Applied Superconductivity:

Josephson Effect and Superconducting Electronics

Manuscript to the Lectures during WS 2003/2004, WS 2005/2006, WS 2006/2007, WS 2007/2008, WS 2008/2009, and WS 2009/2010

Prof. Dr. Rudolf Gross

and

Dr. Achim Marx

Walther-Meißner-Institut
Bayerische Akademie der Wissenschaften
and
Lehrstuhl für Technische Physik (E23)
Technische Universität München

Walther-Meißner-Strasse 8
D-85748 Garching
Rudolf.Gross@wmi.badw.de

Contents

List of Figures

List of Tables

Part I

Foundations of the Josephson Effect

Chapter 1

Macroscopic Quantum Phenomena

1.1 The Macroscopic Quantum Model

One of the main principles of quantum mechanics is the fact that physical quantities such as energy or momentum are, under certain conditions, quantized. That is, they can have only discrete values. However, for a long time it was believed that quantization is relevant only for microscopic objects such as nuclei, atoms or molecules. Indeed, if we are considering the behavior of macroscopic objects consisting of a large number of atoms, quantization effects cannot be observed, although every single atom obeys the laws of quantum mechanics. This is due to the fact that thermal motion masks quantum regularities. However, for a number of phenomena, in particular superconductivity, it has been found that it is possible to observe macroscopic quantization. That is, we can observe quantization of parameters that characterize macroscopic objects (for example the flux through a superconducting ring of macroscopic dimension) many orders of magnitude larger than microscopic objects like atoms. As will be discussed in the following, this is caused by the fact that the electron system in a superconductor is highly correlated due to coherence effects. Then we have to consider all superconducting electrons as a single quantum mechanical entity.

1.1.1 Coherent Phenomena in Superconductivity

Although superconductivity has been discovered already in 1911,[1] it took many decades until a modern concept for the superconducting state has been developed. The main milestones along the way towards a deeper understanding of superconductivity have been the discovery of the ***Meißner-Ochsenfeld effect***[2] by **Walther Meißner** and **Robert Ochsenfeld** in 1933, the development of the phenomenological theories by **Fritz** and **Heinz London**[3] and **V.L. Ginzburg** and **L.D. Landau**[4] and finally the creation of the microscopic BCS theory by **J. Bardeen, L.N. Cooper** and **J.R. Schrieffer**[5,6] with later important

[1] H. Kammerlingh Onnes, *The resistance of pure mercury at helium temperatures*, Communication from the Physical Laboratory at the University of Leiden, Nos. **120b**, **122b**, and **124c** (1911).

[2] Walther Meißner, Robert Ochsenfeld, *Ein neuer Effekt bei Eintritt der Supraleitfähigkeit*, Naturwissenschaften **21** (44), 787–788 (1933).

[3] F. London, H. London, *The Electromagnetic Equations of the Superconductor*, Proc. Roy. Soc. Lond. **A 149**, 71 (1935); see also F. London, *Superfluids*, Wiley, New York (1950).

[4] V.L. Ginzburg, L.D. Landau, *On the theory of superconductivity*, Zh. Eksp. Teor. Fiz. **20**, 1064 (1950).

[5] J. Bardeen, L.N. Cooper, and J.R. Schrieffer, *Microscopic Theory of Superconductivity*, Phys. Rev. **106**, 162–164 (1957).

[6] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Theory of Superconductivity*, Phys. Rev. **108**, 1175–1205 (1957).

contributions by **N.N. Bogolyubov**[7] and **L.P. Gor'kov**[8] as well as by **A.A. Abrikosov**[9].

A great deal of our knowledge on superconductivity can be obtained from phenomenological models (e.g. electrodynamics of superconductors) such as the ***London-Laue-theory*** developed by Fritz and Heinz London, but also by Max von Laue. However, these models have just been formulated to agree with the fundamental observations, namely perfect conductance, the Meißner-Ochsenfeld effect and the thermodynamic nature of the superconducting transition. That is, the models do not show us how these phenomena are related to each other. Historically, the theoretical predictions and experimental observation of coherent phenomena in superconductors such as flux quantization[10,11] have been proven to be the key for the final formulation of the macroscopic quantum concept of superconductivity. It was already realized in 1935 by **Fritz London** that the phenomenon of superconductivity cannot be understood in terms of classical concepts. By 1948 he was able to derive the ***London equations*** from more fundamental ideas, if the superelectron fluid was treated as a quantum mechanical entity. London made this development, since he realized that

> superconductivity is an inherently quantum phenomenon manifesting itself on a macroscopic scale.

We know that although quantum mechanics has replaced Newtonian mechanics as the appropriate physical theory, the classical laws are very good approximations on length scales much larger than atomic dimensions. Therefore, it is not evident on first sight why quantum mechanics is required to describe the properties of a macroscopic superconductor. However, superconductivity is like the coherent light emitted by a laser. There is no way to describe the phenomenon by the laws of classical physics alone. The reason for that is that superconductivity is a ***macroscopic quantum phenomenon*** and this is precisely the reason why we can observe the unusual quantum phenomena on a macroscopic scale.

The macroscopic quantum model of superconductivity is based on the hypothesis that there is a macroscopic wave function $\psi(\mathbf{r},t)$, which describes the behavior of the whole ensemble of superconducting electrons. Of course, this hypothesis can be justified by the microscopic theory of superconductivity (BCS-theory). This theory is based on the idea that in superconducting metals there is an attractive force between electrons near the Fermi level. At temperatures below the critical temperature T_c this attractive force creates a new quantum state differing from the Fermi sea of a normal metal. Roughly we can say that a small portion of the electrons close to the Fermi level are bound to Cooper pairs. In the simplest case, the internal motion of the pairs has no orbital angular momentum (symmetric s-state) and consequently Pauli's principle requires that the two spins must be in a singlet (antisymmetric) spin state. In contrast to binding of two atoms to a molecule, the orbital state of the pair has a much larger radius typically between 10 nm and 1 μm so that the individual pairs overlap strongly in space and consequently the binding turns cooperative. In particular, the binding energy of any pair depends on how many other pairs have condensed and, furthermore, the center of mass motion of the pairs is so strongly correlated that each pair resides in the same state with the same center of mass motion.[12] It is this state which we are describing by a macroscopic wave function and which gives the system its superfluid properties. For example, the center of mass motion can be described by the wave function

[7] N.N. Bogoliubov *A new method in the theory of superconductivity*, Zh. Eksp. Teor. Fiz. **34**, 58 (1958).

[8] see e.g. A.A. Abrikosov, L.P. Gor'kov, I.E. Dzyaloshinskii in *Quantum Field Theoretical Models in Statistical Physics*, Pergamon Press, London (1965).

[9] A.A. Abrikosov, Zh. Eksp. Teor. Fiz. **32**, 1141 (1957)

[10] R. Doll, M. Nähbauer, *Experimental Proof of Magnetic Flux Quantization in a Superconducting Ring*, Phys. Rev. Lett. **7**, 51 (1961)

[11] B.S. Deaver, W.M. Fairbank, *Experimental Evidence for Quantized Flux in Superconducting Cylinders*, Phys. Rev. Lett. **7**, 43 (1961).

[12] This is possible since the pairs represent bosons. We then can say that these bosons undergo a Bose-Einstein condensation, i.e. many pairs condense in the same quantum state like in the condensation which occurs for purely statistical reasons in an ideal Bose gas.

Walther Meißner (1882 -1974):

Walther Meißner was born on December 16, 1882 in Berlin.

He studied mechanical engineering at the Technische Hochschule Berlin Charlottenburg from 1901 - 1904 as well as mathematics and physics at the University of Berlin from 1904 - 1906. In 1907 he finished his Ph.D. in the group of Max Planck. Then he joined the National Bureau of Standards at Berlin. From 1922 - 1925 he set up a helium liquifier which was the third one worldwide. In 1933 he discovered the perfect diamagnetism in superconductors together with Robert Ochsenfeld. Today, this effect therefore is denoted as Meißner-Ochsenfeld effect.
In 1934 he was offered a full professor position at the Technische Hochschule München. After the second world war Walther Meißner was the first president of the Bavarian Academy of Sciences and founded in 1946 the Commission for Low Temperature Research. The laboratories of this commission first were at Herrsching close to Munich. In 1965, the new Central Institute for Low Temperature Research was build on the research campus at Garching. On the occasion of Walther-Meißner's 100. birthday this institute was renamed Walther-Meißner-Institute in 1982.

Walther Meißner died on November 15, 1974 in Munich.

$\psi(\mathbf{r},t) = \psi_0 \exp(\imath\theta(\mathbf{r},t)) = \psi_0 \exp(\imath \mathbf{k}_s \cdot \mathbf{r} - \imath\omega t)$ with every pair having the same momentum $\hbar\mathbf{k}_s$ or pair velocity $\mathbf{v}_s = \hbar\mathbf{k}/m^\star$.

We note that the macroscopic quantum model can not only be applied to charged superfluids but also to uncharged superfluids such as superfluid ^{4}He and ^{3}He, or Bose-Einstein condensates.[13,14] The development and understanding of the macroscopic quantum model of superconductivity requires the sound knowledge of quantum mechanics. Therefore, in the following we briefly review the most fundamental concepts of quantum mechanics that have direct relevance to superconductivity.

Schrödinger's Equation

In 1900 **Max Planck** introduced the concept of quantization to explain the radiation emitted by a black body at a given temperature. In order to explain the experimental observations he had to abandon the classical concept that radiation can be emitted in arbitrarily small quanta. Instead he had to postulate that the electromagnetic field can exchange energy only in certain discrete amounts. Planck did not feel comfortable with this postulation and considered it as a mathematical trick. However, **Albert Einstein** considered Planck's departure from classical physics as something far more fundamental. In 1905 he postulated that electromagnetic radiation has to be considered as a collection of particles known as ***photons***. A single photon of a known angular frequency ω represents the smallest amount of energy namely

$$\boxed{E \;=\; \hbar\omega \qquad \text{with} \qquad \hbar \;=\; 1.054\,571\,596(82)\times 10^{-34}\,\text{J s}\;.} \tag{1.1.1}$$

that can be radiated from a black body. That is, Planck has not merely postulated a mathematical trick but discovered a very fundamental characteristic of nature.

[13] D. Einzel, *Supraleitung und Suprafluidität*, in Lexikon der Physik, Spektrum Akademischer Verlag, Heidelberg, Berlin (2000).
[14] D. Einzel, *Superfluids*, Encyclopedia of Mathematical Physics (2005).

Later on, **Louis de Broglie**[15] realized in 1924 that, just by reasons of symmetry, it should be possible to describe classical particles as waves, in the same way as classical waves can be described as particles. In his doctoral thesis he introduced the concept of ***matter waves***, which was completely outside normal experience that time. Today, there is overwhelming experimental evidence for the ***wave-particle duality*** and we are used to this concept. The ***de Broglie relations*** linking the particle quantities energy E and momentum $\mathbf{p}$ to the wave quantities frequency ω and wave vector $\mathbf{k}$, resp. wavelength λ, are given by

$$\boxed{E \;=\; \hbar\omega \;,} \tag{1.1.2}$$

$$\boxed{\mathbf{p} \;=\; \hbar\,\mathbf{k} = \frac{h}{\lambda}\,\widehat{\mathbf{k}} \;.} \tag{1.1.3}$$

with $h = 2\pi\cdot\hbar = 6.6262\times 10^{-34}$Js and $\widehat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$. In both equations the constant of proportionality is related to the Planck's constant. It is due to the small value of Planck's constant that we usually do not notice quantum effects in our macroscopic world.

Based on these ideas we have to find an equation of motion for a quantum system playing the role of Newton's equation of motion for a classical system. Based on the existence of matter waves, **Erwin Schrödinger** initiated in 1926 the development of the wave mechanics by using the analogy to wave optics. According to Schrödinger a quantum particle can be described by the complex wave function $\Psi(\mathbf{r},t)$, where in analogy to wave optics the relations

$$\Psi(\mathbf{r},t) \;=\; A\;\exp[\imath(\mathbf{k}\cdot\mathbf{r}-\omega t)] \;=\; A\;\exp[\frac{\imath}{\hbar}(\mathbf{p}\cdot\mathbf{r}-Et)] \tag{1.1.4}$$

are valid.

We first consider the case of a free particle without any spin effects for which the potential energy $E_{\rm pot}=0$ and hence $E = E_{\rm kin}$. According to the analogy to wave optics it is natural to start with the wave equation

$$\nabla^2\Psi - \frac{1}{v_{\rm ph}^2}\frac{\partial^2\Psi}{\partial t^2} \;=\; 0 \tag{1.1.5}$$

for waves with the phase velocity $v_{\rm ph}$. For stationary problems, for which $\mathbf{p}$ and E do not vary with time, the wave function can be separated into two parts depending on space and time. Thus we can write

$$\Psi(\mathbf{r},t) \;=\; \Psi(\mathbf{r},0)\;\exp(-\imath\omega t)\;. \tag{1.1.6}$$

Using this Ansatz in the above wave equation and by using

$$k^2 = \frac{\omega^2}{v_{\rm ph}^2} \;=\; \frac{p^2}{\hbar^2} = \frac{2mE_{\rm kin}}{\hbar^2} = \frac{2m\omega}{\hbar} \tag{1.1.7}$$

we obtain the expression

$$\nabla^2\Psi = -k^2\Psi \;=\; -\frac{2mE_{\rm kin}}{\hbar^2}\,\Psi\;. \tag{1.1.8}$$

[15] **Louis Victor Prince de Broglie**, born on August 15, 1892 in Dieppe, died on March 19, 1987, Nobel Prize in Physics 1929.

In the more general case the particle can move in the potential V. If the potential is conservative, we can attribute each position a potential energy $E_{\rm pot}$ with the total energy $E = E_{\rm kin} + E_{\rm pot}$ being constant. With $E_{\rm kin} = E - E_{\rm pot}$ we obtain from (1.1.8) the ***stationary Schrödinger equation***

$$\boxed{\left(-\frac{\hbar^2}{2m}\nabla^2 + E_{\rm pot}\right)\Psi(\mathbf{r}) \quad = \quad E\Psi(\mathbf{r}) \ .} \tag{1.1.9}$$

By using the time derivative of (1.1.6) we obtain

$$\imath\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) \quad = E_{\rm kin}\,\Psi(\mathbf{r},t) \ . \tag{1.1.10}$$

Then, with (1.1.8) we obtain the time dependent equation

$$\boxed{\imath\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) \quad = \quad -\frac{\hbar^2}{2m}\nabla^2\,\Psi(\mathbf{r},t) \ .} \tag{1.1.11}$$

For non-stationary problems (i.e. $E_{\rm kin} = E_{\rm kin}(t)$ and $p = p(t)$) we can no longer express $\partial^2\Psi/\partial t^2$ by $-\omega^2\Psi$ and hence derive the wave equation for matter waves. Schrödinger postulated that even in the case of a time dependent potential energy the equation

$$\boxed{\imath\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) \quad = \quad \left(-\frac{\hbar^2}{2m}\nabla^2 + E_{\rm pot}(\mathbf{r},t)\right)\Psi(\mathbf{r},t)} \tag{1.1.12}$$

is valid. This general ***time dependent Schrödinger equation*** was noted by Schrödinger for the first time in 1926. Up to now this equation has been confirmed in a huge number of experiments and represents the basic equation of quantum mechanics. Often the Schrödinger equation is written as $\imath\hbar\frac{\partial\Psi}{\partial t} = \widehat{H}\,\Psi$ with $\widehat{H} \equiv -\frac{\hbar^2}{2m}\nabla^2 + E_{\rm pot}(\mathbf{r},t)$ being the ***Hamilton operator***.

