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**Exercise 9**

Exercise to the Lecture

# Superconductivity and Low Temperature Physics I
## WS 2021/2022

## 4 Microscopic Theory

### 4.7 The Bogoliubov Quasiparticles – Dispersion Relation

**Exercise:**

The energy spectrum of the Bogoliubov quasiparticles is given by

$$E_{\mathbf{k}} \;=\; \sqrt{\xi_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}\,,$$

where $\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu = (\hbar^2 k^2/2m) - (\hbar^2 k_{\mathrm{F}}^2/2m)$ is the normal electron energy relative to the chemical potential ($\mu = \hbar^2 k_{\mathrm{F}}^2/2m$ at $T = 0$). We consider the dispersion of the Bogoliubov quasiparticles for an isotropic gap $\Delta_{\mathbf{k}} = \Delta$.

(a) Discuss the energy spectrum of the Bogoliubov quasiparticles close to the Fermi wave number $k_{\mathrm{F}}$.

(b) How does the effective mass $m_{\mathrm{qp}}$ of the Bogoliubov quasiparticles compare to that of free electrons?

(c) Discuss the group velocity

$$v_{\mathbf{k}} \;=\; \frac{1}{\hbar}\frac{\partial E_{\mathbf{k}}}{\partial k}$$

close to the Fermi wave number.

**Solution:**

(a) Close the the Fermi wave number we can use the approximation

$$\xi_{\mathbf{k}} = \frac{\hbar^2 k^2}{2m} - \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2 (k_F + \delta k)^2}{2m} - \frac{\hbar^2 k_F^2}{2m}, \tag{1}$$

where $\delta k = k - k_F$ is the deviation from the Fermi wave number. Considering only the regime close to $k_F$ we can neglect quadratic terms in $\delta k$ and obtain

$$\xi_{\mathbf{k}} = \frac{\hbar^2 k_F \delta k}{m} = v_F \hbar \delta k\,. \tag{2}$$

Here, $v_F = \hbar k_F / m$ is the Fermi velocity. With (2) the expression for the quasiparticle energy reads as

$$\begin{aligned} E_{\mathbf{k}} &= \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2} = \sqrt{v_F^2 \hbar^2 \delta k^2 + \Delta^2} \\ &= \Delta \sqrt{1 + \frac{v_F^2}{\Delta^2} \hbar^2 \delta k^2}\,. \end{aligned} \tag{3}$$

For small $\delta k$ we can use the approximation $\sqrt{1+x} \simeq 1 + \frac{1}{2}x$ and obtain

$$E_{\mathbf{k}} \simeq \Delta \left(1 + \frac{v_F^2}{2\Delta^2} \hbar^2 \delta k^2 \right) = \Delta + \frac{\hbar^2 (k - k_F)^2}{2(\Delta / v_F^2)}\,. \tag{4}$$

This dispersion relation of the Bogoliubov quasiparticles is shown in Fig. 1 together with that of electrons and holes in a normal metal. It is similar to that of rotons in superfluid $^4$He. We see that close to the Fermi momentum $\hbar k_F$ the Bogoliubov quasiparticles have a parabolic dispersion. Note that for the energy of normal electrons and holes relative to the chemical potential we have $\xi_{\mathbf{k},h} = -\xi_{\mathbf{k},e}$. The reason is that in order to generate a hole we have to remove an electron. Since removing an electron means to bring it to the chemical potential, we have $\xi_{\mathbf{k},h} = \mu - \varepsilon_{\mathbf{k},e} = -\xi_{\mathbf{k},e}$, since $\varepsilon_{\mathbf{k},e} = \mu + \xi_{\mathbf{k},e}$.

