi_3\rangle \;=\; |10\rangle = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \tag{I.13}$$

$$|\phi_4\rangle \;=\; |11\rangle = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \ . \tag{I.14}$$

These four states are also the basis states of a ***quantum two-bit system***, which is given by the superposition of these basis states

$$|\Psi(t)\rangle \;=\; c_{00}(t)|00\rangle + c_{01}(t)|01\rangle + c_{10}(t)|10\rangle + c_{11}(t)|11\rangle \;=\; \begin{pmatrix}c_{00}\\c_{01}\\c_{10}\\c_{11}\end{pmatrix} \ . \tag{I.15}$$

Similar as for the one-qubit system the four coefficients are complex, continuous and have to satisfy the normalization condition

$$\langle\Psi(t)|\Psi(t)\rangle \;=\; |c_{00}(t)|^2 + |c_{01}(t)|^2 + |c_{10}(t)|^2 + |c_{11}(t)|^2 \;=\; 1 \ . \tag{I.16}$$

We see that the two-qubit state is a superposition of four basis states and therefore can be represented as a unit vector in a four-dimensional Hilbert space $\mathscr{H}_4$. Note that for a n-qubit state the number of coefficients increases to 2^n.

If we are performing measurements on a two-qubit state, we are perturbing the qubit state. The results A and B with the respective probabilities $P(A)$ and $P(B)$ of successive measurements of the first and second qubit are summarized in Table E.1.

We can now consider special states. If we assume for example that two of the four coefficients are zero, we obtain the following results for the measurement of the first (A) and the second qubit (B):

	$c_{00}=0$	$c_{01}=0$	$c_{10}=0$	$c_{11}=0$
$c_{00}=0$	—	$A\equiv 1$	$B\equiv 1$	$B\equiv 1-A$
$c_{01}=0$	$A\equiv 1$	—	$B\equiv A$	$B\equiv 0$
$c_{10}=0$	$B\equiv 1$	$B\equiv A$	—	$A\equiv 0$
$c_{11}=0$	$B\equiv 1-A$	$B\equiv 0$	$A\equiv 0$	—

	measurement of 1. qubit			measurement of 2. qubit	
A	$P(A)$	remaining state	B	$P(B)$	remaining state
0	$\|c_{00}\|^2+\|c_{01}\|^2$	$\|\Psi'\rangle = \frac{c_{00}\|00\rangle+c_{01}\|01\rangle}{\sqrt{\|c_{00}\|^2+\|c_{01}\|^2}}$	0	$\frac{\|c_{00}\|^2}{\|c_{00}\|^2+\|c_{01}\|^2}$	$\|00\rangle$
			1	$\frac{\|c_{01}\|^2}{\|c_{00}\|^2+\|c_{01}\|^2}$	$\|01\rangle$
1	$\|c_{10}\|^2+\|c_{11}\|^2$	$\|\Psi'\rangle = \frac{c_{10}\|10\rangle+c_{11}\|11\rangle}{\sqrt{\|c_{10}\|^2+\|c_{11}\|^2}}$	0	$\frac{\|c_{10}\|^2}{\|c_{10}\|^2+\|c_{11}\|^2}$	$\|10\rangle$
			1	$\frac{\|c_{11}\|^2}{\|c_{10}\|^2+\|c_{11}\|^2}$	$\|11\rangle$

Table E.1: Successive measurements on a two-qubit state showing the results A and B with the corresponding probabilities $P(A)$ and $P(B)$ and the remaining state after the measurement.

These results directly follow from Table E.1. If for example $c_{00} = c_{01} = 0$, the probability for the measurement result $A = 0$ is $P(A) = |c_{00}|^2 + |c_{01}|^2 = 0$. That is, that in a measurement we obtain always the result $A = 1$.

II Entanglement

Entanglement is a new kind of correlations between two subsystems of a quantum system, which does not exist in classical physics (or classical probability). The term is a translation of the German ***"Verschränktheit"***, coined by **Erwin Schrödinger** in 1935.[1] Both notations reflect well the efforts of understanding such correlations in classical terms. However, from the point of view of quantum theory such correlations are rather straightforward and, in fact, ubiquitous.

Some correlations between quantum systems can be understood completely in classical terms: Suppose that two subsystems are prepared by two independent devices, whose operation may depend on the output of some classical random generator, which they both receive. In this case the source of the correlations is simply the classical random generator, and states produced in this way are called "classically correlated" or "separable". The density operator of such a state is a convex combination of tensor products of density operators. All other states are called "entangled". A simple example is a pure state, which happens not to be a product state. Since a pure state cannot be non-trivially decomposed into a convex combination of any other states, it also cannot be decomposed into products states, so it is not classically correlated. The fact that entangled states are not some bizarre but expendable feature of quantum mechanics but lead to observable effects, is shown most directly by Bell's inequality. It is easy to show that these inequalities are satisfied by every classically correlated state, but they have been found violated in a series of now

Figure E.3: Entanglement – an artist's view.

[1] E. Schrödinger, *Die gegenwärtige Situation in der Quantenmechanik*, Die Naturwissenschaften **23** 807-812, 823-828, and 844-849 (1935).

famous experiments.[2] Hence, these experiments directly confirm the existence of entangled states.

In the theory of Quantum Information entanglement is viewed as a resource needed to perform otherwise impossible tasks of information processing or computation. There is a variety of tasks for which entanglement plays an important role and, correspondingly, a variety of quantitative measures of entanglement. For pure states most of these reduce to the von Neumann entropy of the restricted density operators. This is a quantitative version of a crucial special feature of quantum mechanics, namely that pure states of composite systems may be mixed when restricted to a subsystem, as measured by the von Neumann entropy.

For mixed states there are many quantitative notions of entanglement, some of which are provably different. Probably only a few such quantities will turn out to be useful as the theory develops. But it is much too early to say which the interesting ones are.

