Walther Meißner Institut

BAdW | BAYERISCHE AKADEMIE DER WISSENSCHAFTEN

Technische Universität München

Superconductivity and Low Temperature Physics I

Lecture Notes
Winter Semester 2021/2022

R. Gross

Chapter 5

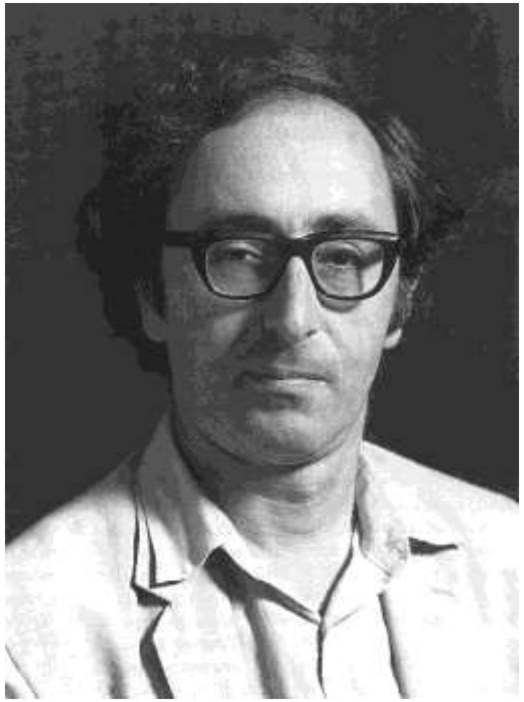
Josephson Effects

Superconductivity and Low Temperature Physics I

Lecture No. 10
23 December 2021

R. Gross

5. Josephson Effects



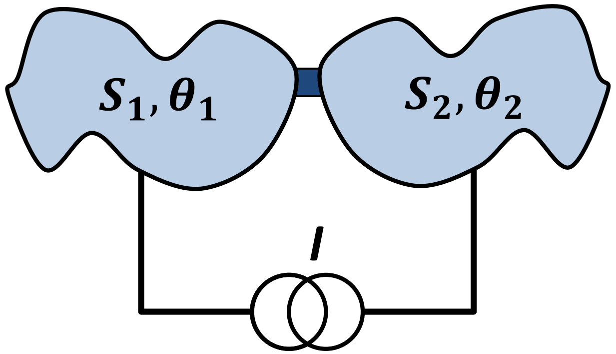
Brian David Josephson
born 04.01.1940

What happens if we weakly couple two superconductors ?

Possible new effects in superconductive tunnelling, Physics Letters **1**(7), 251-253 (1962)

5.1 Josephson Effects (cf. 3.2.3)

- ***what happens if we weakly couple two superconductors?***
  - coupling by *tunneling barriers, point contacts, normal conducting layers, etc.*
  - do they form a bound state such as a molecule?
  - if yes, what is the binding energy?



- **B.D. Josephson** in 1962
  (Nobel Prize in physics with Esaki and Giaever in 1973)

→ Cooper pairs can tunnel through thin insulating barrier (T = transmission amplitude for single charge carriers)

expectation: tunneling probability for pairs $\propto (|T|^2)^2$ ➔ extremely small $\sim (10^{-4})^2$

Josephson: tunneling probability for pairs $\propto |T|^2$
coherent tunneling of pairs („***tunneling of macroscopic wave function***“)

predictions:

- ➢ ***finite supercurrent at zero applied voltage***
- ➢ ***oscillation of supercurrent at constant applied voltage***

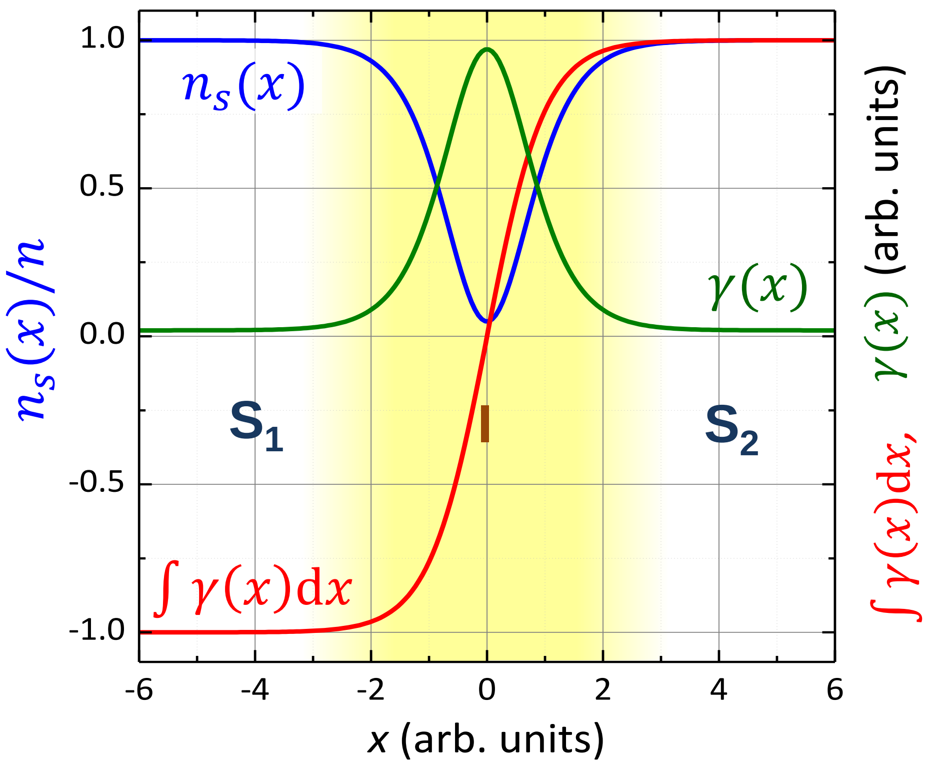
} ***Josephson effects***

- ➢ ***finite binding energy of coupled SCs = Josephson coupling energy***

- **coupling is weak** → supercurrent density between S_1 and S_2 is small → $|\psi|^2 = n_s$ is not changed in S_1 and S_2
- supercurrent density depends on gauge invariant phase gradient:

$$\mathbf{J}_s(\mathbf{r},t) = \frac{q_s n_s(\mathbf{r},t)\hbar}{m_s}\left\{\nabla\theta(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t)\right\} = \frac{q_s n_s(\mathbf{r},t)\hbar}{m_s}\ \gamma(\mathbf{r},t)$$

- **simplifying assumptions:**
  - current density is spatially homogeneous
  - $\gamma(\mathbf{r},t)$ varies negligibly in S_1 and S_2
  - $\mathbf{J}_s$ is equal in electrodes and junction area
    → γ in S_1 and S_2 much smaller than in insulator I

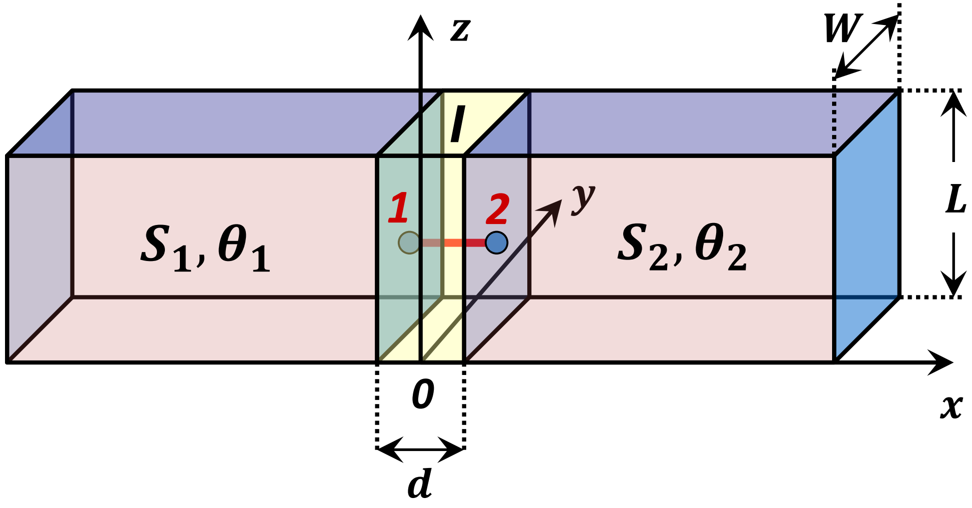


- **approximation:**
  - replace gauge invariant phase gradient γ by ***gauge invariant phase difference φ***:

$$\varphi(\mathbf{r},t) = \int_1^2 \gamma(\mathbf{r},t)\cdot\,\mathrm{d}\ell = \int_1^2\left(\nabla\theta(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t)\right)\cdot\mathrm{d}\ell = \theta_2(\mathbf{r},t) - \theta_1(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot\mathrm{d}\ell$$

5.1 Josephson Effects (cf. 3.2.3)

first Josephson equation:



- we *expect:* $J_s = J_s(\varphi)$

  $$J_s(\varphi) = J_s(\varphi + n \cdot 2\pi)$$

- for $J_s = 0$: phase difference must be zero:

  $$J_s(0) = J_s(n \cdot 2\pi) = 0$$

$$\Rightarrow \quad \boxed{J_s(\varphi) = J_c \sin\varphi + \sum_{m=2}^{\infty} J_{c,m} \sin(m\varphi)}$$

J_c = crititical or maximum Josephson current density

general formulation of **1st Josephson equation**: ***current-phase relation***

- in most cases: we have to keep only 1st term (especially for weak coupling):

  $$\boxed{J_s(\varphi) = J_c \sin\varphi}$$ **1st Josephson equation**

- *generalization* to ***spatially inhomogeneous*** supercurrent density:

  $$J_s(y,z) = J_c(y,z)\ \sin\varphi\,(y,z)$$

derived by Josephson for SIS junctions

supercurrent density J_s varies sinusoidally with phase difference $\varphi = \theta_2 - \theta_1$ w/o external potentials

second Josephson equation (for spatially homogeneous junction)

- take time derivative of the gauge invariant phase difference $\varphi(t) = \theta_2(t) - \theta_1(t) - \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A}(t)\cdot \mathrm{d}\ell$

$$\frac{\partial\varphi(t)}{\partial t} = \frac{\partial\theta_2(t)}{\partial t} - \frac{\partial\theta_1(t)}{\partial t} - \frac{2\pi}{\Phi_0}\frac{\partial}{\partial t}\int_1^2 \mathbf{A}(t)\cdot \mathrm{d}\ell$$

- substitution of the energy-phase relation $\hbar\frac{\partial\theta(t)}{\partial t} = -\left\{\frac{1}{2n_s}\Lambda\mathbf{J}_s^2(t) + q_s\phi_{\mathrm{el}}(\mathbf{r},t)\right\}$ gives:

$$\frac{\partial\varphi(t)}{\partial t} = -\frac{1}{\hbar}\left(\frac{\Lambda}{2n_s}\left[\mathbf{J}_s^2(2) - \mathbf{J}_s^2(1)\right] + q_s\left[\phi_{\mathrm{el}}(2) - \phi_{\mathrm{el}}(1)\right]\right) - \frac{2\pi}{\Phi_0}\frac{\partial}{\partial t}\int_1^2 \mathbf{A}(t)\cdot \mathrm{d}\ell$$

- supercurrent density across the junction is *continuous* $(\mathbf{J}_s(1) = \mathbf{J}_s(2))$:

$$\frac{\partial\varphi(t)}{\partial t} = \frac{2\pi}{\Phi_0}\int_1^2 \left(-\nabla\phi_{\mathrm{el}} - \frac{\partial\mathbf{A}(t)}{\partial t}\right)\cdot \mathrm{d}\ell$$

(term in parentheses = electric field)

$$\Rightarrow \quad \boxed{\frac{\partial\varphi(t)}{\partial t} = \frac{2\pi}{\Phi_0}\underbrace{\int_1^2 \mathbf{E}(t)\cdot \mathrm{d}\ell}_{\text{voltage drop } V} = \frac{2\pi}{\Phi_0}V(t) = \frac{q_s V(t)}{\hbar}}$$

2nd Josephson equation: *voltage – phase relation*

5.1 Josephson Effects (cf. 3.2.3)

- for a constant voltage V across the junction:

$$\frac{\partial\varphi(t)}{\partial t} = \frac{2\pi}{\Phi_0}V = \frac{q_s V}{\hbar}$$

integration yields: $\varphi(t) = \varphi_0 + \frac{2\pi}{\Phi_0}V \cdot t = \varphi_0 + \frac{q_s}{\hbar}V \cdot t$

phase difference increases linearly in time

- supercurrent density J_s oscillates at the Josephson frequency $\nu = V/\Phi_0$:

$$J_s(\varphi(t)) = J_c \sin\varphi(t) = J_c \sin\left(\frac{2\pi}{\Phi_0}V \cdot t\right)$$

$$\frac{\nu}{V} = \frac{\omega/2\pi}{V} = \frac{1}{\Phi_0} = 483.597\,9\,\frac{\mathrm{MHz}}{\mu\mathrm{V}}$$

➔ **Josephson junction = *voltage controlled oscillator***

- **applications:**
  – Josephson voltage standard
  – microwave sources
  – ….