Probability Currents

In the last section we have seen that the Schrödinger equation governs the evolution of the wave function Ψ in space and time. The wave function Ψ is somehow descriptive of the quantum system but its interpretation is not obvious. On the first sight we might consider it as a quantum field similar to the fields encountered in electromagnetism. However, this is not the case. It is evident that Ψ cannot be a real scalar function as a result of the factor i in the Schrödinger equation. Therefore, if Ψ is a scalar function, it must have both real ***and*** imaginary parts. For a plane wave this implies

$$\Psi(\mathbf{r},t) \quad = \quad \Psi_0\,\mathrm{e}^{\imath(\mathbf{k}\cdot\mathbf{r}-\omega t)} \ . \tag{1.1.13}$$

In contrast, electromagnetic fields always can be represented as the real ***or*** imaginary part of a complex expression:

$$\mathbf{E}(\mathbf{r},t) \quad = \quad \Re\left\{\mathbf{E}_0\,\mathrm{e}^{\imath(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right\} \ . \tag{1.1.14}$$

Erwin Schrödinger (1887 -1961), Nobel Price in Physics 1933:

Erwin Schrödinger was born on August 12, 1887, in Vienna. He was a highly gifted man with a broad education. After having finished his chemistry studies, he devoted himself for years to Italian painting. After this he took up botany, which resulted in a series of papers on plant phylogeny. Schrödinger's wide interests dated from his school years at the Gymnasium, where he not only had a liking for the scientific disciplines, but also appreciated the severe logic of ancient grammar and the beauty of German poetry. (What he abhorred was memorizing of data and learning from books.)
From 1906 to 1910 he was a student at the University of Vienna, during which time he came under the strong influence of Fritz Hasenöhrl, who was Boltzmann's successor. It was in these years that Schrödinger acquired a mastery of eigenvalue problems in the physics of continuous media, thus laying the foundation for his future great work. Hereafter, as assistant to Franz Exner, he, together with his friend K. W. F. Kohlrausch, conducted practical work for students (without himself, as he said, learning what experimenting was). During the First World War he served as an artillery officer.
In 1920 he took up an academic position as assistant to Max Wien, followed by positions at Stuttgart (extraordinary professor), Breslau (ordinary professor), and at the University of Zurich (replacing von Laue) where he settled for six years. In later years Schrödinger looked back to his Zurich period with great pleasure - it was here that he enjoyed so much the contact and friendship of many of his colleagues, among whom were Hermann Weyl and Peter Debye. It was also his most fruitful period, being actively engaged in a variety of subjects of theoretical physics. His papers at that time dealt with specific heats of solids, with problems of thermodynamics (he was greatly interested in Boltzmann's probability theory) and of atomic spectra; in addition, he indulged in physiological studies of colour (as a result of his contacts with Kohlrausch and Exner, and of Helmholtz's lectures). His great discovery, Schrödinger's wave equation, was made at the end of this epoch-during the first half of 1926.
It came as a result of his dissatisfaction with the quantum condition in Bohr's orbit theory and his belief that atomic spectra should really be determined by some kind of eigenvalue problem. For this work he shared with Dirac the Nobel Prize for 1933.
In 1927 Schrödinger moved to Berlin as Planck's successor. Germany's capital was then a center of great scientific activity and he enthusiastically took part in the weekly colloquies among colleagues, many of whom "exceeding him in age and reputation". With Hitler's coming to power (1933), however, Schrödinger decided he could not continue in Germany. He came to England and for a while held a fellowship at Oxford. In 1934 he was invited to lecture at Princeton University and was offered a permanent position there, but did not accept. In 1936 he was offered a position at University of Graz, which he accepted only after much deliberation and because his longing for his native country outweighed his caution. With the annexation of Austria in 1938, he was immediately in difficulty because his leaving Germany in 1933 was taken to be an unfriendly act. Soon afterwards he managed to escape to Italy, from where he proceeded to Oxford and then to University of Ghent. After a short stay he moved to the newly created Institute for Advanced Studies in Dublin, where he became Director of the School for Theoretical Physics. He remained in Dublin until his retirement in 1955.
All this time Schrödinger continued his research and published many papers on a variety of topics, including the problem of unifying gravitation and electromagnetism, which also absorbed Einstein and which is still unsolved; (he was also the author of the well-known little book “What is Life?”, 1944). He remained greatly interested in the foundations of atomic physics. Schrödinger disliked the generally accepted dual description in terms of waves and particles, with a statistical interpretation for the waves, and tried to set up a theory in terms of waves only. This led him into controversy with other leading physicists.
After his retirement he returned to an honored position in Vienna. He died on the 4th of January, 1961, after a long illness.

The fact that the quantum wave function is necessarily a complex quantity does not generate any problems with mathematics. However, recalling that Ψ_0 is just the amplitude of a plane wave we see from Schrödinger's equation that the absolute phase of this quantity cannot be arbitrarily. This is quite astonishing. Usually we do not discuss the absolute phase, since it does not change the physics of the problem. Schrödinger's equation in contrast seems to suggest that the absolute phase is not arbitrary but rather a measurable quantity with physical significance. To remove this problem, **Max Born** was proposing that the absolute square of the wavefunction corresponds to the probability $\rho(\mathbf{r},t)$ of a quantum object to be at the location $\mathbf{r}$ at time t. That is, we can write

$$\rho(\mathbf{r},t) = |\Psi(\mathbf{r},t)|^2 = \Psi^\star(\mathbf{r},t)\Psi(\mathbf{r},t) \tag{1.1.15}$$

$$\int \Psi^\star(\mathbf{r},t)\Psi(\mathbf{r},t)\, dV = 1 \ . \tag{1.1.16}$$

Here, (1.1.16) represents a normalization condition, since the probability to find the particle somewhere in space must be unity at all times.

We now will discuss how $\rho(\mathbf{r},t)$ evolves in space and time. In order to do so we perform the following steps:

- We first multiply the Schrödinger equation from the left side by $\Psi^\star(\mathbf{r},t)$.
- We next multiply the complex conjugate of the Schrödinger equation from the left side by $\Psi(\mathbf{r},t)$.
- We then subtract both equations from each other.

In this way we obtain

$$\frac{\partial}{\partial t}(\Psi^\star\Psi) + \frac{\hbar}{2m\imath}(\Psi^\star\triangle\Psi - \Psi\triangle\Psi^\star) = 0 \ . \tag{1.1.17}$$

Because any scalar function f and vector field $\mathbf{F}$ obeys the vector identity

$$\boldsymbol{\nabla}\cdot(f\mathbf{F}) = f\,\boldsymbol{\nabla}\cdot\mathbf{F} + \mathbf{F}\cdot\boldsymbol{\nabla}f$$

and $\triangle \equiv \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$ we can rewrite (1.1.17) to

$$\frac{\partial}{\partial t}(\Psi^\star\Psi) + \boldsymbol{\nabla}\cdot\left[\frac{\hbar}{2m\imath}(\Psi^\star\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^\star)\right] = 0 \ . \tag{1.1.18}$$

This equation has the from of a continuity equation for the probability

$$\boxed{\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J}_\rho = 0 \ ,} \tag{1.1.19}$$

where the ***probability current*** $\mathbf{J}_\rho$ is defined as

$$\boxed{\mathbf{J}_\rho \equiv \frac{\hbar}{2m\imath}(\Psi^\star\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^\star) = \Re\left\{\Psi^\star\frac{\hbar}{\imath m}\boldsymbol{\nabla}\Psi\right\} = \Re\left\{\Psi^\star\frac{\widehat{\mathbf{p}}}{m}\Psi\right\} \ .} \tag{1.1.20}$$

Equation 1.1.19 is the desired expression describing the evolution of the probability in space and time. It gives the local constraint on ρ, whereas (1.1.16) gives the global constraint on ρ. Eq.(1.1.20) says that the probability of a quantum objects at a certain point cannot change instantaneously but rather has to change in a continuous fashion by the flow of a probability current. Then, expression (1.1.19) can be viewed as a statement of the ***conservation of probability***.

It is obvious that (1.1.19) resembles the familiar expression for the conservation of charge. However, this similarity is only mathematical. Whereas the electrical current is a real, physical measurable quantity the probability current is only a theoretical construct. It is not possible to measure $\mathbf{J}_\rho$ for a single particle.

The probability current (1.1.20) describes the probabilistic flow of a quantum object which is subjected to forces varying in space and time. However, it still does not describe the situation we are interested in namely the motion of a charged particle in an electromagnetic field. This is caused by the fact that a charged particle moving in an electromagnetic field is subjected to forces depending on the motion of the particle itself. To find $\mathbf{J}_\rho$ for this situation we first have to find the appropriate form of the Schrödinger equation. In order to do so we start with the general classical equation of motion

$$\frac{d}{dt}\mathbf{p} = -\boldsymbol{\nabla} V \ . \tag{1.1.21}$$

Here, $\mathbf{p}$ is the canonical momentum and V the externally applied potential, which are used in writing the total energy of the system. We will see that this formalism is useful for the considered case, since the electromagnetic field represents a non-conservative potential making the formulation of energy relationships difficult.

We begin with the classical equation of motion for a particle of charge q and mass m in an electromagnetic field given by the ***Lorentz's law***

$$m\frac{d\mathbf{v}}{dt} = q\left[\mathbf{E} + (\mathbf{v}\times\mathbf{B})\right] \ . \tag{1.1.22}$$

In order to obtain the Schrödinger equation for this problem we first rewrite (1.1.22) into the form suggested by (1.1.21). The first step is to express the field quantities $\mathbf{E}$ and $\mathbf{B}$ in terms of potentials. According to ***Gauss's law*** the flux density $\mathbf{B}$ always can be written as

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} \ , \tag{1.1.23}$$

where $\mathbf{A}$ is a ***vector potential***. $\mathbf{A}$ can be used to write ***Faraday's law*** $\boldsymbol{\nabla}\times\mathbf{E} = -\partial\mathbf{B}/\partial t$ as

$$\boldsymbol{\nabla}\times\left(\mathbf{E} + \frac{\partial\mathbf{A}}{\partial t}\right) = 0 \ . \tag{1.1.24}$$

Using the fact that the curl of the gradient of any single-valued scalar field ϕ is zero, (1.1.24) is equivalent to the statement

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \boldsymbol{\nabla}\phi \ . \tag{1.1.25}$$

Then, Lorentz's law in terms of theses potentials can be written as

$$m\frac{d\mathbf{v}}{dt} = -q\left(\boldsymbol{\nabla}\phi + \frac{\partial\mathbf{A}}{\partial t} - \mathbf{v}\times(\nabla\times\mathbf{A})\right) \ . \tag{1.1.26}$$

In order to bring this equation into the form of (1.1.21) we have to group all time derivatives together. Using the chain rule of differentiation

$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{A} \tag{1.1.27}$$

we obtain

$$\frac{d}{dt}(m\mathbf{v}+q\mathbf{A}) = -q\left[\boldsymbol{\nabla}\phi-(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{A}-\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A})\right] \; . \tag{1.1.28}$$

Now eq. (1.1.28) is close to the desired form and we can suspect that the ***canonical momentum*** is given by

$$\boxed{\mathbf{p} = m\mathbf{v}+q\mathbf{A} \; .} \tag{1.1.29}$$

To verify this we must be able to express the right hand side of (1.1.28) as the gradient of a scalar function. We therefore rewrite it in terms of the canonical momentum and obtain

$$\frac{d\mathbf{p}}{dt} = -q\boldsymbol{\nabla}\phi+\frac{q}{m}(\mathbf{p}\cdot\boldsymbol{\nabla})\,\mathbf{A}-\frac{q^2}{m}(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{A}+\frac{q}{m}\mathbf{p}\times(\boldsymbol{\nabla}\times\mathbf{A})-\frac{q^2}{m}\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{A}) \; . \tag{1.1.30}$$

Using the vector identities

$$\mathbf{a}\times(\boldsymbol{\nabla}\times\mathbf{b}) = \boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{b})-(\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b}-(\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a}-\mathbf{b}\times(\boldsymbol{\nabla}\times\mathbf{a}) \tag{1.1.31}$$

$$\mathbf{a}\times(\boldsymbol{\nabla}\times\mathbf{a}) = \frac{1}{2}\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{a})-(\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{a} \tag{1.1.32}$$

we can rewrite (1.1.30) as

$$\frac{d\mathbf{p}}{dt} = -q\boldsymbol{\nabla}\phi+\frac{q}{m}\boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{A})-\frac{q^2}{2m}\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{A})-\frac{q}{m}(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{p})-\frac{q}{m}\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{p}) \; . \tag{1.1.33}$$

At this point we have to recall that we are using a set of independently specified variables $(\mathbf{r},\mathbf{p})$ to describe the problem. Therefore, the spatial derivative of the canonical momentum is zero and we obtain

$$\boxed{\frac{d\mathbf{p}}{dt} = -\boldsymbol{\nabla}\left\{q\phi-\frac{q}{m}(\mathbf{p}\cdot\mathbf{A})+\frac{q^2}{2m}(\mathbf{A}\cdot\mathbf{A})\right\} \; .} \tag{1.1.34}$$

That is, we have achieved our initial goal of writing Lorentz's law in the generic form of eq. (1.1.21).

We briefly will discuss the physical meaning of (1.1.34). First, the canonical momentum $\mathbf{p}$ given by (1.1.29) is composed of two parts. The first part, $m\mathbf{v}$, is the usual ***kinetic momentum*** and is associated with the momentum in elementary mechanics. The second part, $q\mathbf{A}$, is denoted as the ***field momentum***, which is a direct result of the charge of the particle. Any change of the velocity of the particle produces an electromagnetic field that must be considered self-consistently. Therefore, the generalized potential of the problem

$$V = q\phi-\frac{q}{m}(\mathbf{p}\cdot\mathbf{A})+\frac{q^2}{2m}(\mathbf{A}\cdot\mathbf{A}) \tag{1.1.35}$$

is not only a function of space and time but also of the canonical momentum. In this way the interaction of the externally applied field and the induced current created by the motion of the charged particle is accounted for self-consistently.

The next step in obtaining the Schrödinger equation is to use the expressions for **p** and V to write down the total energy:

$$E = E_{\rm kin} + E_{\rm pot} = \frac{\mathbf{p}\cdot\mathbf{p}}{2m} + \left\{ q\phi - \frac{q}{m}(\mathbf{p}\cdot\mathbf{A}) + \frac{q^2}{2m}(\mathbf{A}\cdot\mathbf{A}) \right\} . \tag{1.1.36}$$

This purely classical expression can be rewritten as

$$E = \frac{1}{2m}(\mathbf{p}-q\mathbf{A})\cdot(\mathbf{p}-q\mathbf{A}) + q\phi . \tag{1.1.37}$$

The last step is to replace energy and momentum by the corresponding quantum mechanical operators

$$E \Rightarrow \imath\hbar\frac{\partial}{\partial t} \qquad\qquad \mathbf{p} \Rightarrow -\imath\hbar\boldsymbol{\nabla} . \tag{1.1.38}$$

Using these expression we expect the quantum form of the Lorentz's law to be

$$\boxed{\imath\hbar\frac{\partial\Psi}{\partial t} = \frac{1}{2m}\left(\frac{\hbar}{\imath}\boldsymbol{\nabla} - q\mathbf{A}\right)^2 \Psi + q\phi\,\Psi .} \tag{1.1.39}$$

This equation corresponds to the Schrödinger equation of a charge particle in an electromagnetic field. We finally can use now eq. (1.1.39) to derive the probability current of a charged particle in an electromagnetic field to

$$\boxed{\mathbf{J}_\rho = \Re\left\{\Psi^\star\left(\frac{\hbar}{\imath m}\boldsymbol{\nabla} - \frac{q}{m}\mathbf{A}\right)\Psi\right\} = \Re\left\{\Psi^\star\,\frac{\widehat{\mathbf{p}}}{m}\,\Psi\right\} .} \tag{1.1.40}$$

In deriving this equation we have to take into account that ϕ represents a portion of the applied potential field and therefore is a real quantity. We will see in the following section that Eq. (1.1.40) is the central expression in the quantum mechanical description of superconductivity.

1.1.2 Macroscopic Quantum Currents in Superconductors

After having recalled some basic concepts of quantum mechanics we can apply these concept to superconductors. Before doing so let us first consider the situation in a normal conductor. Of course, the electrons in a normal metal move according to the laws of quantum mechanics. In the usual approximation of weakly or non-interacting particles this motion can be described in terms of the ordinary Schrödinger equation

$$\imath\hbar\frac{\partial\Psi}{\partial t} = \widehat{H}\,\Psi , \tag{1.1.41}$$

where

$$\Psi(\mathbf{r},t) = \Psi_0(\mathbf{r},t)\,\mathrm{e}^{\imath\theta(\mathbf{r},t)} \tag{1.1.42}$$

is the complex wave function of a particle. According to quantum mechanics $|\Psi|^2$ can be interpreted as the probability density of the particles. In the stationary situation $|\Psi|^2$ can be assumed constant and $\widehat{H}$ can be replaced by the energy E of the particle. As a result we can write

$$\hbar \frac{\partial \theta}{\partial t} = -E \ . \tag{1.1.43}$$

That is, the quantum specific character is reduced to that of the wave function phase θ.

