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Exercise 4

Exercise to the Lecture

Superconductivity and Low Temperature Physics I
WS 2021/2022

3 Phenomenological Models of Superconductivity

3.1 Magnetization of Superconductors

Exercise:

The relation between magnetic flux density $\mathbf{B}$, magnetic field $\mathbf{H}$ and magnetization $\mathbf{M}$ is given by

$$\mathbf{B} \;=\; \mu_0(\mathbf{H}+\mathbf{M}) \;=\; \mu_0(\mathbf{H}+\chi\mathbf{H}) \;=\; \mu_0\mathbf{H}(1+\chi) \;\; .$$

Due to its perfect diamagnetism, we have $\chi = -1$ for a superconductor. In the most simple case of an infinitely long cylinder with external magnetic field $\mathbf{H}_{\rm ext} = \mathbf{B}_{\rm ext}/\mu_0$ applied parallel to the cylinder we can neglect demagnetization effects. However, in the case of samples with a finite demagnetization factor N, we have to consider the so-called macroscopic magnetic field given by

$$\mathbf{H}_{\rm mac} \;=\; \mathbf{H}_{\rm ext}+\mathbf{H}_{\rm N} \;=\; \mathbf{H}_{\rm ext}-N\mathbf{M} \;\; .$$

In addition to the external field we have to take into account the demagnetization field $\mathbf{H}_{\rm N}$ originating from magnetic charges on the surface of the magnetized sample. For a particular sample shape, the demagnetization factor N is obtained by solving the boundary problem $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ and $\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J}$. For a long cylinder or a thin planar disk parallel to the applied magnetic field $\mathbf{H}_{\rm ext}$ we have $N = 0$. For a sphere, $N = 1/3$ and for a cylinder and planar disk perpendicular to the applied field we have $N = 1/2$ and $N = 1$, respectively.

(a) How does N qualitatively look like for a cylinder? Which sample geometry is most appropriate for a quantitative measurement of the magnetization of a superconductor?

(b) Calculate $\mathbf{H}_{\rm mac}$, $\mathbf{B}_i$ and $\mathbf{M}$ for a sphere as a function of χ and $\mathbf{H}_{\rm ext}$.

(c) How could a realistic measuring configuration for the determination of χ look like? How can we obtain χ for a homogeneous isotropic sample with demagnetization factor N from the measured quantities?

(d) Discuss the meaning of **H** and **B** by considering a cylindrical superconducting sample located in a long solenoid. Neglect demagnetization effects in this discussion.

(e) Consider a superconducting sphere in the Meißner state. What is the value of the magnetic flux density $\mathbf{B}_{\text{ext}}(r=R)$ right outside this sphere at the poles and the equator?

(f) Calculate the magnetic flux density outside a superconducting sphere in the Meißner state in spherical coordinates by using

$$\mathbf{B}(r \geq R) \quad = \quad \mu_0 \mathbf{H}_{\text{ext}} + |\mu_0 \mathbf{H}_{\text{ext}}| \frac{R^3}{2} \boldsymbol{\nabla} \left(\frac{\cos\theta}{r^2} \right) \quad .$$

Solution:

Demagnetization and demagnetization factor: Before discussing the solution we first define the demagnetization factor N. The macroscopic magnetic field $\mathbf{H}_{\text{mac}}$ for a magnetized piece of material is identical to the externally applied magnetic field $\mathbf{H}_{\text{ext}}$ only if the permeable material completely fills in the volume of the external field (e.g. for a ring-shaped or infinitely long solenoid). If in contrast the materials fills the field volume only along a limited part of the magnetic field lines, magnetic poles are forming at the free ends of the inserted piece of material, which are the starting point of field lines. Depending on the sign of the magnetization **M** or to be precise the magnetic susceptibility χ, these field lines weaken or strengthen the original field $\mathbf{H}_{\text{ext}}$ (cf. Fig. 1). This phenomenon is called ***demagnetization***. The size of the demagnetization on the one hand depends on the magnetization **M** induced in the permeable material and on the other on the geometric shape of the considered piece of material. In general, both the magnitude and direction of the magnetization, as well as the demagnetization factor determined by the geometrical shape, are complicated functions of the spatial coordinate. Only for the special case of a homogeneously magnetized sample having the shape of a prolate ellipsoid also the demagnetization field is homogeneous. In this case we can describe the impact of the demagnetization simply by the demagnetization field

