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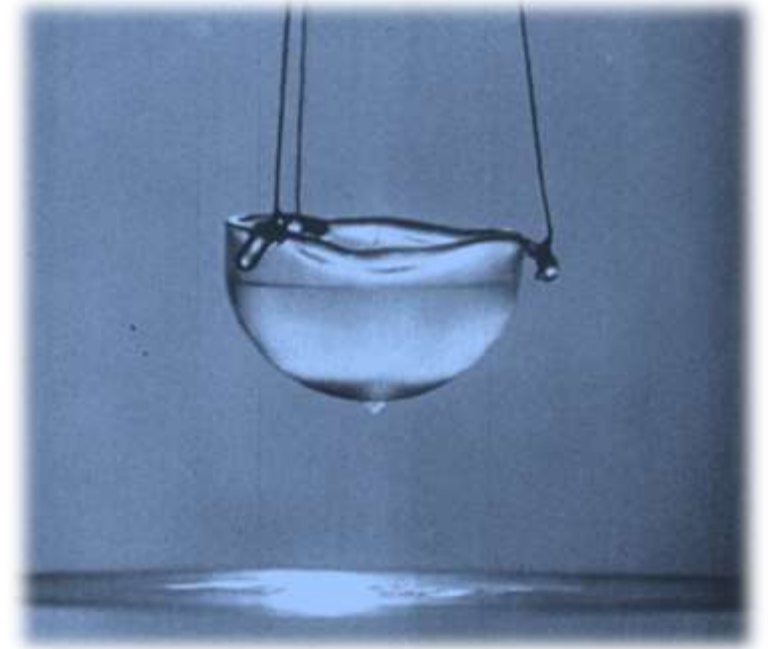
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**TUM**

# Superconductivity and Low Temperature Physics II



**Lecture Notes  
Summer Semester 2022**

**R. Gross  
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# Chapter 1

## Quantum Liquids

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### I.6 $^3\text{He}$ / $^4\text{He}$ mixtures

# I.1 Foundations and General Properties

- **Satyendranath Bose (1924)**

- letter to Einstein: proposes new method to derive Planck's law for spectral density of electromagnetic radiation emitted by a black body
- Bose describes the radiation emitted by a black body in thermal equilibrium by a gas on non-interacting identical particles  
→ *photons*
- description is in perfect agreement with Einstein's explanation of the photoelectric effect
- Einstein immediately recognized that this description can be transferred to many other systems:
  - (i) gases of identical particles with finite mass,
  - (ii) phonons
- Einstein strongly supported the publication of Bose's results, he translated the paper to German

Satyendranath Bose: [Plancks Gesetz und Lichtquantenhypothese](#).  
In: *Zeitschrift für Physik*. 26 (1924), S. 178-181.



**Satyendranath Bose (around 1926)**

# I.1 Foundations and General Properties

What happens if you cool down a gas of **non-interacting** bosons ?

Below a certain temperature the particles feel each other due to their finite de Broglie wavelength, although they are non-interacting !



phase transition driven by statistics of particles  
(not by their interactions)

## Bose-Einstein Condensation

thermodynamic phase transition in a gas of non-interacting particles:  
**driven by statistics of particles and not by interactions**

# I.1 Foundations and General Properties

- 1924: prediction of **Bose-Einstein Condensation (BEC)** by Einstein
- 1995: first observation of BEC on alkali atoms by



Eric A. Cornell



Wolfgang Ketterle



Carl E. Wieman

## *Nobel Prize in Physics 2001*

"for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates"

→ long period between theoretical prediction and experimental realization:

***high density, ultra-cold gas of non-interacting atoms required***

→ problem: gas liquefies or solidifies due to finite interactions

→ more simple to realize:

***interacting quantum gas, e.g. superfluid  $^4\text{He}$***

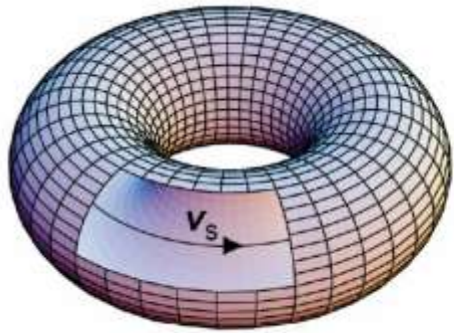
# I.1 Foundations and General Properties

- nomenclature

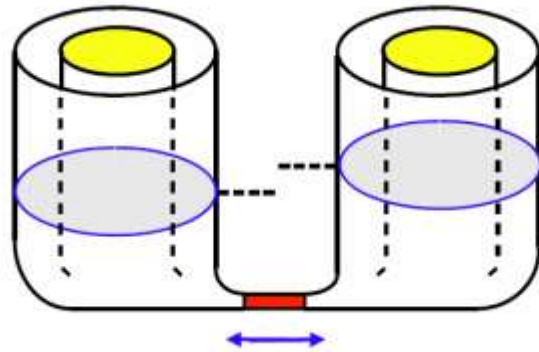
- system of **non-interacting** particles: **classical gas** e.g. classical ideal gas
  - if quantum fluctuations dominate: **quantum gas** e.g. Bose-Einstein Condensate, Fermi gas
- system of **interacting** particles: **classical liquid** e.g. water, liquid helium-4
  - if quantum fluctuations dominate: **quantum liquid** e.g. superfluid helium-4, Fermi liquid



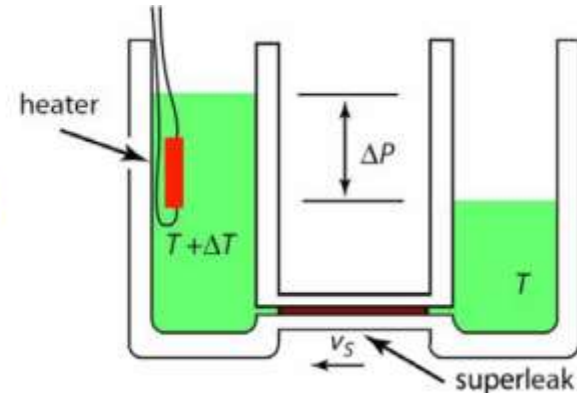
# I.1.1 Quantum Liquids



supercurrents

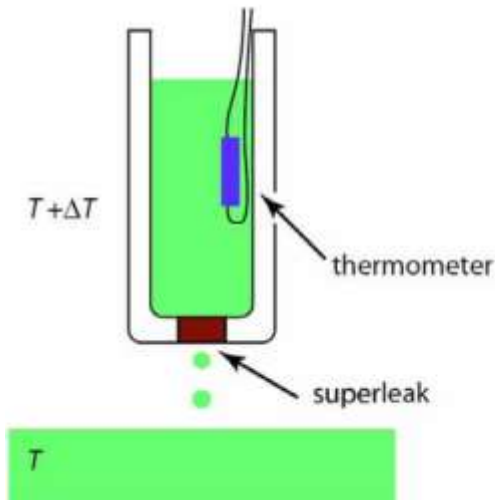


U-tube oscillation

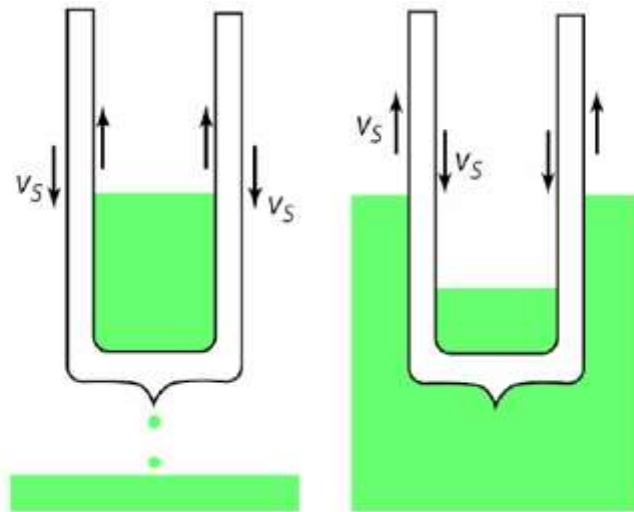


thermomech. effects

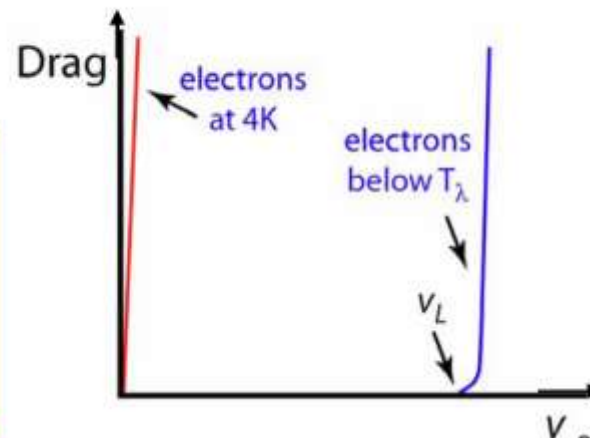
quantum liquids & condensates show fascinating properties → discussed in this lecture



superleaks



beaker film flow



critical velocity



# I.1.1 Quantum Liquid

- classical liquids and solids – characteristic energy scales:

characteristic energies: (1) average interaction energy:  $\langle V \rangle$  or  $\langle E_{\text{pot}} \rangle$   
 (2) thermal (kinetic) energy:  $k_B T$  or  $\langle E_{\text{kin}} \rangle$

criterion for liquid/solid: *liquid*:  $k_B T \gg \langle V \rangle$   
*solid*:  $k_B T \ll \langle V \rangle$

- characteristic length scales in a gas:

- particle distance:  $d = n^{-1/3}$   
 - interaction range:  $r_0 \simeq a$  (scattering length  $a$ )

- de Broglie wavelength:

$$\lambda_T = \sqrt{\frac{h^2}{2\pi m k_B T}} = \sqrt{\frac{2\pi \hbar^2}{m k_B T}}$$

$$\frac{\hbar^2 k_T^2}{2m} \simeq k_B T \quad k_T = \frac{2\pi}{\lambda_T}$$

$$\frac{h^2}{2m \lambda_T^2} \simeq k_B T \rightarrow \lambda_T \simeq \left( \frac{h^2}{2m k_B T} \right)^{1/2}$$

**classical non-interacting gas:**  $d \gg r_0$ , since atoms interact only during collisions

# I.1.1 Quantum Liquids

- calculation of the average thermal wavelength of a statistical ensemble (1)

we consider a classical gas of  $N$  particles:

→ probability for a specific particular configuration at temperature  $T$  is given by Boltzmann distribution

$$P(\mathbf{p}_1, \dots, \mathbf{p}_N, \mathbf{r}_1, \dots, \mathbf{r}_N) = \frac{1}{Z_N} \exp(-\beta \mathcal{H}) \quad \beta = 1/k_B T, \mathcal{H} = \text{Hamiltonian}$$

$$Z_N = \frac{1}{N!} \int d^3 p_1, \dots, d^3 p_N d^3 r_1, \dots, d^3 r_N \exp(-\beta \mathcal{H})$$

classical partition function,  
 $\frac{1}{N!}$  results from fact that particles cannot be distinguished

partition function can be factorized into two terms (related to kinetic and potential energy)

$$Z_N = \underbrace{\frac{1}{N!} \prod_i \int d^3 p_i \exp(-p_i^2 / 2mk_B T)}_{\substack{\text{kinetic energy contribution} \\ \text{product of independent factors} \\ [(2\pi mk_B T)^{3/2}]^N}} \cdot \underbrace{\int d^3 r_1, \dots, d^3 r_N \exp\left(-\frac{1}{2}\beta \sum_{i \neq j} V(\mathbf{r}_i - \mathbf{r}_j)\right)}_{\text{potential energy contribution}}$$

(momenta of particles are statistically independent)

# I.1.1 Quantum Liquids

- calculation of the average thermal wavelength of a statistical ensemble (2)

probability that particle has momentum in volume  $d^3p$  of phase space

$$P(\mathbf{p})d^3p = (2\pi mk_B T)^{3/2} \exp(-p_i^2 / 2mk_B T) d^3p$$

fraction of particles with momentum between  $p$  and  $p + dp$

$$P_{\text{MB}}(p)dp = \frac{4\pi p^2}{\underbrace{(2\pi mk_B T)^{3/2}}_{p_T^3}} \exp(-p^2 / 2mk_B T) dp$$

Maxwell-Boltzmann distribution

the most likely momentum of the particles is  $p = \sqrt{2mk_B T}$ , the average momentum is  $\bar{p} = \sqrt{8mk_B T / \pi}$

→ typical value of the **thermal de Broglie wavelength**

$$\lambda_T = \frac{h}{p_T} = \sqrt{\frac{h^2}{2\pi mk_B T}} = \sqrt{\frac{2\pi \hbar^2}{mk_B T}}$$

# I.1.1 Quantum Liquids

- calculation of the average thermal wavelength of a statistical ensemble (3)

the wave number  $k$  is given by

$$p_T = \int_{-\infty}^{\infty} dp \exp\left(-\frac{p^2/2m}{k_B T}\right)$$

$$p_T = \sqrt{2\pi m k_B T}$$

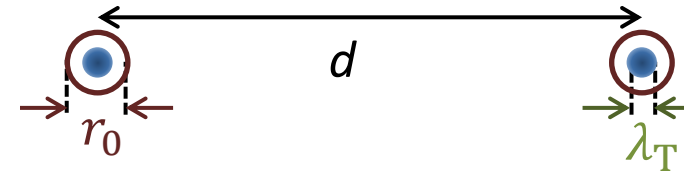


$$\lambda_T = \frac{h}{p_T} = \frac{h}{\sqrt{2\pi m k_B T}}$$

# I.1.1 Quantum Liquids

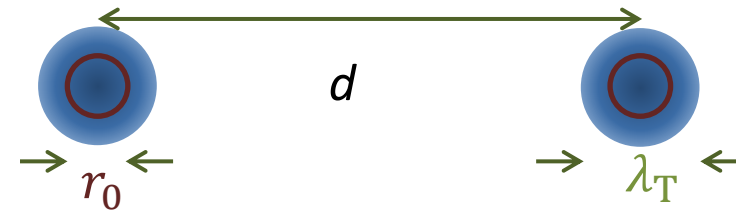
- cooling down a classical gas  $\rightarrow \lambda_T = \sqrt{\frac{h^2}{2\pi m k_B T}}$  increases and we can distinguish three regimes

(1) classical regime:  $\lambda_T \ll r_0 \ll d$



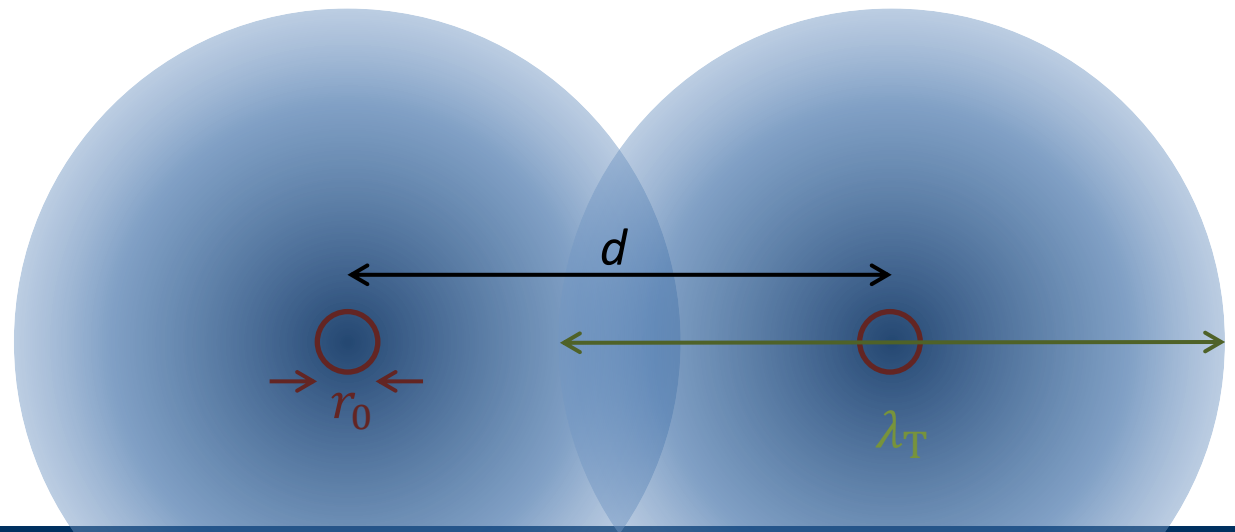
(2) quantum collisions:  $r_0 \ll \lambda_T \ll d$

*collisions can no longer be treated classically*



(3) quantum degeneracy:  $r_0 \ll d \leq \lambda_T$

*all degrees of freedom of the gas must be treated quantum mechanically*



# I.1.1 Quantum Liquids

- define new energy scale related to  $d = \lambda_T$

## (3) zero-point energy $E_{zp}$

with - particle density  $n = N/V$ ,  $d = n^{-1/3}$   
 - de Broglie wavelength  $\lambda_T = d$

→ momentum uncertainty  $\Delta p = \hbar \Delta k = \frac{h}{\lambda_T} = \frac{h}{d} = hn^{1/3}$

→ **zero-point energy**

$$E_{zp} = \frac{\Delta p^2}{2m} = \frac{h^2}{2md^2} = \frac{h^2 n^{2/3}}{2m} = \pi k_B T_{zp}$$

characteristic temperature

for an ideal gas the energy  $E = \pi k_B T$  is assumed; the factor  $\pi$  results from the partition function

# I.1.1 Quantum Liquids

- **classification of liquids** (interacting liquids with  $\langle V \rangle > 0$ ) :

(a) classical liquid:  $T_{zp} \ll T_{\text{melting}} \ll T$  (quantum fluctuation play no role)

(b) quantum liquid:  $T_{\text{melting}} < T < T_{zp}$  (quantum fluctuation dominate)

proportional to  $\langle V \rangle$  or  $\langle E_{\text{pot}} \rangle$

quantum parameter:

$$\Lambda \equiv \frac{\langle E_{zp} \rangle}{\langle E_{\text{pot}} \rangle}$$

quantum liquid:

$$\Lambda > 1$$

classical liquid:

$$\Lambda < 1$$



# I.1.1 Quantum Liquids

- **classification of liquids** (non-interacting liquids:  $\langle V \rangle = 0$ ) :

(a) classical liquid:  $T_{zp} < T$

(b) quantum liquid:  $T < T_{zp}$

$$T_{zp} = \frac{h^2 n^{2/3}}{2\pi m k_B} = \frac{h^2}{2\pi m k_B d^2} \iff T = \frac{h^2}{2\pi m k_B \lambda_T^2}$$

(a) classical liquid:  $\lambda_T \leq n^{-1/3} \simeq d \rightarrow n \cdot \lambda_T^3 \leq 1$

(a) quantum liquid:  $\lambda_T \geq n^{-1/3} \simeq d \rightarrow n \cdot \lambda_T^3 \geq 1$

average particle distance

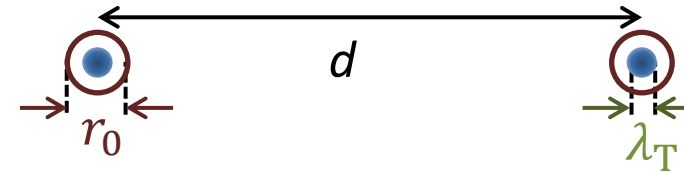
e.g. electrons @ 300 K:  $\lambda_T = 4.2 \text{ nm}$ ,  $d \simeq 0.2 \text{ nm}$  for Cu ( $n \simeq 8.5 \times 10^{28} \text{ m}^{-3}$ )

**→ particles become indistinguishable → degenerate quantum liquid**

# I.1.1 Quantum Liquids

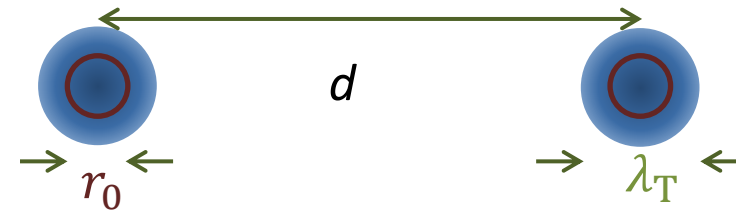
- cooling down a classical gas  $\rightarrow \lambda_T = \sqrt{\frac{h^2}{2\pi m k_B T}}$  increases and we can distinguish three regimes

(1) classical regime:  $\lambda_T \ll r_0 \ll d$



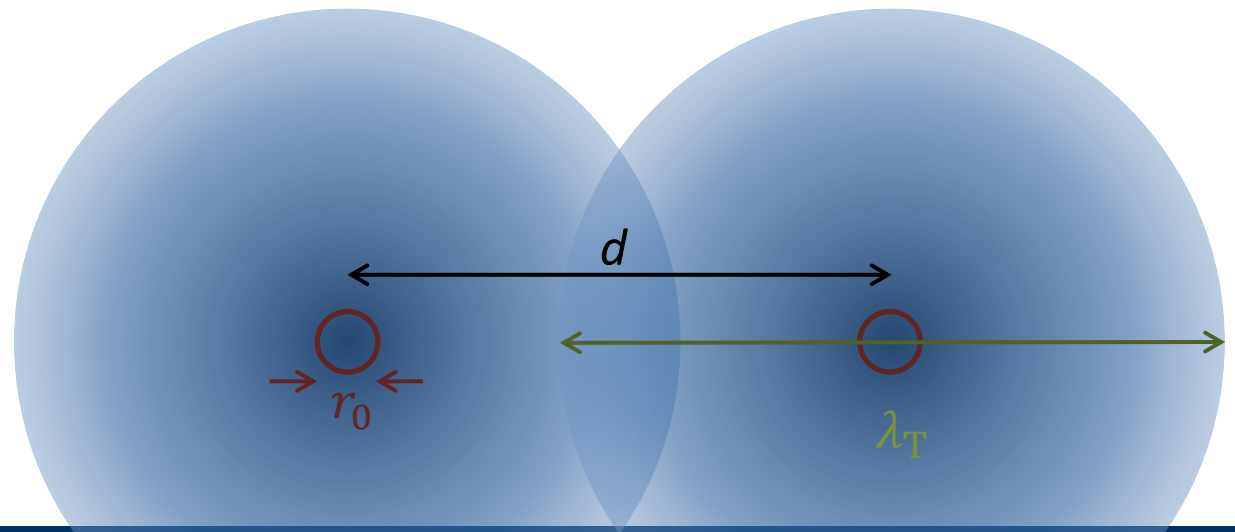
(2) quantum collisions:  $r_0 \ll \lambda_T \ll d$

*collisions can no longer be treated classically*

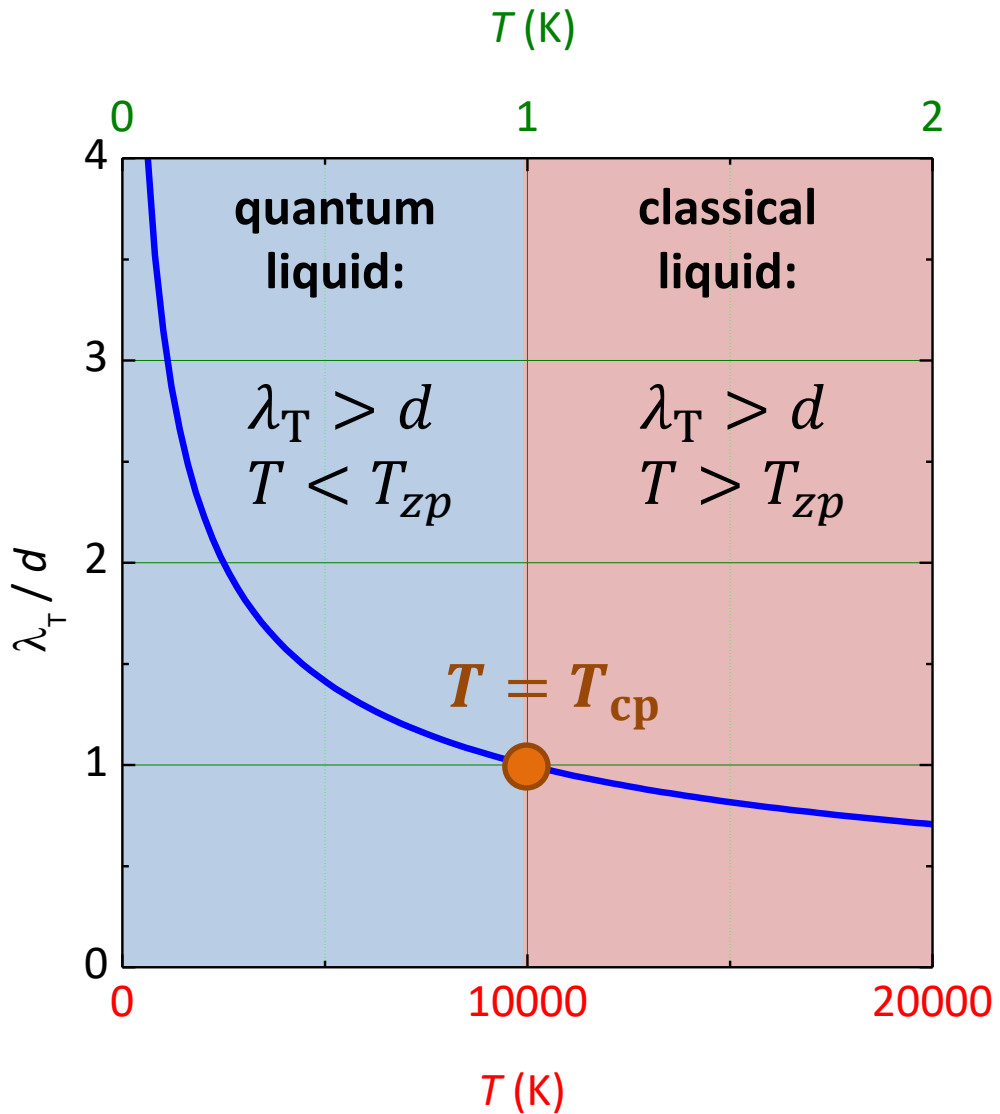


(3) quantum degeneracy:  $r_0 \ll d \leq \lambda_T$

*all degrees of freedom of the gas must be treated quantum mechanically*



# I.1.1 Quantum Liquids



←  $^4\text{He}$ :  $m = 6.65 \times 10^{-27}$  kg  
 $n \simeq 4 \times 10^{27}$  m<sup>-3</sup>

$$\lambda_T = \sqrt{\frac{h^2}{2\pi m k_B T}} \propto \frac{1}{\sqrt{T}}$$

←  $e^-$ :  $m = 9.11 \times 10^{-31}$  kg  
 $n \simeq 3.6 \times 10^{27}$  m<sup>-3</sup>