(b) The effective mass of the Bogoliubov quasiparticles is given by the curvature of their dispersion relation (4). We find

$$m_{\mathrm{qp}} = \left( \frac{1}{\hbar^2} \frac{\partial^2 E_{\mathbf{k}}}{\partial k^2} \right)^{-1} = \frac{\Delta}{v_F^2}\,. \tag{5}$$

To compare this effective mass with the free electron mass $m$ we can rewrite (5) and obtain the reduced effective mass

$$\frac{m_{\mathrm{qp}}}{m} = \frac{\Delta}{m v_F^2} = \frac{\Delta}{2E_F}\,, \tag{6}$$

where $E_F = \frac{1}{2} m v_F^2$ is the Fermi energy. Typically, $\Delta / E_F \sim 10^{-4} - 10^{-3}$ for metallic superconductors. Therefore, the effective mass of the Bogoliubov quasiparticles is orders of magnitude smaller than the free electron mass.

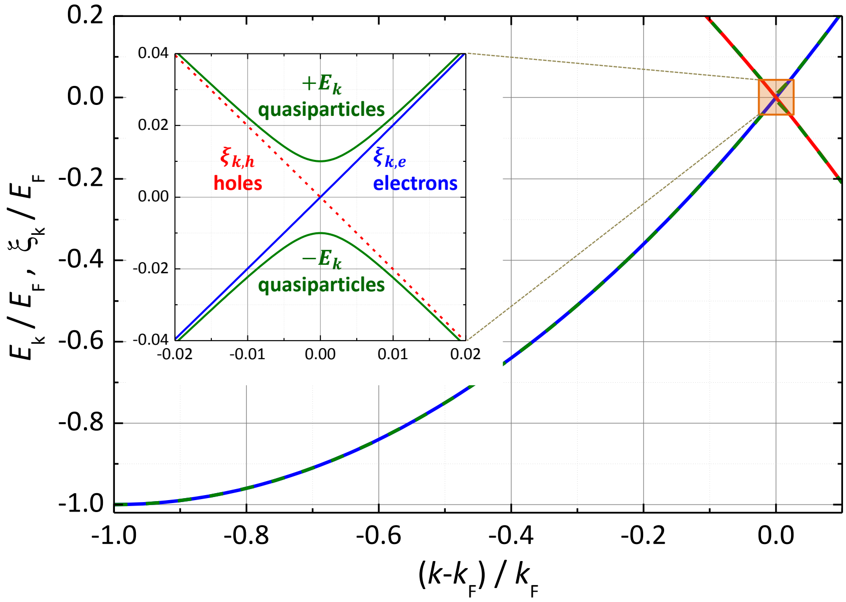


**Figure 1:** Energy dispersion of Bogoliubov quasiparticles (olive) close to the Fermi momentum calculated for $\Delta/E_{\rm F} = 0.01$. For comparison, the energy dispersion $\xi_{\mathbf{k},e}$ of free electrons (blue) and holes $\xi_{\mathbf{k},h} = -\xi_{\mathbf{k},e}$ (red) is also plotted. Due to the small value of $\Delta/E_{\rm F}$ differences are seen only in the regime very close to $k_{\rm F}$ shown in the inset.

(c) We can use eqs. (2) to (4) to derive the group velocity

$$\begin{aligned} v_{\mathbf{k}} &= \frac{1}{\hbar}\frac{\partial E_{\mathbf{k}}}{\partial k} = \frac{1}{2\hbar}\frac{2\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\hbar v_{\rm F} = v_{\rm F}\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}} \\ &= v_{\rm F}\frac{v_{\rm F}\hbar(k-k_{\rm F})}{\Delta + \frac{\hbar^2(k-k_{\rm F})^2}{2m_{\rm qp}}} = \frac{\hbar(k-k_{\rm F})}{\frac{\Delta}{v_{\rm F}^2} + \frac{\hbar^2(k-k_{\rm F})^2}{2m_{\rm qp}v_{\rm F}^2}} . \end{aligned} \tag{7}$$

Considering only the regime close to the Fermi momentum we can neglect terms of order $\mathcal{O}\left([k-k_{\rm F}]^2\right)$. In this approximation the group velocity is given by

$$v_{\mathbf{k}} = \frac{\hbar(k-k_{\rm F})}{m_{\rm qp}} + \mathcal{O}\left([k-k_{\rm F}]^2\right) \simeq \frac{\hbar(k-k_{\rm F})}{m_{\rm qp}} . \tag{8}$$

As already expected from Fig. 1 we obtain a vanishing group velocity at $k = k_{\rm F}$ due to the vanishing slope of the dispersion curve $E_{\mathbf{k}}(k-k_{\rm F})$. This can be understood by recalling that for $k = k_{\rm F}$ the Bogoliubov quasiparticles consist of an equal superposition ($|u_{\mathbf{k}}^2| = |v_{\mathbf{k}}^2| = 1/2$) of an electron and a hole with opposite momentum. For $k < k_{\rm F}$, they are more hole-like with negative group velocity. For $k > k_{\rm F}$, they are more electron-like with positive group velocity.