As an example we consider the situations where the results A and B of a measurement on a two-qubit state are correlated. This is for example the case for the following normalized two-qubit states, which we obtain for $c_{01} = c_{10} = 0$ and $c_{00} = c_{11} = 0$:

$$\frac{c_{00}(t)|00\rangle + c_{11}(t)|11\rangle}{\sqrt{|c_{00}|^2 + |c_{11}|^2}} \quad \text{or} \quad \frac{c_{01}(t)|01\rangle + c_{10}(t)|10\rangle}{\sqrt{|c_{01}|^2 + |c_{10}|^2}} \quad . \tag{II.17}$$

It is obvious that by measuring the quantum state of the first qubit of these states we also fix the quantum state of the second qubit. Such states are called ***Bell states*** or ***Einstein-Podolsky-Rosen (EPR) pairs***.[3] They represent ***entangled states***.

In order to discuss entanglement a little bit more, we consider two quantum systems (such as two photon or two spins). If these two systems are not coupled, the wavefunction of the total systems is just given by the product of the two wavefunctions of the subsystems:

$$|0\rangle \cdot |1\rangle \;=\; |10\rangle \quad \text{or} \quad |1\rangle \cdot |0\rangle \;=\; |01\rangle \ . \tag{II.18}$$

If there is a finite interaction between the subsystems, we obtain a coupling which is causing linear combinations of the wavefunction in (E.II.18). A well known example is

$$|\Psi\rangle \;=\; \frac{1}{\sqrt{2}} \left(|01\rangle - |10\rangle \right) \ , \tag{II.19}$$

which corresponds to a spin singlet state for a spin system. Such linear combination of the product states is called ***entanglement***. The EPR pairs discussed above represent entangled states. An important mathematical property of entangled states is the fact that they cannot be expressed as a product of the basis states. The important physical property of entangled states is the fact that the measurement of the one-qubit state is fixing the measurement result of the other. We will discuss in section III how we can produce entangled states by one- and two-qubit operations.

[2] A. Aspect, P. Grangier, G. Roger, *Experimental tests of realistic local theories via Bell's Theorem*, Phys. Rev. Lett. **47**, 460-463 (1981); see also Phys. Rev. Lett. **49**, 91-94 (1982); Phys. Rev. Lett. **49**, 1804-1807 (1982).

[3] A. Einstein, B. Podolsky, N. Rosen, *Can quantum mechanical description of physical reality be complete?*, Phys. Rev. **47** (1935).

III Qubit Operations

III.1 Unitarity

If we discuss possible manipulations of the qubit state we have to take into account the normalization condition (E.I.16). That is, during the time evolution of the qubit states we have to satisfy the normalization of the state. With the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\,|\Psi\rangle \;=\; \mathscr{H}\,|\Psi\rangle \tag{III.20}$$

the time evolution of the state can be expressed as

$$|\Psi(t)\rangle \;=\; \exp\left(-\frac{\imath}{\hbar}\mathscr{H}t\right)\,|\Psi(0)\rangle \;=\; \mathscr{U}(t)\,|\Psi(0)\rangle\ . \tag{III.21}$$

Since we have to preserve normalization, we obtain

$$\langle\Psi(t)|\Psi(t)\rangle \;=\; \langle\Psi(0)|\mathscr{U}^\dagger(t)\mathscr{U}(t)|\Psi(0)\rangle \;=\; 1\ . \tag{III.22}$$

That is, we obtain the unitary condition

$$\mathscr{U}^\dagger(t)\mathscr{U}(t) \;=\; 1 \quad\rightarrow\quad \mathscr{U}^\dagger \;=\; \mathscr{U}^{-1}\ . \tag{III.23}$$

We see that qubit operation in general have to be achieved with $n\times n$ unitary matrices with unit determinant. These matrices are forming the $\mathrm{SU}(n)$ group. For a single-qubit we have to deal with the 2×2 matrices of the $\mathrm{SU}(2)$ group.[4]

III.2 Single Qubit Operations

We use the spin-1/2 model system to discuss single-qubit operations. With the control fields B_z and B_x, which may be time dependent, the qubit Hamiltonian can be written in the spin-1/2 notation as

$$\mathscr{H} \;=\; -\mathscr{H}_z\,\mathbf{Z}-\mathscr{H}_x\,\mathbf{X} \;=\; -\frac{\hbar}{2}\gamma B_z\,\mathbf{Z}-\frac{\hbar}{2}\gamma B_x\,\mathbf{X}\ , \tag{III.24}$$

where γ is the gyromagnetic ratio and the Pauli matrices in the space states $|\uparrow\rangle$ and $\downarrow\rangle$ are given by

$$\vec{\sigma} \;=\; \{\mathbf{X},\mathbf{Y},\mathbf{Z}\} \;=\; \left\{\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix},\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}\right\}\ . \tag{III.25}$$

It is evident that the field B_z results in an energy splitting of the basis states $|\uparrow\rangle$ and $\downarrow\rangle$ proportional to the applied magnetic field but does not mix these states. The magnetic field B_x in contrast results in a mixing of the basis state

A single-qubit operation can be performed, for example, by turning on the control field $B_x(t)$ for a time interval τ. As a result of this operation the quantum state evolves according to the unitary transformation

$$\mathscr{U}_x(\theta) \;=\; \exp\left(\frac{\imath\gamma B_x\tau}{2}\,\mathbf{X}\right) \;=\; \begin{pmatrix} \cos\frac{\theta}{2} & \imath\sin\frac{\theta}{2}\\ \imath\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix} \;=\; \cos\frac{\theta}{2}\,\mathbf{1}+\imath\sin\frac{\theta}{2}\,\mathbf{X}\ , \tag{III.26}$$

[4]Note that unitarity is the Hilbert space equivalent of rotation matrix orthogonality (isomorphism $\mathrm{SU}(2)\leftrightarrow\mathrm{SO}(3)$.

where $\theta \equiv \gamma B_x \tau = \omega_x \tau$. For example, by proper choice of the time span τ we can achieve $\theta = \pi$ or $\theta = \pi/2$. This produces a spin flip (NOT operation) or an equal weight superposition of the spin states, respectively.

