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Exercise 10

Exercise to the Lecture

Superconductivity and Low Temperature Physics I
WS 2021/2022

4 Microscopic Theory

4.9 Particle Current Density

Exercise:

Within Boltzmann transport theory the particle current density in a metal can be written as

$$\mathbf{J} = \frac{1}{V}\sum_{\mathbf{k}\sigma}\frac{\hbar\mathbf{k}}{m}\left[f\left(\varepsilon_{\mathbf{k}}\right) - f_0\left(\varepsilon_{\mathbf{k}}\right)\right] = \frac{1}{V}\sum_{\mathbf{k}\sigma}\frac{\hbar\mathbf{k}}{m}\left[-\frac{\partial f_0\left(\varepsilon_{\mathbf{k}}\right)}{\partial\varepsilon_{\mathbf{k}}}\,\delta\varepsilon_{\mathbf{k}}\right] = n\mathbf{v}$$

$$\text{with } f_0\left(\varepsilon_{\mathbf{k}}\right) = \frac{1}{\mathrm{e}^{(\varepsilon_{\mathbf{k}}-\mu)/k_{\mathrm{B}}T}+1} \; .$$

Here, n is the particle density, $\mathbf{v}$ the particle drift velocity, $\varepsilon_{\mathbf{k}} = \hbar^2k^2/2m$, $\delta\varepsilon_{\mathbf{k}} = \hbar\mathbf{k}\cdot\mathbf{v}$, and $f_0\left(\varepsilon_{\mathbf{k}}\right)$ the thermal equilibrium Fermi-Dirac distribution.

(a) Use the above definition of the particle current density to derive the particle density in a normal metal.

(b) Calculate the density n^{qp} of Bogoliubov quasiparticles in a superconductor (normal fluid density).

(c) Calculate the density n^{s} of the paired electrons in a superconductor (superfluid density density).

(d) Use the superfluid density to discuss the temperature dependence of the London penetration depth λ_{L} close to the transition temperature T_c.

Solution:

(a) The i^{th} component ($i = 1, 2, 3$) of the particle current density can be written as

$$\begin{aligned} J_i &= \frac{1}{V}\sum_{\mathbf{k}\sigma}\frac{\hbar k_i}{m}\left[-\frac{\partial f_0\left(\varepsilon_{\mathbf{k}}\right)}{\partial \varepsilon_{\mathbf{k}}}\hbar k_j v_j\right] \\ &= n_{ij} v_j \ . \end{aligned} \tag{1}$$

With this result we can define the particle current density in a normal metal as

$$n_{ij} = \frac{1}{V}\sum_{\mathbf{k}\sigma} y_{\mathbf{k}} \frac{\hbar^2 \mathbf{k}\mathbf{k}}{m} \ , \tag{2}$$

where $\mathbf{kk}$ is the dyadic product and

$$y_{\mathbf{k}} = -\frac{\partial f_0\left(\varepsilon_{\mathbf{k}}\right)}{\partial \varepsilon_{\mathbf{k}}} = \frac{1}{2k_{\mathrm{B}}T}\frac{1}{\cosh(\varepsilon_{\mathbf{k}}/k_{\mathrm{B}}T)+1} = \frac{1}{4k_{\mathrm{B}}T}\frac{1}{\cosh^2(\varepsilon_{\mathbf{k}}/k_{\mathrm{B}}T)} \ . \tag{3}$$

To evaluate (2) we have to convert the summation into an integration. We replace $\varepsilon_{\mathbf{k}}$ by $\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu$ in (3) and obtain

