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Superconductivity and Low Temperature Physics I



Lecture Notes
Winter Semester 2023/2024

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Chapter 5

Josephson Effects

5. Josephson Effects

5.1 Josephson Equations

5.1.1 SIS Josephson Junction

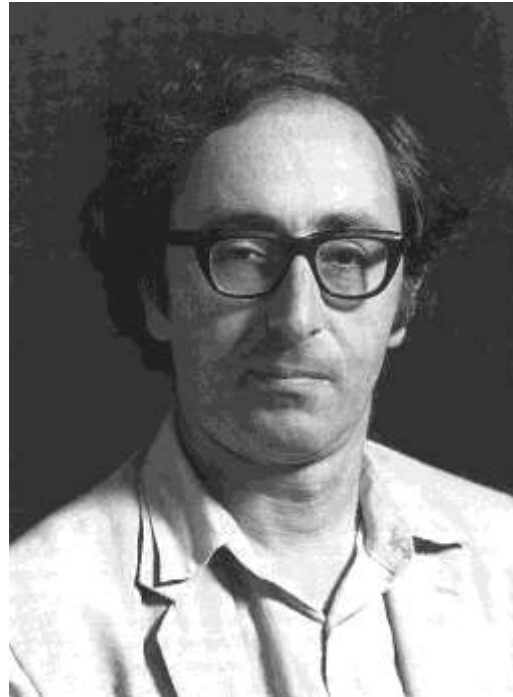
5.1.2 Ambegaokar-Baratoff relation

5.2 Josephson Coupling Energy

5.2.1 Josephson Junction with applied current

5.3 Applications of the Josephson Effect

5 Josephson Effects



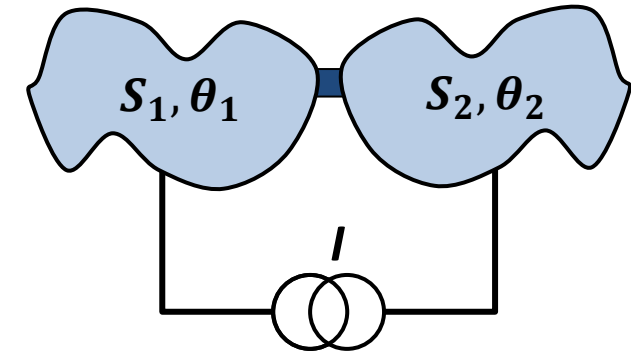
Brian David Josephson
born 04.01.1940

What happens if we weakly couple two superconductors ?

*Possible new effects in superconductive tunnelling, Physics Letters **1**(7), 251-253 (1962)*

5.1 Josephson Effects (cf. 3.2.3)

- *what happens if we weakly couple two superconductors?*
 - coupling by *tunneling barriers, point contacts, normal conducting layers, etc.*
 - do they form a bound state such as a molecule?
 - if yes, what is the binding energy?
- **B.D. Josephson** in 1962
(Nobel Prize in physics with Esaki and Giaever in 1973)



→ Cooper pairs can tunnel through thin insulating barrier (T = transmission amplitude for single charge carriers)

expectation: tunneling probability for pairs $\propto (|T|^2)^2 \rightarrow$ extremely small $\sim (10^{-4})^2$

Josephson: tunneling probability for pairs $\propto |T|^2$
coherent tunneling of pairs („*tunneling of macroscopic wave function*“)

predictions:

- *finite supercurrent at zero applied voltage*
 - *oscillation of supercurrent at constant applied voltage*
 - *finite binding energy of coupled SCs = Josephson coupling energy*
- } *Josephson effects*

5.1 Josephson Effects (cf. 3.2.3)

- **coupling is weak** \rightarrow supercurrent density between S_1 and S_2 is small $\rightarrow |\psi|^2 = n_s$ is not changed in S_1 and S_2
- supercurrent density depends on gauge invariant phase gradient:

$$\mathbf{J}_s(\mathbf{r}, t) = \frac{q_s n_s(\mathbf{r}, t) \hbar}{m_s} \left\{ \nabla \theta(\mathbf{r}, t) - \frac{q_s}{\hbar} \mathbf{A}(\mathbf{r}, t) \right\} = \frac{q_s n_s(\mathbf{r}, t) \hbar}{m_s} \gamma(\mathbf{r}, t)$$

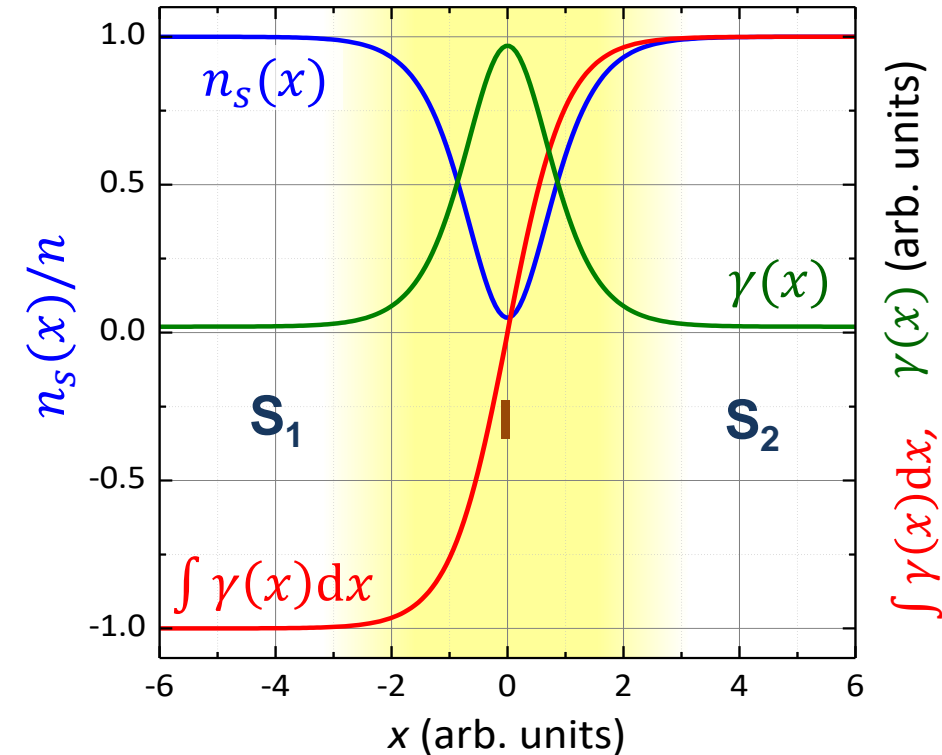
- **simplifying assumptions:**

- current density is spatially homogeneous
- $\gamma(\mathbf{r}, t)$ varies negligibly in S_1 and S_2
- \mathbf{J}_s is equal in electrodes and junction area
 $\rightarrow \gamma$ in S_1 and S_2 much smaller than in insulator I

- **approximation:**

- replace gauge invariant phase gradient γ by ***gauge invariant phase difference*** φ :

$$\varphi(\mathbf{r}, t) = \int_1^2 \gamma(\mathbf{r}, t) \cdot d\ell = \int_1^2 \left(\nabla \theta(\mathbf{r}, t) - \frac{q_s}{\hbar} \mathbf{A}(\mathbf{r}, t) \right) \cdot d\ell = \theta_2(\mathbf{r}, t) - \theta_1(\mathbf{r}, t) - \frac{2\pi}{\Phi_0} \int_1^2 \mathbf{A}(\mathbf{r}, t) \cdot d\ell$$



5.1 Josephson Effects (cf. 3.2.3)

first Josephson equation:

- we expect: $J_S = J_S(\varphi)$
 $J_S(\varphi) = J_S(\varphi + n \cdot 2\pi)$
- for $J_S = 0$: phase difference must be zero:
 $J_S(0) = J_S(n \cdot 2\pi) = 0$



$$J_S(\varphi) = J_C \sin \varphi + \sum_{m=2}^{\infty} J_{C,m} \sin(m\varphi)$$

J_C = critical or maximum Josephson current density

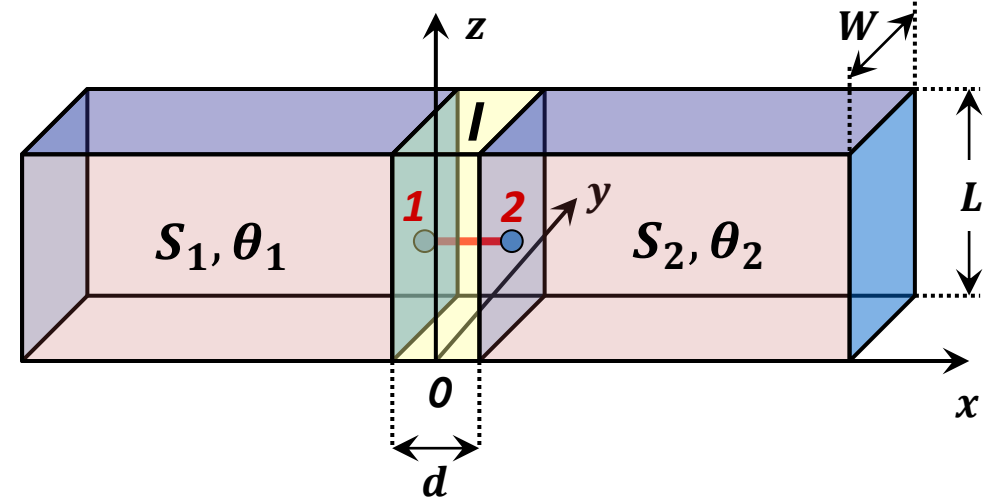
general formulation of **1st Josephson equation**: *current-phase relation*

- in most cases: we have to keep only 1st term (especially for weak coupling):

$$J_S(\varphi) = J_C \sin \varphi \quad \text{1st Josephson equation}$$

- generalization to **spatially inhomogeneous** supercurrent density:

$$J_S(y, z) = J_C(y, z) \sin \varphi(y, z)$$



derived by Josephson for SIS junctions

supercurrent density J_S varies sinusoidally with phase difference $\varphi = \theta_2 - \theta_1$ w/o external potentials

5.1 Josephson Effects (cf. 3.2.3)

second Josephson equation (for spatially homogeneous junction)

- take time derivative of the gauge invariant phase difference $\varphi(t) = \theta_2(t) - \theta_1(t) - \frac{2\pi}{\Phi_0} \int_1^2 \mathbf{A}(t) \cdot d\ell$

$$\frac{\partial \varphi(t)}{\partial t} = \frac{\partial \theta_2(t)}{\partial t} - \frac{\partial \theta_1(t)}{\partial t} - \frac{2\pi}{\Phi_0} \frac{\partial}{\partial t} \int_1^2 \mathbf{A}(t) \cdot d\ell$$

- substitution of the energy-phase relation $\hbar \frac{\partial \theta(t)}{\partial t} = - \left\{ \frac{1}{2n_s} \Lambda \mathbf{J}_s^2(t) + q_s \phi_{el}(\mathbf{r}, t) \right\}$ gives:

$$\frac{\partial \varphi(t)}{\partial t} = -\frac{1}{\hbar} \left(\frac{\Lambda}{2n_s} [\mathbf{J}_s^2(2) - \mathbf{J}_s^2(1)] + q_s [\phi_{el}(2) - \phi_{el}(1)] \right) - \frac{2\pi}{\Phi_0} \frac{\partial}{\partial t} \int_1^2 \mathbf{A}(t) \cdot d\ell$$

- supercurrent density across the junction is *continuous* ($\mathbf{J}_s(1) = \mathbf{J}_s(2)$):

$$\frac{\partial \varphi(t)}{\partial t} = \frac{2\pi}{\Phi_0} \int_1^2 \left(-\nabla \phi_{el} - \frac{\partial \mathbf{A}(t)}{\partial t} \right) \cdot d\ell \quad (\text{term in parentheses} = \text{electric field})$$



$$\frac{\partial \varphi(t)}{\partial t} = \frac{2\pi}{\Phi_0} \underbrace{\int_1^2 \mathbf{E}(t) \cdot d\ell}_{\text{voltage drop } V} = \frac{2\pi}{\Phi_0} V(t) = \frac{q_s V(t)}{\hbar}$$

2nd Josephson equation: voltage – phase relation

5.1 Josephson Effects (cf. 3.2.3)

- for a constant voltage V across the junction:

$$\frac{\partial \varphi(t)}{\partial t} = \frac{2\pi}{\Phi_0} V = \frac{q_s V}{\hbar} \quad \text{integration yields: } \varphi(t) = \varphi_0 + \frac{2\pi}{\Phi_0} V \cdot t = \varphi_0 + \frac{q_s}{\hbar} V \cdot t$$

phase difference increases linearly in time

- supercurrent density J_s oscillates at the Josephson frequency $\nu = V/\Phi_0$:

$$J_s(\varphi(t)) = J_c \sin \varphi(t) = J_c \sin \left(\frac{2\pi}{\Phi_0} V \cdot t \right)$$

$$\frac{\nu}{V} = \frac{\omega/2\pi}{V} = \frac{1}{\Phi_0} = 483.5979 \frac{\text{MHz}}{\mu\text{V}}$$