Switching on $B_z(t)$ for a time interval τ produces another basic single bit operation, namely a phase shift between $|\uparrow\rangle$ and $\downarrow\rangle$. The unitary operation reads as

$$\mathscr{U}_z(\varphi) \quad = \quad \exp\left(\frac{\imath\gamma B_z \tau}{2}\,\mathbf{Z}\right) = \begin{pmatrix} e^{\imath\varphi/2} & 0 \\ 0 & e^{-\imath\varphi/2} \end{pmatrix} , \tag{III.27}$$

where $\varphi \equiv \gamma B_z \tau = \omega_z \tau$. Note that with a sequence of these x- and z-rotations any unitary transformation of the qubit state can be achieved, that is, every position on the Bloch sphere can be accessed. There is no need to turn on B_y.

III.3 Two Qubit Operations

A two-qubit operation on two qubits i and j is induced by switching on a coupling $J^{ij}(t)$ for a time interval τ. According to (9.2.3) the coupling term can be expressed as

$$\mathscr{H}(t) \quad = \quad \sum_{i \neq j} J^{ij}_{\alpha\beta}(t)\, \vec{\sigma}^{\,i}_{\alpha} \vec{\sigma}^{\,j}_{\beta} , \tag{III.28}$$

where the summation over the state (e.g. spin) indices α, β is implied. As an example we discuss the **XY** coupling of two spins: [5]

$$\mathscr{H}(t) \quad = \quad J^{ij}(t)\, \vec{\sigma}_{\alpha} \vec{\sigma}_{\beta} \;=\; J^{ij}(\mathbf{XX}+\mathbf{YY}) \;=\; 2J^{ij} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{III.29}$$

In the basis $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\uparrow\rangle$ the result is described by the unitary operator

$$\mathscr{U}^{ij}(\gamma) \quad = \quad \exp\left(\frac{\imath 2 J^{ij}\tau}{\hbar}(\mathbf{XX}+\mathbf{YY})\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\delta & \imath\sin\delta & 0 \\ 0 & \imath\sin\delta & \cos\delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{III.30}$$

where $\delta = 2J^{ij}\tau/\hbar = 2\omega_{ij}\tau$. For $\delta = \pi/2$ the operation leads to a swap (exchange) of the states $|\uparrow\downarrow\rangle$ ($|10\rangle$) and $|\downarrow\uparrow\rangle$ ($|01\rangle$) and an additional multiplication by $\imath$. In contrast, for $\delta = \pi/4$ the operation

[5] Note that for a **YY** coupling we obtain

$$\mathscr{H}(t) = J^{ij}\mathbf{YY} = J^{ij} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and for a **ZZ** coupling

$$\mathscr{H}(t) = J^{ij}\mathbf{ZZ} = J^{ij} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

transforms the state $|\uparrow\downarrow\rangle$ into an entangled state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + \imath|\downarrow\uparrow\rangle)$ (this is equivalent to the $\sqrt{\imath\mathrm{SWAP}}$ gate, see IV).

It is evident that the qubit operations must be realized by unitary operators ($UU^\dagger = U^\dagger U = 1$). First, the normalization condition must be valid for the qubit state after the operation. Therefore, the absolute value of the determinant of the matrix must be unity. In this way we rotate the qubit vector on the Bloch sphere without changing its length. Second, the operation must be reversible, that is the matrix must be invertable. Note that classical computation is not reversible since heat is dissipated during the operations thereby making the computation thermodynamically irreversible. This is not possible for quantum computers, since the superposition of the quantum states must be maintained during the whole computational process. If heat would be dissipated in an uncontrolled way, the coherence of the quantum state would be lost.

We note that so far we only considered the sudden switching of $B^i_{z,x}(t)$ or $J^{ij}(t)$. This is called an ***non-adiabatic process***. However, one can also use other techniques to implement single or two qubit operations. For example, one can induce Rabi oscillations between different states of a qubit or a qubit pair by ac resonance signals. Furthermore, one can perform adiabatic manipulations of the qubits Hamiltonian to exchange different eigenstates with the occupations remaining unchanged.

IV Quantum Logic Gates

In the previous subsection we have shown how we can use unitary operators to realize manipulations of one- and two-qubit states. The details of the physical realization of an unitary operation such as the application of a magnetic field pulse or the way how one couples two qubits of course depend on the specific model system that is considered (e.g. spin, superconducting phase qubit, etc.). Quantum information theory, on the other hand, discusses quantum computation in a treatment that is independent of the physical system used to implement quantum computation. Here, the quantum algorithms are built out of standard single- and two-qubit gates. In the following we will discuss a several of them. In order to implement a quantum algorithm on a physical system we have to know how to express these standard gates in terms of the unitary operations specific to a physical system.

IV.1 Single-Bit Gates

We first consider ***classical single-bit gates***. As shown in Fig. E.4, a single-bit gate acting on the binary variable x is transferring this variable to the $f(x)$, which is again a binary variable:

$$x \quad \rightarrow \quad f(x) \quad \text{with} \quad x \in \{0,1\} \quad \text{and} \quad f(x) \in \{0,1\} \ . \tag{IV.31}$$

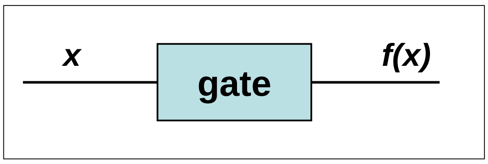

Figure E.4: The classical-single bit gate.