$$\mathbf{H}_N \quad = \quad -N\mathbf{M} \,. \tag{1}$$

For the macroscopic magnetic field $\mathbf{H}_{\text{mac}}$ we then obtain

$$\mathbf{H}_{\text{mac}} \quad = \quad \mathbf{H}_{\text{ext}} + \mathbf{H}_{\text{N}} \;=\; \mathbf{H}_{\text{ext}} - N\mathbf{M} \quad .$$

The demagnetization factor N only depends on the geometrical shape of the homogeneously magnetized body. In general, it is given by a second rank tensor

$$\begin{pmatrix} H_{N,x} \\ H_{N,y} \\ H_{N,z} \end{pmatrix} \quad = \quad - \begin{pmatrix} N_{xx} & N_{xy} & N_{xz} \\ N_{yx} & N_{yy} & N_{yz} \\ N_{zx} & N_{zy} & N_{zz} \end{pmatrix} \begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} \quad .$$

Along the principal axes of the prolate ellipsoid $\mathbf{H}_N$ and **M** are collinear. Then, in the component representation there are only three different demagnetization factors N_a, N_b and N_c for the field components along the direction of the three principal axes a, b and c. If we consider a prolate ellipsoid and if his principal axes coincide with the axes of the cartesian coordinate systems used in the discussion, the demagnetization tensor becomes diagonal and the diagonal elements have to satisfy the condition

$$N_{xx} + N_{yy} + N_{zz} \quad = \quad N_a + N_b + N_c \;=\; 1 \quad .$$

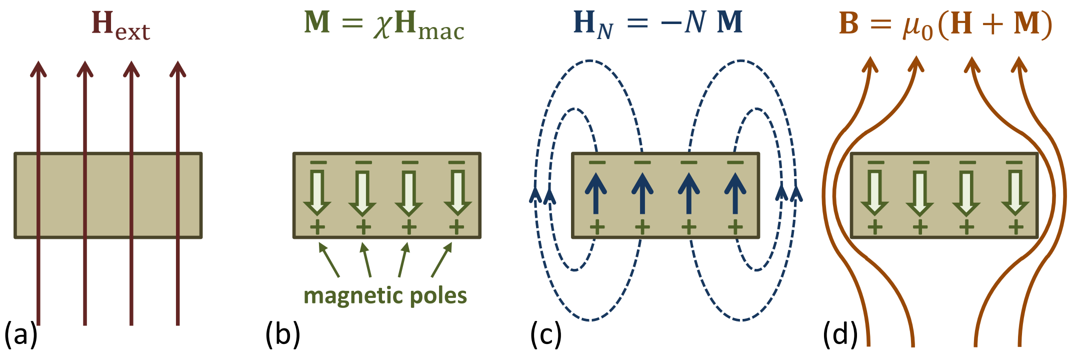

Figure 1: (a) A diamagnetic sample with magnetic susceptibility $\chi < 0$ is brought into a homogeneous magnetic field $\mathbf{H}_{\text{ext}}$. (b) In the sample a finite magnetization $\mathbf{M} = \chi \mathbf{H}_{\text{mac}}$ is induced which is antiparallel to the local magnetic field due to $\chi < 0$. (c) Because of the finite sample size magnetic poles are appearing on the sample surface, resulting in the demagnetization field $\mathbf{H}_N$. The latter is antiparallel to the magnetization and its magnitude depends on the sample geometry. (d) For a perfect diamagnet ($\chi = -1$) the magnetic flux density $\mathbf{B}_i = \mu_0(\mathbf{H}_{\text{mac}} + \mathbf{M})$ vanishes inside the sample. The flux density outside the sample results from the superposition of the spatially homogeneous applied magnetic field $\mathbf{H}_{\text{ext}}$ and the stray field due to the magnetic surface poles.

