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Exercise 1

Exercises to the Lecture

Superconductivity and Low Temperature Physics I
WS 2023/2024

1 Basic Properties of Superconductors

1.1 AC Conductivity of Normal Metals and Superconductors

Exercise:

We consider a normal metal with volume V containing N charge carriers of mass m_n resulting in the charge carrier density $n = N/V$. The charge current density $\mathbf{J}_q$ is connected to the drift velocity $\mathbf{v}_n$ by $\mathbf{J}_q = nq_n\mathbf{v}_n$, where $q_n = -e$ for electrons. In the presence of an electric field $\mathbf{E}(t)$ drift velocity $\mathbf{v}_n(t)$ follows the Drude relaxation equation

$$m_n \left(\frac{\partial}{\partial t} + \Gamma_n \right) \mathbf{v}_n(t) \quad = \quad q_n \mathbf{E}(t)$$

with the momentum relaxation rate $\Gamma_n = 1/\tau_n$.

We also consider a superconducting metal which we describe by a simple two-fluid model. That is, we assume that at finite temperatures there are quasiparticle excitations, forming a normal fluid of density n_n. Their drift velocity $\mathbf{v}_n$ again follows the Drude relaxation equation

$$m_n \left(\frac{\partial}{\partial t} + \Gamma_n \right) \mathbf{v}_n \quad = \quad q_n \mathbf{E} \ .$$

In addition, we have a superfluid component formed by superconducting charge carries of density $\widetilde{n}_s$, charge q_s and mass m_s. Their drift velocity $\mathbf{v}_s$ follows the analogue relaxation equation

$$m_s \left(\frac{\partial}{\partial t} + \Gamma_s \right) \mathbf{v}_s \quad = \quad q_s \mathbf{E} \ ,$$

where $\Gamma_s \to 0$ due to the dissipationless flow of the superfluid component. The total charge current density is given by the sum of the normal and superfluid component

$$\mathbf{J} \quad = \quad \mathbf{J}_s + \mathbf{J}_n \ = \ q_s \widetilde{n}_s \mathbf{v}_s + q_n n_n \mathbf{v}_n$$

Usually, in metals we have $q_n = -e$, $m_n = m$ (m = electron mass), $q_s = -2e$, $m_s = 2m$ and a pair density $\widetilde{n}_s = n_s/2$ ($n_s = n - n_n$, with n the total electron density) and we obtain

$$\mathbf{J} \;=\; -e(n_s\mathbf{v}_s + n_n\mathbf{v}_n)$$

within the two fluid description. The minus sign expresses the fact that the drift velocity of the electrons is antiparallel to the technical current density. According to London theory, the drift velocity of the superfluid component, $\mathbf{v}_s = \frac{\hbar}{m_s}\boldsymbol{\nabla}\theta - \frac{q_s}{m_s}\mathbf{A}$, is determined by the gradient of the phase of the macroscopic wave function describing the superconducting condensate and the vector potential $\mathbf{A}$.

(a) Calculate the time dependent charge current density $\mathbf{J}_q(t)$ in a normal metal for the case of a harmonic time dependence of the electric field $\mathbf{E}(t) = \mathbf{E}_0\exp(-\imath\omega t)$ and derive an expression for the complex ac conductivity $\sigma(\omega) = \delta\mathbf{J}_q/\delta\mathbf{E}$ for the case $t/\tau_n \gg 1$.

(b) Calculate the complex ac conductivity $\sigma(\omega)$ in a superconductor, assuming again a harmonic time dependence $\mathbf{E}(t) = \mathbf{E}_0\mathrm{e}^{-\imath\omega t}$ and $\mathbf{J}(t) = \mathbf{J}_0\mathrm{e}^{-\imath\omega t}$ and the constitutive relation $\mathbf{J} = \sigma\mathbf{E}$.

(c) Calculate the real and imaginary part of the complex ac conductivity of a superconductor, $\sigma = \sigma' + \imath\sigma''$, and derive the contributions σ_n' (σ_s') and σ_n'' (σ_s'') of the normal (superfluid) component.

