WALTHER–MEIßNER–INSTITUT
Bayerische Akademie der Wissenschaften
Lehrstuhl für Technische Physik E23, Technische Universität München
Prof. Dr. Rudolf Gross
Tel.: +49 (0)89 289 14249
E-mail: rudolf.gross@wmi.badw.de

17 November 2022

Exercise 5

Exercise to the Lecture

Superconductivity and Low Temperature Physics I
WS 2022/2023

3 Phenomenological Models of Superconductivity

3.2 Critical Current Density in the Meißner State

Exercise:

We consider a type-I or type-II superconductor with thermodynamic critical field $\mathbf{B}_{\rm cth}$ and lower critical field $\mathbf{B}_{\rm c1}$, respectively, in the Meißner state and discuss the value of the critical current density $\mathbf{J}_c$ that can be derived from London, Ginzburg-Landau and BCS theory.

(a) Calculate the critical current density $\mathbf{J}_c^{\rm London}$ of a long superconducting cylinder by using London theory.

(b) How does this value compare to the value $\mathbf{J}_c^{\rm GL}$ derived from Ginzburg-Landau theory for a superconducting wire with diameter small compared to the London penetration depth? What is the origin of the different results obtained from London and Ginzburg-Landau theory?

(c) BCS theory predicts an average reduction of energy density of $\frac{1}{4}D(E_{\rm F})\Delta^2$ due to the formation of Cooper pairs in the superconducting state. Here, $D(E_{\rm F})$ is the density of states for both spin directions and Δ the BCS energy gap. Use this result to estimate the BCS critical current density $\mathbf{J}_c^{\rm BCS}$.

(d) Calculate the critical current I_c of an Al wire with diameter $d = 1\,$cm. Use the critical current density $J_c \simeq 2 \times 10^{11}\,$A/m^2 and the London penetration depth $\lambda_{\rm L} = 50\,$nm of Al. How does this current compare to the maximum current that can be fed through a Cu wire of the same diameter?

Solution:

(a) The 2. London equation $\nabla \times (\Lambda \mathbf{J}_s) + \mathbf{b} = 0$ with the London coefficient $\Lambda = m_s/n_s q_s^2$ provides the relation between the supercurrent density $\mathbf{J}_s$ and the local magnetic flux density $\mathbf{b}(\mathbf{r})$. Due to the flux expulsion in the Meißner state the magnetic field and, hence, the supercurrent density is restricted to a thin surface layer of thickness λ_L with the London penetration depth $\lambda_\mathrm{L} = \sqrt{\Lambda/\mu_0}$. In order to estimate the critical supercurrent density J_c in the Meißner phase we consider a superconducting cylinder with radius $R \gg \lambda_\mathrm{L}$. If we drive a supercurrent along this cylinder (z-direction) it generates a circular magnetic field b_φ in the (r,φ)-plane perpendicular to the cylinder axis. From the Maxwell equation $\nabla \times \mathbf{b} = \mu_0 \mathbf{J}_s$ we obtain[1] $\partial b_\varphi(r)/\partial r \simeq \mu_0 J_{s,z}$. As the field generated by the transport current is shielded in the Meißner state, we can write $b_\varphi(R-r) = b_\varphi^\mathrm{surface} \mathrm{e}^{-(R-r)/\lambda_\mathrm{L}}$ and, hence, $J_{s,z} = b_\varphi^\mathrm{surface}/\mu_0\lambda_\mathrm{L}$. The critical supercurrent density is reached if the magnetic field on the surface of the cylinder becomes equal to the critical field B_c. Then, according to London theory the the critical current density is given by

$$J_c^\mathrm{London} = \frac{B_\mathrm{c}}{\mu_0 \lambda_\mathrm{L}} . \tag{1}$$

Note that this expression is valid both for type-I and type-II superconductors in the Meißner state. Here, $B_\mathrm{c} = B_\mathrm{cth}$ for type-I and $B_\mathrm{c} = B_\mathrm{c1}$ for type-II superconductors. For aluminium with $B_\mathrm{cth} \simeq 10\,\mathrm{mT}$ and $\lambda_\mathrm{L} \simeq 50\,\mathrm{nm}$ we obtain $J_c^\mathrm{London} \simeq 2 \times 10^{11}\,\mathrm{A/m^2}$.