The important point is that in normal metals (1.1.43) does not result in quantum correlations for the macroscopic variables because the electrons obey Fermi-Dirac statistics and their energies can never be exactly equal. Therefore, according to (1.1.43) the temporal evolution of the phases of the particle wave functions differs for all particles. That is, the phases are uniformly distributed and since all macroscopic quantities are sums over all the particles the phases drop out in these quantities.

We now discuss that this is not the case for superconductors. In superconductors bound pairs of electrons (Cooper pairs) are formed with opposite momenta and spins in the simplest case. These pairs with zero net spin obey the Bose-Einstein statistics and therefore can occupy the lowest energy state at low temperatures. As a result, their rates $\partial\theta/\partial t$ are identical. Furthermore, the Cooper pairs have a relatively large size of the order of 10 to 1000 nm, which is much larger than the typical distance between the pairs. Therefore, the wave functions of the individual pairs are strongly overlapping. As a result of these two factors, all the pairs are forming a ***phase-locked*** state that can be described by a single wave function Ψ, which is frequently denoted as the order parameter. In this situation the phases do not drop out during summation over all particles and therefore macroscopic variables, in particular current, can depend on the phase θ, which changes in a quantum manner under the action of an electromagnetic field. This quantum dependence leads not only to the zero resistance of superconductors and the Meißner-Ochsenfeld effect but also to specific coherent effects such as the ***flux quantization*** and the ***Josephson effect***.

The qualitative discussion of the previous paragraph shows that the situation for superconductors is somehow similar to the situation in atoms. For the latter, the fact that electrons can orbit the nucleus without decaying and letting the atom collapse could not be explained classically. Only quantum mechanics provided the necessary framework. In the same way, the infinite conductivity of superconductors cannot be explained classically. If one would try to do so, one has to postulate that the superelectrons do not scatter. Although the results of this assumption are consistent with the experimental fact, it seems to us arbitrary to postulate an infinite scattering time. In the same way as Schrödinger's equation provides the explanation for stable microscopic currents created by orbiting electrons, it was hypothesized by **Fritz London** that the macroscopic currents in superconductors might be explained in a similar way. This was the starting point of the ***macroscopic quantum model*** of superconductivity.

The central hypothesis behind the macroscopic quantum model of superconductivity can be stated as follows:

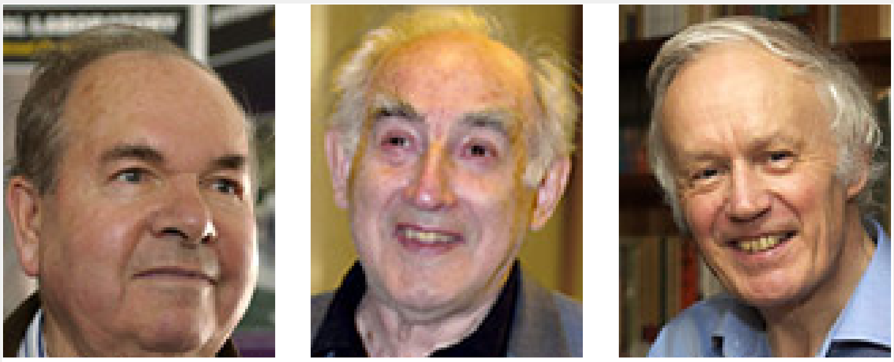
There exists a macroscopic quantum wave function

$$\psi(\mathbf{r},t) = \psi_0(\mathbf{r},t)\, \mathrm{e}^{\imath\theta(\mathbf{r},t)} \tag{1.1.44}$$

that describes the behavior of the entire ensemble of superelectrons in a superconductor.

The motivation for this assumption is that superconductivity is a ***coherent phenomenon*** of all superelectrons. This situation is analogous to the quantum description of electromagnetism. According to wave-particle dualism we can envision a photon as a quantum particle. Then, when a large number of photons interact coherently such as in a laser, the entire collection of these quantum particles can be adequately described in terms of an electromagnetic field with amplitude and phase. As we will see, the macroscopic wave function ψ is a field-like quantity that similarly describes the whole ensemble of superelectrons.

The Nobel Prize in Physics 2003 has been given to Alexei A. Abrikosov, Vitaly L. Ginzburg, and Anthony J. Leggett *for their pioneering contributions to the theory of superconductors and superfluids*.



Alexei A. Abrikosov Vitaly L. Ginzburg Anthony J. Leggett

Alexei A. Abrikosov, born 1928 in Moscow, the former Soviet Union, American (and Russian) citizen. Doctor's degree in physics in 1951 at the Institute for Physical Problems, Moscow. Distinguished Argonne Scientist, Argonne National Laboratory, Argonne, Illinois, USA.

Vitaly L. Ginzburg, born 1916 in Moscow, Russia (Russian citizen). Doctor's degree in physics at the University of Moscow. Former Head of the Theory Group at the P.N. Lebedev Physical Institute, Moscow, Russia.

Anthony J. Leggett, born 1938 in London, England (British and American citizen). Doctor's degree in physics in 1964 at the University of Oxford. MacArthur Professor at the University of Illinois at Urbana-Champaign, USA.

We have now to examine the consequences of postulating a macroscopic wave function describing the whole ensemble of superelectrons. We first discuss the meaning of $|\psi|^2$. For a single quantum particle described by the wave function Ψ the absolute square $|\Psi(\mathbf{r},t)|^2$ has been interpreted as the probability to find the particle at the location $\mathbf{r}$ at a given time t. As a result we have the global constraint or normalization condition $\int \Psi^\star \Psi \, dV = 1$ (compare (1.1.16)) stating that the particle exists somewhere in space at any time. Along this line it is natural to assume that the wave function ψ representing the whole ensemble of superelectrons satisfies the normalization condition

$$\boxed{\int \psi^\star(\mathbf{r},t)\psi(\mathbf{r},t)\, dV \quad = \quad N_s^\star} \tag{1.1.45}$$

$$\boxed{|\psi(\mathbf{r},t)|^2 \; = \; \psi^\star(\mathbf{r},t)\psi(\mathbf{r},t) \quad = \quad n_s^\star(\mathbf{r},t)} \tag{1.1.46}$$

Here, $n_s^\star(\mathbf{r},t)$ is the local density and $N_s^\star$ the total number of superconducting electrons. Condition (1.1.45) says that if we are searching the whole space we have to find all superelectrons. Since the superelectrons are discrete objects, there must be of course a sufficiently large density in order to make the definition of a local density sense. This concern is similar to the situation in fluid mechanics. Although we know that all fluids consist of discrete atoms or molecules, it is convenient to describe the system by a local fluid density. Due to the analogy to fluid mechanics the collection of superelectrons is often referred to as a ***charged superfluid***. Therefore, the theoretical description of superconductors (charged superfluid) and uncharged superfluids such as superfluid helium has many similarities. Indeed, the Nobel Prize in Physics 2003 was given to **Vitaly L. Ginzburg, Alexei A. Abrikosov** and **Anthony J. Leggett** for their pioneering contributions to the theory of superconductors and superfluids.

Note that the macroscopic quantum model of superconductivity does not explain the microscopic origin of superconductivity, which is not discussed here. That is, it does not explain the microscopic origin of the attractive interaction of electrons in a solid resulting in the formation of Cooper pairs. This mechanism, which may be different for the classical metallic superconductors, the novel high temperature superconductors or the heavy fermion superconductors is not relevant for the macroscopic quantum phenomena discussed in the following. The only relevant issue is the possibility to describe the superelectron fluid as a quantum mechanical entity irrespective of the detailed pairing mechanism.

The analogy to superfluids is powerful to establish an intuitive picture about the macroscopic quantum model. Instead of a single particle wave function describing the probabilities for single particles we are now considering so many quantum objects that we have a wave function describing the actual location of a complete subset of the whole ensemble. Then, the continuity equation (1.1.19) for the probability becomes a continuity equation for the condensate density $n_s^\star$. Furthermore, we do not have a probability flow $\mathbf{J}_\rho$ but rather a flow of particles, which is nothing more than a physical current. Following (1.1.40) we can immediately write down the macroscopic quantum current density $\mathbf{J}_s$ in an electromagnetic field, which is equivalent to the supercurrent density. We only have to multiply the ensemble probability current describing the particle flux by the charge $q^\star$ of the superelectron:

$$\boxed{\begin{aligned} \mathbf{J}_s &= q^\star\, \Re\left\{\Psi^\star\left(\frac{\hbar}{m^\star \imath}\nabla - \frac{q^\star}{m^\star}\mathbf{A}\right)\Psi\right\} \\ &= \frac{q^\star\hbar}{2m^\star\imath}\left(\psi^\star\nabla\psi - \psi\nabla\psi^\star\right) - \frac{q^{\star 2}}{2m^\star}\psi\psi^\star\,\mathbf{A}\ . \end{aligned}} \tag{1.1.47}$$

Here, $m^\star$ is the mass of the superelectrons. This expression can be brought in a more useful form by making some assumption on ψ. According our discussion the macroscopic wavefunction $\psi(\mathbf{r},t)$ obeys the Schrödinger-like equation for the ensemble in an electromagnetic field:

$$\imath\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = \frac{1}{2m^\star}\left(\frac{\hbar}{\imath}\nabla - q^\star\mathbf{A}(\mathbf{r},t)\right)^2\psi(\mathbf{r},t) + q^\star\phi(\mathbf{r},t)\,\psi(\mathbf{r},t)\ . \tag{1.1.48}$$

Because of the phase factor $\imath$ in this expression the macroscopic wave function ψ in the same way as the microscopic one is a complex quantity. Therefore, $\psi(\mathbf{r},t)$ is of the form

$$\psi(\mathbf{r},t) = \sqrt{n_s^\star(\mathbf{r},t)}\,\mathrm{e}^{\imath\theta(\mathbf{r},t)}\ . \tag{1.1.49}$$

Here, we have used an amplitude satisfying the condition that the absolute square of the wave function is equivalent to the density of superelectrons. We also note that θ is a real function representing the phase of the complex number.

Substitution of (1.1.49) into the expression (1.1.47) for the supercurrent we obtain the supercurrent equation

$$\boxed{\mathbf{J}_s = q^\star n_s^\star(\mathbf{r},t)\left\{\frac{\hbar}{m^\star}\nabla\theta(\mathbf{r},t) - \frac{q^\star}{m^\star}\mathbf{A}(\mathbf{r},t)\right\}\ .} \tag{1.1.50}$$

Since a current density always can be written as $\mathbf{J}_s = q^\star n_s^\star \mathbf{v}_s$, the expression in parentheses corresponds to the velocity of the superelectrons:

$$\mathbf{v}_s \equiv \frac{\hbar}{m^\star}\nabla\theta(\mathbf{r},t) - \frac{q^\star}{m^\star}\mathbf{A}(\mathbf{r},t)\ . \tag{1.1.51}$$

Additional Topic:
Gauge Invariance

We briefly address a rather technical point, which is however of particular importance for superconductors. We have to recall that quantities such as $\mathbf{A}$, ϕ or θ describe physical variables but are not themselves observable. We also can find formal transformations for these quantities that have no effect on the observable quantities such as $\mathbf{B}$, $\mathbf{E}$, or $\mathbf{J}_s$. Such transformations are called ***gauge transformations***. Unobservable quantities such as $\mathbf{A}$, ϕ or θ, which change in a well defined way under a gauge transformation, are denoted as ***gauge covariant***.

Let us consider eq.(1.1.50). It states that the supercurrent $\mathbf{J}_s$ only depends on the phase of the macroscopic wave function and the vector potential. That is, the observable quantity $\mathbf{J}_s$ is related to two quantities, which cannot be determined directly from the experiment. Moreover, because any single-valued scalar field f satisfies the condition $\nabla \times (\nabla f) = 0$, we know that for an arbitrary scalar function χ

$$\mathbf{B} \;=\; \nabla \times \mathbf{A} = \nabla \times (\mathbf{A} + \nabla\chi) \ . \tag{1.1.52}$$

That is, there exists an infinite number of possible vector potentials that can describe the correct magnetic flux density. This suggests that we can obtain a well defined value for the measurable quantity $\mathbf{J}_s$ only if we can measure both θ and $\mathbf{A}$, which have been introduced only for mathematical convenience and are not physical observables.

The way out of this dilemma is to recognize the fact that the relation between phase and vector potential is not arbitrary but fixed. In this way we can measure the supercurrent but still are not able to determine θ ***and*** $\mathbf{A}$. That is, we demand that expression (1.1.50) is independent of the special choice of $\mathbf{A}$. The specific choice of $\mathbf{A}$ is usually referred to as the gauge and, hence, we have to make the expression for the supercurrent ***gauge invariant***. Mathematically this is straightforward. Suppose we define a new vector potential $\mathbf{A}'$ as

$$\mathbf{A}' \;\equiv\; \mathbf{A} + \nabla\chi \ . \tag{1.1.53}$$

Then, according to (1.1.52) this new vector potential correctly gives the magnetic flux density. In addition, the new vector potential also must correctly describe the electric field. Therefore, we define a new scalar potential ϕ' so that the electric field is given by

$$\mathbf{E} \;=\; -\frac{\partial \mathbf{A}'}{\partial t} - \nabla\phi' \ . \tag{1.1.54}$$

Comparing this expression to the original expression $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi$ (1.1.25) we see that the two scalar potentials are related by

$$\phi' \;\equiv\; \phi - \frac{\partial\chi}{\partial t} \ . \tag{1.1.55}$$

According to (1.1.53) and (1.1.55) we can separately specify the temporal and spatial dependence of a scalar function χ to generate new sets of the scalar and the vector potentials which still describe the original electric and magnetic fields.

By rewriting (1.1.48) in terms of the new potentials with the new wave function $\psi'(\mathbf{r},t) = \sqrt{n_s^\star(\mathbf{r},t)}\, e^{\imath\theta'(\mathbf{r},t)}$ we can easily derive a new expression for the supercurrent density given by

$$\mathbf{J}_s \;=\; q^\star n_s^\star(\mathbf{r},t) \left\{ \frac{\hbar}{m^\star}\nabla\theta'(\mathbf{r},t) - \frac{q^\star}{m^\star}\mathbf{A}'(\mathbf{r},t) \right\} \ . \tag{1.1.56}$$

Since the supercurrent, the experimentally measurable quantity, must be the same in (1.1.56) and (1.1.50), we have to satisfy the condition

$$\theta' \quad = \quad \theta + \frac{q^\star}{m^\star}\chi \ . \tag{1.1.57}$$

This in turn results in

$$\psi'(\mathbf{r},t) \quad = \quad \psi(\mathbf{r},t)\, \mathrm{e}^{\imath(q^\star/\hbar)\chi} \ . \tag{1.1.58}$$

That is, the same scalar function χ is changing both the phase and the vector potential. In this way the supercurrent always has the same value and can be measured regardless the specific gauge chosen. The important conclusion that can be drawn is that the expression for the supercurrent is gauge invariant and therefore we do no longer be concerned about this issue.

From the expression (1.1.56) and (1.1.50) for the supercurrent density we obtain the condition

$$\nabla\theta' - \frac{q^\star}{\hbar}\mathbf{A}' \quad = \quad \nabla\theta - \frac{q^\star}{\hbar}\mathbf{A} \ . \tag{1.1.59}$$

We therefore can introduce a ***gauge invariant phase gradient***

$$\gamma \quad = \quad \nabla\theta - \frac{q^\star}{\hbar}\mathbf{A} \ = \ \nabla\theta - 2\pi\frac{(-2e)}{h}\mathbf{A} \ = \ \nabla\theta - \frac{2\pi}{\Phi_0}\mathbf{A} \ , \tag{1.1.60}$$

where we have used $q^\star = -2e$ and

$$\Phi_0 \quad = \quad \frac{h}{|q^\star|} \ = \ \frac{h}{2e} \tag{1.1.61}$$

is the flux quantum. We see that the supercurrent is then given by

$$\boxed{\mathbf{J}_s \quad = \quad \frac{q^\star n_s^\star \hbar}{m^\star}\gamma \ = \ \frac{\hbar}{q^\star \Lambda}\gamma \ ,} \tag{1.1.62}$$

where

$$\Lambda \quad \equiv \quad \frac{m^\star}{n_s^\star q^{\star 2}} \tag{1.1.63}$$

is the ***London coefficient*** and

$$\lambda_L \quad \equiv \quad \sqrt{\frac{m^\star}{\mu_0 n_s^\star q^{\star 2}}} \tag{1.1.64}$$

the ***London penetration depth***. We see that the supercurrent density is proportional to the gauge invariant phase gradient.[16]

[16]Note that $\nabla\theta - \frac{q^\star}{\hbar}\mathbf{A}$ cannot be written as $\nabla\gamma$, that is, as the gradient of a gauge invariant phase. In this case we would have $\mathbf{A} \propto \nabla\theta - \nabla\gamma$ and hence $\nabla \times \mathbf{A} = \mathbf{B} = 0$.