Solution:

(a) We start with the relaxation equation

$$\left(\frac{\partial}{\partial t} + \Gamma_n\right)\mathbf{v}_n(t) \;=\; q_n\frac{\mathbf{E}(t)}{m_n} \tag{1}$$

and use $n_n = n$, $q_n = -e$, and $m_n = m$, resulting in $\mathbf{J}_q(t) = -en\mathbf{v}_n(t)$. The relaxation equation then reads as

$$\left(\frac{\partial}{\partial t} + \Gamma_n\right)J_q(t) \;=\; \frac{ne^2}{m}\,\mathbf{E}(t)\;. \tag{2}$$

The general solution of this first order differential equation is given by

$$\mathbf{J}_q(t) \;=\; \mathbf{J}_q(0)\mathrm{e}^{-\frac{t}{\tau_n}} + \frac{ne^2}{m}\mathrm{e}^{-\frac{t}{\tau_n}}\underbrace{\int_0^t dt'\mathbf{E}(t')\mathrm{e}^{\frac{t'}{\tau_n}}}_{(*)}\;. \tag{3}$$

Assuming a harmonic electric field $\mathbf{E}(t) = \mathbf{E}_0\exp(-\imath\omega t)$ we can evaluate the term (*):

$$(*) \;=\; \mathbf{E}_0\int_0^t dt'\;\mathrm{e}^{\left(-\imath\omega+\frac{1}{\tau_n}\right)t'} \;=\; \frac{\mathbf{E}_0}{-\imath\omega+\frac{1}{\tau_n}}\left[\mathrm{e}^{\left(-\imath\omega+\frac{1}{\tau_n}\right)t} - 1\right]\;. \tag{4}$$

With this result equation (3) reads as

$$\begin{aligned}\mathbf{J}_q(t) &= \mathbf{J}_q(0)\mathrm{e}^{-\frac{t}{\tau_n}} + \frac{ne^2}{m}\frac{\mathbf{E}_0}{-\imath\omega+\frac{1}{\tau_n}}\mathrm{e}^{-\frac{t}{\tau_n}}\left[\mathrm{e}^{\left(-\imath\omega+\frac{1}{\tau_n}\right)t} - 1\right]\\ &= \left[\mathbf{J}_q(0) - \frac{ne^2}{m\left(-\imath\omega+\frac{1}{\tau_n}\right)}\mathbf{E}_0\right]\mathrm{e}^{-\frac{t}{\tau_n}} + \frac{ne^2}{m\left(-\imath\omega+\frac{1}{\tau_n}\right)}\mathbf{E}(t)\;.\end{aligned} \tag{5}$$

In this expression we can identify the following quantity as the complex ac conductivity:

$$\sigma(\omega) = \frac{ne^2}{m\left(-\imath\omega + \frac{1}{\tau_n}\right)} = \frac{ne^2\tau}{m\,(1-\imath\omega\tau_n)} = \frac{\sigma_0}{1-\imath\omega\tau_n} \tag{6}$$

with the Drude conductivity $\sigma_0 = \frac{ne^2\tau_n}{m}$ of a normal metal. Evidently, using $\sigma(\omega)$ we can rewrite the charge current density in the limit $t/\tau_n \gg 1$ as

$$\mathbf{J}_q(t) = \left[\mathbf{J}_q(0) - \sigma(\omega)\mathbf{E}_0\right] \mathrm{e}^{-\frac{t}{\tau_n}} + \sigma(\omega)\mathbf{E}(t) \overset{t \gg \tau}{=} \sigma(\omega)\mathbf{E}(t) \;. \tag{7}$$

(b) With $q_n = -e$, $q_s = -2e$, $m_n = m$ and $m_s = 2m$ we can express the current densities of the normal and superfluid component as $\mathbf{J}_s = -en_s\mathbf{v}_s$, $\mathbf{J}_n = -en_n\mathbf{v}_n$. Using the relaxation equations

$$m_n \left(\frac{\partial}{\partial t} + \Gamma_n\right) \mathbf{v}_n(t) = q_n \mathbf{E}(t) \tag{8}$$

$$m_s \left(\frac{\partial}{\partial t} + \Gamma_s\right) \mathbf{v}_s(t) = q_s \mathbf{E}(t)\;,\;\;, \tag{9}$$