**degenerate:**  
indistinguishable particles

**non-degenerate:**  
distinguishable particles

# I.1.1 Quantum Fluids

- quantum statistics:

photons, mesons,  $^4\text{He}$ -atoms, ...

integral spin

$$S = 0, \hbar, 2\hbar, 3\hbar, \dots$$

electrons, nucleons,  $^3\text{He}$ -atoms, ...

half-integral spin

$$S = \hbar/2, 3\hbar/2, 5\hbar/2, \dots$$

indistinguishability of identical particles:

*symmetry of many-particle wavefunction with respect to particle exchange*

$$\begin{aligned} \wp \Psi(1,2) &\equiv \Psi(2,1) = e^{i\alpha} \Psi(1,2) \\ \wp \Psi(2,1) &\equiv \Psi(1,2) = (e^{i\alpha})^2 \Psi(1,2) \end{aligned} \quad \Rightarrow \quad e^{i\alpha} = \pm 1$$

- bosons** (integral spin)

$$\Psi_{AB}(1,2) = \frac{1}{\sqrt{2}} [\Psi_A(1)\Psi_B(2) + \Psi_A(2)\Psi_B(1)]$$

- fermions** (semi-integral spin)

$$\Psi_{AB}(1,2) = \frac{1}{\sqrt{2}} [\Psi_A(1)\Psi_B(2) - \Psi_A(2)\Psi_B(1)]$$

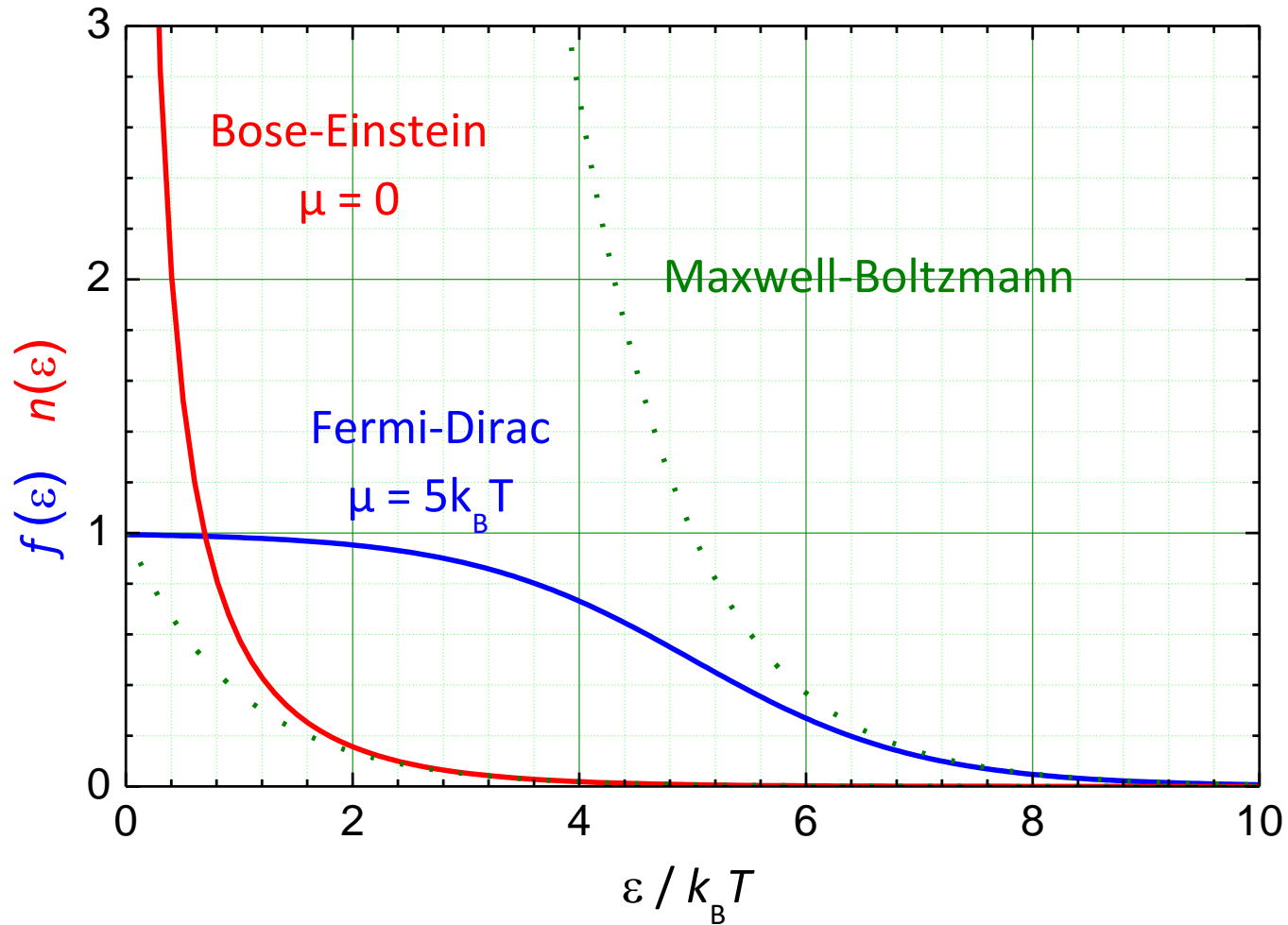
- Pauli principle:** *two fermions cannot occupy the same quantum state*

# I.1.1 Quantum Fluids

- example: two particles (1,2) in quantum states A and B:

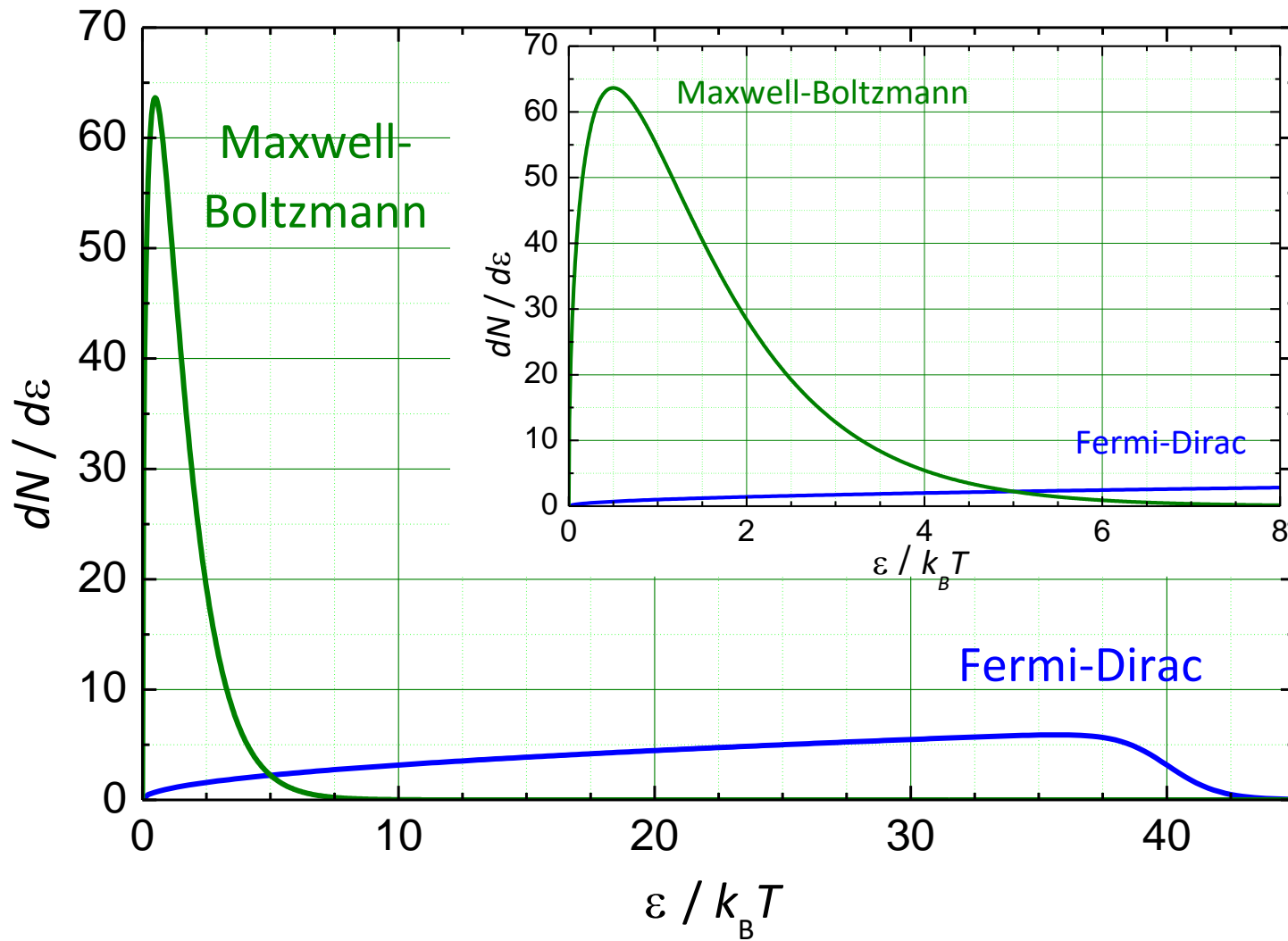
classical:	4 possibilities double-occupancy ratio: <b>1/2</b>	$A(1)A(2)$ $A(1)B(2)$	$B(1)B(2)$ $B(1)A(2)$
bosons:	3 possibilities double-occupancy ratio: <b>2/3</b>	$A(1)A(2)$ $\frac{1}{\sqrt{2}} \{A(1)B(2) + A(2)B(1)\}$ $B(1)B(2)$	
fermions:	1 possibility double-occupancy ratio: <b>0</b>	$\frac{1}{\sqrt{2}} \{A(1)B(2) - A(2)B(1)\}$	
<b>bosons:</b>	tendency to occupy the same quantum state		
<b>fermions:</b>	multiple occupancy of the same quantum state is forbidden		

# I.1.1 Quantum Fluids



**Bose-Einstein, Fermi-Dirac and Maxwell-Boltzmann distributions** as a function of normalized energy. For the Bose-Einstein distribution we assumed  $\mu = 0$ , for the Fermi-Dirac distribution  $\mu = 5k_B T$  (for a metal  $\mu$  is several eV corresponding to temperatures of several 10 000 K). For the Maxwell-Boltzmann distributions we also assumed  $\mu = 0$  and  $5k_B T$  so that the classical distribution coincides with the quantum mechanical distributions for large  $\varepsilon \gg k_B T$ .

# I.1.1 Quantum Fluids

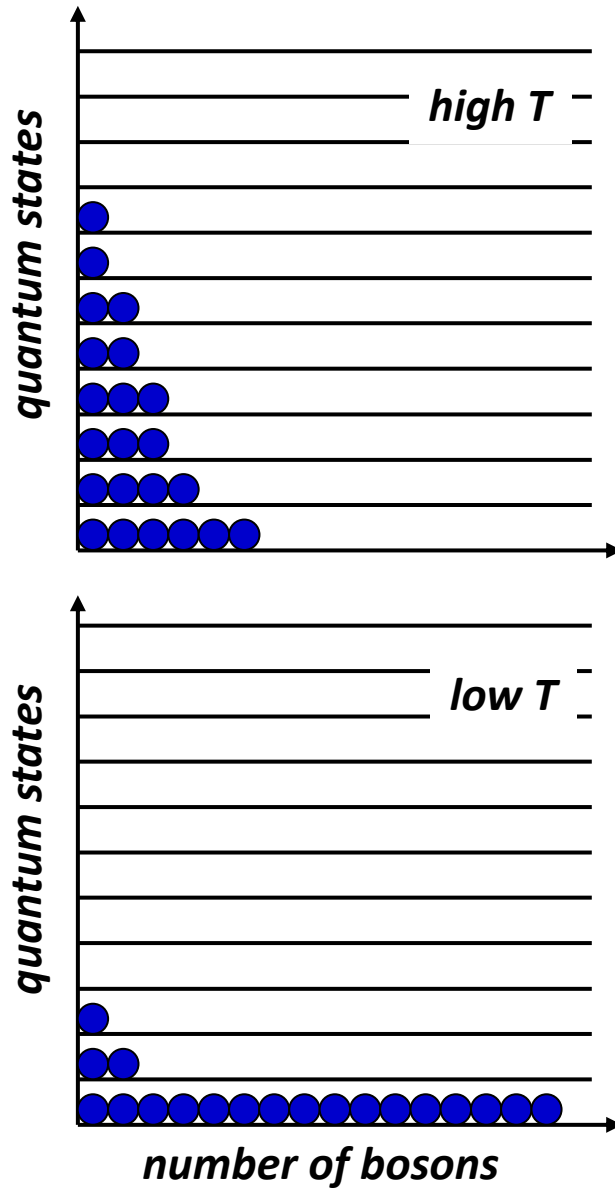


particle number per energy interval: 
$$\frac{dN}{d\epsilon} = D(\epsilon) \cdot f(\epsilon)$$

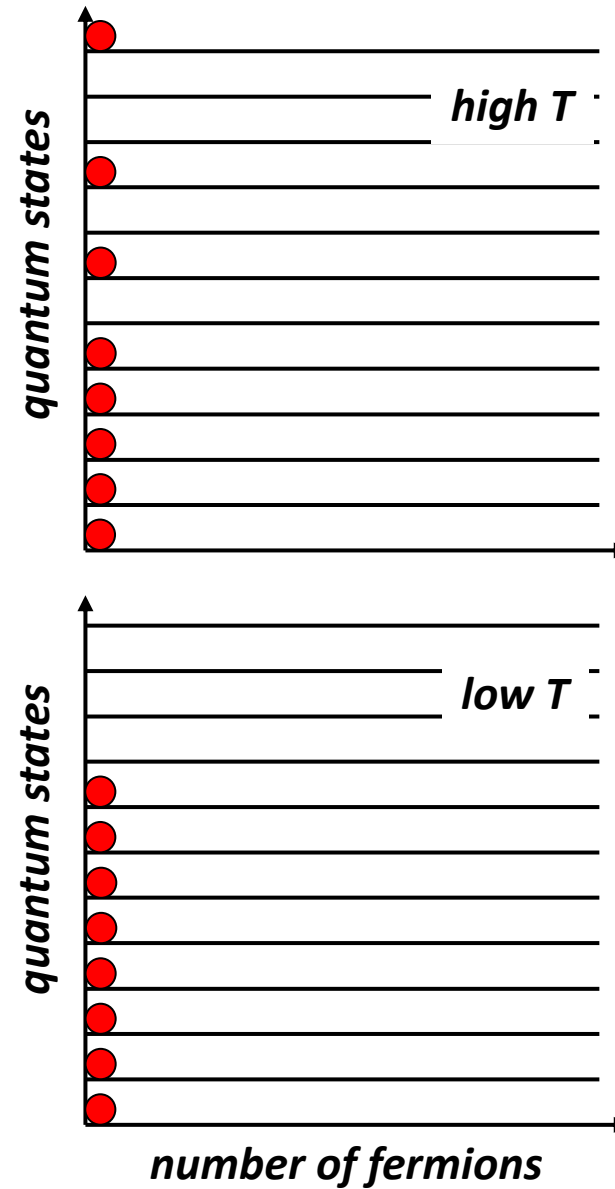


# I.1.1 Quantum Fluids

*boson*



*fermions*



# I.1.1 Quantum Fluids

- well-known quantum fluids:

- 1. liquid helium:**

  - small mass and  $d$  → high  $T_{zp}$  (a few K)
  - small  $\langle V \rangle$  → liquid down to  $T = 0$
  - $^4\text{He}$ : spin 0 → Bose liquid
  - $^3\text{He}$ : spin  $\frac{1}{2}$  → Fermi liquid
  
- 2. electrons in metals:**

  - “non-interacting” electron gas
  - very small electron mass, higher density as for He
    - very high  $T_{zp}$  (a few  $10^4$  K)
    - $T_{zp} \gg T_{\text{melting}}$
  - spin  $\frac{1}{2}$  → Fermi liquid
  
- 3. nuclear matter:**

  - fermions in nuclei, neutron stars, ...
  - very high density → very high  $T_{zp}$  (a few  $10^{12}$  K)
  - spin  $\frac{1}{2}$  → Fermi liquid

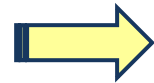
# I.1.1 Quantum Fluids

	mass (kg)	$n$ (m <sup>-3</sup> )	$T_{zp}$ (K)	statistics
<sup>3</sup> He	$5.01 \times 10^{-27}$	$1.63 \times 10^{28}$	6.50	Fermi
<sup>4</sup> He	$6.65 \times 10^{-27}$	$2.18 \times 10^{28}$	5.93	Bose
electrons in Na	$9.11 \times 10^{-31}$	$2.65 \times 10^{28}$	$4.94 \times 10^4$	Fermi
nuclear matter	$1.67 \times 10^{-27}$	$1.95 \times 10^{38}$	$1 \times 10^{12}$	Fermi

$$E_{zp} = \frac{\Delta p^2}{2m} = \frac{h^2}{2md^2} = \frac{h^2 n^{2/3}}{2m} = \pi k_B T_{zp}$$

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- I.2.1 Bose-Einstein Condensation
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- I.5.1 normal fluid  $^3\text{He}$
- I.5.2 solid  $^3\text{He}$  and Pomeranchuk effect
- I.5.3 superfluid  $^3\text{He}$

### I.6 $^3\text{He}$ / $^4\text{He}$ mixtures

# I.1.2 Helium

- *history:  $^4\text{He}$*

*application of  $^4\text{He}$  in low temperature physics*

- soon after discovery on earth, race for the liquefaction of He
  - main opponents: **Heike Kamerlingh Onnes** and **James Dewar**
  - Onnes extracted 360 l of He gas out of Monazit sand from North Carolina
- **1908**: first liquefaction: July 10, 1908
- **1925**: first liquefaction of He in Germany (March 7, 1925: 4.2 K, 200 ml) by **Walther Meißner**, 3<sup>rd</sup> system world-wide besides Leiden and Toronto
- **1932**: Linde supplies first industrial He liquefier to University of Charkov, Ukraine
- **1947**: first commercial liquefier by **S.C. Collins**



**Heike Kamerlingh  
Onnes**  
\* 21. 09. 1853  
† 21. 02. 1926



**James Dewar**  
\* 20. 09. 1842  
† 27.03.1923



SIR JAMES DEWAR

Sir James Dewar, (1842-1923) was a chemist and physicist, best known for his work with low-temperature phenomena. Dewar was born in Kincardine, Scotland, and educated at the University of Edinburgh. He was professor of experimental natural philosophy at the University of Cambridge, England, in 1875 and professor of chemistry at the Royal Institution of Great Britain in 1877, where he was appointed director of the Davy-Faraday Research Laboratory.

Dewar developed structural formulas for benzene (1867). He studied the specific heat of hydrogen and was the first person to produce hydrogen in liquid form (1898) and to solidify it (1899). He constructed a machine for producing liquid oxygen in quantity (1891). He invented the Dewar flask or thermos (1892) and co-invented cordite (1889), a smokeless gunpowder, with Sir Frederick Abel. His discovery (1905) that cooled charcoal can be used to help create high vacuums later proved useful in atomic physics. Dewar was knighted in 1904.

The Dewar flask or vacuum flask/bottle is a container for storing hot or cold substances, i.e. liquid air. It consists of two flasks, one inside the other, separated by a vacuum. The vacuum greatly reduces the transfer of heat, preventing a temperature change. The walls are usually made of glass because it is a poor conductor of heat; its surfaces are usually lined with a reflective metal to reduce the transfer of heat by radiation. Dewar used silver. The whole fragile flask rests on a shock-absorbing spring within a metal or plastic container, and the air between the flask and the container provides further insulation. The common thermos bottle is an adaptation of the Dewar flask. Dewar invented the Dewar flask in 1892 to aid him in his work with liquid gases.

The vacuum flask was not manufactured for commercial/home use until 1904, when two German glass blowers formed Thermos GmbH. They held a contest to rename the vacuum flask and a resident of Munich submitted "Thermos", which came from the Greek word "Therme" meaning "hot."