Let us briefly check whether eq. (1) is in agreement with Ampère's law. For a cylinder with total transport current I the magnetic field on the cylinder surface is given by $b_\varphi(R) = \mu_0 I/2\pi R$. If we equate this to the critical field, we obtain the critical current

$$I_c = 2\pi R\, B_\mathrm{c}/\mu_0 . \tag{2}$$

Obviously this expression differs from eq. (1). However, for a comparison we have to relate I_c to $J_c = I_c/A_\mathrm{eff}$. Since the transport current flows only in a thin surface layer of thickness λ_L, the effective area is given by $A_\mathrm{eff} = 2\pi R \lambda_\mathrm{L}$ and we again obtain the result (2) for the critical current density.

(b) A basic assumption of London theory is a spatially constant density of the superconducting charge carriers. In particular, London theory neglects the fact that the density of the

[1] The Maxwell equation in cylindrical coordinates reads as

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(r b_\varphi) - \frac{\partial b_r}{\partial \varphi}\right] = \mu_0 J_{s,z} .$$

Since $\partial b_r/\partial\varphi = 0$ for the considered configuration we obtain

$$\left[\frac{b_\varphi(r)}{r} + \frac{\partial b_\varphi(r)}{\partial r}\right] = \mu_0 J_{s,z}$$

For $R \gg \lambda_\mathrm{L}$, the first term in the brackets is of the order of b_φ/R on the cylinder surface and therefore much smaller than the second term which is of the order $b_\varphi/\lambda_\mathrm{L}$. Neglecting the first term for simplicity we have

$$\frac{\partial b_\varphi(r)}{\partial r} = \mu_0 J_{s,z}$$

with the solution $b_\varphi(R-r) = b_\varphi^\mathrm{surface}\mathrm{e}^{-(R-r)/\lambda_\mathrm{L}}$ for the boundary condition $b_\varphi(0) = b_\varphi^\mathrm{surface}$.

superconducting charge carriers may depend on the magnitude of the supercurrent density. This rough approximation is removed in Ginzburg-Landau theory where the decay of the order parameter amplitude with increasing supercurrent density is explicitly taken into account.

We now use Ginzburg-Landau theory to derive the critical current density of a quasi-onedimensional superconductor which can be realized by a very thin wire with diameter $d \ll \xi_{\rm GL}(T)$. Since variations of the order parameter amplitude are allowed only on the characteristic length scale $\xi_{\rm GL}(T)$, we can assume that there are no gradients in the radial direction. By further assuming that the wire material is homogeneous and the current density is the same everywhere along the wire, we can also safely assume that there are no gradients of the order parameter amplitude along the wire. In this case we can express the order parameter by $\Psi(\mathbf{r}) = |\Psi| e^{\imath\theta(\mathbf{r})}$ and we obtain from the second Ginzburg-Landau equation

$$\mathbf{J}_s = \frac{q_s}{m_s}|\Psi|^2 \left(\hbar\boldsymbol{\nabla}\theta - q_s\mathbf{A}\right) = q_s|\Psi|^2\mathbf{v}_s \, . \tag{3}$$

On the other hand, we can rewrite the normalized 1. Ginzburg-Landau equation

$$\frac{\hbar^2}{2m_s\alpha}\left(\frac{1}{\imath}\boldsymbol{\nabla} - \frac{q_s}{\hbar}\mathbf{A}\right)^2 \psi + \psi - |\psi|^2\psi = 0 \, . \tag{4}$$

to

$$-\frac{\xi_{\rm GL}^2 m_s^2}{\hbar^2}\underbrace{\left(\frac{\hbar}{m_s}\boldsymbol{\nabla}\theta - \frac{q_s}{m_s}\mathbf{A}\right)^2}_{=v_s^2} \psi + \psi - |\psi|^2\psi = 0 \, . \tag{5}$$

