WALTHER–MEIßNER–INSTITUT
Bayerische Akademie der Wissenschaften
Lehrstuhl für Technische Physik E23, Technische Universität München
Prof. Dr. Rudolf Gross
Tel.: +49 (0)89 289 14249
E-mail: rudolf.gross@wmi.badw.de

23 November 2023

Exercise 6

Exercise to the Lecture

Superconductivity and Low Temperature Physics I
WS 2023/2024

4 Microscopic Theory

4.1 Second Quantization Formalism: The Quantum Harmonic Oscillator

Exercise:

The second quantization formalism is used to describe quantum many-body systems. In this formalism, the quantum many-body states are represented in the so-called Fock state basis. Fock states are constructed by filling up each single-particle state with a certain number of identical particles. The second quantization formalism introduces the creation and annihilation operators to construct and handle the Fock states. They provide useful tools to the study of the quantum many-body theory.

The second quantization formalism is also known as the canonical quantization in quantum field theory, in which the fields (such as the wave functions of matters) are upgraded into field operators. This is analogous to first quantization, where the physical quantities (such as position and momentum) are upgraded into operators. The key ideas of the second quantization formalism were introduced in 1927 by Dirac and were further developed by Fock and Jordan.

In this exercise we use the second quantization formalism to describe the quantum harmonic oscillator. In our discussion we use the dimensionless (first quantization) operators for position and momentum,

$$\hat{X} \;=\; \sqrt{\frac{m\omega}{\hbar}}\,\hat{x}, \qquad\qquad \hat{P} \;=\; \sqrt{\frac{1}{m\omega\hbar}}\,\hat{p}\,.$$

(a) Derive the expression for the Hamiltonian of the quantum harmonic oscillator in terms of the annihilation operator $\hat{a}$ and the creation operator $\hat{a}^\dagger$.

(b) Use the commutation relations for the position and momentum operators to show the identities

$$\left[\hat{a},\hat{a}^\dagger\right] \;=\; 1, \qquad [\hat{a},\hat{a}] \;=\; 0, \qquad \left[\hat{a}^\dagger,\hat{a}^\dagger\right] \;=\; 0$$

as well as

$$\left[\hat{a}\hat{a}^\dagger, \hat{a}\right] = -\hat{a}, \qquad \left[\hat{a}\hat{a}^\dagger, \hat{a}^\dagger\right] = \hat{a}^\dagger .$$

(c) Discuss the action of the creation and annihilation operators on the Fock states and show the identities

$$\hat{a}|\varphi_n\rangle = \sqrt{n}|\varphi_{n-1}\rangle, \quad \hat{a}^\dagger|\varphi_n\rangle = \sqrt{n+1}|\varphi_{n+1}\rangle \quad n = 0, 1, 2, 3, \ldots .$$

(d) Use the eigenfunction $|\varphi_n\rangle$ with the eigenenergies E_n and show the following identities:

$$\hat{a}|\varphi_0\rangle = 0, \quad \langle\hat{n}\rangle = \langle\hat{a}^\dagger\hat{a}\rangle = n, \quad \hat{n}|\varphi_{n-1}\rangle = (n-1)|\varphi_{n-1}\rangle, \quad \hat{n}|\varphi_{n+1}\rangle = (n+1)|\varphi_{n-1}\rangle .$$

Solution:

(a) The creation and annihilation operators for bosons were originally constructed in the context of the quantum harmonic oscillator as the raising and lowering operators. Later on, they have been generalized to the field operators in the quantum field theory. The creation and annihilation operators add or remove a particle from a many-body system. Applying the creation (annihilation) operator to a first quantized many-body wave function adds (deletes) a single-particle state from the wave function in a symmetrized way depending on the particle statistics. All the second quantized Fock states can be constructed by applying the creation operators to the vacuum state repeatedly. The creation and annihilation operators are fundamental to the quantum many-body theory. Every many-body operator – such as the Hamiltonian of the many-body system – can be expressed in terms of them.