The quasiparticle dispersion curve approaches that of the free electrons and holes moving away from $k_{\rm F}$ and its maximum slope is given by the Fermi velocity. Then, we can use eq. (8) to estimate the maximum range $(k-k_{\rm F})$ in which the Bogoliubov quasiparticles live. We obtain

$$\delta k = k - k_{\rm F} \simeq \frac{m_{\rm qp} v_{\rm F}}{\hbar} = \frac{\Delta}{\hbar v_{\rm F}} . \tag{9}$$

Using the uncertainty relation $\delta k \cdot \delta x \geq 1$, we can estimate the spatial range of the Bogoliubov quasiparticles to

$$\delta x \simeq \frac{1}{\delta k} = \frac{\hbar v_{\rm F}}{\Delta} . \tag{10}$$

This estimate is close to the BCS coherence length $\xi_0 = \hbar v_{\rm F}/\pi\Delta$. Obviously, the Bogoliubov quasiparticles are superpositions of electrons and holes living in a volume $\sim \xi_0^3$.

## 4.8 Spin Susceptibility of BCS Superconductors

**Exercise:**

The spin polarization of the electrons in a normal metal is defined as $\delta n = \delta N/V = n_\uparrow - n_\downarrow$, i.e., by the difference in the density of spin-up and spin-down electrons. In a spin singlet superconductor, the condensate has vanishing total spin. Therefore, in order to evaluate $\delta n$ we only have to consider the difference $\delta n$ for the thermal excitations out of the superconducting ground state, the so-called Bogoliubov quasiparticles. This difference can be expressed in terms of a shifted Fermi-Dirac distribution as

$$\begin{aligned} \delta N &= \sum_{\mathbf{k}\sigma} \sigma\, \delta f^{\mathrm{loc}}_{\mathbf{k}\sigma}, \qquad \delta f^{\mathrm{loc}}_{\mathbf{k}\sigma} = f(E_\mathbf{k} + \delta E_{\mathbf{k}\sigma}) - f^0_\mathbf{k} \\ f(E_\mathbf{k}) &= \frac{1}{\mathrm{e}^{E_\mathbf{k}/k_\mathrm{B}T} + 1}, \qquad E_\mathbf{k} = \sqrt{\xi^2_\mathbf{k} + \Delta^2}\ . \end{aligned}$$

Here, $\Delta$ is the energy gap (for simplicity we assume an isotropic energy gap $\Delta_\mathbf{k} = \Delta$), $\sigma = \pm 1$ and $\delta E_{\mathbf{k}\sigma} = -\sigma\mu_\mathrm{B}\mu_0 H_\mathrm{ext}$ the Zeeman energy due to an external magnetic field. Show that the modification of the Pauli spin susceptibility $\chi_\mathrm{P}$ of a normal metal has the following form in the superconducting state:

$$\chi(T) = \chi_\mathrm{P} Y(T) \quad \text{with} \quad Y(T) = \int_0^\infty \frac{dx}{\cosh^2 \sqrt{x^2 + \left(\frac{\Delta(T)}{2k_\mathrm{B}T}\right)^2}}\ .$$

Here, $Y(T)$ is the so-called Yosida function.