1.1.3 The London Equations

The two ***London equations*** formulated by **Fritz London** to describe the behavior of superconductors based on classical physics can be easily derived from the expression (1.1.50) for the supercurrent density by assuming $n_s^\star = const$. Note that (1.1.50) includes the cases where the superelectron density varies in space and time.

By using the London coefficient $\Lambda = \frac{m^\star}{n_s^\star q^{\star 2}}$ we can rewrite (1.1.50) as

$$\Lambda \mathbf{J}_s \;=\; -\left\{\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q^\star}\nabla\theta(\mathbf{r},t)\right\} \; . \tag{1.1.65}$$

Second London Equation and Meißner-Ochsenfeld-Effect

By taking the curl of this expression we obtain the ***second London equation***

$$\boxed{\nabla \times (\Lambda \mathbf{J}_s) \;=\; -\nabla \times \mathbf{A} \;=\; -\mathbf{B} \; ,} \tag{1.1.66}$$

Taking the curl of the Maxwell equation $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_s$ we obtain $\nabla \times \nabla \times \mathbf{B} = \nabla \times \mu_0 \mathbf{J}_s$. With the vector identity $\nabla \times \nabla \times \mathbf{B} = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$ and using $\nabla \cdot \mathbf{B} = 0$ and the second London equation we arrive at

$$\nabla^2 \mathbf{B} \;=\; \frac{\mu_0}{\Lambda}\mathbf{B} = \frac{1}{\lambda_L^2}\mathbf{B} \; . \tag{1.1.67}$$

This equation describes the Meißner-Ochsenfeld effect.[17,18] An applied field decays exponentially inside a superconductor with the characteristic decay length given by the London penetration depth λ_L. For example, near a plane surface extending in the yz-plane the magnetic field B_z parallel to the z-direction decays exponentially with x as $B_z(x) = B_{z,0}\,\mathrm{e}^{-x/\lambda_L}$ (see Fig. 1.1). With $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_s$ we obtain also an exponential decay of the supercurrent density flowing in y-direction as $J_{s,y}(x) = \frac{B_{z,0}}{\mu_0 \lambda_L}\,\mathrm{e}^{-x/\lambda_L} = \frac{H_{z,0}}{\lambda_L}\,\mathrm{e}^{-x/\lambda_L} = J_{y,0}\,\mathrm{e}^{-x/\lambda_L}$.

First London Equation and Perfect Conductivity

In order to get the first London equation from (1.1.50) we have to take the partial derivative with respect to time:

$$\frac{\partial}{\partial t}(\Lambda \mathbf{J}_s) \;=\; -\left\{\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} - \frac{\hbar}{q^\star}\nabla\left(\frac{\partial \theta(\mathbf{r},t)}{\partial t}\right)\right\} \; . \tag{1.1.68}$$

From the Schrödinger-like equation (1.1.48) we obtain for $n_s^\star = const$

$$-\hbar\frac{\partial \theta}{\partial t} \;=\; \frac{1}{2n_s^\star}\Lambda \mathbf{J}_s^2 + q^\star \phi \; . \tag{1.1.69}$$

[17] **Walther Meißner**, born on December 16, 1882 in Berlin, died on November 15, 1974 in Munich.

[18] **Robert Ochsenfeld**, born on May 18, 1901 in Helberhausen, died on December 5, 1993 in Helberhausen.

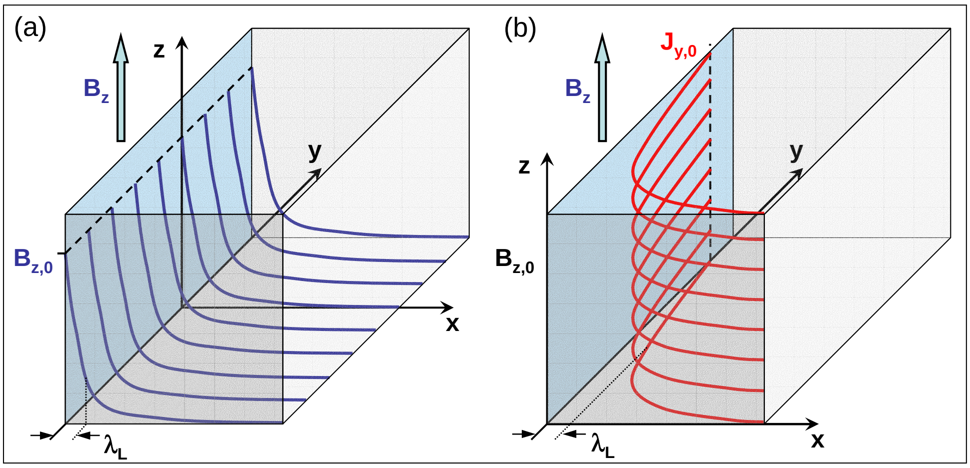


Figure 1.1: Exponential decay of the magnetic field $\mathbf{B}$ (a) and the supercurrent density $\mathbf{J}_s$ (b) with distance x into a bulk superconductor.

This expression is known as the ***energy-phase relationship***, since the first term on the right hand side represents the kinetic energy ($\frac{1}{2}m^\star v_s^2$) and the second the potential energy. By substituting (1.1.69) into (1.1.68) and using $\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla\phi$ we obtain the ***first London equation***

$$\boxed{\frac{\partial}{\partial t}(\Lambda\mathbf{J}_s) \;=\; \mathbf{E} - \frac{1}{n_s^\star q^\star}\nabla\left(\frac{1}{2}\Lambda\mathbf{J}_s^2\right) \;.} \tag{1.1.70}$$

As discussed below the second term on the right hand side usually can be neglected. Then, the 1. London equation reads $\frac{\partial}{\partial t}(\Lambda\mathbf{J}_s) = \mathbf{E}$. We hence see that for a time-independent supercurrent the electric field inside the superconductor vanishes and we therefore have a dissipationless current.[19]

The fact that a supercurrent flowing in a superconductor does not decay in time is interesting and not obvious at first sight. Therefore, we briefly consider processes that could cause a decay. In order to do so we consider the Fermi sphere which is shown in Fig. 1.2 in two-dimensions in the k_xk_y-plane. The allowed $\mathbf{k}$-states are represented by points. At $T = 0$, all states within the Fermi sphere (Fermi circle in two dimensions) are occupied. Without any current the Fermi sphere is centered around the origin (dashed line). However, for a finite current e.g. in x-direction it is slightly shifted along the k_x-direction by δk_x. In the normal state of a metal the charge carriers can relax into states of lower energy, where the Pauli principle has to be taken into account (see Fig. 1.2a). Since there is a large variety of possible scattering processes, the system rapidly will relax into the situation with the Fermi sphere centered around the origin, i.e. the current will relax rapidly. In contrast, in the superconducting state all the Cooper pairs have the same center of mass momentum. Therefore, they can be scattered only around the sphere as shown in Fig. 1.2b. However, these scattering processes do not result in a shift of the center of the Fermi sphere and hence in a decay of the current. That is we have a non-decaying supercurrent. Note that other scattering processes are only possible if we destroy the Cooper pairs what however requires the supply of its binding energy.

We emphasize that the two London equations, although closely related to each other, are independent and neither can be deduced from the other. For example, if we take the time derivative of the 2. London

[19] Based on this result we can derive the 1. London equation by a simple classical consideration. For a dissipationless motion of charge carriers we can neglect the friction and can write the equation of motion as $m^\star\dot{\mathbf{v}}_s = q^\star\mathbf{E}$. With $\mathbf{J}_s = n_s^\star q^\star\mathbf{v}_s$ we find $\mathbf{E} = \frac{m^\star}{q^\star n_s^\star}\dot{\mathbf{J}}_s = \frac{\partial}{\partial t}(\Lambda\mathbf{J}_s)$, i.e., the 1. London equation. However, one has to take into account that in this derivation we already have made use of the existence of a dissipationless current, which is an essential result of the 1. London equation.

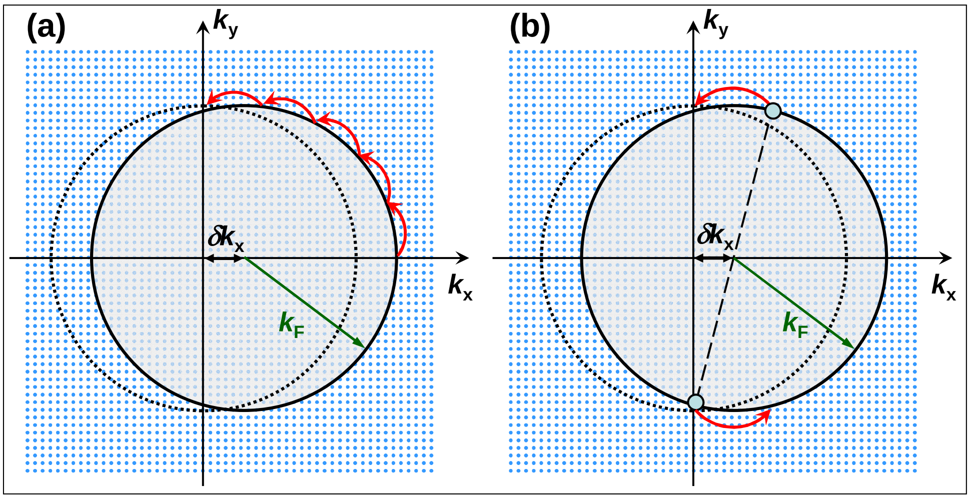


Figure 1.2: Intuitive picture for the decay of a current in the normal state (a) and the superconducting state (b) of a metal. In (b) the electrons are correlated to Cooper pairs relative to the center of the shifted Fermi sphere since they all have the same center of mass velocity.

equation (1.1.66) to derived the 1. London equation (1.1.70), we cannot fix the term $\nabla\phi$ on the right hand side of (1.1.70) with certainty. That is, although it is obvious that the screening currents on the surface of the superconductor must flow resistanceless because they do not decay, we cannot prove that $\mathbf{E} = 0$ inside the superconductor from (1.1.66) alone. On the other hand, by taking the curl of (1.1.70) we can get the time derivative of (1.1.66) but not (1.1.66) itself. Using the time derivative of Ampère's rule, we further can get the time derivative of (1.1.67). By integration over time we then can deduce that changes in $\mathbf{B}$ are screened from the bulk of the superconductor. However, eq.(1.1.66) is stronger since it implies that not only changes in $\mathbf{B}$ but the field $\mathbf{B}$ itself is screened. This is the difference between a perfect conductor, which is screening changes in $\mathbf{B}$, and a superconductor (perfect diamagnet), which also screens $\mathbf{B}$.

Additional Topic:
Linearized 1. London Equation

Usually, the first London equation is given as

$$\frac{\partial}{\partial t}(\Lambda \mathbf{J}_s) \quad = \quad \mathbf{E} \ , \tag{1.1.71}$$

that is, without the second term on the right hand side of (1.1.70) which contains the kinetic energy of the superelectrons. Since this term does not contain $\hbar$, it is not of quantum mechanical origin.

In order to discuss the origin of the extra term in (1.1.70) we use the vector identity

$$\mathbf{a} \times (\nabla \times \mathbf{a}) \quad = \quad \frac{1}{2}\nabla(\mathbf{a}\cdot\mathbf{a}) - (\mathbf{a}\cdot\nabla)\mathbf{a} \tag{1.1.72}$$

to write $\frac{1}{2}\nabla \mathbf{J}_s^2 = \mathbf{J}_s \times (\nabla \times \mathbf{J}_s) + (\mathbf{J}_s \cdot \nabla)\mathbf{J}_s$. Then, by using the second London equation we can rewrite (1.1.70) as

$$\frac{\partial}{\partial t}(\Lambda \mathbf{J}_s) \quad = \quad \mathbf{E} - \frac{1}{n_s^\star q^\star}(\mathbf{J}_s \cdot \nabla)(\Lambda \mathbf{J}_s + \frac{1}{n_s^\star q^\star}(\mathbf{J}_s \times \mathbf{B}) \ . \tag{1.1.73}$$

Fritz London (1900 -1954):

Fritz London was born on March 7, 1900 in Breslau, Germany (now Wroclaw, Poland). He was a German-American physicist who, with Walter Heitler, devised (1927) the first quantum mechanical treatment of the hydrogen molecule. London was educated at the universities of Bonn, Frankfurt, Göttingen, Munich (Ph.D., 1921), and Paris. He was a Rockefeller research fellow at Zürich and Rome and a lecturer at the University of Berlin. From 1933 to 1936 he was a research fellow at the University of Oxford and then went to the University of Paris as master and director of research.
In 1939 he immigrated to the United States to become professor of theoretical chemistry at Duke University, Durham, N.C., and from 1953 he was James B. Duke professor of chemical physics there. He became a U.S. citizen in 1945. His publications include two volumes on Superfluids (1950, 1954).
London's theory of the chemical binding of homopolar molecules marked the beginning of modern quantum mechanical treatment of the hydrogen molecule and is considered one of the most important advances in modern chemistry. With his brother, Heinz London, he developed (1935) the phenomenological theory of superconductivity, providing a new foundation for the understanding of molecular forces and clarifying the connection between pure quantum phenomena and many of the most striking facts of chemistry.

London died on March 30,1954, at Durham, N.C., USA.

With $\frac{d}{dt}(\Lambda \mathbf{J}_s) = \frac{\partial}{\partial t}(\Lambda \mathbf{J}_s) + (\mathbf{v}_s \cdot \nabla)(\Lambda \mathbf{J}_s)$ and $\mathbf{J}_s = n_s^\star q^\star \mathbf{v}_s$ we obtain

$$m^\star \frac{d\mathbf{v}_s}{dt} = q^\star \mathbf{E} + q^\star \mathbf{v}_s \times \mathbf{B} \ , \qquad (1.1.74)$$

This expression corresponds to the Lorentz's law. From this we can conclude that

- the nonlinear first London equation results from the Lorentz's law and the second London equation. Therefore, (1.1.70) must be the exact form of the expression describing the phenomenon of zero dc resistance in superconductors.
- The first London equation is derived using the second London equation. This shows that the Meißner-Ochsenfeld effect is the more fundamental property of superconductors than the vanishing dc resistance.[20]

Since the nonlinear first London equation represents the correct expression the question arises, whether conclusions derived from the more frequently used linear London equation (1.1.71) are wrong. Fortunately, the answer is no, since in most cases the nonlinear term can be neglected. However, one has to be aware that we cannot always neglect the nonlinear term. In some cases it even plays an important role.

In order to elucidate the relevance of the nonlinear term in the first London equation we can state the following:

1. Situations dealing with superconductors in an applied magnetic field (e.g. slab in a parallel field) we usually can treat without using the first London equation. Electric fields are not important in this case. They are derived using Faraday's law after having calculated the magnetic fields and the corresponding currents. The first London equation only would yield small corrections to the supercurrent density distribution.

[20] Since the vanishing dc resistance has been discovered already in 1911, whereas the Meißner-Ochsenfeld effect was found only in 1933, the phenomenon was not denoted as "superdiamagnetism" but as superconductivity.

2. It is evident from eq.(1.1.70) that the nonlinear term can always be neglected if

$$|\mathbf{E}| \gg \left|\frac{1}{n_s^\star q^\star}\boldsymbol{\nabla}\left(\Lambda\mathbf{J}_s^2\right)\right| . \tag{1.1.75}$$

Assuming that the spatial variation of the supercurrent occurs on a length scale ℓ, we have $\boldsymbol{\nabla}\cdot\mathbf{J}_s \sim J_s/\ell$. Then, the condition can be written as

$$|\mathbf{E}| \gg |\mathbf{v}_s|\left|\frac{\Lambda\mathbf{J}_s}{\ell}\right| . \tag{1.1.76}$$

With the same assumption we obtain

$$\left|\frac{\Lambda\mathbf{J}_s}{\ell}\right| \sim |\mathbf{B}| \tag{1.1.77}$$

from the second London equation. As a result, we obtain the condition

$$|\mathbf{E}| \gg |\mathbf{v}_s|\,|\mathbf{B}| \tag{1.1.78}$$

for neglecting the nonlinear term. This is not surprising, since this condition is equivalent to the usually made assumption that the magnetic contribution to the Lorentz's law can be neglected compared the electric one in deriving the first London equation.