The Hamiltonian of the quantum harmonic oscillator can be expressed in terms of the dimensionless (first quantization) position and momentum operators $\hat{X}$ and $\hat{P}$ as

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega\hat{x}^2}{2} = \frac{\hbar\omega}{2}\left(\hat{P}^2 + \hat{X}^2\right) . \tag{1}$$

Using the commutator $[\hat{P}, \hat{X}] = \hat{P}\hat{X} - \hat{X}\hat{P}$ we obtain

$$\begin{aligned}
\mathcal{H} &= \frac{\hbar\omega}{2}\left(\hat{P}^2 + \hat{X}^2\right) \\
&= \frac{\hbar\omega}{2}\left(\hat{X}^2 + \hat{P}^2 + \imath[\hat{X}, \hat{P}] - \imath[\hat{X}, \hat{P}]\right) \\
&= \frac{\hbar\omega}{2}\left(\hat{X}^2 + \hat{P}^2 + \imath\hat{X}\hat{P} - \imath\hat{P}\hat{X} - \imath[\hat{X}, \hat{P}]\right) \\
&= \frac{\hbar\omega}{2}\left(\left[\hat{X} - \imath\hat{P}\right]\left[\hat{X} + \imath\hat{P}\right] - \imath[\hat{X}, \hat{P}]\right) .
\end{aligned} \tag{2}$$

We can now define the creation and annihilation operators

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\hat{X} + \imath\hat{P}\right), \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{X} - \imath\hat{P}\right) \tag{3}$$

yielding

$$\hat{X} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right), \qquad \hat{P} = \frac{1}{\sqrt{2}}\left(\hat{a} - \hat{a}^\dagger\right) . \tag{4}$$

With these expressions and the commutator $[\hat{X}, \hat{P}] = \frac{1}{\hbar}[\hat{x}, \hat{p}] = \imath$ we can rewrite eq. (2) to

$$\mathcal{H} \;=\; \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) \tag{5}$$

with the number operator

$$\hat{n} \;=\; \hat{a}^\dagger \hat{a}\,. \tag{6}$$

(b) We use the commutation relations for the position and momentum operators

$$[\hat{x}, \hat{p}] \;=\; \imath\hbar, \qquad [\hat{x}, \hat{x}] \;=\; 0, \qquad [\hat{p}, \hat{p}] \;=\; 0 \tag{7}$$

and the dimensionless operators for position and momentum,

$$\hat{X} \;=\; \sqrt{\frac{m\omega}{\hbar}}\,\hat{x}, \qquad\qquad \hat{P} \;=\; \sqrt{\frac{1}{m\omega\hbar}}\,\hat{p}\,. \tag{8}$$

We obtain

$$\left[\hat{X}, \hat{P}\right] \;=\; \sqrt{\frac{m\omega}{\hbar}}\sqrt{\frac{1}{m\omega\hbar}}\underbrace{[\hat{x}, \hat{p}]}_{=\imath\hbar} \;=\; \imath \tag{9}$$

and

$$\begin{aligned}
\left[\hat{a}, \hat{a}^\dagger\right] &= \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} \\
&= \frac{1}{2}\left\{\left(\hat{X} + \imath\hat{P}\right)\left(\hat{X} - \imath\hat{P}\right) - \left(\hat{X} - \imath\hat{P}\right)\left(\hat{X} + \imath\hat{P}\right)\right\} \\
&= \frac{\imath}{2}\left\{\left(\hat{P}\hat{X} - \hat{X}\hat{P}\right) - \left(\hat{X}\hat{P} - \hat{P}\hat{X}\right)\right\} \\
&= \frac{\imath}{2}\underbrace{\left[\hat{P}, \hat{X}\right]}_{=-\imath/2} - \frac{\imath}{2}\underbrace{\left[\hat{X}, \hat{P}\right]}_{=\imath/2} \\
&= 1
\end{aligned} \tag{10}$$

The identities $[\hat{a}, \hat{a}] = 0$ and $\left[\hat{a}^\dagger, \hat{a}^\dagger\right] = 0$ can be shown in the same way.
Using eq. (10) we further obtain

$$[\hat{n}, \hat{a}] \;=\; \left[\hat{a}^\dagger\hat{a}, \hat{a}\right] \;=\; \left(\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger\right)\hat{a} \;=\; \underbrace{\left[\hat{a}^\dagger, \hat{a}\right]}_{-1}\hat{a} \;=\; -\hat{a} \tag{11}$$

$$\left[\hat{n}, \hat{a}^\dagger\right] \;=\; \left[\hat{a}^\dagger\hat{a}, \hat{a}^\dagger\right] \;=\; \hat{a}^\dagger\left(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\right) \;=\; \hat{a}^\dagger\underbrace{\left[\hat{a}, \hat{a}^\dagger\right]}_{+1} \;=\; \hat{a}^\dagger \tag{12}$$

and

$$\hat{n}^\dagger \;=\; \left(\hat{a}^\dagger\hat{a}\right)^\dagger \;=\; \hat{a}^\dagger\hat{a}^{\dagger\dagger} \;=\; \hat{a}^\dagger\hat{a} \;=\; \hat{n}\,. \tag{13}$$

This shows that the particle number operator is Hermitian and self-adjoint. Therefore, the eigenvalues of $\hat{n}$ are real. They are the occupation number n.