# I.1.2 Helium

- *history:  $^3\text{He}$*

- **1933**: discovery by *Oliphant, Kinsey* and *Rutherford* (they believed that  $^3\text{He}$  is unstable, disproven by *Alvarez* and *Cornog* in 1939)
- **1947**: first experiments with gaseous  $^3\text{He}$
- **1949**: first condensation of  $^3\text{He}$  gas (*Sydoriak and Co.*)
- **1956**: theory of Fermi liquids (*L.D. Landau*, **Nobel Prize in Physics 1962**)
- **1960s**: first experiments with the Fermi liquid  $^3\text{He}$  (*various groups*)
- **1963**: first theory of superfluid  $^3\text{He}$  (*Anderson and Morel, Balian and Werthamer*)
- **1971**: discovery of superfluid phase transition at 2 mK (*Lee, Osheroff, Richardson*, **Nobel Prize in Physics 1996**)
- **1975**: extension of BCS theory to triplet pairing (*Leggett*, **Nobel Prize in Physics 2003**)
- **1977**: first experiments on superfluid  $^3\text{He}$  in Germany at WMI (*Eska, Schuberth, Uhlig*)



# I.1.2 Helium

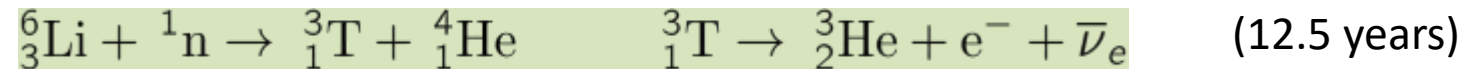
- *He isotops:  $^4\text{He}$  and  $^3\text{He}$*

- $^4\text{He}$ : - obtained out of natural gas (e.g. sources in USA, Poland, Algeria contain up to 10%)

- $L = 0, S = 0$ , nuclear spin  $I = 0 \rightarrow$  *boson*

- $^3\text{He}$ : - natural amount of  $^3\text{He}$  in  $^4\text{He}$  is only 0.14 ppm

- artificial generation of  $^3\text{He}$  by nuclear reaction



- $^3\text{He}$  is very expensive, several 1000 € per liter

- $L = 0, S = 0$ , nuclear spin  $I = 1/2 \rightarrow$  *fermion*

- despite their simple electronic structure, the liquids of both He isotopes show
  - *a rich variety of exotic phenomena*
  - *pronounced differences due to different quantum statistics*



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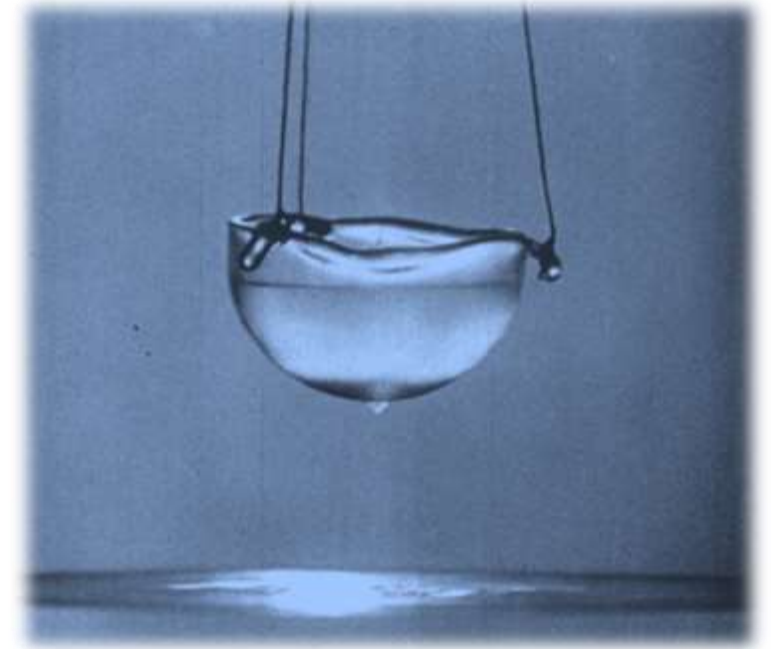
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# Superconductivity and Low Temperature Physics II




**Lecture No. 2  
05 May 2022**

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© Walther-Meißner-Institut**

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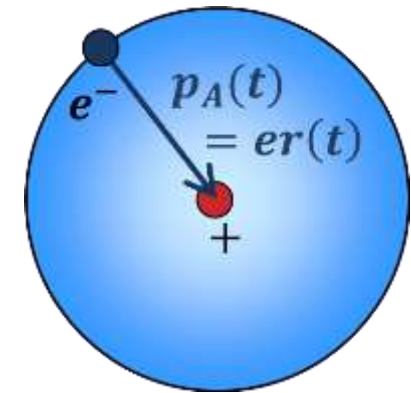
# I.1.3 Van der Waals Bonding

- **neutral atom**, average dipole moment vanishes:  $\langle \mathbf{p}_A \rangle = \mathbf{0}$

→ dipole moment induced by electric field:  $\mathbf{p}_A^{\text{ind}} \propto \mathbf{E}$

proportionality constant is **polarizability**  $\alpha_A$  of atom A:

$$\mathbf{p}_A^{\text{ind}} = \alpha_A \mathbf{E}$$



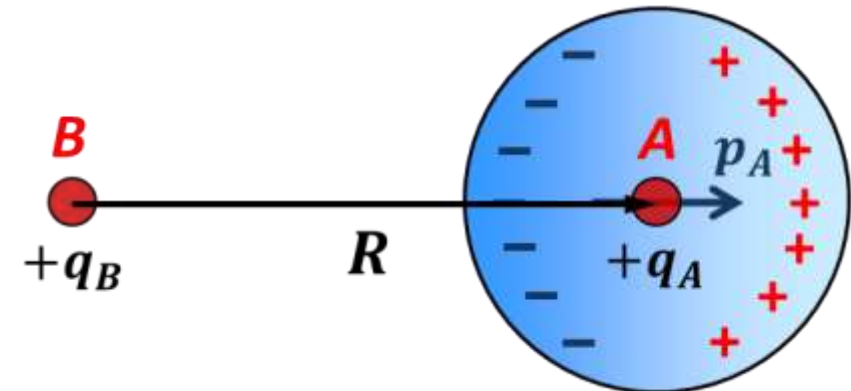
- **assumption:**

$\mathbf{E}$ -field is generated by charge  $q_B$  of ion B at distance  $R$

$$\mathbf{p}_A^{\text{ind}} = \alpha_A \mathbf{E}_{B,\text{ion}} = \alpha_A \frac{q_B}{4\pi\epsilon_0 R} \hat{\mathbf{R}}$$

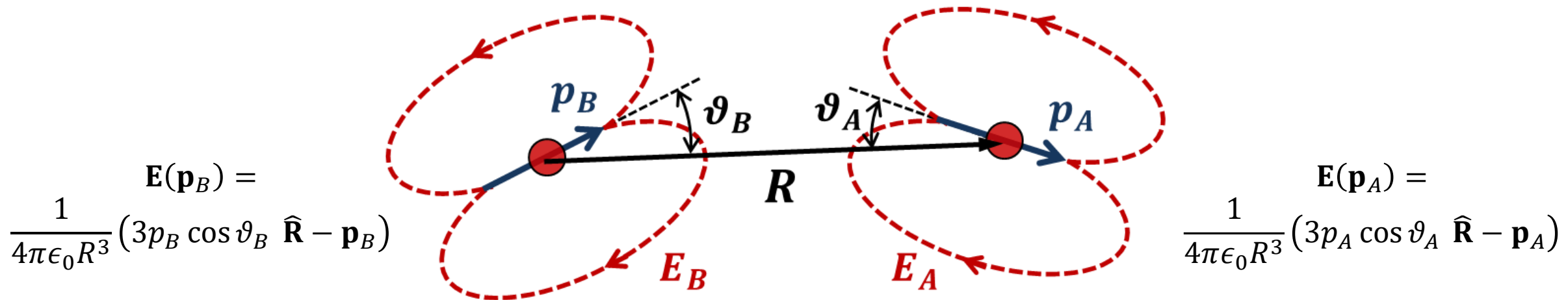
- potential energy of atom A

$$E_{\text{pot}} = -\mathbf{p}_A^{\text{ind}} \cdot \mathbf{E}_{B,\text{ion}} = -(\alpha_A \mathbf{E}_{B,\text{ion}}) \cdot \mathbf{E}_{B,\text{ion}} \propto \mathbf{E}_{B,\text{ion}}^2$$



# I.1.3 Van der Waals Bonding

- **assumption:**
  - field  $\mathbf{E}(\mathbf{p}_B)$  at position of atom A is generated by atom B with dipole moment  $\mathbf{p}_B$
  - $\mathbf{E}(\mathbf{p}_B)$  induces dipole moment  $\mathbf{p}_A$  of atom A, which in turn generates field  $\mathbf{E}(\mathbf{p}_A)$  at position of atom B



- **Interaction energy** (same atoms:  $p_A = p_B = p$ ,  $\alpha_A = \alpha_B = \alpha$ )

$$E_{\text{pot}}^{\text{dd}}(R) = \frac{1}{4\pi\epsilon_0} \frac{p_A p_B - 3(\hat{\mathbf{R}} \cdot \mathbf{p}_A)(\hat{\mathbf{R}} \cdot \mathbf{p}_B)}{R^3}$$



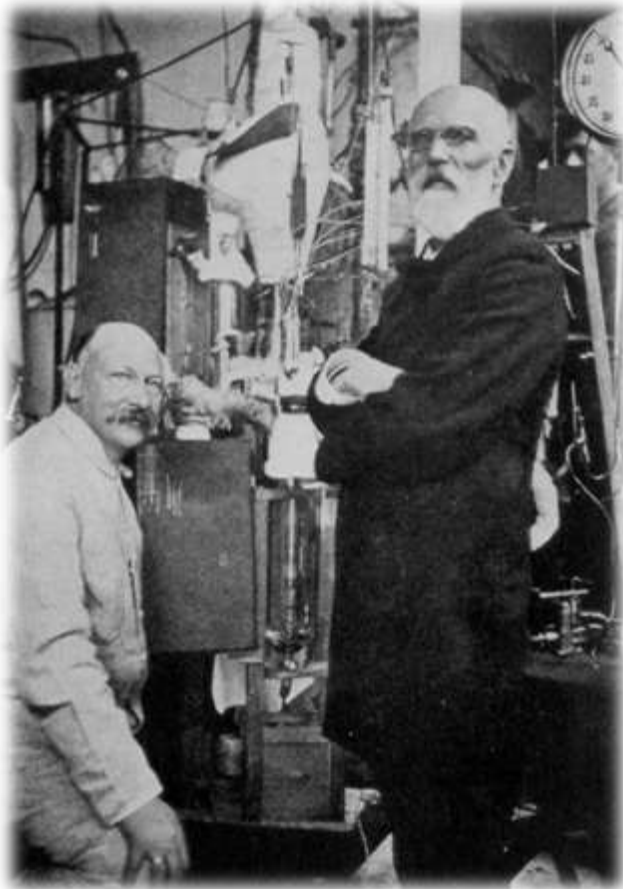
$$E_{\text{pot}}^{\text{dd}}(R) \propto -\frac{p_A p_B}{R^3} \propto -\frac{p}{R^3} \frac{\alpha p}{R^3} = -\frac{\alpha p^2}{R^6}$$

$$E_{\text{pot}}^{\text{dd}}(R) = -C \frac{\alpha}{R^6}$$

*induced dipole-dipole interaction*

- **attractive interaction**
- weak
- short range

# I.1.3 Van der Waals Bonding



*Van der Waals (right) and Heike Kamerlingh Onnes in Leiden with the helium 'liquefactor' (1908)*



*Johannes Diderik Van der Waals (1837 - 1923)  
Nobel Prize in Physics (1910)  
"for his work on the equation of state for gases and liquids"*

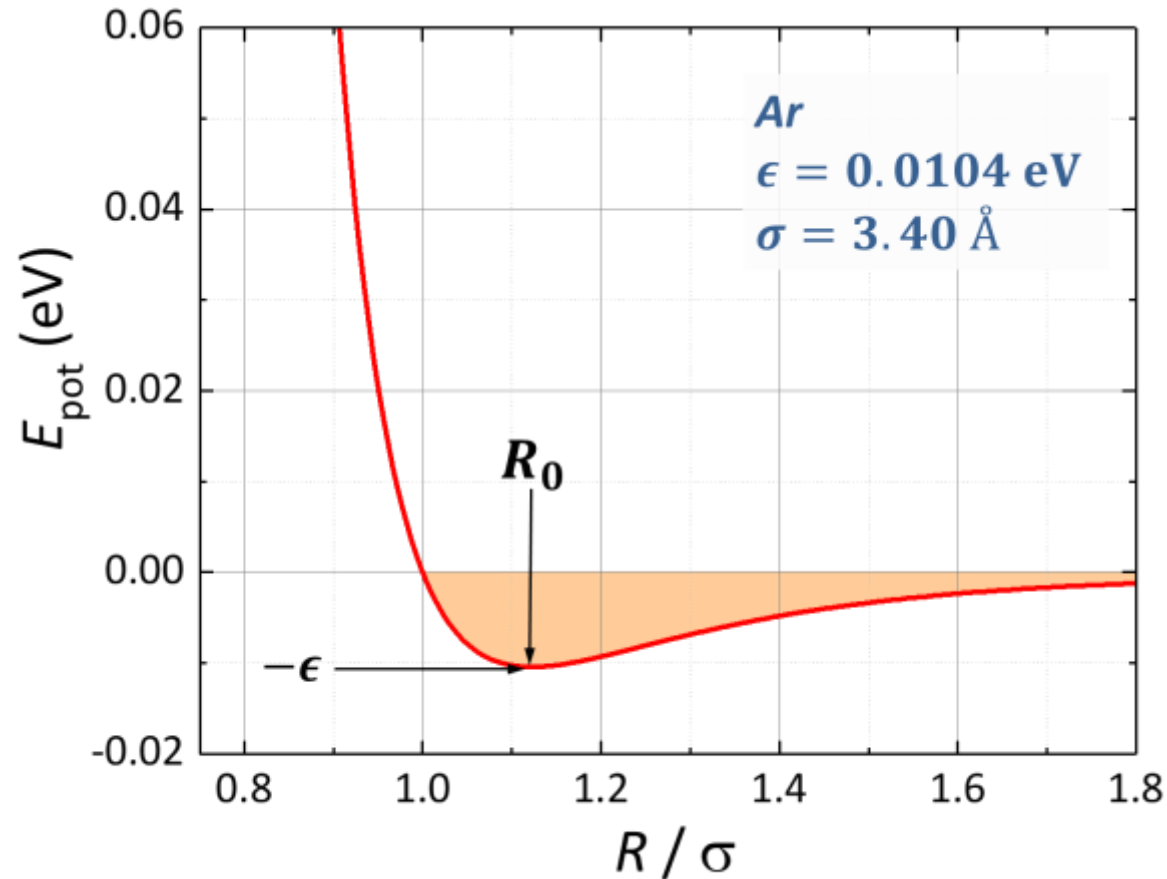


# I.1.3 Van der Waals Bonding

- *interaction potential* with short range repulsion due to Pauli principle:

$$E_{\text{pot}}(R) = 4\epsilon \left[ \left(\frac{\sigma}{R}\right)^{12} - \left(\frac{\sigma}{R}\right)^6 \right]$$

*Lennard-Jones potential*



	Ne	Ar	Kr	Xe
$\epsilon$ (eV)	0.0031	0.0104	0.0141	0.0200
$\sigma$ (Å)	2.74	3.40	3.65	3.98
$R_0/\sigma$	1.14	1.11	1.10	1.09

11.6 K  $\Leftrightarrow$  1 meV

➔ *Van der Waals bonding is very weak !!*



# I.1.3 Van der Waals Bonding

determination of **equilibrium distance** of atoms in lattice: → minimize total potential energy

$$U_{\text{tot}} = \frac{N}{2} \sum_{i \neq j} 4\epsilon \left[ \left( \frac{\sigma}{r_{ij}} \right)^{12} - \left( \frac{\sigma}{r_{ij}} \right)^6 \right]$$

$\Sigma$  = binding energy of a single atom  $j$  with all other atoms  $i$ .  
Total binding energy is obtained by multiplication with  $N/2$ .

we rewrite

$$U_{\text{tot}}(R) = 2N\epsilon \left[ A_{12} \left( \frac{\sigma}{R} \right)^{12} - A_6 \left( \frac{\sigma}{R} \right)^6 \right]$$

$R$ : nearest neighbor distance

with  $A_k = \sum_{i \neq j} \frac{1}{\alpha_{ij}^k}, \quad k = 6, 12$  (lattice sum)

$r_{ij} = \alpha_{ij}R$   
 $k$ : number of nearest neighbors

equilibrium distance:  $\partial U_{\text{tot}}(R)/\partial R = 0$

$$R_0 = \left( 2 \frac{A_{12}}{A_6} \right)^{1/6} \cdot \sigma \quad R_0 = 1.09 \cdot \sigma$$

Kristallstruktur	$A_6$	$A_{12}$
fcc	14.4539	12.1319
hcp	14.4549	12.1323
bcc	12.253	9.114

$$U_{\text{tot}}(R_0) = -N\epsilon \begin{cases} 8.61016 & \text{fcc - structure} \\ 8.61107 & \text{hcp - structure} \end{cases}$$

hcp has lowest energy but fcc is found experimentally for noble gases

- effect of zero point fluctuations
- more detailed band structure calculation is required

(modification of Van der Waals binding energy due to overlap of excited atomic states with neighboring atoms in crystal structure)

# I.1.4 Zero Point Fluctuations

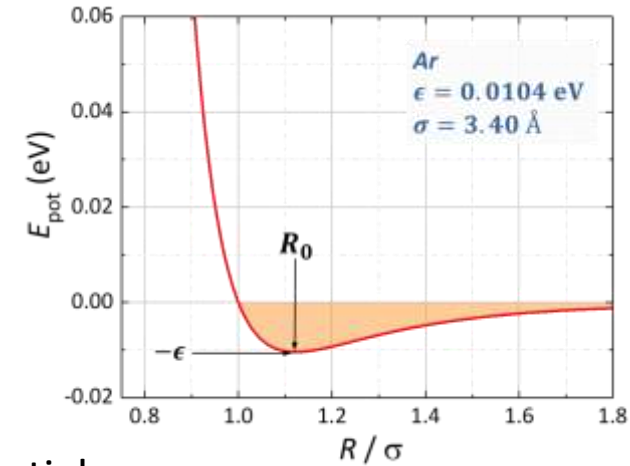
- estimate of *zero-point fluctuation amplitude* for harmonic oscillator:

$$E_{\text{tot}} = E_{\text{kin}} + E_{\text{pot}}$$

- for  $x = x_{\text{max}}$ :

$$E_{\text{kin}} = 0, \quad E_{\text{pot}} = \frac{1}{2} k x_{\text{max}}^2 = \frac{1}{2} M \omega^2 x_{\text{max}}^2 \quad \omega^2 = k/M$$

spring constant  $k$  is given by second derivative of the interaction potential



- by equating  $\frac{1}{2} k x_{\text{max}}^2$  with the ground state energy  $\frac{1}{2} \hbar \omega$  of the harmonic oscillator, we obtain

$$\Rightarrow x_{\text{max}}^2 = \frac{\hbar}{M\omega} = \frac{\hbar}{\sqrt{kM}}$$

for He:  $x_{\text{max}} \approx 0.3 - 0.4$  times the lattice constant  $a$

$$a \approx \left( \frac{V_{\text{mol}}}{N_A} \right)^{1/3} \quad V_{\text{mol}} = \text{mole volume}, \quad N_A = \text{Avogadro number}$$

(*Lindeman* criterion for melting of solid:  $x_{\text{max}} \approx 0.3 \cdot a$ )

- Helium stays liquid even at  $T = 0$  due to large zero point fluctuations caused by
- (i) weak Van der Waals forces (small  $k$ )
  - (ii) small mass  $M$

# I.1.4 Zero Point Fluctuations

- we can use uncertainty relation to estimate the *kinetic energy* due to zero point fluctuations:

$$\Delta p \cdot \Delta x \simeq \hbar \quad \Rightarrow \quad \Delta p \sim \frac{\hbar}{\Delta x} \approx \frac{\hbar}{0.3 a} \approx \frac{h}{a} \quad a = \text{lattice constant}$$

$$E_{\text{zp}} = \frac{\Delta p^2}{2m} \approx \frac{h^2}{2Ma^2} \approx 18 \text{ K}$$

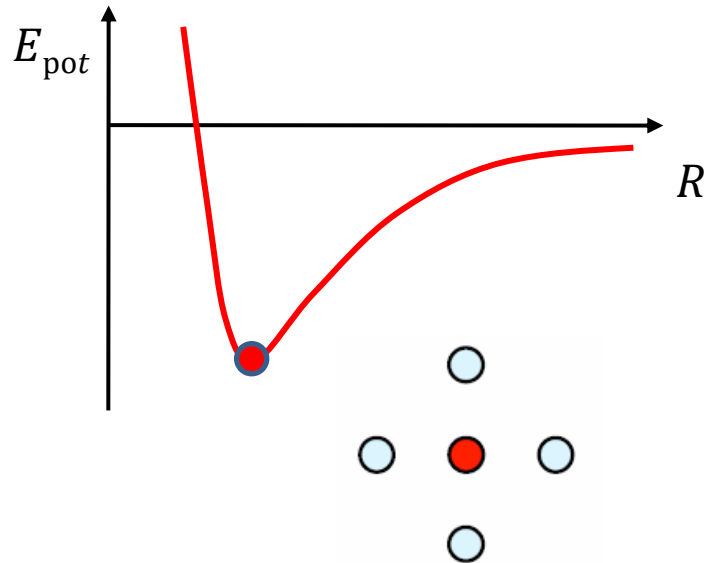
- large quantum mechanical zero-point energy due to small mass  $M$
- kinetic energy due to zero point fluctuations is comparable to potential energy due to Van der Waals interaction

*due to strong influence of quantum fluctuations, helium stays liquid down to temperatures, where  $\lambda_T \geq d \simeq a$*

→ *Helium becomes a **quantum liquid***

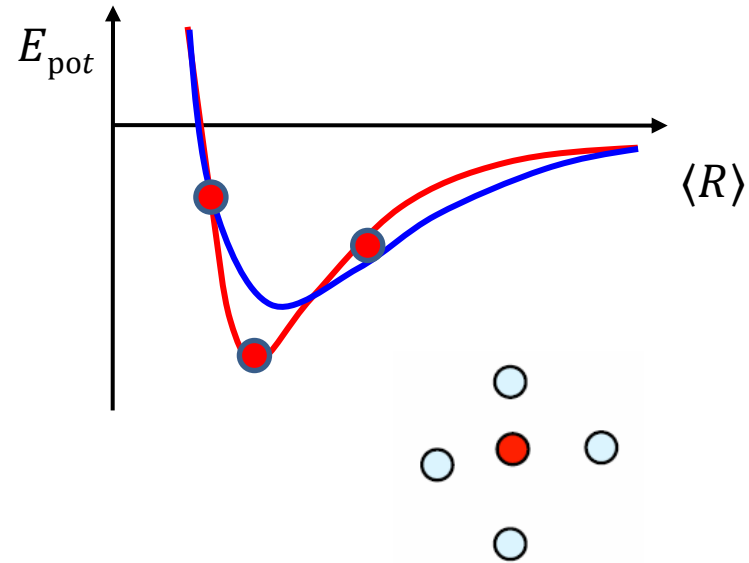
# I.1.4 Zero Point Fluctuations

- total potential energy for solid and liquid He



**solid:** regular lattice

- same distance between atoms
- $E_{\text{pot}}(R) \approx \text{potential } U_{\text{tot}}(R)$



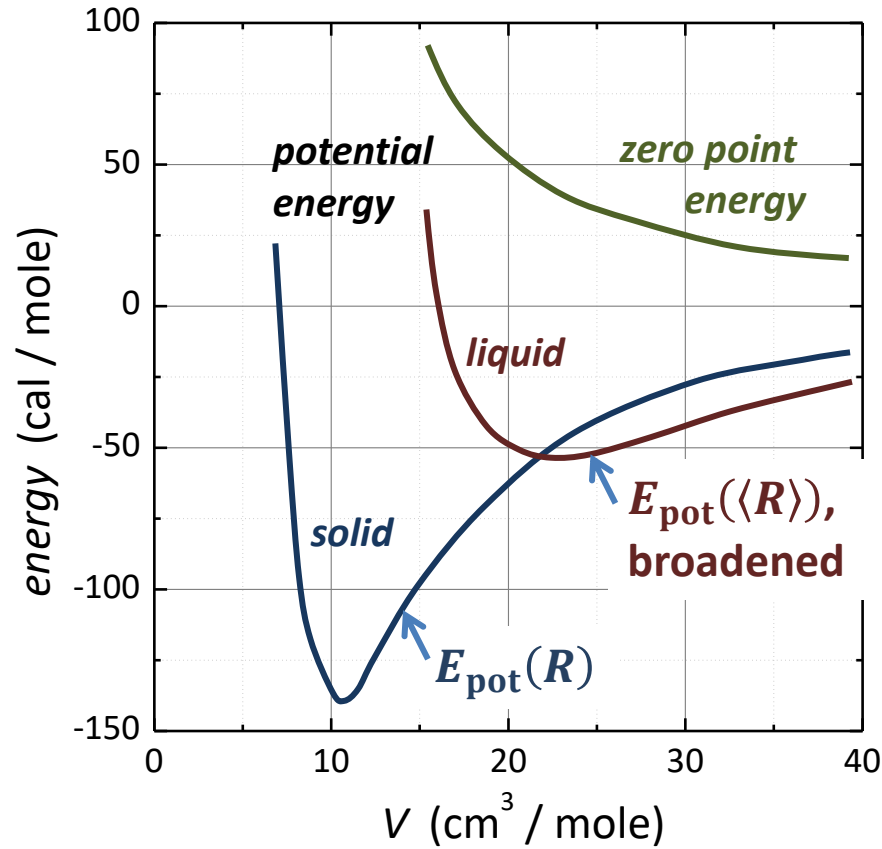
**liquid:** disordered system

- varying distance between atoms
- $E_{\text{pot}}$  is smeared out due to averaging over various distances
- reduced depth and increased width of  $E_{\text{pot}}(\langle R \rangle)$