Additional Topic:
The London Gauge

Although we usually keep all expressions gauge invariant, i.e. they hold in all gauges, in some cases it is convenient to use a special gauge. If the macroscopic wavefunction is single valued (this is the case for a simply connected superconductor containing no flux) we can choose $\chi(\mathbf{r},t)$ so that $\theta = \theta' - \frac{q^\star}{m^\star}\chi = 0$ everywhere. This is sometimes called the rigid gauge, since the phase of ψ never changes and therefore $\hbar\boldsymbol{\nabla}\theta = 0$ when we switch on a magnetic field or introduce a finite supercurrent. Hence (1.1.65) reads

$$\Lambda\mathbf{J}_s = -\mathbf{A}(\mathbf{r},t) . \tag{1.1.79}$$

Frequently we also have $\boldsymbol{\nabla}\cdot\mathbf{J}_s = 0$ (this is the case when no supercurrent is converted into a normal current or vice versa). Then we obtain from (1.1.79)

$$\boldsymbol{\nabla}\cdot\mathbf{A} = 0 \qquad \text{and} \qquad A_n = \Lambda\, J_{sn} . \tag{1.1.80}$$

Here, the second relation is just a boundary condition. A_n is the component of $\mathbf{A}$ normal to the boundary and J_{sn} is the supercurrent density normal to the boundary. A vector potential which satisfies the conditions (1.1.80) is said to be in the ***London gauge***. Together with $\boldsymbol{\nabla}\times\mathbf{A} = \mathbf{B}$ the conditions of the London gauge fix $\mathbf{A}$ unambiguously. One can easily show that in the London gauge the vector potential obeys the equations $\boldsymbol{\nabla}^2\mathbf{A} = 0$ and $\boldsymbol{\nabla}^2\mathbf{A} = \mathbf{A}/\lambda_L^2$ outside and inside the superconductor, respectively.

We briefly use the London gauge to compare the perfect diamagnetism of a superconductor to the diamagnetism of a single atom. When we are placing an atom in a uniform magnetic field $\mathbf{B}$, the electrons perform a Larmor precession with angular frequency $eB/2m_e$ giving the atom a diamagnetic moment. We can describe the uniform field by the vector potential $\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r}$, which is in the London gauge. For weak fields we can neglect the effect of the magnetic field on the wavefunctions (for the superconductor we say the wavefunction is rigid). Then, the formal angular momentum quantum numbers stay the same and locally the canonical momentum is not changed. However, since $\mathbf{p} = m_e\mathbf{v} - e\mathbf{A}$, the electrons of the atom have now an extra local velocity $e\mathbf{A}/m_e$ in the same way as in the superconductor. For the atom the velocity is $(e\mathbf{B}/2m_e) \times \mathbf{r}$, i.e. the expected precession velocity. The only difference between the atom and the superconductor is the fact that for the atom the diamagnetic screening current is much too small to screen the applied field on the atomical length scale, whereas in the superconductor it is strong enough to restrict the field to a region within a penetration depth λ_L from the surface.

1.2 Flux Quantization

In the previous section we have shown that the macroscopic quantum model of superconductivity is consistent with the phenomenological laws deduced using classical reasoning. We now examine the quantum mechanical consequences of the model. The first example is ***fluxoid quantization*** in multiply connected superconductors (see Fig. 1.3).

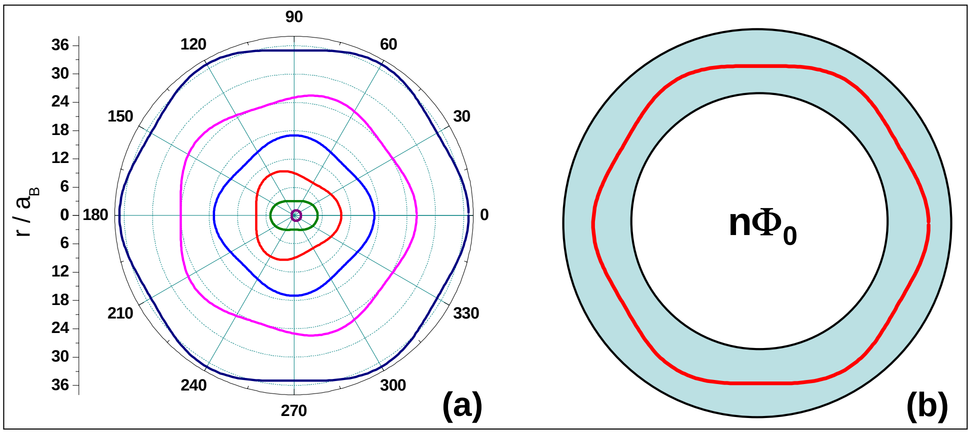


Figure 1.3: Stationary quantum states: (a) The standing electron wave around the nucleus of an atom at $r = 0$ resulting in the Bohr-Sommerfeld quantization of the angular momentum. (b) The standing wave of the macroscopic wave function representing the superconducting state in a superconducting cylinder resulting in flux quantization.

We start our discussion with a simple ***Gedanken*** experiment. We take a superconducting ring and generate (just by magnetic induction) a supercurrent in the ring. Since the superconductor has zero dc resistance this supercurrent should be stable, that is, the considered system is in a stationary state. Of course we can change this state by changing the induction process generating the supercurrent. Classically we would expect that in this way we can generate arbitrary supercurrents in the ring. However, after having learnt that we have to consider the superconductor as a macroscopic quantum system, we have to revise the expectation. The quantum mechanical treatment of microscopic systems teaches us that stationary states are determined by certain quantum conditions. Within Bohr's model for atoms the stationary states for the electrons are determined by the quantization condition for the angular momentum. As shown in Fig. 1.3a this is equivalent to the requirement that the electron wave is not interfering destructively. In the same way we expect a stationary state for the supercurrent along the ring only if the macroscopic wave function describing the whole ensemble of superelectrons is not interfering destructively (see Fig. 1.3b). Therefore, we immediately expect a quantization condition. This has been first supposed by **Fritz London**.[21] He came to the conclusion that the magnetic flux enclosed by a superconducting ring can only have discrete values given by multiples of a flux quantum Φ_0^L. London suggested the value

$$\Phi_0^L \;=\; \frac{h}{e} \simeq 4\times 10^{-15}\mathrm{Vs}\ . \tag{1.2.1}$$

London derived this value for the flux quantum, since he presumed that single electrons are carrying the supercurrent. The fact that Cooper pairs are carrying the supercurrent became clear only after the development of the BCS theory in 1957.[22]

Before discussing the experimental observation of flux quantization, we use the macroscopic quantum model of superconductivity to mathematically derive the quantization condition. For simplicity we will

[21] F. London, ***Superfluids***, Wiley, New York (1950).
[22] J. Bardeen, L.N. Cooper, J.R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

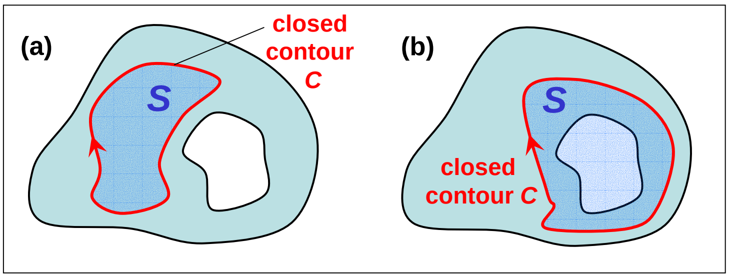


Figure 1.4: Different possibilities for closed contours within a superconducting medium: (a) The path is in a simply connected superconducting region. (b) The path is in a multiply connected region.

assume a homogeneous and isotropic superconductor. We start with the expression (1.1.65) for the supercurrent density

$$\Lambda \mathbf{J}_s = -\left\{\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q^\star}\nabla\theta(\mathbf{r},t)\right\} \tag{1.2.2}$$

and integrate this expression around a closed contour C. From ***Stoke's theorem*** we know that

$$\oint_C \mathbf{A}\cdot d\mathbf{l} = \int_S (\nabla\times\mathbf{A})\cdot d\mathbf{s} = \int_S \mathbf{B}\cdot d\mathbf{s} \ , \tag{1.2.3}$$

where S is the surface defined by the closed contour C (see Fig. 1.4) and $\mathbf{B}$ is the flux density associated with the vector potential $\mathbf{A}$. Using Stoke's theorem we can rewrite (1.2.2) as

$$\oint_C (\Lambda\mathbf{J}_s)\cdot d\mathbf{l} + \int_S \mathbf{B}\cdot d\mathbf{s} = \frac{\hbar}{q^\star}\oint_C \nabla\theta\cdot d\mathbf{l} \ . \tag{1.2.4}$$

We first evaluate the integral on the right hand side of (1.2.4). We know from the vector calculus that the integral of the gradient of a scalar function along the path defined by points $\mathbf{r}_1$ and $\mathbf{r}_2$ is given by

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \nabla\theta\cdot d\mathbf{l} = \theta(\mathbf{r}_2,t) - \theta(\mathbf{r}_1,t) \ . \tag{1.2.5}$$

We see that if $\mathbf{r}_1 \to \mathbf{r}_2$ such that a closed path is formed the integral is zero. However, in general this is not true, since the specific value of the phase of ψ is not well defined. Indeed there exists an infinite amount of possible phase values, because for integer values of n all values $\theta_n = \theta_0 + 2\pi n$ give the same value of

$$\psi(\mathbf{r},t) = \sqrt{n_s^\star}e^{\imath(\theta_0+2\pi n)} \ . \tag{1.2.6}$$

That is, although the macroscopic wavefunction ψ is well defined, this is not the case for the phase:

$$\theta(\mathbf{r},t) = \theta_0(\mathbf{r},t) + 2\pi n \ . \tag{1.2.7}$$

The phase is specified only within modulo 2π of its principal value θ_0 ranging in the interval $[-\pi,\pi]$. Since θ_0 is single valued, we obtain for the integral of the phase gradient along a closed contour

$$\oint_C \nabla\theta\cdot d\mathbf{l} = \lim_{\mathbf{r}_2\to\mathbf{r}_1}[\theta(\mathbf{r}_2,t) - \theta(\mathbf{r}_1,t)] = 2\pi n \ . \tag{1.2.8}$$

With this result (1.2.4) becomes

$$\boxed{\oint_C (\Lambda \mathbf{J}_s) \cdot d\mathbf{l} + \int_S \mathbf{B} \cdot d\mathbf{s} \;=\; n\,\frac{h}{q^\star} = n\,\Phi_0} \tag{1.2.9}$$

with the ***flux quantum***

$$\boxed{\Phi_0 \equiv \frac{h}{|q^\star|} = \frac{h}{2e} \;=\; 2.067\,833\,636(81) \times 10^{-15}\mathrm{Vs}\ .} \tag{1.2.10}$$

Here, we have set $|q^\star| = 2e$ in giving a value for the flux quantum and have replace n by $-n$ in (1.2.9) with no loss of generality. The quantity Φ_0 represents the flux quantum and is the smallest amount of flux included by the closed contour line.

The consequences of (1.2.9) can be easily seen by considering Fig. 1.4:

1. We first consider the case (a), where the surface S defined by the contour C is in a simply connected superconducting region (see Fig. 1.4a). We have to recall that we are performing the integration along the closed contour by imagining a line integration between two points $\mathbf{r}_1$ and $\mathbf{r}_2$ including the limit $\mathbf{r}_2 \rightarrow \mathbf{r}_1$. Since (1.2.9) holds for all contour lines, we also have to include the case where the size of the contour just has shrunk to zero. In this case both integrals in (1.2.9) vanish and we find $n = 0$ for the simply connected superconductor.[23] This result is expected since the condition $n = 0$ just yields the integral form of the second London equation.

2. We next consider the case of a multiply connected superconductor as illustrated in Fig. 1.4b. Here, the important point is that the surface S defined by the contour C now contains both superconducting and nonsuperconducting regions (in the most simple case the nonsuperconducting region is just a hole). Thus, if we are closing the line integral by applying the limit $\mathbf{r}_2 \rightarrow \mathbf{r}_1$, we somehow have built in a "memory" in our path: We know that we have enclosed a nonsuperconducting region into the contour. In other words, the phases at the points $\mathbf{r}_2$ and $\mathbf{r}_1$ are now distinct. Therefore, although the principal value of the two phases is the same, the difference between them is $2\pi\, n$.

1.2.1 Flux and Fluxoid Quantization

Fluxoid Quantization

The left-hand side of (1.2.9) is denoted as the ***fluxoid*** and hence this equation is a statement of ***fluxoid quantization***. Note that the externally applied magnetic flux is not necessarily quantized. However, we have to take into account ***both*** the external applied flux ***and*** the flux generated by the induced supercurrent in our calculation. Then, the total flux threading the multiply connected superconductor cannot be arbitrary but must have discrete values corresponding to integer multiples of the flux quantum.

Flux Quantization

We now discuss a superconducting cylinder as sketched in Fig. 1.5. We assume that the wall of the cylinder is much thicker than the London penetration depth λ_L. If we apply a small magnetic field (much

[23] Of course this is only true, if there are no singularities in the supercurrent density $\mathbf{J}_s$ or the flux density $\mathbf{B}$.

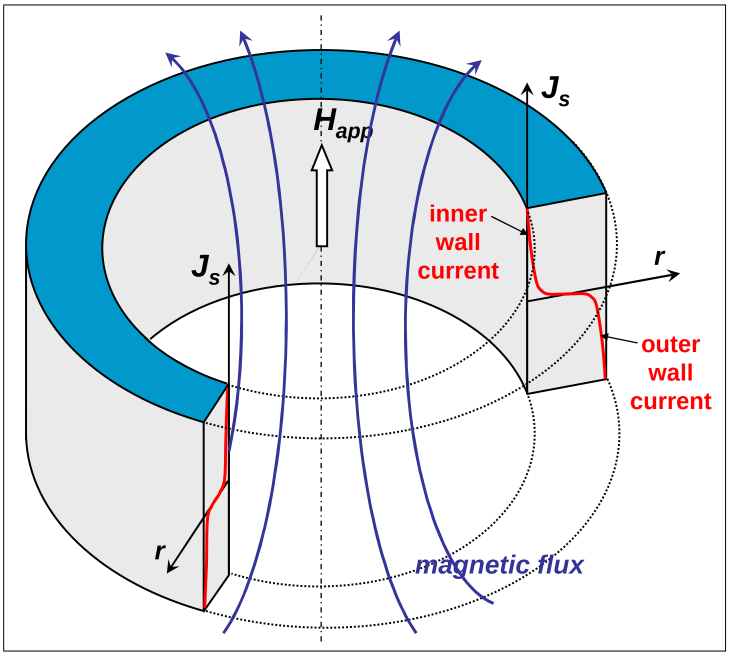


Figure 1.5: Sketch of a superconducting cylinder in the presence of an applied magnetic field along the cylinder axis. Also shown is the expected supercurrent distribution in the wall of the cylinder. For a wall thickness much larger than the superconducting screening length λ_L, the supercurrent density in the center of the wall is negligible.

smaller than the critical field of the superconducting material) after cooling down the cylinder below the transition temperature of the superconducting material, no flux will thread the superconducting cylinder. There are screening currents flowing on the outer surface of the cylinder wall screening the magnetic field.

The more interesting case is obtained by applying the magnetic field during cooling down. In this case below the transition temperature a screening current flows on the outer surface of the cylinder to expel the applied magnetic field from the superconducting material. In the same way, a screening current with opposite direction is flowing on the inner surface of the cylinder to keep the applied magnetic field outside the superconducting material.

We now use (1.2.9) to analyze the amount of flux trapped in the superconducting cylinder. In the classical case the currents flowing on the inner surface would be constraint only by Ampère's law. Then, classically we could trap arbitrary amounts of magnetic flux in the cylinder by simply varying the magnetic field applied during cooling down. However, in an exact quantum mechanical treatment we have to satisfy also the fluxoid quantization condition (1.2.9). Since the thickness of the superconducting material is much larger than the London penetration depth λ_L, we can choose a closed contour deep inside the superconducting material, where in very good approximation we have $\mathbf{J}_s = 0$. The fluxoid quantization condition then simplifies to

$$\int\limits_S \mathbf{B} \cdot d\mathbf{s} \quad = \quad n\,\Phi_0 \ . \qquad (1.2.11)$$

That means, if we are removing the applied magnetic field after cooling down, the magnetic flux trapped in the cylinder exactly is an integer multiple of the flux quantum. Therefore, (1.2.11) can be considered as a statement of ***flux quantization***.