Josephson coupling energy E_J: binding energy of two coupled superconductors

$$\frac{E_J}{A} = \int_0^{t_0} J_s\, V\, \mathrm{d}t = \int_0^{t_0} J_c \sin\varphi \left(\frac{\Phi_0}{2\pi}\frac{\partial\varphi}{\partial t}\right) \mathrm{d}t = \frac{\Phi_0 J_c}{2\pi}\int_0^{\varphi} \sin\varphi' \; \mathrm{d}\varphi'$$

with $\varphi(0) = 0$ and $\varphi(t_0) = \varphi$
$A =$ junction area

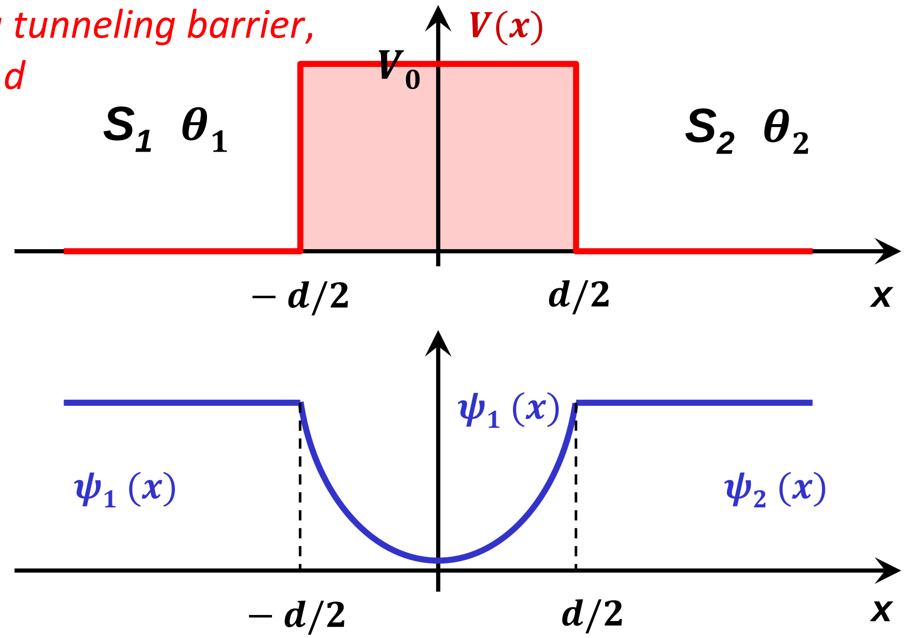
integration yields:

$$\boxed{\frac{E_J}{A} = \frac{\Phi_0 J_c}{2\pi}(1 - \cos\varphi)}$$

Josephson coupling energy (per junction area)

5.1.1 Superconducting Tunnel Junctions

Josephson effect in superconducting tunnel junctions



- **derive the Josephson equations**
  - starting point is time-dependent Schrödinger equation

$$\imath\hbar \frac{\partial\psi(\mathbf{r},t)}{\partial t} = E\,\psi(\mathbf{r},t)$$

  - S_1 and S_2 are described by macroscopic wave functions

$$\psi_1(\mathbf{r},t) = \psi_{01}\,\mathrm{e}^{\imath\theta_1(\mathbf{r},t)} \qquad n_{s1} = |\psi_{01}|^2$$

$$\psi_2(\mathbf{r},t) = \psi_{02}\,\mathrm{e}^{\imath\theta_2(\mathbf{r},t)} \qquad n_{s2} = |\psi_{02}|^2$$

  - finite coupling between S_1 and S_2 is introduced by small perturbation T (tunnel coupling)

$$\imath\hbar \frac{\partial\psi_1(\mathbf{r},t)}{\partial t} = E_1\,\psi_1(\mathbf{r},t) + T_{\mathrm{LR}}\psi_2(\mathbf{r},t) = +e\Delta\phi\,\psi_1(\mathbf{r},t) + T_{\mathrm{LR}}\psi_2(\mathbf{r},t)$$

$$\imath\hbar \frac{\partial\psi_2(\mathbf{r},t)}{\partial t} = E_2\,\psi_2(\mathbf{r},t) + T_{\mathrm{RL}}\psi_1(\mathbf{r},t) = -e\Delta\phi\,\psi_2(\mathbf{r},t) + T_{\mathrm{RL}}\psi_1(\mathbf{r},t)$$

$$\Delta\phi = \frac{E_1 - E_2}{|q_s|} = \frac{E_1 - E_2}{2e} = \frac{1}{2}\frac{E_1 - E_2}{e}$$

– by inserting the wave functions ψ_1, ψ_2 into the time-dependent Schrödinger equation we obtain for the **imaginary part**:

$$\frac{\partial \theta_1(t)}{\partial t} = -\frac{T}{\hbar}\sqrt{\frac{n_{s2}}{n_{s1}}}\cos\varphi(t) - \frac{e\Delta\phi}{\hbar}$$

$$\frac{\partial \theta_2(t)}{\partial t} = -\frac{T}{\hbar}\sqrt{\frac{n_{s1}}{n_{s2}}}\cos\varphi(t) + \frac{e\Delta\phi}{\hbar}$$

we use $T_{\mathrm{LR}} = T_{\mathrm{RL}} = T$ and $\varphi = \theta_2 - \theta_1$,

– for the **real part** we obtain:

$$\frac{\partial n_{s1}(t)}{\partial t} = +\frac{2T}{\hbar}\sqrt{n_{s1}n_{s2}}\sin\varphi(t)$$

$$\frac{\partial n_{s2}(t)}{\partial t} = -\frac{2T}{\hbar}\sqrt{n_{s1}n_{s2}}\sin\varphi(t)$$

we see that $\frac{\partial n_{s1}(\mathbf{r},t)}{\partial t} = -\frac{\partial n_{s2}(\mathbf{r},t)}{\partial t}$ $\Rightarrow$ ***conservation of particle number***

– supercurrent density:

$$J_s^{1\to 2} = \frac{2e}{A}\frac{\partial n_{s1}(t)}{\partial t}$$

$$J_s^{2\to 1} = \frac{2e}{A}\frac{\partial n_{s2}(t)}{\partial t}$$

$$\Rightarrow \quad \boxed{J_s = J_s^{1\to 2} - J_s^{2\to 1} = \frac{4eT}{\hbar A}\frac{\partial n_{s2}(t)}{\partial t}\sqrt{n_{s1}n_{s2}}\sin\varphi(t) = J_c\sin\varphi(t)}$$

1st Josephson equation
(current-phase relation)

5.1.1 Superconducting Tunnel Junctions

– the 2nd Josephson equation is obtained from the gauge invariant phase difference $\varphi(\mathbf{r},t) = \theta_2(\mathbf{r},t) - \theta_1(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot \mathrm{d}\ell$

$$\frac{\partial\varphi(t)}{\partial t} = \frac{\partial\theta_2(t)}{\partial t} - \frac{\partial\theta_1(t)}{\partial t} - \frac{2e}{\hbar}\frac{\partial}{\partial t}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot \mathrm{d}\ell$$

$$\frac{\partial\theta_1(t)}{\partial t} = -\frac{T}{\hbar}\sqrt{\frac{n_{s2}}{n_{s1}}}\cos\varphi(t) - \frac{e\Delta\phi}{\hbar}$$

$$\frac{\partial\theta_2(t)}{\partial t} = -\frac{T}{\hbar}\sqrt{\frac{n_{s1}}{n_{s2}}}\cos\varphi(t) + \frac{e\Delta\phi}{\hbar}$$

$$\frac{\partial\varphi(t)}{\partial t} = \frac{2e\Delta\phi}{\hbar} - \frac{2e}{\hbar}\frac{\partial}{\partial t}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot \mathrm{d}\ell$$

for $n_{s1} = n_{s2}$

$$\frac{\partial\varphi(t)}{\partial t} = \frac{2e}{\hbar}\underbrace{\int_1^2 \left[-\nabla\phi - \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t)\right]\cdot \mathrm{d}\ell}_{=V}$$

$$\boxed{\frac{\partial\varphi(t)}{\partial t} = \frac{2e}{\hbar}V = \frac{2\pi}{\Phi_0}V}$$

2nd Josephson equation
(voltage-phase relation)

calculation of the maximum Josephson current density: How does T_{LR} depend on height and thickness of barrier?

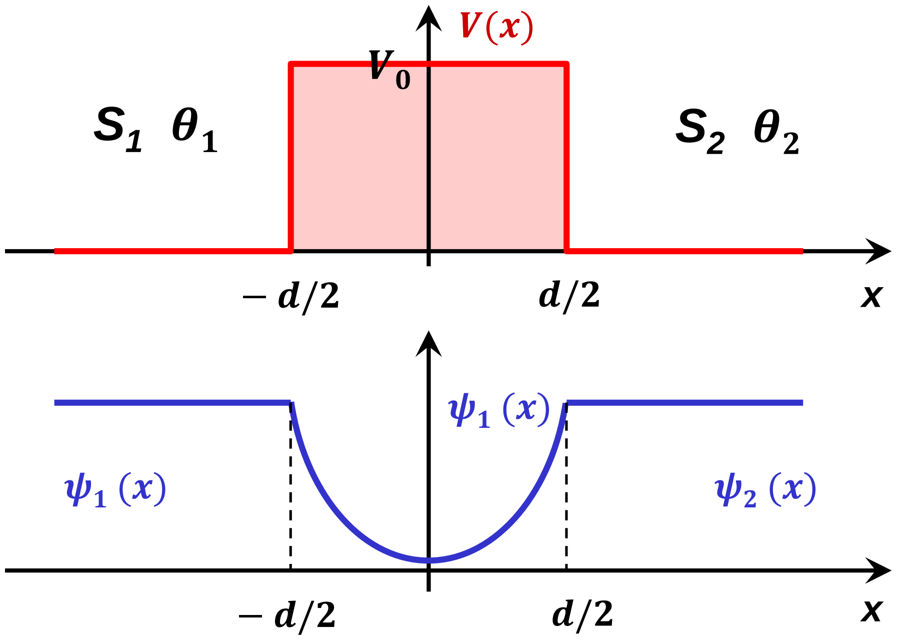
- calculation by the ***wave matching method***

solve time-independent Schrödinger equation for S_1, S_2 and barrier region and match solution at interfaces

$$-\frac{\hbar^2}{2m_s}\nabla^2\psi(\mathbf{r}) = (E_0 - V_0)\psi(\mathbf{r})$$

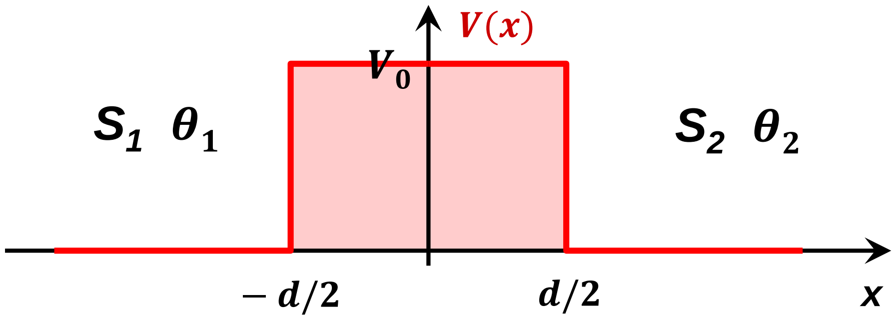
with $\psi(\mathbf{r}) = \psi_0(\mathbf{r})\, e^{\imath\theta(\mathbf{r})}$

insulating tunneling barrier,
thickness *d*



5.1.1 Superconducting Tunnel Junctions

$$-\frac{\hbar^2}{2m_s}\nabla^2\psi(\mathbf{r}) = (E_0 - V_0)\psi(\mathbf{r})$$



- ***assumption***:

  homogeneous barrier and supercurrent flow → **1D problem**

- ***solutions***:

