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Superconductivity and Low Temperature Physics I

Lecture Notes
Winter Semester 2022/2023

R. Gross

Chapter 5

Josephson Effects

5. Josephson Effects

5.1 Josephson Equations

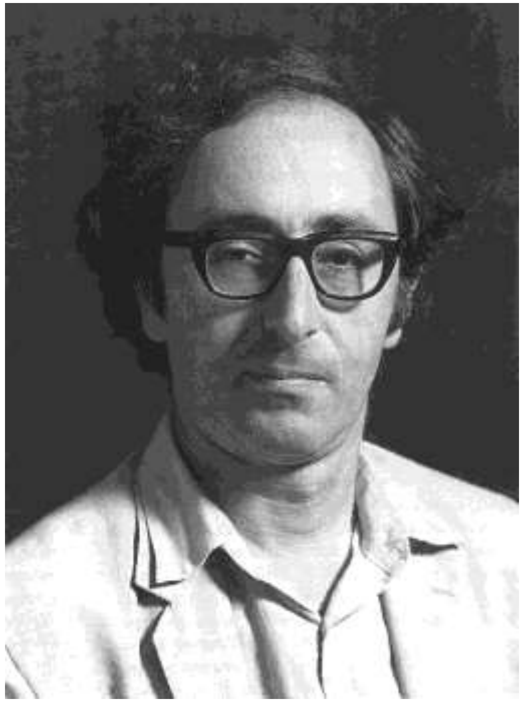
5.1.1 SIS Josephson Junction

5.1.2 Ambegaokar-Baratoff relation

5.2 Josephson Coupling Energy

5.2.1 Josephson Junction with applied current

5.3 Applications of the Josephson Effect



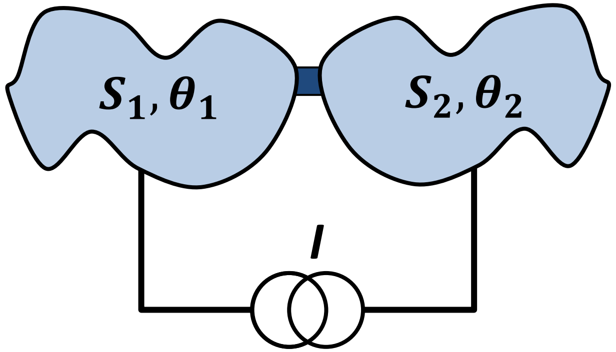
Brian David Josephson
born 04.01.1940

What happens if we weakly couple two superconductors ?

Possible new effects in superconductive tunnelling, Physics Letters **1**(7), 251-253 (1962)

5.1 Josephson Effects (cf. 3.2.3)

- ***what happens if we weakly couple two superconductors?***
  - coupling by *tunneling barriers, point contacts, normal conducting layers, etc.*
  - do they form a bound state such as a molecule?
  - if yes, what is the binding energy?



- **B.D. Josephson** in 1962
  (Nobel Prize in physics with Esaki and Giaever in 1973)

→ Cooper pairs can tunnel through thin insulating barrier (T = transmission amplitude for single charge carriers)

expectation: tunneling probability for pairs $\propto (|T|^2)^2$ ➔ extremely small $\sim (10^{-4})^2$

Josephson: tunneling probability for pairs $\propto |T|^2$
coherent tunneling of pairs („***tunneling of macroscopic wave function***")

predictions:

- ➢ ***finite supercurrent at zero applied voltage***
- ➢ ***oscillation of supercurrent at constant applied voltage***

} ***Josephson effects***

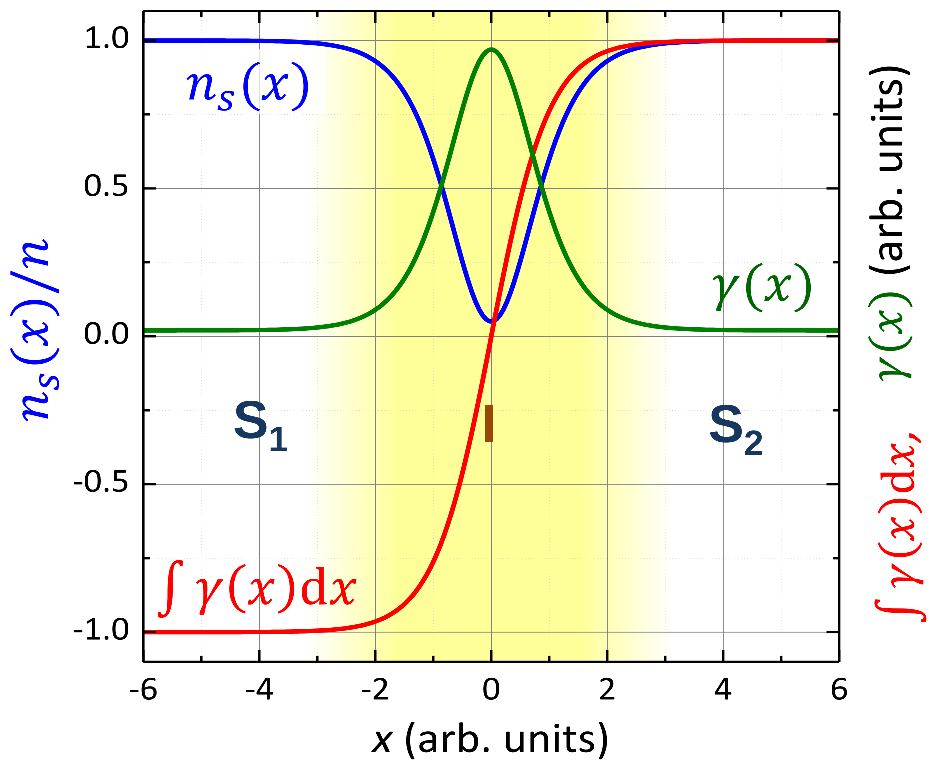
- ➢ ***finite binding energy of coupled SCs = Josephson coupling energy***

5.1 Josephson Effects (cf. 3.2.3)

- **coupling is weak** → supercurrent density between S_1 and S_2 is small → $|\psi|^2 = n_s$ is not changed in S_1 and S_2
- supercurrent density depends on gauge invariant phase gradient:

$$\mathbf{J}_s(\mathbf{r},t) = \frac{q_s n_s(\mathbf{r},t)\hbar}{m_s}\left\{\nabla\theta(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t)\right\} = \frac{q_s n_s(\mathbf{r},t)\hbar}{m_s}\;\gamma(\mathbf{r},t)$$

- **simplifying assumptions:**
  - current density is spatially homogeneous
  - $\gamma(\mathbf{r},t)$ varies negligibly in S_1 and S_2
  - $\mathbf{J}_s$ is equal in electrodes and junction area
    → γ in S_1 and S_2 much smaller than in insulator I

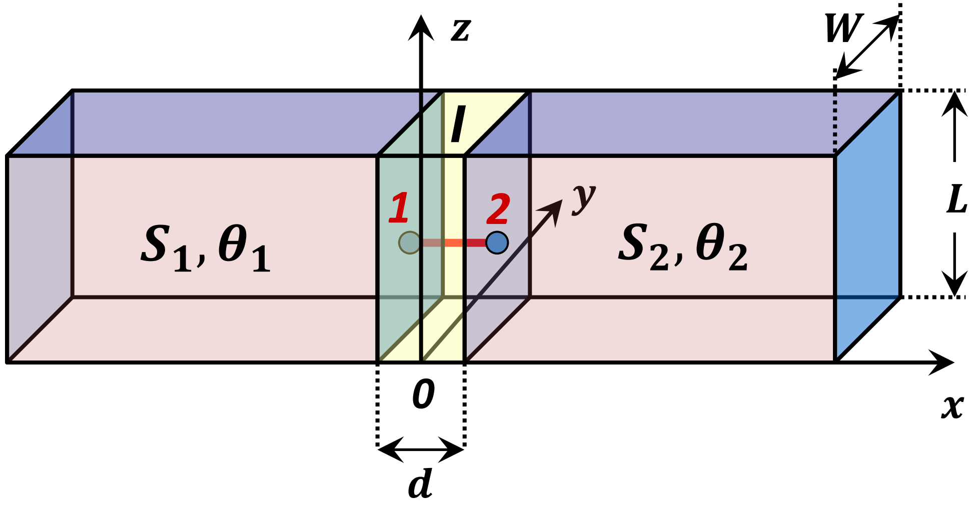


- **approximation:**
  - replace gauge invariant phase gradient γ by ***gauge invariant phase difference φ***:

$$\varphi(\mathbf{r},t) = \int_1^2 \gamma(\mathbf{r},t)\cdot\,\mathrm{d}\ell = \int_1^2\left(\nabla\theta(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t)\right)\cdot\mathrm{d}\ell = \theta_2(\mathbf{r},t) - \theta_1(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot\mathrm{d}\ell$$

5.1 Josephson Effects (cf. 3.2.3)

first Josephson equation:



- we *expect:* $J_s = J_s(\varphi)$

  $$J_s(\varphi) = J_s(\varphi + n \cdot 2\pi)$$

- for $J_s = 0$: phase difference must be zero:

  $$J_s(0) = J_s(n \cdot 2\pi) = 0$$

$$\Rightarrow \quad J_s(\varphi) = J_c \sin\varphi + \sum_{m=2}^{\infty} J_{c,m} \sin(m\varphi)$$

J_c = crititical or maximum Josephson current density

general formulation of **1st Josephson equation**: ***current-phase relation***

- in most cases: we have to keep only 1st term (especially for weak coupling):

  $$J_s(\varphi) = J_c \sin\varphi$$

  1st Josephson equation

- *generalization* to ***spatially inhomogeneous*** supercurrent density:

  $$J_s(y,z) = J_c(y,z) \ \sin\varphi\,(y,z)$$

derived by Josephson for SIS junctions

supercurrent density J_s varies sinusoidally with phase difference $\varphi = \theta_2 - \theta_1$ w/o external potentials

5.1 Josephson Effects (cf. 3.2.3)

second Josephson equation (for spatially homogeneous junction)

- take time derivative of the gauge invariant phase difference $\varphi(t) = \theta_2(t) - \theta_1(t) - \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A}(t)\cdot \mathrm{d}\boldsymbol{\ell}$

$$\frac{\partial\varphi(t)}{\partial t} = \frac{\partial\theta_2(t)}{\partial t} - \frac{\partial\theta_1(t)}{\partial t} - \frac{2\pi}{\Phi_0}\frac{\partial}{\partial t}\int_1^2 \mathbf{A}(t)\cdot \mathrm{d}\boldsymbol{\ell}$$

- substitution of the energy-phase relation $\hbar\frac{\partial\theta(t)}{\partial t} = -\left\{\frac{1}{2n_s}\Lambda\mathbf{J}_s^2(t) + q_s\phi_{\mathrm{el}}(\mathbf{r},t)\right\}$ gives:

$$\frac{\partial\varphi(t)}{\partial t} = -\frac{1}{\hbar}\left(\frac{\Lambda}{2n_s}\left[\mathbf{J}_s^2(2) - \mathbf{J}_s^2(1)\right] + q_s\left[\phi_{\mathrm{el}}(2) - \phi_{\mathrm{el}}(1)\right]\right) - \frac{2\pi}{\Phi_0}\frac{\partial}{\partial t}\int_1^2 \mathbf{A}(t)\cdot \mathrm{d}\boldsymbol{\ell}$$

- supercurrent density across the junction is *continuous* $(\mathbf{J}_s(1) = \mathbf{J}_s(2))$:

$$\frac{\partial\varphi(t)}{\partial t} = \frac{2\pi}{\Phi_0}\int_1^2 \left(-\boldsymbol{\nabla}\phi_{\mathrm{el}} - \frac{\partial\mathbf{A}(t)}{\partial t}\right)\cdot \mathrm{d}\boldsymbol{\ell}$$

(term in parentheses = electric field)

⇒

$$\frac{\partial\varphi(t)}{\partial t} = \frac{2\pi}{\Phi_0}\underbrace{\int_1^2 \mathbf{E}(t)\cdot \mathrm{d}\boldsymbol{\ell}}_{\text{voltage drop } V} = \frac{2\pi}{\Phi_0}V(t) = \frac{q_s V(t)}{\hbar}$$

2nd Josephson equation: *voltage – phase relation*

voltage drop V

5.1 Josephson Effects (cf. 3.2.3)

- for a constant voltage V across the junction:

$$\frac{\partial\varphi(t)}{\partial t} = \frac{2\pi}{\Phi_0}V = \frac{q_s V}{\hbar}$$

integration yields: $$\varphi(t) = \varphi_0 + \frac{2\pi}{\Phi_0}V\cdot t = \varphi_0 + \frac{q_s}{\hbar}V\cdot t$$

phase difference increases linearly in time

- supercurrent density J_s oscillates at the Josephson frequency $\nu = V/\Phi_0$:

$$J_s(\varphi(t)) = J_c \sin\varphi(t) = J_c \sin\left(\frac{2\pi}{\Phi_0}V\cdot t\right)$$

$$\frac{\nu}{V} = \frac{\omega/2\pi}{V} = \frac{1}{\Phi_0} = 483.597\,9\ \frac{\mathrm{MHz}}{\mu\mathrm{V}}$$

➔ **Josephson junction = *voltage controlled oscillator***

- **applications:**
  - Josephson voltage standard
  - microwave sources
  - ….