(c) With the the creation operator $\hat{a}^\dagger$ we can construct the Fock state $|\varphi_{n+1}\rangle$ by adding a boson to $|\varphi_n\rangle$, that is, $\hat{a}^\dagger|\varphi_n = c|\varphi_{n+1}\rangle$. In the same way, with the annihilation operator $\hat{a}$ we can construct the Fock state $|\varphi_{n-1}\rangle$ by removing a boson to $|\varphi_n\rangle$, that is, $\hat{a}|\varphi_n = d|\varphi_{n-1}\rangle$. In the following, we determine the factors c and d. Doing so, we make use of the commutator (10).

Using eq. (10) we obtain for the factor c to

$$|c|^2 \;=\; |\langle\varphi_{n+1}|c^\star c|\varphi_{n+1}\rangle \;=\; \langle\varphi_n|\hat{a}\hat{a}^\dagger|\varphi_n\rangle \;=\; \langle\varphi_n|\hat{a}^\dagger\hat{a}+1|\varphi_n\rangle \;=\; n+1\,, \tag{14}$$

that is, $c = \sqrt{n+1}\mathrm{e}^{\imath\phi}$. The phase ϕ can be neglected, so that $c = \sqrt{n+1}$. Similarly, using eq. (10) we obtain for the factor d to

$$|d|^2 \;=\; |\langle\varphi_{n-1}|d^\star d|\varphi_{n-1}\rangle \;=\; \langle\varphi_n|\hat{a}^\dagger\hat{a}|\varphi_n\rangle \;=\; n\,, \tag{15}$$

that is, $d = \sqrt{n}\mathrm{e}^{\imath\phi}$. Again, neglecting the phase ϕ yields $d = \sqrt{n}$.

The creation operator raises the boson occupation number by 1. Therefore all the occupation number states $|\varphi_n\rangle$ can be constructed by the boson creation operator from the vacuum state $|\varphi_0\rangle = |0\rangle$ as

$$|\varphi_n\rangle \;=\; = \frac{1}{\sqrt{n}}\hat{a}^\dagger|\varphi_{n-1}\rangle \;=\; \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n|\varphi_0\rangle\,, \tag{16}$$

(d) Using $\hat{a}|\varphi_n\rangle = \sqrt{n}|\varphi_{n-1}\rangle$ and $\hat{a}^\dagger|\varphi_n\rangle = \sqrt{n+1}|\varphi_{n+1}\rangle$ we obtain

$$\begin{aligned}\langle\varphi_n|\hat{a}^\dagger\hat{a}|\varphi_n\rangle &= (\hat{a}|\varphi_n\rangle)^\star\,(\hat{a}|\varphi_n\rangle)\\ &= \left(\sqrt{n}|\varphi_{n-1}\rangle\right)^\star\left(\sqrt{n}|\varphi_{n-1}\rangle\right)\\ &= n\,\langle\varphi_{n-1}|\varphi_{n-1}\rangle\\ &= n\end{aligned} \tag{17}$$

due to the normalization of the eigenstates $|\varphi_n\rangle$. Furthermore, we obtain

$$\hat{a}|\varphi_0\rangle \;=\; \sqrt{0}|\varphi_0\rangle \;=\; 0 \tag{18}$$

and

$$\begin{aligned}\hat{n}|\varphi_{n-1}\rangle \;=\; \hat{a}^\dagger\hat{a}|\varphi_{n-1}\rangle &= \left(\hat{a}\hat{a}^\dagger-1\right)|\varphi_{n-1}\rangle\\ &= \sqrt{n}\,(\hat{a}|\varphi_n\rangle) - |\varphi_{n-1}\rangle\\ &= (n-1)\,|\varphi_{n-1}\rangle\end{aligned} \tag{19}$$

as well as

$$\hat{n}|\varphi_{n+1}\rangle \;=\; \hat{a}^\dagger\hat{a}|\varphi_{n+1}\rangle \;=\; \sqrt{n+1}\left(\hat{a}^\dagger|\varphi_n\rangle\right) \;=\; (n+1)\,|\varphi_n\rangle\,. \tag{20}$$

With these results we obtain the energy spectrum of the Hamiltonian (5) to

$$\mathcal{H}|\varphi_n\rangle \;=\; \hbar\omega\,\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)|\varphi_n\rangle \;=\; E_n|\varphi_n\rangle\,, \tag{21}$$

with $E_n = \hbar\omega\,\left(n+\frac{1}{2}\right)$.