**Why does He not become solid at low T ??**

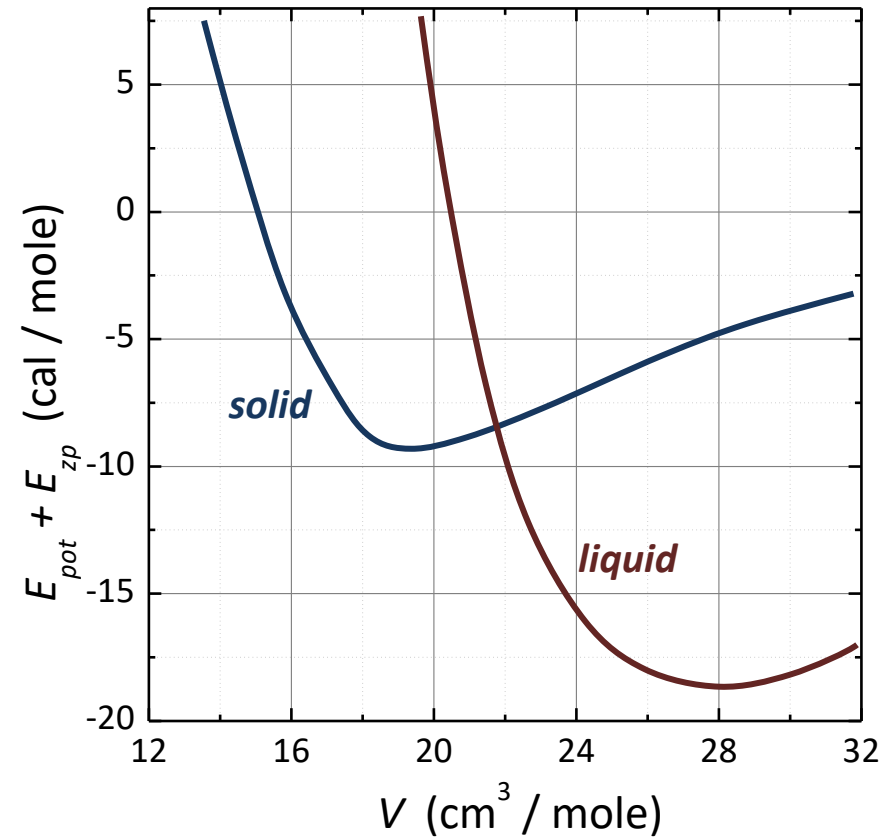
# I.1.4 Zero Point Fluctuations

- zero point fluctuations play a crucial role



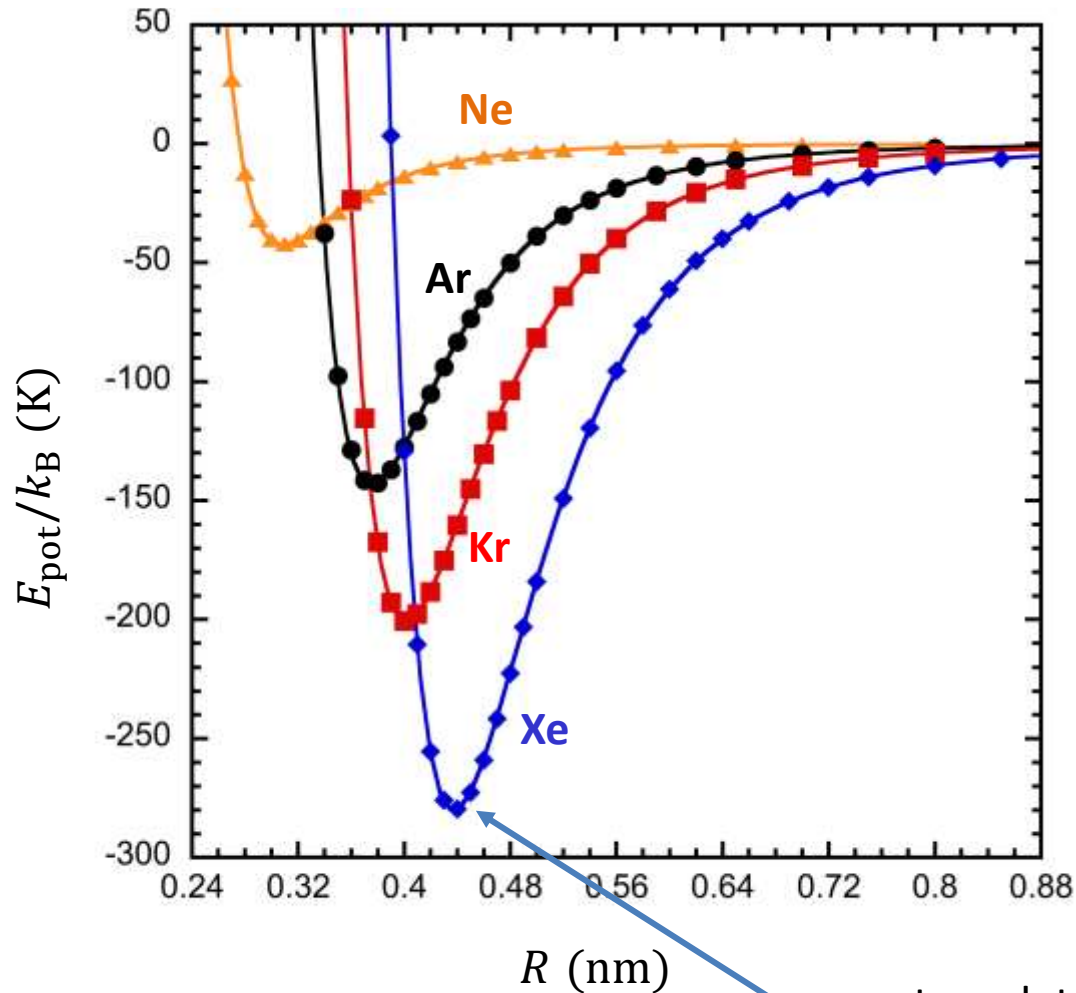
- *without quantum fluctuations:*  
<sup>4</sup>He would be solid at a molar volume of **10 cm<sup>3</sup>**

*for liquid helium:*  
 contribution of  $E_{\text{zp}}$  smaller due to broadened  $E_{\text{pot}}$



- *with quantum fluctuations:*  
<sup>4</sup>He stays liquid with molar volume of **28 cm<sup>3</sup>**

# I.1.4 Zero Point Fluctuations



Comparison of *ab initio* potential energies for Ne, Ar, Kr, Xe (symbols) with SAAP calculations (solid lines)

U. K. Deiters et al., J. Chem. Phys. **150**, 134504 (2019)  
<https://doi.org/10.1063/1.5085420>

comparison of  $^4\text{He}$  and Ne:

	He	Ne
potential minimum $\varepsilon$	1.03 meV	3.94 meV
interatomic distance $R_0$	0.265 nm	0.296 nm

1 meV = 11.6 K

curvature determines spring constant:  $k = \frac{1}{2} \frac{\partial^2 E_{\text{pot}}(R)}{\partial R^2} \Big|_{R=R_0} = \frac{36\varepsilon}{R_0^2}$  (for fcc)

# I.1.4 Zero Point Fluctuations

quantum parameter:

$$\Lambda \equiv \frac{\langle E_{zp} \rangle}{\langle E_{pot} \rangle}$$

quantum liquid:

$$\Lambda > 1$$

liquid	Xe	Kr	Ar	Ne	H <sub>2</sub>	<sup>4</sup> He	<sup>3</sup> He
$\Lambda$	0.06	0.10	0.19	0.59	1.73	2.64	3.05

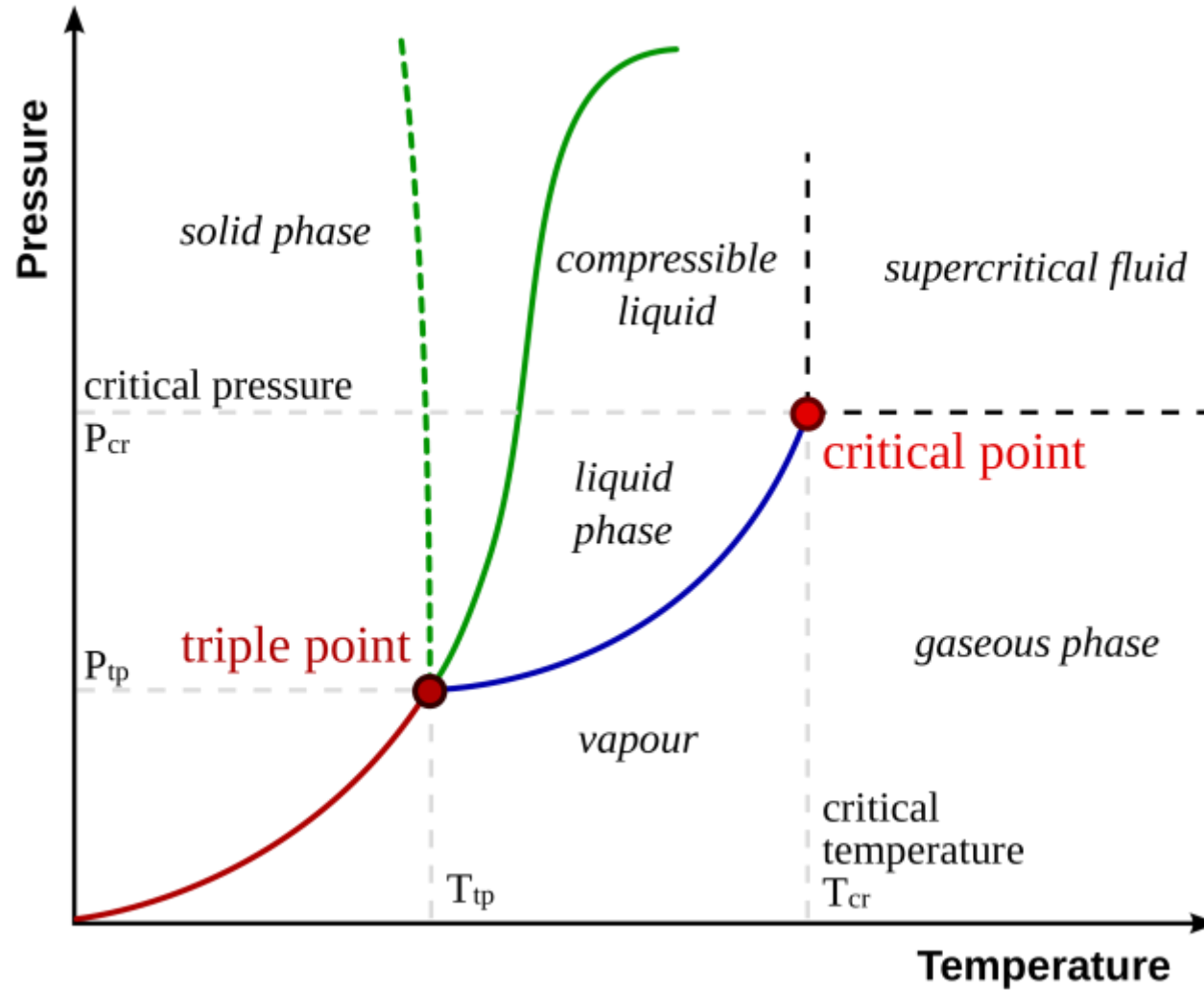
*Large  $E_{zp}$ , but also large  $E_{pot}$*      
 *identical  $E_{pot}$  but different  $E_{zp}$*

mass of <sup>3</sup>He smaller than of <sup>4</sup>He

- ➔ even stronger effect of quantum fluctuations
  - ➔ larger quantum parameter
  - ➔ larger molar volume
- <sup>3</sup>He: **40 cm<sup>3</sup>/mole**, <sup>4</sup>He: **28 cm<sup>3</sup>/mole**

# I.1.6 pT phase diagram of $^4\text{He}$

- general nomenclature, definitions



<https://commons.wikimedia.org/wiki/File:Phase-diag2.svg>



# I.1.5 Helium under Pressure

– **enthalpy:**  $H = U + pV$   
 $H = \text{min?}$

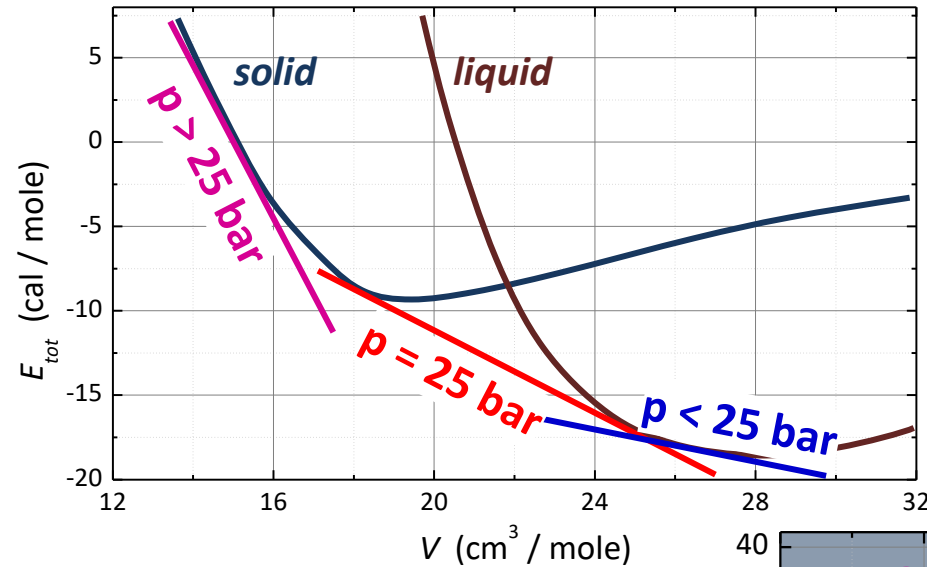
– use of isenthalpy lines:  
 $H = \text{const.}$

– with  $U = H - pV$

→ straight lines with slope  $-p$

– **lowest isenthalpy:**

→ tangent with slope  $-p$



$p > 25 \text{ bar: solid}$

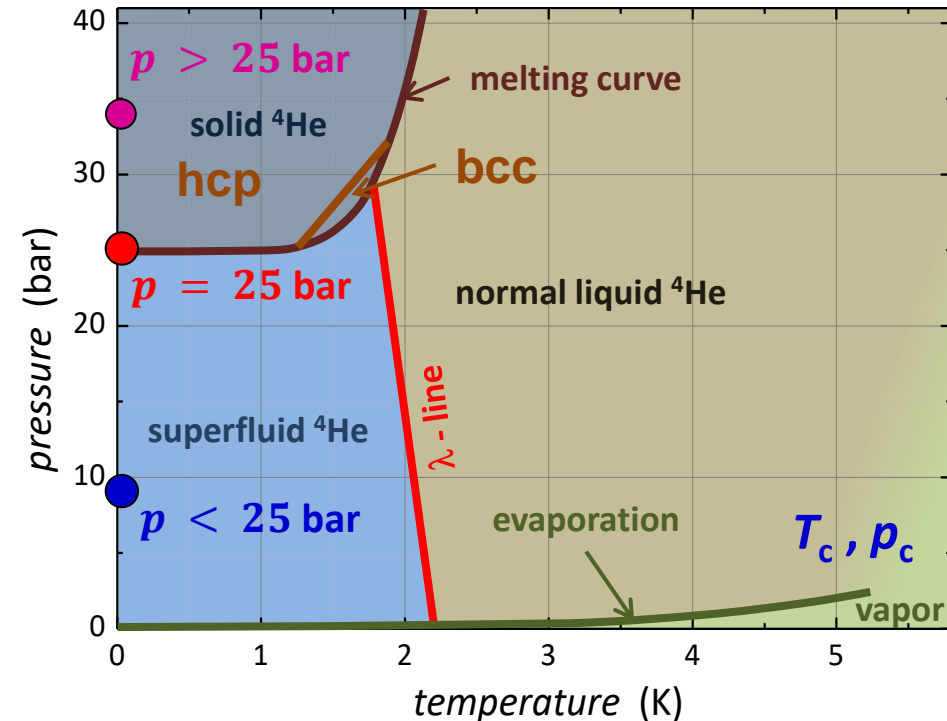
$p = 25 \text{ bar: coexistence}$

$p < 25 \text{ bar: liquid}$

$$H = U + pV$$

$$dH = \underbrace{dU}_{Tds - pdv} + pdV + Vdp$$

$$dH = TdS + Vdp$$



# I.1.6 pT phase diagram of $^4\text{He}$

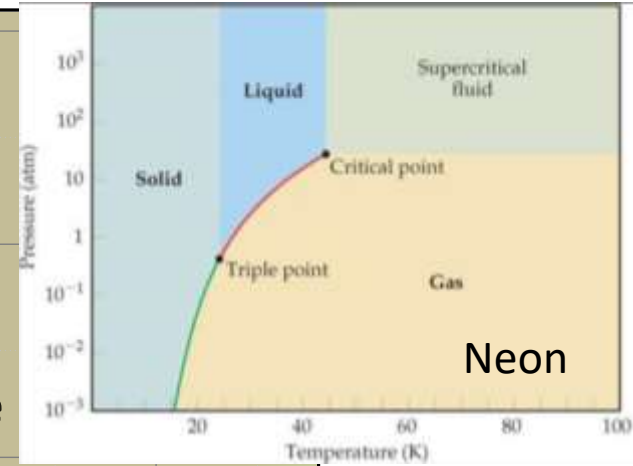
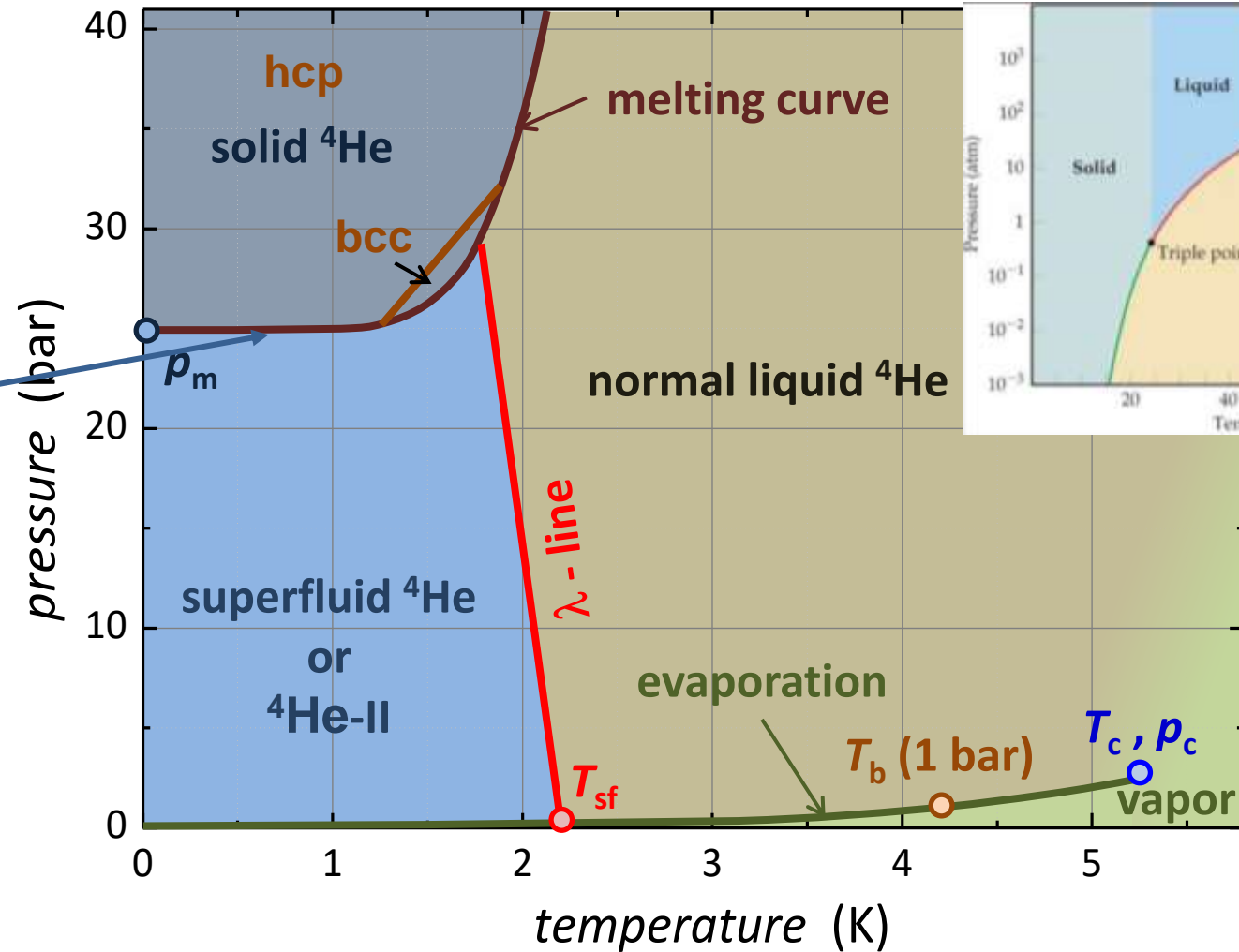
- Clausius-Clapeyron:**

$$\frac{\partial p_m}{\partial T} = \frac{\Delta S_m}{\Delta V_m}$$

$$\frac{\partial p_m}{\partial T} = \frac{\Delta S_m}{\Delta V_m} \approx 0 \quad @ \quad T \rightarrow 0$$

$$\Rightarrow \Delta S_m \rightarrow 0 \quad \text{for} \quad T \rightarrow 0$$

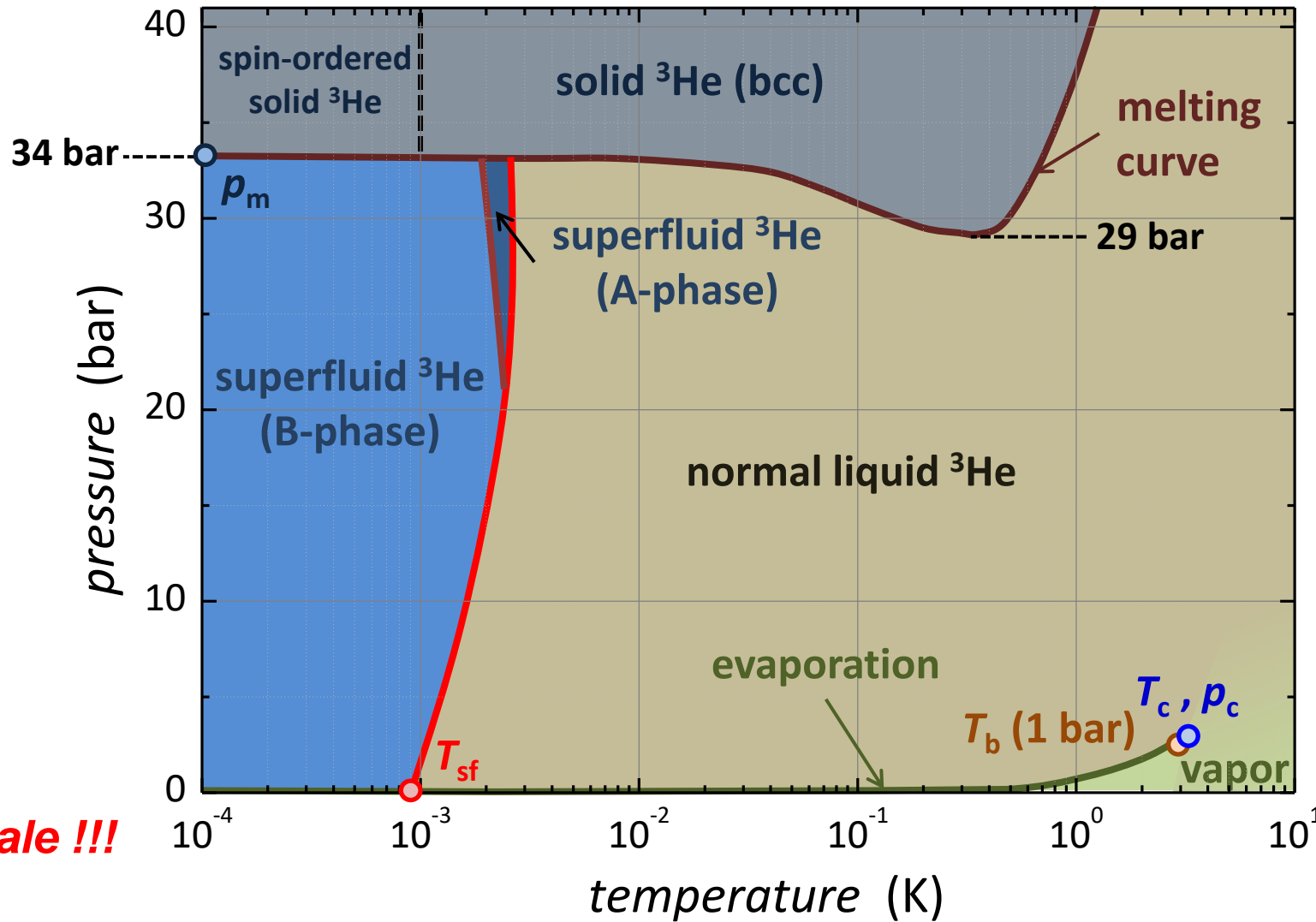
(agrees with 3<sup>rd</sup> Law of thermodynamics)



Phase diagram of  $^4\text{He}$  at low temperatures.  $^4\text{He}$  remains liquid at zero temperature if the pressure is below 2.5 MPa (approximately 25 atmospheres) → **no triple point**.

The liquid has a phase transition to a **superfluid phase**, also known as He-II, at the temperature of 2.17 K (at vapor pressure). The solid phase has either hexagonal close packed (hcp) or body centered cubic (bcc) symmetry.

# I.1.6 pT phase diagram of $^3\text{He}$



**log T scale !!!**

- big difference in value of  $T_{sf}$
- normal/superfluid transition dominated by quantum effects
- quantum statistics matters!! (bosons vs. Fermions)
- nature of superfluid phase in  $^4\text{He}$  and  $^3\text{He}$  is different

- phase diagram of  $^3\text{He}$ : note the logarithmic temperature scale. There are **two superfluid phases of  $^3\text{He}$ , A and B.**
- the line within the solid phase indicates a transition between spin-ordered and spin disordered structures (at low and high temperatures, respectively).