➔ **Josephson junction = voltage controlled oscillator**

- **applications:**
 - Josephson voltage standard
 - microwave sources
 -

5.1 Josephson Effects (cf. 3.2.3)

Josephson coupling energy E_J : binding energy of two coupled superconductors

$$\frac{E_J}{A} = \int_0^{t_0} J_s V dt = \int_0^{t_0} J_c \sin \varphi \left(\frac{\Phi_0}{2\pi} \frac{\partial \varphi}{\partial t} \right) dt = \frac{\Phi_0 J_c}{2\pi} \int_0^{\varphi} \sin \varphi' d\varphi'$$

with $\varphi(0) = 0$ and $\varphi(t_0) = \varphi$
 $A =$ junction area

integration yields:

$$\frac{E_J}{A} = \frac{\Phi_0 J_c}{2\pi} (1 - \cos \varphi)$$

Josephson coupling energy (per junction area)

5.1.1 Superconducting Tunnel Junctions

Josephson effect in superconducting tunnel junctions

- derive the Josephson equations

- starting point is time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = E \psi(\mathbf{r}, t)$$

- S_1 and S_2 are described by macroscopic wave functions

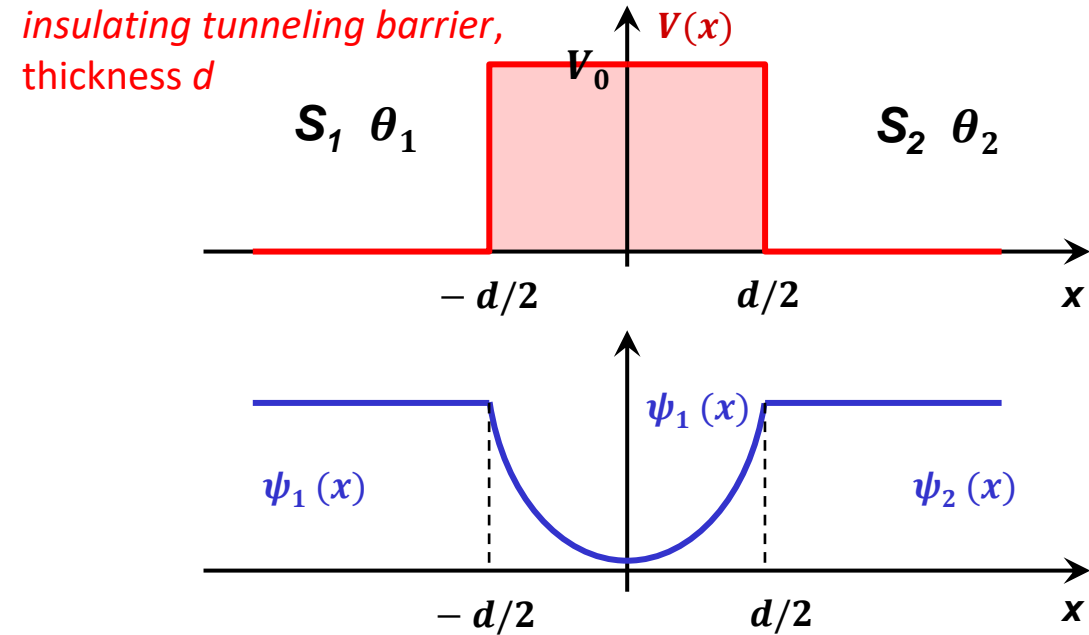
$$\psi_1(\mathbf{r}, t) = \psi_{01} e^{i\theta_1(\mathbf{r}, t)} \quad n_{s1} = |\psi_{01}|^2$$

$$\psi_2(\mathbf{r}, t) = \psi_{02} e^{i\theta_2(\mathbf{r}, t)} \quad n_{s2} = |\psi_{02}|^2$$

- **finite coupling** between S_1 and S_2 is introduced by small coupling constant C (tunnel coupling)

$$i\hbar \frac{\partial \psi_1(\mathbf{r}, t)}{\partial t} = E_1 \psi_1(\mathbf{r}, t) + C_{LR} \psi_2(\mathbf{r}, t) = +e\Delta\phi \psi_1(\mathbf{r}, t) + C_{LR} \psi_2(\mathbf{r}, t)$$

$$i\hbar \frac{\partial \psi_2(\mathbf{r}, t)}{\partial t} = E_2 \psi_2(\mathbf{r}, t) + C_{RL} \psi_1(\mathbf{r}, t) = -e\Delta\phi \psi_2(\mathbf{r}, t) + C_{RL} \psi_1(\mathbf{r}, t)$$



$$\Delta\phi = \frac{E_1 - E_2}{|q_s|} = \frac{E_1 - E_2}{2e} = \frac{1}{2} \frac{E_1 - E_2}{e}$$

5.1.1 Superconducting Tunnel Junctions

- by inserting the wave functions ψ_1, ψ_2 into the time-dependent Schrödinger equation we obtain for the **imaginary part**:

$$\frac{\partial \theta_1(t)}{\partial t} = -\frac{C}{\hbar} \sqrt{\frac{n_{s2}}{n_{s1}}} \cos \varphi(t) - \frac{e\Delta\phi}{\hbar}$$

$$\frac{\partial \theta_2(t)}{\partial t} = -\frac{C}{\hbar} \sqrt{\frac{n_{s1}}{n_{s2}}} \cos \varphi(t) + \frac{e\Delta\phi}{\hbar}$$

we use $C_{LR} = C_{RL} = C$ and $\varphi = \theta_2 - \theta_1$,

- for the **real part** we obtain:

$$\frac{\partial n_{s1}(t)}{\partial t} = +\frac{2C}{\hbar} \sqrt{n_{s1}n_{s2}} \sin \varphi(t)$$

$$\frac{\partial n_{s2}(t)}{\partial t} = -\frac{2C}{\hbar} \sqrt{n_{s1}n_{s2}} \sin \varphi(t)$$

we see that $\frac{\partial n_{s1}(\mathbf{r},t)}{\partial t} = -\frac{\partial n_{s2}(\mathbf{r},t)}{\partial t} \Rightarrow$ **conservation of particle number**

- supercurrent density:**

$$J_s^{1 \rightarrow 2} = \frac{2e}{A} \frac{\partial N_{s1}(t)}{\partial t}$$

$$J_s^{2 \rightarrow 1} = \frac{2e}{A} \frac{\partial N_{s2}(t)}{\partial t}$$



$$J_s = J_s^{1 \rightarrow 2} - J_s^{2 \rightarrow 1} = \frac{4eC}{\hbar A} \sqrt{N_{s1}N_{s2}} \sin \varphi(t) = J_c \sin \varphi(t)$$

$$N_{s1,s2} = V_{1,2} n_{s1,s2}$$

1st Josephson equation
(current-phase relation)

5.1.1 Superconducting Tunnel Junctions

- the 2nd Josephson equation is obtained from the gauge invariant phase difference $\varphi(\mathbf{r}, t) = \theta_2(\mathbf{r}, t) - \theta_1(\mathbf{r}, t) - \frac{2\pi}{\Phi_0} \int_1^2 \mathbf{A}(\mathbf{r}, t) \cdot d\ell$

$$\frac{\partial \varphi(t)}{\partial t} = \frac{\partial \theta_2(t)}{\partial t} - \frac{\partial \theta_1(t)}{\partial t} - \frac{2e}{\hbar} \frac{\partial}{\partial t} \int_1^2 \mathbf{A}(\mathbf{r}, t) \cdot d\ell$$

$$\frac{\partial \varphi(t)}{\partial t} = \frac{2e\Delta\phi}{\hbar} - \frac{2e}{\hbar} \frac{\partial}{\partial t} \int_1^2 \mathbf{A}(\mathbf{r}, t) \cdot d\ell \quad \text{for } n_{s1} = n_{s2}$$

$$\frac{\partial \varphi(t)}{\partial t} = \frac{2e}{\hbar} \underbrace{\int_1^2 \left[-\nabla\phi - \frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}, t) \right] \cdot d\ell}_{=V}$$



$$\frac{\partial \varphi(t)}{\partial t} = \frac{2e}{\hbar} V = \frac{2\pi}{\Phi_0} V$$

2nd Josephson equation
(voltage-phase relation)

$$\frac{\partial \theta_1(t)}{\partial t} = -\frac{C}{\hbar} \sqrt{\frac{n_{s2}}{n_{s1}}} \cos \varphi(t) - \frac{e\Delta\phi}{\hbar}$$

$$\frac{\partial \theta_2(t)}{\partial t} = -\frac{C}{\hbar} \sqrt{\frac{n_{s1}}{n_{s2}}} \cos \varphi(t) + \frac{e\Delta\phi}{\hbar}$$

5.1.1 Superconducting Tunnel Junctions

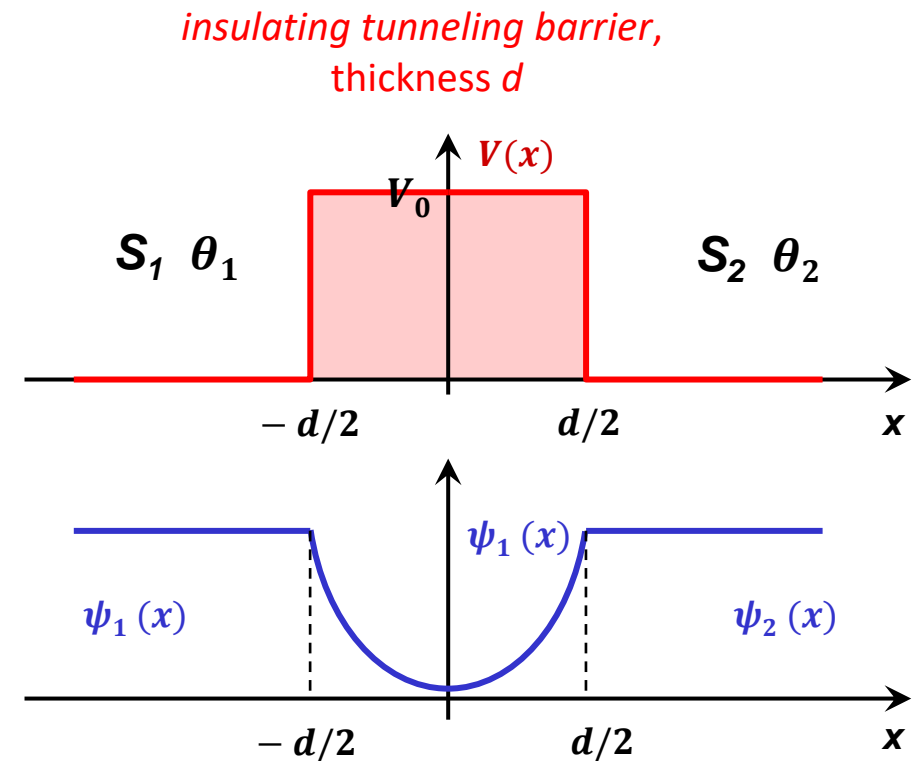
calculation of the maximum Josephson current density: How does T_{LR} depend on height and thickness of barrier?

- calculation by the **wave matching method**

solve time-independent Schrödinger equation for S_1, S_2 and barrier region and match solution at interfaces

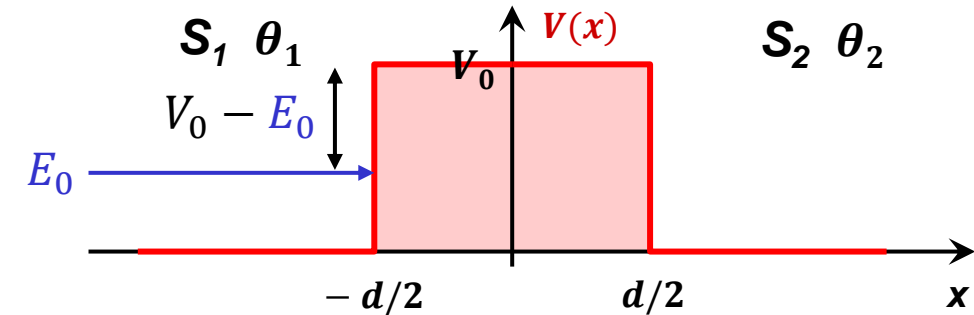
$$-\frac{\hbar^2}{2m_s} \nabla^2 \psi(\mathbf{r}) = (E_0 - V_0) \psi(\mathbf{r})$$

with $\psi(\mathbf{r}) = \psi_0(\mathbf{r}) e^{i\theta(\mathbf{r})}$



5.1.1 Superconducting Tunnel Junctions

$$-\frac{\hbar^2}{2m_s} \nabla^2 \psi(\mathbf{r}) = (E_0 - V_0) \psi(\mathbf{r})$$



- **assumption:**
homogeneous barrier and supercurrent flow → **1D problem**

- **solutions:**

- in superconductors: $\psi_{1,2}(x) = \psi_{01,02} e^{i\theta_{1,2}(x)} = \sqrt{n_{s1,s2}} e^{i\theta_{1,2}(x)}$ (macroscopic wave function)

- in insulator: sum of decaying and growing exponentials $\psi(x) = A \cosh(\kappa x) + B \sinh(\kappa x)$

- characteristic decay constant: $\kappa = \sqrt{2m_s(V_0 - E_0)/\hbar^2}$ for $E_0 < V_0$

- coefficients A and B are determined by the boundary conditions at $x = \pm d/2$:

$$\psi(x = -d/2) = \sqrt{n_{s1}} e^{i\theta_1} \quad \psi(x = +d/2) = \sqrt{n_{s2}} e^{i\theta_2}$$

$n_{1,2}, \theta_{1,2}$: Cooper pair density and wave function phase at the boundaries $x = \pm d/2$

$$\Rightarrow \sqrt{n_{s1}} e^{i\theta_1} = A \cosh(\kappa d/2) - B \sinh(\kappa d/2) \quad \sqrt{n_{s2}} e^{i\theta_2} = A \cosh(\kappa d/2) + B \sinh(\kappa d/2)$$

