

Exercise to the Lecture

Superconductivity and Low Temperature Physics I
WS 2022/2023**4 Microscopic Theory****4.9 Particle Current Density****Exercise:**

Within Boltzmann transport theory the particle current density in a metal can be written as

$$\mathbf{J} = \frac{1}{V} \sum_{\mathbf{k}\sigma} \frac{\hbar\mathbf{k}}{m} [f(\varepsilon_{\mathbf{k}}) - f_0(\varepsilon_{\mathbf{k}})] = \frac{1}{V} \sum_{\mathbf{k}\sigma} \frac{\hbar\mathbf{k}}{m} \left[-\frac{\partial f_0(\varepsilon_{\mathbf{k}})}{\partial \varepsilon_{\mathbf{k}}} \delta\varepsilon_{\mathbf{k}} \right] = n\mathbf{v}$$
$$\text{with } f_0(\varepsilon_{\mathbf{k}}) = \frac{1}{e^{(\varepsilon_{\mathbf{k}} - \mu)/k_{\text{B}}T} + 1} .$$

Here, n is the particle density, \mathbf{v} the particle drift velocity, $\varepsilon_{\mathbf{k}} = \hbar^2 k^2 / 2m$, $\delta\varepsilon_{\mathbf{k}} = \hbar\mathbf{k} \cdot \mathbf{v}$, and $f_0(\varepsilon_{\mathbf{k}})$ the thermal equilibrium Fermi-Dirac distribution.

- (a) Use the above definition of the particle current density to derive the particle density in a normal metal.
- (b) Calculate the density n^{qp} of Bogoliubov quasiparticles in a superconductor (normal fluid density).
- (c) Calculate the density n^{s} of the paired electrons in a superconductor (superfluid density density).
- (d) Use the superfluid density to discuss the temperature dependence of the London penetration depth λ_{L} close to the transition temperature T_c .

Solution:

(a) The i^{th} component ($i = 1, 2, 3$) of the particle current density can be written as

$$\begin{aligned} J_i &= \frac{1}{V} \sum_{\mathbf{k}\sigma} \frac{\hbar k_i}{m} \left[-\frac{\partial f_0(\varepsilon_{\mathbf{k}})}{\partial \varepsilon_{\mathbf{k}}} \hbar k_j v_j \right] \\ &= n_{ij} v_j . \end{aligned} \quad (1)$$

With this result we can define the particle current density in a normal metal as

$$n_{ij} = \frac{1}{V} \sum_{\mathbf{k}\sigma} y_{\mathbf{k}} \frac{\hbar^2 \mathbf{k} \mathbf{k}}{m} , \quad (2)$$

where $\mathbf{k} \mathbf{k}$ is the dyadic product and

$$y_{\mathbf{k}} = -\frac{\partial f_0(\varepsilon_{\mathbf{k}})}{\partial \varepsilon_{\mathbf{k}}} = \frac{1}{2k_B T} \frac{1}{\cosh(\varepsilon_{\mathbf{k}}/k_B T) + 1} = \frac{1}{4k_B T} \frac{1}{\cosh^2(\varepsilon_{\mathbf{k}}/k_B T)} . \quad (3)$$

To evaluate (2) we have to convert the summation into an integration. We replace $\varepsilon_{\mathbf{k}}$ by $\zeta_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu$ in (3) and obtain