5.1 Josephson Effects (cf. 3.2.3)

Josephson coupling energy E_J: binding energy of two coupled superconductors

$$\frac{E_J}{A} = \int_0^{t_0} J_s\, V\, \mathrm{d}t = \int_0^{t_0} J_c \sin\varphi \left(\frac{\Phi_0}{2\pi}\frac{\partial\varphi}{\partial t}\right) \mathrm{d}t = \frac{\Phi_0 J_c}{2\pi}\int_0^{\varphi} \sin\varphi'\; \mathrm{d}\varphi'$$

with $\varphi(0) = 0$ and $\varphi(t_0) = \varphi$
$A =$ junction area

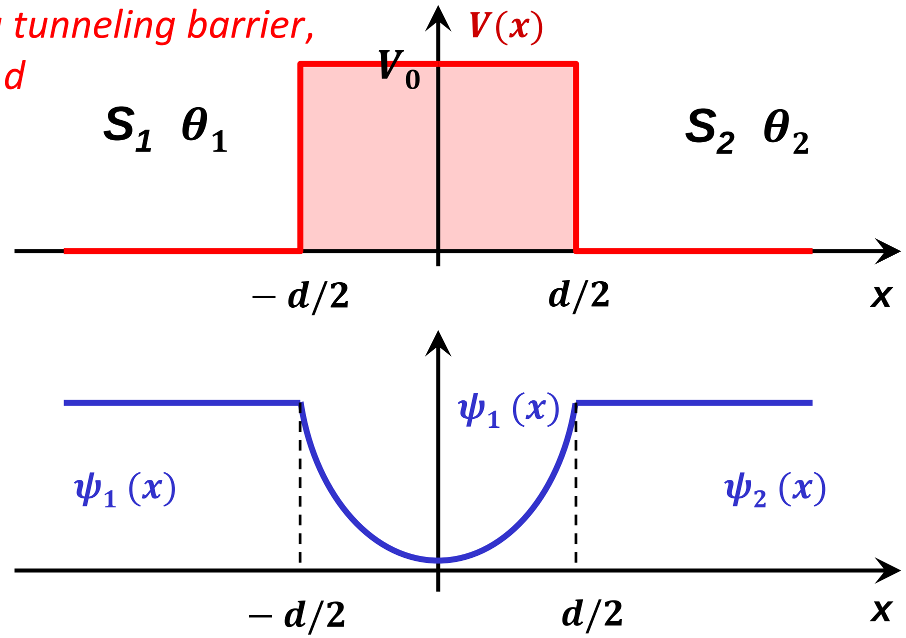
integration yields:

$$\frac{E_J}{A} = \frac{\Phi_0 J_c}{2\pi}(1 - \cos\varphi)$$

Josephson coupling energy (per junction area)

5.1.1 Superconducting Tunnel Junctions

Josephson effect in superconducting tunnel junctions



- **derive the Josephson equations**
  - starting point is time-dependent Schrödinger equation

$$\imath\hbar \frac{\partial \psi(\mathbf{r},t)}{\partial t} = E\,\psi(\mathbf{r},t)$$

  - S_1 and S_2 are described by macroscopic wave functions

$$\psi_1(\mathbf{r},t) = \psi_{01}\, \mathrm{e}^{\imath\theta_1(\mathbf{r},t)} \qquad n_{s1} = |\psi_{01}|^2$$

$$\psi_2(\mathbf{r},t) = \psi_{02}\, \mathrm{e}^{\imath\theta_2(\mathbf{r},t)} \qquad n_{s2} = |\psi_{02}|^2$$

  - finite coupling between S_1 and S_2 is introduced by small perturbation T (tunnel coupling)

$$\imath\hbar \frac{\partial \psi_1(\mathbf{r},t)}{\partial t} = E_1\,\psi_1(\mathbf{r},t) + T_{\mathrm{LR}}\psi_2(\mathbf{r},t) = +e\Delta\phi\,\psi_1(\mathbf{r},t) + T_{\mathrm{LR}}\psi_2(\mathbf{r},t)$$

$$\imath\hbar \frac{\partial \psi_2(\mathbf{r},t)}{\partial t} = E_2\,\psi_2(\mathbf{r},t) + T_{\mathrm{RL}}\psi_1(\mathbf{r},t) = -e\Delta\phi\,\psi_2(\mathbf{r},t) + T_{\mathrm{RL}}\psi_1(\mathbf{r},t)$$

$$\Delta\phi = \frac{E_1 - E_2}{|q_s|} = \frac{E_1 - E_2}{2e} = \frac{1}{2}\frac{E_1 - E_2}{e}$$

5.1.1 Superconducting Tunnel Junctions

- by inserting the wave functions ψ_1, ψ_2 into the time-dependent Schrödinger equation we obtain for the **imaginary part**:

$$\frac{\partial \theta_1(t)}{\partial t} = -\frac{T}{\hbar}\sqrt{\frac{n_{s2}}{n_{s1}}}\cos\varphi(t) - \frac{e\Delta\phi}{\hbar}$$

$$\frac{\partial \theta_2(t)}{\partial t} = -\frac{T}{\hbar}\sqrt{\frac{n_{s1}}{n_{s2}}}\cos\varphi(t) + \frac{e\Delta\phi}{\hbar}$$

we use $T_{\mathrm{LR}} = T_{\mathrm{RL}} = T$ and $\varphi = \theta_2 - \theta_1$,

- for the **real part** we obtain:

$$\frac{\partial n_{s1}(t)}{\partial t} = +\frac{2T}{\hbar}\sqrt{n_{s1}n_{s2}}\sin\varphi(t)$$

$$\frac{\partial n_{s2}(t)}{\partial t} = -\frac{2T}{\hbar}\sqrt{n_{s1}n_{s2}}\sin\varphi(t)$$

we see that $\frac{\partial n_{s1}(\mathbf{r},t)}{\partial t} = -\frac{\partial n_{s2}(\mathbf{r},t)}{\partial t} \Rightarrow$ ***conservation of particle number***

- **supercurrent density**:

$$J_s^{1\to 2} = \frac{2e}{A}\frac{\partial n_{s1}(t)}{\partial t}$$

$$J_s^{2\to 1} = \frac{2e}{A}\frac{\partial n_{s2}(t)}{\partial t}$$

$$J_s = J_s^{1\to 2} - J_s^{2\to 1} = \frac{4eT}{\hbar A}\sqrt{n_{s1}n_{s2}}\sin\varphi(t) = J_c \sin\varphi(t)$$

1st Josephson equation
(current-phase relation)

5.1.1 Superconducting Tunnel Junctions

– the 2nd Josephson equation is obtained from the gauge invariant phase difference $\varphi(\mathbf{r},t) = \theta_2(\mathbf{r},t) - \theta_1(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot \mathrm{d}\ell$

$$\frac{\partial\varphi(t)}{\partial t} = \frac{\partial\theta_2(t)}{\partial t} - \frac{\partial\theta_1(t)}{\partial t} - \frac{2e}{\hbar}\frac{\partial}{\partial t}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot \mathrm{d}\ell$$

$$\frac{\partial\theta_1(t)}{\partial t} = -\frac{T}{\hbar}\sqrt{\frac{n_{s2}}{n_{s1}}}\cos\varphi(t) - \frac{e\Delta\phi}{\hbar}$$

$$\frac{\partial\theta_2(t)}{\partial t} = -\frac{T}{\hbar}\sqrt{\frac{n_{s1}}{n_{s2}}}\cos\varphi(t) + \frac{e\Delta\phi}{\hbar}$$

$$\frac{\partial\varphi(t)}{\partial t} = \frac{2e\Delta\phi}{\hbar} - \frac{2e}{\hbar}\frac{\partial}{\partial t}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot \mathrm{d}\ell \qquad \text{for } n_{s1} = n_{s2}$$

$$\frac{\partial\varphi(t)}{\partial t} = \frac{2e}{\hbar}\underbrace{\int_1^2 \left[-\nabla\phi - \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t)\right]\cdot \mathrm{d}\ell}_{=V}$$

$$\Rightarrow \quad \frac{\partial\varphi(t)}{\partial t} = \frac{2e}{\hbar}V = \frac{2\pi}{\Phi_0}V$$

2nd Josephson equation
(voltage-phase relation)

5.1.1 Superconducting Tunnel Junctions

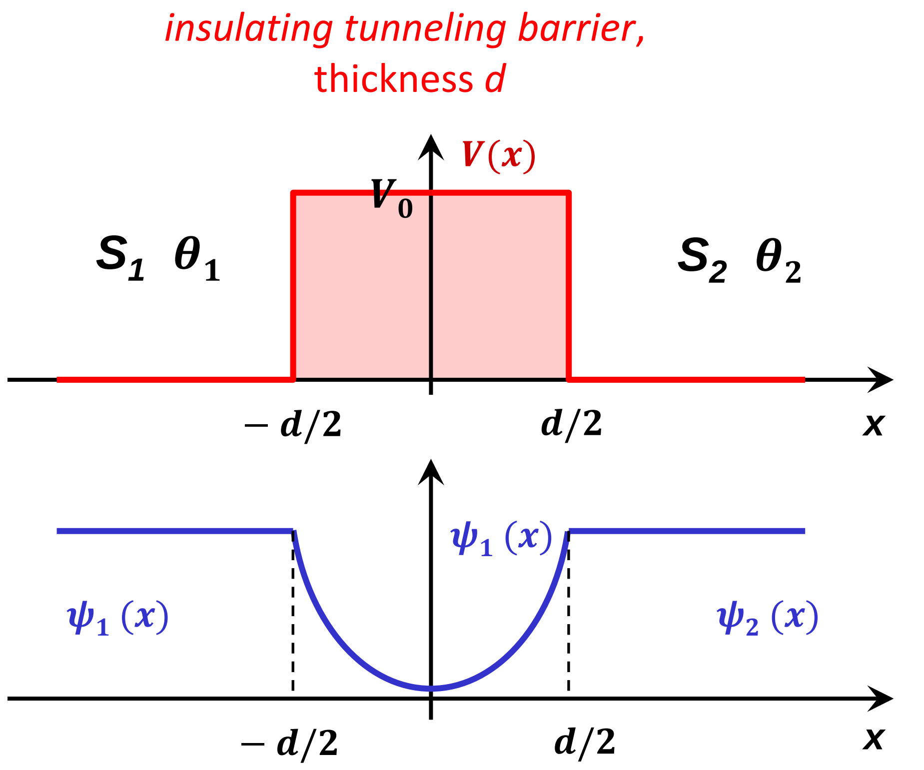
calculation of the maximum Josephson current density: How does T_{LR} depend on height and thickness of barrier?