  - in superconductors: $\psi_{1,2}(x) = \psi_{01,02}\mathrm{e}^{\imath\theta_{1,2}(x)} = \sqrt{n_{s1,s2}}\,\mathrm{e}^{\imath\theta_{1,2}(x)}$ (macrosopic wave function)

  - in insulator: sum of decaying and growing exponentials $\psi(x) = A\cosh(\kappa x) + B\sinh(\kappa x)$

  - characteristic decay constant: $\kappa = \sqrt{2m_s(V_0 - E_0)/\hbar^2}$ for $E_0 < V_0$

- coefficients A and B are determined by the boundary conditions at $x = \pm d/2$:

$$\psi(x = -d/2) = \sqrt{n_{s1}}\,\mathrm{e}^{\imath\theta_1} \qquad \psi(x = +d/2) = \sqrt{n_{s2}}\,\mathrm{e}^{\imath\theta_2}$$

$n_{1,2}, \theta_{1,2}$: Cooper pair density and wave function phase at the boundaries $x = \pm d/2$

$$\Rightarrow \quad \sqrt{n_{s1}}\,\mathrm{e}^{\imath\theta_1} = A\cosh(\kappa d/2) - B\sinh(\kappa d/2) \qquad \sqrt{n_{s2}}\,\mathrm{e}^{\imath\theta_2} = A\cosh(\kappa d/2) + B\sinh(\kappa d/2)$$

5.1.1 Superconducting Tunnel Junctions

- solving for A and B: $$A = \frac{\sqrt{n_{s1}}\,\mathrm{e}^{\imath\theta_1} + \sqrt{n_{s2}}\,\mathrm{e}^{\imath\theta_2}}{\cosh(\kappa d/2)} \qquad B = -\frac{\sqrt{n_{s1}}\,\mathrm{e}^{\imath\theta_1} - \sqrt{n_{s2}}\,\mathrm{e}^{\imath\theta_2}}{2\sinh(\kappa d/2)}$$

- supercurrent density: $$\mathbf{J}_s = \frac{q_s\hbar}{2m_s\imath}\left(\psi\boldsymbol{\nabla}\psi^\star - \psi^\star\boldsymbol{\nabla}\psi\right)$$

- substituting the coefficients A and B (after some lengthy calculation):

$$\mathbf{J}_s = \mathbf{J}_c\sin(\theta_2-\theta_1) = \mathbf{J}_c\sin\varphi$$

current-phase relation

$$\mathbf{J}_c = -\frac{q_s\hbar\kappa}{m_s}\frac{\sqrt{n_{s1}n_{s2}}}{\underbrace{2\sinh(\kappa d/2)\cosh(\kappa d/2)}_{=\sinh(2\kappa d)}} = -\frac{q_s\hbar\kappa}{m_s}\frac{\sqrt{n_{s1}n_{s2}}}{\sinh(2\kappa d)}$$

maximum Josephson current density

- ***real junctions***:

$V_0\;\approx$ few eV $\Rightarrow 1/\kappa < 1\,\mathrm{nm}$, $d \approx$ few nm $\Rightarrow \kappa d \ll 1$, then $\sinh(2\kappa d) \simeq \frac{1}{2}\exp(2\kappa d)$, $q_s = -2e$:

- maximum Josephson current **decays exponentially** with increasing barrier thickness d:

$$\mathbf{J}_c = \frac{2e\hbar\kappa}{m_s}\,2\sqrt{n_{s1}n_{s2}}\exp(-2\kappa d) \qquad q_s = -2e$$

5.1.1 Superconducting Tunnel Junctions

more elaborate theory of tunneling between superconductors

M. H. Cohen, L. M. Falicov, and J. C. Phillips, *Superconductive Tunneling*, Phys. Rev. Lett. **8**, 316-318 (1962).

B. D. Josephson, *Possible new effects in superconductive tunnelling*, Physics Letters **1**(7), 251-253 (1962).

- **open questions**
  - what is the charge crossing the tunneling barrier when a Bogoliubov quasiparticle tunnels from S_1 to S_2
  - what is the role of the coherence factors ?

- **only brief description of theoretical approach**

Hamiltonian: $\mathcal{H} = \mathcal{H}_\mathrm{L} + \mathcal{H}_\mathrm{R} + \mathcal{H}_T$

$\mathcal{H}_T$ = tunneling hamiltonian, small extra term

$$\mathcal{H}_T = \sum_{\mathbf{kq}} T_{\mathrm{LR}} \underbrace{\left(c^\dagger_{\mathbf{k}} c_{\mathbf{q}} + c^\dagger_{\mathbf{q}} c_{\mathbf{k}}\right)}$$

transfer electrons from states $\mathbf{k}$ on left to $\mathbf{q}$ on rhs of the barrier and vice versa

matrix elements, fall of exponentially with barrier thickness d

determination of the tunneling current by calculation of $\langle \dot{\mathcal{N}}_\mathrm{L} \rangle = -\langle \dot{\mathcal{N}}_\mathrm{R} \rangle$ by using the equation of motion for $\dot{\mathcal{N}}_{\mathrm{L,R}}$

$$\imath\hbar \dot{\mathcal{N}}_{\mathrm{L,R}} = \left[\mathcal{N}_{\mathrm{L,R}}, \mathcal{H}\right] = \left[\mathcal{N}_{\mathrm{L,R}}, \mathcal{H}_T\right]$$

as $\mathcal{H}_\mathrm{L}, \mathcal{H}_\mathrm{R}$ commute with $\mathcal{N}_{\mathrm{L},R}$ (conserve particle number)

5.1.1 Superconducting Tunnel Junctions

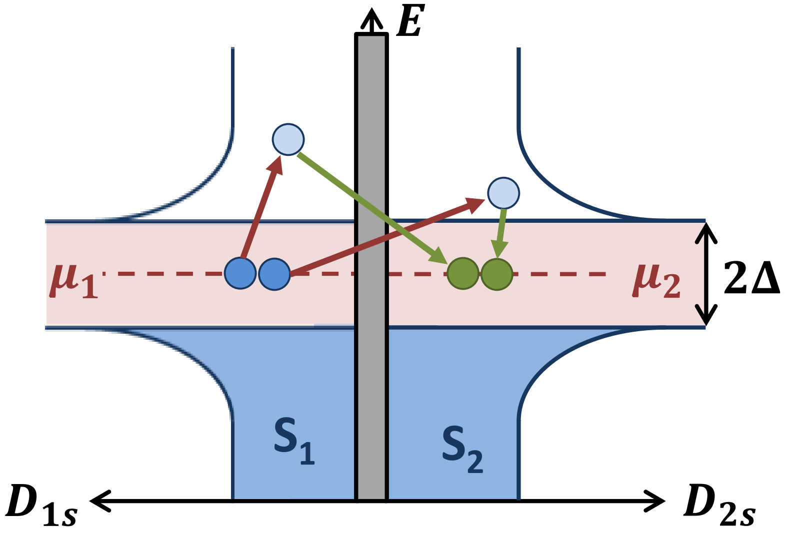
more elaborate theory of tunneling between superconductors

tunneling current: $$\mathcal{I} = q\dot{\mathcal{N}} = \frac{q}{\imath\hbar}[\mathcal{N}, \mathcal{H}_T]$$

q = charge transported per tunneling particle

- NIN-junction: $$\mathcal{I} = \frac{e}{\imath\hbar}\sum_{\mathbf{kq}} T_{\mathbf{kq}}\left(c_{\mathbf{k}}^{\dagger}c_{\mathbf{q}} - c_{\mathbf{q}}^{\dagger}c_{\mathbf{k}}\right)$$ for tunneling between two normal metals

- SIS-junction: we have to replace $c_{\mathbf{k,q}}^{\dagger}c_{\mathbf{k,q}}$ by the Bogoliubov quasiparticle excitation and anihilation operators $\alpha_{\mathbf{k,q}}^{\dagger}, \alpha_{\mathbf{k,q}}, \beta_{\mathbf{k,q}}^{\dagger}, \beta_{\mathbf{k,q}}$



- visualization of Josephson tunneling as a 2 step process
  - **step 1**: blue Cooper pair in S_1 is broken up and one excitation crosses barrier into S_2
  - resulting intermediate state (light blue) is classically forbidden but allowed within uncertainty relation
  - **step 2**: second excitation crosses barrier and recombines with the first forming green Cooper pair

- **remark**:
  Josephson tunneling may be viewed as a second order process and $\propto |T|^4$. However, Josephson was assuming a constant (and not arbitrary) phase difference between initial and final states. Then, quantum mechanical treatment yields a supercurrent $\propto |T|^2$

5.1.1 Superconducting Tunnel Junctions

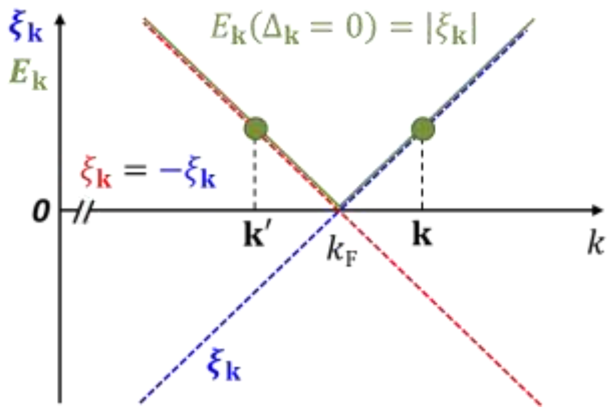
tunneling in SIS junctions at finite voltage – quasiparticle tunneling

evaluation of $\langle \dot{\mathcal{N}} \rangle$ shows that coherence factors are not dropping out (cf. 4.4.2)

$$J_{\mathrm{qp}} \propto -\frac{e|T|^2}{\imath\hbar}\sum_{\mathbf{kq}} \frac{u_{\mathbf{k}}^2 u_{\mathbf{q}}^2 (f_{\mathbf{k}} - f_{\mathbf{q}})}{E_{\mathbf{k}} - E_{\mathbf{q}} + eV}$$

sum has to be taken over electron and hole branches on both sides
➔ **coherence factors** all disappear (see argument given below)
➔ sum → integration: principal part integrals all cancel, only residues at poles are left