$$\begin{aligned} n_{ij} &= \frac{1}{V}\sum_{\mathbf{k}\sigma}\left(-\frac{\partial f_0(\xi_{\mathbf{k}}+\mu)}{\partial \xi_{\mathbf{k}}}\right)\frac{\hbar^2 \mathbf{k}\mathbf{k}}{m} \\ &= \frac{1}{V}\frac{\hbar^2 k_{\mathrm{F}}^2}{m}\underbrace{\int \frac{d\Omega_{\mathbf{k}}}{4\pi}\widehat{\mathbf{k}}\widehat{\mathbf{k}}}_{=\frac{1}{3}\delta_{ij}}\int_{-\mu}^{\infty} d\xi_{\mathbf{k}} D(\mu+\xi_{\mathbf{k}})\left(-\frac{\partial f_0(\mu+\xi_{\mathbf{k}})}{\partial \xi_{\mathbf{k}}}\right) \\ &\simeq \frac{1}{3}\delta_{ij}\underbrace{\frac{D(E_{\mathrm{F}})\hbar^2 k_{\mathrm{F}}^2}{mV}}_{=3n}\frac{1}{4k_{\mathrm{B}}T}\int_{-\mu}^{\infty}\frac{d\xi_{\mathbf{k}}}{\cosh^2\frac{\xi_{\mathbf{k}}}{2k_{\mathrm{B}}T}} \\ &\overset{x=\xi_{\mathbf{k}}/2k_{\mathrm{B}}T}{=} \frac{1}{2}n\delta_{ij}\underbrace{\int_{-\infty}^{\infty}\frac{dx}{\cosh^2 x}}_{=\tanh(\infty)-\tanh(-\infty)=2} = n\delta_{ij} \ . \end{aligned} \tag{4}$$

Here, $\widehat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ and $|\mathbf{k}| \simeq k_{\mathrm{F}}$. Furthermore, we have used $D(\mu+\xi_{\mathbf{k}}) \simeq D(E_{\mathrm{F}}) = const$, since the function $\partial f_{\mathbf{k}}^0/\partial E_{\mathbf{k}}$ is finite only in a narrow energy interval $\sim k_{\mathrm{B}}T$ around the chemical potential μ, and we have set the lower integration limit to $-\infty$, since typically $\mu/2k_{\mathrm{B}}T \gg 1$ for a metal. Obviously we obtain the expected result that the particle density is given by the electron density of the normal metal.

(b) We next consider the superconducting state of a metal and calculate the normal fluid density n^{qp} of the Bogoliubov quasiparticles. To calculate n^{qp} we can use eqs. (2) and (3) but have to replace $\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu$ by the quasiparticle energy $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}$ in (3). With

$Z(k)d^3k = D(E_{\mathbf{k}})dE_{\mathbf{k}} = D(\xi_{\mathbf{k}})d\xi_{\mathbf{k}}$ (conservation of states) we obtain

$$
\begin{aligned}
n_{ij}^{\mathrm{qp}} &= \frac{1}{V}\sum_{\mathbf{k}\sigma} y_{\mathbf{k}} \frac{\hbar^2 \mathbf{k}\mathbf{k}}{m} \\
&= \frac{1}{V}\frac{\hbar^2 k_{\mathrm{F}}^2}{m} \underbrace{\int \frac{d\Omega_{\mathbf{k}}}{4\pi}\widehat{\mathbf{k}}\widehat{\mathbf{k}}}_{=\frac{1}{3}\delta_{ij}} \int_{-\mu}^{\infty} d\xi_{\mathbf{k}} D(\mu+\xi_{\mathbf{k}}) \left(-\frac{\partial f_{\mathbf{k}}}{\partial E_{\mathbf{k}}}\right) \\
&\simeq \frac{1}{3}\delta_{ij} \underbrace{\frac{D(E_{\mathrm{F}})\hbar^2 k_{\mathrm{F}}^2}{mV}}_{=3n} \frac{1}{4k_{\mathrm{B}}T} \int_{-\mu}^{\infty} \frac{d\xi_{\mathbf{k}}}{\cosh^2 \frac{E_{\mathbf{k}}}{2k_{\mathrm{B}}T}} \\
&\overset{x=\xi_{\mathbf{k}}/2k_{\mathrm{B}}T}{=} \frac{1}{2} n\delta_{ij} \underbrace{\int_{-\infty}^{\infty} \frac{dx}{\cosh^2 \sqrt{x^2 + \left(\frac{\Delta(T)}{2k_{\mathrm{B}}T}\right)^2}}}_{=2Y(T)} \\
&= n\delta_{ij}\, Y(T) \quad . \qquad (5)
\end{aligned}
$$

We see that the normal fluid density is given by the normal state particle density multiplied by the Yosida function $Y(T)$ (cf. Fig. 1). The Yosida function is zero at $T = 0$ and continuously increases towards one at $T = T_c$. Therefore, the quasiparticle density decreases from $n_{ij}^{\mathrm{qp}}(T) = n\delta_{ij}$ at $T = T_c$ to $n_{ij}^{\mathrm{qp}}(T) = 0$ at $T = 0$.