Flux Trapping

We briefly discuss the question why the superconducting ring does not expel the magnetic flux but keeps it trapped inside the ring after switching of the magnetic field. The answer can be derived from the 1. London equation saying that the electric field deep inside the body of the superconducting cylinder must be zero, since $\partial \mathbf{J}_s/\partial t = 0$ there. This applies also for situations where the supercurrent in the ring is changing, since this supercurrent is only flowing on the surface within λ_L (see Fig. 1.5). With $\mathbf{E} = -\partial \mathbf{A}/\partial t - \nabla\phi$ and $\nabla\phi = 0$ we can obtain

$$\oint \mathbf{E}\cdot d\mathbf{l} \;=\; -\frac{\partial}{\partial t}\oint \mathbf{A}\cdot d\mathbf{l} = -\frac{\partial}{\partial t}\int_S \mathbf{B}\cdot d\mathbf{s} \;=\; -\frac{\partial\Phi}{\partial t} \;, \tag{1.2.12}$$

where Φ is the magnetic flux enclosed within the contour. Taking the contour deep inside the superconductor we have $\mathbf{E} = 0$ and hence $\frac{\partial\Phi}{\partial t} = 0$. That is, the flux enclosed in the cylinder has to stay constant. The supercurrents flowing on the surface of the cylinder are adjusting themselves so that the flux in the ring is not changing, i.e. the flux stays trapped.

1.2.2 Experimental Proof of Flux Quantization

In 1961 the flux quantization in superconducting cylinders has been experimentally proved by two groups, **R. Doll** and **M. Näbauer** at the Walther-Meißner-Institute in Munich and **B. S. Deaver** and **W. M. Fairbank** at Stanford.[24,25] These very difficult experiments not only have demonstrated the quantization of magnetic flux in a superconducting cylinder but also for the first time proved the existence of Cooper pairs with charge $q^\star = 2e$ thereby confirming the prediction of the microscopic theory developed by **John Bardeen, Leon Cooper** and **Robert Schrieffer** (BCS-theory) in 1957.

The aim of the experiments was to show that the flux enclosed by a superconducting cylinder of wall thickness $d \gg \lambda_L$ can have only discrete values $\Phi_n = n\cdot\Phi_0$. In order to do so different amounts of magnetic flux have to be frozen in during cooling down the cylinder in an applied magnetic field $B_{\rm cool}$ and then the amount of trapped flux has to be measured with a precision much better than a single flux quantum. In order to obtain a large relative change of the magnetic flux from measurement to measurement only small values of $B_{\rm cool}$ have to be used resulting in a small number of trapped flux quanta. Since $\Phi = B\cdot S$, both small applied fields and a small cross-sectional area S, resp. diameter d of the cylinder have to be used. Note that for a cylinder area of $S = 1\,\mathrm{mm}^2$ a flux density B of only about 10^{-9} T is required to generate one flux quantum. This flux density is much smaller than that due to the earth magnetic field $B_e \simeq 2\times10^{-5}$T.

Both experimental groups were using small hollow cylinders with an outer diameter of the order of $10\,\mu$m. For such diameter, a flux density of $B \simeq 2\times10^{-5}$T is required to generate one flux quantum $\Phi_0 = 2\times10^{-15}\mathrm{Tm}^2$. Since this flux density is of the same order of magnitude as the one due to the earth magnetic field, a careful shielding of the earth magnetic field and other perturbing fields was necessary.

In the experiment of Doll and Näbauer a hollow Pb cylinder was used (see Fig. 1.6). The cylinder was obtained by evaporating Pb on a quartz fiber. In this cylinder they were trapping magnetic flux by cooling down the sample below the transition temperature in a small magnetic field applied parallel to the axis of the cylinder. The magnitude of the trapped magnetic flux has been determined by measuring the torque

[24] R. Doll, M. Nähbauer, *Experimental Proof of Magnetic Flux Quantization in a Superconducting Ring*, Phys. Rev. Lett. **7**, 51 (1961)

[25] B.S. Deaver, W.M. Fairbank, *Experimental Evidence for Quantized Flux in Superconducting Cylinders*, Phys. Rev. Lett. **7**, 43 (1961).

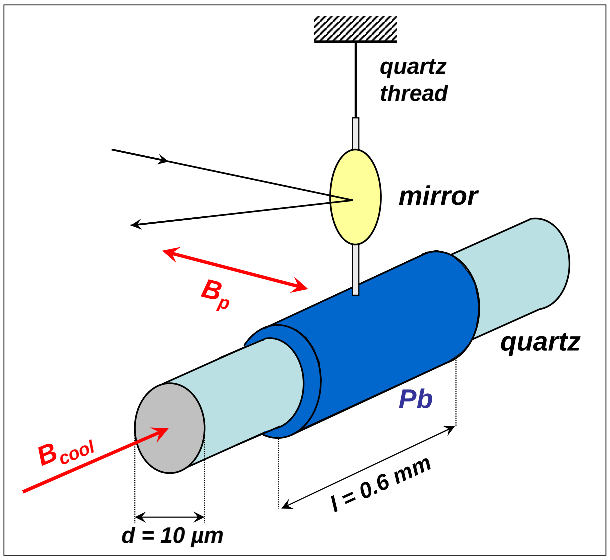


Figure 1.6: Sketch of the experimental configuration used by Doll and Näbauer in 1961 for the determination of the flux quantization in a superconducting cylinder (according to R. Doll, M. Näbauer, Phys. Rev. Lett. **7**, 51 (1961)).

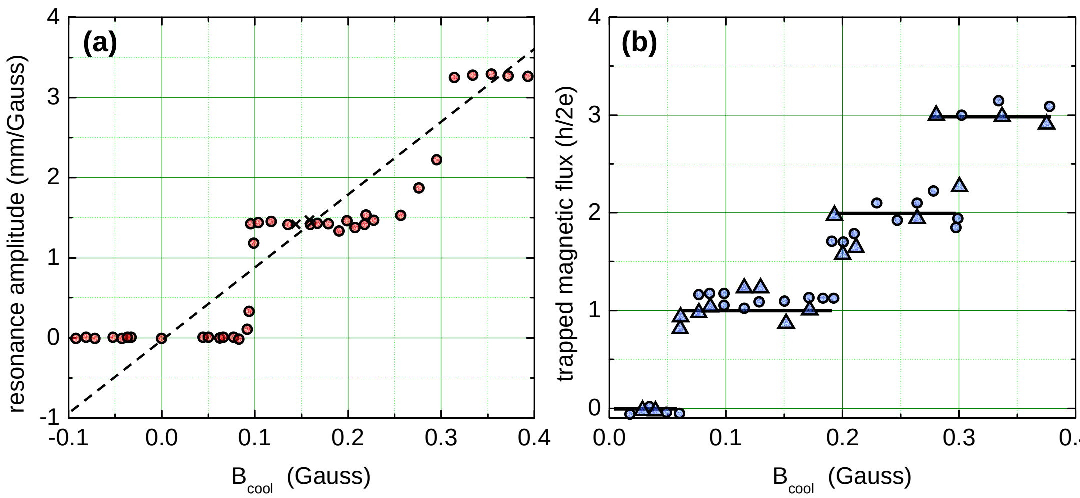


Figure 1.7: Magnetic flux trapped in a superconducting cylinder as a function of the applied magnetic flux density during cooling down the cylinder below the superconducting transition temperature. (a) Experiment by Doll and Näbauer (according to R. Doll, M. Näbauer, Phys. Rev. Lett. **7**, 51 (1961) and Zeitschrift für Physik **169**, 526 (1962)). (b) Experiment by Deaver and Fairbank (according to B. S. Deaver, W. M. Fairbank, Phys. Rev. Lett. **7**, 43 (1961)).

$\mathbf{D} = \boldsymbol{\mu} \times \mathbf{B}_p$ due to a probe field B_p applied perpendicular to the cylinder axis. Here, $\boldsymbol{\mu}$ is the magnetic moment of the trapped magnetic flux. The measurement has been done by hanging up the cylinder using a thin quartz thread. The rotation of the cylinder has been measured by shining light on a mirror attached to the quartz thread. Since the torque is very small, static measurements of the torque could not be performed. Therefore, Doll and Näbauer used a so-called self-resonance method. In this method they were using the small torque created by the probe field B_p for the excitation of a torque vibration of the system. In case of resonance the resulting resonance amplitude is becoming large enough to be measured. The resonance amplitude is proportional to the exciting torque, which in turn is proportional to the flux trapped in the cylinder. Of course, in order to excite the vibration the direction of the probe field $\mathbf{B}_p$ has to be switched at the resonance frequency.

In the experiment by Deaver and Fairbank a tiny Sn tube with a length of about 0.9 mm, an inner diameter of 13 μm and a wall thickness of 1.5 μm was used. The cylinder was vibrated in the axial direction at a frequency of about 100 Hz and the resulting rf signal was detected via a pair of pick-up coils. The experimental results of both experiments are shown in Fig. 1.7. The results of both experiments were essentially identical and convincing. Although the cylinders were cooled down in different magnetic fields, the net magnetic flux trapped in the cylinder always occurred in quantized amounts. In this way the two groups experimentally demonstrated the limitations of a purely classical treatment of superconductivity.

1.2.3 Additional Topic: Rotating Superconductor

We consider the interesting case when a superconducting cylinder is rotating at an angular frequency Ω (see Fig. 1.8). In this case we have to modify our analysis given in Section 1.1.3. We have to rewrite Faraday's law as

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} = \mu_0 n_s^\star q^\star (\mathbf{v}_s - \mathbf{v}_\Omega) = \mu_0 (\mathbf{J}_s - n_s^\star q^\star \mathbf{v}_\Omega) \ . \tag{1.2.13}$$

Here, $\mathbf{J}$ is the net current density and $\mathbf{v}_s - \mathbf{v}_\Omega$ is the velocity of the superfluid relative to the velocity $\mathbf{v}_\Omega$ of the positively charged lattice. We can take the curl of (1.2.13) and take into account that $\nabla \times \mathbf{v}_\Omega = 2\mathbf{\Omega}$ to arrive at

$$\nabla \times \nabla \times \mathbf{B} = \mu_0 \nabla \times \mathbf{J}_s - \mu_0 n_s^\star q^\star \nabla \times \mathbf{v}_\Omega = \frac{\mu_0}{\Lambda} \left(\nabla \times (\Lambda \mathbf{J}_s) - \frac{2m^\star}{q^\star}\,\Omega \right) \ . \tag{1.2.14}$$

With $\nabla \times \nabla \times \mathbf{B} = -\nabla^2 \mathbf{B}$, $\mu_0/\Lambda = 1/\lambda_L^2$, and the 2. London equation $\nabla \times (\Lambda \mathbf{J}_s) = -\mathbf{B}$ we obtain

$$\nabla^2 \mathbf{B} = \frac{\mathbf{B} - \frac{2m^\star}{q^\star}\,\Omega}{\lambda_L^2} = \frac{\mathbf{B} - \mathbf{B}_L}{\lambda_L^2} \ . \tag{1.2.15}$$

Here, $\mathbf{B}_L = \frac{2m^\star}{q^\star}\,\Omega$ is the ***London field***.

The modified screening equation shows that the field inside the superconductor does no longer decay to zero but to the London field $\mathbf{B}_L$. That means that there is a finite field in a rotating superconductor. We can understand this from our discussion on page 22. The additional field is just required to give the electron systems a precession equal to the rotating lattice. For an observer in the reference frame rotating with the superconductor, the effect of this field is to compensate for the effect of the Coriolis force acting on the electrons. That means that for such an observer the dynamics of the electrons appear to be unaffected by the rotation.

Since the London field is spatially uniform, it follows from Ampère's rule that there is no net current flowing deep inside the superconductor. However, if there is no external magnetic field ($\mathbf{B} = 0$), there must be a screening current flowing on the outside surface of the superconductor (see Fig. 1.8).

We now discuss how the flux quantization is affected by the rotation. We have to replace eq.(1.2.9) by

$$\oint_C \Lambda (\mathbf{J}_s - n_s^\star q^\star \mathbf{v}_\Omega) \cdot d\mathbf{l} + \int_S \mathbf{B} \cdot d\mathbf{s} = n\,\Phi_0 \tag{1.2.16}$$

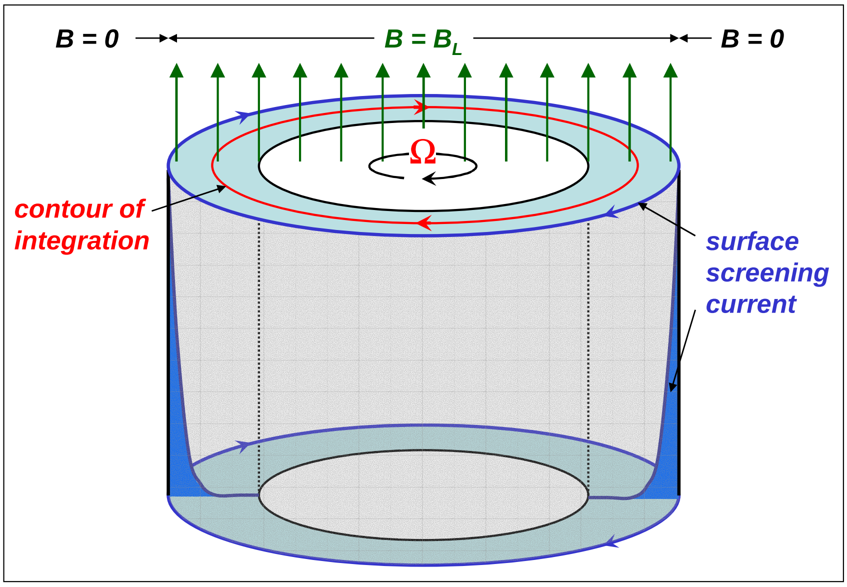


Figure 1.8: Magnetic fields and currents in a superconducting cylinder rotating at angular frequency Ω. The London field $\mathbf{B}_L$ fills the superconductor and the hole. A corresponding screening current flows only on the outer surface of the cylinder and decays exponentially inside the cylinder within the decay length λ_L.

For a contour deep inside the superconductor, where $\mathbf{J}_s = 0$, we obtain[26]

$$\Phi = n\,\Phi_0 + 2\frac{m^\star}{q^\star}\,\mathbf{\Omega}\cdot\mathbf{S} \;, \qquad (1.2.17)$$

where $\mathbf{S}$ is the area enclosed by the contour of integration. We see that the hole contains the usual quantized flux plus a non-quantized term due to the London field filling the hole as well as the bulk superconductor. For $n = 0$ the field in the hole is the same as in the bulk superconductor so that no screening current flows on the inner surface of the superconducting cylinder.[27]

[26]We use $\oint \mathbf{v}_\Omega \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{v}_\Omega)\cdot d\mathbf{s} = 2\mathbf{\Omega}\cdot\mathbf{S}$.

[27]In principle the field inside the hole should be detectable using e.g. a SQUID detector. However, a detailed analysis shows that the SQUID can only detect the relative angular velocities of the cylinder and the SQUID detector making the experimental setup difficult (see e.g. R.M. Brady, IEEE Trans. Magn. **MAG 17**, 861 (1981)).

1.3 Josephson Effect

In section 1.2 we have discussed the fluxoid quantization in multiply connected superconductors as the first consequence of the macroscopic quantum nature of superconductivity. It became evident that this phenomenon is a direct manifestation of the macroscopic quantum coherence of the superconducting state. In this section we discuss the ***Josephson effect*** as a second coherent phenomenon. The Josephson effect has been predicted by **Brian D. Josephson** in 1962 and is equally important as flux quantization.[28] Today, the Josephson effect is used for many applications of superconductivity in electronics, sensors and high frequency devices. The Josephson effect is observed, if two superconductors are weakly connected by an electrical contact. Such contact can be established in many different ways. Prominent examples are tunneling barriers, point contacts or normal conducting layers connecting the two superconducting electrodes. In the following we will denote such contact as ***Josephson junction***. The initial theoretical work by Josephson has considered only superconductor-insulator-superconductor (SIS) junctions.