5.1.1 Superconducting Tunnel Junctions

- solving for A and B:
$$A = \frac{\sqrt{n_{s1}} e^{i\theta_1} + \sqrt{n_{s2}} e^{i\theta_2}}{\cosh(\kappa d/2)} \quad B = -\frac{\sqrt{n_{s1}} e^{i\theta_1} - \sqrt{n_{s2}} e^{i\theta_2}}{2 \sinh(\kappa d/2)} \quad \psi(x) = A \cosh(\kappa x) + B \sinh(\kappa x)$$

- supercurrent density:
$$\mathbf{J}_s = \frac{q_s \hbar}{2m_s l} (\psi \nabla \psi^* - \psi^* \nabla \psi)$$

- substituting the coefficients A and B (after some lengthy calculation):

$$\mathbf{J}_s = \mathbf{J}_c \sin(\theta_2 - \theta_1) = \mathbf{J}_c \sin \varphi$$

current-phase relation

$$\mathbf{J}_c = -\frac{q_s \hbar \kappa}{m_s} \frac{\sqrt{n_{s1} n_{s2}}}{\underbrace{2 \sinh(\kappa d/2) \cosh(\kappa d/2)}_{=\sinh(2\kappa d)}} = -\frac{q_s \hbar \kappa}{m_s} \frac{\sqrt{n_{s1} n_{s2}}}{\sinh(2\kappa d)}$$

maximum Josephson current density

- real junctions:**

$$V_0 \approx \text{few eV} \Rightarrow 1/\kappa < 1 \text{ nm}, \quad d \approx \text{few nm} \Rightarrow \kappa d \ll 1, \quad \text{then } \sinh(2\kappa d) \approx \frac{1}{2} \exp(2\kappa d), \quad q_s = -2e:$$

- maximum Josephson current density **decays exponentially** with increasing barrier thickness

d:

$$\mathbf{J}_c = \frac{2e\hbar\kappa}{m_s} 2\sqrt{n_{s1}n_{s2}} \exp(-2\kappa d) \quad q_s = -2e$$

5.1.1 Superconducting Tunnel Junctions

more elaborate theory of tunneling between superconductors

M. H. Cohen, L. M. Falicov, and J. C. Phillips, *Superconductive Tunneling*, Phys. Rev. Lett. **8**, 316-318 (1962).

B. D. Josephson, *Possible new effects in superconductive tunnelling*, Physics Letters **1**(7), 251-253 (1962)

- **open questions**

- what is the charge crossing the tunneling barrier when a Bogoliubov quasiparticle tunnels from S_1 to S_2
- what is the role of the coherence factors ?

- **only brief description of theoretical approach**

Hamiltonian: $\mathcal{H} = \mathcal{H}_L + \mathcal{H}_R + \mathcal{H}_T$

\mathcal{H}_T = tunneling hamiltonian, small extra term

$$\mathcal{H}_T = \sum_{\mathbf{k}\mathbf{q}} T_{LR} (c_{\mathbf{k}}^\dagger c_{\mathbf{q}} + c_{\mathbf{q}}^\dagger c_{\mathbf{k}})$$

transfer electrons from states \mathbf{k} on left to \mathbf{q} on rhs of the barrier and vice versa

matrix elements, fall of exponentially with barrier thickness d

determination of the tunneling current by calculation of $\langle \dot{\mathcal{N}}_L \rangle = -\langle \dot{\mathcal{N}}_R \rangle$ by using the equation of motion for $\mathcal{N}_{L,R}$

$$i\hbar \dot{\mathcal{N}}_{L,R} = [\mathcal{N}_{L,R}, \mathcal{H}] = [\mathcal{N}_{L,R}, \mathcal{H}_T] \quad \text{as } \mathcal{H}_L, \mathcal{H}_R \text{ commute with } \mathcal{N}_{L,R} \text{ (conserve particle number)}$$

5.1.1 Superconducting Tunnel Junctions

more elaborate theory of tunneling between superconductors

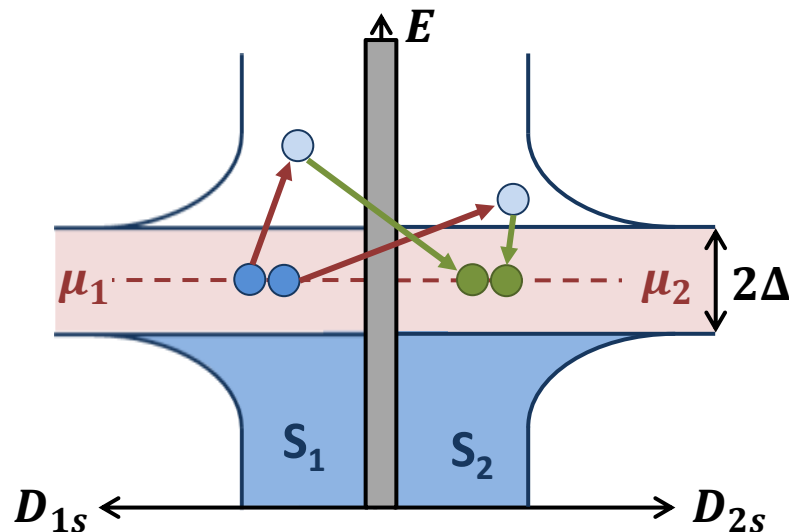
tunneling current:
$$J = q\dot{N} = \frac{q}{i\hbar} [\mathcal{N}, \mathcal{H}_T]$$

q = charge transported per tunneling particle

- NIN-junction:
$$J = \frac{e}{i\hbar} \sum_{\mathbf{kq}} T_{\mathbf{kq}} (c_{\mathbf{k}}^\dagger c_{\mathbf{q}} - c_{\mathbf{q}}^\dagger c_{\mathbf{k}})$$

for tunneling between two normal metals

- SIS-junction: we have to replace $c_{\mathbf{k},\mathbf{q}}^\dagger c_{\mathbf{k},\mathbf{q}}$ by the Bogoliubov quasiparticle excitation and annihilation operators $\alpha_{\mathbf{k},\mathbf{q}}^\dagger, \alpha_{\mathbf{k},\mathbf{q}}, \beta_{\mathbf{k},\mathbf{q}}^\dagger, \beta_{\mathbf{k},\mathbf{q}}$



- visualization of Josephson tunneling as a 2 step process
 - **step 1:** blue Cooper pair in S_1 is broken up and one excitation crosses barrier into S_2
 - resulting intermediate state (light blue) is classically forbidden but allowed within uncertainty relation
 - **step 2:** second excitation crosses barrier and recombines with the first forming green Cooper pair

- remark:** Josephson tunneling may be viewed as a second order process and $\propto |T|^4$. However, Josephson was assuming a constant (and not arbitrary) phase difference between initial and final states. Then, the quantum mechanical treatment yields a supercurrent $\propto |T|^2$

5.1.1 Superconducting Tunnel Junctions

tunneling in SIS junctions at finite voltage – quasiparticle tunneling

evaluation of $\langle \dot{\mathcal{N}} \rangle$ shows that coherence factors are not dropping out (cf. 4.4.2)

$$J_{qp} \propto -\frac{e|T|^2}{i\hbar} \sum_{\mathbf{k}\mathbf{q}} \frac{u_{\mathbf{k}}^2 u_{\mathbf{q}}^2 (f_{\mathbf{k}} - f_{\mathbf{q}})}{E_{\mathbf{k}} - E_{\mathbf{q}} + eV}$$

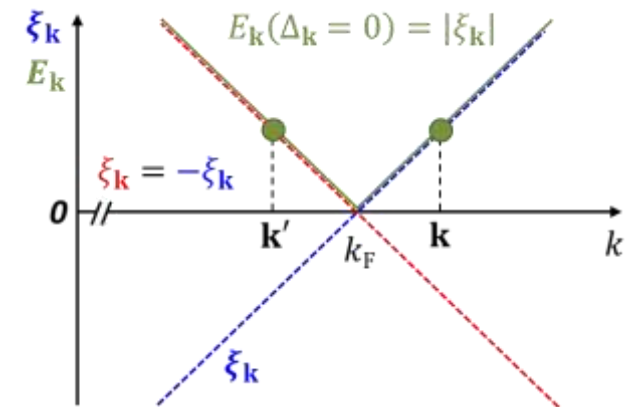
sum has to be taken over electron and hole branches on both sides
 → **coherence factors** all disappear (see argument given below)
 → sum → integration: principal part integrals all cancel, only residues at poles are left

- qualitative argument:

- tunneling from state $|\mathbf{q}\sigma\rangle$ into a state $|\mathbf{k}\sigma\rangle$ is only possible if pair state $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ is empty
 → resulting tunneling probability is $\propto |u_{\mathbf{k}}|^2 |T_{\mathbf{k}\mathbf{q}}|^2$

- for each state $|\mathbf{k}\sigma\rangle$ there exists a state $|\mathbf{k}'\sigma\rangle$ with $E_{\mathbf{k}} = E_{\mathbf{k}'}$ but with $\xi_{\mathbf{k}'} = -\xi_{\mathbf{k}}$

→ resulting tunneling probability is $\propto |u_{\mathbf{k}'}|^2 |T_{\mathbf{k}'\mathbf{q}}|^2 \stackrel{|u(-\xi_{\mathbf{k}})|=|v(\xi_{\mathbf{k}})|}{=} |v_{\mathbf{k}}|^2 |T_{\mathbf{k}'\mathbf{q}}|^2$



total tunneling probability $\propto (|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2) |T_{\mathbf{k}\mathbf{q}}|^2 = |T_{\mathbf{k}\mathbf{q}}|^2$ does not depend on coherence factors

→ *simple „semiconductor model“ for quasiparticle tunneling is applicable*

5.1.1 Superconducting Tunnel Junctions

tunneling in SIS junctions at zero voltage – Josephson tunneling

evaluation of $\langle \dot{\mathcal{N}} \rangle$ shows that coherence factors play an important role

$$J_s \propto -\frac{e|T|^2}{i\hbar} \sum_{\mathbf{k}\mathbf{q}} \frac{u_{\mathbf{k}}v_{\mathbf{k}}u_{\mathbf{q}}v_{\mathbf{q}}(f_{\mathbf{k}} - f_{\mathbf{q}}) e^{i\varphi}}{E_{\mathbf{k}} - E_{\mathbf{q}}}$$

- we have to sum up over electron and hole branches on both sides
- sum → integration: principal part integrals do no longer cancel
- leads to a finite Josephson current density $J_s = J_c \sin \varphi$ at zero voltage with

$$J_c = \frac{2e|T|^2}{\hbar A} D_{s1}(E_F)D_{s2}(E_F) \mathcal{P} \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2 \frac{\Delta_1}{E_1} \frac{\Delta_2}{E_2} \frac{f(E_1)f(E_2)}{E_1 - E_2}$$

\mathcal{P} = principal part

for $\Delta_1 = \Delta_2$:

$$J_c = \frac{\pi}{2eR_n A} \Delta(T) \tanh\left(\frac{\Delta(T)}{2k_B T}\right)$$

5.1.2 Ambegaokar-Baratoff Relation

important result of elaborate tunneling theory

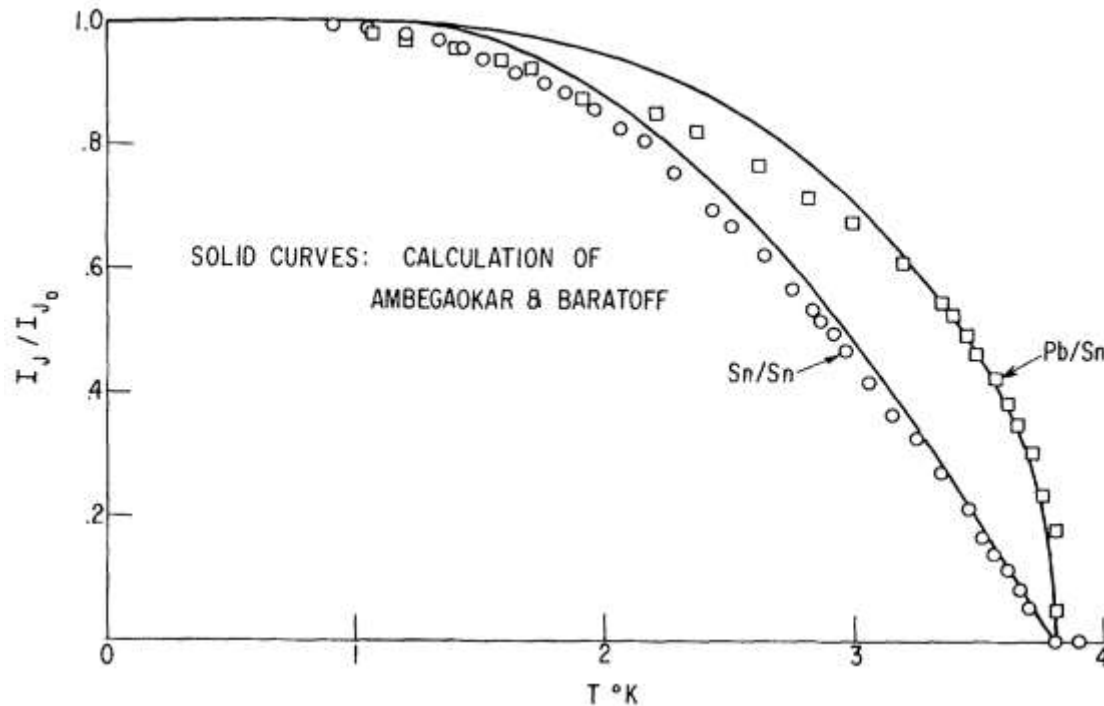
- ratio of maximum Josephson current density J_c and $J_{NIN} = J_{qp}(eV \gg 2\Delta) \propto 1/R_n A = const$

$$\rightarrow \frac{J_c}{J_{NIN}} = J_c R_n A = I_c R_n = const.$$

Ambegaokar-Baratoff relation:

$$I_c R_n = \frac{\pi}{2e} \Delta(T) \tanh\left(\frac{\Delta(T)}{2k_B T}\right)$$