**Solution:**

Since the energy shift $\delta E_{\mathbf{k}\sigma} = -\sigma\mu_\mathrm{B}\mu_0 H_\mathrm{ext}$ due to the applied magnetic field is small, we can do a Taylor expansion of the Fermi-Dirac distribution for the Bogoliubov quasiparticles $f(E_\mathbf{k} + \delta E_{\mathbf{k}\sigma})$ and keep only the leading term in $\delta E_{\mathbf{k}\sigma}$:

$$f(E_\mathbf{k} + \delta E_{\mathbf{k}\sigma}) = f^0_\mathbf{k} + \frac{\partial f^0_\mathbf{k}}{\partial E_\mathbf{k}} \delta E_{\mathbf{k}\sigma} = f^0_\mathbf{k} + \left(-\frac{\partial f^0_\mathbf{k}}{\partial E_\mathbf{k}}\right) \sigma\mu_\mathrm{B}\mu_0 H_\mathrm{ext}\ . \tag{1}$$

The deviation from local equilibrium, $f(E_\mathbf{k} + \delta E_{\mathbf{k}\sigma}) - f^0_\mathbf{k}$, then reads as

$$\delta f^{\mathrm{loc}}_{\mathbf{k}\sigma} = \left(-\frac{\partial f^0_\mathbf{k}}{\partial E_\mathbf{k}}\right) \sigma\mu_\mathrm{B}\mu_0 H_\mathrm{ext} \tag{2}$$

and we obtain the spin polarization $\delta n = \delta N/V$ to

$$\begin{aligned} \delta n &= \frac{1}{V}\sum_{\mathbf{k}\sigma} \sigma \delta f^{\mathrm{loc}}_{\mathbf{k}\sigma} = \frac{1}{V}\sum_{\mathbf{k}\sigma} \sigma \left(-\frac{\partial f^0_\mathbf{k}}{\partial E_\mathbf{k}}\right) \sigma\mu_\mathrm{B}\mu_0 H_\mathrm{ext} \\ &= \frac{\mu_\mathrm{B}\mu_0 H_\mathrm{ext}}{V} \sum_{\mathbf{k}\sigma} \left(-\frac{\partial f^0_\mathbf{k}}{\partial E_\mathbf{k}}\right)\ . \end{aligned} \tag{3}$$

With this result the magnetization can be expressed as

$$M = \mu_{\mathrm{B}}\delta n = \frac{\mu_{\mathrm{B}}^2\mu_0 H_{\mathrm{ext}}}{V}\sum_{\mathbf{k}\sigma}\left(-\frac{\partial f_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}}\right) . \tag{4}$$

With $M = \chi(T)H_{\mathrm{ext}}$ we obtain

$$\chi(T) = \frac{\mu_0\mu_{\mathrm{B}}^2}{V}\sum_{\mathbf{k}\sigma}\left(-\frac{\partial f_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}}\right) . \tag{5}$$

To derive an analytical expression for $\chi(T)$ we have to convert the summation into an integration in $k$- and finally in energy space. With $Z(k)d^3k = D(E_{\mathbf{k}})dE_{\mathbf{k}} = D(\xi_{\mathbf{k}})d\xi_{\mathbf{k}}$ (conservation of states) we obtain

$$\begin{aligned}\frac{\chi(T)}{\mu_0\mu_{\mathrm{B}}^2} &= \frac{1}{V}\int_0^\infty d^3k\, Z(k)\left(-\frac{\partial f_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}}\right) = \frac{1}{V}\int_0^\infty dE_{\mathbf{k}}\, D(E_{\mathbf{k}})\left(-\frac{\partial f_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}}\right)\\ &= \frac{1}{V}\int_\mu^\infty d\xi_{\mathbf{k}}\, D(\mu+\xi_{\mathbf{k}})\left(-\frac{\partial f_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}}\right) .\end{aligned} \tag{6}$$

Here, $Z(k)$ and $D(\xi_{\mathbf{k}})$ are the density of states in $k$- and energy space for both spin directions and $\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu = (\hbar^2k^2/2m) - \mu$ is the energy relative to the chemical potential. Since the function $\partial f_{\mathbf{k}}^0/\partial E_{\mathbf{k}}$ is finite only in a narrow energy interval $\sim k_{\mathrm{B}}T$ around the chemical potential $\mu$, we can use $D(\mu+\xi_{\mathbf{k}}) \simeq D(E_{\mathrm{F}}) = const$ and obtain

$$\frac{\chi(T)}{\mu_0\mu_{\mathrm{B}}^2} = \frac{D(E_{\mathrm{F}})}{V}\int_{-\mu}^\infty d\xi_{\mathbf{k}}\left(-\frac{\partial f_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}}\right) = \frac{D(E_{\mathrm{F}})}{V}\frac{1}{4k_{\mathrm{B}}T}\int_{-\mu}^\infty \frac{d\xi_{\mathbf{k}}}{\cosh^2\frac{E_{\mathbf{k}}}{2k_{\mathrm{B}}T}} \tag{7}$$