# I.1.6 pT phase diagram of $^3\text{He}$

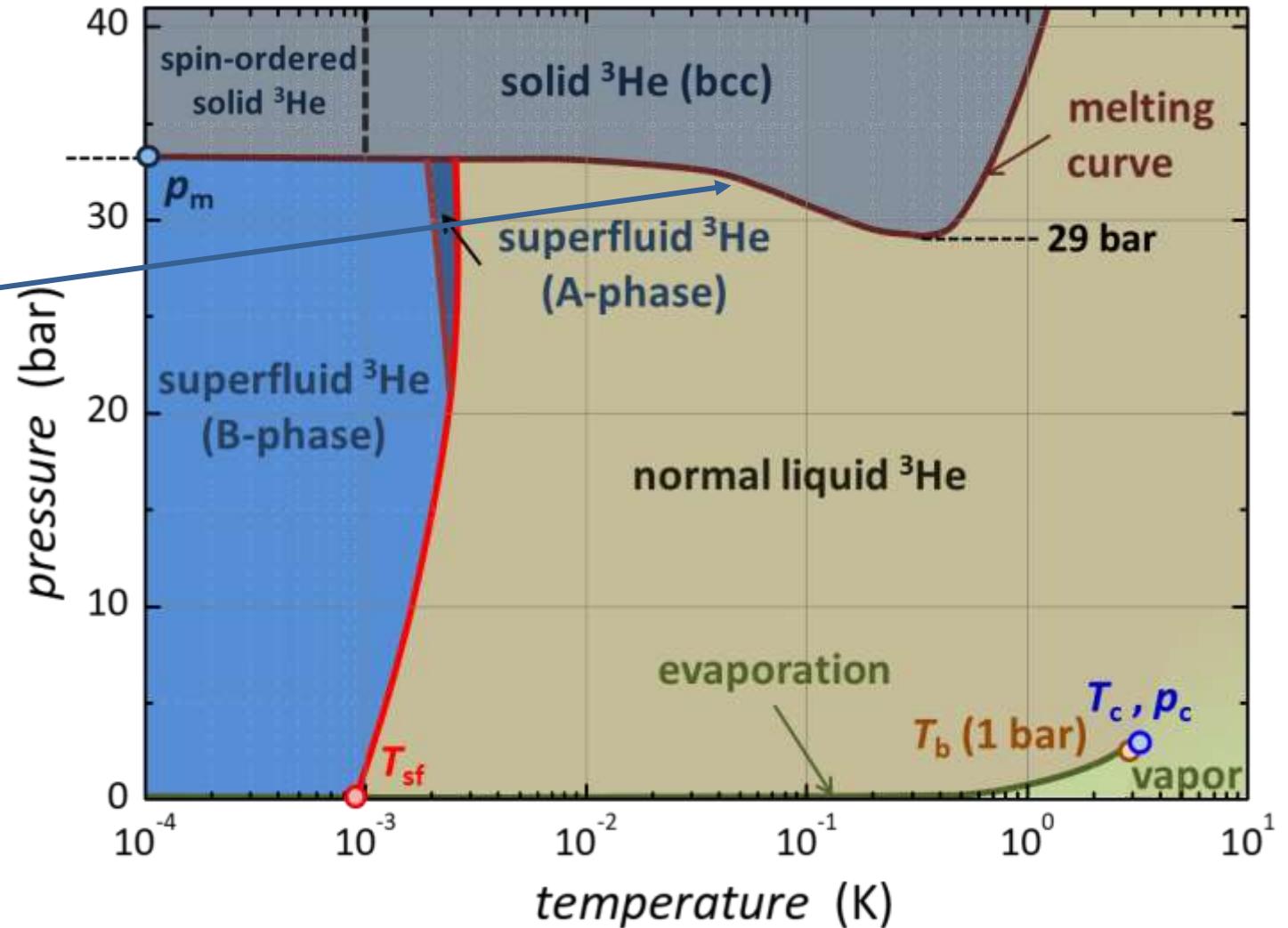
- Clausius-Clapeyron:*

$$\frac{\partial p_m}{\partial T} = \frac{\Delta S_m}{\Delta V_m}$$

$$\frac{\partial p_m}{\partial T} = \frac{\Delta S_m}{\Delta V_m} < 0$$

since volume of liquid state is larger, the entropy of the liquid state must be smaller

- more ordered liquid state!?
- explanation follows later:  
*ordering in momentum space*



	<sup>3</sup> He	<sup>4</sup> He
<b>boiling point <math>T_b</math> (K)</b> (@ $p = 1$ bar)	<b>3.19</b>	<b>4.21</b>
<b>critical temperature <math>T_c</math> (K)</b>	<b>3.32</b>	<b>5.20</b>
<b>critical pressure <math>p_c</math> (bar)</b>	<b>1.16</b>	<b>2.29</b>
<b>superfluid transition temperature <math>T_s</math> (K)</b>	<b>0.0025</b>	<b>2.177</b>
<b>density <math>\rho</math> (g/cm<sup>3</sup>)</b> (for $T \rightarrow 0$ K)	<b>0.076</b>	<b>0.145</b>
<b>density <math>\rho</math> (g/cm<sup>3</sup>)</b> (@ boiling point)	<b>0.055</b>	<b>0.125</b>
<b>classical molar volume <math>V_m</math> (cm<sup>3</sup> /mole)</b> (@ saturated vapor pressure and $T = 0$ K)	<b>12</b>	<b>12</b>
<b>actual molar volume <math>V_m</math> (cm<sup>3</sup> /mole)</b> (@ saturated vapor pressure and $T = 0$ K)	<b>36.84</b>	<b>27.58</b>
<b>melting pressure <math>p_m</math> (bar)</b> (@ $T = 0$ K)	<b>34.39</b>	<b>25.36</b>

*strong difference between superfluid properties of <sup>4</sup>He and <sup>3</sup>He*

*→ different nuclear spin → different quantum statistics*

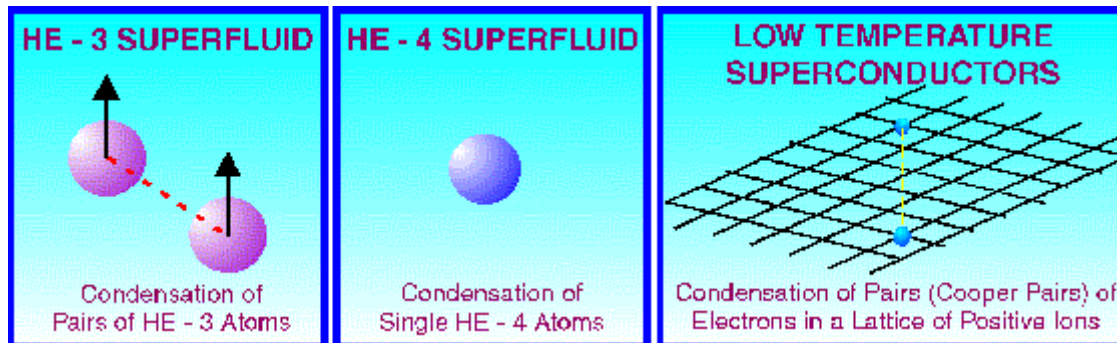
${}^4\text{He}$	$p^+ \uparrow \quad p^+ \downarrow$	$n \uparrow \quad n \downarrow$	$e^- \uparrow \quad e^- \downarrow$	<b>boson</b>
${}^3\text{He}$	$p^+ \uparrow \quad p^+ \downarrow$	$n \uparrow$	$e^- \uparrow \quad e^- \downarrow$	<b>fermion</b>

*different nuclear spin:*  
 ${}^4\text{He}: I = 0$   
 ${}^3\text{He}: I = 1/2$

*same chemical properties*

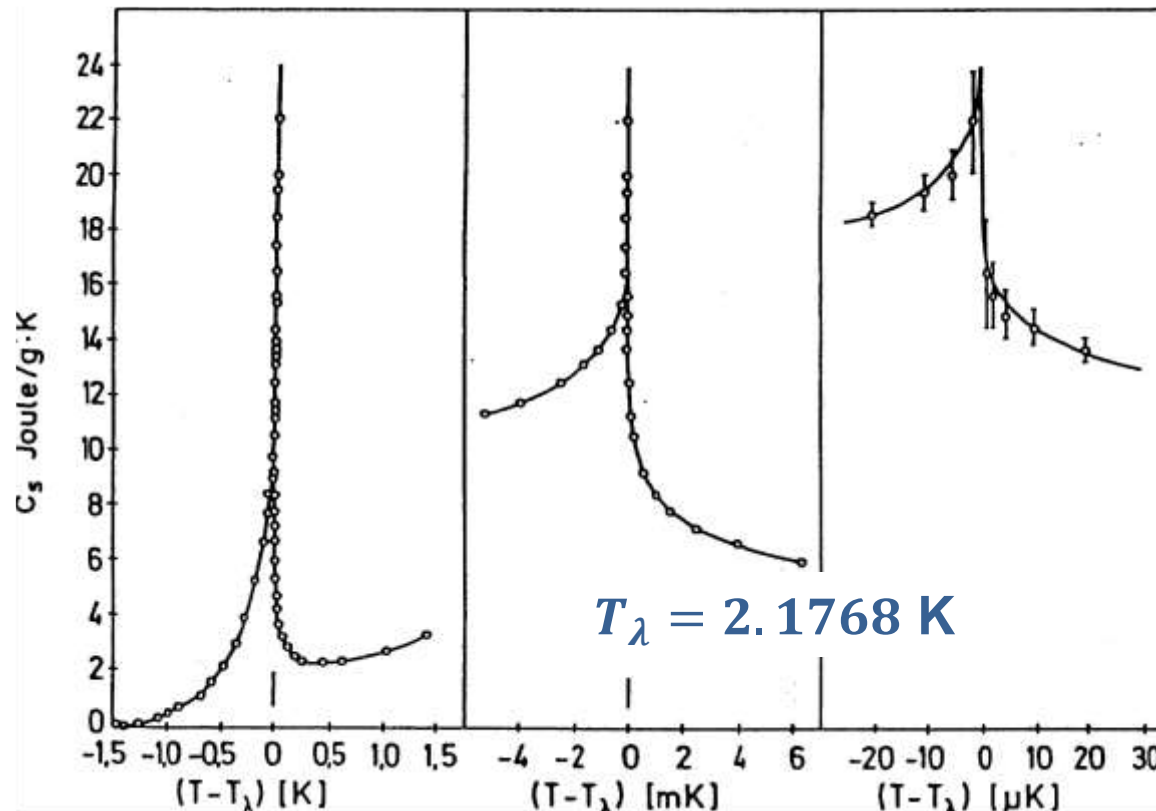
*different statistics*

- ${}^3\text{He}$ : *pairing similar to superconductivity in metals*  
*weak attractive interaction between Fermions* → „Cooper pairs“
- ${}^4\text{He}$ : *Bose-Einstein condensation of Bosons* without „pairing interaction“  
 → *no attractive interaction required*



# I.1.8 Specific Heat of $^4\text{He}$ – Lambda Point

- first measurements by *Dana* and *Kamerlingh Onnes* in 1923
  - anomalous increase of  $C_p$  at about  $T_\lambda = 2.177$  K
  - in publication (1926) some data points were omitted, since they were believed to result from exp. artefacts
- confirmation by *Clusius* and *Keesom* in 1932 (introduced denotation **He I** and **He II** → **phase transition**)
  - **He I: normal fluid, He II: superfluid**



specific heat:

$$C_{p,V} \equiv \left( \frac{\partial U}{\partial T} \right)_{p,V}$$

observation: specific heat diverges at  $T = T_\lambda$   
 → 1<sup>st</sup> order phase transition  
 → shape: **Lambda point**

free energy  $F$ : **kink**

$$(F = U - TS \Rightarrow dF = -SdT - pdV + \mu dN \text{ with } dU = TdS - pdV + \mu dN)$$

entropy  $S \propto (-\partial F / \partial T)_{V,N}$ : **jump**

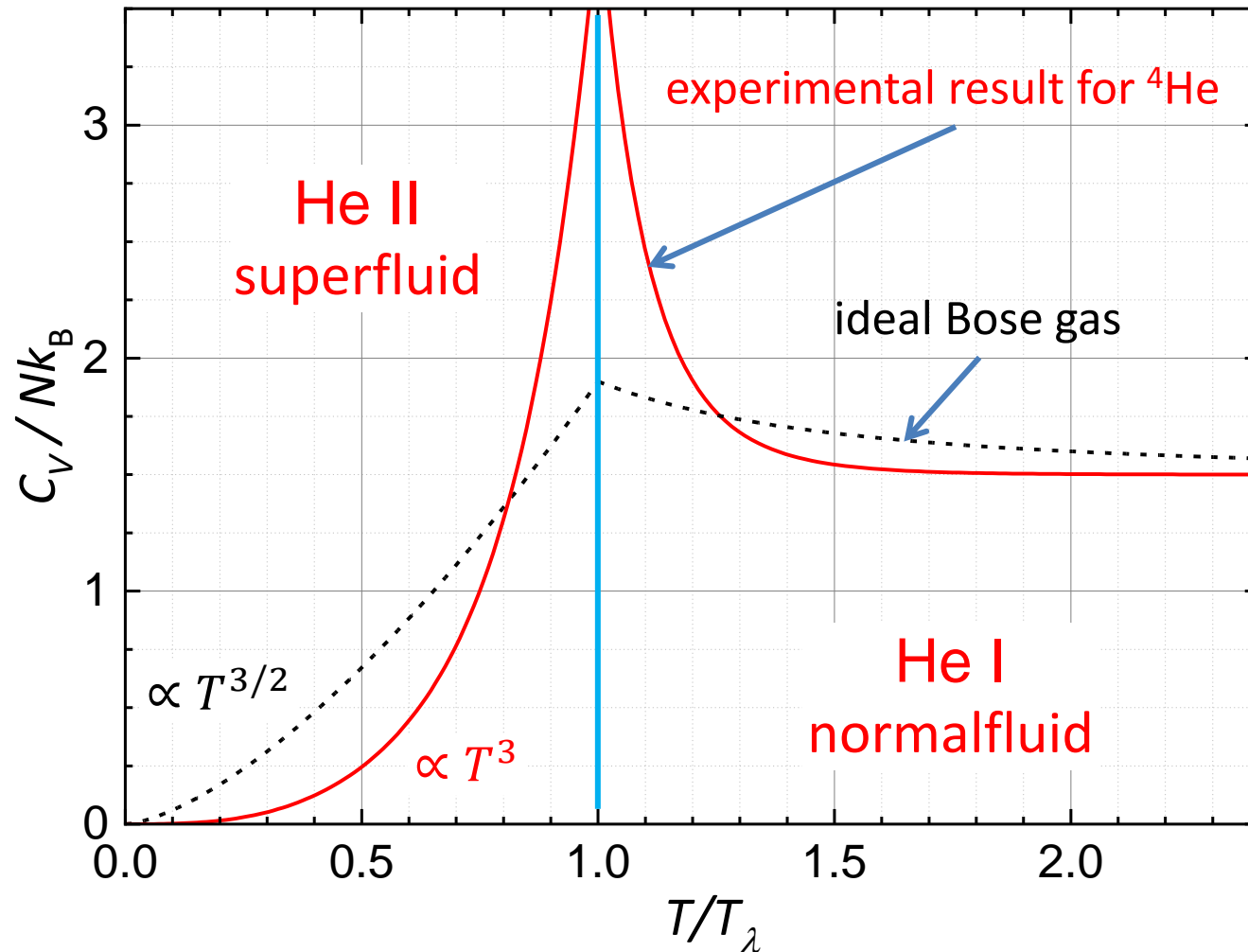
spec. heat  $C_V \propto (-\partial^2 F / \partial T^2)_{V,N}$ : **divergence**

Buckingham, M. J. and Fairbank, W. M., "The Nature of the Lambda Transition", in Progress in Low Temperature Physics III, 1961.



# I.1.8 Specific Heat of $^4\text{He}$ – Lambda Point

- comparison of specific heat of  $^4\text{He}$  and that of an ideal Bose-Einstein condensate



2<sup>nd</sup> order phase transition is expected for ideal Bose-Einstein condensate

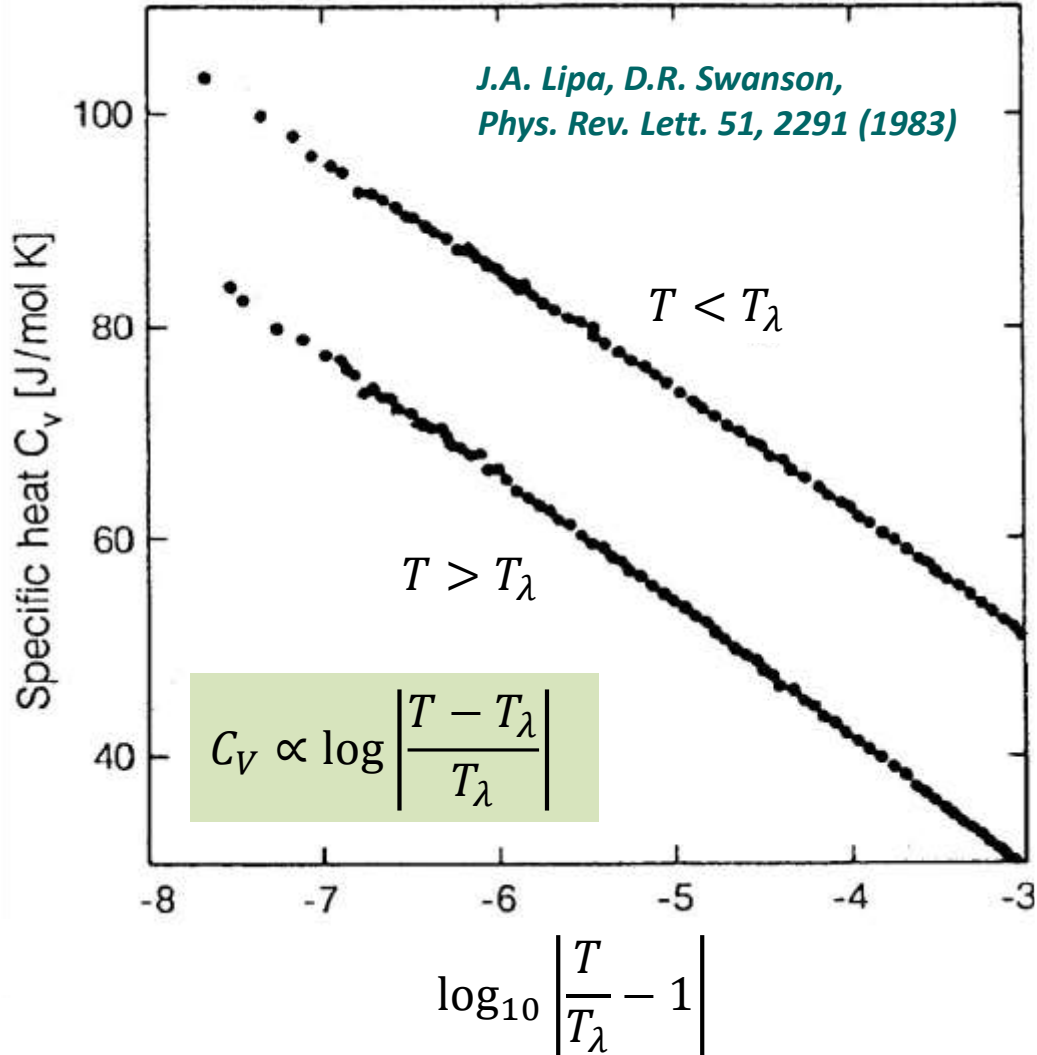
→ no divergence of  $C_V$  at  $T = T_\lambda$

measured behavior of  $^4\text{He}$  near  $T_\lambda$  is more complicated

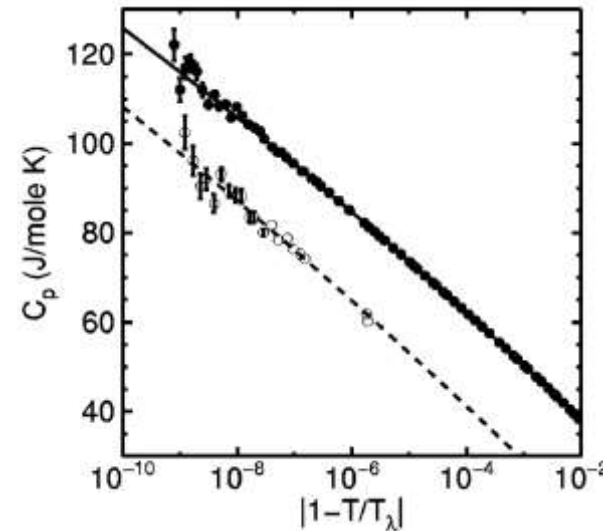
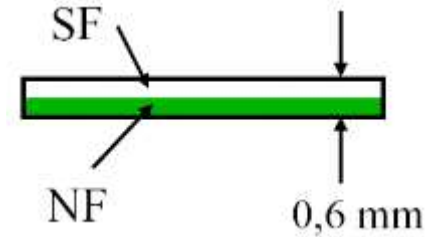
(detailed discussion follows later)



# I.1.8 Specific Heat of $^4\text{He}$ – Precision Measurement



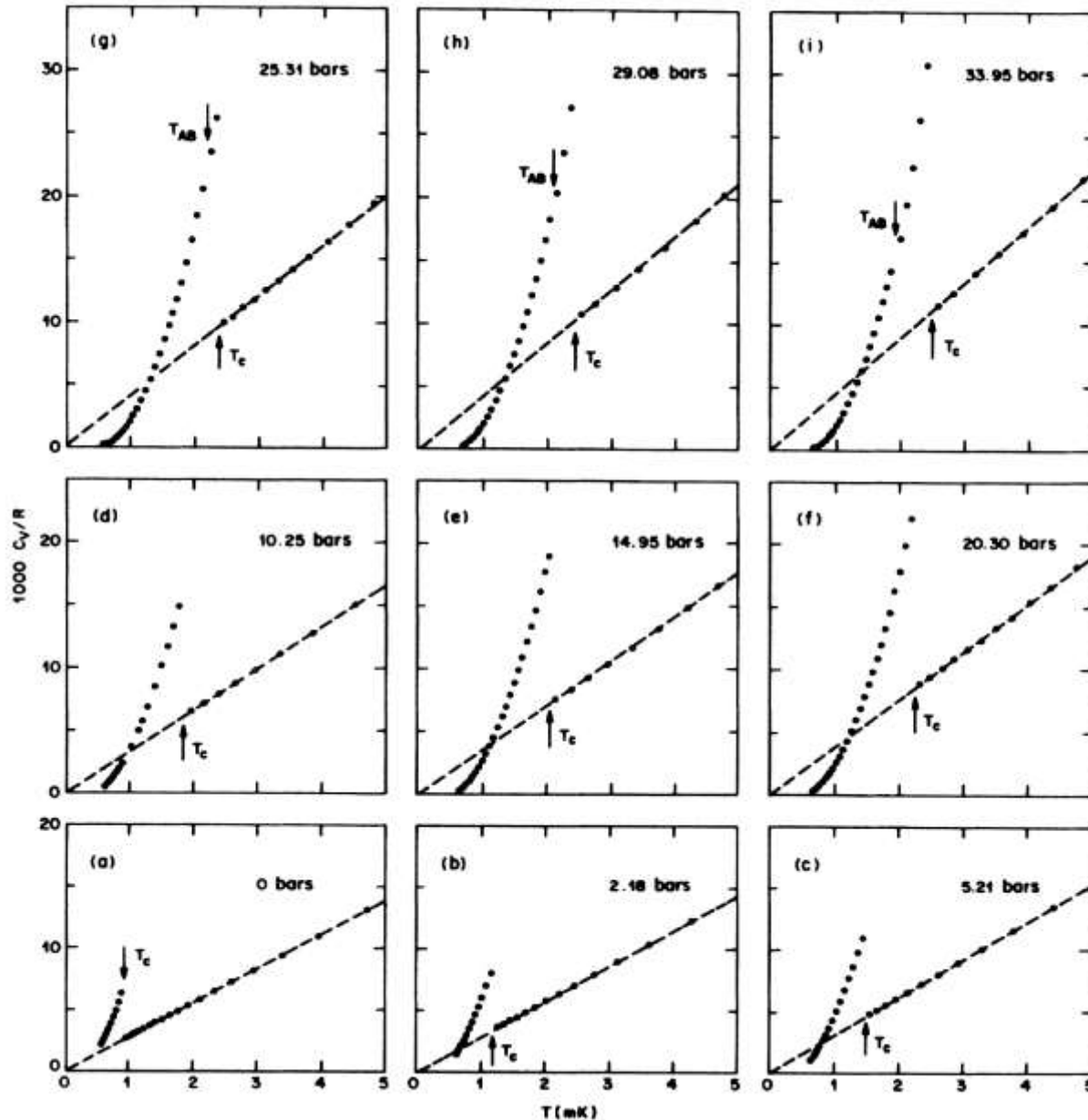
- specific heat is measured with very high temperature resolution ( $\approx$  nK)
- extreme purity of liquid He is important
  - *model system for testing physics of phase transitions*
- experimental problem:
  - hydrostatic pressure
  - space experiments



Specific heat of liquid helium in zero gravity very near the lambda point  
 J. A. Lipa et al., Phys. Rev. B 68, 174518 (2003)