the total current density $\mathbf{J} = \mathbf{J}_s + \mathbf{J}_n$ can be expressed as

$$\mathbf{J}(t) = \left(\frac{n_s}{-\imath\omega + \Gamma_s} + \frac{n_n}{-\imath\omega + \Gamma_n}\right) \frac{e^2}{m}\,\mathbf{E} \equiv \sigma(\omega)\mathbf{E}(t) \tag{10}$$

with the complex ac conductivity of the superconductor

$$\sigma(\omega) = \sigma_s(\omega) + \sigma_n(\omega) = \frac{e^2}{m}\left(\frac{n_s}{-\imath\omega + \Gamma_s} + \frac{n_n}{-\imath\omega + \Gamma_n}\right) \;. \tag{11}$$

In order to determine σ_s we have to evaluate

$$\begin{aligned} \frac{n_s e^2}{m} \lim_{\Gamma_s \to 0} \frac{1}{-\imath\omega + \Gamma_s} &= \frac{n_s e^2}{m} \lim_{\Gamma_s \to 0} \frac{1/\Gamma_s}{1 - \imath\omega/\Gamma_s} \\ &= \frac{n_s e^2}{m} \lim_{\Gamma_s \to 0} \left(\frac{1/\Gamma_s}{1+(\omega/\Gamma_s)^2} + \imath \frac{\omega/\Gamma_s^2}{1+(\omega/\Gamma_s)^2}\right) \\ &= \frac{n_s e^2}{m} \left(\pi\delta(\omega) + \frac{\imath}{\omega}\right) \;. \end{aligned} \tag{12}$$

With this result we obtain

$$\sigma(\omega) = \underbrace{\frac{n_s e^2}{m}\left[\pi\delta(\omega) + \frac{\imath}{\omega}\right]}_{\text{superfluid component}} + \underbrace{\frac{n_n e^2}{m}\frac{1}{-\imath\omega + \Gamma_n}}_{\text{normal component}} \;. \tag{13}$$

(c) Separating expression (13) into real and imaginary part, $\sigma(\omega) = \sigma'(\omega) + \imath\sigma''(\omega)$, yields for the normal component

$$\sigma_n'(\omega) = \frac{n_n e^2 \tau_n}{m} \frac{1}{1+(\omega\tau_n)^2} = \frac{1}{\Lambda_s}\frac{n_n}{n_s}\frac{\tau_n}{1+(\omega\tau_n)^2} \tag{14}$$

$$\sigma_n''(\omega) = \frac{n_n e^2 \tau_n}{m} \frac{\omega\tau_n}{1+(\omega\tau_n)^2} = \frac{1}{\Lambda_s}\frac{n_n}{n_s}\tau_n \frac{\omega\tau_n}{1+(\omega\tau_n)^2} \;. \tag{15}$$

Here we have used $\Lambda_n \equiv m/n_n e^2 = \Lambda_s n_s/n_n$ with the London coefficient $\Lambda_s \equiv m_s/\widetilde{n}_s q_s^2 = m/n_s e^2 = \mu_0 \lambda_{\rm L}^2$ ($\lambda_{\rm L}$ is the London penetration depth).

For the superfluid component we obtain

$$\sigma_s'(\omega) = \frac{n_s e^2}{m}\,\pi\delta(\omega) = \frac{1}{\Lambda_s}\,\pi\delta(\omega) \tag{16}$$

$$\sigma_s''(\omega) = \frac{n_s e^2}{m}\frac{1}{\omega} = \frac{1}{\omega\Lambda_s}\,. \tag{17}$$

For the total conductivity we then obtain

$$\begin{aligned}\sigma'(\omega) &= \sigma_s'(\omega) + \sigma_n'(\omega) = \frac{1}{\Lambda_s}\left[\pi\delta(\omega) + \frac{n_n}{n_s}\frac{\tau_n}{1+(\omega\tau_n)^2}\right] \\ &\overset{\omega\tau_n\ll 1}{\simeq} \frac{1}{\omega\Lambda_s}\left[\omega\pi\delta(\omega) + \frac{n_n}{n_s}\,\omega\tau_n\right]\end{aligned} \tag{18}$$

$$\begin{aligned}\sigma''(\omega) &= \sigma_s''(\omega) + \sigma_n''(\omega) = \frac{1}{\Lambda_s}\left[\frac{1}{\omega} + \frac{1}{\omega}\frac{n_n}{n_s}\frac{(\omega\tau_n)^2}{1+(\omega\tau_n)^2}\right] \\ &\overset{\omega\tau_n\ll 1}{\simeq} \frac{1}{\omega\Lambda_s}\left[1 + \frac{n_n}{n_s}\,(\omega\tau_n)^2\right]\,.\end{aligned} \tag{19}$$