4.2 Coherent Bosonic and Fermionic States

Exercise:

In quantum mechanics, a coherent state is a specific quantum state of the quantum harmonic oscillator. Its dynamics most closely resembles the oscillating behaviour of a classical harmonic oscillator. In 1926, Erwin Schrödinger was the first to construct a coherent state while searching for solutions of the Schrödinger equation that satisfy the correspondence principle. Later on, in the quantum theory of light (quantum electrodynamics) and other bosonic quantum field theories, coherent states were introduced by Roy J. Glauber in 1963. Here, the coherent state of a field describes an oscillating field, the closest quantum state to a classical sinusoidal wave such as a continuous laser wave.

The discussion of the properties of coherent states in bosonic systems is a good exercise, preparing for the discussion of the coherent superconducting ground state in the microscopic BCS theory.

(a) Construct quantum states of a bosonic system (quantum harmonic oscillator) such that the predictions of quantum mechanics agree in the limit of large quantum numbers with the results of classical mechanics (classical harmonic oscillator). Show that such states are coherent superpositions of all stationary states $|\varphi_n\rangle$. Such states are called coherent states.

(b) Determine $\langle\hat{n}\rangle$, $\langle\hat{n}^2\rangle$, and $\Delta N = \sqrt{\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2}$ for the coherent state constructed in (a) and interpret the obtained results.

(c) Use the coherent state derived for a bosonic system to derive a coherent state for a fermionic system.

Solution:

(a) We start with the classical equation of motion of a one-dimensional harmonic oscillator with mass m and angular frequency ω:

$$\frac{d}{dt}x(t) = \frac{1}{m}p(t) \tag{1}$$

$$\frac{d}{dt}p(t) = -m\omega^2 x(t)\,. \tag{2}$$

Using the dimensionless parameter $\bar{x} = \sqrt{\frac{m\omega}{\hbar}}\, x$ and $\bar{p} = \sqrt{\frac{1}{m\omega\hbar}}\, p$ we can rewrite the equation of motion as

$$\frac{d}{dt}\bar{x}(t) = \omega\bar{p}(t) \tag{3}$$

$$\frac{d}{dt}\bar{p}(t) = -\omega\bar{x}(t)\,. \tag{4}$$

Combining the two parameters $\bar{x}$ and $\bar{p}$ in a single complex parameter

$$\alpha(t) = \frac{1}{\sqrt{2}}\left[\bar{x}(t) + \imath\bar{p}(t)\right] \tag{5}$$

the equations (3) and (4) can be put together to

$$\frac{d}{dt}\alpha(t) \quad = \quad -\imath\omega\alpha(t) \tag{6}$$

with the solution

$$\alpha(t) \quad = \quad \alpha(0)\,\mathrm{e}^{-\imath\omega t} \quad \text{with} \quad \alpha(0) \quad = \quad \frac{1}{\sqrt{2}}\left[\bar{x}(0) + \imath\bar{p}(0)\right]\,. \tag{7}$$

The interpretation of this result is straightforward. Equation (7) describes the circular motion in the complex plane with angular frequency $-\omega$ and radius $|\alpha_0|$. The classical energy of the system is given by

$$H \quad = \quad \frac{\hbar\omega}{2}\,\left([\bar{x}(0)]^2 + [\bar{p}(0)]^2\right) \;=\; \hbar\omega|\alpha(0)|^2\,. \tag{8}$$

For a macroscopic harmonic oscillator, $|\alpha(0)| \gg 1$. Therefore, its energy is much bigger than $\hbar\omega$.