Prominent examples for the classical single-bit gate are the NOT, IDENTITY or RESET gates:

$$\begin{aligned} &\mathrm{NOT} & f(x) &= \mathrm{NOT}(x) = 1 - x & \text{(IV.32)}\\ &\mathrm{IDENTITY} & f(x) &= \mathrm{IDENTITY}(x) = x & \text{(IV.33)}\\ &\mathrm{RESET} & f(x) &= \mathrm{RESET}(x) = 0 \ . & \text{(IV.34)} \end{aligned}$$

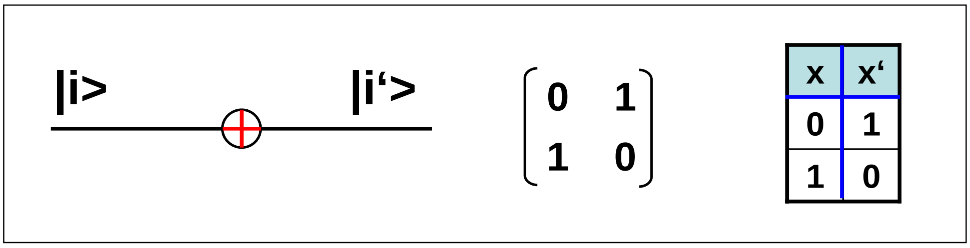

Figure E.5: The quantum NOT gate with the corresponding unitary matrix and the truth table.

Discussing the ***quantum realization of single-bit gates*** we have to use the unitary transformations discussed in the previous subsection:

Rotations about the x-axis are obtained by

$$\mathcal{U}_x(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & \imath\sin\frac{\theta}{2} \\ \imath\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} = \cos\frac{\theta}{2}\,\mathbf{1} + \imath\sin\frac{\theta}{2}\,\mathbf{X} \; . \tag{IV.35}$$

We see that a rotation about the x-axis by an arbitrary angle θ interpolates between the classical gates IDENTITY and NOT. The quantum NOT gate (see Fig. E.5)

$$\mathrm{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbf{X} = \mathrm{e}^{-\imath\frac{\pi}{2}}\mathcal{U}_x(\pi) \tag{IV.36}$$

permutes the basis vectors $|0\rangle \longrightarrow |1\rangle$ and $|1\rangle \longrightarrow |0\rangle$. We see that it can be realized (up to an unimportant overall phase factor) by the unitary operation (x-rotation) of (E.III.26) with a properly chosen time interval τ resulting in $\theta \equiv B_x\tau/\hbar = \pi$: $\mathcal{U}_x(\theta = \pi) = \imath \cdot \mathrm{NOT}$.

In contrast to classical computation in quantum logic there is a logic gate called $\sqrt{\mathrm{NOT}}$, that when applied twice produces the NOT gate:

$$\sqrt{\mathrm{NOT}} = \frac{1}{\sqrt{2\imath}}\begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+\imath & -1+\imath \\ -1+\imath & 1+\imath \end{pmatrix} = \sqrt{\mathbf{X}} = \mathrm{e}^{-\imath\frac{\pi}{4}}\mathcal{U}_x\left(\frac{\pi}{2}\right) \tag{IV.37}$$

This gate is also obtained by the unitary operation (x-rotation) of (E.III.26) with $\theta \equiv B_x\tau/\hbar = \pi/2$, more precisely $\mathcal{U}_x(\theta = \pi/2) = \sqrt{\imath} \cdot \sqrt{\mathrm{NOT}}$.

Rotations of a one-qubit state about the z-axis are obtained by the unitary operation (compare (E.III.27))

$$\mathcal{U}_z(\varphi) = \begin{pmatrix} \mathrm{e}^{\imath\varphi/2} & 0 \\ 0 & \mathrm{e}^{-\imath\varphi/2} \end{pmatrix} = \mathrm{e}^{\imath\frac{\varphi}{2}}\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-\imath\varphi} \end{pmatrix} \; . \tag{IV.38}$$

The action on a qubit results in a relative phase shift φ

$$\mathcal{U}_z(\varphi)|0\rangle = |0\rangle \tag{IV.39}$$
$$\mathcal{U}_z(\varphi)|1\rangle = \mathrm{e}^{-\imath\varphi}|1\rangle \; . \tag{IV.40}$$

Special cases are the **Z** gate

$$\mathbf{Z} = e^{-\imath\frac{\pi}{2}}\mathcal{U}_z(\pi) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{IV.41}$$

the **S** gate

$$\mathbf{S} = \sqrt{\mathbf{Z}} = \begin{pmatrix} 1 & 0 \\ 0 & \imath \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\imath\frac{\pi}{2}} \end{pmatrix} , \tag{IV.42}$$

and the **T** gate

$$\mathbf{T} = \sqrt{\mathbf{S}} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\imath\frac{\pi}{4}} \end{pmatrix} . \tag{IV.43}$$

The Hadamard gate is another important, essentially quantum mechanical, single bit gate defined as

$$\mathrm{H} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{\mathbf{X}+\mathbf{Z}}{\sqrt{2}} . \tag{IV.44}$$

This gate, which is composed of a y- and z-rotation transforms the basis vectors into superpositions:

$$\mathrm{H}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle \quad \text{and} \quad \mathrm{H}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle = |-\rangle) . \tag{IV.45}$$

The Hadamard gate is used to prepare a specific initial state. When applied to the ground state $|00\ldots0\rangle$, it provides an equally weighted superposition of all basis states:

$$\mathrm{H}\otimes\cdots\otimes\mathrm{H}|0\ldots0\rangle = \frac{1}{2^{N/2}}\sum_{a_1\ldots a_N=0,1}|a_1\cdots a_N\rangle . \tag{IV.46}$$

The terms in the sum can be viewed as binary representations of all integers from 0 up to 2^N-1. Therefore, the state (E.IV.46) represents a superposition of all these integers. When this state is used as an input state for a quantum algorithm, it represents 2^N classical inputs. Due to the linearity of the quantum time evolution these inputs are processed simultaneously and the output is a superposition of 2^N classical results. This massive ***quantum parallelism*** is a key property of quantum computation and is responsible for the exponential speedup of certain quantum algorithms.