In practice, we often use demagnetization factors also for the description of the internal magnetic fields of bodies (e.g. cylinders, disks) which are no prolate ellipsoids and therefore do not have a homogeneous demagnetization field. In this case, the shape of the body is approximated by a prolate ellipsoid in a first approximation.

(a) We consider a cylinder which we approximate by a prolate ellipsoid to estimate its demagnetization factor for certain limiting cases. For a cylinder with $a \geq b = c$ and a parallel to $\mathbf{B}_{\text{ext}}$ according to (2) the demagnetization factor has to range between $N_a = N_{xx} = 0$ for $a \gg b, c$ and $N_a = N_{xx} = 1/3$ for $a = b = c$. For a cylinder with $a \geq b = c$ and a perpendicular to $\mathbf{B}_{\text{ext}}$ the corresponding demagnetization factor must range between $N_b = N_{yy} = N_c = N_{zz} = 1/3$ for $a = b = c$ and $N_b = N_{yy} = N_c = N_{zz} = 1/2$ for $a \gg b, c$. If for the latter case $b \neq c$, then $N_b \neq N_c$. For $c \ll b$ and c parallel to $\mathbf{B}_{\text{ext}}$, we have $N_c = N_{zz} = 0$; for c perpendicular to $\mathbf{B}_{\text{ext}}$ we obtain $N_b = N_{yy} = 1$. For quantitative measurements of the magnetization a sample geometry with $N = 0$ is optimal. This is realized by a long and thin cylinder with $a \gg b, c$ and $\mathbf{B}_{\text{ext}}$ parallel to a.

We easily see that due to $a = b = c$ for a sphere we obtain $N_{xx} = N_{yy} = N_{zz} = N = 1/3$ from (2). A planar thin plate with c perpendicular to the plate we can approximate by a prolate ellipsoid with $a, b \gg c$. For $\mathbf{B}_{\text{ext}}$ parallel to the plate we obtain $N_{xx} = N_{yy} = 0$, for $\mathbf{B}_{\text{ext}}$ perpendicular to the plate we have $N_{zz} = 1$.

(b) If we bring a magnetically isotropic sphere with magnetic susceptibility χ into a homogeneous external magnetic field $\mathbf{H}_{\text{ext}}$, we magnetize the sphere (cf. Fig. 2). However, the magnetization is not proportional to $\mathbf{H}_{\text{ext}}$, but to the macroscopic magnetic field $\mathbf{H}_{\text{mac}}$:

$$\mathbf{M} = \chi \mathbf{H}_{\text{mac}} = (\mu - 1)\mathbf{H}_{\text{mac}} \,. \tag{2}$$

The finite magnetization of the sphere results in magnetic poles on its surface leading to a demagnetization field $\mathbf{H}_N$. Depending on the direction of $\mathbf{M}$, that is, of the magnetic susceptibility χ, the demagnetization fields result in an increase or decrease of the externally