We now search for a quantum state for which at any time t the expectation values $\langle\hat{X}\rangle$, $\langle\hat{P}\rangle$ and $\langle\mathcal{H}\rangle$ are practically the same as the classical values $\bar{x}$, $\bar{p}$ and H of the classical motion. For an arbitrary quantum state $|\varphi(t)\rangle$ the temporal evolution of the matrix element $\langle\hat{a}\rangle(t) = \langle\varphi(t)|\hat{a}|\varphi(t)\rangle$ is given by the equation of motion

$$\imath\hbar\frac{d}{dt}\langle\hat{a}\rangle(t) \quad = \quad \langle[\hat{a},\mathcal{H}]\rangle(t)\,. \tag{9}$$

With

$$[\hat{a},\mathcal{H}]\rangle \quad = \quad \hbar\omega\,\left[\hat{a},\hat{a}^\dagger\hat{a}\right] \;=\; \hbar\omega\;\hat{a} \tag{10}$$

we obtain

$$\imath\hbar\frac{d}{dt}\langle\hat{a}\rangle(t) \quad = \quad \hbar\omega\;\hat{a}(t)\,, \tag{11}$$

that is,

$$\langle\hat{a}\rangle(t) \quad = \quad \langle\hat{a}\rangle(0)\,\mathrm{e}^{-\imath\omega t}\,. \tag{12}$$

The time evolution of $\langle\hat{a}^\dagger\rangle(t) = \langle\varphi(t)|\hat{a}^\dagger|\varphi(t)\rangle$ follows the complex conjugate equation

$$\langle\hat{a}^\dagger\rangle(t) \quad = \quad \langle\hat{a}^\dagger\rangle(0)\,\mathrm{e}^{\imath\omega t} \;=\; (\langle\hat{a}\rangle)^\star(0)\,\mathrm{e}^{\imath\omega t}\,. \tag{13}$$

Equations (12) and (13) correspond to the classical result (7). Comparing these expressions we see that the classical and quantum mechanical result are the same at any time if we set

$$\langle\hat{a}\rangle(0) \quad = \quad \alpha(0)\,, \tag{14}$$

where $\alpha(0)$ is the complex parameter characterizing the classical motion. Since the quantum mechanical result has to approximate the classical motion as good as possible, the quantum state first has to satisfy the condition

$$\langle\varphi(0)|\hat{a}|\varphi(0)\rangle \quad = \quad \alpha(0)\,. \tag{15}$$

Furthermore, the quantum mechanical expectation value

$$\langle \mathcal{H} \rangle \;=\; \hbar\omega \left(\langle \hat{a}^\dagger \hat{a} \rangle (0) + \frac{1}{2} \right) \tag{16}$$

must be equal to the classical energy (8). Neglecting the term $\hbar\omega/2$, which is of purely quantum mechanical origin, we obtain the second condition to

$$\langle \hat{a}^\dagger \hat{a} \rangle (0) \;=\; |\alpha(0)|^2 \tag{17}$$

or equivalently

$$\langle \varphi(0) | \hat{a}^\dagger \hat{a} | \varphi(0) \rangle \;=\; |\alpha(0)|^2 \,. \tag{18}$$

In the following we show that the conditions (14) and (18) are sufficient to determine the normalized state vector $|\varphi(0)\rangle$.

We introduce the operator

$$\hat{b}(\alpha(0)) \;=\; \hat{a} - \alpha(0) \,. \tag{19}$$

With this definition we have

$$\begin{aligned} \hat{b}^\dagger(\alpha(0))\hat{b}(\alpha(0)) \;&=\; (\hat{a}^\dagger - \alpha(0)^\star)(\hat{a} - \alpha(0)) \\ &=\; \hat{a}^\dagger \hat{a} - \alpha(0)\hat{a}^\dagger - \alpha(0)^\star \hat{a} + \alpha(0)^\star \alpha(0) \end{aligned} \tag{20}$$

and furthermore, using the condition (14) and (18), we obtain

$$\begin{aligned} \langle \varphi(0) | \hat{b}^\dagger(\alpha(0))\hat{b}(\alpha(0)) | \varphi(0) \rangle \;&=\; \underbrace{\langle \varphi(0) | \hat{a}^\dagger \hat{a} | \varphi(0) \rangle}_{|\alpha(0)|^2} - \underbrace{\alpha(0) \langle \varphi(0) | \hat{a}^\dagger | \varphi(0) \rangle}_{\alpha(0)\alpha(0)^\star} \\ &\quad - \underbrace{\alpha(0)^\star \langle \varphi(0) | \hat{a} | \varphi(0) \rangle}_{\alpha(0)^\star \alpha(0)} + \underbrace{\alpha(0)^\star \alpha(0)}_{|\alpha(0)|^2} \\ &=\; 0 \,. \end{aligned} \tag{21}$$