In a superconductor we obviously have two conduction channels in parallel. It is interesting to calculate the crossover frequency at which the normal and superfluid channel carry the same current density. The ratio of the two components is given by

$$\frac{J_s}{J_n} = \frac{\sigma_s E}{\sigma_n E} = \frac{\sigma_s''}{\sigma_n' + \sigma_n''} \overset{\omega\tau_n\ll 1}{\simeq} \frac{\sigma_s''}{\sigma_n'} = \frac{n_n}{n_s}\frac{1}{\omega\tau_n}\,. \tag{20}$$

Assuming that we are far below the transition temperature so that $n_n/n_s \sim 10^{-2}$, the crossover frequency is $\omega/2\pi \sim 10^{-2}/\tau_n \sim 10 - 100\,\mathrm{GHz}$ for a typical momentum relaxation time of $10^{-12} - 10^{-13}$s.

We finally can estimate the dissipated power density

$$P = \rho J^2 = \Re\left(\frac{1}{\sigma}\right) J^2 = \frac{\sigma'}{(\sigma')^2 + (\sigma'')^2}\,J^2 \simeq \frac{\sigma'}{(\sigma'')^2}\,J^2 \propto \frac{n_n}{n_s}\tau_n\omega^2\,. \tag{21}$$

Here we have used the fact that typically $\sigma' \ll \sigma''$. We see that the dissipated power density increases proportional to ω^2 and the normal to superfluid density ratio n_n/n_s.

1.2 Frequency dependent skin depth and dielectric function of a normal metal

Exercise:

We consider a normal metal with volume V containing N electrons of mass m resulting in the electron density $n = N/V$. The charge current density is related to the electric field by Ohm's law, $\mathbf{J}_q = \sigma(\omega)\mathbf{E}$, with the complex ac conductivity

$$\sigma(\omega) = \frac{ne^2}{m\left(-\imath\omega + \frac{1}{\tau}\right)} = \frac{ne^2\tau}{m\,(1-\imath\omega\tau)} = \frac{\sigma_0}{1-\imath\omega\tau} \tag{1}$$

Here, $\sigma_0 = \frac{ne^2\tau}{m}$ is the Drude conductivity and τ the momentum relaxation time.

(a) Use $\sigma(\omega)$ and Maxwell's equations to derive the frequency dependent dielectric function $\widetilde{\epsilon}(\omega)$ of the normal metal.

(b) Calculate the frequency dependent electromagnetic penetration depth (skin depth) $\delta(\omega)$ for the electric and magnetic field.

(c) What is the relation between $\delta(\omega)$ and $\widetilde{\epsilon}(\omega)$?

Solution:

We start by repeating Maxwell's equation which read as (in SI unit, $\epsilon = \mu = 1$)

$$
\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_q \,, \quad \mathbf{D} = \epsilon_0 \mathbf{E} && (2)\\
\boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \,, \quad \mathbf{B} = \mu_0 \mathbf{H} && (3)\\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 && (4)\\
\boldsymbol{\nabla} \cdot \mathbf{D} &= \rho_q \;. && (5)
\end{aligned}
$$