IV.2 Two Bit Gates

We again first consider ***classical two-bit gates***. As shown in Fig. E.6, a two-bit gate acting on the binary variable (x,y) is transferring this variable to $f(x,y)$, which is again a binary variable:

$$(x,y) \rightarrow f(x,y) \quad \text{with} \quad (x,y) \in \{0,1\} \quad \text{and} \quad f(x,y) \in \{0,1\} . \tag{IV.47}$$

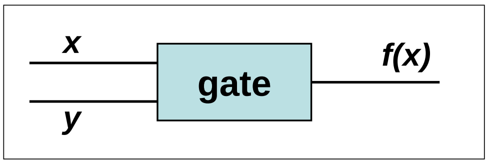

Figure E.6: The classical two-bit gate.

Prominent examples for classical two-bit gates are

$$
\begin{aligned}
f(x,y) &= (x \,\text{AND}\, y) = xy &\text{(IV.48)}\\
f(x,y) &= (x \,\text{NAND}\, y) = \text{NOT}(x \,\text{AND}\, y) = 1 - xy &\text{(IV.49)}\\
f(x,y) &= (x \,\text{OR}\, y) = x + y - xy &\text{(IV.50)}\\
f(x,y) &= (x \,\text{NOR}\, y) = \text{NOT}(x \,\text{OR}\, y) = 1 + xy - x - y &\text{(IV.51)}\\
f(x,y) &= (x \,\text{EQUIV}\, y) = \delta_{xy} &\text{(IV.52)}\\
f(x,y) &= (x \,\text{XOR}\, y) = x \oplus y = \text{NOT}(x \,\text{EQUIV}\, y) = 1 - \delta_{xy}\ . &\text{(IV.53)}
\end{aligned}
$$

The truth table of these operations is given by

(x,y)	AND	NAND	OR	NOR	EQUIV	XOR
0 0	0	1	0	1	1	0
0 1	0	1	1	0	0	1
1 0	0	1	1	0	0	1
1 1	1	0	1	0	1	0

It is evident from Fig. E.6 and the truth table that the two-bit gates discussed so far are ***irreversible gates***. The difference between ***reversible*** and ***irreversible*** gates is shown in Fig. E.7. After the operation we can no longer reverse the operation to determine the input states. That is, information is lost what is resulting in an increase of entropy by $\Delta S = k_B \ln 2$ (**Leo Szillard**, 1929). It can further be shown that not all of the logical gates are required. Only a small ***universal set of gates*** is necessary to construct all other gates. It can be shown that the NAND gate is sufficient to produce all other gates.

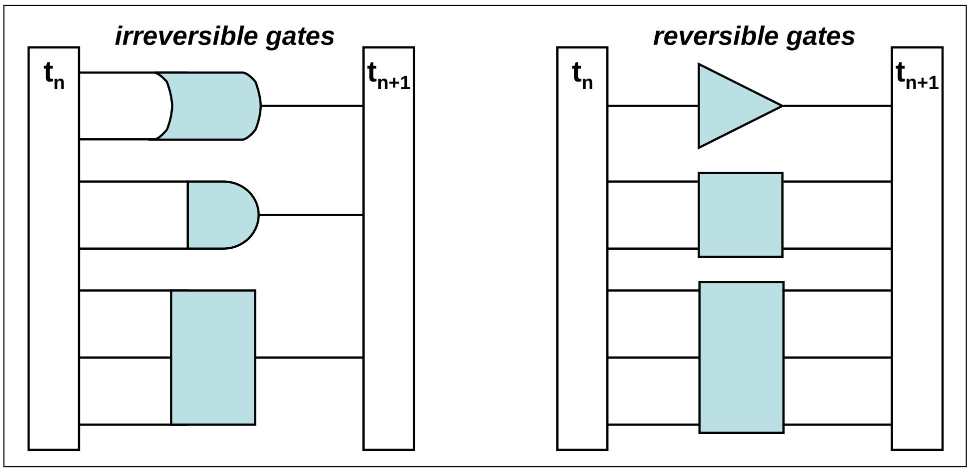

Figure E.7: Irreversible (left) and reversible (right) gates.

In the 1970ies a ***reversible classical logic*** has been established (**Bennett**, 1973). The structure of a classical reversible gate is shown in Fig. E.8. Reversibility is achieved by storage of the input bit x. A typical example is the controlled NOT (CNOT) gate corresponding to a reversible exclusive OR (XOR) gate as shown in Fig. E.9a. The operation of the CNOT gate is defined as:

$$(x,y) \quad \rightarrow \quad \mathrm{CNOT}(x,y) \;=\; (x,\, x \oplus y) \;=\; (x,\, 1-\delta_{xy}) \; . \tag{IV.54}$$

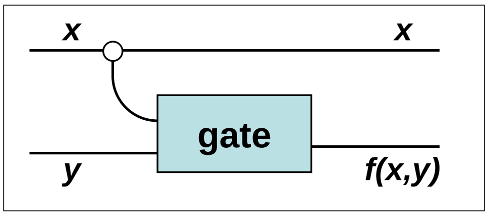

Figure E.8: Reversible classical logic gate.

A further important reversible gate is the SWAP gate that interchanges x and y (see Fig. E.9b):[6]

$$\begin{aligned} (x,y) &\rightarrow (x,\, x\oplus y) \\ (x,x\oplus y) &\rightarrow (x\oplus(x\oplus y), x\oplus y) \;=\; (y, x\oplus y) \\ (y,x\oplus y) &\rightarrow (y,(x\oplus y)\oplus y) \;=\; (y,x) \; . \end{aligned} \tag{IV.55}$$

We next have to discuss ***two-bit quantum gates***. In the same way as for classical two-bit gates there exists a ***universal set of two-qubit quantum gates*** that is required to construct all other gates. It can be shown that the CNOT gate together with the one-qubit rotations **X**, **Y**, **Z**, **S**, **T**, ... discussed above are sufficient to produce all other gates. That is, for the implementation of a quantum computer we only have to realize the CNOT gate and the single qubit rotations. With respect to two-bit quantum gates we therefore have to discuss mainly the CNOT gate.