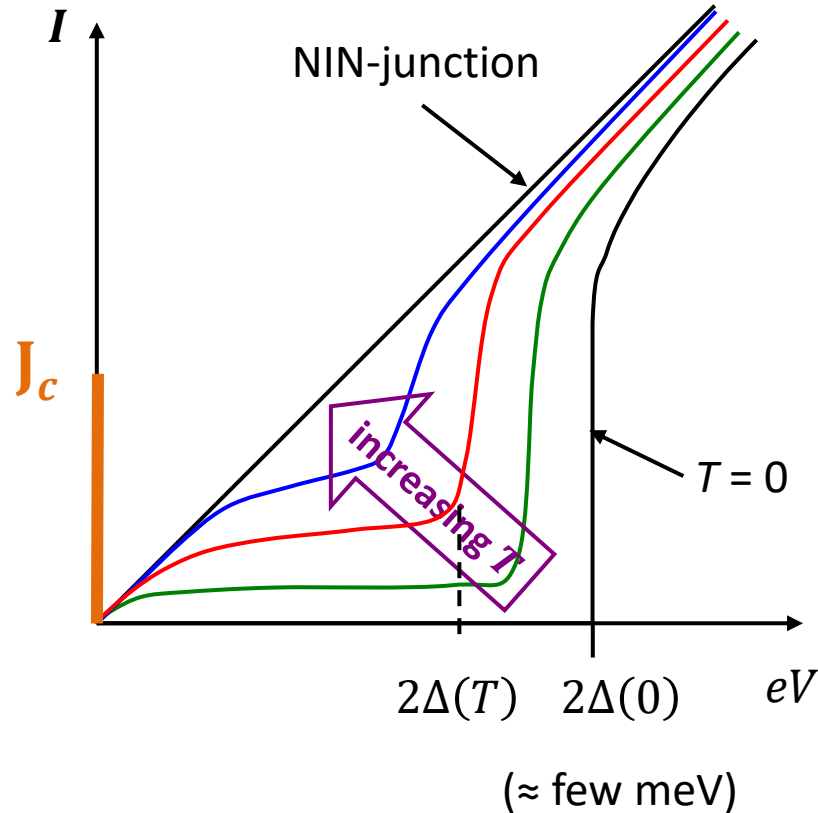
V. Ambegaokar, A. Baratoff, *Tunneling Between Superconductors*, Phys. Rev. Lett. **10**, 486-489 (1963).



M.D. Fiske, Rev. Mod. Phys. **36**, 221-222 (1964)
Temperature and Magnetic Field Dependences of the Josephson Tunneling Current

5.1.2 Ambegaokar-Baratoff Relation

current-voltage characteristics



- quasiparticle tunneling (cf. 4.4.2):

at $eV > 0$

$$\overline{J_s(t)} = J_c \overline{\sin(\varphi(t))} = J_c \overline{\sin\left(\frac{2eV}{\hbar}t\right)} = 0$$

at $eV \gg 2\Delta(T)$

$$J_{qp}(V) \simeq J_{NIN}(V) \propto \frac{1}{R_n A} \cdot \exp(-2\kappa d)$$

R_n = normal resistance $\hat{=}$ resistance of NIN tunnel junction

- Cooper pair tunneling:

at $eV = 0$

$$J_{qp}(V = 0) = 0$$

$$J_c(V = 0) = \frac{e\hbar\kappa}{m_s} 2\sqrt{n_{s1}n_{s2}} \cdot \exp(-2\kappa d)$$

5.2 Josephson Coupling Energy

Josephson coupling energy E_J : binding energy of two coupled superconductors (cf. 3.2.3)

- the two weakly coupled superconductors form “*molecule*” analogous to H_2 molecule
 → what is the **binding energy** of this molecule ?
- consider a JJ with $J_s = 0$ and then **increase junction current from zero to finite value**
 - phase difference has to change → phase change corresponds to finite voltage according to voltage-phase relation
 - external source has to supply energy (to accelerate the superelectrons)
 - stored in kinetic energy of moving superelectrons
 - integral of the supplied power $I \cdot V$ to increase current to $I(\varphi) = I_c \sin \varphi$ (voltage during increase of current)

$$\frac{E_J}{A} = \int_0^{t_0} J_s V dt = \int_0^{t_0} J_c \sin \varphi \left(\frac{\Phi_0}{2\pi} \frac{\partial \varphi}{\partial t} \right) dt = \frac{\Phi_0 J_c}{2\pi} \int_0^{\varphi} \sin \varphi' d\varphi' \quad \text{with } \varphi(0) = 0 \text{ and } \varphi(t_0) = \varphi$$

$A = \text{junction area}$

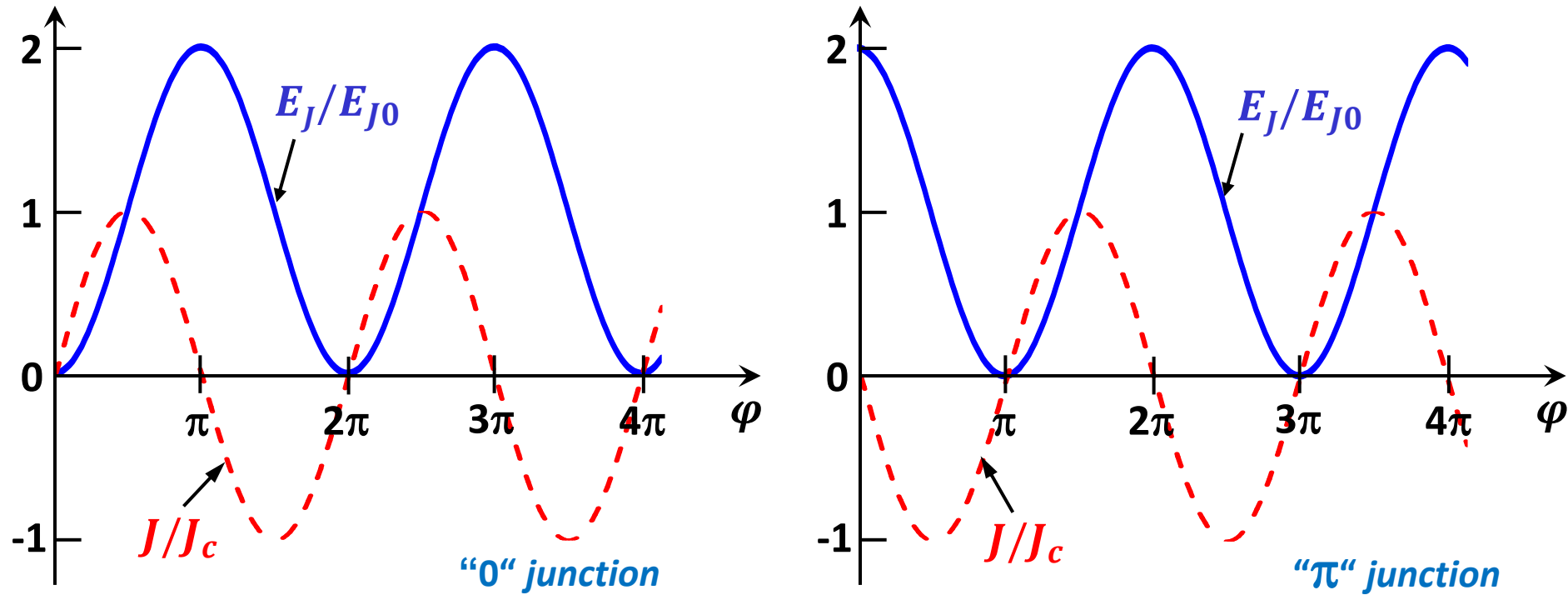
integration yields:

$$\frac{E_J}{A} = \frac{\Phi_0 J_c}{2\pi} (1 - \cos \varphi) = \frac{E_{J0}}{A} (1 - \cos \varphi)$$

Josephson coupling energy (per junction area)

5.2 Josephson Coupling Energy

Josephson coupling energy E_J (cf. 3.2.3)



- **order of magnitude estimate:**

- typically: $I_c = J_c A \sim 1 \text{ mA} \Rightarrow E_{J0} \approx 3 \times 10^{-19} \text{ J}$
- corresponds to thermal energy $k_B T$ for $T \approx 20 \text{ 000 K}$
- junction with very small critical current: $I_c \approx 1 \mu\text{A} \Rightarrow \text{thermal energy} \approx k_B \times 20 \text{ K}$

5.2.1 Josephson Junction with Applied Current

Josephson junction under the action of an external force (applied current)

• **potential energy** E_{pot} of the system under action of external force: $E_{\text{pot}} = E_J - F \cdot x$

- E_J : intrinsic free energy of the junction
- F : generalized force ($F = I$)
- x : generalized coordinate
- ⇒ $F \cdot \partial x / \partial t =$ power flowing into subsystem ($I \cdot V$)
- ⇒ $\partial x / \partial t = V$:

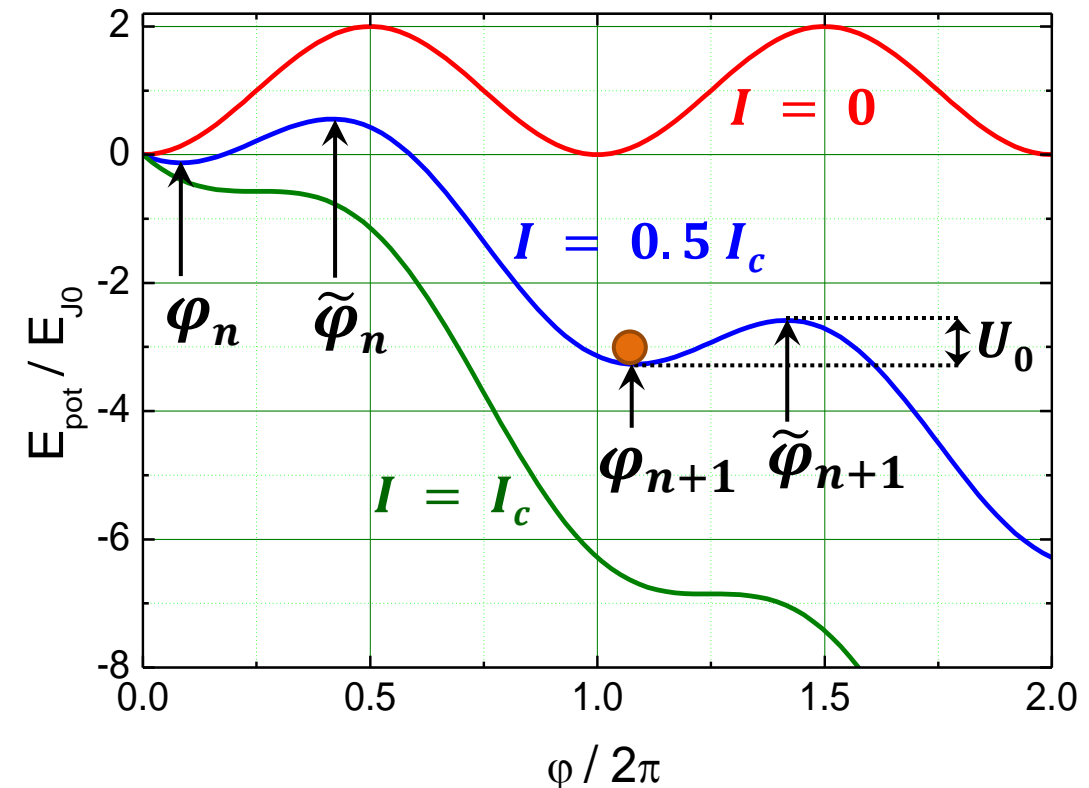
$$x = \int V dt = \frac{\hbar}{2e} \varphi + c = \Phi_0 \frac{\varphi}{2\pi} + c$$

$$E_{\text{pot}}(\varphi) = E_J(\varphi) - I \left(\Phi_0 \frac{\varphi}{2\pi} + c \right)$$

$$E_{\text{pot}}(\varphi) = \underbrace{\frac{\Phi_0 I_c}{2\pi}}_{E_{J0}} \left(1 - \cos \varphi - \frac{I}{I_c} \varphi \right) + \tilde{c}$$

tilted washboard potential

stable minima at φ_n , unstable maxima at $\tilde{\varphi}_n$,
states for different n are equivalent



• **junction dynamics: motion of “phase particle” φ in tilted washboard potential** (not discussed here)