→ **key result: second order phase transition with logarithmic singularity**

# I.1.8 Specific Heat of $^3\text{He}$



$C_V(T)$  similar to a metallic superconductor

- $T > T_c$ :  
linear  $C_V(T)$  of Fermi liquid

$$C_V(T) = \gamma T$$

$$\gamma = \frac{3}{2} \frac{n}{E_F} \frac{\pi^2}{3} k_B^2 \quad (\text{Sommerfeld, 1927})$$

- $T = T_c$ :  
jump due to superfluid transition
- $T < T_c$ :  
reduction of  $C_V$  due to condensation of Fermions (similar to superconductors)

Source:

Greywall, Phys. Rev. B 33, 7520 (1986)

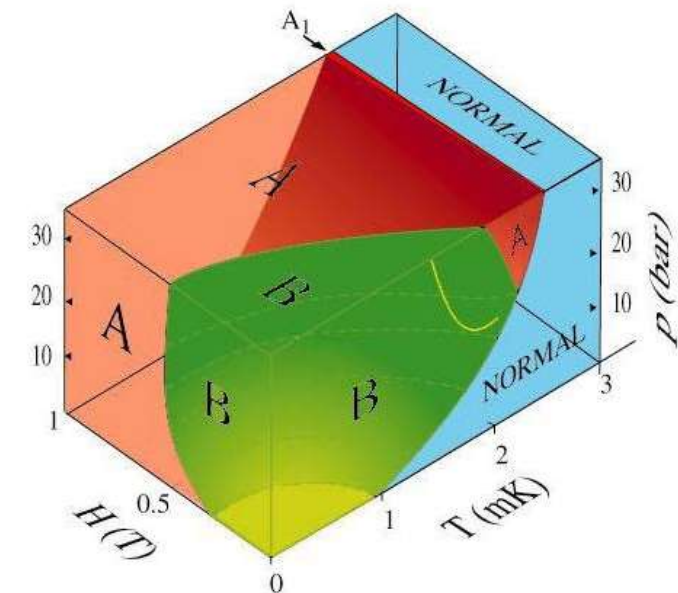
$^3\text{He}$  specific heat and thermometry at millikelvin temperatures

# I.1 Summary of Experimental Observations

- specific heat of  $^4\text{He}$  is large:  $C_V \approx \mathbf{1 \text{ J/g K}}$  (compared to about  $10^{-5} \text{ J/g K}$  of Cu,  $C_V \propto n \cdot T/T_F$ )
- latent heat of evaporation of  $^4\text{He}$  is small ( $\mathbf{20.59 \text{ J/g}}$ ), but large compared to specific heat of other materials  
 → *application as cooling liquid*
- strong anomaly of specific heat at  $T_\lambda$ , origin unknown for more than 30 years  
 → originates from *Bose-Einstein condensation* (bosonic character of  $^4\text{He}$ , interacting bosonic system)
- shape of anomaly resembles greek letter  $\lambda$  → *Lambda point*
- below  $T_\lambda$  entropy and specific heat decrease rapidly with decreasing  $T$   
 → *condensation in momentum space*
- for  $^4\text{He}$  temperature variation of  $C_V$  below  $T_\lambda$  determined by collective excitations in superfluid  
 (*phonons, rotons* – discussion later)

# I.1 Summary of Experimental Observations

- for  $^4\text{He}$  phase transition at  $T_\lambda \rightarrow$  Lambda point is due to **bosonic character of  $^4\text{He}$**   
 → no such transition was expected for  $^3\text{He}$  due to nuclear spin  $I = 1/2$  (**fermionic character of  $^3\text{He}$** )
- only after the development of BCS theory in 1957 (condensation of Cooper pairs into macroscopic quantum state), the **search for the superfluid phase of  $^3\text{He}$**  has been intensified
- 1963: **Theory of superfluid  $^3\text{He}$**  (Anderson and Morel, Balian and Werthamer)
- 1971: after the temperature regime below 5 mK has been accessed, the **superfluid phase of  $^3\text{He}$  has been found** by Osheroff, Richardsen and Lee in NMR experiment  
*D.D. Osheroff, R.C. Richardsen, D.M. Lee, Phys. Rev. Lett. 28, 885 (1972)*
- there is not only a single superfluid phase, but three, depending on temperature, pressure and applied magnetic field
- 1977: first experiment with  $^3\text{He}$  in Germany at WMI (Eska, Schuberth, Uhlig, ....)



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# I.2 $^4\text{He}$ as Ideal Bose Gas

- basic assumption

$^4\text{He}$  is treated as a non-interacting gas of bosons

we already have learnt that there are Van der Waals interactions

- treatment of  $^4\text{He}$  as an ideal bosonic gas is only a ***first order approximation***
- interactions lead to deviations from behavior of ideal gas of non-interacting bosons

# I.2 $^4\text{He}$ as Ideal Bose Gas

- high temperatures:

average distance  $\bar{R}$  of atoms in gas is large compared to **de Broglie wavelength**  $\lambda_T = h/p \propto 1/\sqrt{T}$

→ **classical gas with Maxwell-Boltzmann statistics**

thermal de Broglie wave length:

$$\lambda_T = \sqrt{\frac{h^2}{2\pi m k_B T}} = \sqrt{\frac{2\pi \hbar^2}{m k_B T}}$$

compare I.1.1

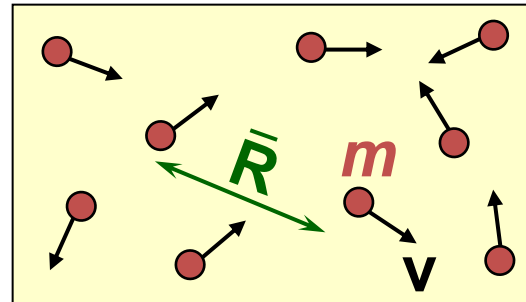
- low temperatures:

- quantum mechanical description, as soon as average particle distance  $\bar{R} = \lambda_T$

- classical gas of distinguishable particles turns into quantum soup of indistinguishable identical particles

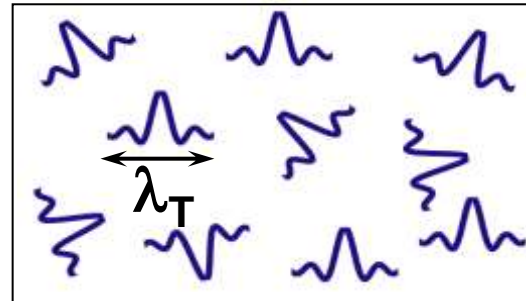
→ **quantum gas with Bose or Fermi statistics**

→ **Bose-Einstein condensation for bosons**



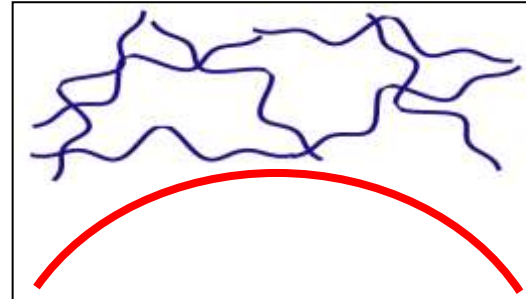
high temperature:  
 thermal velocity  $v_T$   
 density  $\frac{N}{V} = n \approx \frac{1}{\bar{R}^3}$

**billiard balls**



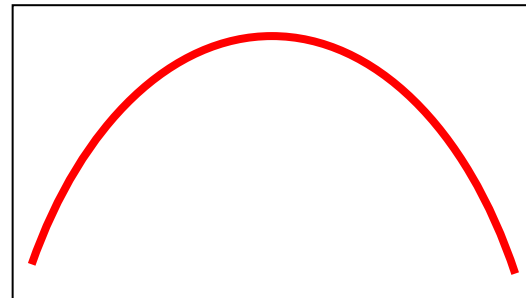
low temperature:  
 de Broglie wavelength  $\lambda_T$   
 $\lambda_T = h/mv_T \propto 1/\sqrt{T}$

**wave packets**



$T = T_{\text{BEC}}$ :  
 Bose-Einstein condensation  
 $\lambda_T \approx \bar{R}$

**wave packets overlap**



$T \rightarrow 0$ :  
 pure Bose-Einstein condensate

**„Super“-mater wave**



## Bose - Einstein Condensation

# I.2.1 Bose-Einstein Condensation

- phase transition occurs for *interactionless bosonic particles*
- originates from the *indistinguishability of identical particles*



Albert Einstein und Satyendranath Bose

*proposed in 1924 that below a characteristic temperature  $T_B$  a macroscopic number of bosons can condense into the state of lowest energy*

experimental confirmation in 1995



Eric A. Cornell



Wolfgang Ketterle

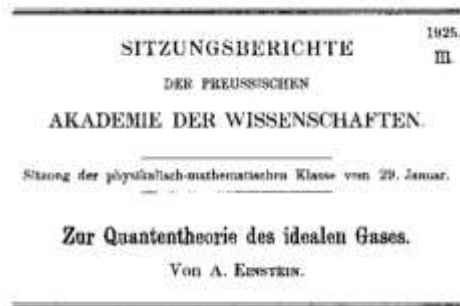


Carl E. Wieman

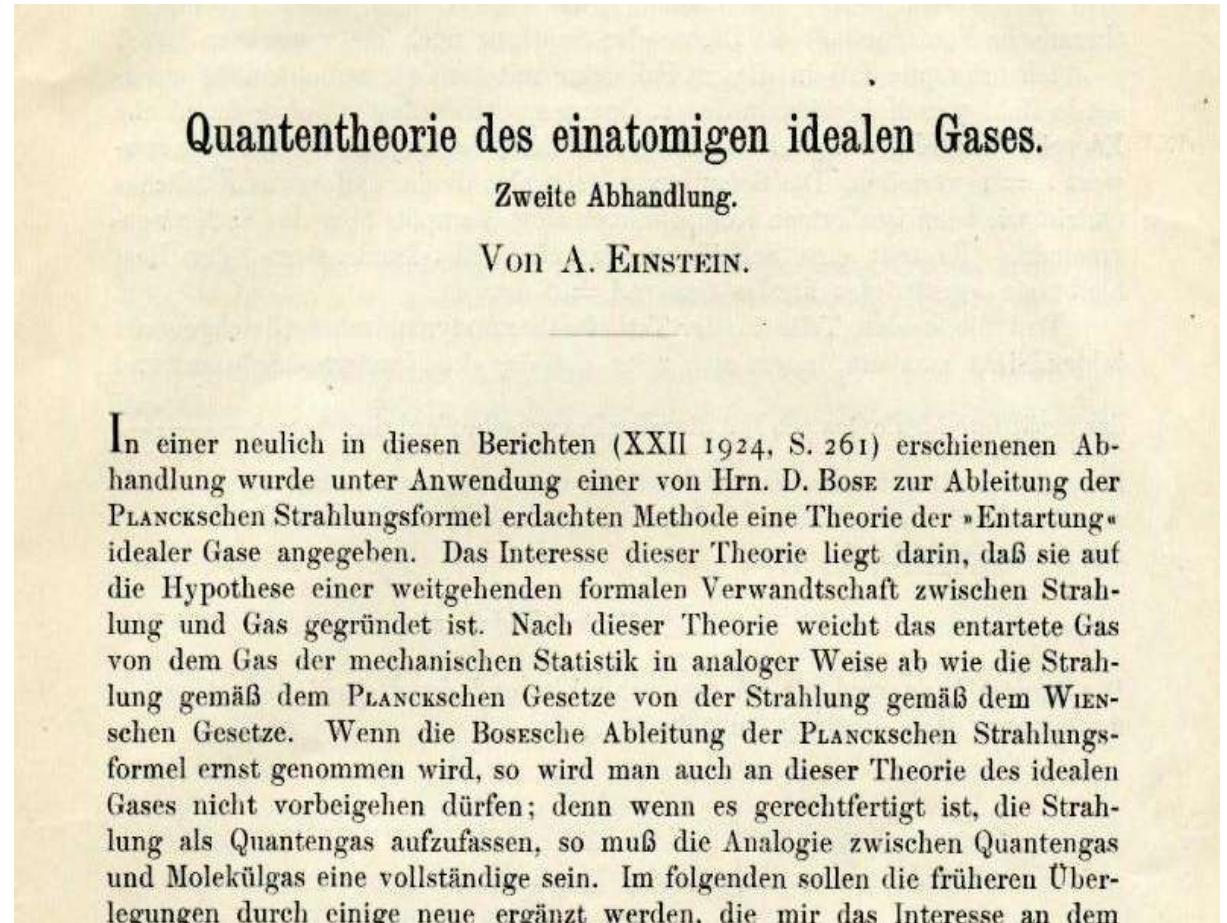
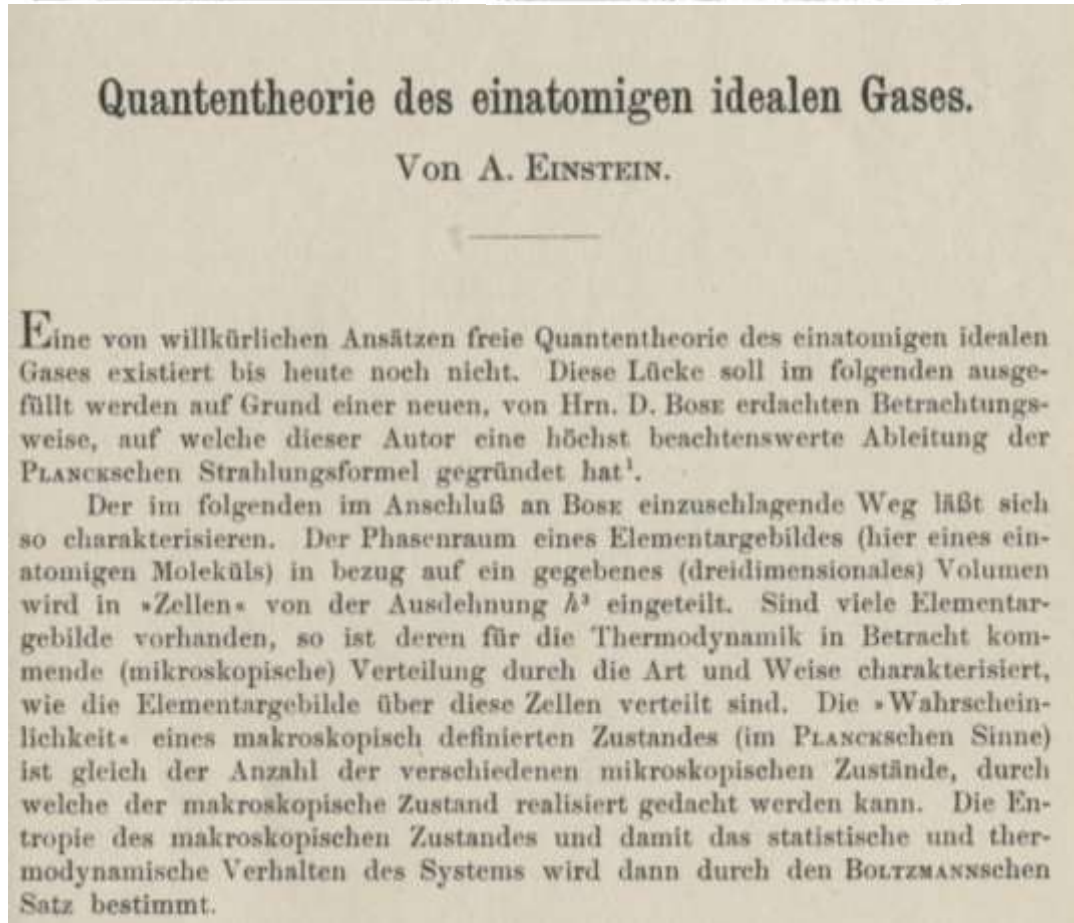


**Nobel Prize in Physics 2001**  
*"for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms, and for early fundamental studies of the condensates"*

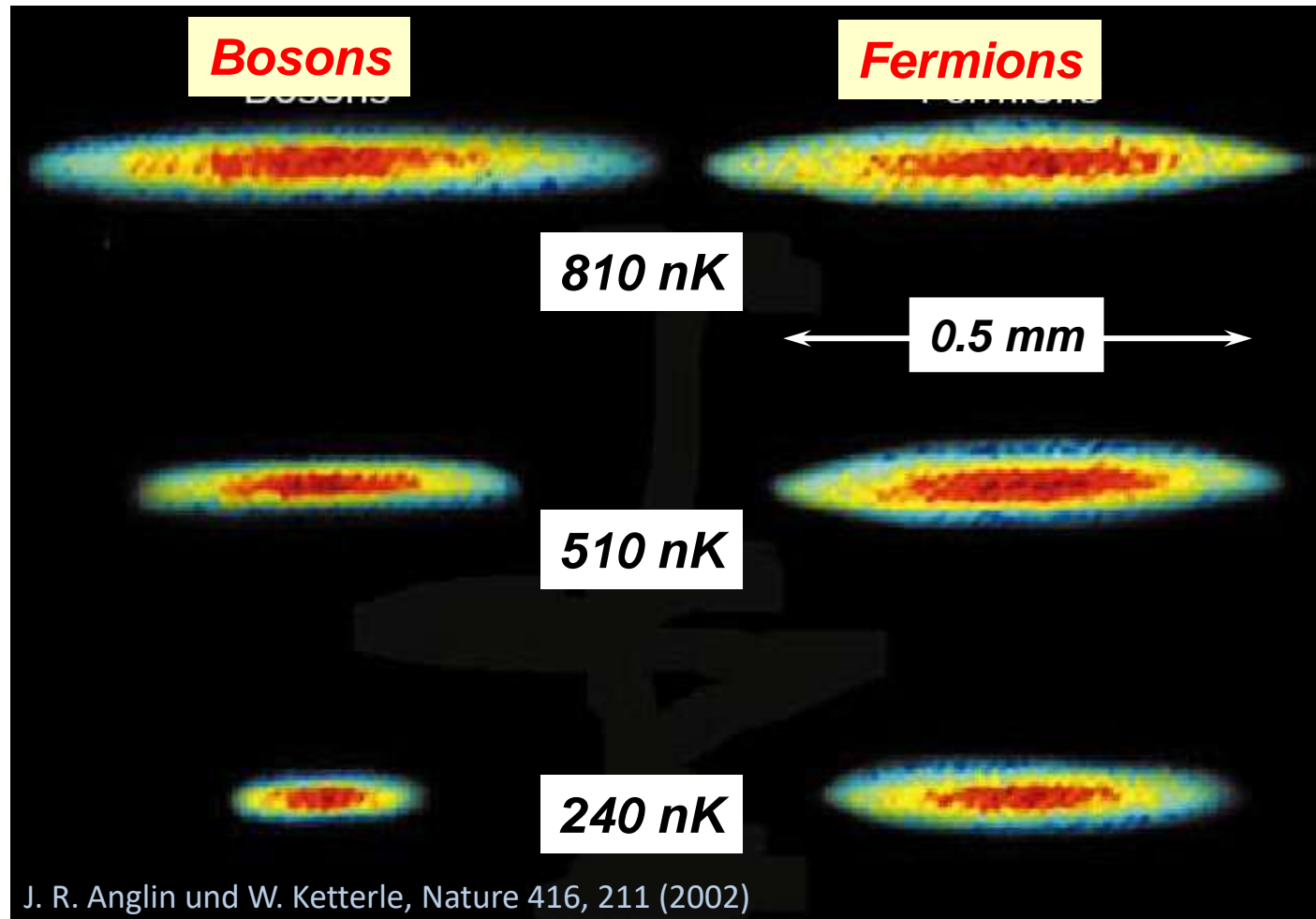




A. Einstein.  
 Quantentheorie des einatomigen Gases.  
 SITZUNGSBERICHTE DER PREUSSISCHEN AKADEMIE DER WISSENSCHAFTEN  
 PHYSIKALISCH-MATHEMATISCHE KLASSE 3, 1925.



# I.2.1 Bose-Einstein Condensation



*Pauli exclusion principle*  
 does not allow further  
 compression of fermions in  
 momentum space  
 at low  $T$

# I.2.1 Bose-Einstein Condensation

- degenerate multiparticle system: Bosons and Fermions (cf. I.1.1)

- at given density  $n$ :  
particle occupies cube with  $d = 1/n^{1/3}$

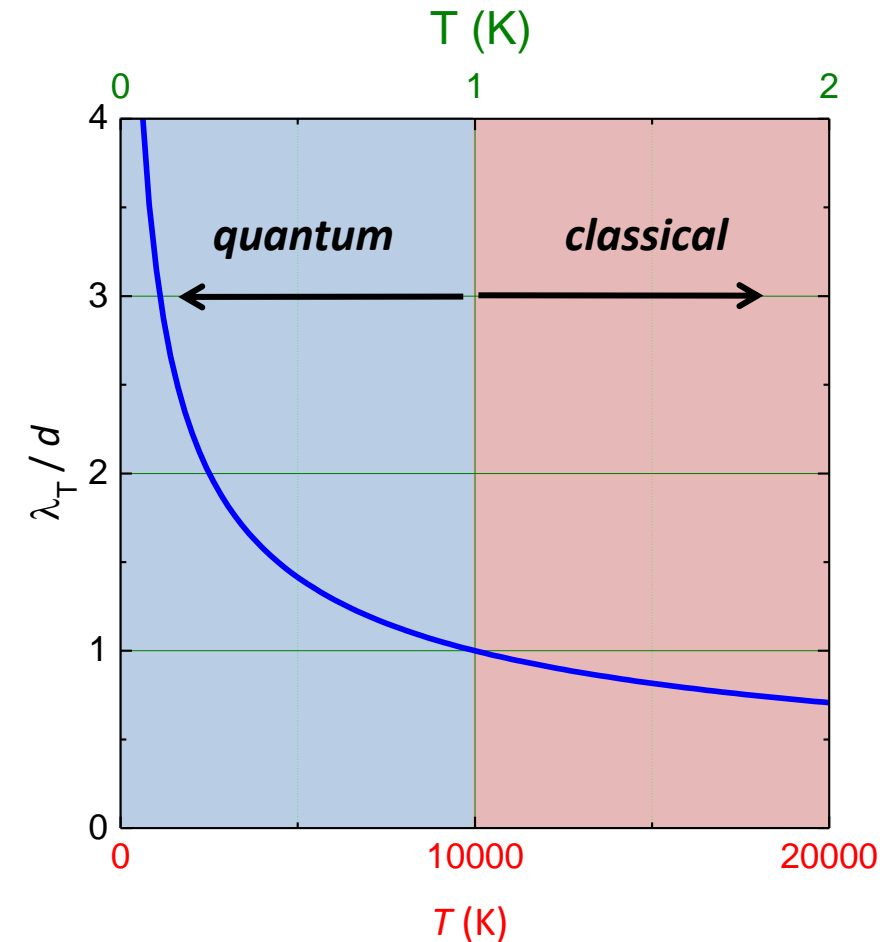
- classical liquid:**  $\lambda_T \leq d$   
*particles are distinguishable*

- quantum liquid:**  $\lambda_T \geq d$   
*particles are indistinguishable*  
→ *degenerate quantum liquid*

- Bosons:** ( ${}^4\text{He}$ , ...)  
→ *Bose-Einstein condensation*

- Fermions:** ( ${}^3\text{He}$ , e, p, n, ...)  
→ *Fermi sea*

${}^4\text{He}: n = 4 \cdot 10^{21} \text{ cm}^{-3}$ ,



$e^-: n = 3.6 \cdot 10^{21} \text{ cm}^{-3}$

# I.2.1 Bose-Einstein Condensation

- cooling down atoms

$$\lambda_T^3/d^3$$

	temperature	density (1/cm <sup>3</sup> )	phase space density
atomic oven	500 K	10 <sup>-1</sup>	10 <sup>-13</sup>
Laser cooling	50 μK	10 <sup>11</sup>	10 <sup>-6</sup>
evaporation cooling	500 nK	10 <sup>14</sup>	2.612
Bose-Einstein condensation		10 <sup>15</sup>	10 <sup>7</sup>

$$\lambda_T = \sqrt{\frac{h^2}{2\pi m k_B T}} = \sqrt{\frac{2\pi \hbar^2}{m k_B T}} \propto \frac{1}{\sqrt{mT}}$$

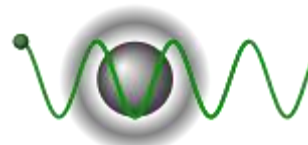
# I.2.1.1 Laser Cooling

- **Laser cooling:**  
techniques in which atomic and molecular samples are cooled down to near absolute zero **through the interaction with one or more laser fields**
- **Laser cooling techniques**
  - Doppler cooling (Schawlow, Hänsch)  
T.W. Hänsch, and A.L. Schawlow, *Cooling of gases by laser radiation*, Optics Communications, 13, 68 (1975)
  - other techniques
    - Sisyphus cooling
    - Resolved sideband cooling
    - Velocity selective coherent population trapping (VSCPT)
    - Anti-Stokes inelastic light scattering (typically in the form of fluorescence or Raman scattering)
    - Cavity mediated cooling
    - Sympathetic cooling
- **Nobel Price in Physics 1997:**  
**„for the development of methods to cool and trap atoms with laser light”**  
for **Steven Chu**, **Claude Cohen-Tannoudji** und **William D. Phillips**