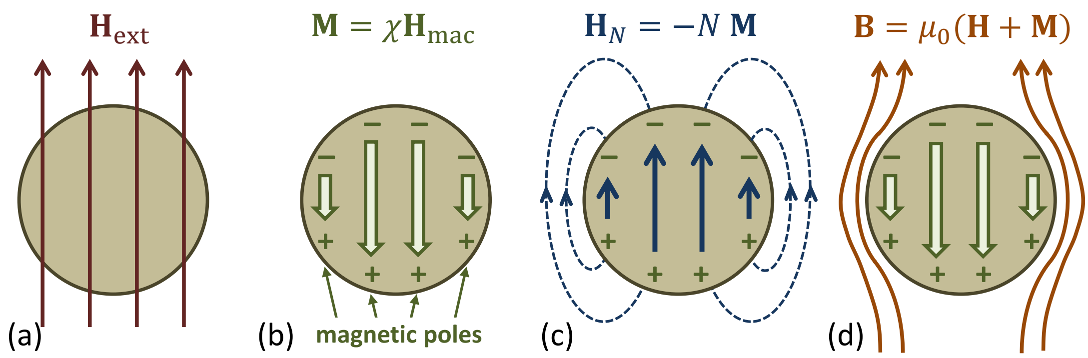

Figure 2: (a) A superconducting sphere with magnetic susceptibility $\chi = -1$ is brought into a homogeneous external magnetic field $\mathbf{H}_{\text{ext}}$. (b) A homogeneous magnetization $\mathbf{M} = \chi\mathbf{H}_{\text{mac}} = -\frac{3}{2}\mathbf{H}_{\text{ext}}$ is induced in the sample which is antiparallel to the locally acting magnetic field due to $\chi = -1$. (c) The magnetic poles with area density $\sigma_m = \mathbf{M} \cdot \widehat{\mathbf{e}}_n$ appearing on the sample surface ($\widehat{\mathbf{e}}_n$ is the unit vector perpendicular the the surface of the sphere), result in the demagnetization field $\mathbf{H}_N = -N\mathbf{M} = \frac{1}{3}\frac{3}{2}\mathbf{H}_{\text{ext}} = \frac{1}{2}\mathbf{H}_{\text{ext}}$ antiparallel to the magnetization. (d) The magnetic flux density $\mathbf{B}_i = \mu_0(\mathbf{H}_{\text{mac}} + \mathbf{M})$ vanishes inside the sample. The flux density outside the sample can be viewed as resulting from the the superposition of the homogeneous applied magnetic field and the stray field due to the magnetic surface poles.

applied magnetic field $\mathbf{H}_{\text{ext}}$ (cf. Fig. 2). For the macroscopic magnetic field we obtain

$$\begin{aligned}\mathbf{H}_{\text{mac}} &= \mathbf{H}_{\text{ext}} + \mathbf{H}_N \\ &= \mathbf{H}_{\text{ext}} - N\mathbf{M} = \mathbf{H}_{\text{ext}} - N\chi\mathbf{H}_{\text{mac}} = \mathbf{H}_{\text{ext}} - N(\mu - 1)\mathbf{H}_{\text{mac}} .\end{aligned} \tag{3}$$

Resolving for $\mathbf{H}_{\text{mac}}$ yields

$$\mathbf{H}_{\text{mac}} = \frac{1}{1 + N\chi}\mathbf{H}_{\text{ext}} = \frac{1}{1 + N(\mu - 1)}\mathbf{H}_{\text{ext}} . \tag{4}$$

We see that for a superconducting sphere ($N = 1/3$, $\chi = -1$) we have the macroscopic field $\mathbf{H}_{\text{mac}} = \frac{3}{2}\mathbf{H}_{\text{ext}}$. Inserting (4) into (2) results in

$$\mathbf{M} = \chi\mathbf{H}_{\text{mac}} = \frac{\chi}{1 + N\chi}\mathbf{H}_{\text{ext}} = \frac{(\mu - 1)}{1 + N(\mu - 1)}\mathbf{H}_{\text{ext}} . \tag{5}$$

For a superconducting sphere ($N = 1/3$, $\chi = -1$) we then obtain $\mathbf{M} = -\frac{3}{2}\mathbf{H}_{\text{ext}}$. Without taking into account for demagnetization effects, apparently we would determine the susceptibility to $\widetilde{\chi} = -3/2$. With the expressions for $\mathbf{H}_{\text{mac}}$ and $\mathbf{M}$, the magnetic flux density $\mathbf{B}_i$ inside the sample is obtained to

$$\mathbf{B}_i = \frac{1 + \chi}{1 + N\chi}\mathbf{B}_{\text{ext}} = \frac{\mu}{1 + N(\mu - 1)}\mathbf{B}_{\text{ext}} . \tag{6}$$

For a superconducting sphere ($N = 1/3$, $\chi = -1$) we have $\mathbf{B}_i = 0$ as expected for a perfectly diamagnetic material (cf. Fig. 2).