With the substitution $x = \xi_{\mathbf{k}}/2k_{\mathrm{B}}T$ and $N_{\mathrm{F}} = D(E_{\mathrm{F}})/V$ we finally obtain

$$\chi(T) = \underbrace{\mu_0\mu_{\mathrm{B}}^2N_{\mathrm{F}}}_{=\chi_{\mathrm{P}}}\ \frac{1}{2}\underbrace{\int_{-\infty}^{\infty}\frac{dx}{\cosh^2\sqrt{x^2+\left(\frac{\Delta(T)}{2k_{\mathrm{B}}T}\right)^2}}}_{=2Y(T)} . \tag{8}$$

Here we have set the lower integration limit to $-\infty$ since typically $\mu/2k_{\mathrm{B}}T \gg 1$ for metals.

We see that the spin susceptibility $\chi(T)$ of a BCS superconductor is given by the product of the temperature independent normal state Paul spin susceptibility $\chi_{\mathrm{P}}$ and the Yosida function $Y(T)$:

$$\chi_{\mathrm{BCS}}(T) = \mu_0\mu_{\mathrm{B}}^2N_{\mathrm{F}}\cdot Y(T) = \chi_{\mathrm{P}}\cdot Y(T) \tag{9}$$

$$\text{with } Y(T) = \int_0^\infty \frac{dx}{\cosh^2\sqrt{x^2+\left(\frac{\Delta(T)}{2k_{\mathrm{B}}T}\right)^2}} . \tag{10}$$

The Yosida function is plotted in Fig. 2 as a function of $\Delta(T)/k_{\mathrm{B}}T$.

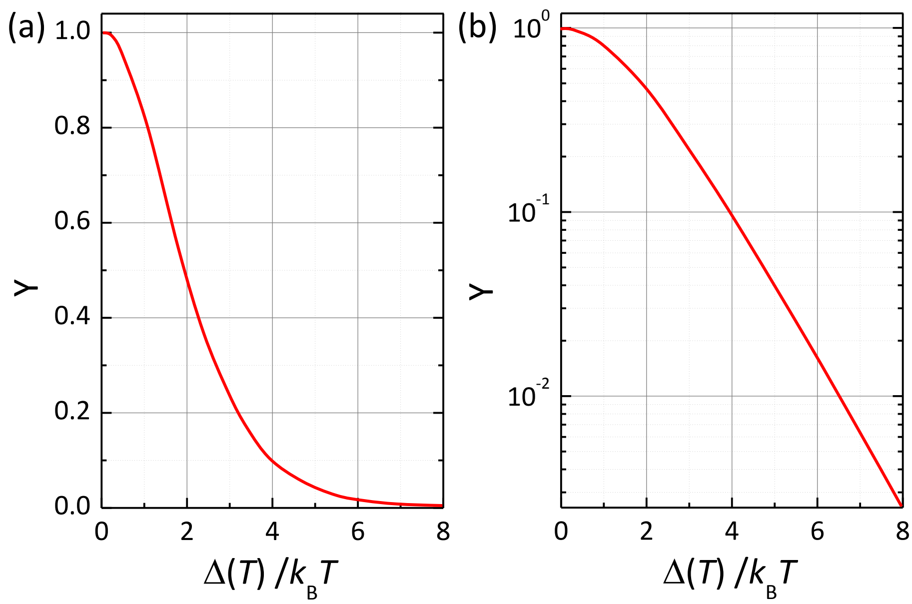


**Figure 2:** Yosida function plotted versus $\Delta(T)/k_\mathrm{B}T$ using (a) a linear and (b) logarithmic scale.