- calculation by the ***wave matching method***

solve time-independent Schrödinger equation for S_1, S_2 and barrier region and match solution at interfaces

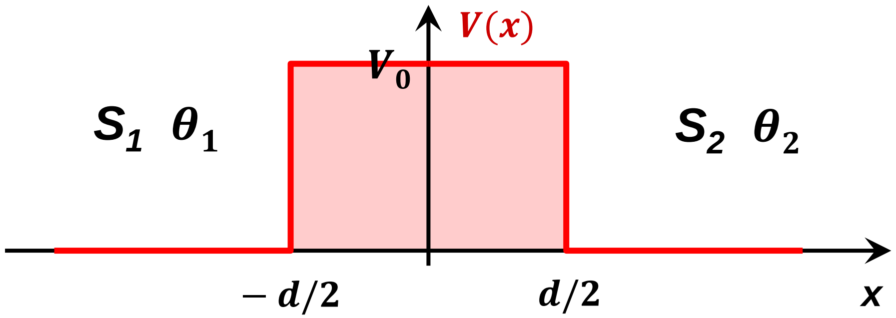
$$-\frac{\hbar^2}{2m_s}\nabla^2\psi(\mathbf{r}) = (E_0 - V_0)\psi(\mathbf{r})$$

with $\psi(\mathbf{r}) = \psi_0(\mathbf{r})\, e^{\imath\theta(\mathbf{r})}$



5.1.1 Superconducting Tunnel Junctions

$$-\frac{\hbar^2}{2m_s}\,\nabla^2\psi(\mathbf{r}) = (E_0 - V_0)\psi(\mathbf{r})$$



- ***assumption***:

  homogeneous barrier and supercurrent flow → **1D problem**

- ***solutions***:

  - in superconductors: $\psi_{1,2}(x) = \psi_{01,02}\mathrm{e}^{\imath\theta_{1,2}(x)} = \sqrt{n_{s1,s2}}\,\mathrm{e}^{\imath\theta_{1,2}(x)}$ (macrosopic wave function)
  - in insulator: sum of decaying and growing exponentials $\psi(x) = A\cosh(\kappa x) + B\sinh(\kappa x)$
  - characteristic decay constant: $\kappa = \sqrt{2m_s(V_0 - E_0)/\hbar^2}$ for $E_0 < V_0$

- coefficients A and B are determined by the boundary conditions at $x = \pm d/2$:

$$\psi(x = -d/2) = \sqrt{n_{s1}}\,\mathrm{e}^{\imath\theta_1} \qquad \psi(x = +d/2) = \sqrt{n_{s2}}\,\mathrm{e}^{\imath\theta_2}$$

$n_{1,2}, \theta_{1,2}$: Cooper pair density and wave function phase at the boundaries $x = \pm d/2$

$$\Rightarrow \quad \sqrt{n_{s1}}\,\mathrm{e}^{\imath\theta_1} = A\cosh(\kappa d/2) - B\sinh(\kappa d/2) \qquad \sqrt{n_{s2}}\,\mathrm{e}^{\imath\theta_2} = A\cosh(\kappa d/2) + B\sinh(\kappa d/2)$$

5.1.1 Superconducting Tunnel Junctions

- solving for A and B: $$A = \frac{\sqrt{n_{s1}}\,\mathrm{e}^{\imath\theta_1} + \sqrt{n_{s2}}\,\mathrm{e}^{\imath\theta_2}}{\cosh(\kappa d/2)} \qquad B = -\frac{\sqrt{n_{s1}}\,\mathrm{e}^{\imath\theta_1} - \sqrt{n_{s2}}\,\mathrm{e}^{\imath\theta_2}}{2\sinh(\kappa d/2)}$$

- supercurrent density: $$\mathbf{J}_s = \frac{q_s\hbar}{2m_s\imath}\left(\psi\boldsymbol{\nabla}\psi^\star - \psi^\star\boldsymbol{\nabla}\psi\right)$$

- substituting the coefficients A and B (after some lengthy calculation):

$$\mathbf{J}_s = \mathbf{J}_c \sin(\theta_2 - \theta_1) = \mathbf{J}_c \sin\varphi$$

current-phase relation

$$\mathbf{J}_c = -\frac{q_s\hbar\kappa}{m_s}\frac{\sqrt{n_{s1}n_{s2}}}{\underbrace{2\sinh(\kappa d/2)\cosh(\kappa d/2)}_{=\sinh(2\kappa d)}} = -\frac{q_s\hbar\kappa}{m_s}\frac{\sqrt{n_{s1}n_{s2}}}{\sinh(2\kappa d)}$$

maximum Josephson current density

- ***real junctions***:
  $V_0 \approx$ few eV $\Rightarrow 1/\kappa < 1\,\mathrm{nm}$, $d \approx$ few nm $\Rightarrow \kappa d \ll 1$, then $\sinh(2\kappa d) \simeq \frac{1}{2}\exp(2\kappa d)$, $q_s = -2e$:

- maximum Josephson current density **decays exponentially** with increasing barrier thickness d:

$$\mathbf{J}_c = \frac{2e\hbar\kappa}{m_s}\,2\sqrt{n_{s1}n_{s2}}\,\exp(-2\kappa d) \qquad q_s = -2e$$

5.1.1 Superconducting Tunnel Junctions

M. H. Cohen, L. M. Falicov, and J. C. Phillips, *Superconductive Tunneling*, Phys. Rev. Lett. **8**, 316-318 (1962).

B. D. Josephson, *Possible new effects in superconductive tunnelling*, Physics Letters **1**(7), 251-253 (1962).

more elaborate theory of tunneling between superconductors

- **open questions**
  - what is the charge crossing the tunneling barrier when a Bogoliubov quasiparticle tunnels from S_1 to S_2
  - what is the role of the coherence factors ?

- **only brief description of theoretical approach**

Hamiltonian: $\mathcal{H} = \mathcal{H}_\mathrm{L} + \mathcal{H}_\mathrm{R} + \mathcal{H}_T$

$\mathcal{H}_T$ = tunneling hamiltonian, small extra term

$$\mathcal{H}_T = \sum_{\mathbf{kq}} T_\mathrm{LR} \underbrace{\left(c_\mathbf{k}^\dagger c_\mathbf{q} + c_\mathbf{q}^\dagger c_\mathbf{k}\right)}$$

transfer electrons from states $\mathbf{k}$ on left to $\mathbf{q}$ on rhs of the barrier and vice versa

matrix elements, fall of exponentially with barrier thickness d

determination of the tunneling current by calculation of $\langle \dot{\mathcal{N}}_\mathrm{L} \rangle = -\langle \dot{\mathcal{N}}_\mathrm{R} \rangle$ by using the equation of motion for $\dot{\mathcal{N}}_\mathrm{L,R}$

$$\imath\hbar\dot{\mathcal{N}}_\mathrm{L,R} = \left[\mathcal{N}_\mathrm{L,R}, \mathcal{H}\right] = \left[\mathcal{N}_\mathrm{L,R}, \mathcal{H}_T\right]$$

as $\mathcal{H}_\mathrm{L}, \mathcal{H}_\mathrm{R}$ commute with $\mathcal{N}_{\mathrm{L},R}$ (conserve particle number)

5.1.1 Superconducting Tunnel Junctions

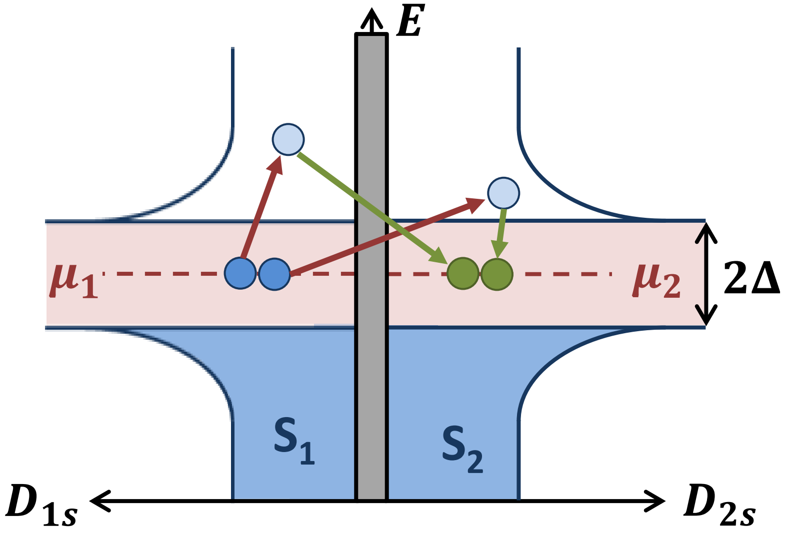
more elaborate theory of tunneling between superconductors

tunneling current: $\mathcal{I} = q\dot{\mathcal{N}} = \frac{q}{\imath\hbar}[\mathcal{N}, \mathcal{H}_T]$

q = charge transported per tunneling particle

- NIN-junction: $\mathcal{I} = \frac{e}{\imath\hbar}\sum_{\mathbf{kq}} T_{\mathbf{kq}}\left(c_{\mathbf{k}}^{\dagger}c_{\mathbf{q}} - c_{\mathbf{q}}^{\dagger}c_{\mathbf{k}}\right)$ for tunneling between two normal metals

- SIS-junction: we have to replace $c_{\mathbf{k},\mathbf{q}}^{\dagger}c_{\mathbf{k},\mathbf{q}}$ by the Bogoliubov quasiparticle excitation and anihilation operators $\alpha_{\mathbf{k},\mathbf{q}}^{\dagger}, \alpha_{\mathbf{k},\mathbf{q}}, \beta_{\mathbf{k},\mathbf{q}}^{\dagger}, \beta_{\mathbf{k},\mathbf{q}}$



- visualization of Josephson tunneling as a 2 step process
  - **step 1**: blue Cooper pair in S_1 is broken up and one excitation crosses barrier into S_2
  - resulting intermediate state (light blue) is classically forbidden but allowed within uncertainty relation
  - **step 2**: second excitation crosses barrier and recombines with the first forming green Cooper pair

- **remark**:
  Josephson tunneling may be viewed as a second order process and $\propto |T|^4$. However, Josephson was assuming a constant (and not arbitrary) phase difference between initial and final states. Then, quantum mechanical treatment yields a supercurrent $\propto |T|^2$

tunneling in SIS junctions at finite voltage – quasiparticle tunneling

evaluation of $\langle\dot{\mathcal{N}}\rangle$ shows that coherence factors are not dropping out (cf. 4.4.2)

$$J_{\mathrm{qp}} \propto -\frac{e|T|^2}{\imath\hbar}\sum_{\mathbf{kq}}\frac{u_{\mathbf{k}}^2 u_{\mathbf{q}}^2 (f_{\mathbf{k}} - f_{\mathbf{q}})}{E_{\mathbf{k}} - E_{\mathbf{q}} + eV}$$

sum has to be taken over electron and hole branches on both sides
➔ **coherence factors** all disappear (see argument given below)
➔ sum → integration: principal part integrals all cancel, only residues at poles are left

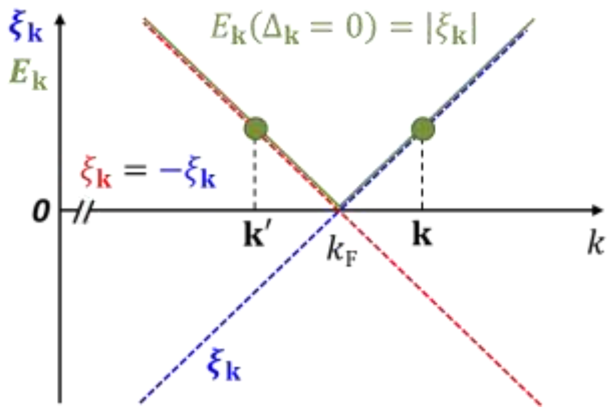
- qualitative argument:

  - tunneling from state $|\mathbf{q}\sigma\rangle$ into a state $|\mathbf{k}\sigma\rangle$ is only possible if pair state $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ is empty

    → resulting tunneling probability is $\propto |u_{\mathbf{k}}|^2\,|T_{\mathbf{kq}}|^2$

  - for each state $|\mathbf{k}\sigma\rangle$ there exists a state $|\mathbf{k}'\sigma\rangle$ with $E_{\mathbf{k}} = E_{\mathbf{k}'}$ but with $\xi_{\mathbf{k}'} = -\xi_{\mathbf{k}}$

    → resulting tunneling probability is $\propto |u_{\mathbf{k}'}|^2\,|T_{\mathbf{k}'\mathbf{q}}|^2 \underbrace{=}_{|u(-\xi_{\mathbf{k}})|=|v(\xi_{\mathbf{k}})|} |v_{\mathbf{k}}|^2\,|T_{\mathbf{k}'\mathbf{q}}|^2$



total tunneling probability $\propto \left(|\mathbf{u_k}|^2 + |\mathbf{v_k}|^2\right)|\mathbf{T_{kq}}|^2 = |\mathbf{T_{kq}}|^2$ **does not depend on coherence factors**