$$\begin{aligned} n_{ij} &= \frac{1}{V} \sum_{\mathbf{k}\sigma} \left(-\frac{\partial f_0(\zeta_{\mathbf{k}} + \mu)}{\partial \zeta_{\mathbf{k}}} \right) \frac{\hbar^2 \mathbf{k} \mathbf{k}}{m} \\ &= \frac{1}{V} \frac{\hbar^2 k_F^2}{m} \underbrace{\int \frac{d\Omega_{\mathbf{k}}}{4\pi} \widehat{\mathbf{k}} \widehat{\mathbf{k}}}_{=\frac{1}{3}\delta_{ij}} \int_{-\mu}^{\infty} d\zeta_{\mathbf{k}} D(\mu + \zeta_{\mathbf{k}}) \left(-\frac{\partial f_0(\mu + \zeta_{\mathbf{k}})}{\partial \zeta_{\mathbf{k}}} \right) \\ &\simeq \frac{1}{3} \delta_{ij} \underbrace{\frac{D(E_F) \hbar^2 k_F^2}{mV}}_{=3n} \frac{1}{4k_B T} \int_{-\mu}^{\infty} \frac{d\zeta_{\mathbf{k}}}{\cosh^2 \frac{\zeta_{\mathbf{k}}}{2k_B T}} \\ &\stackrel{x=\zeta_{\mathbf{k}}/2k_B T}{=} \frac{1}{2} n \delta_{ij} \underbrace{\int_{-\infty}^{\infty} \frac{dx}{\cosh^2 x}}_{=\tanh(\infty) - \tanh(-\infty) = 2} = n \delta_{ij} . \end{aligned} \quad (4)$$

Here, $\widehat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ and $|\mathbf{k}| \simeq k_F$. Furthermore, we have used $D(\mu + \zeta_{\mathbf{k}}) \simeq D(E_F) = \text{const}$, since the function $\partial f_{\mathbf{k}}^0 / \partial E_{\mathbf{k}}$ is finite only in a narrow energy interval $\sim k_B T$ around the chemical potential μ , and we have set the lower integration limit to $-\infty$, since typically $\mu/2k_B T \gg 1$ for a metal. Obviously we obtain the expected result that the particle density is given by the electron density of the normal metal.

(b) We next consider the superconducting state of a metal and calculate the normal fluid density n^{qp} of the Bogoliubov quasiparticles. To calculate n^{qp} we can use eqs. (2) and (3) but have to replace $\zeta_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu$ by the quasiparticle energy $E_{\mathbf{k}} = \sqrt{\zeta_{\mathbf{k}}^2 + \Delta^2}$ in (3). With

$Z(k)d^3k = D(E_{\mathbf{k}})dE_{\mathbf{k}} = D(\zeta_{\mathbf{k}})d\zeta_{\mathbf{k}}$ (conservation of states) we obtain

$$\begin{aligned}
n_{ij}^{\text{qp}} &= \frac{1}{V} \sum_{\mathbf{k}\sigma} y_{\mathbf{k}} \frac{\hbar^2 \mathbf{k}\mathbf{k}}{m} \\
&= \frac{1}{V} \frac{\hbar^2 k_{\text{F}}^2}{m} \underbrace{\int \frac{d\Omega_{\mathbf{k}}}{4\pi} \widehat{\mathbf{k}}\widehat{\mathbf{k}}}_{=\frac{1}{3}\delta_{ij}} \int_{-\mu}^{\infty} d\zeta_{\mathbf{k}} D(\mu + \zeta_{\mathbf{k}}) \left(-\frac{\partial f_{\mathbf{k}}}{\partial E_{\mathbf{k}}} \right) \\
&\simeq \frac{1}{3} \delta_{ij} \underbrace{\frac{D(E_{\text{F}})\hbar^2 k_{\text{F}}^2}{mV}}_{=3n} \frac{1}{4k_{\text{B}}T} \int_{-\mu}^{\infty} \frac{d\zeta_{\mathbf{k}}}{\cosh^2 \frac{E_{\mathbf{k}}}{2k_{\text{B}}T}} \\
&\stackrel{x=\zeta_{\mathbf{k}}/2k_{\text{B}}T}{=} \frac{1}{2} n \delta_{ij} \underbrace{\int_{-\infty}^{\infty} \frac{dx}{\cosh^2 \sqrt{x^2 + \left(\frac{\Delta(T)}{2k_{\text{B}}T}\right)^2}}}_{=2Y(T)} \\
&= n \delta_{ij} Y(T) . \tag{5}
\end{aligned}$$

We see that the normal fluid density is given by the normal state particle density multiplied by the Yosida function $Y(T)$ (cf. Fig. 1). The Yosida function is zero at $T = 0$ and continuously increases towards one at $T = T_c$. Therefore, the quasiparticle density decreases from $n_{ij}^{\text{qp}}(T) = n\delta_{ij}$ at $T = T_c$ to $n_{ij}^{\text{qp}}(T) = 0$ at $T = 0$.