For electrons with charge $q = -e$ the charge density is given by $\rho_q = -en = -eN/V$ and the charge current density by $\mathbf{J}_q = qn\mathbf{v} = -en\mathbf{v}$. We see that the direction of motion of the electrons is antiparallel to the technical charge current density. Maxwell's equations are supplemented by the constitutive relation (Ohm's law)

$$
\mathbf{J}_q = \boldsymbol{\sigma} \cdot \mathbf{E} \;, \tag{6}
$$

linking the charge current density $\mathbf{J}_q$ with the electric field $\mathbf{E}$ via the electric conductivity $\boldsymbol{\sigma}$ (which in the most general case is a second rank tensor). Applying $\boldsymbol{\nabla} \cdot \ldots$ on both sides of eq. (2) yields

$$
\underbrace{\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{H})}_{=0} = \frac{\partial}{\partial t} \underbrace{(\boldsymbol{\nabla} \cdot \mathbf{D})}_{=\rho_q} + \boldsymbol{\nabla} \cdot \mathbf{J}_q \;, \tag{7}
$$

immediately leading to the continuity equation

$$
\frac{\partial \rho_q}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J}_q = 0 \tag{8}
$$

for the charge density ρ_q.

Applying $\boldsymbol{\nabla} \times \ldots$ on both sides of eq. (2) yields

$$
\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{H}) &= -\boldsymbol{\nabla}^2 \mathbf{H} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{H}) = -\left(\boldsymbol{\nabla}^2 - \boldsymbol{\nabla} : \boldsymbol{\nabla}\right) \mathbf{H} \\
-\left(\boldsymbol{\nabla}^2 - \boldsymbol{\nabla} : \boldsymbol{\nabla}\right) \mathbf{H} &= \frac{\partial}{\partial t} \boldsymbol{\nabla} \times \mathbf{D} + \boldsymbol{\nabla} \times \mathbf{J}_q \\
&= \epsilon_0 \frac{\partial}{\partial t} \boldsymbol{\nabla} \times \mathbf{E} + \boldsymbol{\nabla} \times \mathbf{J}_q \\
&\overset{(3)}{=} -\epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} + \boldsymbol{\nabla} \times \mathbf{J}_q \\
&= -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{H}}{\partial t^2} + \boldsymbol{\nabla} \times \mathbf{J}_q \;. && (9)
\end{aligned}
$$

Using $\mu_0\epsilon_0 = 1/c^2$, this finally results in

$$\left[\nabla^2 - \nabla : \nabla - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{B} \;=\; -\mu_0 \nabla \times \mathbf{J}_q \quad . \tag{10}$$

We will see below that we can interpret this expression as the screening equation for the magnetic field, allowing us to derive the frequency dependent penetration depth of the magnetic field.

Finally, applying $\nabla \times \ldots$ on both sides of eq. (3) yields

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{E}) &= -\frac{\partial}{\partial t}\nabla \times \mathbf{B} = -\nabla^2\mathbf{E} + \nabla(\nabla \cdot \mathbf{E}) \\
-\left(\nabla^2 - \nabla : \nabla\right)\mathbf{E} &= -\mu_0 \frac{\partial}{\partial t}\nabla \times \mathbf{H} \\
&= -\mu_0\epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial \mathbf{J}_q}{\partial t} \quad .
\end{aligned} \tag{11}$$

Again, using $\mu_0\epsilon_0 = 1/c^2$ we can rewrite this equation as

$$\left[\nabla^2 - \nabla : \nabla - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{E} \;=\; \mu_0 \frac{\partial \mathbf{J}_q}{\partial t} \quad . \tag{12}$$

(a) Using Ohm's law $\mathbf{J}_q = \sigma(\omega)\mathbf{E}$ and $\partial\mathbf{D}/\partial t = -\imath\omega\epsilon_0\epsilon(\omega)\mathbf{E}$ [we assume a harmonic time dependence $\mathbf{D}(t) = \mathbf{D}_0 \exp(-\imath\omega t)$] we can rewrite eq. (2) as

$$\nabla \times \mathbf{H} \;=\; \mathbf{J}_q + \frac{\partial \mathbf{D}}{\partial t} \;=\; \sigma(\omega)\mathbf{E} - \imath\omega\epsilon_0\epsilon(\omega)\mathbf{E} \tag{13}$$

We can write the r.h.s. of eq. (13) as $\widetilde{\sigma}(\omega)\mathbf{E}$ using the generalized conductivity

$$\widetilde{\sigma}(\omega) \;=\; \sigma(\omega) - \imath\omega\epsilon_0\epsilon(\omega) \overset{\epsilon(\omega)=1}{=} \sigma(\omega) - \imath\omega\epsilon_0 \quad . \tag{14}$$