→ ***simple „semiconductor model“ for quasiparticle tunneling is applicable***

M. H. Cohen, L. M. Falicov, and J. C. Phillips, *Superconductive Tunneling*, Phys. Rev. Lett. **8**, 316-318 (1962).

tunneling in SIS junctions at zero voltage – Josephson tunneling

evaluation of $\langle \dot{\mathcal{N}} \rangle$ shows that coherence factors play an important role

$$J_s \propto -\frac{e|T|^2}{\imath\hbar} \sum_{\mathbf{kq}} \frac{u_{\mathbf{k}} v_{\mathbf{k}} u_{\mathbf{q}} v_{\mathbf{q}} \left(f_{\mathbf{k}} - f_{\mathbf{q}}\right) \mathrm{e}^{\imath\varphi}}{E_{\mathbf{k}} - E_{\mathbf{q}}}$$

- we have to sum up over electron and hole branches on both sides
- sum → integration: principal part integrals do no longer cancel
- leads to a finite Josephson current density $J_s = J_c \sin\varphi$ at zero voltage with

$$J_\mathrm{c} = \frac{2e|T|^2}{\hbar\, A} D_{s1}(E_\mathrm{F}) D_{s2}(E_\mathrm{F})\, \mathcal{P} \int_{-\infty}^{\infty} \mathrm{d}\xi_1 \int_{-\infty}^{\infty} \mathrm{d}\xi_2 \frac{\Delta_1}{E_1} \frac{\Delta_2}{E_2} \frac{f(E_1) f(E_2)}{E_1 - E_2}$$

$\mathcal{P}$ = principal part

for $\Delta_1 = \Delta_2$:

$$J_c = \frac{\pi}{2eR_nA} \Delta(T) \tanh\left(\frac{\Delta(T)}{2k_\mathrm{B}T}\right)$$

5.1.2 Ambegaokar-Baratoff Relation

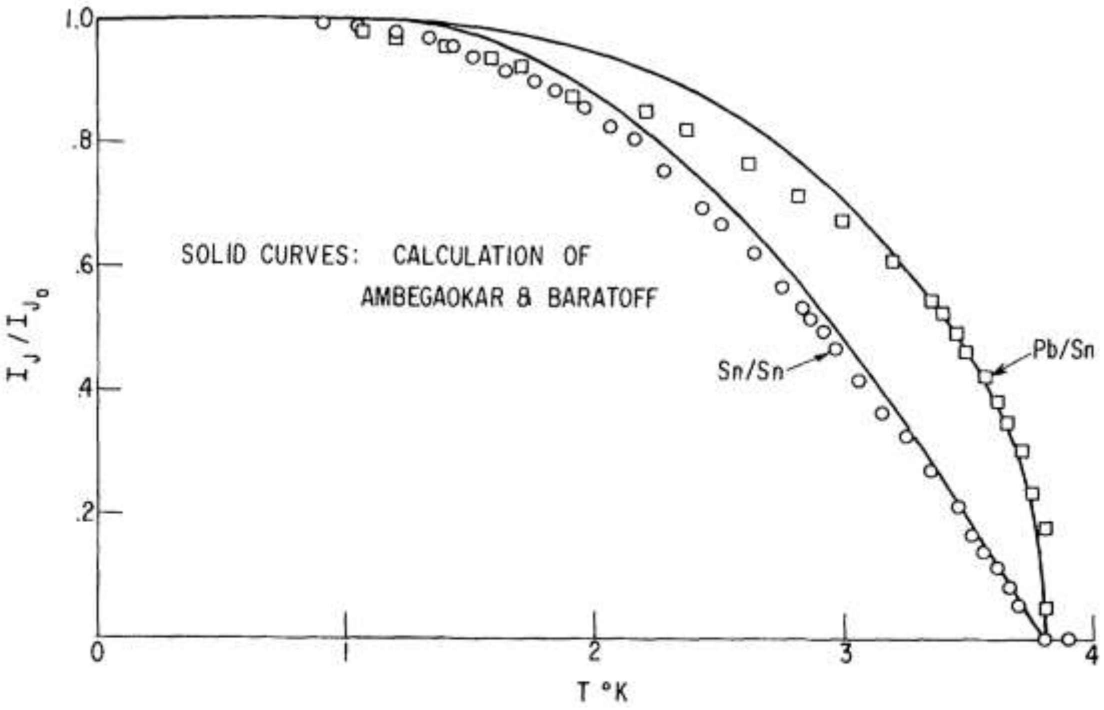
important result of elaborate tunneling theory

- ratio of maximum Josephson current density J_c and $J_{\mathrm{NIN}} = J_{\mathrm{qp}}(eV \gg 2\Delta) \propto 1/R_n A = const$

➔ $\frac{J_c}{J_{\mathrm{NIN}}} = J_c R_n A = I_c R_n = const.$

Ambegaokar-Baratoff relation:

$$I_c R_n = \frac{\pi}{2e}\Delta(T)\tanh\left(\frac{\Delta(T)}{2k_\mathrm{B}T}\right)$$

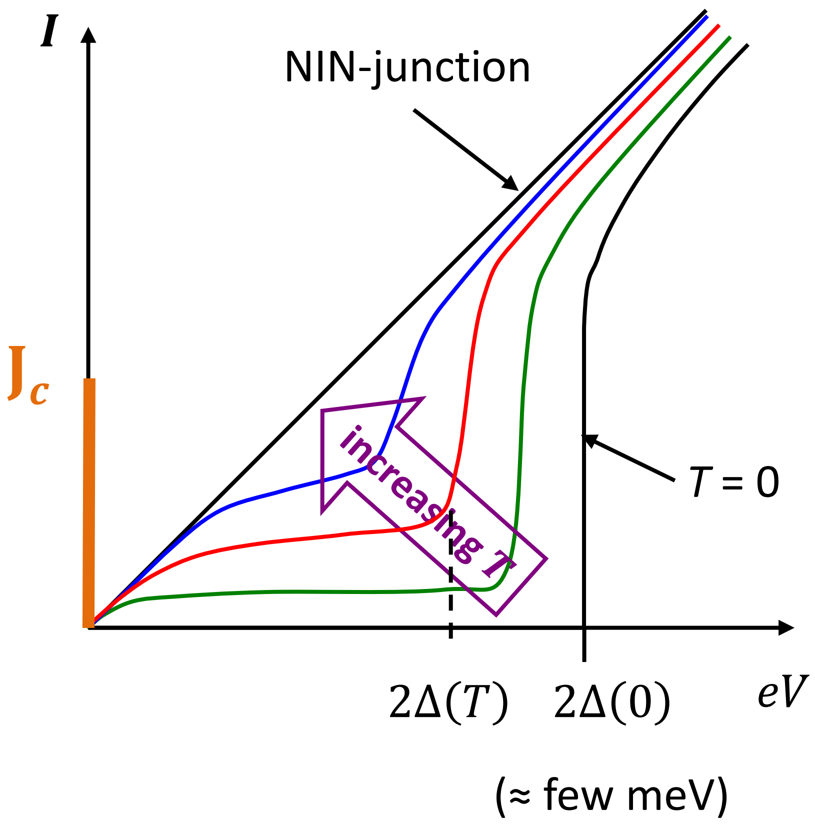


V. Ambegaokar, A. Baratoff, *Tunneling Between Superconductors*, Phys. Rev. Lett. **10**, 486-489 (1963).

M.D. Fiske, Rev. Mod. Phys. **36**, 221–222 (1964)
Temperature and Magnetic Field Dependences of the Josephson Tunneling Current

5.1.2 Ambegaokar-Baratoff Relation

current-voltage characteristics



- **quasiparticle tunneling (cf. 4.4.2)**:

at $eV > 0$

$$\overline{\mathbf{J}_s(t)} = \mathbf{J}_c\ \overline{\sin(\varphi(t))} = \mathbf{J}_c\ \overline{\sin\left(\frac{2eV}{\hbar}t\right)} = 0$$

at $eV \gg 2\Delta(T)$

$$\mathbf{J}_{\mathrm{qp}}(V) \simeq \mathbf{J}_{\mathrm{NIN}}(V) \propto \frac{1}{R_n A} \cdot \exp(-2\kappa d)$$

R_n = normal resistance ≙ resistance of NIN tunnel junction

- **Cooper pair tunneling**:

at $eV = 0$

$$\mathbf{J}_{\mathrm{qp}}(V=0) = 0$$

$$\mathbf{J}_{\mathrm{c}}(V=0) = \frac{e\hbar\kappa}{m_s} 2\sqrt{n_{s1}n_{s2}} \cdot \exp(-2\kappa d)$$

5.2 Josephson Coupling Energy

Josephson coupling energy E_J: binding energy of two coupled superconductors (cf. 3.2.3)

- the two weakly coupled superconductors form "***molecule***" analogous to H_2 molecule
  → what is the ***binding energy*** of this molecule ?

- consider a JJ with $\mathbf{J}_s = 0$ and then ***increase junction current from zero to finite value***
  - phase difference has to change → phase change corresponds to finite voltage according to voltage-phase relation
  - external source has to supply energy (to accelerate the superelectrons)
  - stored in kinetic energy of moving superelectrons
  - integral of the supplied power $I \cdot V$ to increase current to $I(\varphi) = I_c \sin\varphi$ (voltage during increase of current)

$$\frac{E_J}{A} = \int_0^{t_0} J_s \, V \, \mathrm{d}t = \int_0^{t_0} J_c \sin\varphi \left(\frac{\Phi_0}{2\pi}\frac{\partial\varphi}{\partial t}\right) \mathrm{d}t = \frac{\Phi_0 J_c}{2\pi}\int_0^{\varphi} \sin\varphi' \, \mathrm{d}\varphi'$$

with $\varphi(0) = 0$ and $\varphi(t_0) = \varphi$
$A =$ junction area

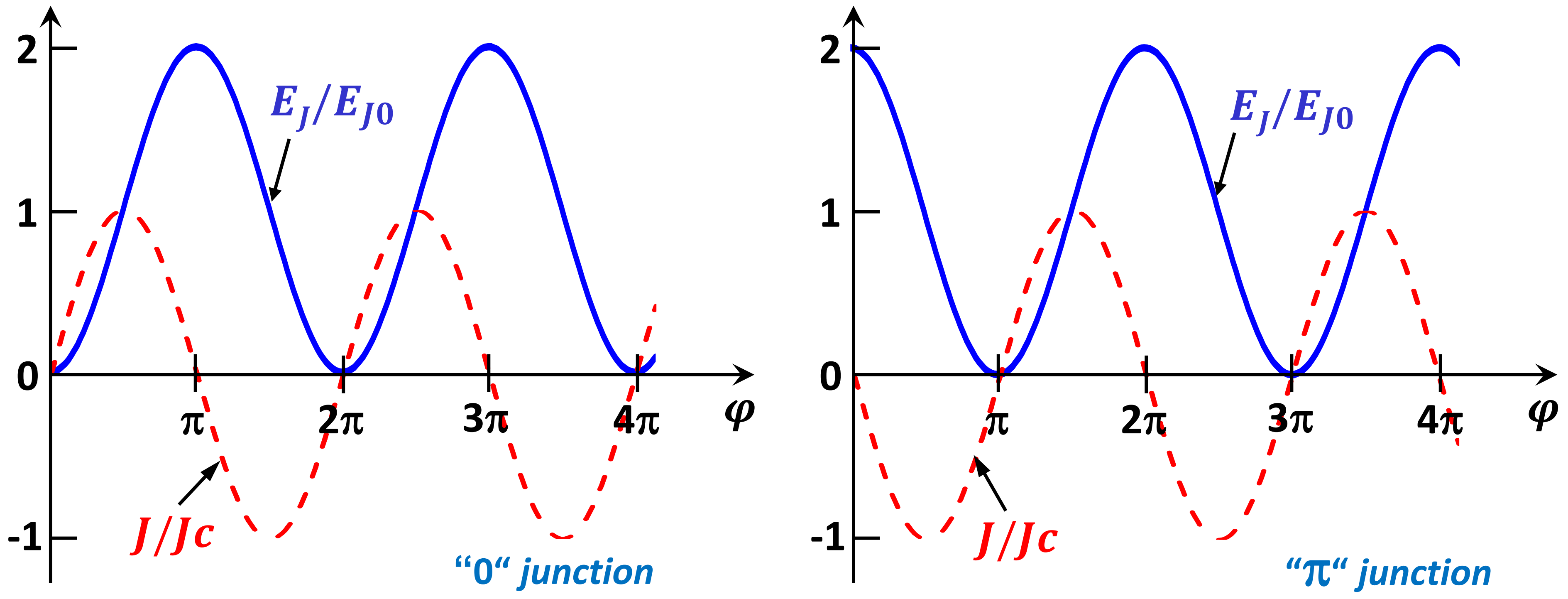
integration yields:

$$\frac{E_J}{A} = \frac{\Phi_0 J_c}{2\pi}(1 - \cos\varphi) = \frac{E_{J0}}{A}(1 - \cos\varphi)$$