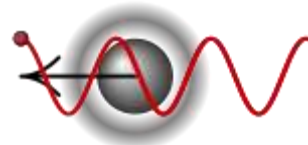


# I.2.1.1 Laser Cooling – Doppler Cooling

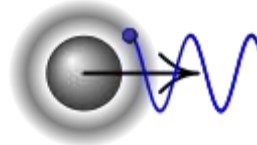
- in **Doppler cooling**, the frequency of *light is tuned slightly below an electronic transition in the atom*
- because the light is detuned to the "red" (i.e., at lower frequency) of the transition, the atoms will absorb more photons if they move towards the light source, due to the Doppler effect



– *a stationary atom sees the laser neither red- nor blue-shifted and does not absorb the photon*



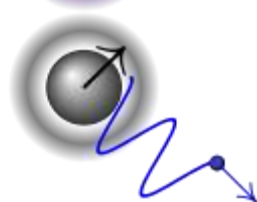
– *an atom moving away from the laser sees it red-shifted and does not absorb the photon*



– *an atom moving towards the laser sees it blue-shifted and absorbs the photon, slowing the atom*



– *the photon excites the atom, moving an electron to a higher quantum state.*

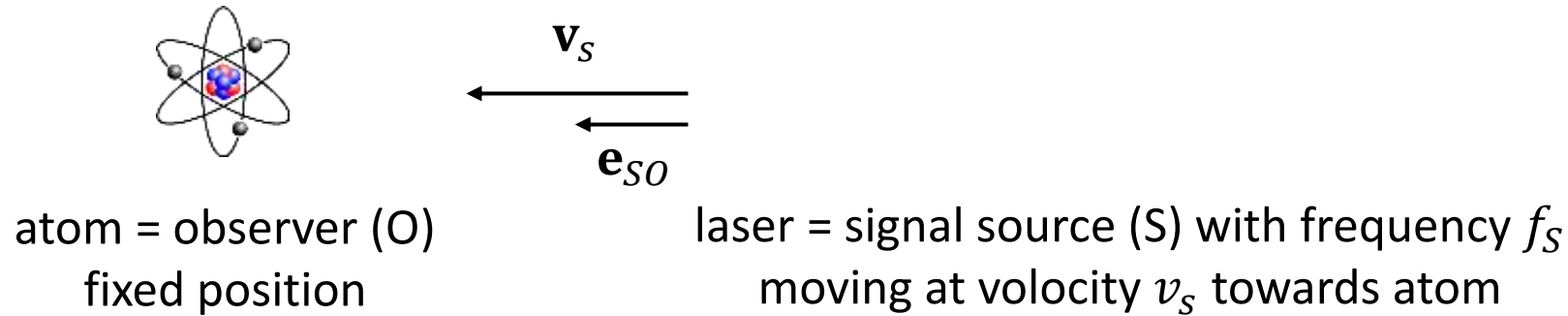


– *the atom re-emits a photon, as its direction is random, there is no net change in momentum over many absorption-emission cycles*



# I.2.1.1 Laser Cooling – Doppler Cooling

- **Doppler shift** (how does the moving atom see the laser frequency)



$$f_o = \frac{f_s}{1 - \frac{v_s \cdot e_{s0}}{c}}$$

laser frequency seen by the atom is blue-shifted,  
if laser and atom move towards each other

→ red detuned laser light becomes resonant if atom moves towards the laser

- **momentum/velocity change of atom due to photon absorption**

$$\frac{\Delta p}{p} = \frac{\hbar k_{ph}}{mv} = \frac{\Delta v}{v} \quad \Rightarrow \quad \Delta v = \frac{\hbar k_{ph}}{m}$$

# I.2.1.1 Laser Cooling – Doppler Cooling

- *atom-laser detuning with atom at rest:*

$$\delta_0 = \omega_{atom} - \omega_{laser}$$

- *atom-laser detuning with atom moving at velocity v:*

$$\delta_{Doppler} = 2\pi(f_0 - f_s) = 2\pi f_0 \frac{\mathbf{v} \cdot \mathbf{e}_{SO}}{c} = 2\pi f_0 \frac{\hbar \mathbf{k} \cdot \mathbf{v}}{\hbar \omega_0} = \mathbf{k} \cdot \mathbf{v}$$

$$f_0 = \frac{f_s}{1 - \frac{\mathbf{v}_s \cdot \mathbf{e}_{SO}}{c}}$$

effective detuning between atom and laser

$$\delta_{eff} = \delta_0 - \delta_{Doppler} = \delta_0 - \mathbf{k} \cdot \mathbf{v}$$

we use:

$$p = \frac{E}{c}, \quad c = \frac{E}{p} = \frac{\hbar \omega}{mv} = \frac{\hbar \omega}{\hbar k}$$

- *force acting on the atom:*

$$\mathbf{F} = \hbar \mathbf{k} \gamma = \hbar \mathbf{k} \frac{\Gamma}{2} \frac{(I/I_0)}{1 + I/I_0 + \left(\frac{2\delta_{eff}}{\Gamma}\right)^2}$$

large intensity  $I \gg I_0$ :

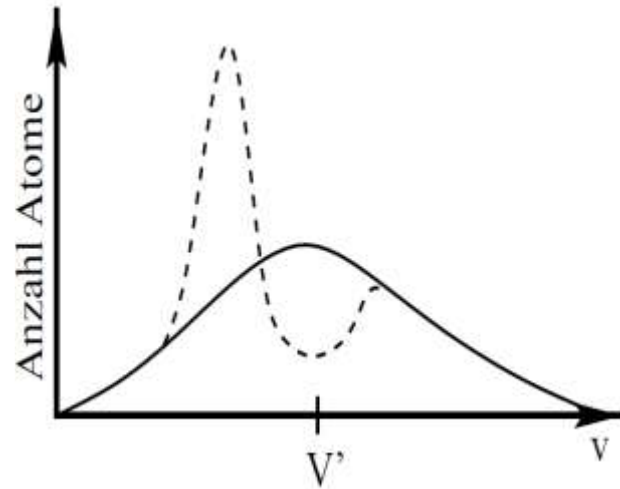
$$\mathbf{F} \rightarrow \hbar \mathbf{k} \frac{\Gamma}{2}$$

$\gamma$  = absorption rate

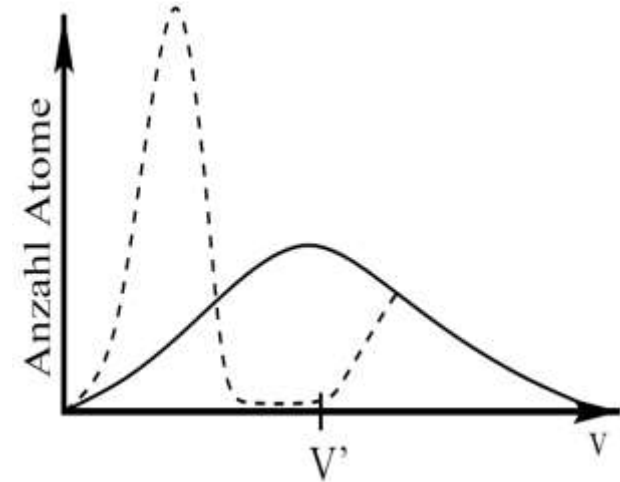
$\Gamma$  = line width of atom transition,  
spontaneous emission rate

$I_0$  = saturation intensity

# I.2.1.1 Laser Cooling – Doppler Cooling



- velocity distribution of an atom beam before and after (dashed line) the interaction with a laser beam of fixed frequency



- velocity distribution of an atom beam before and after (dashed line) the interaction with a laser beam of continuously decreasing frequency

# I.2.1.1 Laser Cooling – Doppler Cooling

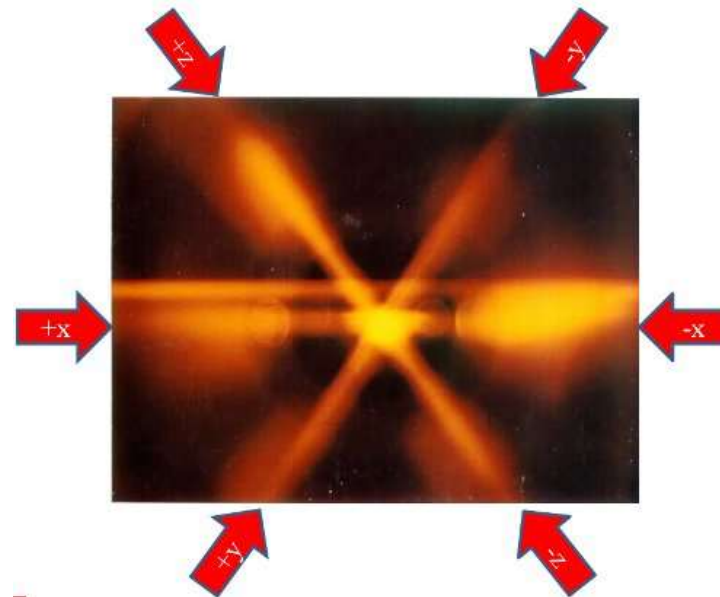
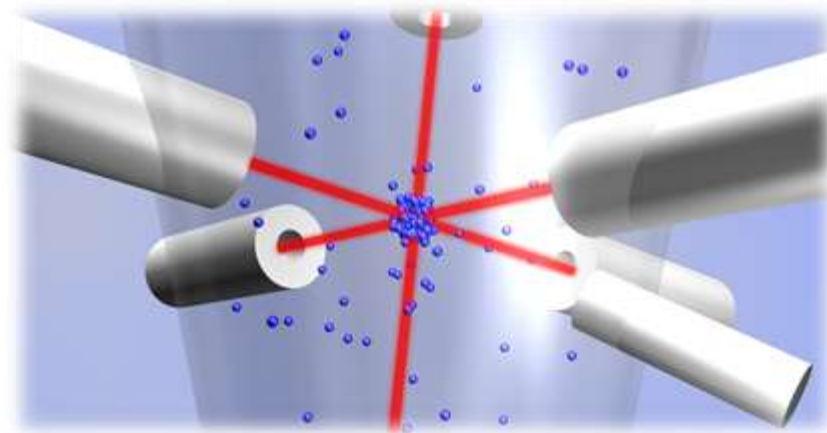
- *achievable minimum temperature in Doppler cooling:*
  - spontaneous absorption and emission are random processes
    - the average force acting on the atoms is statistically fluctuating
    - Brownian motion in phase space
  - minimum temperature is given by the compensation of heating effect due to the random absorption and emission processes and maximum cooling rate

$$k_B T_{\text{Doppler}} = \frac{\hbar \Gamma}{2}$$

- e.g.  $T_{\text{Doppler}} = 240 \mu\text{K}$  for Na corresponding to average velocity of 1 km/h  
 $T_{\text{Doppler}} = 140 \mu\text{K}$  for Rb

# I.2.1.1 Laser Cooling – Optical Molasses

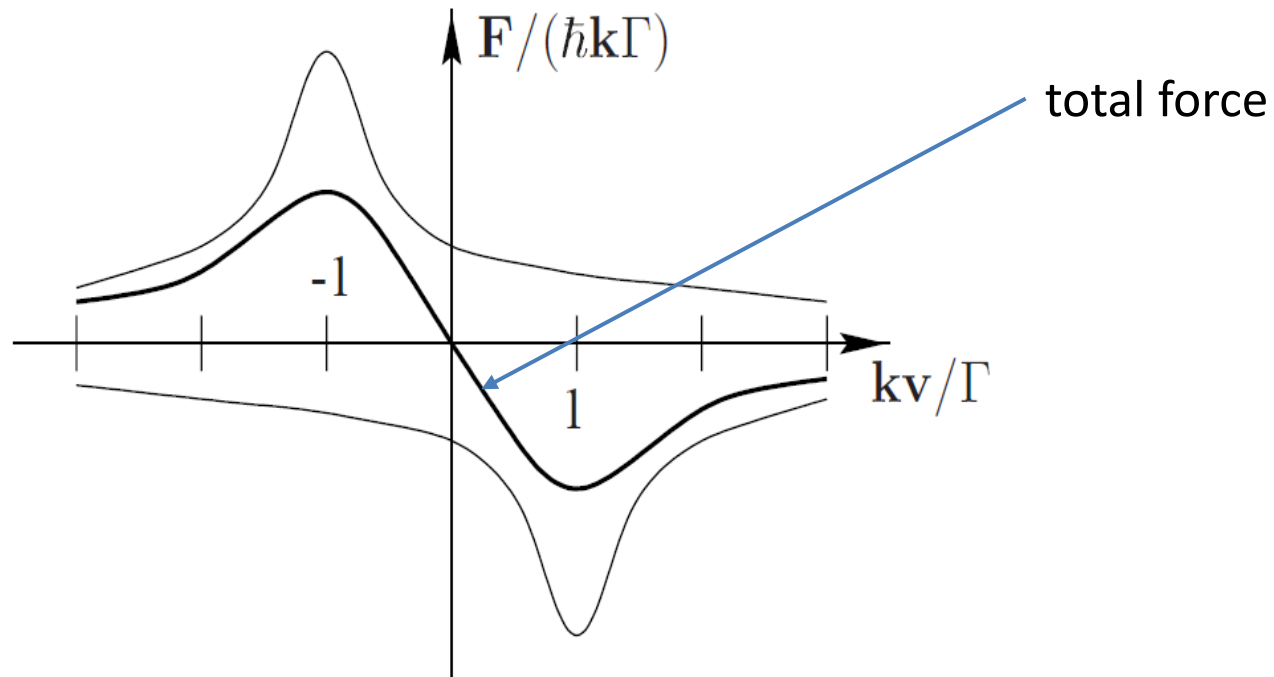
- **optical molasses** (Steven Chu, 1985)
- an **optical molasses** consists of 3 pairs of counter-propagating circularly polarized laser beams intersecting in the region where the atoms are present
- while for simple Doppler cooling the minimum temperature for Na atoms is about  $250 \mu\text{K}$ , optical molasses can cool the atoms down to  $40 \mu\text{K}$ , i.e. an order of magnitude colder.



# I.2.1.1 Laser Cooling – Optical Molasses

- force acting by two laser with opposite directions

$$\mathbf{F} = +\hbar\mathbf{k}\frac{\Gamma}{2}\frac{(I/I_0)}{1 + \frac{I}{I_0} + \left(\frac{2(\delta_0 - \mathbf{k} \cdot \mathbf{v})}{\Gamma}\right)^2} - \hbar\mathbf{k}\frac{\Gamma}{2}\frac{(I/I_0)}{1 + \frac{I}{I_0} + \left(\frac{2(\delta_0 + \mathbf{k} \cdot \mathbf{v})}{\Gamma}\right)^2}$$





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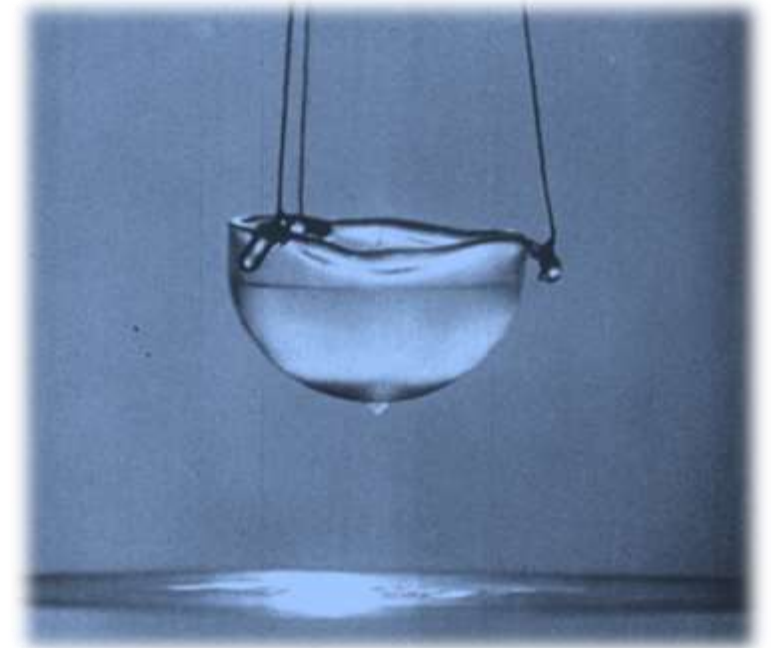


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Technische  
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# Superconductivity and Low Temperature Physics II



**Lecture No. 3**  
**12 May 2022**

**R. Gross**  
**© Walther-Meißner-Institut**

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### I.6 $^3\text{He}$ / $^4\text{He}$ mixtures





# I.2.2 Ideal Bose Gas

- consider a gas of  $N$  ( $\approx 10^{22}$ ) bosons of mass  $m$  in a cube of volume  $V = L^3$  ( $\approx 1 \text{ cm}^3$ ), canonical ensemble (given  $T, V, N$ )

- eigen energies:

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m} \left(\frac{\pi}{L}\right)^2 n^2 \quad \text{with } n^2 = n_x^2 + n_y^2 + n_z^2$$

- $\Delta\varepsilon/k_B T \leq 10^{-14}$

→ level spacing small compared to thermal energy (e.g. He @ 10 K)  
 → no influence on experimental result

- occupation probability:

$$n_k = \frac{1}{\exp\left(\frac{\varepsilon_k - \mu}{k_B T}\right) - 1}$$

### Bose-Einstein distribution

$\mu$  = chemical potential

#### important note:

- $\mu \leq \varepsilon_k$  for all  $k$  to avoid negative occupation probabilities
- usually one sets  $\varepsilon_0 = 0$  for the lowest energy state

- energy and particle number:

$$U(T, V, \mu) = \sum_k \varepsilon_k n_k$$

$$N(T, V, \mu) = \sum_k n_k$$

# I.2.2 Ideal Bose Gas

- derivation of Bose-Einstein distribution (1)

- we use the single spin density of states of free particles in 3D:  $D(\varepsilon) = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\varepsilon}, \quad N(\varepsilon) = \frac{D(\varepsilon)}{V}$

- total entropy of the particle gas:  $S = k_B \ln \Omega$      $\Omega =$  number of the available states at given total energy  $E$

- $\Omega$  is determined by the number of possibilities to distribute the  $N$  particles on the available states
  - we divide the  $k$ -space into spherical shells with the shell  $s$  containing  $M_s$  states
  - the number of possibilities to put  $N_s$  particles on  $M_s$  states is given by

$$\Omega_s = \frac{(N_s + M_s - 1)!}{N_s! (M_s - 1)!} \quad (\text{we use } M_s = D(\varepsilon)\delta\varepsilon_s \text{ for the number of available states in the shell } s)$$

- the total number  $\Omega$  of possibilities is obtained by multiplying the numbers  $\Omega_s$

$$\Omega = \prod_s \Omega_s = \prod_s \frac{(N_s + M_s - 1)!}{N_s! (M_s - 1)!}$$

# I.2.2 Ideal Bose Gas

- derivation of Bose-Einstein distribution (2)

- we use the Stirling approximation  $\ln N! = N \ln N - N$  and for  $N_s, \Omega_s \gg 1$  we obtain

$$S = k_B \ln \Omega = k_B \sum_s [(N_s + M_s) \ln(N_s + M_s) - N_s \ln N_s - M_s \ln M_s]$$

- in thermal equilibrium the particles are distributed on the shells to maximize  $S$  under the boundary conditions  $N = \sum_s N_s = \text{const.}$  and  $U = \sum_s \varepsilon_s N_s = \text{const.}$

- we maximize  $S$  for  $N = \text{const.}$  and  $U = \text{const.}$  we obtain

$$(dU = TdS + \mu dN)$$

$$\frac{\partial S}{\partial N_s} - \frac{1}{T} \frac{\partial U}{\partial N_s} + \frac{\mu}{T} \frac{\partial N}{\partial N_s} = 0 = \frac{\partial S}{\partial N_s} - k_B \beta \frac{\partial U}{\partial N_s} + k_B \beta \mu \frac{\partial N}{\partial N_s} \quad \beta = 1/k_B T$$

(Lagrangian multipliers)

- performing the differentiation we obtain

$$\ln(N_s + M_s) - \ln N_s - \beta \varepsilon_s + \beta \mu = 0 \quad \Rightarrow \quad N_s = \frac{1}{\exp(\beta(\varepsilon_s - \mu)) - 1} M_s$$

- average particle number occupying state in shell  $s$

$$n_{\text{BE}} = \left[ \exp\left(\frac{\varepsilon_s - \mu}{k_B T}\right) - 1 \right]^{-1}$$

# I.2.2 Ideal Bose Gas

- how to get rid of the summation over  $k$  ?

$$\sum_k \dots \Rightarrow \int Z(k) d^3k = \frac{V}{(2\pi)^3} \int d^3k$$

$$\sum_k \dots \Rightarrow \int D(\varepsilon) d\varepsilon = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int \sqrt{\varepsilon} d\varepsilon$$

$$U(T, V, \mu) = \sum_k \varepsilon_k n_k \Rightarrow \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int \frac{\varepsilon^{3/2}}{\exp\left(\frac{\varepsilon - \mu}{k_B T}\right) - 1} d\varepsilon$$

$$N(T, V, \mu) = \sum_k n_k \Rightarrow \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int \frac{\varepsilon^{1/2}}{\exp\left(\frac{\varepsilon - \mu}{k_B T}\right) - 1} d\varepsilon$$

for free bosons with dispersion  $\varepsilon_k = \frac{\hbar^2 k^2}{2m}$ :

$$Z(k) = \frac{V}{(2\pi)^3}$$

$$D(\varepsilon) = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\varepsilon}$$

Note:

we assume that the states are not degenerate, otherwise we have to multiply with degeneracy  $g_k$

→ we assume  $g_k = 1$

**crucial point:** the integral is bounded!! If the particle number  $N$  is too large of the temperature too low, the equation can no longer be satisfied

# I.2.2 Ideal Bose Gas

- evaluation of the equation for the particle number

$$N(T, V, \mu) = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\varepsilon^{1/2}}{\exp\left(\frac{\varepsilon - \mu}{k_B T}\right) - 1} d\varepsilon = \frac{V}{4\pi^2} \left(\frac{2mk_B T}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\gamma e^{-x}}{1 - \gamma e^{-x}} x^{1/2} dx$$

with  $x = \varepsilon/k_B T = \beta\varepsilon$  and **fugacity**  $\gamma = \exp\left(\frac{\mu}{k_B T}\right)$

Gamma function:  $\Gamma(t) = \int_0^\infty e^{-y} y^{t-1} dy$

- to solve the integral we use the series development  $\frac{\gamma e^{-x}}{1 - \gamma e^{-x}} = \gamma e^{-x} (1 + \gamma e^{-x} + \gamma^2 e^{-2x} + \dots) = \sum_{p=1}^\infty \gamma^p e^{-px}$
- plugging in this expression we can perform the integration by using

$$\int_0^\infty e^{-px} x^{1/2} dx = \frac{1}{p^{3/2}} \underbrace{\int_0^\infty e^{-y} y^{1/2} dy}_{\Gamma(3/2)} = \frac{1}{p^{3/2}} \frac{\sqrt{\pi}}{2} \Rightarrow \int_0^\infty \frac{\gamma e^{-x}}{1 - \gamma e^{-x}} x^{1/2} dx = \frac{\sqrt{\pi}}{2} \underbrace{\sum_{p=1}^\infty \frac{\gamma^p}{p^{3/2}}}_{g_{3/2}(\gamma)}$$

Gamma function:  $\Gamma(t) = \int_0^\infty e^{-y} y^{t-1} dy$

$$\Rightarrow N(T, V, \mu) = V \underbrace{\left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2}}_{1/\lambda_T^3} g_{3/2}(\gamma) = \frac{V}{\lambda_T^3} g_{3/2}(\gamma)$$

$$n(T, \mu) \lambda_T^3 = g_{3/2}(\gamma)$$

# I.2.2 Ideal Bose Gas

- analysis of the function  $g_{3/2}(\gamma)$

$$g_{3/2}(\gamma) = \sum_{p=1}^{\infty} \frac{\gamma^p}{p^{3/2}}$$

- $|\gamma| < 1$ : the series converges
- $|\gamma| > 1$ : the series diverges
- $|\gamma| = 1$ : the series converges and has the value  $g_{3/2}(1)$

$$g_{3/2}(1) = \sum_{p=1}^{\infty} \frac{1}{p^{3/2}} = \zeta\left(\frac{3}{2}\right) = 2.612$$

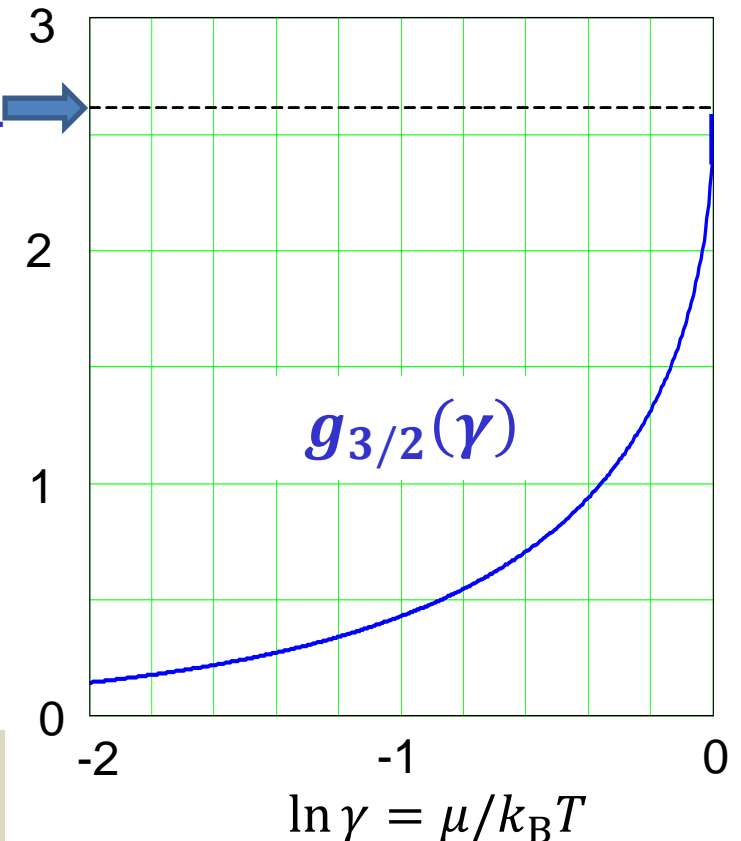
$$\zeta(3/2) = 2.61237 \dots$$

$$\zeta(s) = \sum_{p=1}^{\infty} \frac{1}{p^s}$$

(Rieman Zeta function)

$$\frac{dg_{3/2}(z)}{dz} = \frac{1}{z} \sum_{p=1}^{\infty} \frac{\gamma^p}{p^{1/2}}$$

slope diverges for  $\gamma = 1$



**important fact:**  $g_{3/2}(\gamma)$  has maximum value 2.61 for  $\gamma \rightarrow 1$ , resp.  $\ln \gamma \rightarrow 0$   
particle number  $N$  is bounded to maximum value !!