Suppose we consider a SIS contact between two identical superconductors as sketched in Fig. 1.9. For a normal metal-insulator- normal metal (NIN) tunnel junction it is well known that for a thin enough tunneling barrier the normal electrons can tunnel through the barrier with the tunneling current density decaying exponentially with increasing barrier thickness. Now, for a SIS tunnel junction at zero temperature we do no longer have normal electrons at the Fermi level. Therefore, we expect that there is no tunneling current as long as the applied voltage V is smaller than twice the energy gap voltage $2\Delta/e$. For $eV \geq 2\Delta$ Cooper pairs can be broken up into to normal electrons, which in turn can tunnel through the barrier. As will be discussed later, this effect is indeed observed.

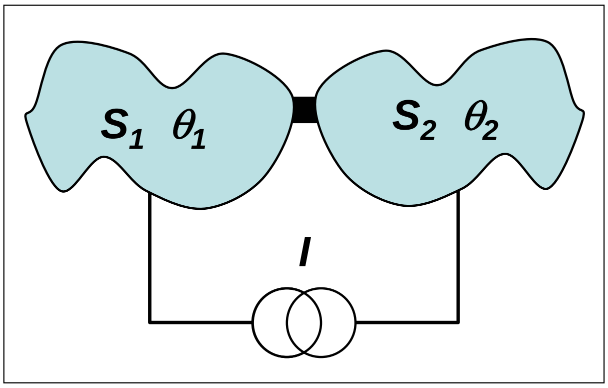
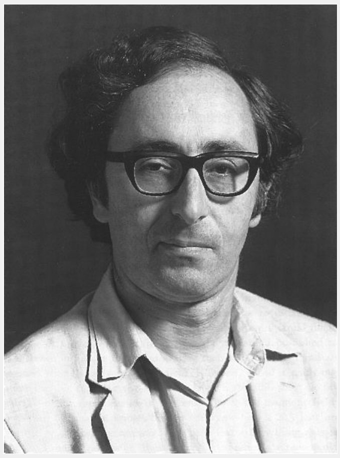


Figure 1.9: The Josephson effect: coherent phenomenon for two weakly coupled superconductors S_1 and S_2 with wave functions $\psi_1 = \sqrt{n_s^{\star 1}}e^{\imath\theta_1}$ and $\psi_2 = \sqrt{n_s^{\star 2}}e^{\imath\theta_2}$.

We also have to ask ourselves, whether or not it is possible for Cooper pairs to tunnel through a thin insulating barrier. The consensus in 1962 was that such events would not happen often enough to be measurable. The reason for that seems to be evident. Even the probability of a single electron to tunnel through a thin barrier is very small. Typically, the tunneling probability is $p_t \lesssim 10^{-4}$. Then, for a Cooper pair one would expect a probability p_t^2, which again is orders of magnitude smaller. However, in 1962 Brian Josephson changed this common reasoning. He discovered that the probability of a Cooper pair tunneling through the barrier is the same as that for a single electron. The reason is that the tunneling of Cooper pairs is a coherent process. That is, we should not consider the Cooper pair as two incoherent electron waves leaking through the barrier. Instead, it is the macroscopic wave function describing

[28] Brian D. Josephson, *Possible new effects in superconductive tunnelling*, Phys. Lett. **1**, 251–253 (1962).
For his theoretical predictions of the properties of a supercurrent through a tunnel barrier, in particular those phenomena which are generally known as the Josephson effects, Brian Josephson received the Nobel Price in physics in 1973 together with **Leo Esaki** and **Ivar Giaever**.

Brian David Josephson (born 1940):



Brian David Josephson was born on January 4, 1940 in Cardiff, Glamorgan, Wales. He is a British physicist whose discovery of the Josephson effect while a 22-year-old graduate student won him a share (with Leo Esaki and Ivar Giaever) of the 1973 Nobel Prize for Physics.
He entered Trinity College, Cambridge, in pursuit of an education in physics and received his bachelor's (1960) and master's and Ph.D. degrees (1964) there, publishing his first work while still an undergraduate. It dealt with certain aspects of the special theory of relativity and the Mößbauer effect. He was elected a fellow of Trinity College in 1962. He was a brilliant and assured student; one former lecturer recalled a special need for precision in any presentation to a class that included Josephson – otherwise, the student would confront the instructor politely after class and explain the mistake.
While still an undergraduate, Josephson became interested in superconductivity, and he began to explore the properties of a junction between two superconductors that later came to be known as a Josephson junction. Josephson extended earlier work in tunneling, the phenomenon by which electrons functioning as radiated waves can penetrate solids, done by Esaki and Giaever. He showed theoretically that tunneling between two superconductors could have very special characteristics, e.g., flow across an insulating layer without the application of a voltage. If a voltage is applied, the current stops flowing and oscillates at high frequency. This was the Josephson effect. Experimentation confirmed it, and its confirmation in turn reinforced the earlier BCS theory of superconductor behavior. Applying Josephson's discoveries with superconductors, researchers at International Business Machines Corporation had assembled by 1980 an experimental computer switch structure, which would permit switching speeds faster than those possible with conventional silicon-based chips, increasing data processing capabilities by a vast amount.
He went to the United States to be a research professor at the University of Illinois in 1965-66 and in 1967 returned to Cambridge as assistant director of research. He was appointed reader in physics in 1972 and professor of physics in 1974. He was elected a fellow of the Royal Society in 1970.
A few years before the Nobel award, Josephson grew interested in the possible relevance of Eastern mysticism to scientific understanding. In 1980 he and V.S. Ramachandran published an edited transcript of a 1978 international symposium on consciousness at Oxford under the title ***Consciousness and the Physical World***.

the entire ensemble of superconducting electrons that is tunneling through the barrier. Only one year later, **Philip W. Anderson** and **John M. Rowell**[29] had experimentally confirmed the prediction of Brian Josephson, which again is a direct consequence of the macroscopic quantum nature of the superconducting state.

We can consider two weakly coupled superconductors as sketched in Fig. 1.9 also as a molecule. In the same way as we obtain molecular binding due to the overlap of the wavefunctions of the electrons of two hydrogen atoms, the overlap of the macroscopic wavefunctions of the two superconductors results in a finite binding energy. This binding energy is called the ***Josephson coupling energy*** and will be derived later in chapter 2.

1.3.1 The Josephson Equations

In deriving the Josephson equations we follow general arguments introduced by **L.D. Landau** and **E.M. Lifschitz**.[30]

[29] P. W. Anderson and J. M. Rowell, *Probable Observation of the Josephson Superconducting Tunneling Effect*, Phys. Rev. Lett. **10**, 230–232 (1963).

[30] L.D. Landau, E.M. Lifschitz, *Lehrbuch der Theoretischen Physik*, Bd. IX, Akademie-Verlag, Berlin (1980).

First Josephson Equation: current-phase relation

We first speculate what determines the supercurrent between two weakly connected superconductors. It certainly can depend on the Cooper pair densities $|\psi_1|^2 = n^\star_{s,1}$ and $|\psi_2|^2 = n^\star_{s,2}$ in the junction electrodes. However, since the coupling between the two superconductors is weak and, hence, the supercurrent density between them is small, we can assume that the supercurrent density between the two junction electrodes is not changing $|\psi|^2$. However, although the amplitude of the wave functions in the electrodes does not play a role, the supercurrent density certainly is expected to depend on the phase of the wave functions.

We already have learnt that the supercurrent density in a bulk superconductor depends on the gauge invariant phase gradient (compare (1.1.50) and (1.1.62)) as

$$J_s(\mathbf{r},t) = \frac{q^\star n_s^\star \hbar}{m^\star}\left[\nabla\theta(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\mathbf{A}(\mathbf{r},t)\right] = \frac{q^\star n_s^\star \hbar}{m^\star}\gamma(\mathbf{r},t) \ . \tag{1.3.1}$$

We simplify our discussion by assuming that (i) the current density can be considered homogeneous. We will see later that this assumption can be always made if the junction area is small enough. We are now applying expression (1.3.1) to the case of two weakly connected superconductors. Doing so, we further assume that (ii) the phase gradient γ is varying negligibly in the superconducting electrodes (see Fig. 1.10). This is always a good assumption as long as the Cooper pairs density $n_s^\star$ in the electrodes is much larger than in the coupling region as shown in Fig. 1.10. Since $\mathbf{J}_s$ is the same in the electrodes and the junction area (current conservation), according to (1.3.1) the gauge-invariant phase gradient is negligibly small in the electrodes compared to the junction region. Then, we can replace the gauge invariant phase gradient $\gamma = \nabla\theta - \frac{2\pi}{\Phi_0}\mathbf{A}$ just by the ***gauge-invariant phase difference*** φ given by

$$\boxed{\begin{aligned}\varphi(\mathbf{r},t) &= \int_1^2 \gamma(\mathbf{r},t) = \int_1^2\left(\nabla\theta - \frac{2\pi}{\Phi_0}\mathbf{A}\right)\cdot d\mathbf{l}\\ &= \theta_2(\mathbf{r},t) - \theta_1(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot d\mathbf{l}\ .\end{aligned}} \tag{1.3.2}$$

We note that the integration path is along the direction of current (see Fig. 1.11). That is, for a SIS-type Josephson junction the path is across the insulating barrier form the superconductor one with phase θ_1 to the superconductor 2 with phase θ_2. For the geometry of Fig. 1.11 the integration path is from $-d/2$ to $+d/2$ and the differential line element is $d\mathbf{l}$.

According to (1.3.1) we expect the supercurrent density J_s to be a function only of φ, that is, $J_s = J_s(\varphi)$. Actually, according to (1.3.1) we expect $J_s \propto \varphi$. However, we have to take into account that any phase change of 2π in the wave functions of the junction electrodes results in the same wave function $\psi_{1,2}$. From this we can conclude that $J_s(\varphi)$ should not be a linear but a 2π-periodic function:

$$J_s(\varphi) = J_s(\varphi + n\,2\pi) \ . \tag{1.3.3}$$

Finally, in the absence of any current the phase gradient must be zero and both electrodes form a single superconductor with a common phase. That is, in this case we have $\theta_1 = \theta_2$ and hence

$$J_s(0) = J_s(n\cdot 2\pi) = 0 \ . \tag{1.3.4}$$

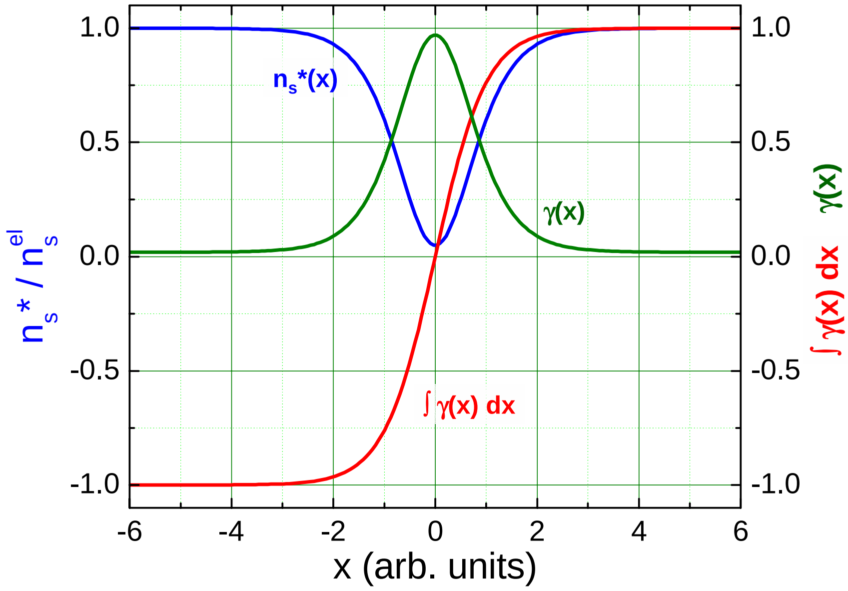


Figure 1.10: Sketch of the variation of the Cooper pair density $n_s^\star$ and the gauge-invariant phase gradient γ across a one-dimensional Josephson junction extending in x-direction. Also shown is the integral $\int \gamma dx$ of the gauge invariant phase gradient.

Summarizing our discussion we can conclude that the supercurrent density between the two junction electrodes in the most general case should have the form[31]

$$J_s(\varphi) \;=\; J_c \sin\varphi + \sum_{m=2}^{\infty} J_m \sin(m\varphi) \ . \tag{1.3.5}$$

Here, J_c is the ***critical or maximum Josephson current density***, which is determined by the coupling strength between the two junction electrodes. Expression (1.3.5) is the general formulation of the ***1. Josephson equation***. It also is denoted as the ***current-phase relation***, since it relates the supercurrent density to the phase difference. Rigorous theoretical treatment shows that in most cases (in particular in the case of weak coupling) the second term on the right hand side can be neglected. Then, (1.3.5) obtains the form

$$\boxed{J_s(\varphi) \;=\; J_c \sin\varphi \ , \qquad \text{(1. Josephson equation)}} \tag{1.3.6}$$

which was derived by Josephson in his original paper for the particular case of an insulating barrier. Here, due to the rapid decay of the wave function in the insulator, weak coupling was explicitly assumed. We will discuss this case in more detail in section 1.3.2.

The basic essence of (1.3.6) can be summarized as follows:

> The supercurrent density through a Josephson junction varies sinusoidally with the phase difference $\varphi = \theta_2 - \theta_1$ across the junction in the absence of any scalar and vector potentials.

[31] In general we could write J_s as a Fourier series of sine and cosine terms. However, eq.(1.3.4) requires that all the coefficients of the cosine terms vanish. The same conclusion can be drawn from the requirement of time reversal symmetry. Upon inversion of time, the Josephson current flows in opposite direction. Furthermore, since the time evolution of the macroscopic wavefunction is given by $\exp(-\imath\omega t)$, inversion of time requires that we are inverting the sign of the phase of the wavefunction. Then, time-invariance of the Josephson current requires that $J_s(\gamma) = -J_s(-\gamma)$ what excludes all cosine terms in a Fourier series.

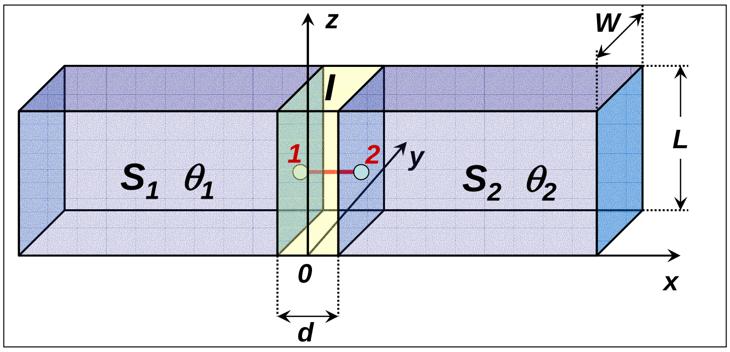


Figure 1.11: Sketch of a superconductor-insulator-superconductor (SIS) Josephson junction with a current source driving a current through the junction.