- qualitative argument:
  - ➢ tunneling from state $|\mathbf{q}\sigma\rangle$ into a state $|\mathbf{k}\sigma\rangle$ is only possible if pair state $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ is empty
    → resulting tunneling probability is $\propto |u_{\mathbf{k}}|^2 \left|T_{\mathbf{kq}}\right|^2$
  - ➢ for each state $|\mathbf{k}\sigma\rangle$ there exists a state $|\mathbf{k}'\sigma\rangle$ with $E_{\mathbf{k}} = E_{\mathbf{k}'}$ but with $\xi_{\mathbf{k}'} = -\xi_{\mathbf{k}}$
    → resulting tunneling probability is $\propto |u_{\mathbf{k}'}|^2 \left|T_{\mathbf{k}'\mathbf{q}}\right|^2 \underbrace{=}_{|u(-\xi_{\mathbf{k}})|=|v(\xi_{\mathbf{k}})|} |v_{\mathbf{k}}|^2 \left|T_{\mathbf{k}'\mathbf{q}}\right|^2$



total tunneling probability $\propto \left(|\mathbf{u}_{\mathbf{k}}|^2 + |\mathbf{v}_{\mathbf{k}}|^2\right)\left|\mathbf{T}_{\mathbf{kq}}\right|^2 = \left|\mathbf{T}_{\mathbf{kq}}\right|^2$ **does not depend on coherence factors**

→ ***simple „semiconductor model" for quasiparticle tunneling is applicable***

M. H. Cohen, L. M. Falicov, and J. C. Phillips, *Superconductive Tunneling*, Phys. Rev. Lett. 8, 316-318 (1962).

tunneling in SIS junctions at zero voltage – Josephson tunneling

evaluation of $\langle\dot{\mathcal{N}}\rangle$ shows that coherence factors play an important role

$$J_\mathrm{s} \propto -\frac{e|T|^2}{\imath\hbar}\sum_{\mathbf{kq}} \frac{u_\mathbf{k} v_\mathbf{k} u_\mathbf{q} v_\mathbf{q} \left(f_\mathbf{k} - f_\mathbf{q}\right) \mathrm{e}^{\imath\varphi}}{E_\mathbf{k} - E_\mathbf{q}}$$

- we have to sum up over electron and hole branches on both sides
- sum → integration: principal part integrals do no longer cancel
- leads to a finite Josephson current density $J_s = J_c \sin\varphi$ at zero voltage with

$$J_\mathrm{c} = \frac{2e|T|^2}{\hbar\, A} D_{s1}(E_\mathrm{F}) D_{s2}(E_\mathrm{F})\, \mathcal{P} \int_{-\infty}^{\infty} \mathrm{d}\xi_1 \int_{-\infty}^{\infty} \mathrm{d}\xi_2 \frac{\Delta_1}{E_1}\frac{\Delta_2}{E_2}\frac{f(E_1) f(E_2)}{E_1 - E_2}$$

$\mathcal{P}$ = principal part

for $\Delta_1 = \Delta_2$:

$$J_c = \frac{\pi}{2eR_nA}\Delta(T)\tanh\left(\frac{\Delta(T)}{2k_\mathrm{B}T}\right)$$

5.1.2 Ambegaokar-Baratoff Relation

important result of elaborate tunneling theory

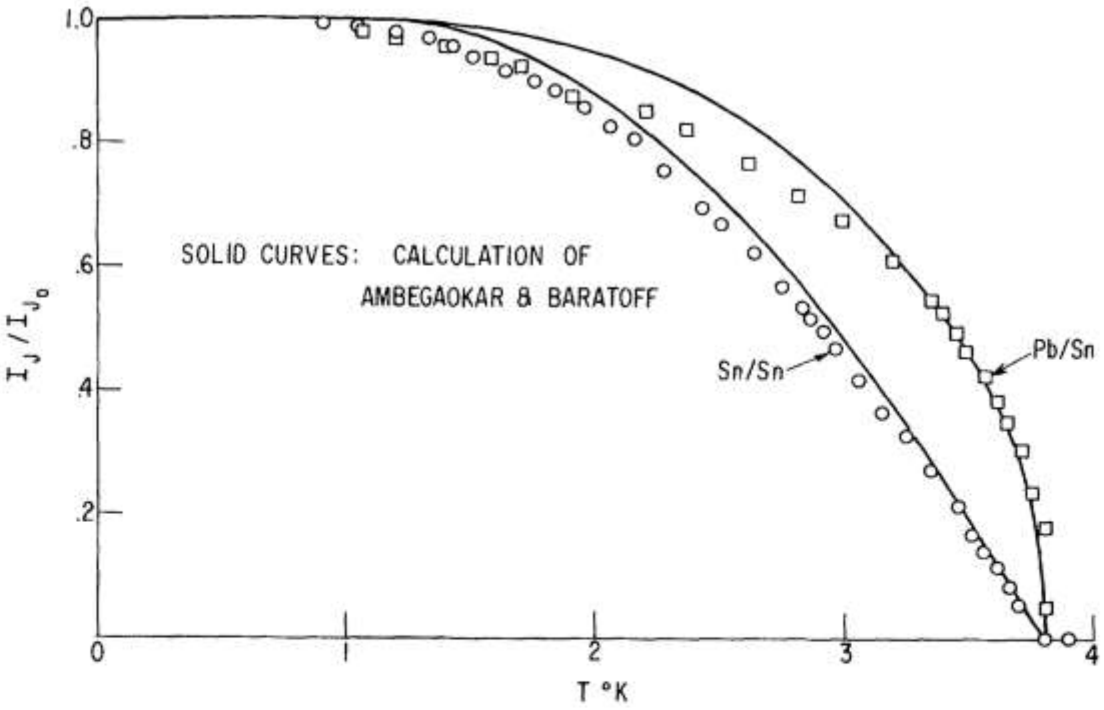
- ratio of maximum Josephson current density J_c and $J_{\rm NIN} = J_{\rm qp}(eV \gg 2\Delta) \propto 1/R_nA = const$

$$\Rightarrow \quad \frac{J_c}{J_{\rm NIN}} = J_cR_nA = I_cR_n = const.$$

Ambegaokar-Baratoff relation:

$$I_cR_n = \frac{\pi}{2e}\Delta(T)\tanh\left(\frac{\Delta(T)}{2k_{\rm B}T}\right)$$

V. Ambegaokar, A. Baratoff, *Tunneling Between Superconductors*, Phys. Rev. Lett. **10**, 486-489 (1963).

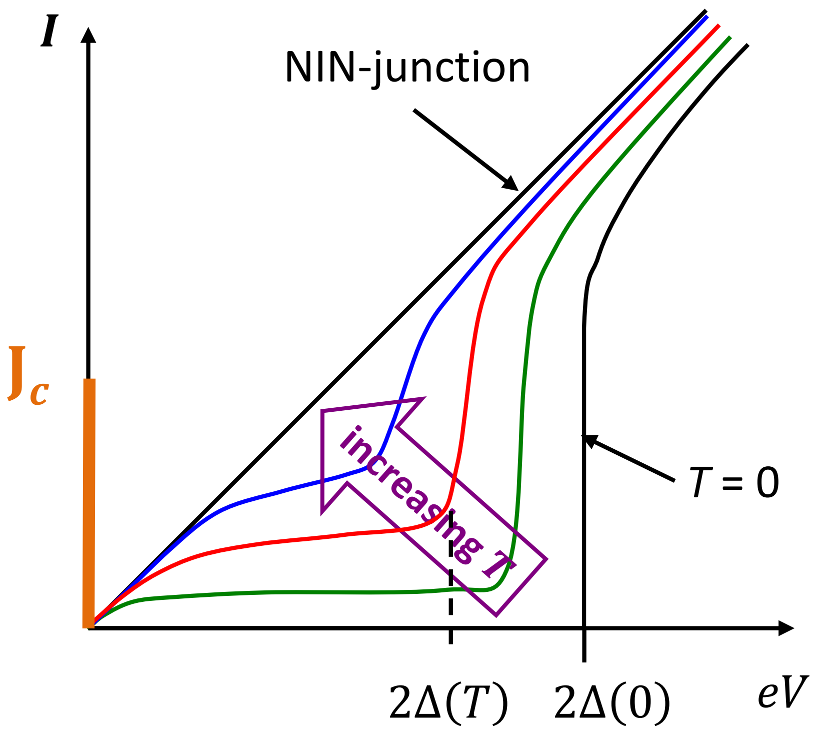


M.D. Fiske, Rev. Mod. Phys. **36**, 221–222 (1964)
Temperature and Magnetic Field Dependences of the Josephson Tunneling Current

R. Gross and A. Marx , © Walther-Meißner-Institut (2004 - 2021)

5.1.2 Ambegaokar-Baratoff Relation

current-voltage characteristics



- **quasiparticle tunneling (cf. 4.4.2)**:

at $eV > 0$

$$\overline{\mathbf{J}_s(t)} = \mathbf{J}_c\ \overline{\sin(\varphi(t))} = \mathbf{J}_c\ \overline{\sin\left(\frac{2eV}{\hbar}t\right)} = 0$$

at $eV \gg 2\Delta(T)$

$$\mathbf{J}_{\mathrm{qp}}(V) \simeq \mathbf{J}_{\mathrm{NIN}}(V) \propto \frac{1}{R_n A} \cdot \exp(-2\kappa d)$$

R_n = normal resistance ≙ resistance of NIN tunnel junction

- **Cooper pair tunneling**:

at $eV = 0$

$$\mathbf{J}_{\mathrm{qp}}(V = 0) = 0$$

$$\mathbf{J}_{\mathrm{c}}(V = 0) = \frac{e\hbar\kappa}{m_s} 2\sqrt{n_{s1}n_{s2}} \cdot \exp(-2\kappa d)$$

5.2 Josephson Coupling Energy

Josephson coupling energy E_J: binding energy of two coupled superconductors (cf. 3.2.3)

- the two weakly coupled superconductors form “***molecule***” analogous to H_2 molecule
  → what is the ***binding energy*** of this molecule ?

- consider a JJ with $\mathbf{J}_s = 0$ and then ***increase junction current from zero to finite value***
  – phase difference has to change → phase change corresponds to finite voltage according to voltage-phase relation
  – external source has to supply energy (to accelerate the superelectrons)
  – stored in kinetic energy of moving superelectrons
  – integral of the supplied power $I \cdot V$ to increase current to $I(\varphi) = I_c \sin\varphi$ (voltage during increase of current)

$$\frac{E_J}{A} = \int_0^{t_0} J_s\, V\, \mathrm{d}t = \int_0^{t_0} J_c \sin\varphi \left(\frac{\Phi_0}{2\pi}\frac{\partial\varphi}{\partial t}\right) \mathrm{d}t = \frac{\Phi_0 J_c}{2\pi}\int_0^{\varphi} \sin\varphi' \; \mathrm{d}\varphi'$$

with $\varphi(0) = 0$ and $\varphi(t_0) = \varphi$
$A =$ junction area

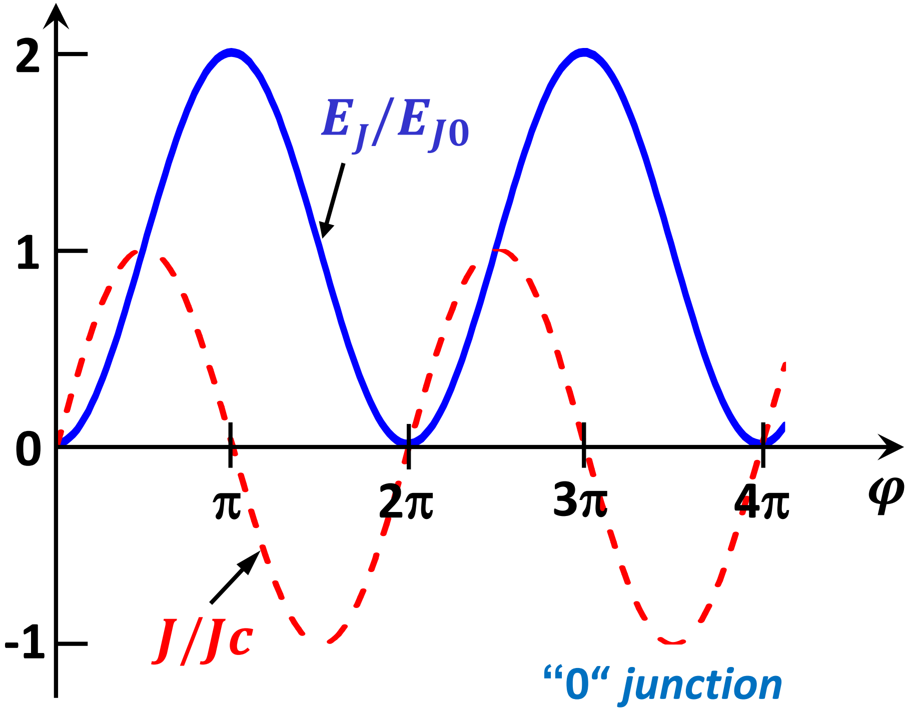
integration yields:

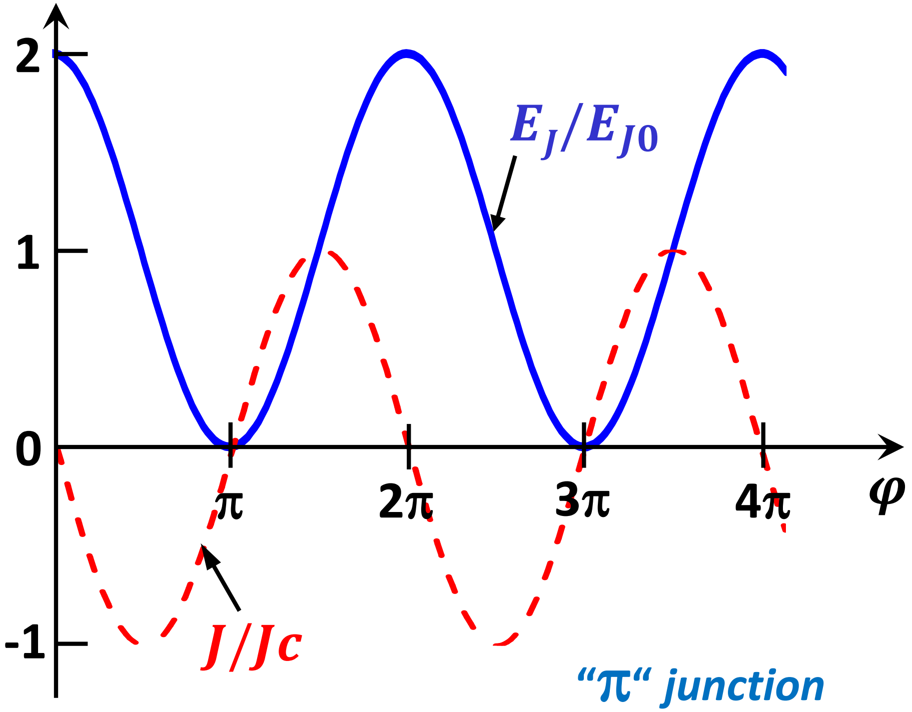
$$\frac{E_J}{A} = \frac{\Phi_0 J_c}{2\pi}(1 - \cos\varphi) = \frac{E_{J0}}{A}\,(1 - \cos\varphi)$$

Josephson coupling energy (per junction area)

5.2 Josephson Coupling Energy

Josephson coupling energy E_{J} (cf. 3.2.3)





- **order of magnitude estimate:**
  - typically: $I_c = J_c A \sim 1\ \mathrm{mA} \;\Rightarrow\; E_{J0} \simeq 3 \times 10^{-19}\ \mathrm{J}$
  - corresponds to thermal energy $k_{\mathrm{B}}T$ for $T \simeq 20\,000\ \mathrm{K}$
  - junction with very small critical current: $I_c \simeq 1\ \mu A \;\Rightarrow$ thermal energy $\simeq k_{\mathrm{B}} \times 20\ \mathrm{K}$

5.2.1 Josephson Junction with Applied Current

Josephson junction under the action of an external force (applied current)

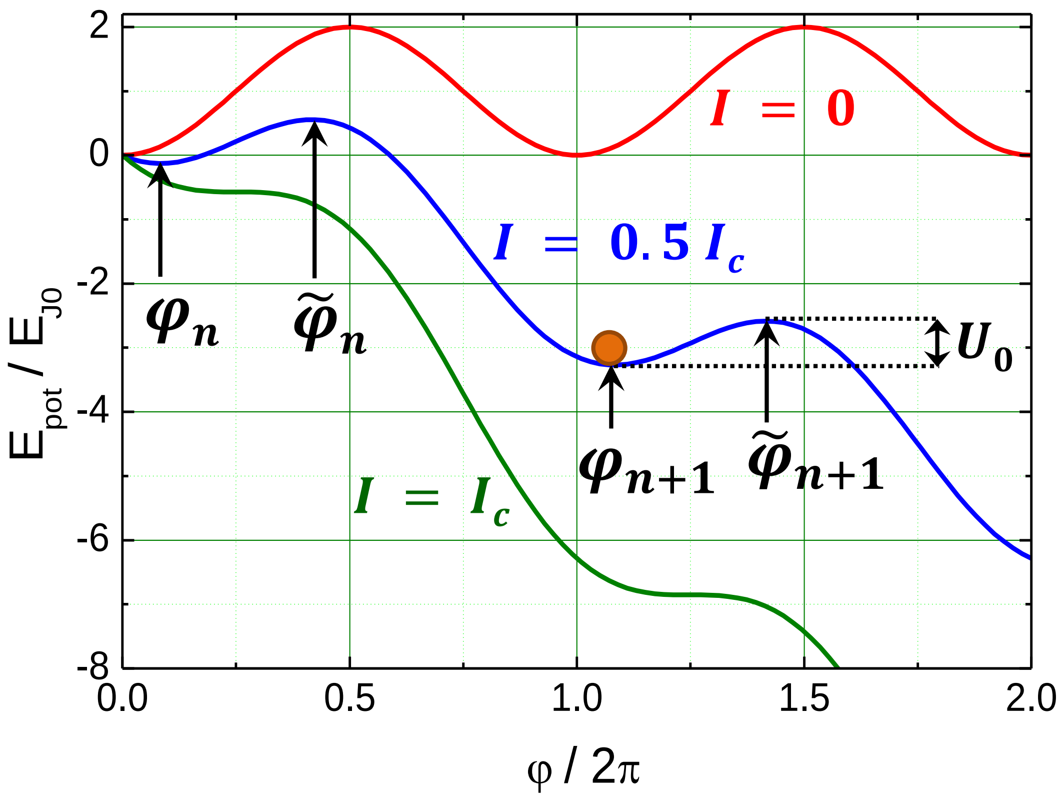
- ***potential energy $E_{\rm pot}$*** of the system under action of external force: $\boldsymbol{E_{\rm pot} = E_J - F \cdot x}$
  - E_J: intrinsic free energy of the junction
  - F: generalized force $(F = I)$
  - x: generalized coordinate
    $\Rightarrow F \cdot \partial x/\partial t$ = power flowing into subsystem $(I \cdot V)$
    $\Rightarrow \partial x/\partial t = V$:

$$x = \int V\, dt = \frac{\hbar}{2e}\,\varphi + c = \Phi_0 \frac{\varphi}{2\pi} + c$$

$$E_{\rm pot}(\varphi) = E_J(\varphi) - I\left(\Phi_0 \frac{\varphi}{2\pi} + c\right)$$

$$E_{\rm pot}(\varphi) = \underbrace{\frac{\Phi_0 I_c}{2\pi}}_{E_{J0}}\left(1 - \cos\varphi - \frac{I}{I_c}\varphi\right) + \tilde{c}$$

tilted washboard potential
stable minima at φ_n, unstable maxima at $\tilde{\varphi}_n$,
states for different n are equivalent



- **junction dynamics**: **motion of “phase particle” φ in tilted washboard potential** (not discussed here)

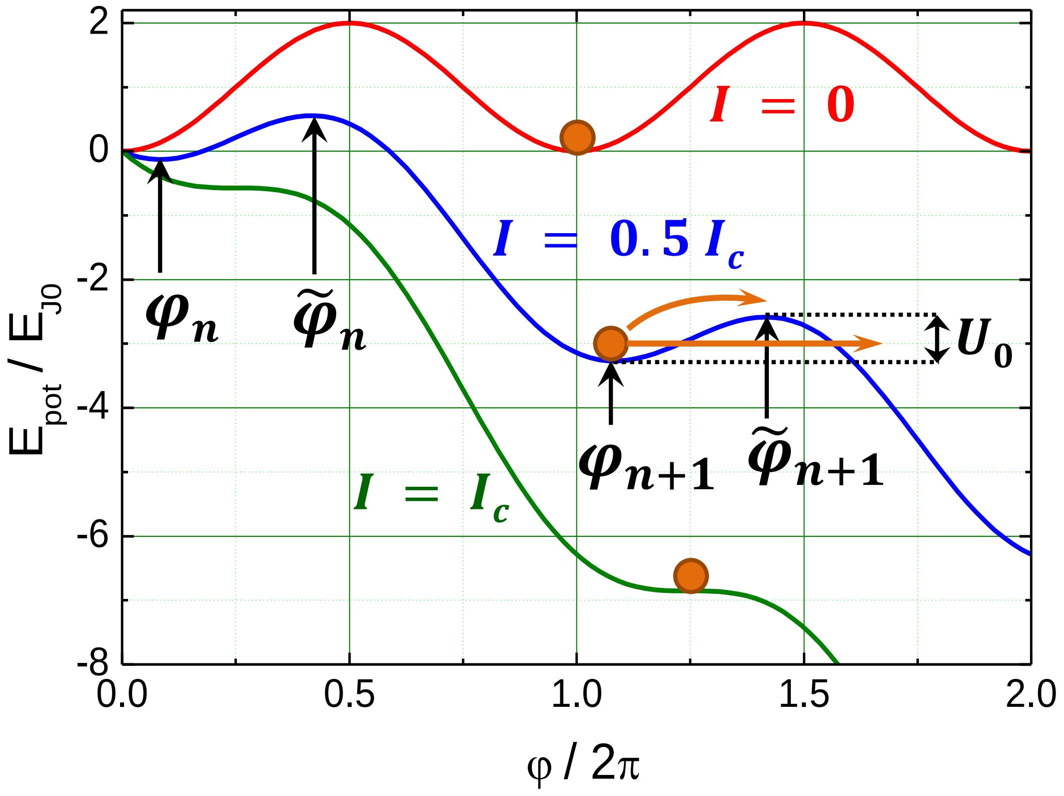
5.2.1 Josephson Junction with Applied Current

$|I| \leq I_c$: ***constant phase difference***: $\varphi = \varphi_n = \arcsin(I/I_c) + 2\pi n$

→ zero voltage state / ordinary (S) state

$|I| > I_c$: ***phase difference increases with time***: $\varphi = \varphi(t)$

→ finite voltage state / running phase state



- thermally activated phase slippage
- quantum tunneling of phase

5.3 Applications of the Josephson

large number of applications in analog and digital electronics

➔ detailed discussion in lecture „***Applied Superconductivity***“

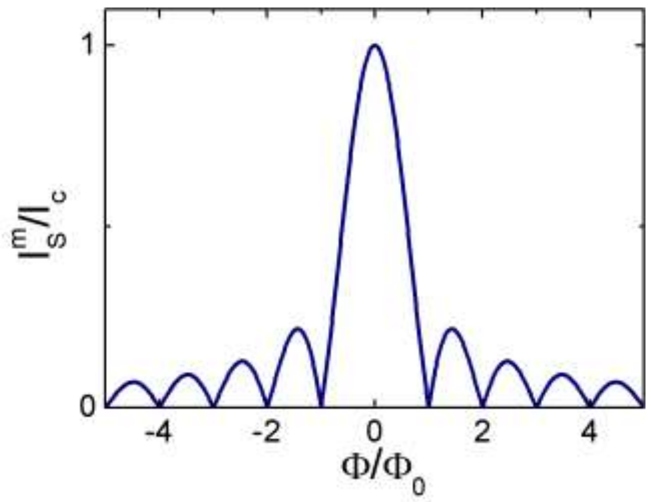
- $I_s^m = I_s^m(B)$:
  → ***magnetic field sensors (SQUIDs)***