(c) A realistic measuring configuration is sketched in Fig. 3. For the measurement of the sample magnetization we use a detection coil (not shown) inside the solenoid generating the applied magnetic field. Ideally the detection coil tightly encloses the sample (large filling factor) and is connected to a voltmeter. Letting the applied magnetic field $\mathbf{H}_{\rm ext}(t) = \mathbf{B}_{\rm ext}(t)/\mu_0$ oscillate slowly – e.g. at a frequency of a few ten Hertz (quasistatic limit) – allows us to determine the change in the flux density $\dot{\mathbf{B}}_i(t)$ inside the sample by measuring the voltage $V_{\rm ind}(t) = -\partial\Phi_i/\partial t \propto -\partial B_i/\partial t$ induced in the detection coil. Hence, after a calibration process the integral $\int V_{\rm ind}(t)dt \propto -B_i$ can directly be related to $\mathbf{B}_i(t)$. With the above expression for $\mathbf{B}_i$ we can determine the magnetic susceptibility χ of the sample if we know its demagnetization factor. Resolving eq. (6) for χ results in

$$\chi \;=\; \frac{B_i - B_{\rm ext}}{B_{\rm ext} - NB_i}\,. \tag{7}$$

We note that the above equations are only exact for homogeneous isotropic materials and sample shapes (e.g. prolate ellipsoid, infinitely thin plate, infinitely long cylinder) resulting in a homogeneous magnetization parallel to the applied magnetic field. For more complex sample shapes and in the presence of a finite magnetic anisotropy the corresponding expressions become much more complicated.

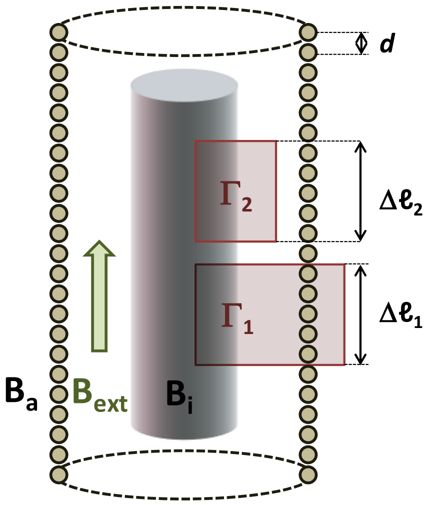

Figure 3: Cylindrical superconducting sample inside an infinitely long solenoid.

(d) In the following we use the configuration shown in Fig. 3 to provide a clear distinction between $\mathbf{H}$ and $\mathbf{B}$. With $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$ we can write

$$\boldsymbol{\nabla}\times\mathbf{B} \;=\; \mu_0(\boldsymbol{\nabla}\times\mathbf{H} + \boldsymbol{\nabla}\times\mathbf{M}) \;=\; \mu_0(\mathbf{J}_{\rm sp} + \mathbf{J}_{\rm m})\,. \tag{8}$$

We see that the curl of $\mathbf{B}$ is related to the total current density, which is given as the sum of the usual "external current density" $\mathbf{J}_{\rm sp}$ – e.g. flowing in the electrical conductor forming the solenoid – and the "magnetization current density" $\mathbf{J}_m$, which is related to the internal currents in the magnetized material. In contrast, the curl of $\mathbf{H}$ is related only to the external current density $\mathbf{J}_{\rm sp}$. The latter is provided by an external current source and is flowing in the solenoid generating the external magnetic field. For the integral over the area S enclosed by the closed path $\Gamma = \partial S$ we therefore obtain

$$\int\limits_S d\mathbf{S}\cdot\boldsymbol{\nabla}\times\mathbf{B} \;=\; \oint\limits_{\partial S}\mathbf{B}\cdot d\ell \;=\; \mu_0\int\limits_S d\mathbf{S}\cdot\left(\mathbf{J}_{\rm sp}+\mathbf{J}_{\rm m}\right) \tag{9}$$

$$\int\limits_S d\mathbf{S}\cdot\boldsymbol{\nabla}\times\mathbf{H} \;=\; \oint\limits_{\partial S}\mathbf{H}\cdot d\ell \;=\; \int\limits_S d\mathbf{S}\cdot\mathbf{J}_{\rm sp}\;\;. \tag{10}$$