Josephson coupling energy (per junction area)

5.2 Josephson Coupling Energy

Josephson coupling energy E_J (cf. 3.2.3)



- **order of magnitude estimate:**
  - typically: $I_c = J_c A \sim 1\ \mathrm{mA} \Rightarrow \quad E_{J0} \simeq 3 \times 10^{-19}\ \mathrm{J}$
  - corresponds to thermal energy $k_\mathrm{B}T$ for $T \simeq 20\,000\ \mathrm{K}$
  - junction with very small critical current: $I_c \simeq 1\ \mu A \ \Rightarrow$ thermal energy $\simeq k_\mathrm{B} \times 20\ \mathrm{K}$

5.2.1 Josephson Junction with Applied Current

Josephson junction under the action of an external force (applied current)

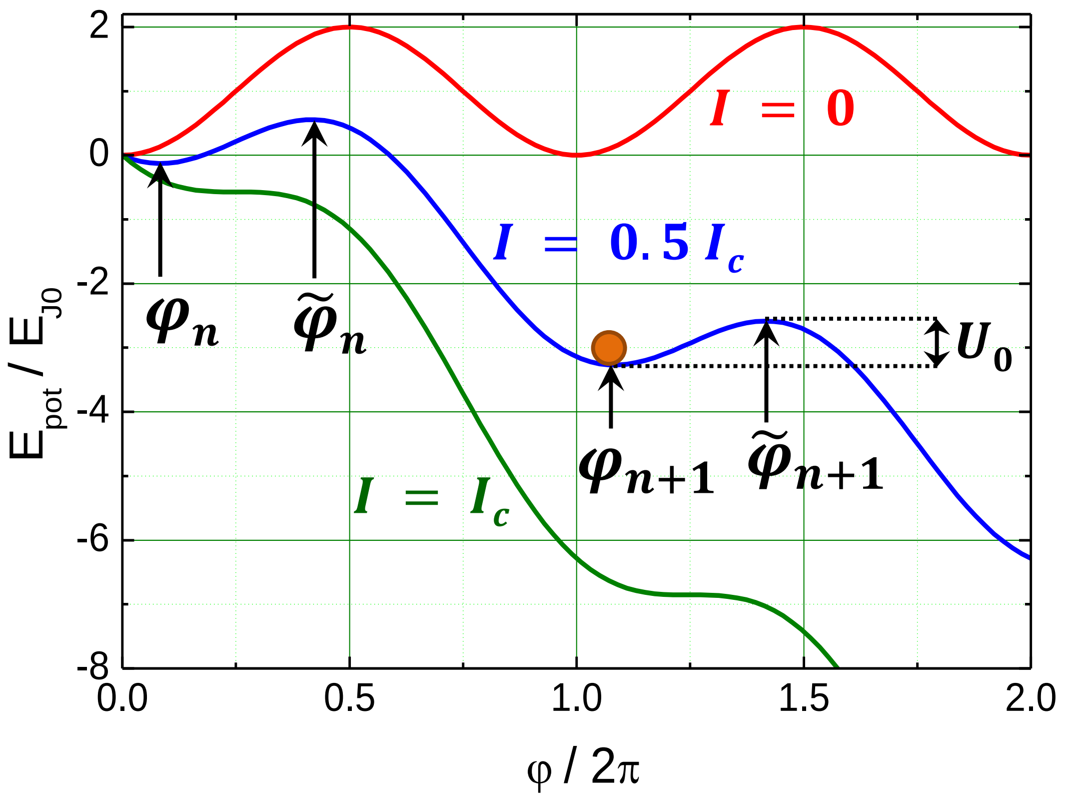
- ***potential energy*** E_{pot} of the system under action of external force: $E_{\mathrm{pot}} = E_J - F \cdot x$
  - E_J: intrinsic free energy of the junction
  - F: generalized force $(F = I)$
  - x: generalized coordinate
    $\Rightarrow F \cdot \partial x/\partial t$ = power flowing into subsystem $(I \cdot V)$
    $\Rightarrow \partial x/\partial t = V$:

$$x = \int V\, dt = \frac{\hbar}{2e}\varphi + c = \Phi_0 \frac{\varphi}{2\pi} + c$$

$$E_{\mathrm{pot}}(\varphi) = E_J(\varphi) - I\left(\Phi_0 \frac{\varphi}{2\pi} + c\right)$$

$$E_{\mathrm{pot}}(\varphi) = \underbrace{\frac{\Phi_0 I_c}{2\pi}}_{E_{J0}}\left(1 - \cos\varphi - \frac{I}{I_c}\varphi\right) + \tilde{c}$$

tilted washboard potential
stable minima at φ_n, unstable maxima at $\tilde{\varphi}_n$,
states for different n are equivalent



- **junction dynamics**: motion of "phase particle" φ in tilted washboard potential (not discussed here)

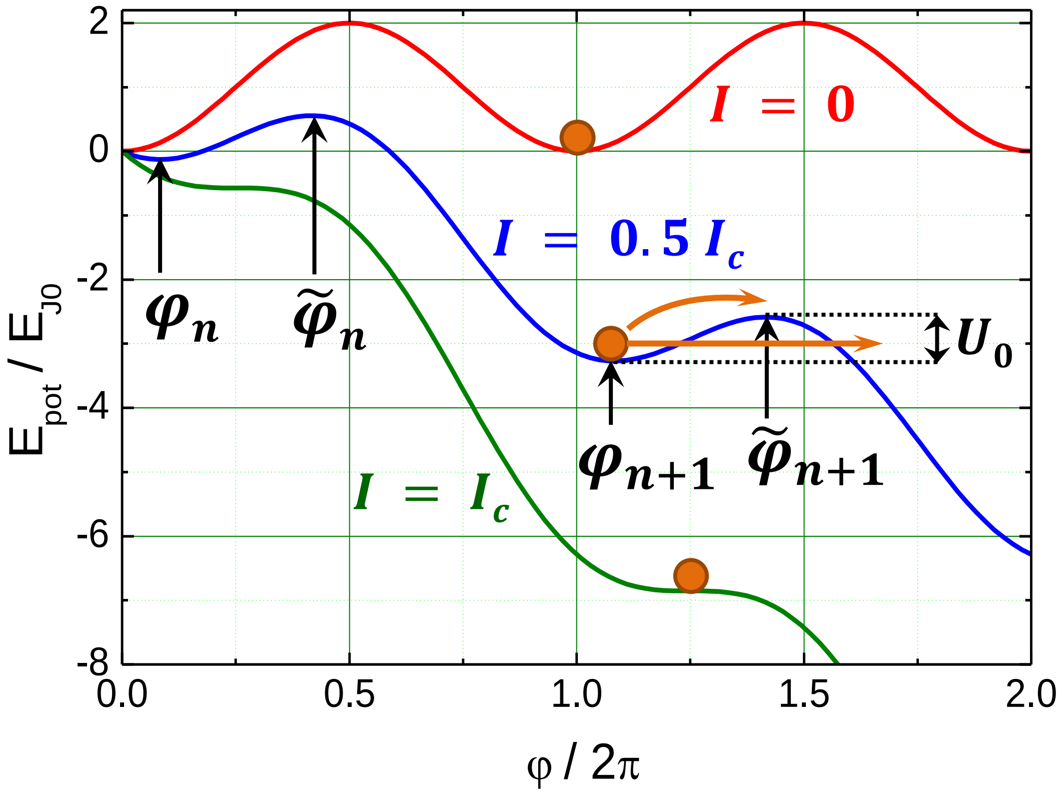
5.2.1 Josephson Junction with Applied Current

$|I| \leq I_c$: ***constant phase difference***: $\varphi = \varphi_n = \arcsin(I/I_c) + 2\pi n$

→ zero voltage state / ordinary (S) state

$|I| > I_c$: ***phase difference increases with time***: $\varphi = \varphi(t)$

→ finite voltage state / running phase state



- thermally activated phase slippage
- quantum tunneling of phase

5.3 Applications of the Josephson

large number of applications in analog and digital electronics

→ detailed discussion in lecture „*Applied Superconductivity*"

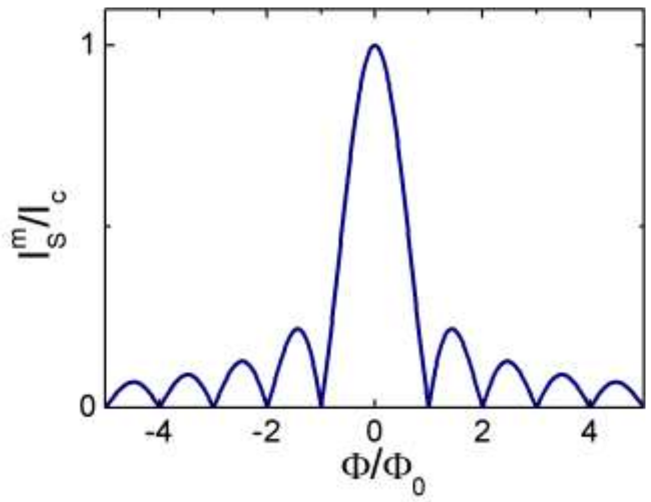
- $I_s^m = I_s^m(B)$:
  → *magnetic field sensors (SQUIDs)*

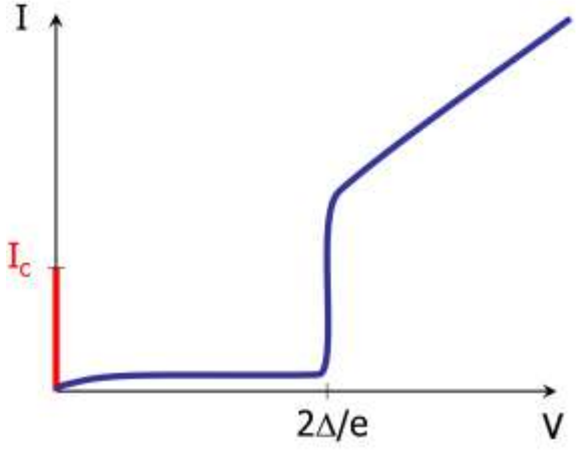
- $\beta_C \gg 1$ (**hysteretic** IVC)
  → **bistability**: zero/voltage state
  → *switching devices, Josephson computer, fast DACs*

- 2nd Josephson equation
  → *voltage controlled oscillator, voltage standard*

- **nonlinear** IVC
  → *mixers up to THz, oscillators*

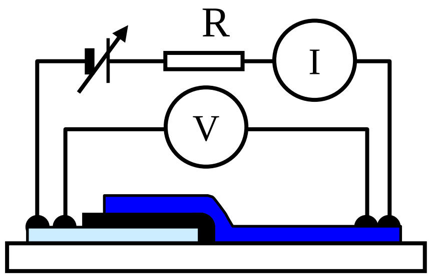
- **macroscopic quantum** behavior
  → *superconducting qubits*