This immediately leads to

$$|\psi|^2 = \left|\frac{\Psi}{\Psi_0}\right|^2 = \left(1 - \frac{m_s^2\xi_{\rm GL}^2 v_s^2}{\hbar^2}\right) = \left(1 - \frac{\frac{1}{2}m_s v_s^2}{|\alpha|}\right) \, . \tag{6}$$

The function $|\psi|^2(v_s)$ is shown in Fig. 1. Because α is the condensation energy per Cooper pair, we see that the reduction of $|\psi|^2$ with increasing v_s is just proportional to the ratio of kinetic energy and condensation energy of the superconducting electrons. This leads us to the intuitively expected conclusion that the additional kinetic energy of the superconducting electrons moving at v_s results in a reduction of condensation energy and, hence, the order parameter.

Equation (6) leads to the supercurrent density $\mathbf{J}_s$ (cf. Fig. 1)

$$\mathbf{J}_s = q_s|\Psi|^2\mathbf{v}_s = q_s|\Psi_0|^2\left(1 - \frac{m_s^2\xi_{\rm GL}^2 v_s^2}{\hbar^2}\right)\mathbf{v}_s \, . \tag{7}$$

The maximum value of the supercurrent density, the critical current density $J_c^{\rm GL}$ of the wire, is obtained from $\partial J_s/\partial v_s = 0$ to

$$J_c^{\rm GL} = \frac{2}{3\sqrt{3}}\frac{\hbar q_s}{m_s\xi_{\rm GL}}|\Psi_0|^2 = \frac{\Phi_0}{3\sqrt{3}\pi\mu_0\lambda_{\rm L}^2(T)\xi_{\rm GL}(T)} \, . \tag{8}$$

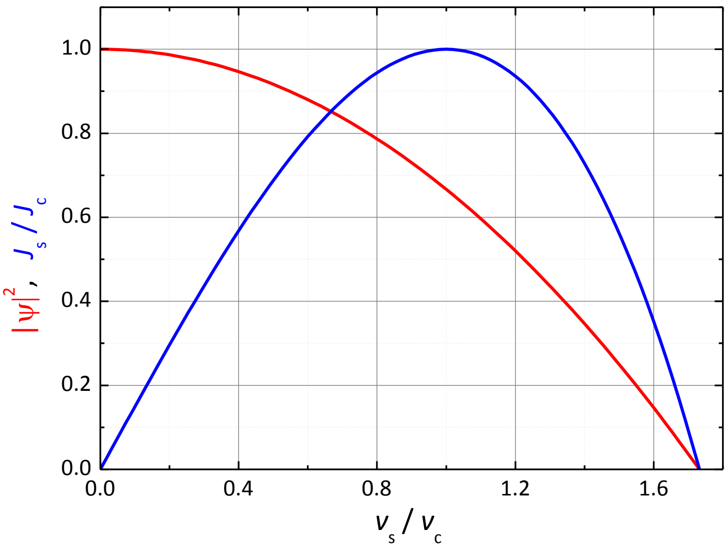

Figure 1: Variation of the superconducting order parameter $|\psi|^2 = |\Psi/\Psi_0|^2$ and the supercurrent density J_s with the velocity v_s of the superconducting electrons.

Here, we have used the flux quantum $\Phi_0 = h/q_s$ and the London penetration depth $\lambda_L^2(T) = m_s/\mu_0|\Psi_0|^2 q_s^2$. Close to T_c, we expect with the temperature dependencies

$$\lambda_L(T) = \frac{\lambda_{GL}(0)}{\sqrt{1-\frac{T}{T_c}}} \tag{9}$$

$$\xi_{GL}(T) = \frac{\xi_{GL}(0)}{\sqrt{1-\frac{T}{T_c}}} \,. \tag{10}$$

the temperature dependence

$$J_c^{GL} \propto (1-T/T_c)^{3/2} \,, \tag{11}$$

what is in good agreement with experimental results.