From this we can conclude that

$$\hat{b}(\alpha(0)) | \varphi(0) \rangle \;=\; 0 \tag{22}$$

and, hence, according to (19)

$$\hat{a} | \varphi(0) \rangle \;=\; \alpha(0) | \varphi(0) \rangle \,. \tag{23}$$

Conversely, if the normalized state vector $|\varphi(0)\rangle$ satisfies (23), then obviously the conditions (14) and (18) are satisfied. The main result is that the quasi-classical state $|\varphi(0)\rangle$, belonging to the classical motion characterized by the parameter $\alpha(0)$, is an eigenstate of the annihilation operator $\hat{a}$ with the eigenvalue $\alpha(0)$. In the following we denote the eigenstate of $\hat{a}$ with eigenvalue α by $|\alpha\rangle$, resulting in the eigenvalue equation

$$\hat{a} | \alpha \rangle \;=\; \alpha | \alpha \rangle \,. \tag{24}$$

Next we have to find the solution of this eigenvalue equation. We do this by expressing $|\alpha\rangle$ as a series expansion of $|\varphi_n\rangle$:

$$| \alpha \rangle \;=\; \sum_n c_n(\alpha) | \varphi_n \rangle \,. \tag{25}$$

This yields

$$\hat{a}|\alpha\rangle \;=\; \sum_n c_n(\alpha)\sqrt{n}|\varphi_{n-1}\rangle\,. \tag{26}$$

Inserting this expression into eq. (24) results in

$$\sum_n c_n(\alpha)\sqrt{n}|\varphi_{n-1}\rangle \;=\; \alpha\sum_n c_n(\alpha)|\varphi_n\rangle\,, \tag{27}$$

that is,

$$c_{n+1}(\alpha) \;=\; \frac{\alpha}{\sqrt{n+1}}\,c_n(\alpha)\,. \tag{28}$$

With this relation, by recursion we can express all coefficients $c_n(\alpha)$ by $c_0(\alpha)$:

$$c_n(\alpha) \;=\; \frac{\alpha^n}{\sqrt{n!}}\,c_0(\alpha)\,. \tag{29}$$

We see that all coefficients $c_n(\alpha)$ are fixed by $c_0(\alpha)$. Therefore, the state vector $|\alpha\rangle$ is unambiguously defined despite a numerical factor. Assuming that $|\alpha\rangle$ is normalized, the coefficients $c_n(\alpha)$ have to satisfy the condition

$$\sum_n |c_n(\alpha)|^2 \;=\; |c_0(\alpha)|^2\sum_n\frac{|\alpha|^{2n}}{n!} \;=\; |c_0(\alpha)|^2\,\mathrm{e}^{|\alpha|^2} \;=\; 1\,, \tag{30}$$

that is,

$$c_0(\alpha) \;=\; \mathrm{e}^{-|\alpha|^2/2}\,. \tag{31}$$

With this result the solution of the eigenvalue equation is finally obtained to

$$|\alpha\rangle \;=\; \mathrm{e}^{-|\alpha|^2/2}\sum_n\frac{\alpha^n}{\sqrt{n!}}|\varphi_n\rangle\,. \tag{32}$$

Using eq. (16) we can rewrite this expression as

$$\begin{aligned}|\alpha\rangle \;&=\; \mathrm{e}^{-|\alpha|^2/2}\sum_n\frac{\alpha^n}{\sqrt{n!}}\frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}|\varphi_0\rangle \;=\; \mathrm{e}^{-|\alpha|^2/2}\sum_n\frac{\left(\alpha\hat{a}^\dagger\right)^n}{n!}\,|\varphi_0\rangle\\ &=\; \mathrm{e}^{-|\alpha|^2/2}\,\mathrm{e}^{\left(\alpha\hat{a}^\dagger\right)}|\varphi_0\rangle\,.\end{aligned} \tag{33}$$

Here, $|\varphi_0\rangle = |0\rangle$ is the vacuum state. In summary, we have constructed a quantum state such that the predictions of quantum mechanics agree in the limit of large quantum numbers with the results of classical mechanics. We see that this state is a coherent superposition of all stationary states $|\varphi_n\rangle$. We call this state a coherent state.