- (c) Since the total particle number is conserved on going from the normal to the superconducting state, the density of the paired electrons in the superconducting state (superfluid

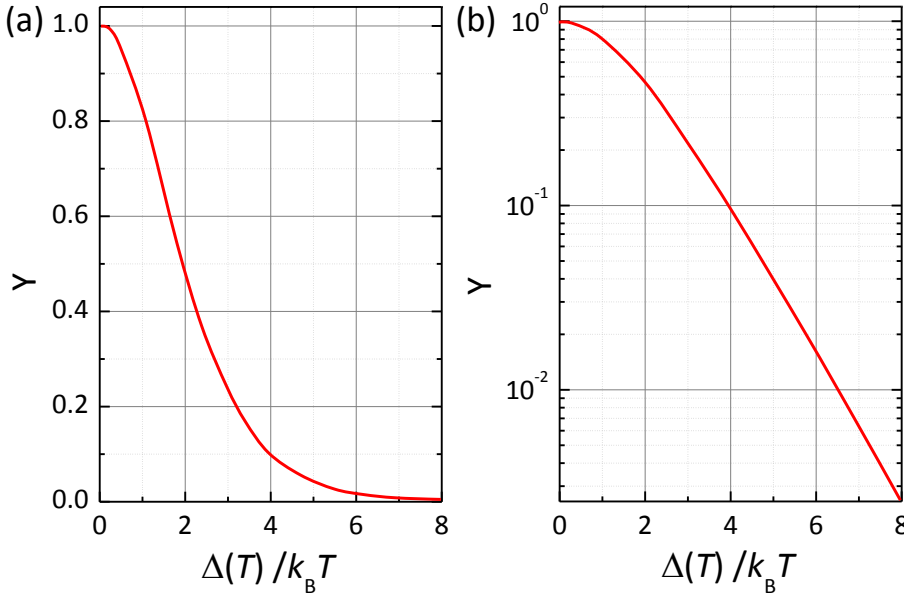


Figure 1: Yosida function plotted versus $\Delta(T)/k_{\text{B}}T$ using (a) a linear and (b) logarithmic scale.

density density) is given by

$$\begin{aligned} n_{ij}^s(T) &= n - n_{ij}^{\text{qp}}(T) = n [1 - Y(T)] \delta_{ij} \\ &= n \left[1 - \int_0^\infty \frac{dx}{\cosh^2 \sqrt{x^2 + \left(\frac{\Delta(T)}{2k_B T}\right)^2}} \right] \delta_{ij} . \end{aligned} \quad (6)$$

(d) We can use the temperature dependence of the Yosida function to discuss the temperature dependence of the superfluid density and, in turn, the London penetration depth

$$\lambda_L(T) = \sqrt{\frac{m_s}{\mu_0 n^s(T) q_s^2}} = \frac{\lambda_L(0)}{\sqrt{1 - Y(T)}} . \quad (7)$$

For $T \simeq T_c$ ($\Delta(T) \rightarrow 0$), we can approximate the temperature dependence of the Yosida function by

$$\lim_{T \rightarrow T_c} Y(T) = 1 - 2 \left(1 - \frac{T}{T_c} \right) \quad (8)$$

and obtain

$$\lim_{T \rightarrow T_c} \lambda_L(T) = \frac{\lambda_L(0)}{\sqrt{2 \left(1 - \frac{T}{T_c} \right)}} . \quad (9)$$

We see that $\lambda_L(T)$ diverges for $T \rightarrow T_c$. This result is obvious since a normal metal cannot screen stationary magnetic fields.

For $T \rightarrow 0$, the Yosida function shows a thermally activated behavior

$$\lim_{T \rightarrow 0} Y(T) = \sqrt{\frac{2\pi\Delta(T)}{k_B T}} e^{-\frac{\Delta(T)}{k_B T}} , \quad (10)$$

as can be seen in Fig. 1(b). Since $Y(T \ll T_c) \ll 1$, according to (7) the London penetration depth shows a very weak temperature dependence in the temperature regime well below T_c .