Before describing the quantum CNOT gate we first introduce the more general controlled U (CU) gate shown in Fig. E.10:

$$\mathrm{CU}\,|ij\rangle \quad = \quad \mathrm{CU}\,|i\rangle \otimes |j\rangle \;=\; |i\rangle \otimes \{\delta_{i0}|j\rangle + \delta_{i1}\mathscr{U}\,|j\rangle\} \; . \tag{IV.56}$$

[6] Proof that $x \oplus (x \oplus y) \equiv y$:

(x,y)	$x\oplus y$	$x\oplus(x\oplus y)$	y
0 0	0	0	0
0 1	1	1	1
1 0	1	0	0
1 1	0	0	1

Proof that $(x \oplus y) \oplus y \equiv x$:

(x,y)	$x\oplus y$	$(x\oplus y)\oplus y$	x
0 0	0	0	0
0 1	1	0	0
1 0	1	1	1
1 1	0	1	1

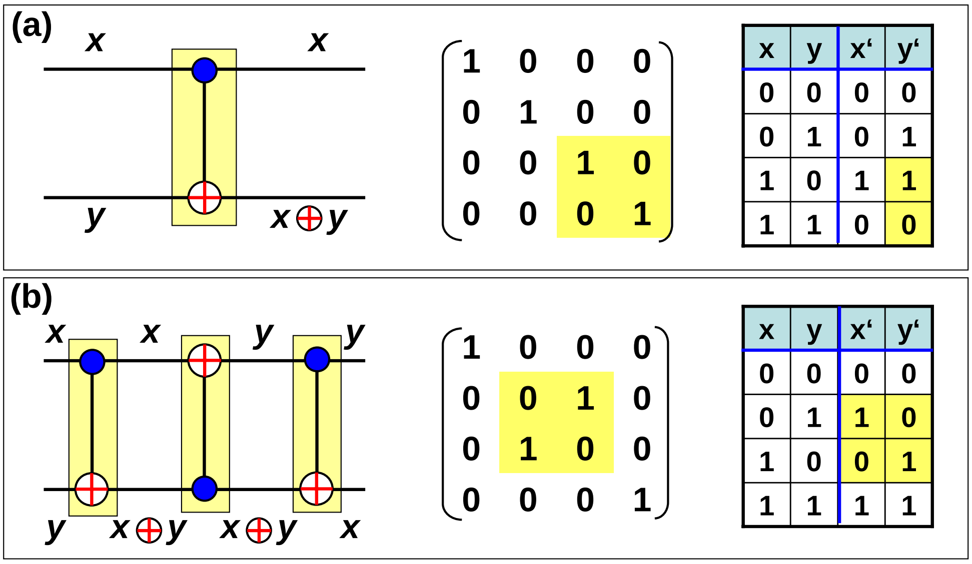

Figure E.9: The reversible XOR (CNOT) gate (a) and the SWAP gate (b) with the corresponding matrix and truth tables.

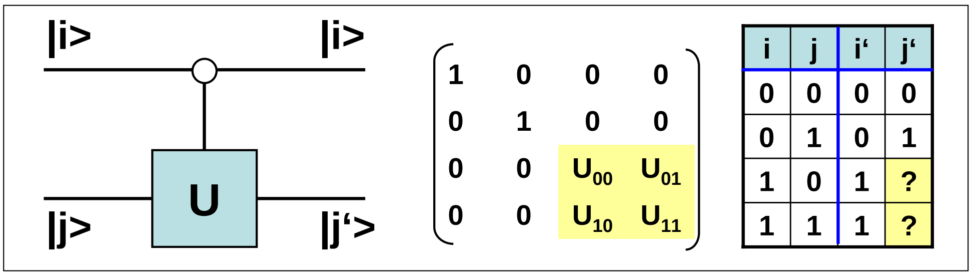

Figure E.10: The controlled U gate.

In a 4×4 matrix representations the CU gate can be expressed as

$$\mathscr{U} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathscr{U}_{00} & \mathscr{U}_{01} \\ 0 & 0 & \mathscr{U}_{10} & \mathscr{U}_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \mathscr{U} \end{pmatrix} . \tag{IV.57}$$

The controlled NOT gate represents a special case of the CU gate with $\mathscr{U}$ being the quantum NOT or $\mathbf{X}$ gate.[7] With (E.IV.36) we obtain

[7] The CNOT gate is therefore also called the CX gate.
Claim:

$$\text{CNOT}\,|ij\rangle \;=\; \text{CNOT}\,|i\rangle \otimes |j\rangle \;=\; |i\rangle \otimes \{\delta_{i0}|j\rangle + \delta_{i1}\text{NOT}\,|j\rangle\} \;.$$

Proof:

$$\begin{aligned} \text{CNOT}\,|i\rangle \otimes |j\rangle \;&=\; |i\rangle \otimes |1-\delta_{ij} \\ &=\; |0\rangle \otimes \{\delta_{00}|j\rangle + \delta_{01}\text{NOT}|j\rangle\} \;=\; |0\rangle \otimes |j\rangle \qquad \text{for } i = 0 \\ &=\; |1\rangle \otimes \{\delta_{10}|j\rangle + \delta_{11}\text{NOT}|j\rangle\} \;=\; |1\rangle \otimes \text{NOT}|j\rangle \qquad \text{for } i = 1 \;. \end{aligned}$$

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{X} \end{pmatrix} . \tag{IV.58}$$

The CNOT gate flips the second qubits only if the first qubits is in the $|1\rangle$ state.