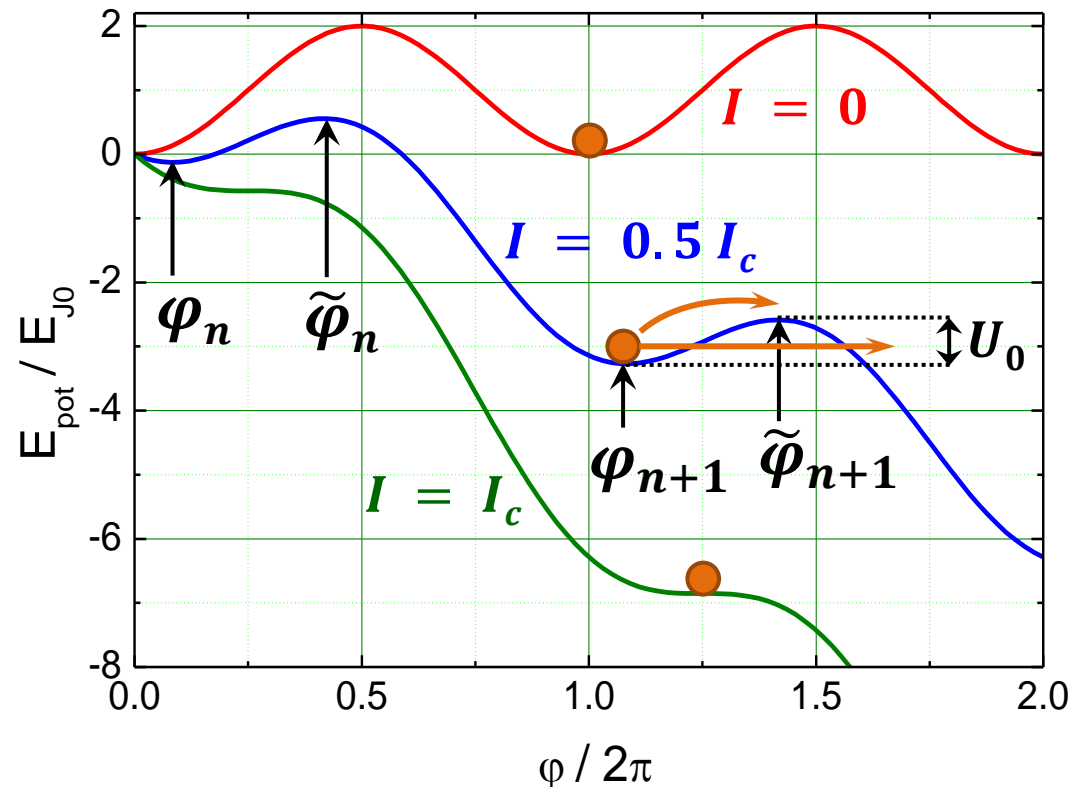
5.2.1 Josephson Junction with Applied Current

$|I| \leq I_c$: **constant phase difference**: $\varphi = \varphi_n = \arcsin(I/I_c) + 2\pi n$

→ zero voltage state / ordinary (S) state

$|I| > I_c$: **phase difference increases with time**: $\varphi = \varphi(t)$

→ finite voltage state / running phase state



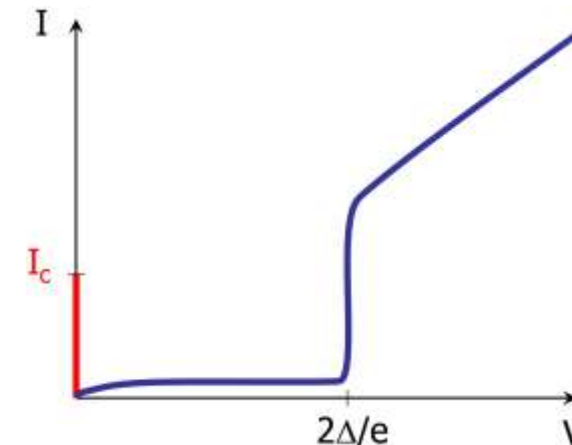
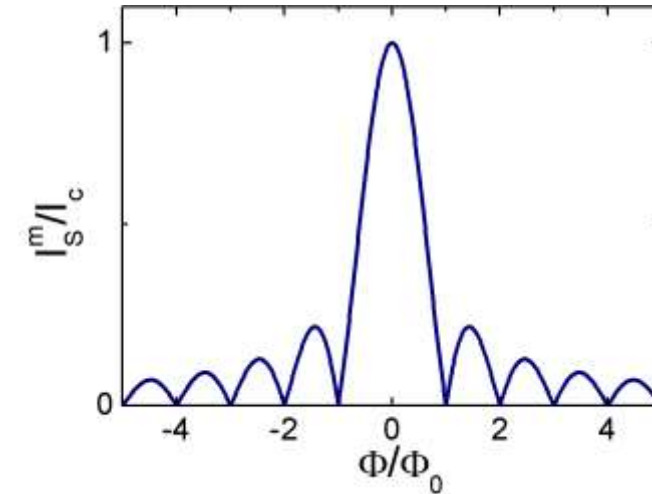
- thermally activated phase slippage
- quantum tunneling of phase

5.3 Applications of the Josephson

large number of applications in analog and digital electronics

→ detailed discussion in lecture „*Applied Superconductivity*“

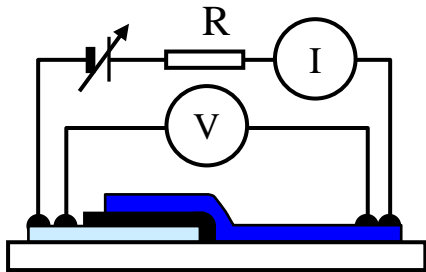
- $I_S^m = I_S^m(B)$:
→ *magnetic field sensors (SQUIDs)*
- $\beta_C \gg 1$ (hysteretic IVC)
→ **bistability**: zero/voltage state
→ *switching devices, Josephson computer, fast DACs*
- 2nd Josephson equation
→ *voltage controlled oscillator, voltage standard*
- nonlinear IVC
→ *mixers up to THz, oscillators*
- macroscopic quantum behavior
→ *superconducting qubits*



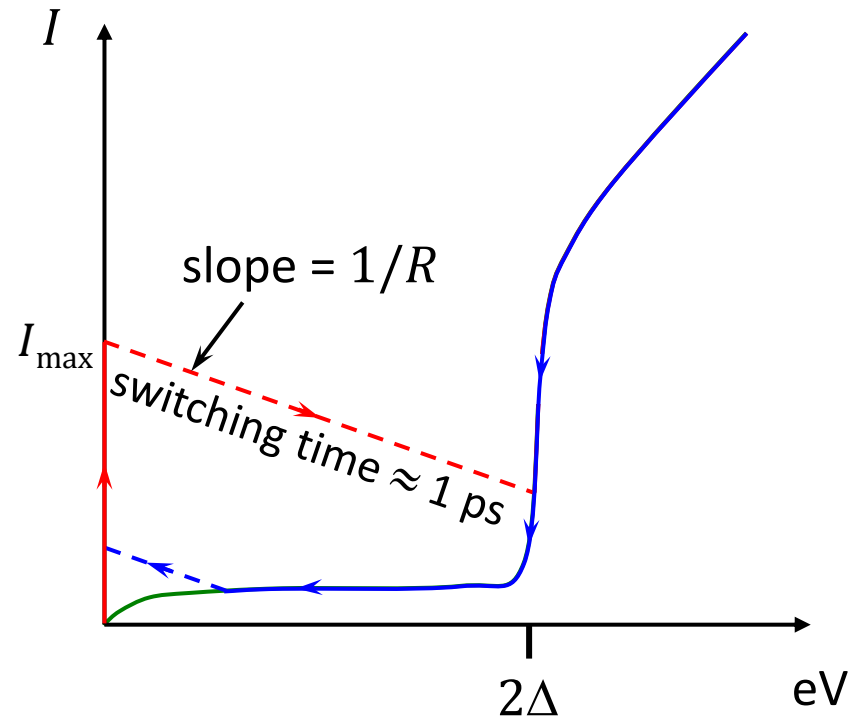
5.3 Applications of the Josephson Effect

Josephson junction as fast switching device

- $V = 0$: Josephson current
- $V \neq 0$: quasiparticle current



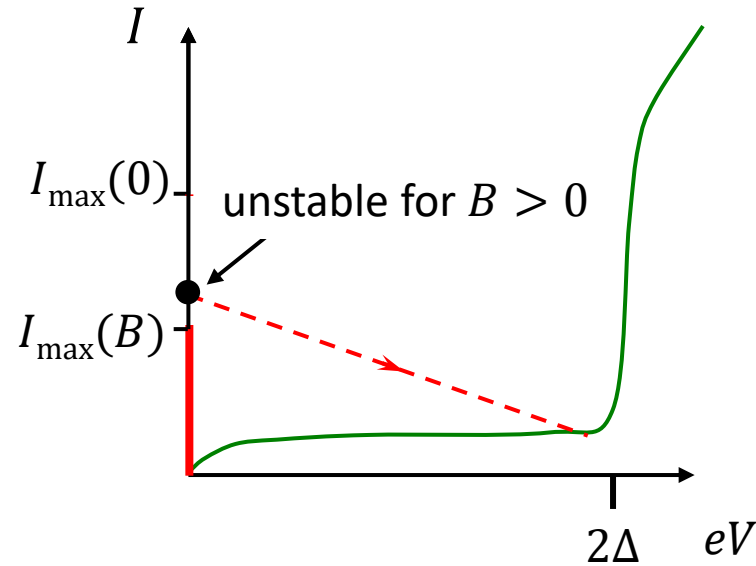
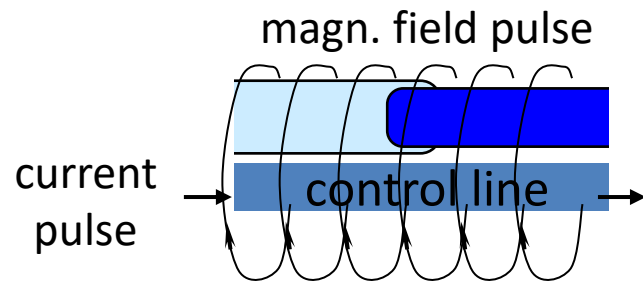
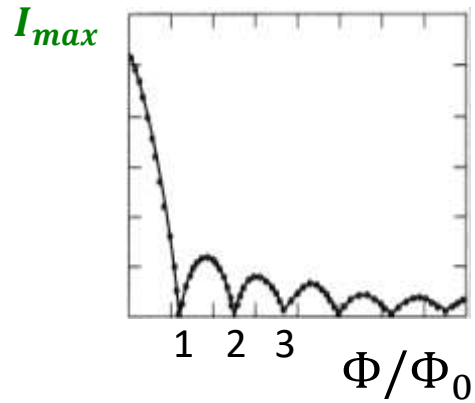
- hysteresis:
 - fast switching device
 - very low power consumption
 - ⇒ **Josephson digital electronics**



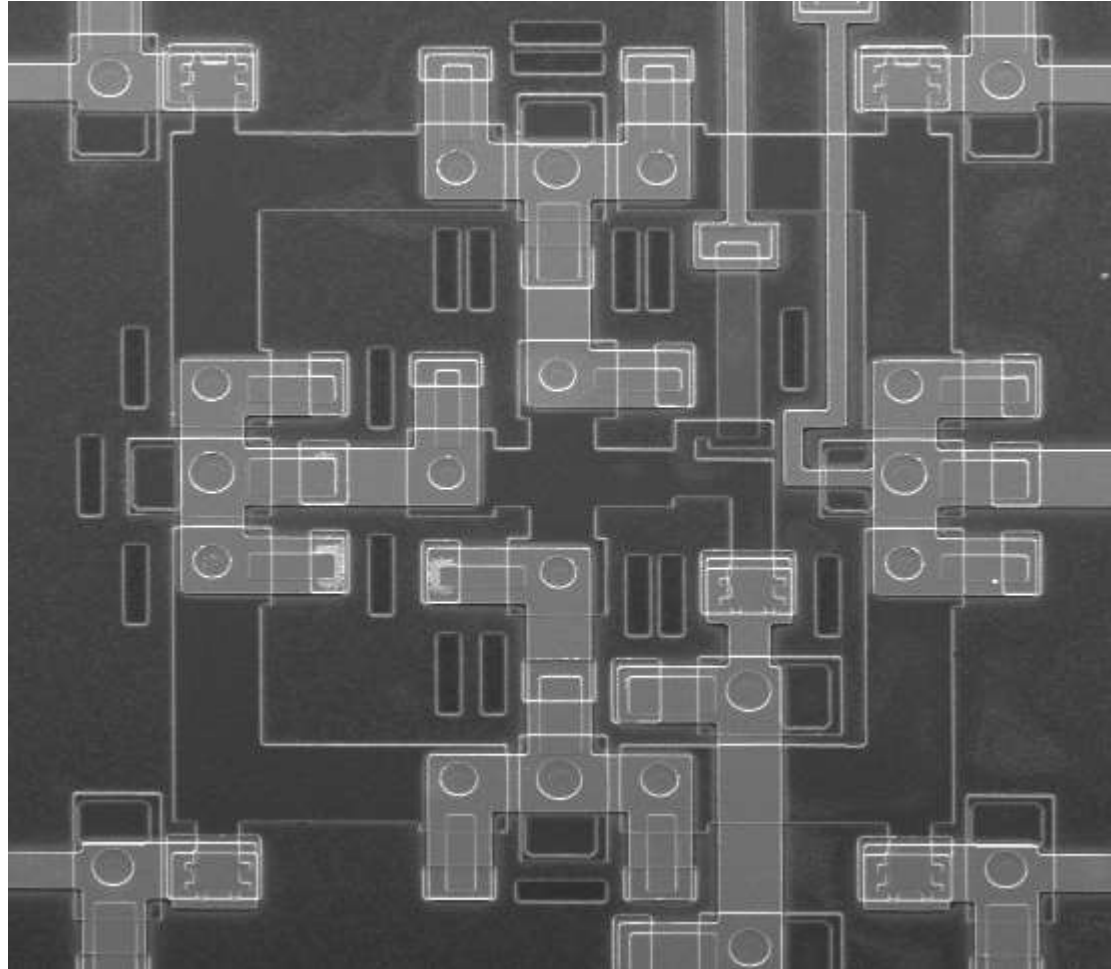
5.3 Applications of the Josephson Effect

principle of switching element:

- magnetic field dependence of the maximum Josephson current



5.3 Applications of the Josephson Effect

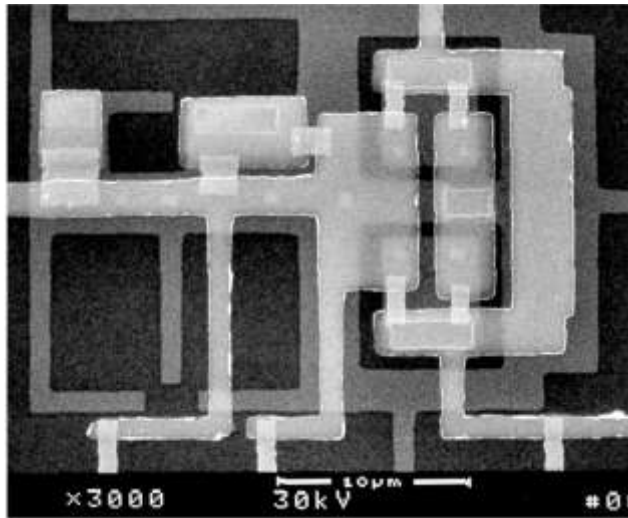
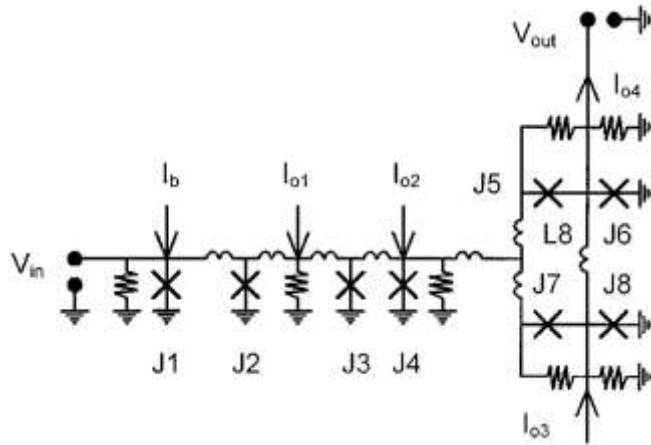


SEM micrograph of a universal asynchronous DR RSFQ (rapid single flux quantum) logic gate

B. Dimov et al.,
Universal asynchronous RSFQ gate for realization of Boolean functions of dual-rail binary variables
[Journal of Physics Conference Series](#) 43(1), 1183 (2006)

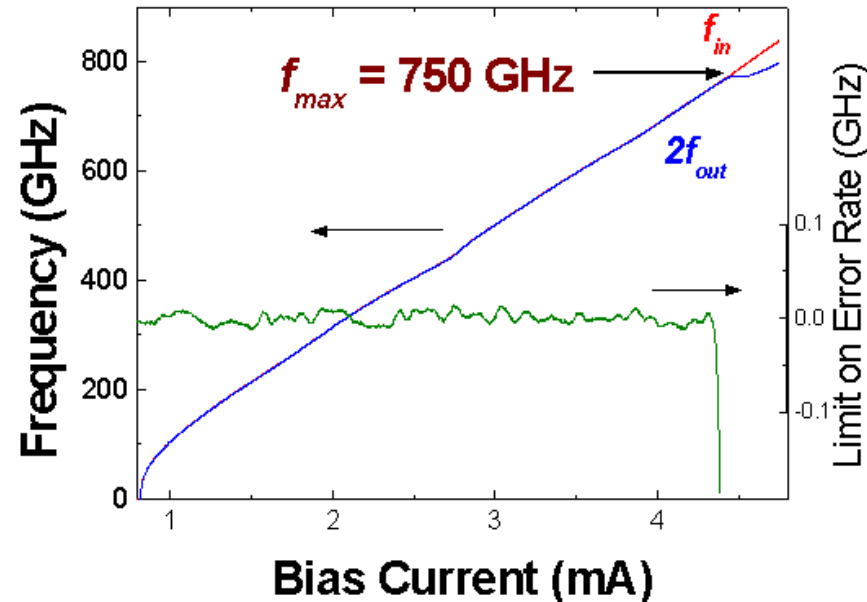
5.3 Applications of the Josephson Effect

superconductor digital frequency divider operating up to 750 GHz



Dividers

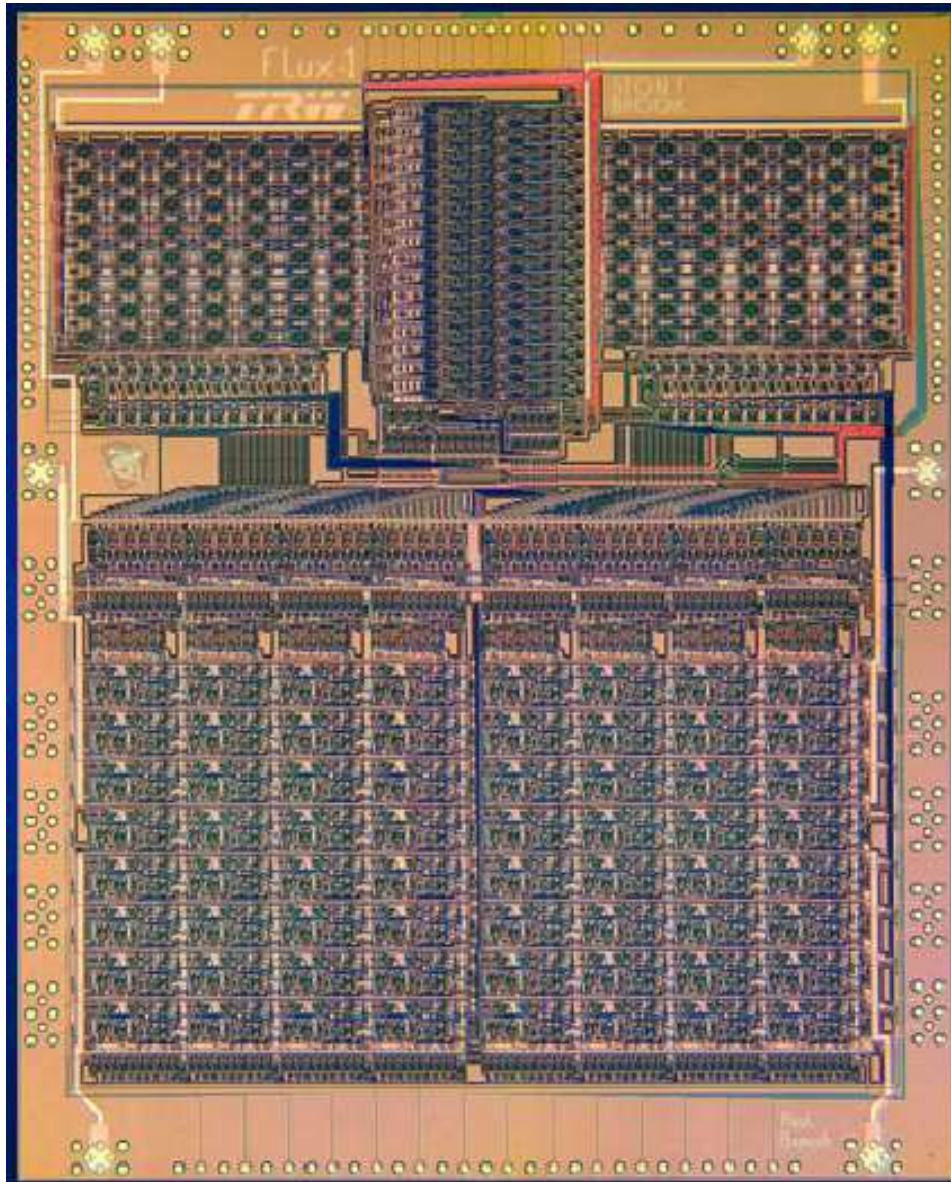
	RSFQ	Semi-conductor
Frequency	750 GHz	60 GHz
Power Dissipation	1.5 μ W	0.5 W



W. Chen, A. Rylyakov, V. Patel, J. Lukens, K. Likharev,
 "Superconductor digital frequency divider operating up to 750 GHz,"
 Appl. Phys. Lett. **73**, 2817 (1998)

- problem: integration of large number of JJs ($> 10^5$) with high yield and small parameter spread

5.3 Applications of the Josephson Effect



Stony Brook

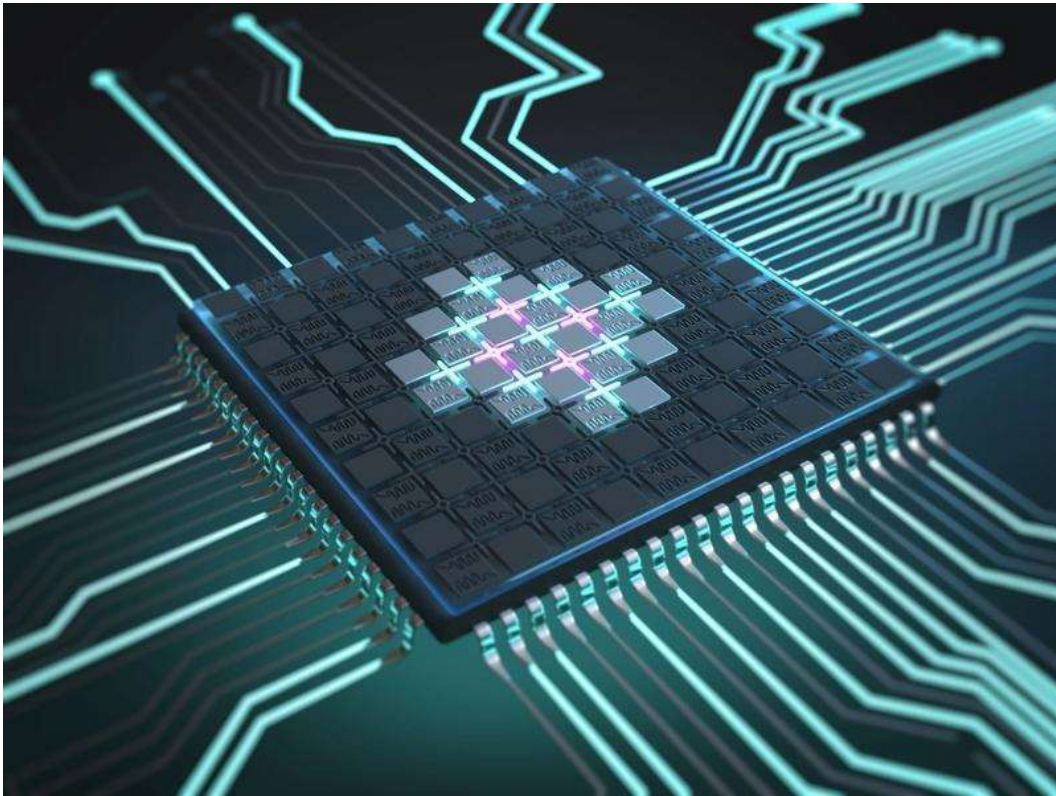
FLUX-1

- the first RSFQ MPU
- 8 bit ALU array
- 16 word instruction memory
- 70,000 JJs
- 14 mW
- 20-22 GHz @ $F = 2.0 \text{ um}$
($\Rightarrow 120\text{-}140 \text{ GHz @ } 0.3 \text{ um}$)
- TRW's 4-metal process