Josephson junction as fast switching device

- $V = 0$: Josephson current
- $V \neq 0$: quasiparticle current

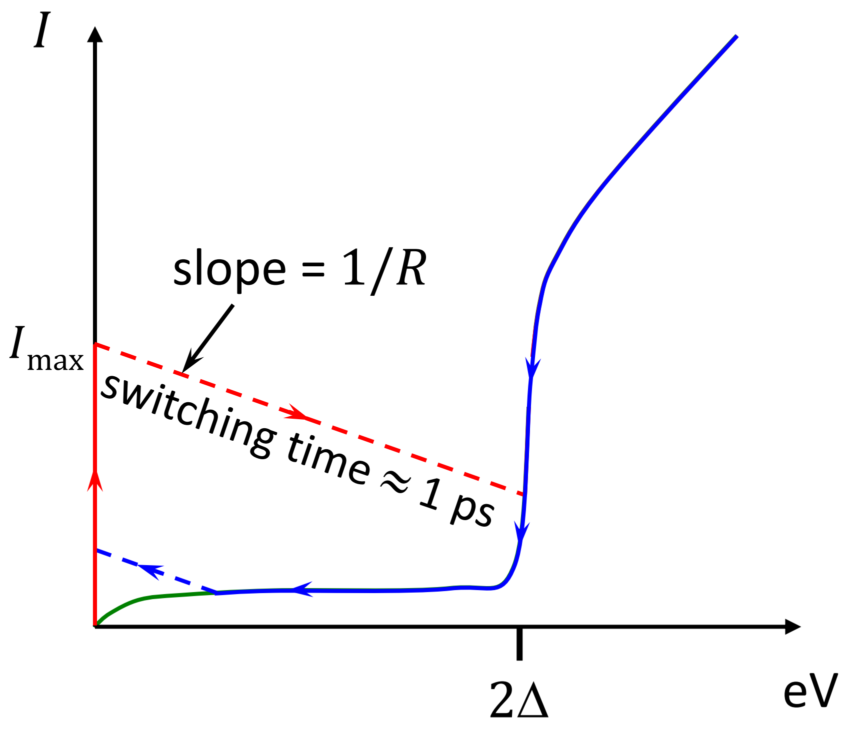


- hysteresis:

  fast switching device

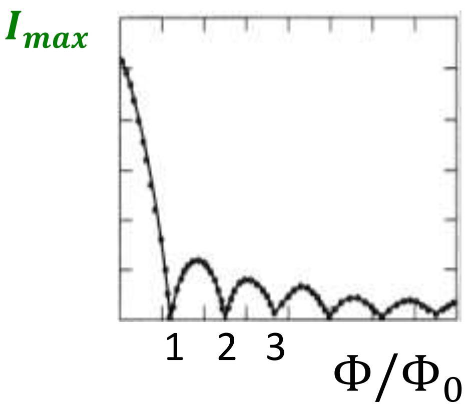
  very low power consumption

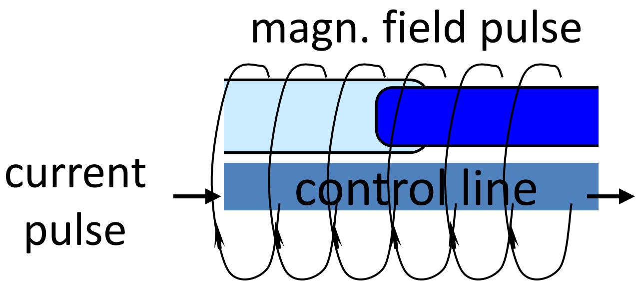
  $\Rightarrow$ **Josephson digital electronics**

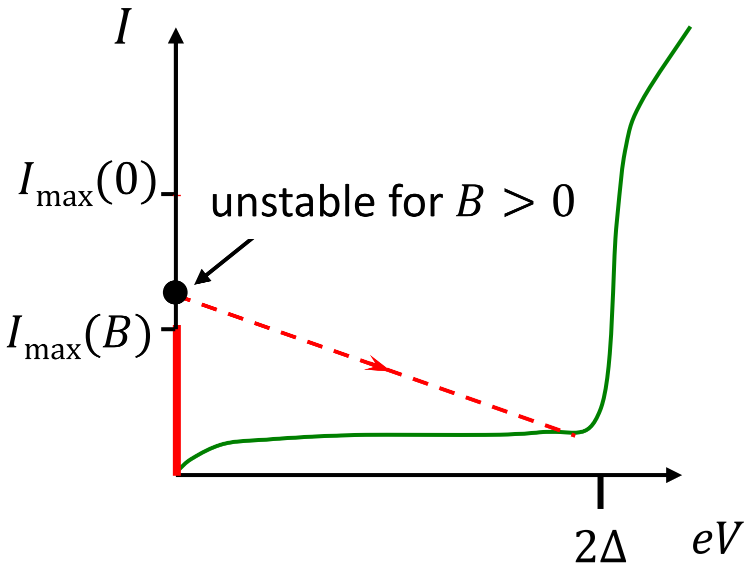


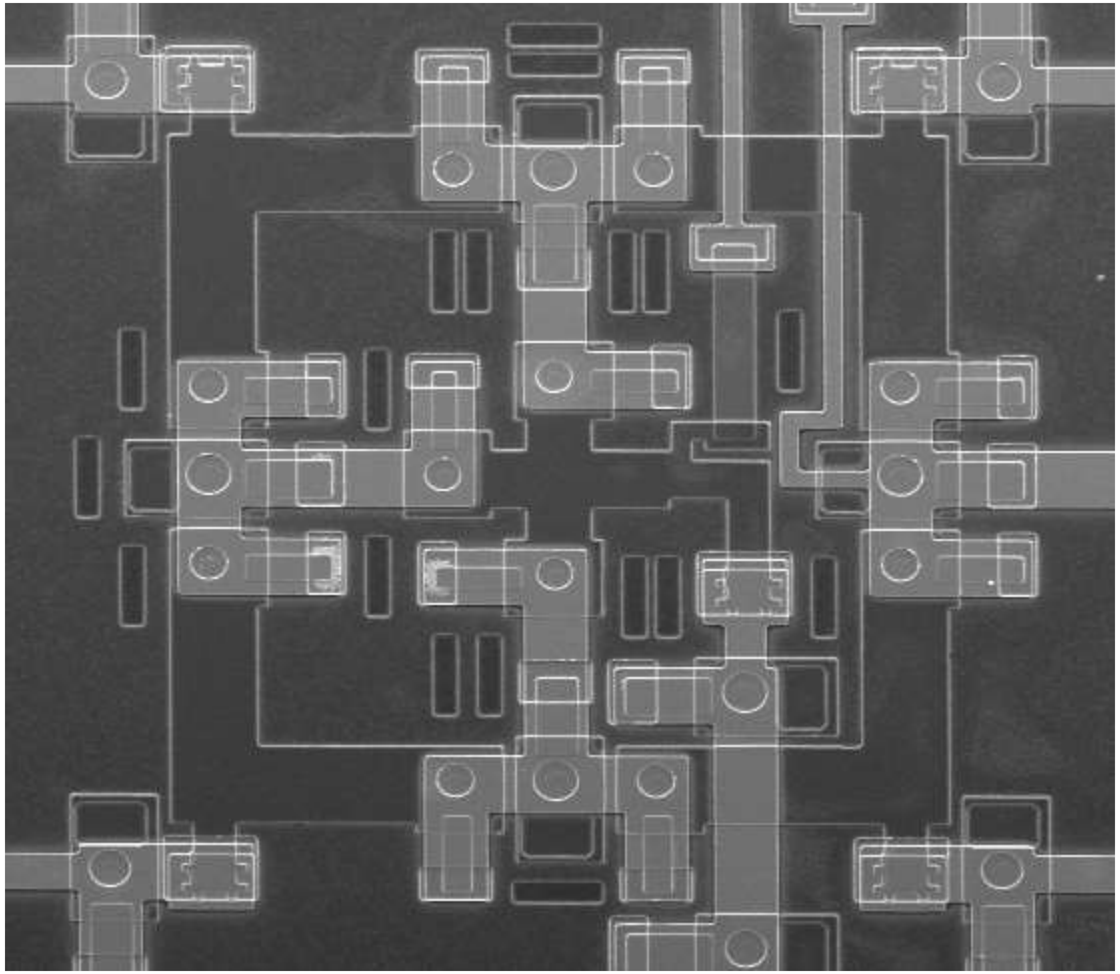
principle of switching element:

- magnetic field dependence of the maximum Josephson current





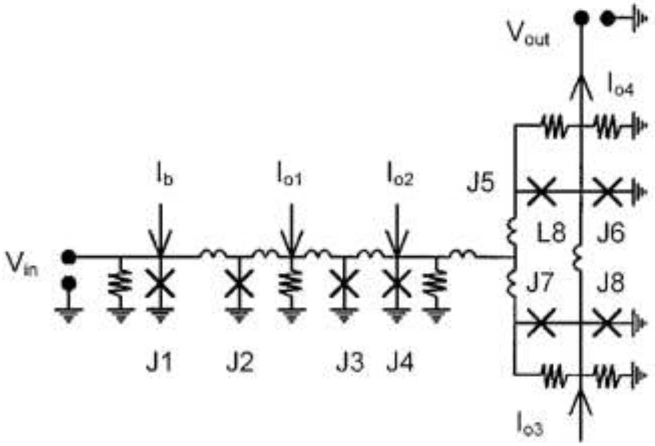


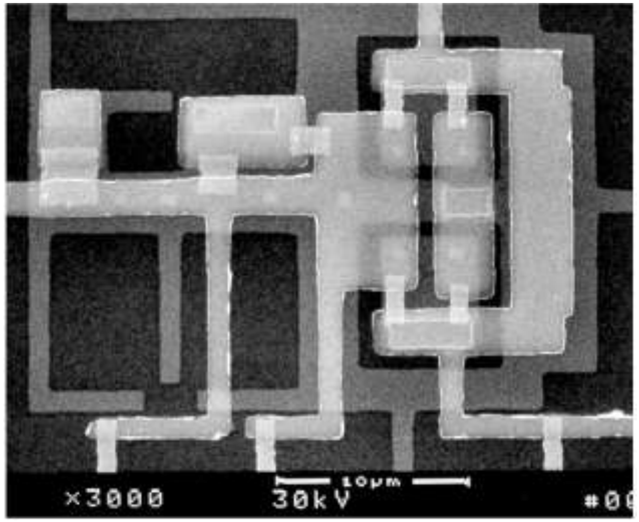


SEM micrograph of a universal asynchronous DR RSFQ (rapid single flux quantum) logic gate

B. Dimov et al.,
Universal asynchronous RSFQ gate for realization of Boolean functions of dual-rail binary variables
Journal of Physics Conference Series 43(1), 1183 (2006)

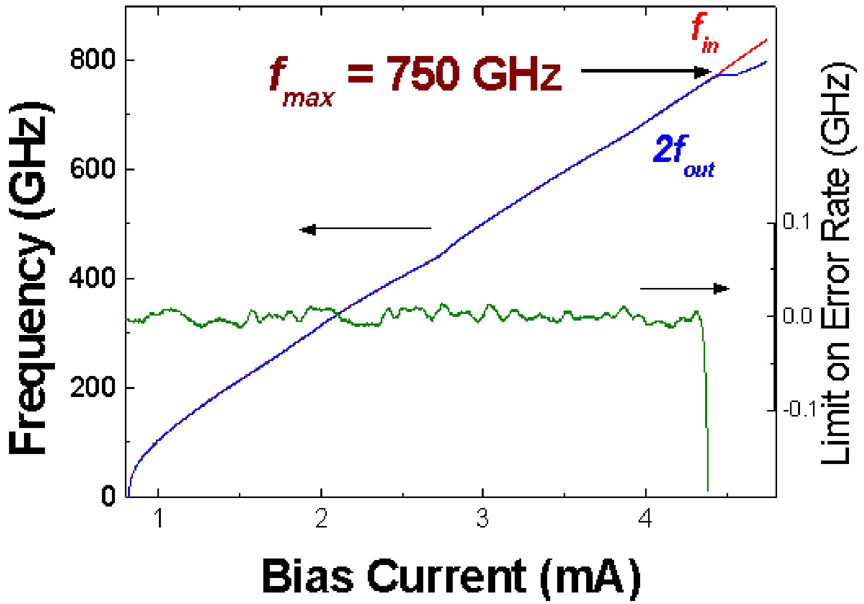
superconductor digital frequency divider operating up to 750 GHz