(b) We now determine the expectation values $\langle\hat{n}\rangle$, $\langle\hat{n}^2\rangle$, and the standard deviation $\Delta N = \sqrt{\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2}$ for the coherent state $|\alpha\rangle$. For the number operator we obtain

$$\begin{aligned}\langle\hat{n}\rangle \;&=\; \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle \;=\; |\alpha|^2\\ &=\; \mathrm{e}^{-|\alpha|^2}\sum_n\frac{|\alpha|^{2n}}{n!}\langle\varphi_n|\hat{a}^\dagger\hat{a}|\varphi_n\rangle\\ &=\; \sum_n n\,\mathrm{e}^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}\,.\end{aligned} \tag{34}$$

We see that the we do not have a fixed n. Therefore, $|\alpha\rangle$ is not an eigenfunction of the number operator $\hat{n}$. Rather the expectation value $\langle\hat{n}\rangle$ yields an average n with the probability distribution

$$P_n(\alpha) \;=\; |c_n(\alpha)|^2 \;=\; \frac{|\alpha|^{2n}}{n!}\,\mathrm{e}^{-|\alpha|^2} \tag{35}$$

corresponding to the Poisson distribution. Considering an oscillator in the state $|\alpha\rangle$, this means that a measurement of energy yields the value $E_n = (n+1/2)\hbar\omega$ with the probability P_n. Analyzing the probability distribution by differentiating with respect to the modulus $|\alpha|$ of the complex quantity $\alpha = |\alpha|\mathrm{e}^{\imath\phi}$, we see that $P_n(\alpha)$ has a maximum at $\langle n\rangle = |\alpha|^2 = N$. Differentiating with respect to the phase ϕ shows that N and ϕ are conjugate variables leading to the uncertainty relation

$$\Delta N\,\Delta\phi \;\geq\; \frac{1}{2}\,. \tag{36}$$

In order to derive the width ΔN of the probability distribution we need the expectation value

$$\begin{aligned}\langle\hat{n}^2\rangle &= \langle\alpha|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\alpha\rangle \\ &= \langle\alpha|\hat{a}^\dagger(\hat{a}^\dagger\hat{a}+1)\hat{a}|\alpha\rangle \;=\; |\alpha|^4+|\alpha|^2\,.\end{aligned} \tag{37}$$

With this result we obtain

$$\Delta N \;=\; \sqrt{\langle\hat{n}^2\rangle-\langle\hat{n}\rangle^2} \;=\; \sqrt{|\alpha|^4+|\alpha|^2-|\alpha|^4} \;=\; |\alpha| \tag{38}$$

and

$$\frac{\Delta N}{\langle\hat{n}\rangle} \;=\; \frac{|\alpha|}{|\alpha|^2} \;=\; \frac{1}{|\alpha|} \;\ll\; 1\,. \tag{39}$$

Evidently, for large $|\alpha|$ the width ΔN becomes large. That means that for the construction of the quasi-classical state we have to superpose a large number of stationary states $|\varphi_n\rangle$. However, the relative standard deviation $\Delta N/\langle\hat{n}\rangle = \Delta N/N$ becomes small leading to a sharp maximum in the probability distribution.

(c) In (a) we constructed the coherent state $|\alpha\rangle$ from bosonic wave functions $|\varphi_n\rangle$. The construction of such a coherent wave function from fermionic wave functions was a big achievement of BCS theory. Since the particle number in this case is huge – we typically have about 10^{22} electrons for a volume of about $1\,\mathrm{cm}^3$ –, also $\Delta N \sim 10^{11}$ is huge while $\Delta\phi \sim 10^{-11}$ is very small. Moreover, the relative standard deviation $\Delta N/N \sim 10^{-11}$ is also vanishingly small leading to a very sharp probability distribution $P_n(\alpha)$.