A further example is the ***controlled phase gate***

$$\text{C} - \varphi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-\imath\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-\imath\varphi}\mathcal{U}_z(\varphi) \end{pmatrix} , \tag{IV.59}$$

which shifts the phase of state $|1\rangle$ of the second qubit when the first qubit is in the state $|1\rangle$.

As a further important two bit quantum gate we discuss the SWAP gate, which is produced by three CNOT gates in series:

$$\text{SWAP} = \text{CNOT}_{12} \cdot \text{CNOT}_{21} \cdot \text{CNOT}_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{IV.60}$$

and the $\sqrt{\imath\text{SWAP}}$ gate, "square root of the complex SWAP operation"

$$\sqrt{\imath\text{SWAP}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{\imath}{\sqrt{2}} & 0 \\ 0 & \frac{\imath}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{IV.61}$$

The result of the SWAP operation is

$$\text{SWAP}|i,j\rangle = |j,i\rangle . \tag{IV.62}$$

That is, the SWAP operation results in an exchange of the two input qubits $|i\rangle$ and $|j\rangle$. In contrast, the $\sqrt{\imath\text{SWAP}}$ operation transforms the state $|10\rangle$ into an entangled state $\frac{1}{\sqrt{2}}(|10\rangle + \imath|01\rangle)$.

The two-qubit gates can be realized by two-qubit operations as described in section III. We only consider the Hadamard and the CNOT gate. The Hadamard gate can be performed up to an overall phase factor as a sequence of the elementary operations $\mathcal{U}_x$ and $\mathcal{U}_z$

$$\text{H} \propto \mathcal{U}_x(\theta = \frac{\pi}{4})\, \mathcal{U}_z(\varphi = \frac{\pi}{4})\, \mathcal{U}_x(\theta = \frac{\pi}{4}) . \tag{IV.63}$$

However, it also can be performed faster by simultaneous switching of B_x and B_z (compare (E.III.26), (E.III.27) and (E.IV.44)):

$$\mathrm{H} \;\propto\; \exp\left(-\imath\frac{\pi}{2}\,\frac{\mathbf{X}+\mathbf{Z}}{\sqrt{2}}\right) \ . \tag{IV.64}$$

The CNOT operation can be implemented by a combination of two-qubit gates $\mathscr{U}^{ij}$ (see (E.III.30)) and several single-qubit gates:

$$\begin{aligned}\mathrm{CNOT} \;\propto\;& \mathscr{U}_x^2\left(\frac{\pi}{2}\right)\mathscr{U}_z^2\left(-\frac{\pi}{2}\right)\mathscr{U}_x^2(-\pi)\,\mathscr{U}^{ij}\left(-\frac{\pi}{2}\right)\\ &\times\mathscr{U}_x^1\left(-\frac{\pi}{2}\right)\mathscr{U}^{ij}\left(\frac{\pi}{2}\right)\mathscr{U}_z^1\left(-\frac{\pi}{2}\right)\mathscr{U}_z^2\left(-\frac{\pi}{2}\right) \ .\end{aligned} \tag{IV.65}$$

We see that it takes quite a number of elementary gates to perform the CNOT operation and optimization is required.

V The No-Cloning Theorem

The term ***cloning*** in the quantum context, coined in the short paper by **Wooters** and **Zurek**,[8] reflects rather well the idea that there is a blueprint for quantum systems from which all its properties could be derived. However, the existence of a ***Quantum Copier***, which would take one quantum system as input and produce two systems of the same kind, both of them indistinguishable from the input, is ruled out by the ***no-cloning theorem***. So far, the ***no-cloning theorem*** has been stated only in a rather weak form, forbidding only ***exact*** cloning. Stronger forms give more detailed information: there is a finite error necessarily made by any putative cloner, and explicit bounds can be placed on this error.

Note that in classical systems cloning is easily possible. A special property of the classical CNOT operation is the fact that is can be used to copy bits:

$$\mathrm{SWAP}(x,0) \;=\; (x,x) \ . \tag{V.66}$$

We can now try to use the quantum CNOT gate to make a copy of the single qubit state

$$|\Psi\rangle \;=\; a|0\rangle + b|1\rangle \ . \tag{V.67}$$

With the two-qubit input

$$|\Psi,0\rangle \;=\; |\Psi\rangle\otimes|0\rangle \;=\; a|00\rangle + b|10\rangle \tag{V.68}$$

we obtain the following output after the quantum CNOT operation

$$\mathrm{CNOT}|\Psi,0\rangle \;=\; \mathrm{CNOT}(a|00\rangle + b|10\rangle) \;=\; a|00\rangle + b|11\rangle \ . \tag{V.69}$$

The copy of $|\Psi\rangle$ is however

$$|\Psi,\Psi\rangle \;=\; |\Psi\rangle\otimes|\Psi\rangle \;=\; a^2|00\rangle + b^2|11\rangle + ab|01\rangle + ab|10\rangle \ . \tag{V.70}$$

That is,

$$\mathrm{CNOT}|\Psi,0\rangle \;\neq\; |\Psi,\Psi\rangle \ . \tag{V.71}$$

This result is called the ***no-cloning theorem*** that says that an unknown quantum state cannot be copied.

[8] W.K. Wootters and W.H. Zurek, *A single quantum cannot be cloned*, Nature **299**, 802 (1982).

VI Quantum Complexity

We have learnt that the quantum state of an n-qubit system is a vector in the 2^n dimensional Hilbert space. As an example, the state $|01001110\rangle$ is a basis vector in the 2^8 dimensional Hilbert space. In order to transform arbitrary quantum state $|\Psi\rangle$ into the new state $|\Psi'\rangle$ a unitary transformation $\mathscr{U}$ is required:

$$|\Psi'\rangle \quad = \quad \mathscr{U}\,|\Psi\rangle \ , \tag{VI.72}$$

where $\mathscr{U}$ is a $2^n \times 2^n$ complex matrix. If we are dealing for example with 100 qubits, a $2^{100} \times 2^{100}$ complex matrix is required ($2^{100} \simeq 10^{30}$). This problem is called ***quantum complexity***.