Dividers

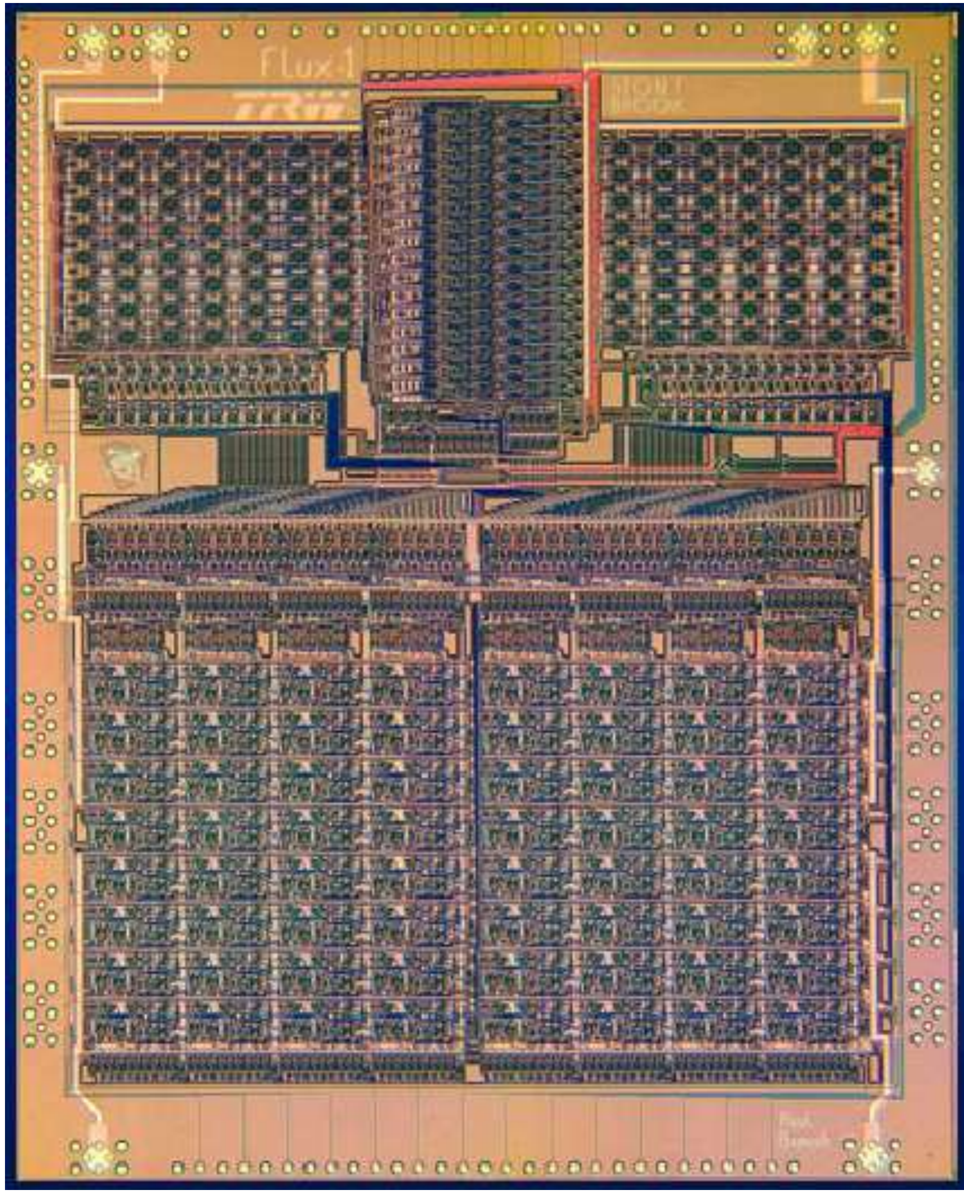
	RSFQ	Semi-conductor
Frequency	750 GHz	60 GHz
Power Dissipation	1.5 μW	0.5 W



W. Chen, A. Rylyakov, V. Patel, J. Lukens, K. Likharev,
"Superconductor digital frequency divider operating up to 750 GHz,"
Appl. Phys. Lett. **73**, 2817 (1998)

- problem: integration of large number of JJs (> 10^5) with high yield and small parameter spead

5.3 Applications of the Josephson Effect



Stony Brook

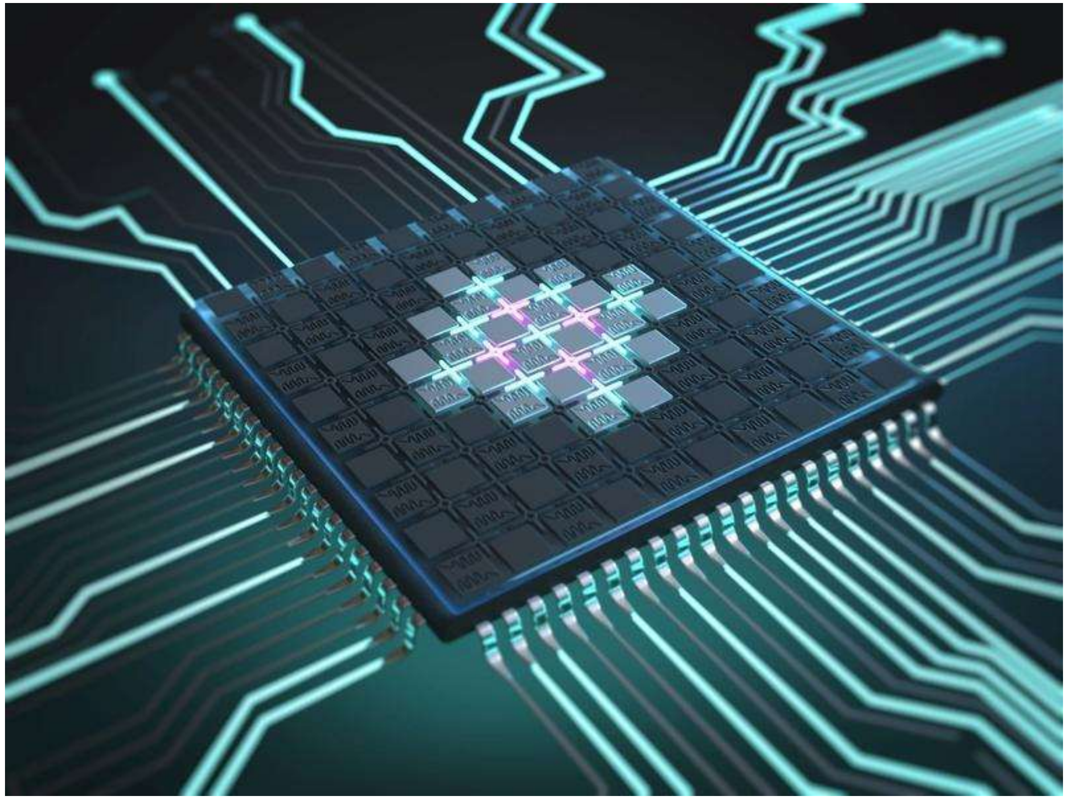
FLUX-1

- the first RSFQ MPU
- 8 bit ALU array
- 16 word instruction memory
- 70,000 JJs
- 14 mW
- 20-22 GHz @ *F* = 2.0 um

  (⇒ 120-140 GHz @ 0.3 um)
- TRW's 4-metal process

www.physics.sunysb.edu/Physics/RSFQ/

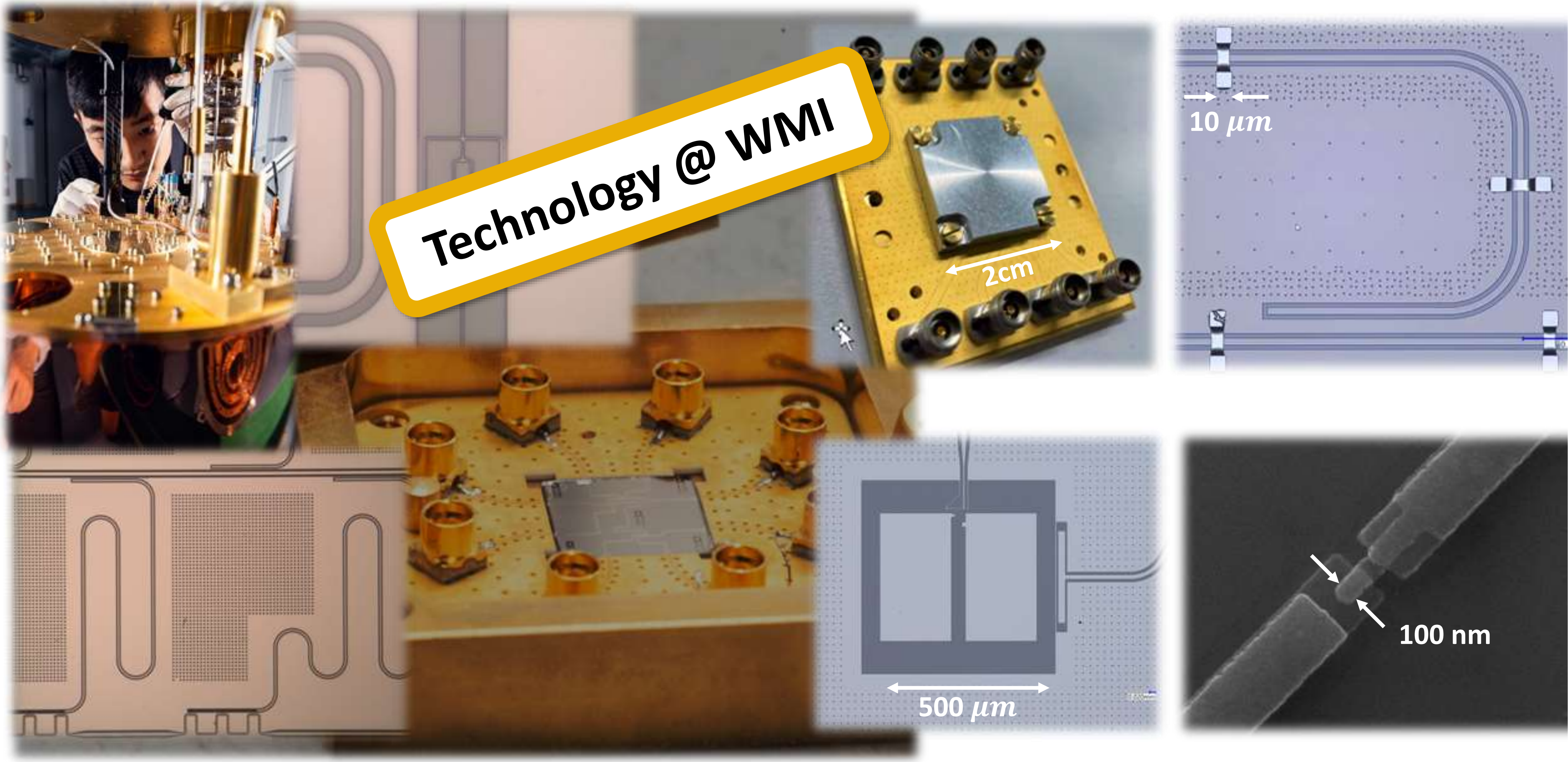
superconducting quantum bits



WMI/MCQST

Chao Song et al.,
Science 365, 574–577 (2019)

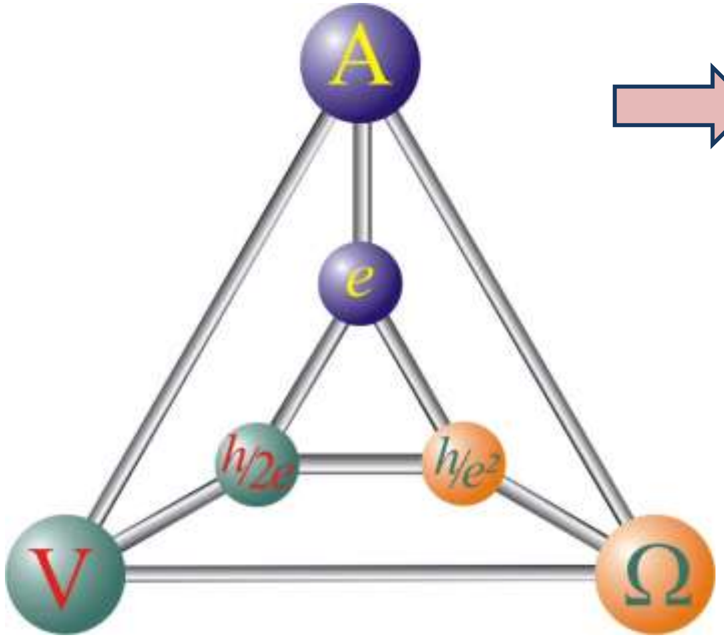
5.3 Applications of the Josephson Effect