Above we have made the assumption of a homogenous supercurrent density allowing a one-dimensional treatment. This assumption can be relaxed by noting that the argument given above still holds if applied ***locally*** to each point (y,z) of the junction area. In particular, we can generalize the critical current density J_c to $J_c(y,z)$ (the junction area extends in the yz-plane, see Fig. 1.11). The current flow is always in x-direction so that there is no divergence of current density. That is, for any given y and z the supercurrent is flowing straight across the junction area. However, the current density given by (1.3.6) may now depend on y and z and we have to generalize the current-phase relation to

$$J_s(y,z,t) \quad = \quad J_c(y,z)\ \sin\varphi(y,z,t) \ . \tag{1.3.7}$$

Second Josephson Equation: voltage-phase relation

In order to derive the 2. Josephson equation we use the time derivative of the gauge invariant phase difference

$$\frac{\partial\varphi}{\partial t} \quad = \quad \frac{\partial\theta_2}{\partial t} - \frac{\partial\theta_1}{\partial t} - \frac{2\pi}{\Phi_0}\frac{\partial}{\partial t}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot d\mathbf{l} \ . \tag{1.3.8}$$

Substitution of the energy-phase relation (1.1.69)

$$-\hbar\frac{\partial\theta}{\partial t} \quad = \quad \frac{1}{2n_s^\star}\Lambda\mathbf{J}_s^2 + q^\star\phi \tag{1.3.9}$$

into (1.3.8) yields

$$\frac{\partial\varphi}{\partial t} \quad = \quad -\frac{1}{\hbar}\left(\frac{\Lambda}{2n_s^\star}\left[\mathbf{J}_s^2(2)-\mathbf{J}_s^2(1)\right] + q^\star\left[\phi(2)-\phi(1)\right]\right) - \frac{2\pi}{\Phi_0}\frac{\partial}{\partial t}\int_1^2 \mathbf{A}\cdot d\mathbf{l} \ . \tag{1.3.10}$$

Since the supercurrent density across the junction is continuous, we can use $\mathbf{J}_s(2) = \mathbf{J}_s(1)$ and obtain with $q^\star = 2e$ and $\Phi_0 = h/2e$

$$\frac{\partial\varphi}{\partial t} \quad = \quad \frac{2\pi}{\Phi_0}\int_1^2\left(-\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}\right)\cdot d\mathbf{l} \ . \tag{1.3.11}$$

Here, we have expressed the difference in the scalar potential ϕ as a line integral of its gradient. Since the term in parentheses is just the electric field (compare (1.1.54)), we can write

$$\boxed{\frac{\partial\varphi}{\partial t} = \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{E}(\mathbf{r},t)\cdot d\mathbf{l} \qquad \text{(2. Josephson equation)}\ .} \tag{1.3.12}$$

This equation is known as the ***second Josephson equation***. We note that $\int_1^2 \mathbf{E}(\mathbf{r},t)\cdot d\mathbf{l}$ corresponds to a voltage drop across the junction resulting in the difference $\Delta\mu = \mu_2 - \mu_1 = eV$ between the electrochemical potentials of the two superconductors. That is, the time derivative of φ is determined by $\Delta\mu$ as expected from the general energy-phase relation. Since $\Delta\mu$ is determined by the voltage drop across the junction, the second Josephson equation also is called the ***voltage-phase relation***. The ***voltage-phase relation*** and the ***current-phase relation*** (1.3.6) together with the expression for the ***gauge-invariant phase difference*** (1.3.2) represent the set of basic equations governing the behavior of Josephson junctions. The fact that $\dot{\varphi} \propto \Delta\mu$ can be viewed as a quantum interference effect of the macroscopic wave function in the two superconducting electrodes.

If a constant voltage V is applied to the Josephson junction, we obtain

$$\frac{\partial\varphi}{\partial t} = \frac{2\pi}{\Phi_0}V\ . \tag{1.3.13}$$

and the phase difference is growing linearly in time:

$$\varphi(t) = \varphi_0 + \frac{2\pi}{\Phi_0}V\cdot t\ . \tag{1.3.14}$$

Then, the Josephson current $I_s(t) = I_c \sin\varphi(t)$ is oscillating at the ***Josephson frequency***

$$\boxed{\frac{\nu}{V} = \frac{\omega}{2\pi V} = \frac{1}{\Phi_0} \simeq 483.597\,898(19)\ \frac{\mathrm{MHz}}{\mu\mathrm{V}}\ .} \tag{1.3.15}$$

We see that the Josephson junctions can be considered as a voltage controlled oscillator that can be used to generate very high frequencies (500 GHz at 1 mV). The linewidth and the power that can be achieved with Josephson junctions will be discussed later, when we are discussing applications of the Josephson effect. By taking two Josephson junctions made of different materials the proportionality constant between frequency and voltage has been compared and found to agree within 2×10^{-16}.[32] More recent experiments even found an agreement in the range of 10^{-19}.

1.3.2 Josephson Tunneling

So far we have not discussed the magnitude of the ***maximum Josephson current density*** J_c. In this subsection we derive an expression of this quantity for the case of Josephson junction with an insulating tunneling barrier of thickness d. That is, we consider the supercurrent density across a superconductor-insulator-superconductor Josephson junction as sketched in Fig. 1.11. To solve the problem we are using

[32] J.S. Tsai, A.K. Jain, J.E. Lukens, Phys. Rev. Lett. **51**, 316 (1983).

the so-called ***wave matching method***. Here, we solve the Schrödinger equation in the three regions, namely the two superconducting electrodes and the insulating barrier. The solutions will contain coefficients that can be determined by matching the solutions at the boundaries between the three regions.

We first start with the wave function in the superconducting electrodes. The supercurrent density at the edges of the junction electrodes at the positions $x = \pm d/2$ is given by the supercurrent density equation (1.1.50)

$$\mathbf{J}_s = q^\star n_s^\star(\mathbf{r},t)\left\{\frac{\hbar}{m^\star}\nabla\theta(\mathbf{r},t) - \frac{q^\star}{m^\star}\mathbf{A}(\mathbf{r},t)\right\} . \tag{1.3.16}$$

We already have found the relationship between the current density at the boundary to the insulator and the phase of the wave functions at each boundary. It is given by the current-phase relation. In order to derive the magnitude of the maximum Josephson current density J_c we make the same assumptions as in section 1.3.1. That is, we assume a uniform tunneling barrier. We further assume that the junction area $L \cdot W$ is small enough, so that the Josephson current density can be assumed uniform within the junction area. It will be discussed later, up to which length scale this approximation is valid.

We start our discussion with the energy-phase relation (1.1.69) for the superconducting electrodes, which directly follows from the Schrödinger equation. In the absence of any electric and magnetic field this equation can be written as

$$-\hbar\frac{\partial\theta}{\partial t} = \frac{1}{2n_s^\star}\Lambda\mathbf{J}_s^2 . \tag{1.3.17}$$

The term on the right hand side corresponds to the kinetic energy $E_0 = m^\star v_s^2/2$ of the moving superelectrons and therefore we can write

$$\frac{\partial\theta}{\partial t} = -\frac{E_0}{\hbar} . \tag{1.3.18}$$

Consequently, the time dependent macroscopic wave function can be written as

$$\psi(\mathbf{r},t) = \psi(\mathbf{r})\, \mathrm{e}^{-\imath(E_0/\hbar)t} , \tag{1.3.19}$$

where $\psi(\mathbf{r})$ is the time independent amplitude of the wave function.

We must now determine the wavefunction within the insulating barrier of thickness d. The barrier height V_0 is assumed larger than E_0. Then, the variation of the potential along the x-direction is given by a step-like function $V(x)$, which is zero outside and V_0 inside the barrier region (see Fig. 1.12). We know that classically for $V_0 > E_0$ the superelectrons cannot penetrate the barrier region. However, quantum mechanically the situation is different. Here, the superelectrons can tunnel through the barrier. In our discussion we consider only elastic processes, that is, the superelectrons maintain their energy. Therefore, the time evolution of the wavefunction is the same outside and inside the barrier and we have to consider only the time independent part. Moreover, since within the barrier we are in a region of constant potential energy V_0, the time dependent Schrödinger-like equation (1.1.48) can be written as the time independent Schrödinger equation

$$-\frac{\hbar^2}{2m^\star}\nabla^2\psi(\mathbf{r}) = (E_0 - V_0)\psi(\mathbf{r}) . \tag{1.3.20}$$

Using now our simplifying assumptions of a homogeneous barrier and homogeneous supercurrent flow we have to consider only the x-dependence, i.e. we have to solve only a one-dimensional problem.

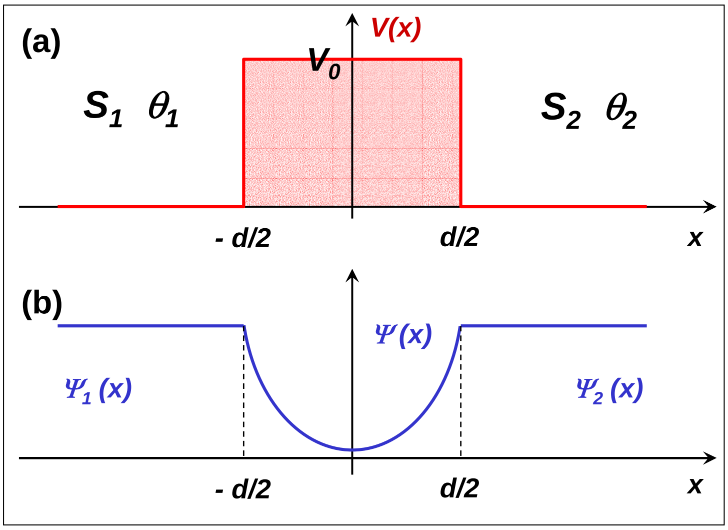


Figure 1.12: (a) Sketch of the model potential $V(x)$ of a superconductor-insulator-superconductor Josephson junction. (b) Time independent part of the wave function.

The solution of (1.3.20) can be written as a sum of decaying and growing exponentials or equivalently as

$$\psi(x) \quad = \quad A\cosh(\kappa x) + B\sinh(\kappa x) \ , \tag{1.3.21}$$

where the ***characteristic decay constant*** κ is determined by the barrier properties and is given by

$$\kappa \quad = \quad \sqrt{\frac{2m^\star(V_0 - E_0)}{\hbar^2}} \ . \tag{1.3.22}$$

The coefficients A and B are determined by the boundary conditions at $x = \pm d/2$:

$$\psi(-d/2) \quad = \quad \sqrt{n_1^\star}\, \mathrm{e}^{\imath\theta_1} \tag{1.3.23}$$

$$\psi(+d/2) \quad = \quad \sqrt{n_2^\star}\, \mathrm{e}^{\imath\theta_2} \ . \tag{1.3.24}$$

Here, $\sqrt{n_{1,2}^\star}$ and $\theta_{1,2}$ are the magnitude and the phase of the wave function at the boundaries $x = \pm d/2$. With these boundary conditions we obtain from (1.3.21)

$$\sqrt{n_1^\star}\, \mathrm{e}^{\imath\theta_1} \quad = \quad A\cosh(\kappa d/2) - B\sinh(\kappa d/2) \tag{1.3.25}$$

$$\sqrt{n_2^\star}\, \mathrm{e}^{\imath\theta_2} \quad = \quad A\cosh(\kappa d/2) + B\sinh(\kappa d/2) \tag{1.3.26}$$

and hence by solving for A and B

$$A \quad = \quad \frac{\sqrt{n_1^\star}\, \mathrm{e}^{\imath\theta_1} + \sqrt{n_2^\star}\, \mathrm{e}^{\imath\theta_2}}{2\cosh(\kappa d/2)} \tag{1.3.27}$$

$$B \quad = \quad -\frac{\sqrt{n_1^\star}\, \mathrm{e}^{\imath\theta_1} - \sqrt{n_2^\star}\, \mathrm{e}^{\imath\theta_2}}{2\sinh(\kappa d/2)} \ . \tag{1.3.28}$$

We now have to recall that for the supercurrent density is given by (compare (1.1.47))

$$\mathbf{J}_s \quad = \quad \frac{q^\star}{m^\star}\, \Re\left\{\psi^\star \left(\frac{\hbar}{\imath}\nabla\right)\psi\right\} \ . \tag{1.3.29}$$

With the wavefunction from (1.3.21) the current density is obtained to

$$\mathbf{J}_s = \frac{q^\star}{m^\star}\kappa\hbar \; \Im\{A^\star B\} \quad . \tag{1.3.30}$$

Substituting (1.3.27) and (1.3.28) in (1.3.30) yields the supercurrent density

$$\boxed{\mathbf{J}_s = \mathbf{J}_c \sin(\theta_2 - \theta_1)} \tag{1.3.31}$$

with the ***maximum Josephson current density***

$$\boxed{\mathbf{J}_c = -\frac{q^\star}{m^\star}\kappa\hbar\frac{\sqrt{n_1^\star n_2^\star}}{2\sinh(\kappa d/2)\cosh(\kappa d/2)} = -\frac{q^\star\hbar\kappa}{m^\star}\frac{\sqrt{n_1^\star n_2^\star}}{\sinh(2\kappa d)} \quad .} \tag{1.3.32}$$

Here we have used $2\sinh(x)\cosh(x) = \sinh(2x)$. We see that from our analysis we not only obtain the value for the maximum Josephson current density but also the current-phase relation, which has been derived from our qualitative discussion above.

In real junctions the barrier height V_0 typically is of the order of a few eV and therefore the decay length $1/\kappa$ less than a nanometer. Since the thickness of the tunnel barrier is usually a few nanometer, we have $\kappa d \gg 1$. In this case we can use the approximation $\sinh(2\kappa d) \simeq \frac{1}{2}\exp(2\kappa d)$. Hence, the maximum Josephson current density decays exponentially with increasing thickness of the tunneling barrier. With $q^\star = -2e$ and $m^\star = 2m$ we obtain

$$\mathbf{J}_c = \frac{e\hbar\kappa}{m} 2\sqrt{n_1^\star n_2^\star} \exp(-2\kappa d) \quad . \tag{1.3.33}$$

We note that the Josephson equations also can be derived by considering two separate quantum mechanical systems that are weakly coupled by a coupling Hamilton operator (transfer Hamiltonian approach). This derivation is discussed in Appendix A

Summary

- Macroscopic quantum model of superconductors:

  Superconductors can be described by a macroscopic wave function

$$\psi(\mathbf{r},t) = \psi_0(\mathbf{r},t)\, e^{\imath\theta(\mathbf{r},t)} \ .$$

  The wave function describes an ensemble of a macroscopic number of superconducting particles of mass $q^\star$ and mass $m^\star$ moving in the electromagnetic potentials ϕ and $\mathbf{A}$.

  In contrast to the usual quantum mechanics interpretation of $|\psi|^2$ as the probability to find a particle at position $\mathbf{r}$ at time t, $|\psi|^2$ is associated with the density $n_s^\star(\mathbf{r},t)$ of the superconducting particles.

- By considering the current density in a superconductor as a quantum mechanical probability current, the current density

$$\mathbf{J}_s = \frac{\hbar n_s^\star q^\star}{m^\star}\left\{\nabla\theta(\mathbf{r},t) - \frac{q^\star}{\hbar}\mathbf{A}(\mathbf{r},t)\right\} = \frac{\hbar n_s^\star q^\star}{m^\star}\left\{\nabla\theta(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\mathbf{A}(\mathbf{r},t)\right\}$$

  is obtained. That is, the current density in a superconductor is proportional to the gauge invariant phase gradient

$$\gamma(\mathbf{r},t) = \nabla\theta(\mathbf{r},t) - \frac{q^\star}{\hbar}\mathbf{A}(\mathbf{r},t) = \nabla\theta(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\mathbf{A}(\mathbf{r},t) \ .$$

- The phenomenological London equations are obtained by taking the time derivative and the curl of the expression for the supercurrent density. In its linearized form they are given by

$$\begin{aligned} \frac{\partial}{\partial t}(\Lambda\mathbf{J}_s) &= \mathbf{E} \qquad \text{(1. London equation)} \\ \nabla\times(\Lambda\mathbf{J}_s) &= -\mathbf{B} \qquad \text{(2. London equation)} \end{aligned} \tag{1.3.34}$$

  Here, $\Lambda = m^\star/n_s^\star q^{\star 2} = \mu_0\lambda_L^2$ is the London parameter and λ_L the London penetration depth. The 1. and 2. London equation phenomenologically describe the perfect conductivity and perfect diamagnetism of superconductors.

- Fluxoid and flux quantization:

  The fluxoid $\oint_C(\Lambda\mathbf{J}_s)\cdot d\mathbf{l} + \int_S \mathbf{B}\cdot d\mathbf{s}$ threading a multiply connected superconductor is quantized in units of the flux quantum $\Phi_0 = 2.068\times 10^{-15}$Vs:

$$\oint_C(\Lambda\mathbf{J}_s)\cdot d\mathbf{l} + \int_S \mathbf{B}\cdot d\mathbf{s} = n\,\Phi_0$$

  If the integration path can be taken in a region where $\mathbf{J}_s = 0$, we arrive at the flux quantization:

$$\Phi = \int_S \mathbf{B}\cdot d\mathbf{s} = n\,\Phi_0$$

- The Josephson effect:

  The current between two weakly coupled superconductors is proportional to the sine of the gauge invariant phase different φ (1. Josephson equation):

$$\begin{aligned}\mathbf{J}_s(\mathbf{r},t) &= \mathbf{J}_c(\mathbf{r},t)\,\sin\varphi(\mathbf{r},t)\\ \varphi(\mathbf{r},t) &= \theta_2(\mathbf{r},t)-\theta_1(\mathbf{r},t)-\frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot d\mathbf{l}\end{aligned}$$

  In the presence of a finite potential difference $\Delta\mu = eV$ between the two superconductors, the gauge invariant phase difference φ changes in time as (2. Josephson equation)

$$\frac{\partial\varphi}{\partial t} = \frac{2eV}{\hbar} = \frac{2\pi}{\Phi_0}V\ .$$

  The Josephson current then oscillates in time as $\mathbf{J}_s = \mathbf{J}_c\sin(\omega t+\varphi_0)$ at a frequency $\omega/2\pi =$ 483.6 GHz/mV.