In the same way we can write the r.h.s. of eq. (13) as $-\imath\omega\epsilon_0\widetilde{\epsilon}(\omega)\mathbf{E}$ using the generalized dielectric function

$$\widetilde{\epsilon}(\omega) \;=\; \epsilon(\omega) + \frac{\imath\sigma(\omega)}{\omega\epsilon_0} \overset{\epsilon(\omega)=1}{=} 1 + \frac{\imath\sigma(\omega)}{\omega\epsilon_0} \quad . \tag{15}$$

With the complex ac conductivity $\sigma(\omega) = \frac{\sigma_0}{1-\imath\omega\tau}$ ($\sigma_0 = \frac{ne^2\tau}{m}$) we obtain

$$\begin{aligned}
\widetilde{\epsilon}(\omega) &= 1 - \frac{1}{\omega^2}\underbrace{\frac{ne^2}{m\epsilon_0}}_{=\omega_\mathrm{p}^2}\frac{-\imath\omega\tau}{1-\imath\omega\tau} \\
&= 1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\frac{-\imath\omega\tau}{1-\imath\omega\tau} \quad \text{with} \;\; \omega_\mathrm{p}^2 = \frac{ne^2}{m\epsilon_0} \quad .
\end{aligned} \tag{16}$$

Here, ω_p is the plasma frequency of the electron system.

(b) Inserting $\mathbf{J}_q = \sigma(\omega)\mathbf{E}$ into eq. (10) yields

$$\begin{aligned}\left[\nabla^2 - \nabla : \nabla + \frac{\omega^2}{c^2}\right]\mathbf{B} &= -\mu_0 \nabla \times \mathbf{J}_q = -\mu_0\sigma(\omega)\nabla \times \mathbf{E} \\ &= \underbrace{-\imath\omega\sigma(\omega)\mu_0}_{=1/\delta^2(\omega)}\mathbf{B} = \frac{\mathbf{B}}{\delta^2(\omega)} , \end{aligned} \tag{17}$$

where[1]

$$\delta^2(\omega) = \frac{1}{-\imath\omega\sigma(\omega)\mu_0} = \underbrace{\frac{m}{\mu_0 n e^2}}_{=\delta_\infty^2}\frac{1-\imath\omega\tau}{-\imath\omega\tau} = \delta_\infty^2\frac{1-\imath\omega\tau}{-\imath\omega\tau} \tag{18}$$

is the penetration depth of the magnetic field (skin depth). Note that the skin depth δ_∞ (collisionless limit: $\tau \to \infty$) can be expressed via the plasma frequency ω_p using the relation $\mu_0\epsilon_0 = 1/c^2$:

$$\delta_\infty^2 = \frac{m}{\mu_0 n e^2} = \frac{c^2}{\omega_\mathrm{p}^2} . \tag{19}$$

We can rewrite eq. (17) in the form of a wave equation for $\mathbf{B}$:

$$\left[\nabla^2 - \nabla : \nabla + \mu_0\epsilon_0\omega^2\underbrace{\left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\frac{-\imath\omega\tau}{1-\imath\omega\tau}\right)}_{=\epsilon(\omega)}\right]\mathbf{B} = 0 . \tag{20}$$

We see that by taking into account the current density $\mathbf{J}_q$ on the r.h.s. of eq. (17) results in the replacement $\epsilon_0\epsilon(\omega) \to \epsilon_0\widetilde{\epsilon}(\omega)$.

For superconductors the complex conductivity in the limit $\omega\tau \ll 1$ usually valid up to GHz frequencies is given by [cf. eq. (21) and (22)]

$$\sigma'(\omega) \simeq \frac{1}{\omega\Lambda_s}\left[\omega\pi\delta(\omega) + \frac{n_n}{n_s}\,\omega\tau_n\right] \tag{21}$$

$$\sigma''(\omega) \simeq \frac{1}{\omega\Lambda_s}\left[1 + \frac{n_n}{n_s}\,(\omega\tau_n)^2\right] . \tag{22}$$