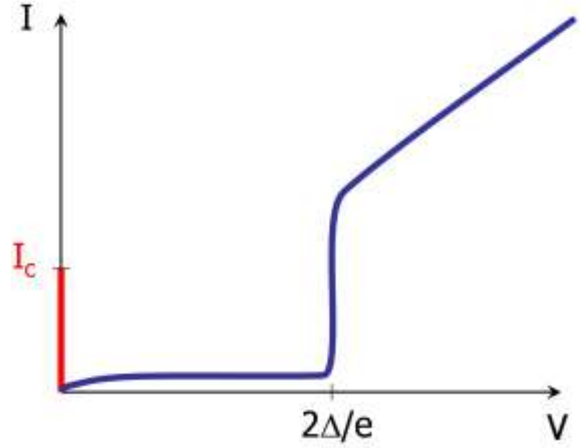
- $\beta_C \gg 1$ (**hysteretic** IVC)
  → **bistability**: zero/voltage state
  → ***switching devices, Josephson computer, fast DACs***

- 2nd Josephson equation
  → ***voltage controlled oscillator, voltage standard***

- **nonlinear** IVC
  → ***mixers up to THz, oscillators***

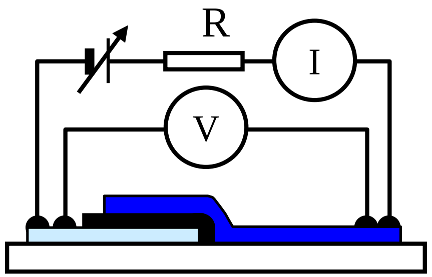
- **macroscopic quantum** behavior
  → ***superconducting qubits***

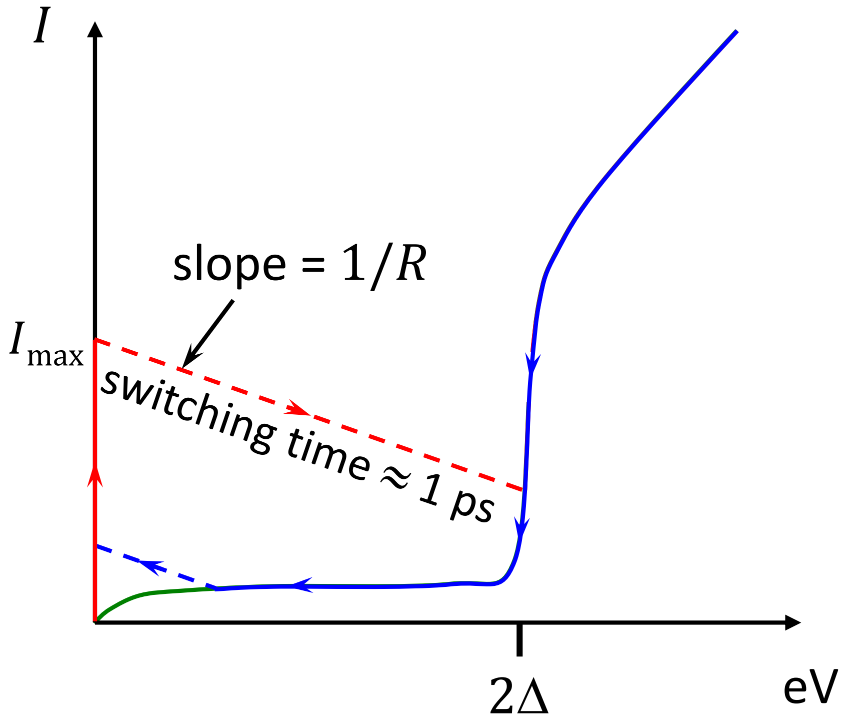




Josephson junction as fast switching device

- $V = 0$: Josephson current
- $V \neq 0$: quasiparticle current





- hysteresis:

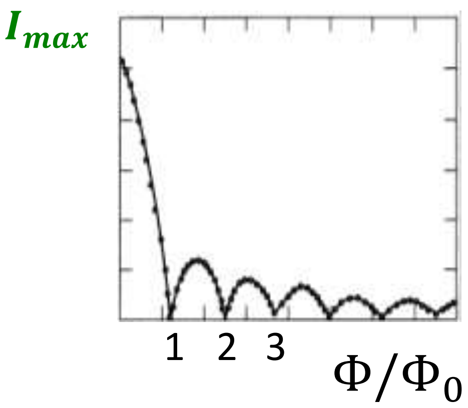
  fast switching device

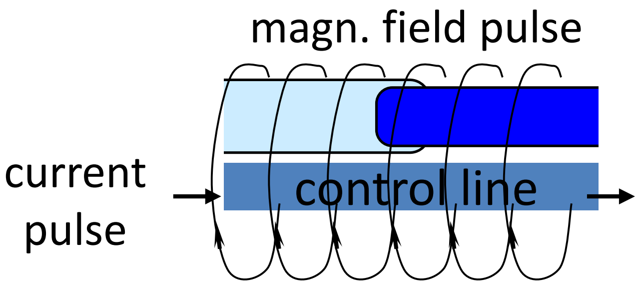
  very low power consumption

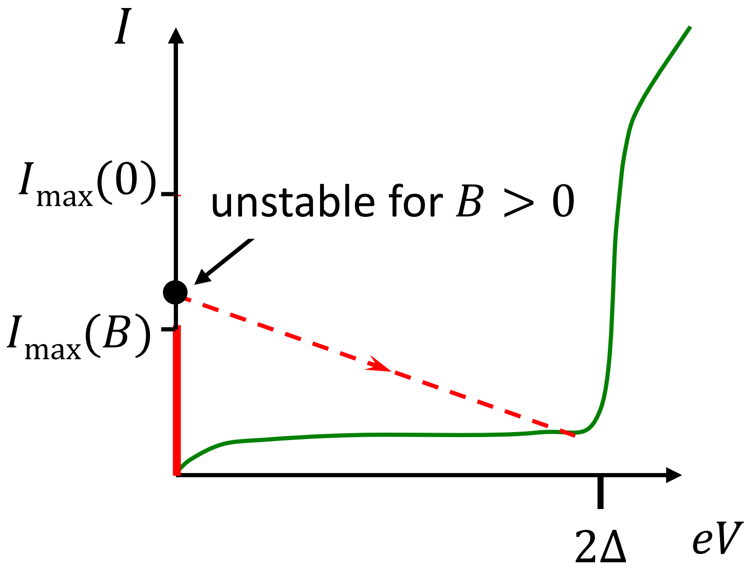
  $\Rightarrow$ **Josephson digital electronics**

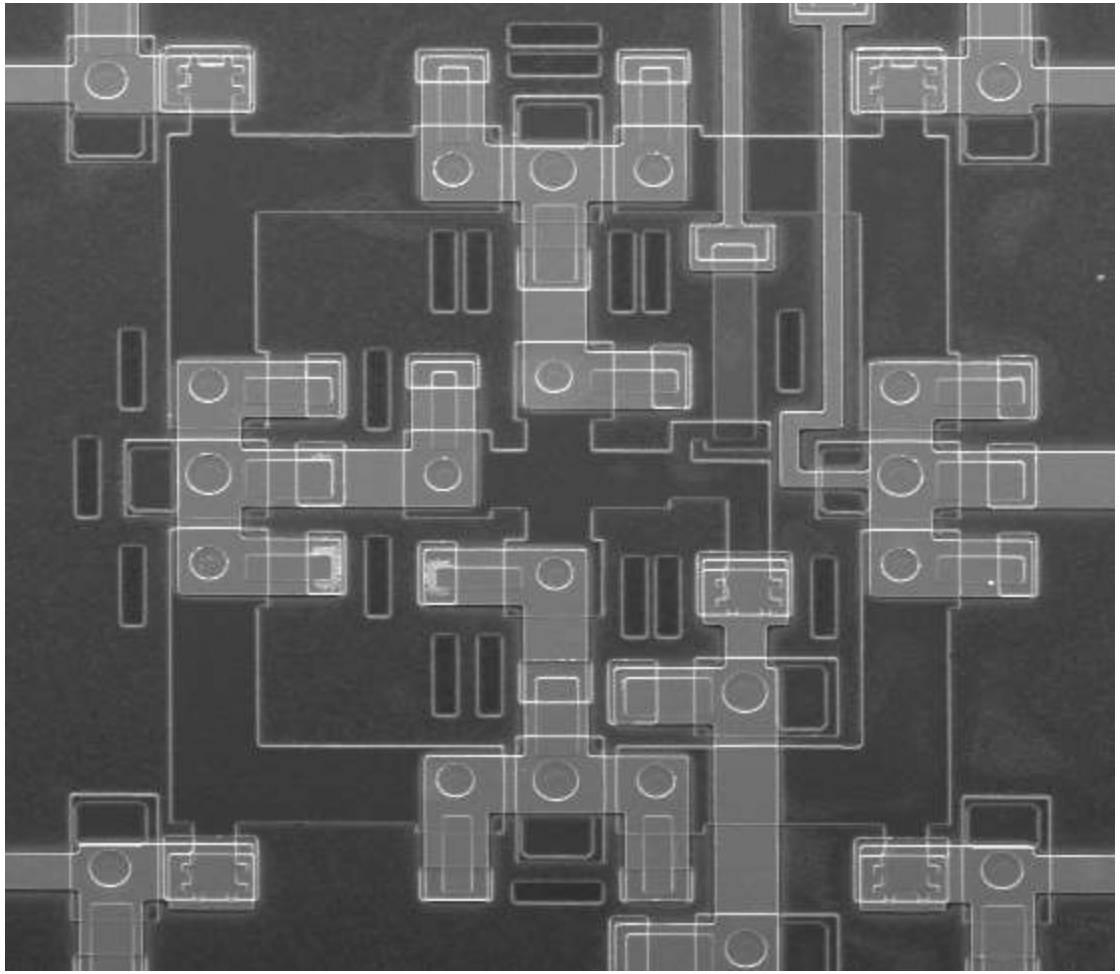
principle of switching element:

- magnetic field dependence of the maximum Josephson current





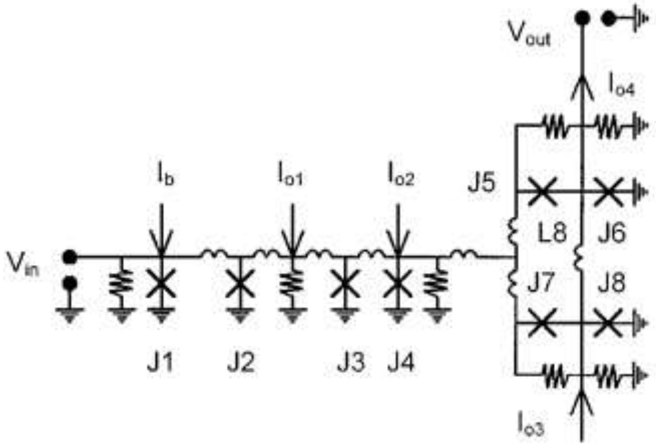


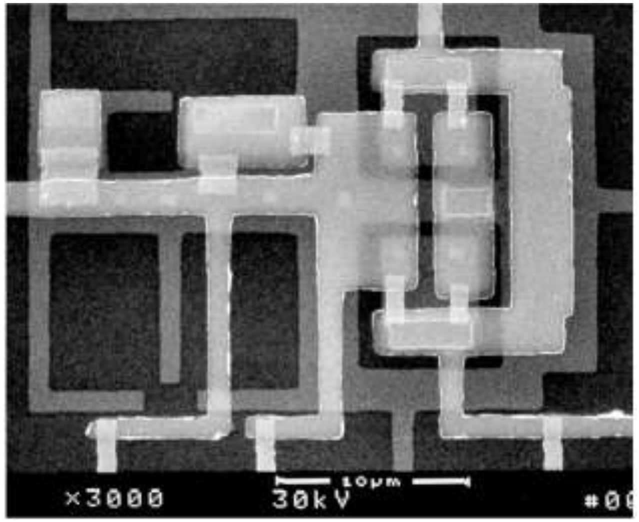


SEM micrograph of a universal asynchronous DR RSFQ (rapid single flux quantum) logic gate