VII The Density Matrix Representation

The density matrix allows the calculation of the expectation values of pure and mixed quantum states. The density matrix of a quantum state is defined as

$$\widehat{\rho} \quad = \quad |\Psi\rangle\langle\Psi| \ . \tag{VII.73}$$

For a simple single-qubit state $|\Psi\rangle = a|0\rangle + b|1\rangle$ we obtain

$$\begin{aligned} |\Psi\rangle\langle\Psi| &= (|\Psi\rangle = a|0\rangle + b|1\rangle) \otimes (a^\star\langle 0| + b^\star\langle 1|) \\ &= aa^\star|0\rangle\langle 0| + bb^\star|1\rangle\langle 1| + ab^\star|0\rangle\langle 1| + ba^\star|1\rangle\langle 0| \\ &= \mathbf{P}_{00} + bb^\star\mathbf{P}_{11} + ab^\star\mathbf{P}_{01} + ba^\star\mathbf{P}_{10} \end{aligned} \tag{VII.74}$$

with the fundamental projection 2D operators

$$\mathbf{P}_{00} \quad = \quad |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{\mathbf{1}+\mathbf{Z}}{2} \tag{VII.75}$$

$$\mathbf{P}_{11} \quad = \quad |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\mathbf{1}-\mathbf{Z}}{2} \tag{VII.76}$$

$$\mathbf{P}_{01} \quad = \quad |0\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{\mathbf{X}+\imath\mathbf{Y}}{2} \tag{VII.77}$$

$$\mathbf{P}_{10} \quad = \quad |1\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{\mathbf{X}-\imath\mathbf{Y}}{2} \ . \tag{VII.78}$$

Rewriting these equation we obtain the density matrix as

$$\widehat{\rho} \quad = \quad \frac{1}{2}(\mathbf{1} + \overrightarrow{v}\cdot\overrightarrow{\sigma}) \tag{VII.79}$$

with the Pauli matrices $\overrightarrow{\sigma}$ and

$$\overrightarrow{v} \quad = \quad \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} a^\star b + b^\star a \\ -\imath(a^\star b - b^\star a) \\ a^\star a - b^\star b \end{pmatrix} \tag{VII.80}$$

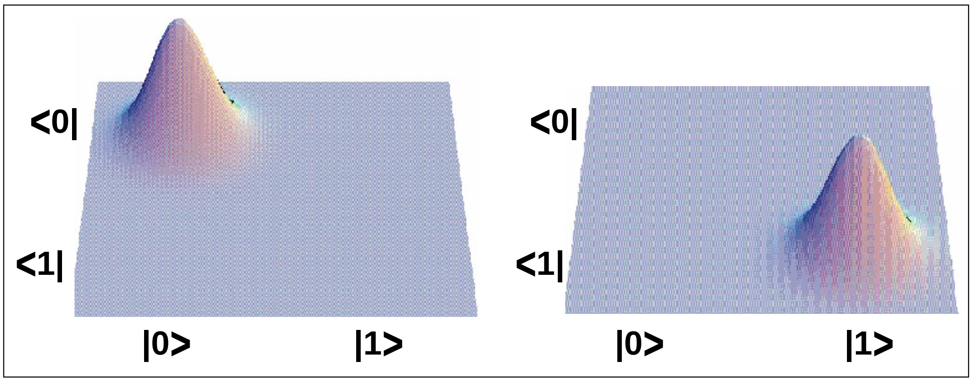

Figure E.11: Graphical representation of the density matrices $\widehat{\rho}_0$ (left) and $\widehat{\rho}_1$ (right) for the pure single qubit states.

For the density matrix of the pure single-qubit states we obtain

$$\widehat{\rho}_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{VII.81}$$

$$\widehat{\rho}_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} . \tag{VII.82}$$

The result is shown in Fig. E.11. We see that for the pure states there is a finite expectation value only for the respective state.

By applying the Hadamard gate we can generate a coherent superposition of the basis states (compare (E.IV.45))

$$\mathrm{H}|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right) = |+\rangle \quad \text{and} \quad \mathrm{H}|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle-|1\rangle = |-\rangle\right) . \tag{VII.83}$$

The corresponding density matrix is

$$\begin{aligned} \widehat{\rho}_+ &= \frac{1}{2}(|0\rangle+|1\rangle)\otimes(\langle 0|+\langle 1|) \\ &= \frac{1}{2}(|0\rangle\langle 0|+|0\rangle\langle 1|+|1\rangle\langle 0|+|1\rangle\langle 0|) \\ &= \frac{1}{2}\,\mathbf{1}+\mathbf{X} . \end{aligned} \tag{VII.84}$$

The result is shown in Fig. E.12. We see that for the coherent superposition of the states achieved by the application of the Hadamard gate we obtain the same expectation value for the four possible configurations, since the Hadamard gate provides an equally weighted superposition of all basis states.

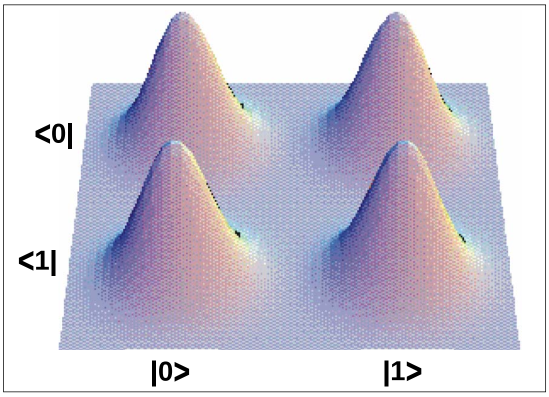

Figure E.12: Graphical representation of the density matrix $\widehat{\rho}_+$ for the coherent superposition of the basis states.