To derive a coherent state $|\beta\rangle$ for a system of fermions (e.g. electrons in a superconductor) we start from the bosonic wave function (33)

$$|\alpha\rangle \;=\; \mathrm{e}^{-|\alpha|^2/2}\,\mathrm{e}^{\left(\alpha\hat{a}^\dagger\right)}|\varphi_0\rangle \tag{40}$$

derived above for the quantum harmonic oscillator. The bosonic creation operator $\hat{a}^\dagger$ has now to be replaced by the sum over the fermionic operators, $\sum_{\mathbf{k}}\hat{c}^\dagger_{\mathbf{k}}$, creating fermionic states with wave vector $\mathbf{k}$ and occupation probability

$$|\beta_{\mathbf{k}}|^2 \;=\; \langle\beta|\hat{c}^\dagger_{\mathbf{k}}\hat{c}_{\mathbf{k}}|\beta\rangle\,. \tag{41}$$

Then, the coherent state corresponding to (40) can be written as

$$
\begin{aligned}
|\beta\rangle &= c\,\exp\left(\sum_{\mathbf{k}} \beta_{\mathbf{k}} \hat{c}_{\mathbf{k}}^{\dagger}\right)|\varphi_0\rangle = c\prod_{\mathbf{k}} \exp\left(\beta_{\mathbf{k}}\hat{c}_{\mathbf{k}}^{\dagger}\right)|\varphi_0\rangle \\
&= c\prod_{\mathbf{k}}\sum_{n} \frac{\left(\beta_{\mathbf{k}}\hat{c}_{\mathbf{k}}^{\dagger}\right)^n}{n!}|\varphi_0\rangle\,,
\end{aligned}
\tag{42}
$$

where c has to be determined by normalizing the wave function. We now have to take into account the important fact that due to Pauli's principle all powers of the creation operator higher than one yield zero. Then (42) can be simplified to

$$
|\beta\rangle = c\prod_{\mathbf{k}}\left(1+\beta_{\mathbf{k}}\hat{c}_{\mathbf{k}}^{\dagger}\right)|\varphi_0\rangle\,. \tag{43}
$$

Here, the ground state $|\varphi_0\rangle$ is the vacuum state $|0\rangle$. The normalization condition

$$
\langle\beta|\beta\rangle = 1 = |c|^2\,\langle\varphi_0|\prod_{\mathbf{k}}(1+\beta_{\mathbf{k}}^{\star}\hat{c}_{\mathbf{k}})\left(1+\beta_{\mathbf{k}}\hat{c}_{\mathbf{k}}^{\dagger}\right)|\varphi_0\rangle \tag{44}
$$

can be satisfied if all individual factors of the product become unity:

$$
1 = |c|^2\,\langle\varphi_0|\,(1+\beta_{\mathbf{k}}^{\star}\hat{c}_{\mathbf{k}})\left(1+\beta_{\mathbf{k}}\hat{c}_{\mathbf{k}}^{\dagger}\right)|\varphi_0\rangle = |c|^2(1+|\beta_{\mathbf{k}}|^2)\,. \tag{45}
$$

With this result we obtain the coherent state of the fermionic system to

$$
|\beta\rangle = \prod_{\mathbf{k}}\left(\frac{1}{\sqrt{1+|\beta_{\mathbf{k}}|^2}} + \frac{\beta_{\mathbf{k}}}{\sqrt{1+|\beta_{\mathbf{k}}|^2}}\hat{c}_{\mathbf{k}}^{\dagger}\right)|\varphi_0\rangle = \prod_{\mathbf{k}}\left(u_{\mathbf{k}}+v_{\mathbf{k}}\hat{c}_{\mathbf{k}}^{\dagger}\right)|\varphi_0\rangle \tag{46}
$$

with the complex probability amplitudes

$$
u_{\mathbf{k}} = \frac{1}{\sqrt{1+|\beta_{\mathbf{k}}|^2}}\,, \qquad v_{\mathbf{k}} = \frac{\beta_{\mathbf{k}}}{\sqrt{1+|\beta_{\mathbf{k}}|^2}}\,. \tag{47}
$$

Note that $|u_{\mathbf{k}}|^2$ and $|v_{\mathbf{k}}|^2$ are the probabilities for the state $|\varphi_{\mathbf{k}}\rangle$ being empty and occupied, respectively, with

$$
|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1\,. \tag{48}
$$

We further note that the BCS ground state is obtained by replacing the creation operator $\hat{c}_{\mathbf{k}}^{\dagger}$ by the pair operator $\hat{P}_{\mathbf{k}}^{\dagger} = \hat{c}_{\mathbf{k}\uparrow}^{\dagger}\hat{c}_{-\mathbf{k}\downarrow}^{\dagger}$, creating a Cooper pair with opposite $\mathbf{k}$ and spin. The probability amplitudes $|u_{\mathbf{k}}|^2$ and $|v_{\mathbf{k}}|^2$ then denote the probabilities for the pair state $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ being empty and occupied, respectively. They are usually called the BCS coherence factors.