B. Dimov et al.,
Universal asynchronous RSFQ gate for realization of Boolean functions of dual-rail binary variables
Journal of Physics Conference Series 43(1), 1183 (2006)

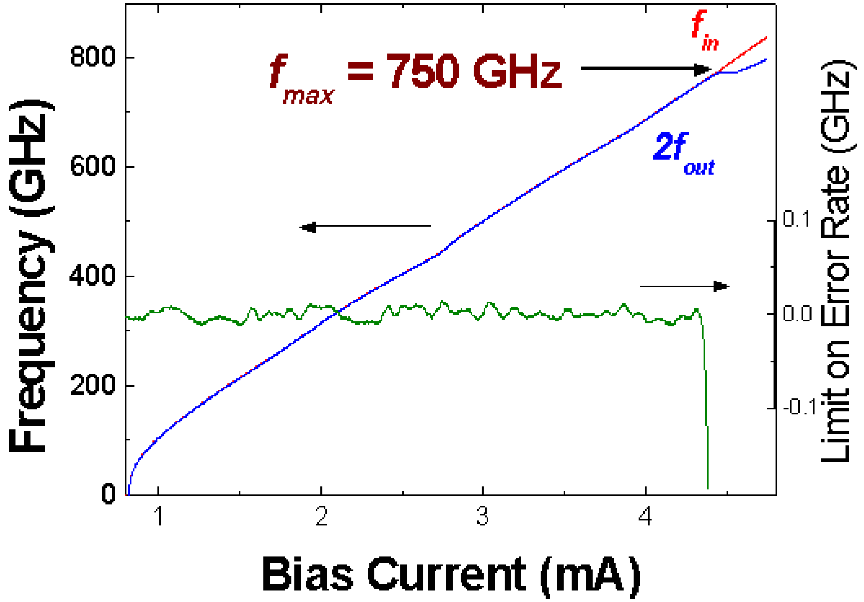
superconductor digital frequency divider operating up to 750 GHz





Dividers

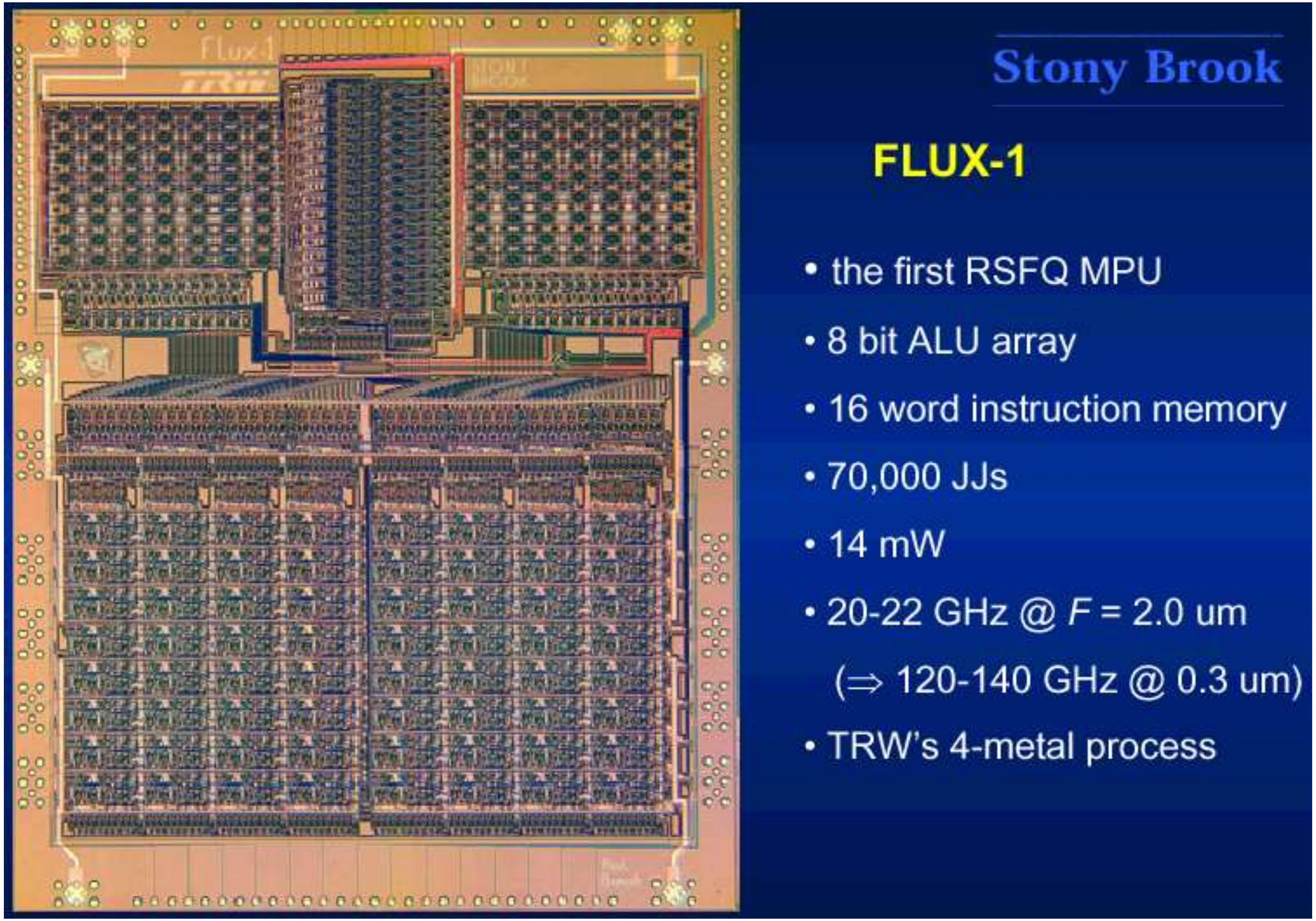
	RSFQ	Semi-conductor
Frequency	750 GHz	60 GHz
Power Dissipation	1.5 μW	0.5 W



W. Chen, A. Rylyakov, V. Patel, J. Lukens, K. Likharev,
"Superconductor digital frequency divider operating up to 750 GHz,"
Appl. Phys. Lett. **73**, 2817 (1998)

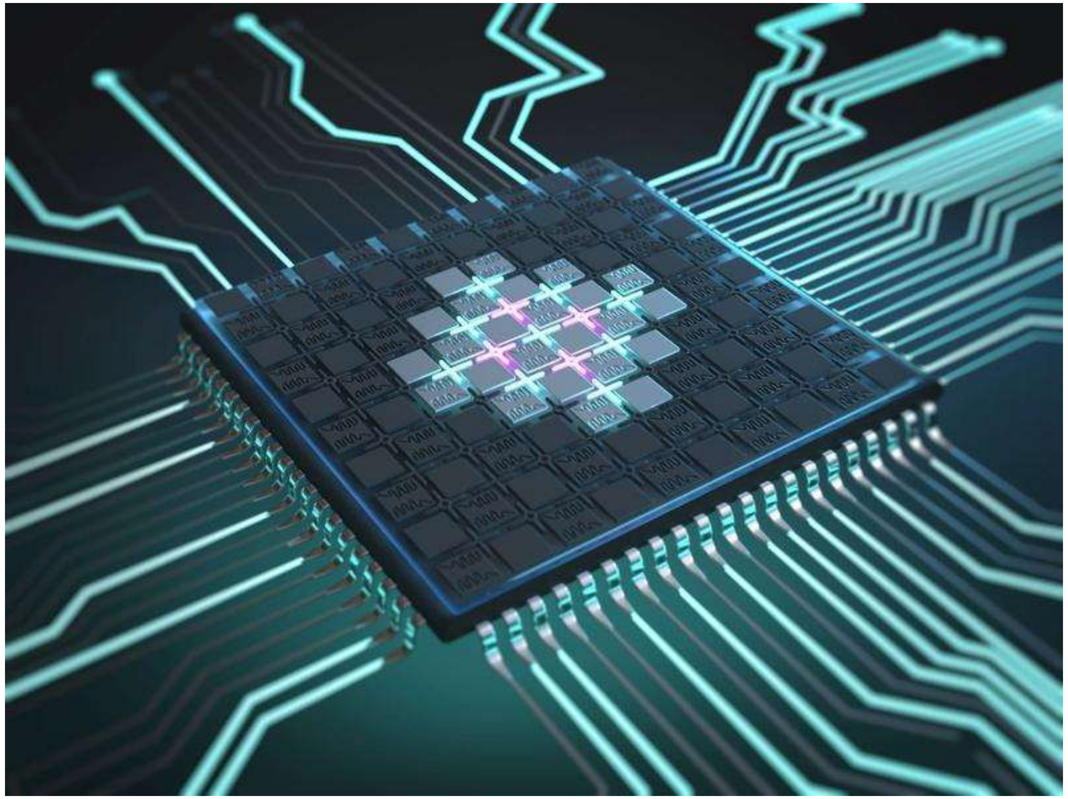
- problem: integration of large number of JJs (> 10^5) with high yield and small parameter spead