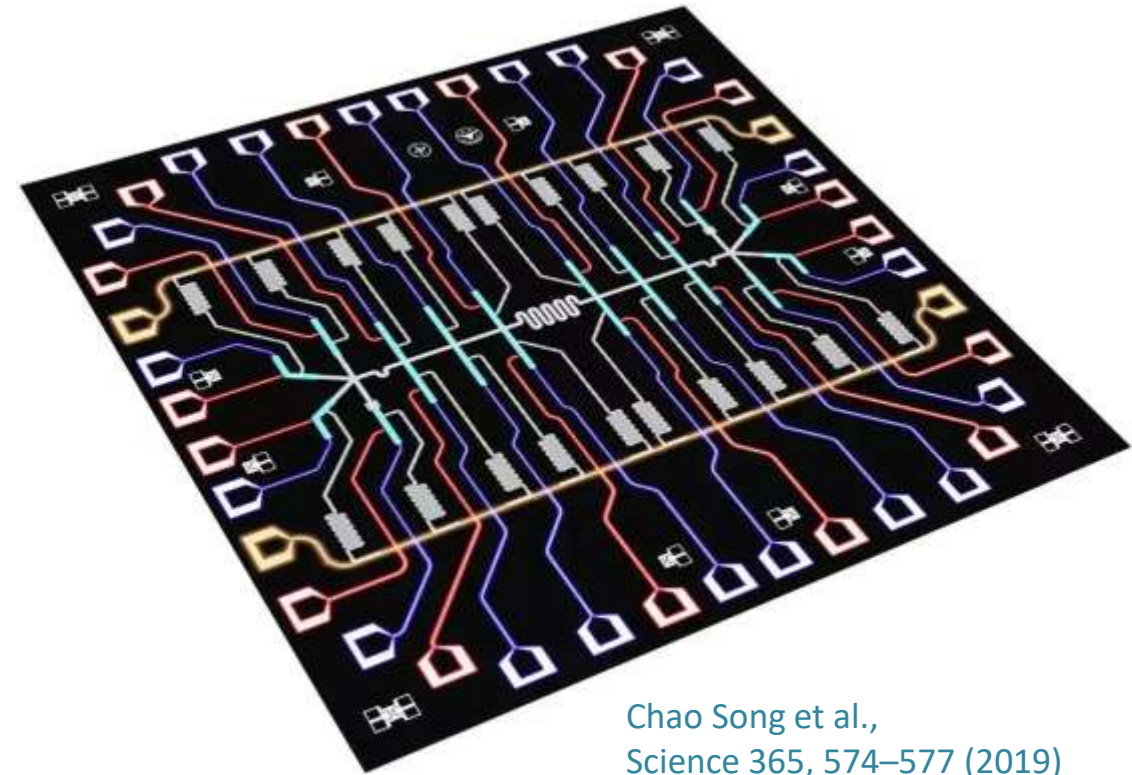
www.physics.sunysb.edu/Physics/RSFQ/

5.3 Applications of the Josephson Effect

superconducting quantum bits

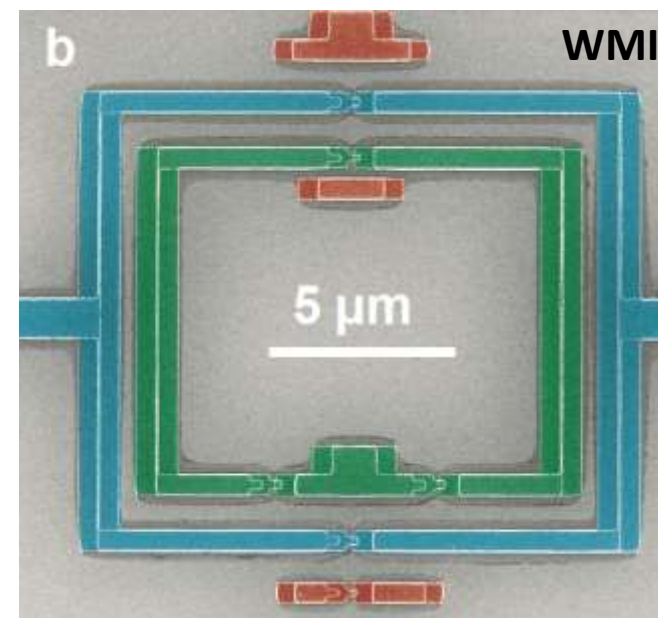
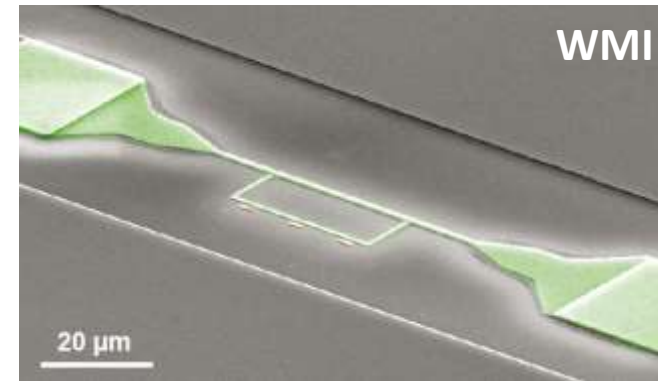
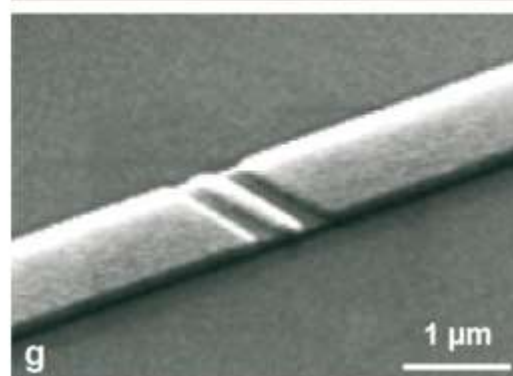
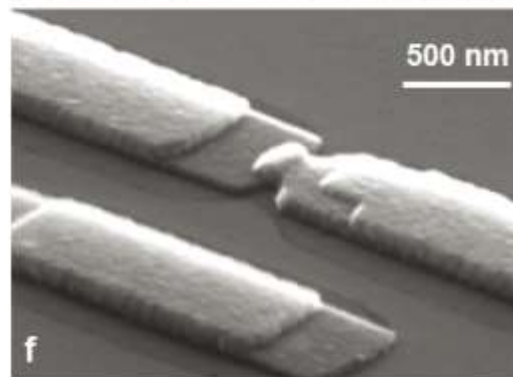
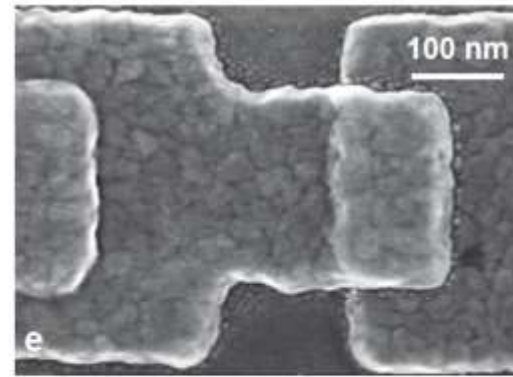
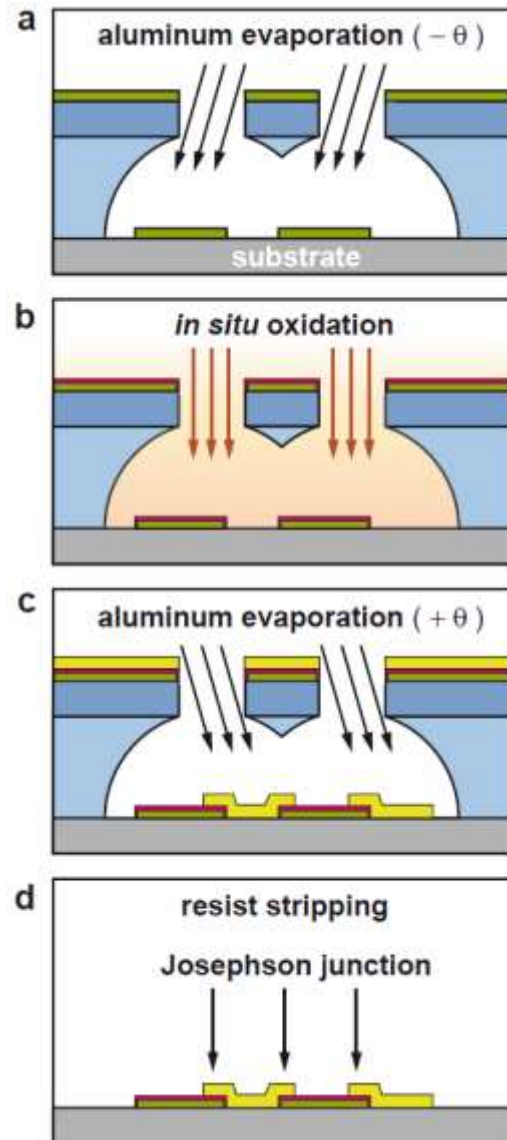


WMI/MCQST



Chao Song et al.,
Science 365, 574–577 (2019)

5.3 Applications of the Josephson Effect



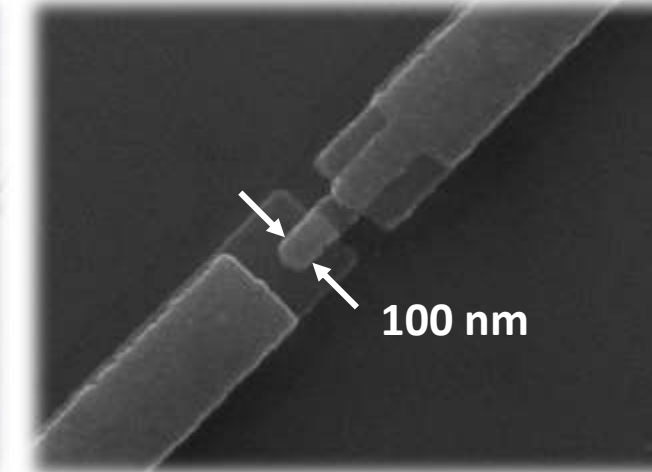
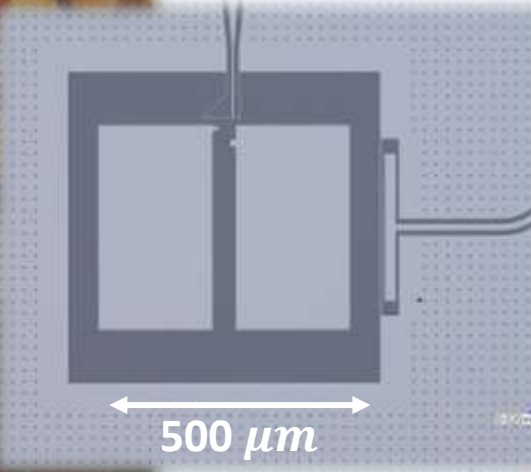
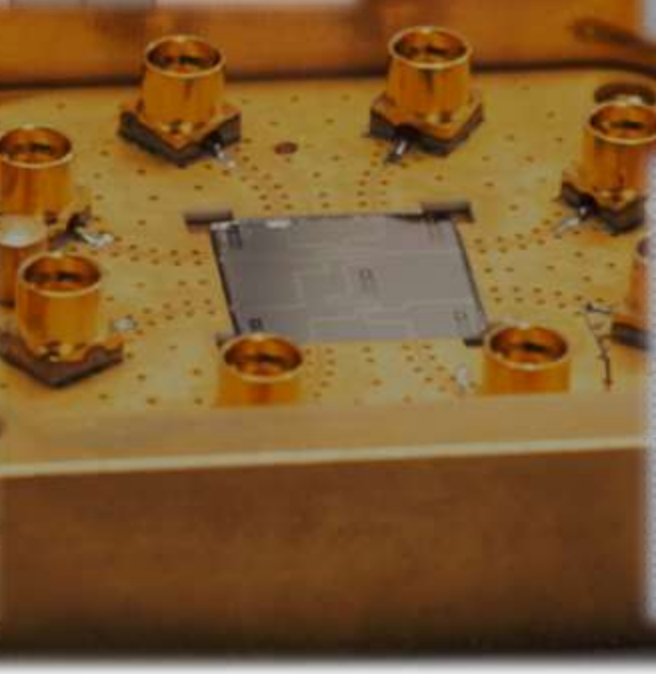
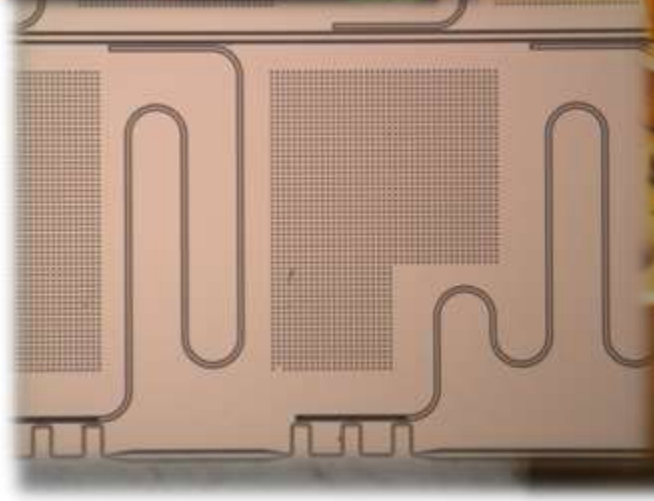
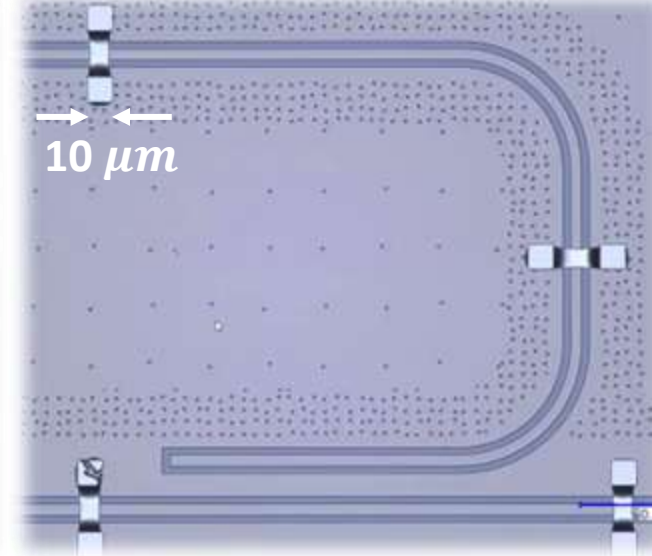
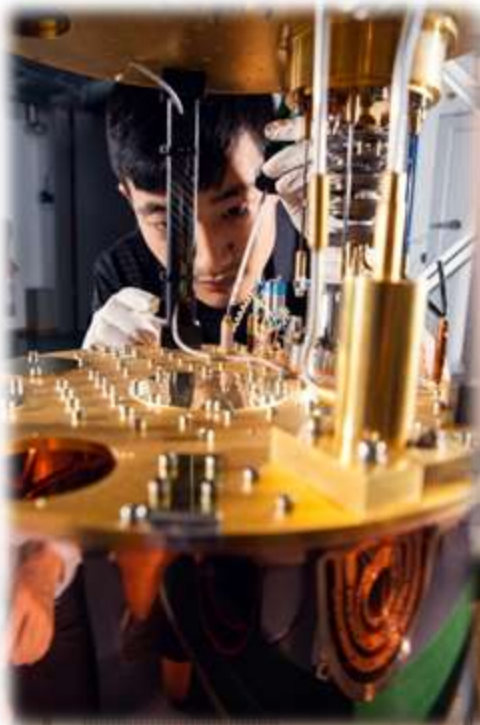
F. Deppe et al., PRB **76**, 214503 (2007)