Using

$$B_{cth}(T) = \frac{\Phi_0}{2\pi\sqrt{2}\xi_{GL}(T)\lambda_L(T)} \,, \tag{12}$$

we can express J_c^{GL} by the thermodynamic critical field as

$$J_c^{GL} = \frac{2\sqrt{2}}{3\sqrt{3}} \frac{B_{cth}}{\mu_0\lambda_L} = 0.544 \frac{B_{cth}}{\mu_0\lambda_L} \,. \tag{13}$$

We see that J_c^{GL} is only about half of the critical current density $J_c^{London} = B_{cth}/\mu_0\lambda_L$ derived above from London theory. This is cause by the fact that in the London theory approximation the reduction of the order parameter amplitude with increasing supercurrent density is not taken into account.

(c) In order to estimate the critical current density of a thin superconducting wire from BCS theory, we have to compare the kinetic energy due to the finite supercurrent with the energy gained by the formation of Cooper pairs. For a first oder estimate we can assume that the critical current density is reached as soon as the kinetic energy of the Cooper pairs is exceeding their binding energy. Therefore, this current density is usually called depairing critical current density.

According to BCS theory the formation of Cooper pairs results in a reduction of the average energy density of $\frac{1}{4}D(E_\mathrm{F})\Delta^2$, where $D(E_\mathrm{F})$ is the density of states for both spin directions and Δ the BCS energy gap. As this gain in energy density corresponds to the condensation energy density $B_\mathrm{cth}^2/2\mu_0$ in GL theory, we can formally express the thermodynamic critical field as $B_\mathrm{cth} = \sqrt{\mu_0 D(E_\mathrm{F})\Delta^2/2}$. By using the above GL result $J_c^\mathrm{GL} = 0.544 B_\mathrm{cth}/\mu_0\lambda_\mathrm{L}$, we can formally write down a BCS critical current density as

$$J_c^\mathrm{BCS} \;=\; 0.544\frac{\Delta}{\lambda_\mathrm{L}}\sqrt{D(E_\mathrm{F})/2\mu_0}\,. \tag{14}$$

(d) In the Meißner state, the critical current in the superconducting wire is flowing only within a thin surface sheet of thickness λ_L. The critical current of the superconducting wire therefore is given by ($J_c^\mathrm{Al} \simeq 2\times 10^{11}\,\mathrm{A/m^2}$, $\lambda_\mathrm{L} = 50\,\mathrm{nm}$, $d = 1\,\mathrm{cm}$)

$$I_c^\mathrm{Al} \;=\; J_\mathrm{c}^\mathrm{Al}\,\pi d\lambda_\mathrm{L} \;=\; 314\,\mathrm{A}\,. \tag{15}$$

The allowed maximum current density in a copper wire (to avoid overheating) is $J_c^\mathrm{Cu} \simeq 10\,\mathrm{A/mm^2}$. Since in contrast to the superconductor the current is allowed to flow homogeneously across the whole cross-sectional area, we obtain the maximum current

$$I_c^\mathrm{Cu} \;=\; J_\mathrm{c}^\mathrm{Cu}\,\pi(d/2)^2 \;=\; 785\,\mathrm{A}\,. \tag{16}$$

We see that the maximum current carrying capacity of the copper wire is actually larger than that of the superconducting wire. This is the reason why type-I superconductors (or type-II superconductors in the Meißner state) are not used in power applications. The restriction of the current flow to a thin surface sheet is lifted for type-II superconductors in the Shubnikov phase. There, the current can flow across the whole cross-sectional area. However, the relevant supercurrent density is no longer the depairing current density but the so-called depinning current density, which typically is at least an order of magnitude smaller.

Note that an improvement of the critical current of type-I superconductors could by obtained by using a large number of thin filaments instead of a single massive conductor. This corresponds to the use of filamentary conductors in high-frequency applications, where the skin effect forces the current to flow in a surface layer with thickness given by the frequency dependent skin depth. However, due to the small value of the London penetration depth (typically 10 to 100 nm) a considerable improvement would require to use of filament with diameter of less than 100 nm. This is technologically very demanding for large wire length.