This results in the total conductivity at finite frequencies

$$\sigma(\omega) = \sigma'(\omega) + \imath\sigma''(\omega) = \frac{1}{\omega\Lambda_s}\left[\frac{n_n}{n_s}\,\omega\tau_n + \imath\left(1 + \frac{n_n}{n_s}\,(\omega\tau_n)^2\right)\right] . \tag{23}$$

At temperature not to close to the transition temperature, $(n_n/n_s)\omega\tau \ll 1$ and we can use the approximation

$$\sigma(\omega) \simeq \frac{\imath}{\omega\Lambda_s} = \frac{\imath}{\omega\mu_0\lambda_\mathrm{L}^2} . \tag{24}$$

[1] Note that for $\omega\tau \ll 1$ we have $\frac{1}{\delta_\infty^2} \simeq \frac{-\imath\omega}{\delta_\infty}$. With $\delta_\infty^2 = m/\mu_0 n e^2$, $\sigma_0 = ne^2\tau/m$ and $\sqrt{-\imath} = \frac{1}{\sqrt{2}}(1-\imath)$ we obtain $\frac{1}{\delta(\omega)} = \sqrt{\frac{\mu_0\sigma_0\omega}{2}}(1-\imath)$.

Inserting this into the expression for the magnetic field penetration depth we obtain

$$\delta^2(\omega) \;=\; \frac{1}{-\imath\omega\sigma(\omega)\mu_0} \;=\; \lambda_{\mathrm{L}}^2 \;. \tag{25}$$

We see that the magnetic field screening length in superconductor up to frequencies satisfying $(n_n/n_s)\omega\tau \ll 1$ is about frequency independent and given by the London penetration depth λ_{L}.

In order to show that the electric field $\mathbf{E}$ also follows a wave equation of the form (20), we insert Ohm's law, $\mathbf{J}_q = \sigma(\omega)\mathbf{E}$, into eq. (12) and obtain

$$\left[\boldsymbol{\nabla}^2 - \boldsymbol{\nabla} : \boldsymbol{\nabla} + \frac{\omega^2}{c^2}\right]\mathbf{E} \;=\; \mu_0\frac{\partial \mathbf{J}_q}{\partial t} \;=\; \underbrace{-\imath\omega\mu_0\sigma(\omega)}_{1/\delta^2(\omega)}\,\mathbf{E} \;=\; \frac{\mathbf{E}}{\delta^2(\omega)} \;. \tag{26}$$

This can be rewritten into an equation identical to (20) despite of the replacement $\mathbf{B} \leftrightarrow \mathbf{E}$:

$$\left[\boldsymbol{\nabla}^2 - \boldsymbol{\nabla} : \boldsymbol{\nabla} + \frac{\omega^2}{c^2}\underbrace{\left(1 + \frac{\imath\sigma(\omega)}{\omega\epsilon_0}\right)}_{=\epsilon(\omega)}\right]\mathbf{E} \;=\; 0 \;. \tag{27}$$

This result shows that also the electric field $\mathbf{E}$ is screened within the same screening length $\delta(\omega)$ as the magnetic field $\mathbf{B}$.

(c) The relation between the dielectric function $\widetilde{\epsilon}(\omega)$ and the electromagnetic penetration depth $\delta(\omega)$ reads

$$\begin{aligned}
\widetilde{\epsilon}(\omega) \;&=\; \epsilon(\omega) - \frac{\omega_{\mathrm{p}}^2}{\omega^2}\,\frac{-\imath\omega\tau}{1-\imath\omega\tau} \;\overset{\epsilon(\omega)=1}{=}\; 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}\,\frac{-\imath\omega\tau}{1-\imath\omega\tau} \\
&=\; 1 - \frac{c^2}{\omega^2}\underbrace{\frac{\omega_{\mathrm{p}}^2}{c^2}}_{=1/\delta_\infty^2}\frac{-\imath\omega\tau}{1-\imath\omega\tau} \;=\; 1 - \frac{c^2}{\omega^2}\underbrace{\frac{1}{\delta_\infty^2}\frac{-\imath\omega\tau}{1-\imath\omega\tau}}_{=1/\delta^2(\omega)} \\
&=\; 1 - \frac{c^2}{\omega^2}\frac{1}{\delta^2(\omega)} \;.
\end{aligned} \tag{28}$$