T. Niemczyk et al., SUST **22**, 034009 (2009)

superconducting flux quantum bits fabricated at WMI

5.3 Applications of the Josephson Effect

Technology @ WMI



5.3 Quantum Computing Start-ups

Dr Jan Goetz | CEO

IQM Finland Oy



IQM: Millionen für Quantencomputer

Co-founder and CEO Jan Goetz studied physics at TUM and did his master and Ph.D. thesis at WMI (2011 - 2017)



- 2018:** starting capital: 11.5 Mio. €
- 2019:** IQM raises 71 Mio. €
- 2022:** IQM's quantum fabrication facility gets a 35 Mio. € boost from the European Investment Bank
- 2022:** IQM raises 128 Mio. € led by World Fund

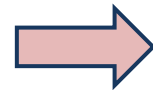
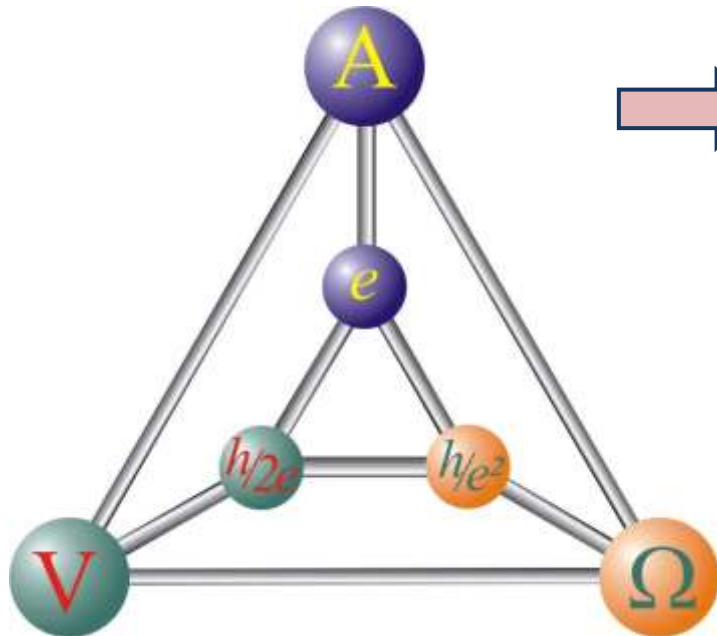
TUM Venture Lab Quantum



5.3 Applications of the Josephson Effect

precise definition of the electrical voltage, current and the resistance by using fundamental quantum effects

- **Josephson effect:** $V = \frac{h}{2e} \cdot f = \Phi_0 \cdot f$ (relation between voltage and time/frequency by flux quantum)
- **Single electron pump:** $I = e \cdot f$ (relation between current and time by charge quantum)
- **Quantum Hall effect:** $V = \frac{h}{e^2} \cdot I = R_K \cdot I$ (relation between voltage and current by quantum resistance, unit = 1 Klitzing)



allows the reproduction of the physical units Volt, Ampère and Ohm with a very high precision and largely uninfluenced by environmental parameters at any place in the world

realization of the Ampère by single electron pump not realized so far at sufficient precision

→ would allow an important experimental test of the consistency of the relations between the fundamental constants illustrated in the “*electrical triangle*”

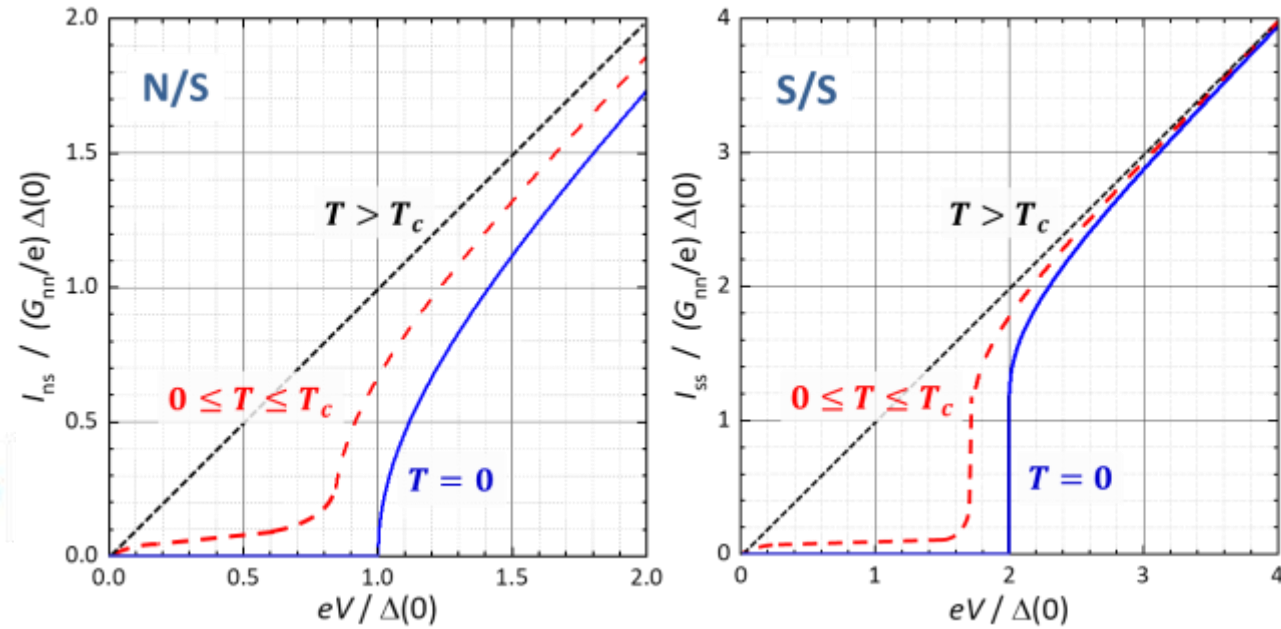
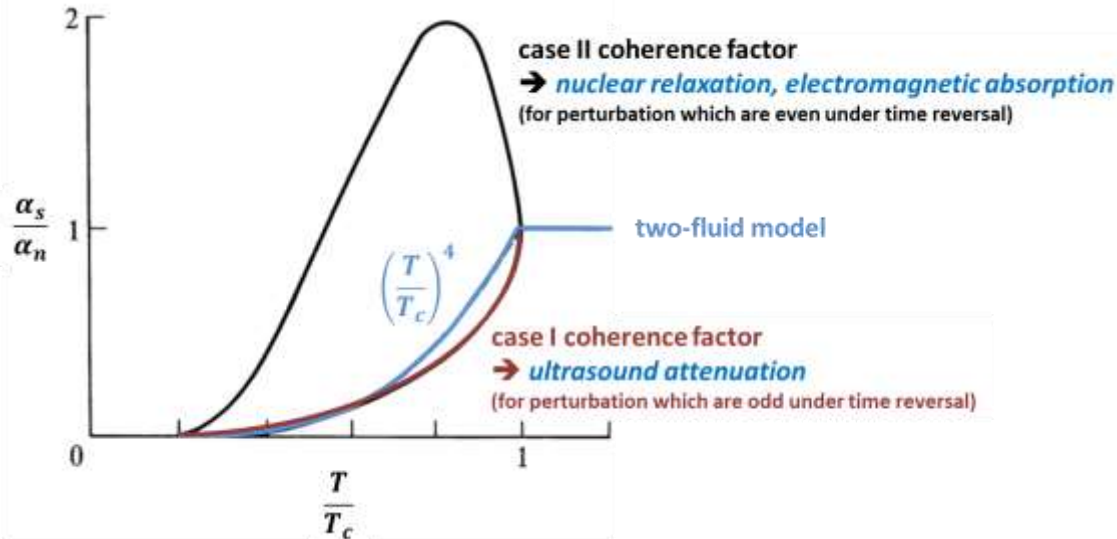
Summary of Lecture No. 10 (1)

- determination of energy gap and DOS by tunneling spectroscopy

$$I_{ns}(V) = \frac{G_{nn}}{e} \int_{-\infty}^{\infty} \frac{D_{s2}(E)}{D_{n2}(E_F)} [f(E) - f(E + eV)] dE \quad \Rightarrow \quad G_{ns}(V) = \frac{\partial I_{ns}(V)}{\partial V} = G_{nn} \frac{D_{s2}(eV)}{D_{n2}(E_F)} \propto D_{s2}(eV) \quad @ T = 0$$

- NIS and SIS tunnel junctions

- BCS coherence factors



Summary of Lecture No. 10 (2)

- **derivation of Josephson equations from tunneling theory**

- starting point is time-dependent Schrödinger equation $i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = E \psi(\mathbf{r}, t)$

$$\psi_{1,2}(\mathbf{r}, t) = \psi_{01,02} e^{i\theta_{1,2}(\mathbf{r}, t)}$$

$$n_{s1,s2} = |\psi_{01,02}|^2$$

- **finite coupling** between S_1 and S_2 is introduced by small coupling constant C (tunnel coupling)

$$i\hbar \frac{\partial \psi_1(\mathbf{r}, t)}{\partial t} = E_1 \psi_1(\mathbf{r}, t) + C_{LR} \psi_2(\mathbf{r}, t) = +e\Delta\phi \psi_1(\mathbf{r}, t) + C_{LR} \psi_2(\mathbf{r}, t)$$

$$i\hbar \frac{\partial \psi_2(\mathbf{r}, t)}{\partial t} = E_2 \psi_2(\mathbf{r}, t) + C_{RL} \psi_1(\mathbf{r}, t) = -e\Delta\phi \psi_2(\mathbf{r}, t) + C_{RL} \psi_1(\mathbf{r}, t)$$

$$\Delta\phi = \frac{E_1 - E_2}{|q_s|} = \frac{E_1 - E_2}{2e} = \frac{1}{2} \frac{E_1 - E_2}{e}$$

$$\Rightarrow J_s = J_s^{1 \rightarrow 2} - J_s^{2 \rightarrow 1} = \frac{4eC}{\hbar A} \sqrt{N_{s1} N_{s2}} \sin \varphi(t) = J_c \sin \varphi(t)$$

1st Josephson equation
(current-phase relation)

$$N_{s1,s2} = V_{1,2} n_{s1,s2}$$

$$C_{LR} = C_{RL} = C$$

- the 2nd Josephson equation is obtained from the gauge invariant phase difference $\varphi(\mathbf{r}, t) = \theta_2(\mathbf{r}, t) - \theta_1(\mathbf{r}, t) - \frac{2\pi}{\Phi_0} \int_1^2 \mathbf{A}(\mathbf{r}, t) \cdot d\ell$.

$$\frac{\partial \varphi(t)}{\partial t} = \frac{\partial \theta_2(t)}{\partial t} - \frac{\partial \theta_1(t)}{\partial t} - \frac{2e}{\hbar} \frac{\partial}{\partial t} \int_1^2 \mathbf{A}(\mathbf{r}, t) \cdot d\ell$$

$$\Rightarrow \frac{\partial \varphi(t)}{\partial t} = \frac{2e}{\hbar} V = \frac{2\pi}{\Phi_0} V$$

2nd Josephson equation
(voltage-phase relation)

$$\frac{\partial \theta_1(t)}{\partial t} = -\frac{C}{\hbar} \sqrt{\frac{n_{s2}}{n_{s1}}} \cos \varphi(t) - \frac{e\Delta\phi}{\hbar}$$

$$\frac{\partial \theta_2(t)}{\partial t} = -\frac{C}{\hbar} \sqrt{\frac{n_{s1}}{n_{s2}}} \cos \varphi(t) + \frac{e\Delta\phi}{\hbar}$$

Summary of Lecture No. 10 (3)

- **maximum Josephson current density J_c**

- can be calculated by e.g. wave matching method

$$J_c = \frac{2e\hbar\kappa}{m_s} 2\sqrt{n_{s,1}n_{s,2}} \exp(-2\kappa d)$$

- Ambegaokar-Baratoff relation:

$$J_c \cdot \frac{R_n}{A} = \frac{\pi}{2e} \Delta(T) \tanh\left(\frac{\Delta(T)}{2k_B T}\right)$$

- **Josephson junction with applied current**

$$E_{\text{pot}}(\varphi) = E_{J0} \left(1 - \cos \varphi - \frac{I}{I_c} \varphi \right) \quad \text{tilted washboard potential}$$

- **many applications in digital and analog electronics**

- magnetic field sensors (SQUIDs)
- switching devices, RSFQ logic, fast DACs
- voltage controlled oscillator, voltage standard
- mixers up to THz frequencies
- superconducting qubits