Here, we have used Stoke's theorem. We now apply this result to the configuration shown in Fig. 3. To keep our discussion simple, we neglect demagnetization effects at the sample ends ($N = 0$), that is, we assume an infinitely long sample. For the integral over the area S enclosed by the closed path $\Gamma_1 = \partial S_1$ we obtain

$$\int_{S_1} d\mathbf{S} \cdot \nabla \times \mathbf{B} \;=\; \oint_{\partial S_1} \mathbf{B} \cdot d\boldsymbol{\ell} \;=\; \mu_0 \int_{S_1} d\mathbf{S} \cdot \left(\mathbf{J}_{\rm sp} + \mathbf{J}_{\rm m}\right) \quad . \tag{11}$$

The current density $\mathbf{J}_{\rm sp}$ is related to the current I_0 flowing in the solenoid, whereas the current density $\mathbf{J}_{\rm m}$ is related to the internal currents resulting in the finite magnetization of the superconductor, that is, to the supercurrent density J_s in the superconductor. This is an important statement. Since the supercurrent density can be considered as a macroscopic current density – in contrast to the microscopic diamagnetic currents flowing on an atomic scale in other diamagnetic materials – it is often forgotten that $\mathbf{J}_s$ is a magnetization current density. Of course, treating $\mathbf{J}_s$ as an external current density leads to paradox results.

With the current I_0 flowing in the solenoid and the winding distance d, eq. (11) yields

$$(B_i - B_a)\,\Delta\ell_1 \;=\; \mu_0 \left[J_s \lambda_{\rm L} \Delta\ell_1 + I_0 \frac{\Delta\ell_1}{d} \right] \quad . \tag{12}$$

Here we have assumed that the supercurrent is flowing only within a surface sheet of thickness $\lambda_{\rm L}$ (London penetration depth). Since we have $B_i = 0$ in the Meißner state and furthermore $B_a = 0$ outside an infinitely long solenoid, we find

$$J_s \;=\; -\frac{I_0}{d\lambda_{\rm L}} \quad . \tag{13}$$

Since for $N = 0$ the macroscopic magnetic field H_i is equal to the externally applied field $H_{\rm ext}$ [compare (4)], we obtain the expected result

$$H_i \;=\; H_{\rm ext} \;=\; \frac{I_0}{d} \quad . \tag{14}$$

Applying Ampère's law to the involved magnetic fields, according to eq. (10) we obtain for the integration path $\Gamma_1 = \partial S_1$

$$\oint_{\partial S_1} \mathbf{H} \cdot d\boldsymbol{\ell} \;=\; \int_{S_1} d\mathbf{S} \cdot \mathbf{J}_{\rm sp} \qquad \Rightarrow \quad H_i \;=\; \frac{I_0}{d} \quad . \tag{15}$$

Again, for $N = 0$ we obtain the expected result $H_i = H_{\rm ext}$. If we instead use the total current density $\mathbf{J}_{\rm sp} + \mathbf{J}_{\rm m}$ on the right hand side of (15) – this would mean that we would not interpret $\mathbf{J}_s$ as a magnetization current density – we would obtain the paradox result

$$H_i \;=\; J_s \lambda_{\rm L} + \frac{I_0}{d} \;=\; 0 \quad . \tag{16}$$

In the same way we would get a paradox result, if we use only the external current density $\mathbf{J}_{\rm sp}$ on the right hand side of (9). Since $B_a = 0$ for an infinitely long solenoid, this would lead to the astonishing result $B_i \neq 0$.