(c) Since the total particle number is conserved on going from the normal to the superconducting state, the density of the paired electrons in the superconducting state (superfluid

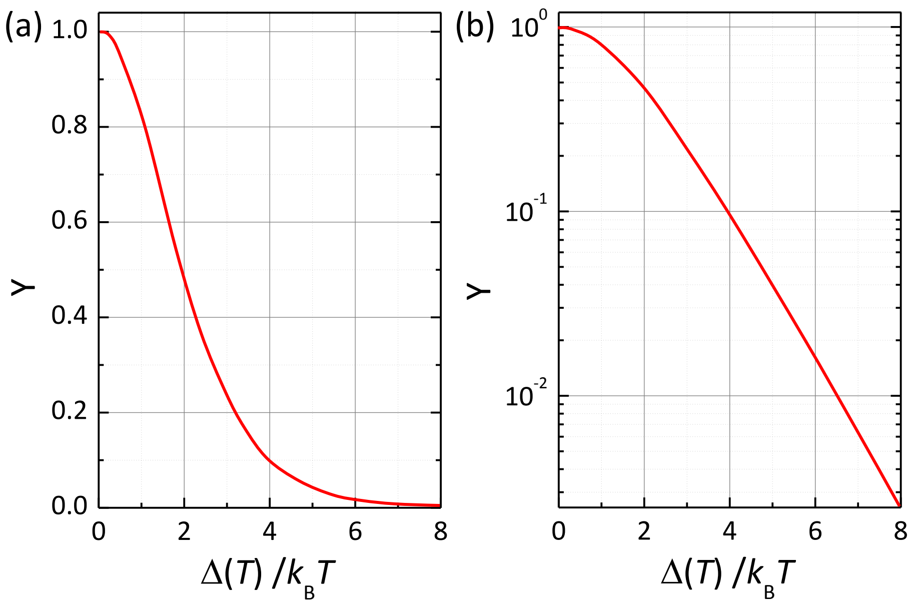

Figure 1: Yosida function plotted versus $\Delta(T)/k_{\mathrm{B}}T$ using (a) a linear and (b) logarithmic scale.

density density) is given by

$$\begin{aligned} n^{\mathrm{s}}_{ij}(T) &= n - n^{\mathrm{qp}}_{ij}(T) \;=\; n\,[1 - Y(T)]\;\delta_{ij} \\ &= n\left[1 - \int\limits_0^\infty \frac{dx}{\cosh^2\sqrt{x^2+\left(\frac{\Delta(T)}{2k_{\mathrm{B}}T}\right)^2}}\right]\;\delta_{ij}\;. \end{aligned} \tag{6}$$

(d) We can use the temperature dependence of the Yosida function to discuss the temperature dependence of the superfluid density and, in turn, the London penetration depth

$$\lambda_{\mathrm{L}}(T) \;=\; \sqrt{\frac{m_s}{\mu_0 n^{\mathrm{s}}(T) q_s^2}} \;=\; \frac{\lambda_{\mathrm{L}}(0)}{\sqrt{1-Y(T)}}\;. \tag{7}$$

For $T \simeq T_c$ ($\Delta(T) \to 0$), we can approximate the temperature dependence of the Yosida function by

$$\lim_{T\to T_c} Y(T) \;=\; 1 - 2\left(1 - \frac{T}{T_c}\right) \tag{8}$$

and obtain

$$\lim_{T\to T_c} \lambda_{\mathrm{L}}(T) \;=\; \frac{\lambda_{\mathrm{L}}(0)}{\sqrt{2\left(1-\frac{T}{T_c}\right)}}\;. \tag{9}$$

We see that $\lambda_{\mathrm{L}}(T)$ diverges for $T \to T_c$. This result is obvious since a normal metal cannot screen stationary magnetic fields.

For $T \to 0$, the Yosida function shows a thermally activated behavior

$$\lim_{T\to 0} Y(T) \;=\; \sqrt{\frac{2\pi\Delta(T)}{k_{\mathrm{B}}T}}\; \mathrm{e}^{-\frac{\Delta(T)}{k_{\mathrm{B}}T}}\;, \tag{10}$$

as can be seen in Fig. 1(b). Since $Y(T \ll T_c) \ll 1$, according to (7) the London penetration depth shows a very weak temperature dependence in the temperature regime well below T_c.