5.3 Applications of the Josephson Effect



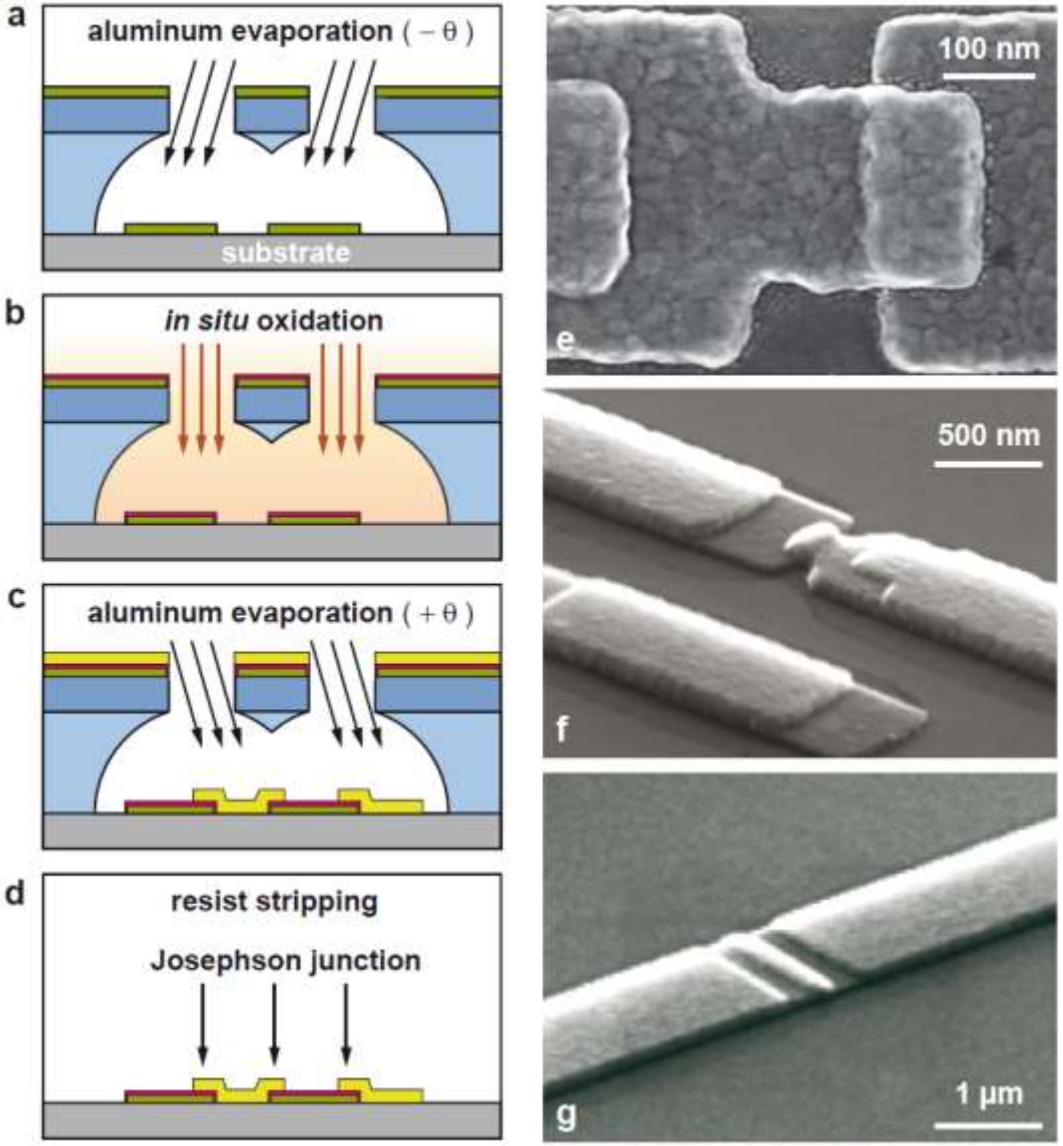
www.physics.sunysb.edu/Physics/RSFQ/

superconducting quantum bits

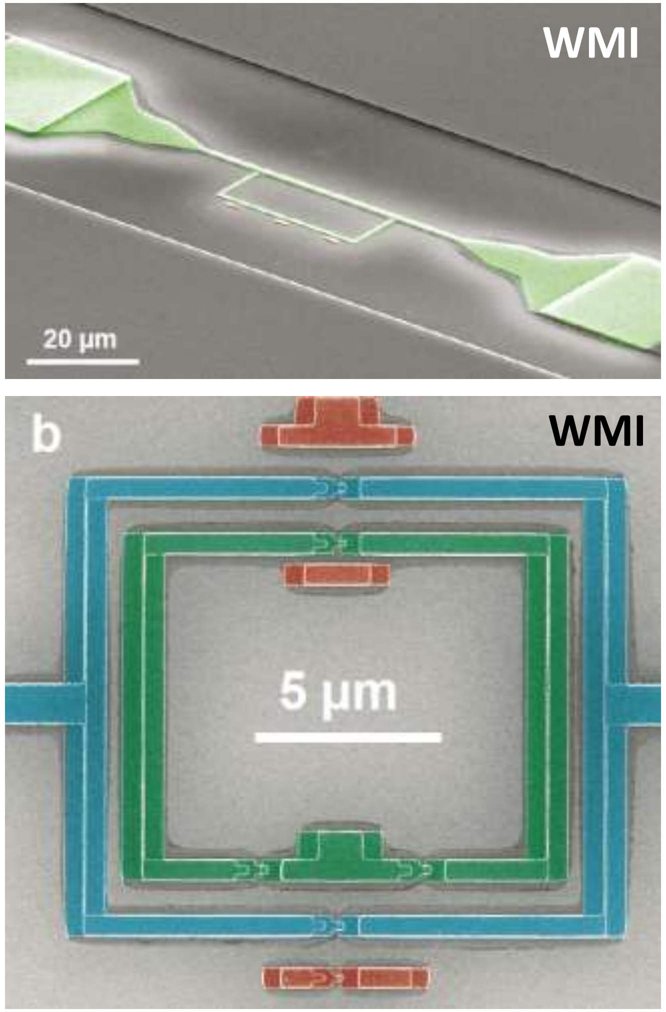


WMI/MCQST

Chao Song et al.,
Science 365, 574–577 (2019)



superconducting flux quantum bits fabricated at WMI

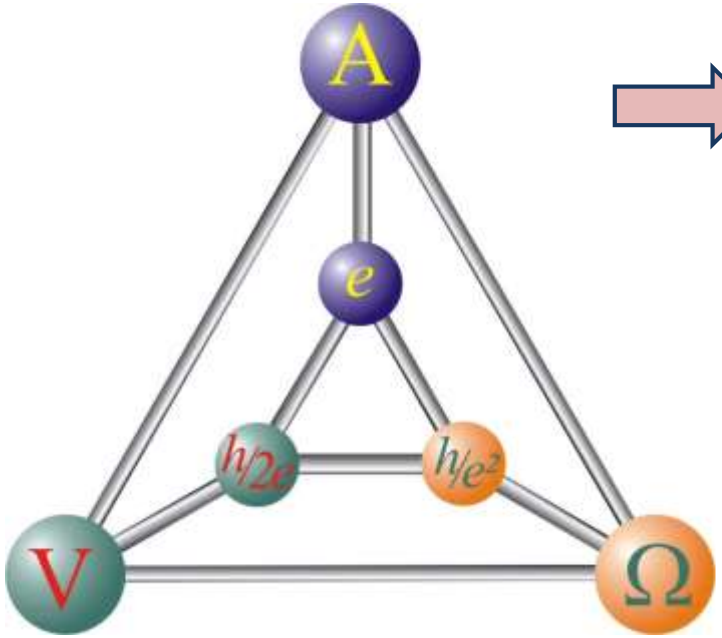


F. Deppe et al., PRB **76**, 214503 (2007)
T. Niemczyk et al., SUST **22**, 034009 (2009)

5.3 Applications of the Josephson Effect

precise definition of the electrical voltage, current and the resistance by using fundamental quantum effects

- ***Josephson effect***: $V = \frac{h}{2e} \cdot f = \Phi_0 \cdot f$ (relation between voltage and time/frequency by flux quantum)
- ***Single electron pump***: $I = e \cdot f$ (relation between current and time by charge quantum)
- ***Quantum Hall effect***: $V = \frac{h}{e^2} \cdot I = R_\mathrm{K} \cdot I$ (relation between voltage and current by quantum resistance, unit = 1 Klitzing)



allows the reproduction of the physical units Volt, Ampère and Ohm with a very high precision and largely uninfluenced by environmental parameters at any place in the world

realization of the Ampère by single electron pump not realized so far at sufficient precision
➔ would allow an important experimental test of the consistency of the relations between the fundamental constants illustrated in the "***electrical triangle***"

Summary of Lecture No. 10 (1)

Macroscopic wave function ψ:

describes ensemble of a macroscopic number of superconducting electrons,
$|\psi|^2 = n_s$ is given by density of superconducting electrons

Current density in a superconductor:

$$\mathbf{J}_s(\mathbf{r},t) = \frac{q_s n_s(\mathbf{r},t)\hbar}{m_s}\left\{\nabla\theta(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t)\right\} = \frac{q_s n_s(\mathbf{r},t)\hbar}{m_s}\left\{\nabla\theta(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\mathbf{A}(\mathbf{r},t)\right\}$$

Gauge invariant phase gradient:

$$\gamma(\mathbf{r},t) = \nabla\theta(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t) = \nabla\theta(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\mathbf{A}(\mathbf{r},t)$$

Phenomenological London equations:

$$(1)\quad \frac{\partial}{\partial t}\left(\Lambda\mathbf{J}_s(\mathbf{r},t)\right) = \mathbf{E} \qquad (2)\quad \nabla\times(\Lambda\mathbf{J}_s) + \mathbf{B} = \mathbf{0} \qquad \Lambda = \frac{m_s}{q_s^2 n_s} = \mu_0\lambda_{\mathrm{L}}^2$$

Fluxoid quantization:

$$\oint_C \Lambda\mathbf{J}_s\cdot d\boldsymbol{\ell} + \int_S \mathbf{B}\cdot\hat{\mathbf{n}}\; dS = n\cdot\frac{h}{q_s} = n\cdot\Phi_0$$

Summary of Lecture No. 10 (2)

Josephson equations: $\mathbf{J}_s(\mathbf{r},t) = \mathbf{J}_c(\mathbf{r},t)\sin\varphi(\mathbf{r},t)$

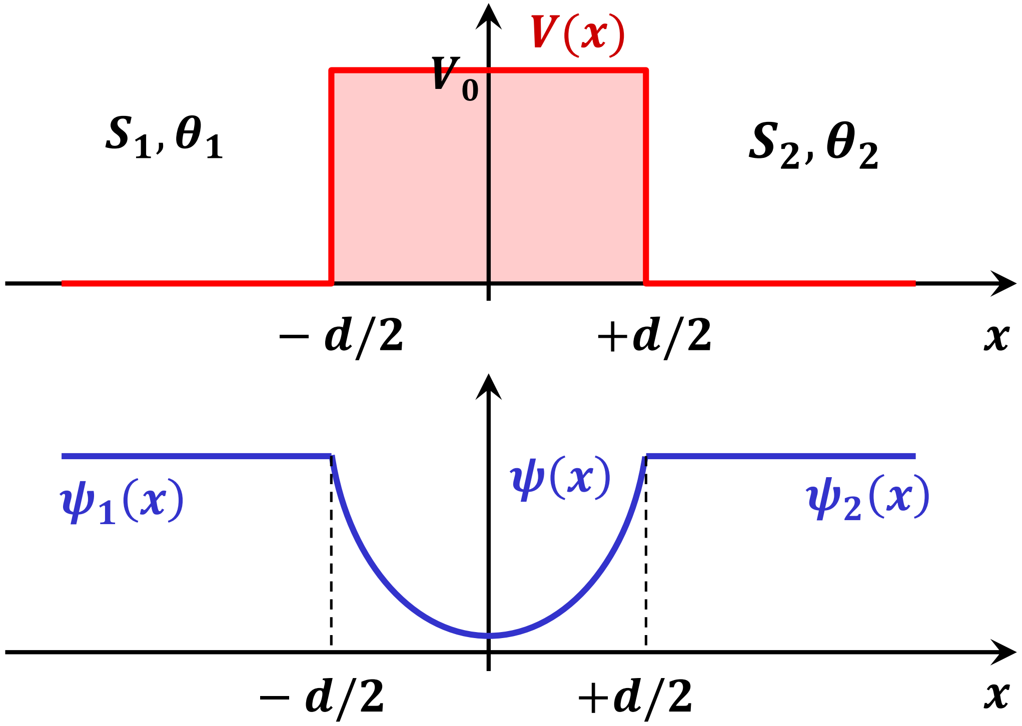
$$\frac{\partial\varphi(t)}{\partial t} = \frac{2\pi}{\Phi_0}V(t) = \frac{q_s V(t)}{\hbar} \qquad \frac{\omega/2\pi}{V} = \frac{1}{\Phi_0} = 483.597\,9\ \frac{\mathrm{MHz}}{\mu\mathrm{V}}$$

Josephson coupling energy: $\frac{E_J}{A} = \frac{\Phi_0 J_c}{2\pi}(1-\cos\varphi) = E_{J0}\,(1-\cos\varphi)$

maximum Josephson current density $\mathbf{J}_c$:

can be calculated by e.g. wave matching method

$$\mathbf{J}_c = \frac{2e\hbar\kappa}{m_s}\,2\sqrt{n_{s,1}n_{s,2}}\ \exp(-2\kappa d) \qquad q_s = -2e$$



Ambegaokar-Baratoff relation:

$$I_c R_n = \frac{\pi}{2e}\Delta(T)\tanh\left(\frac{\Delta(T)}{2k_B T}\right)$$

Summary of Lecture No. 10 (3)

Josephson junction with applied current:

$$E_{\mathrm{pot}}(\varphi) = E_{J0}\left(1 - \cos\varphi - \frac{I}{I_c}\varphi\right)$$ tilted washboard potential

many application in digital and analog electronics

- *magnetic field sensors (SQUIDs)*
- *switching devices, RSFQ logic, fast DACs*
- *voltage controlled oscillator, voltage standard*
- *mixers up to THz frequencies*
- *superconducting qubits*