(e) With the considerations made above we discuss in the following the magnetic flux density outside a superconducting sphere. Because of $\nabla \cdot \mathbf{B} = 0$ we know that the normal component B_n of the flux density on the surface of the sphere must be continuous. This means that $\mathbf{B}(r = R, \theta = 0)$ disappears at the poles of the sphere. According to part (b), the magnetic field inside the sphere is constant and given by

$$\mathbf{H}_{\mathrm{mac}} = \frac{\mathbf{H}_{\mathrm{ext}}}{1 + N\chi} \overbrace{=}^{N=1/3, \chi=-1} \frac{\mathbf{H}_{\mathrm{ext}}}{1 - \frac{1}{3}} = \frac{\mathbf{H}_{\mathrm{ext}}}{2/3} = \frac{3}{2}\mathbf{H}_{\mathrm{ext}} \ . \tag{17}$$

Since the tangential component H_t of $\mathbf{H}$ must be continuous, we obtain the following relation at the equator of the sphere: $\mathbf{B}(r = R, \theta = \pi/2) = \frac{3}{2}\mathbf{B}_{\mathrm{ext}}$.

(f) For the magnetic flux density outside a superconducting sphere one finds the following expression in textbooks:

$$\mathbf{B}(r \geq R) = \mathbf{B}_{\mathrm{ext}} + B_{\mathrm{ext}}\frac{R^3}{2}\nabla\left(\frac{\cos\theta}{r^2}\right) \ . \tag{18}$$

Here, $B_{\mathrm{ext}} = |\mathbf{B}_{\mathrm{ext}}|$. The relation between spherical coordinates (r, θ, φ) and cartesian coordinates (x, y, z) is given by

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \ . \end{aligned} \tag{19}$$

The gradient reads in spherical coordinates as

$$\nabla = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \ . \tag{20}$$

We first have to express $\mathbf{B}_{\mathrm{ext}}$ in spherical coordinates. We obtain

$$\mathbf{B}_{\mathrm{ext}} = B_{\mathrm{ext}}(\mathbf{e}_r\cos\theta - \mathbf{e}_\theta\sin\theta) \ . \tag{21}$$

Because of the symmetry of the problem there is no projection on the $\mathbf{e}_\varphi$-axis. Furthermore, $\mathbf{e}_\theta$ is antiparallel to the $\mathbf{e}_z$-axis, resulting in a negative sign of the $\mathbf{e}_\theta$ term. With the derivatives with respect to r and θ

$$\begin{aligned} \frac{\partial}{\partial r}\left(\frac{\cos\theta}{r^2}\right) &= -\left(\frac{2\cos\theta}{r^3}\right) \\ \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{r^2}\right) &= -\left(\frac{\sin\theta}{r^3}\right) \end{aligned} \tag{22}$$

we obtain

$$\begin{aligned} B_r &= B_{\mathrm{ext}}\cos\theta\left(1 - \frac{R^3}{r^3}\right) \\ B_\theta &= -B_{\mathrm{ext}}\sin\theta\left(1 + \frac{R^3}{2r^3}\right) \ . \end{aligned} \tag{23}$$

For $r = R$, we have $B_r \equiv 0$ and $B_\theta = -\frac{3}{2}B_{\mathrm{ext}}\sin\theta$. The magnetic flux density at the equator is increased by 50% as already discussed above. The magnetic field outside the sphere is a superposition of the applied magnetic field $B_{\text{ext}} = \mu_0 H_{\text{ext}}$ and the field resulting from the magnetic moment $\mathbf{m}$ of the sphere. The latter corresponds to that of a dipole located in the center of the sphere with $\mathbf{m} = (4\pi/3)R^3\mathbf{M} = -2\pi R^3\mathbf{H}_{\text{ext}}$.

Due to $\nabla \times \mathbf{H} = 0$, we can derive $\mathbf{H}$ from a scalar potential, that is, we can write $\mathbf{H} = -\boldsymbol{\nabla}\Phi_m$. With the given boundary conditions we then obtain for the field outside the sphere

$$\Phi_m(r,\theta) = -H_{\text{ext}} \cos\theta \left(r + \frac{R^3}{2r^2} \right) \quad . \tag{24}$$

Obviously, the first term is related to the external magnetic field applied in z-direction and the second to the far field of the dipole, resulting from the superconducting screening currents on the sample surface.