5.3 Applications of the Josephson Effect

precise definition of the electrical voltage, current and the resistance by using fundamental quantum effects

- ***Josephson effect***: $V = \frac{h}{2e} \cdot f = \Phi_0 \cdot f$ (relation between voltage and time/frequency by flux quantum)
- ***Single electron pump***: $I = e \cdot f$ (relation between current and time by charge quantum)
- ***Quantum Hall effect***: $V = \frac{h}{e^2} \cdot I = R_K \cdot I$ (relation between voltage and current by quantum resistance, unit = 1 Klitzing)



allows the reproduction of the physical units Volt, Ampère and Ohm with a very high precision and largely uninfluenced by environmental parameters at any place in the world

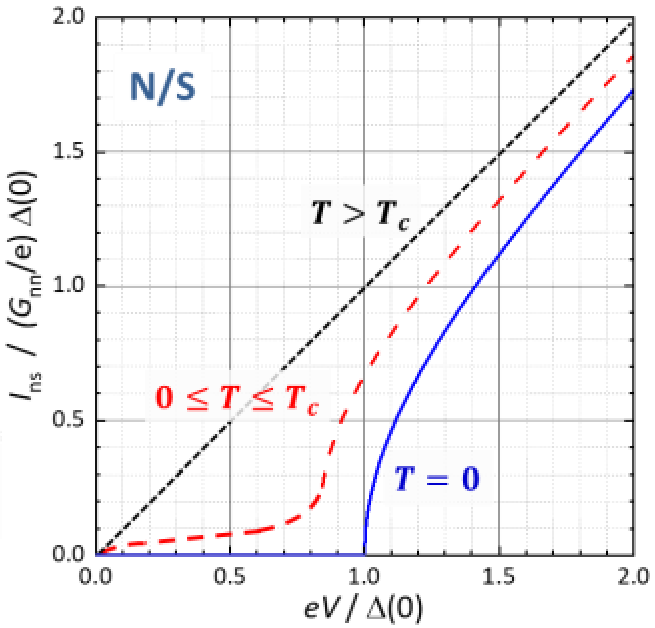
realization of the Ampère by single electron pump not realized so far at sufficient precision
➔ would allow an important experimental test of the consistency of the relations between the fundamental constants illustrated in the "***electrical triangle***"

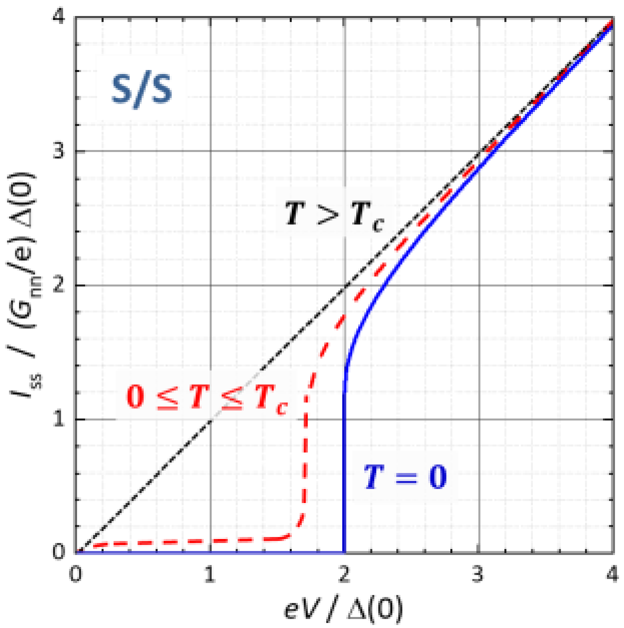
Summary of Lecture No. 10 (1)

- *determination of energy gap and DOS by tunneling spectroscopy*

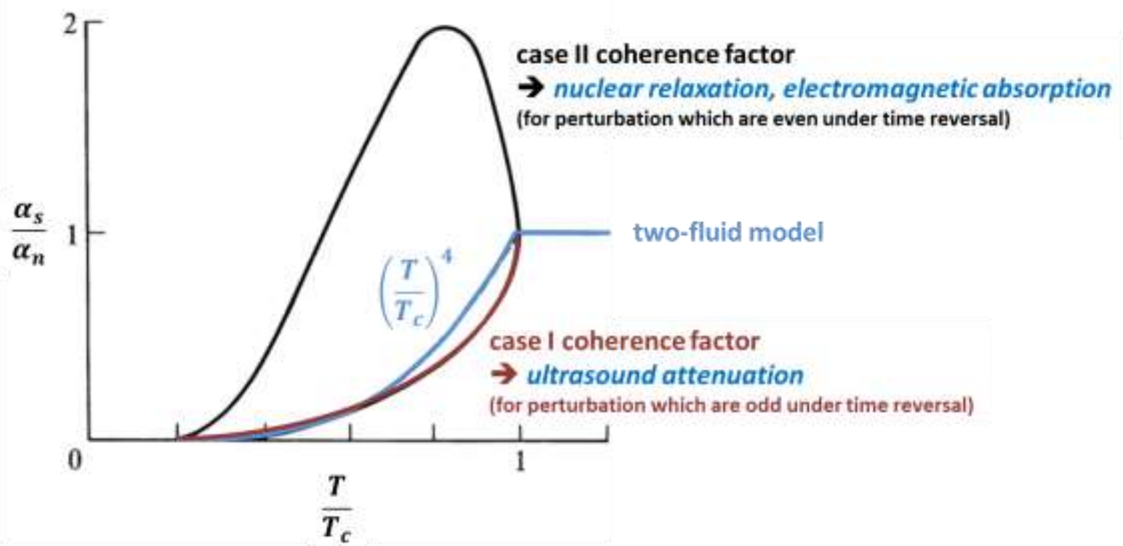
$$I_{ns}(V) = \frac{G_{nn}}{e} \int_{-\infty}^{\infty} \frac{D_{s2}(E)}{D_{n2}(E_F)} \left[f(E) - f(E+eV)\right] \mathrm{d}E \quad \Rightarrow \quad G_{ns}(V) = \frac{\partial I_{ns}(V)}{\partial V} = G_{nn} \frac{D_{s2}(eV)}{D_{n2}(E_F)} \propto D_{s2}(eV) \quad @\, T = 0$$

- *NIS and SIS tunnel junctions*





- *BCS coherence factors*



Summary of Lecture No. 10 (2)

Macroscopic wave function ψ:

describes ensemble of a macroscopic number of superconducting electrons,
$|\psi|^2 = n_s$ is given by density of superconducting electrons

Current density in a superconductor:

$$\mathbf{J}_s(\mathbf{r},t) = \frac{q_s n_s(\mathbf{r},t)\hbar}{m_s}\left\{\nabla\theta(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t)\right\} = \frac{q_s n_s(\mathbf{r},t)\hbar}{m_s}\left\{\nabla\theta(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\mathbf{A}(\mathbf{r},t)\right\}$$

Gauge invariant phase gradient:

$$\gamma(\mathbf{r},t) = \nabla\theta(\mathbf{r},t) - \frac{q_s}{\hbar}\mathbf{A}(\mathbf{r},t) = \nabla\theta(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\mathbf{A}(\mathbf{r},t)$$

Phenomenological London equations:

$$(1)\quad \frac{\partial}{\partial t}\left(\Lambda\mathbf{J}_s(\mathbf{r},t)\right) = \mathbf{E} \qquad (2)\quad \nabla\times(\Lambda\mathbf{J}_s) + \mathbf{B} = \mathbf{0} \qquad \Lambda = \frac{m_s}{q_s^2 n_s} = \mu_0\lambda_\mathrm{L}^2$$

Fluxoid quantization:

$$\oint_C \Lambda\mathbf{J}_s\cdot d\ell + \int_S \mathbf{B}\cdot\hat{\mathbf{n}}\; dS = n\cdot\frac{h}{q_s} = n\cdot\Phi_0$$

Summary of Lecture No. 10 (3)

Josephson equations:

$$\mathbf{J}_s(\mathbf{r},t) = \mathbf{J}_c(\mathbf{r},t)\sin\varphi(\mathbf{r},t)$$

$$\frac{\partial\varphi(t)}{\partial t} = \frac{2\pi}{\Phi_0}V(t) = \frac{q_s V(t)}{\hbar} \qquad \frac{\omega/2\pi}{V} = \frac{1}{\Phi_0} = 483.597\,9\,\frac{\mathrm{MHz}}{\mu\mathrm{V}}$$

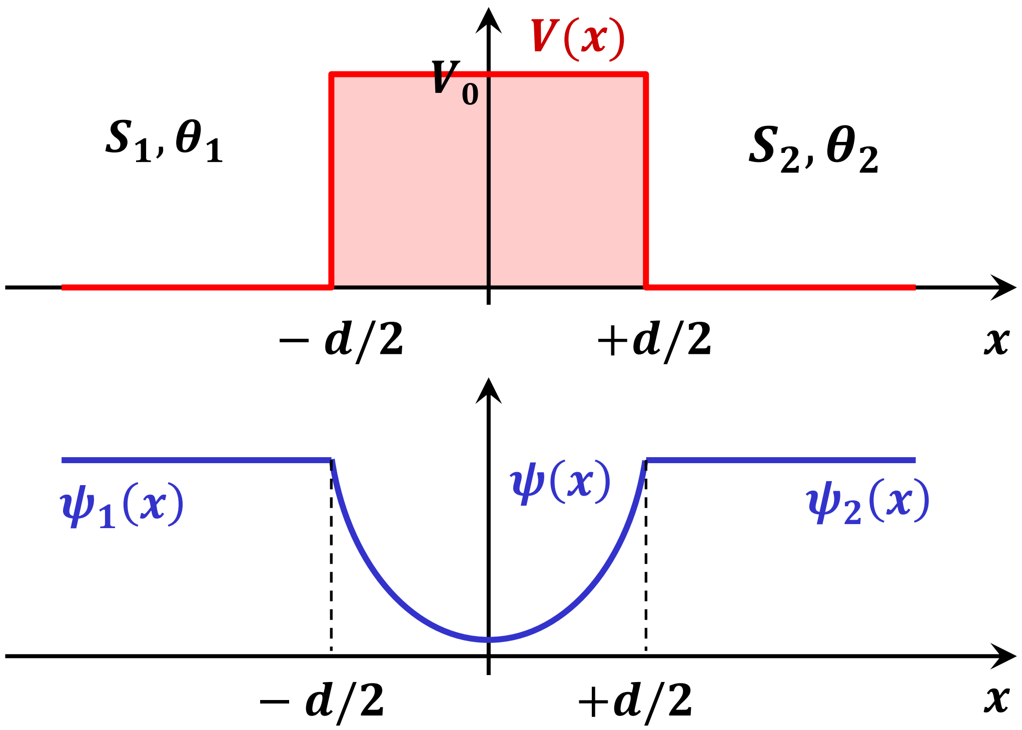
Josephson coupling energy:

$$\frac{E_J}{A} = \frac{\Phi_0 J_c}{2\pi}(1-\cos\varphi) = E_{J0}\,(1-\cos\varphi)$$

maximum Josephson current density $\mathbf{J}_c$:

can be calculated by e.g. wave matching method

$$\mathbf{J}_c = \frac{2e\hbar\kappa}{m_s}\,2\sqrt{n_{s,1}n_{s,2}}\,\exp(-2\kappa d) \qquad q_s = -2e$$



Ambegaokar-Baratoff relation:

$$I_c R_n = \frac{\pi}{2e}\Delta(T)\tanh\left(\frac{\Delta(T)}{2k_B T}\right)$$

Summary of Lecture No. 10 (4)

Josephson junction with applied current:

$$E_{\mathrm{pot}}(\varphi) = E_{J0}\left(1 - \cos\varphi - \frac{I}{I_c}\varphi\right)$$ tilted washboard potential

many application in digital and analog electronics

- *magnetic field sensors (SQUIDs)*
- *switching devices, RSFQ logic, fast DACs*
- *voltage controlled oscillator, voltage standard*
- *mixers up to THz frequencies*
- *superconducting qubits*