# I.2.2 Ideal Bose Gas

- what happens with decreasing  $T$  ?

–  $\lambda_T$  increases  $\propto 1/\sqrt{T}$   $\rightarrow g_{3/2}(\gamma)$  has to increase to keep  $n(T, \mu) = \frac{1}{\lambda_T^3} g_{3/2}(\gamma)$  constant

- **problem:**

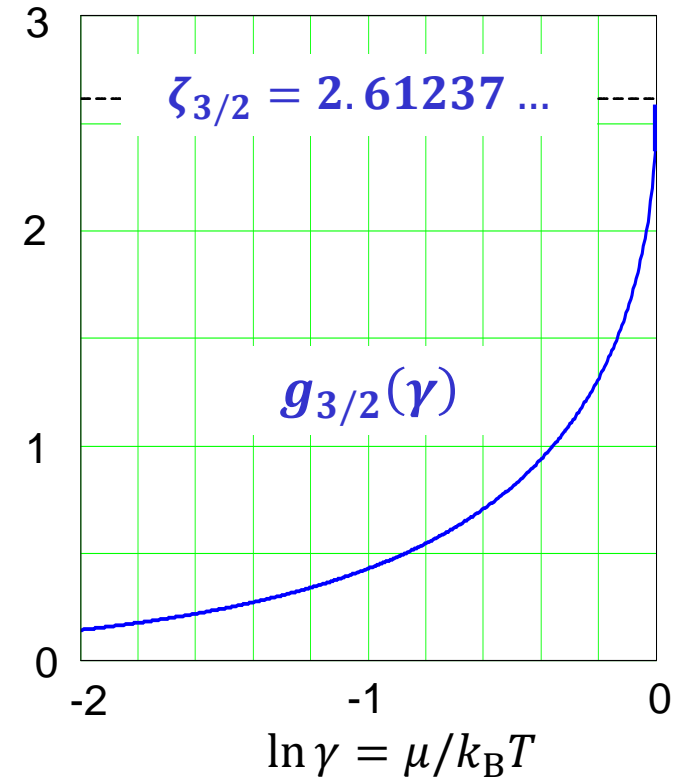
$g_{3/2}(\gamma)$  has maximum value  $g_{3/2}(1) = \zeta(3/2) = 2.61$  for  $\gamma = 1$ , resp.  $\mu = 0$ ,

$\rightarrow g_{3/2}(\gamma = 1)$  is related to **maximum value**  $\lambda_T = \lambda_{T,\text{BEC}}$ ,  
resp. to **minimum temperature**  $T_{\text{BEC}}$

$$\lambda_{T,\text{BEC}}^3 \cdot n = g_{3/2}(\gamma = 1) = \zeta(3/2) = 2.61237 \dots$$

defines the characteristic temperature  $T_{\text{BEC}}$ :

for  $T < T_{\text{BEC}}$  it is not possible to satisfy the equation  $n(T, \mu) = \frac{1}{\lambda_T^3} g_{3/2}(\gamma)$



- with  $\lambda_{T,\text{BEC}} = h/\sqrt{2\pi m k_B T_{\text{BEC}}}$ , we obtain:

$$n = 2.612 \left( \frac{m k_B T_{\text{BEC}}}{2\pi \hbar^2} \right)^{3/2} \quad \Rightarrow \quad T_{\text{BEC}} = \frac{2\pi \hbar^2}{m k_B} \left( \frac{n}{2.612} \right)^{2/3}$$

**Bose-Einstein condensation temperature**

$\Rightarrow \lambda_{T,\text{BEC}}^3 \cdot n \simeq O(1) \rightarrow$  *critterion for the importance of quantum effects or degeneracy*

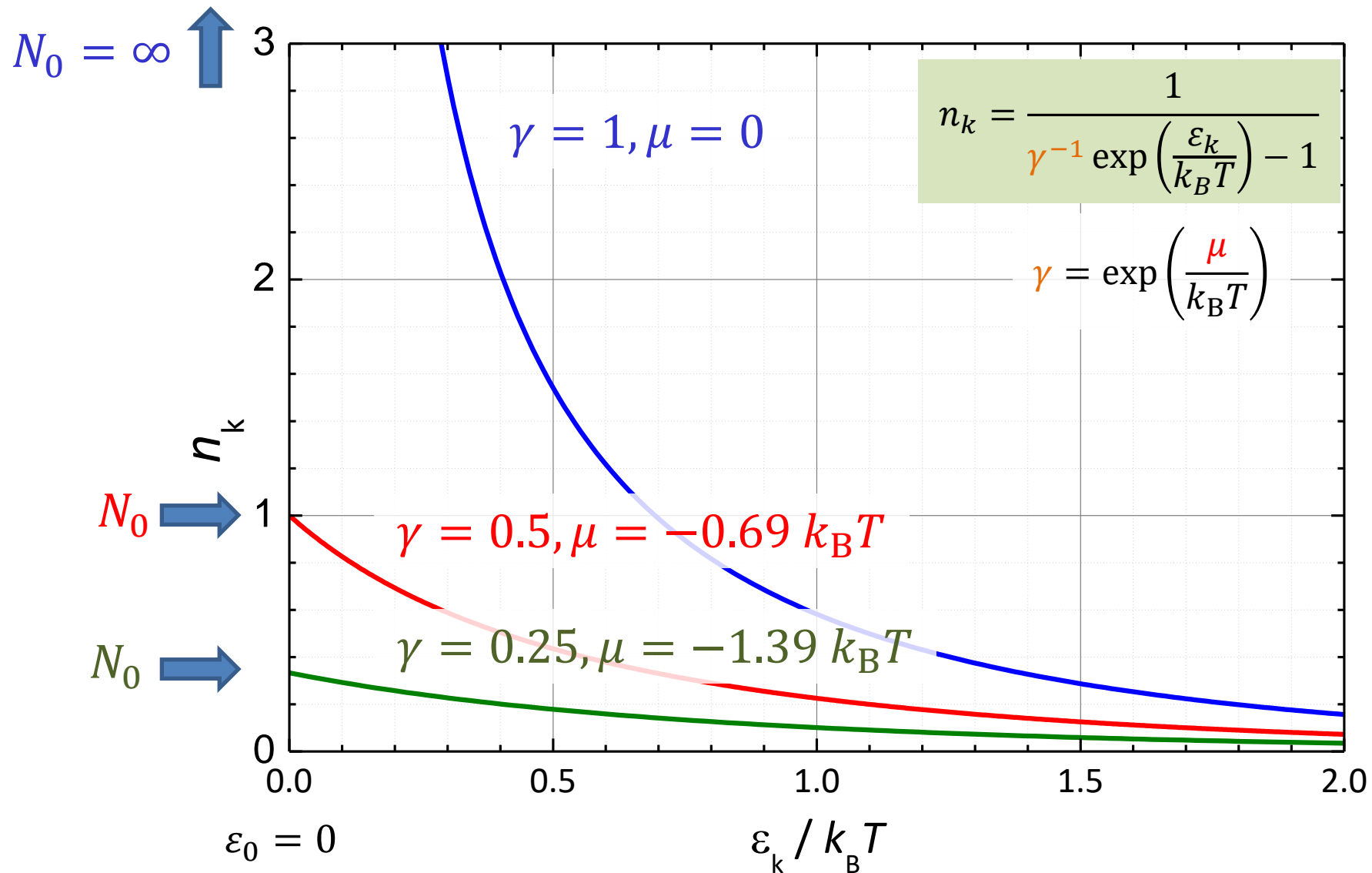
# I.2.2 Ideal Bose Gas

- how to solve the problem ?
  - physical solution of the problem that we cannot satisfy the equation  $n(T, \mu) = \frac{1}{\lambda_T^3} g_{3/2}(\gamma)$  below  $T_{\text{BEC}}$ 
    - we have to put a significant number  $N_0$  of the particles in the ground state  $k = 0$
    - this is in agreement with the Bose-Einstein distribution
    - setting  $\varepsilon(k = 0) = \varepsilon_0 = 0$ , the number of bosons in the ground state is

$$N_0(T, \mu) = \frac{1}{\exp\left(\frac{0 - \mu}{k_B T}\right) - 1} \simeq \frac{1}{1 - \frac{\mu}{k_B T} - 1} = \frac{k_B T}{|\mu|} \rightarrow \infty \text{ for } |\mu| = |\varepsilon_0 - \mu| \rightarrow 0$$

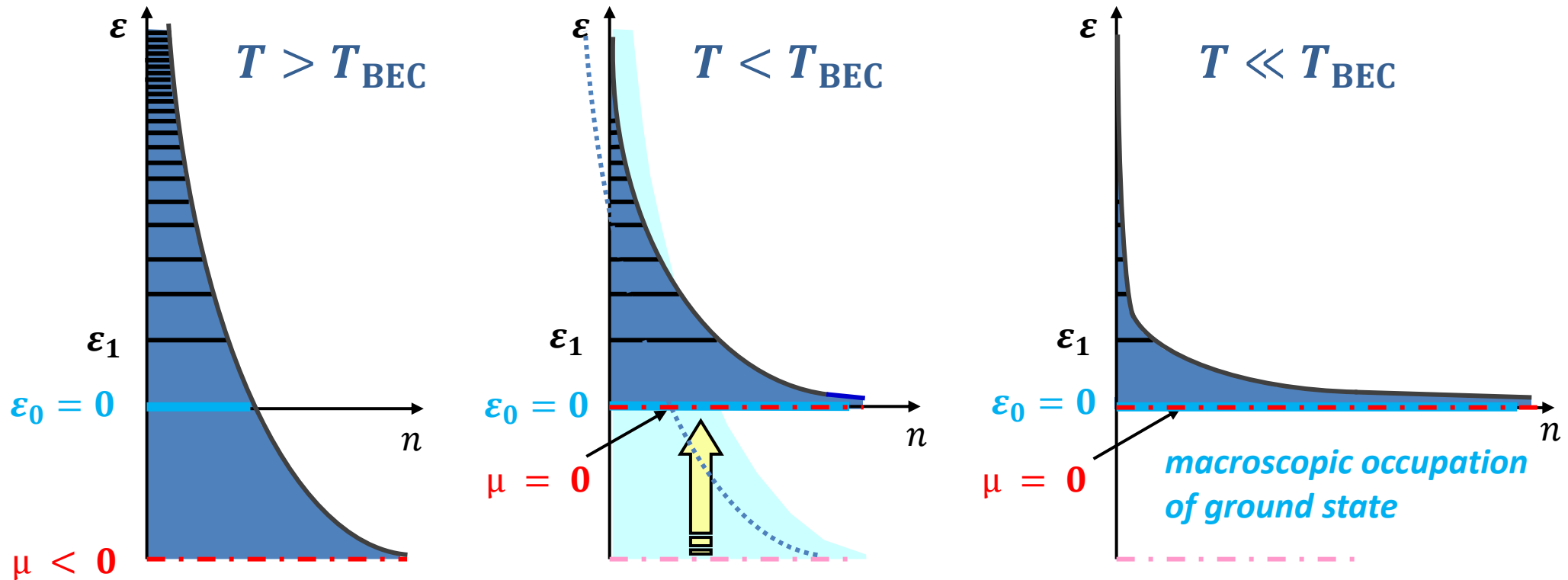


# I.2.2 Ideal Bose Gas



if  $|\epsilon_0 - \mu| = |0 - \mu| = |\mu| \rightarrow 0$ , the occupation number  $N_0 = n(k = 0)$  of the ground state  $\epsilon_0 = 0$  strongly increases

# I.2.2 Ideal Bose Gas



- calculation of  $N_0(T) = N(\epsilon_0, T)$  for  $(\epsilon_0 - \mu) = (0 - \mu) = -\mu \rightarrow 0$ :

$$N_0(T, \mu) = \frac{1}{\exp\left(\frac{\epsilon_0 - \mu}{k_B T}\right) - 1} \approx \frac{1}{1 + \frac{\epsilon_0 - \mu}{k_B T} - 1} = \frac{k_B T}{\epsilon_0 - \mu} \rightarrow \infty \text{ for } \epsilon_0 - \mu \rightarrow 0$$

→  $N_0(T, \mu)$  grows dramatically for  $(\epsilon_0 - \mu) \rightarrow 0$ , that is, for  $\mu \rightarrow 0$

Note:

$$\epsilon_0 - \mu = k_B T \ln\left(1 + \frac{1}{N_0}\right) \approx \frac{k_B T}{N_0}$$

→ difference between ground state and chemical potential is much smaller than  $k_B T$

# I.2.2 Ideal Bose Gas

- detailed analysis of the chemical potential

$$n(T, \mu) = \frac{N(T, V, \mu)}{V} = \left( \frac{mk_B T}{2\pi\hbar^2} \right)^{3/2} g_{3/2}(\gamma) = \frac{1}{\lambda_T^3} g_{3/2}(\gamma) \Rightarrow g_{3/2}(\gamma) = \left( \frac{2\pi\hbar^2}{mk_B T} \right)^{3/2} n(T, \mu)$$

- for high temperatures  $T$  and/or small particle density  $n$  we can use the approximation  $g_{3/2}(\gamma) \simeq \gamma + \dots \simeq e^{\beta\mu} + \dots$

$$\Rightarrow e^{\beta\mu} = \left( \frac{2\pi\hbar^2}{mk_B T} \right)^{3/2} n(T, \mu) \Rightarrow \mu \simeq -\frac{3}{2} k_B T \ln \left( \frac{mk_B T}{2\pi\hbar^2 n^{2/3}} \right)$$

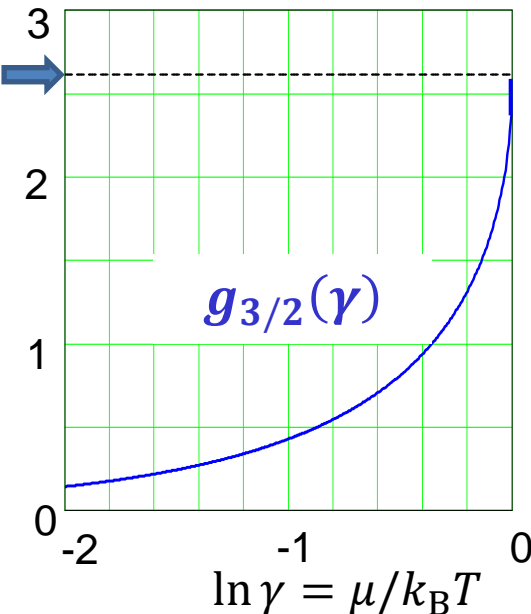
$$\zeta_{3/2} = 2.61237 \dots$$

- for  $T \rightarrow \infty$ :

$\lambda_T \rightarrow 0, \mu \rightarrow -\infty, \gamma \rightarrow 0, \ln \gamma \rightarrow -\infty$  and therefore  $g_{3/2}(\gamma) \rightarrow 0$

- for decreasing  $T$ :

$\lambda_T$  increases,  $\mu/k_B T$  increases  $\rightarrow 0, \gamma$  increases  $\rightarrow 1$  and therefore  $g_{3/2}(\gamma) \rightarrow 2.61$



# I.2.2 Ideal Bose Gas

- what happens if we decrease  $T$  to values below  $T_{\text{BEC}}$ ?

- for  $T < T_{\text{BEC}}$ , there is no solution of equation  $n(T, \mu) = \frac{1}{\lambda_T^3} g_{3/2}(\gamma)$  as  $g_{3/2}(\gamma)$  has maximum value  $g_{3/2}(1)$  for  $\gamma \rightarrow 1$

- for  $T < T_{\text{BEC}}$ , the particle number  $N_0$  in the ground quantum state ( $\varepsilon_0 = 0$ ) must become large (first recognized by Einstein)

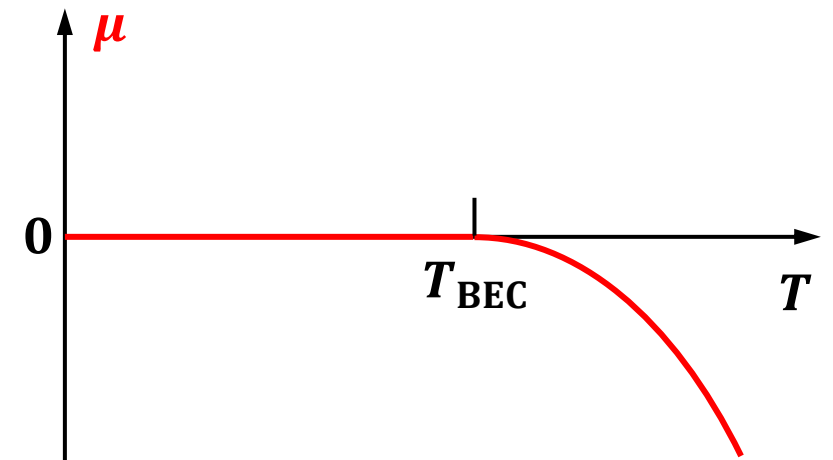
occupation of the ground state  $\varepsilon_0 = 0$ : 
$$N_0 = \frac{1}{\exp(-\mu/k_B T) - 1} \Rightarrow \mu = -k_B T \ln\left(1 + \frac{1}{N_0}\right) \simeq -k_B T \frac{1}{N_0}$$

- from the total particle number  $N$  a macroscopically large number  $N_0$  has to be placed in ground state  $\varepsilon_0$

- macroscopically large means that  $N_0$  is proportional to system volume  $V$

- if the fraction of particles in the ground state becomes finite at  $T = T_{\text{BEC}}$ , then in the thermodynamic limit  $V \rightarrow \infty$  the particle number  $N_0 \rightarrow \infty$  and hence the chemical potential  $\mu \rightarrow 0$

→  $\mu = 0$  for  $T \leq T_{\text{BEC}}$



as  $\mu$  corresponds to the average energy required to add a particle to the systems, the result is evident: we are adding particles only to the ground state  $\varepsilon_0 = 0$

# I.2.2 Ideal Bose Gas

- for  $T \leq T_{\text{BEC}}$ : how are the  $N$  particles distributed among the available states ?

$$- N = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\sqrt{\varepsilon}}{\exp\left(\frac{\varepsilon-\mu}{k_{\text{B}}T}\right)-1} d\varepsilon \text{ yields } \mu \rightarrow 0 \text{ for } T \rightarrow T_{\text{BEC}}$$

$$- \text{for } \mu \rightarrow 0 \text{ (} T \rightarrow T_{\text{BEC}}\text{), we obtain } N_0 = \frac{1}{\exp\left(\frac{\varepsilon_0-\mu}{k_{\text{B}}T}\right)-1} \simeq \frac{k_{\text{B}}T}{|\mu|} \rightarrow \infty$$

$$\varepsilon_0 = \frac{\hbar^2}{2m} \left(\frac{\pi}{L}\right)^2 \propto \frac{1}{V^{2/3}} \propto \frac{1}{N^{2/3}} \rightarrow 0 \text{ for } N \rightarrow \infty \text{ (thermodynamic limit)}$$

below  $T_{\text{BEC}}$ : **macroscopic occupation  $N_0 \gg 1$  of the ground state is required**

- **key result:**

- at  $T \leq T_{\text{BEC}}$  a finite fraction of all particles occupies the ground state
- this phenomenon is called **Bose-Einstein condensation**
- the name „**condensation**“ is used since in analogy to the gaseous-liquid transition the condensed particles do no longer contribute to the pressure (see discussion below)

# I.2.2 Ideal Bose Gas

- for  $T \leq T_{\text{BEC}}$ : occupation of the first excited state

for  $\mu \rightarrow 0$  and  $T \sim 1$  K: level spacing for  $10^{22}$  atoms in a box with  $L = 1$  cm:  $\Delta\varepsilon/k_{\text{B}}T = \varepsilon_1/k_{\text{B}}T \leq 10^{-14}$

$$\Rightarrow n_1(T) = \frac{1}{1 + \exp\left(\frac{\varepsilon_1 - \mu}{k_{\text{B}}T}\right) - 1} \approx \frac{1}{1 + \frac{\varepsilon_1 - \mu}{k_{\text{B}}T} - 1} \approx \frac{k_{\text{B}}T}{\varepsilon_1 - \mu} \approx 10^{14}$$

$n_1$  and all  $n_{k>1}$  are small compared to  $N_0$  and the total particle number  $N = 10^{22}$

(justifies why only the ground state was singled out in the calculation of  $T_{\text{BEC}}$ )

- more detailed calculation:  $n_1/N_0 \propto N^{-1/3}$ , i.e.  $n_1/N_0 \rightarrow 0$  for  $N \rightarrow \infty$

$$\frac{n_1}{N_0} = \frac{k_{\text{B}}T}{\varepsilon_1 - \mu} \cdot \frac{\varepsilon_0 - \mu}{k_{\text{B}}T} = -\frac{\mu}{\varepsilon_1 - \mu} \approx -\frac{\mu}{\varepsilon_1} \approx \frac{k_{\text{B}}T}{N_0 \varepsilon_1} \quad \text{we use } N_0(T, \mu) \approx \frac{k_{\text{B}}T}{\varepsilon_0 - \mu} = \frac{k_{\text{B}}T}{-\mu}$$

for  $T < T_{\text{BEC}}/2$  we can use the crude approximations  $N_0 \simeq N$ ,  $T \simeq T_{\text{BEC}} \propto N^{2/3}$ :

$$\frac{n_1}{N_0} \propto \frac{1}{N^{1/3}} \propto \frac{1}{V^{1/3}} \quad \rightarrow \text{already at } T \simeq T_{\text{BEC}}/2, \text{ we have } N_0 \gg n_1, n_{k>1}$$

- for  $T \rightarrow 0$ :  $\varepsilon_{k>0}/k_{\text{B}}T \rightarrow \infty$  and hence  $n_{k>0} \rightarrow 0$ ,  $\mu/k_{\text{B}}T = 1/N_0 \simeq 0$ : **all bosons occupy ground state**

# I.2.2 Ideal Bose Gas

- for  $T < T_{\text{BEC}}$  we have to treat the ground state  $\varepsilon_0$  separately

$$N = \sum_{\mathbf{k}} \frac{1}{\exp\left(\frac{\varepsilon_{\mathbf{k}} - \mu}{k_B T}\right) - 1} = N_0 + N_{ex} = N_0 + \sum_{\mathbf{k} \neq 0} \left[ \exp\left(\frac{\varepsilon_{\mathbf{k}}}{k_B T}\right) - 1 \right]^{-1} = N_0 + \int_0^{\infty} D(\varepsilon) n(\varepsilon) d\varepsilon$$

↑ *particles in ground state*      ↑ *particles in excited states*

- we obtain for **free bosons**:

$$N_{ex} = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^{\infty} \frac{\varepsilon^{1/2}}{\exp\left(\frac{\varepsilon - \mu}{k_B T}\right) - 1} d\varepsilon = \frac{V}{4\pi^2} \left(\frac{2mk_B T}{\hbar^2}\right)^{3/2} \underbrace{\int_0^{\infty} \frac{\gamma e^{-x}}{1 - \gamma e^{-x}} x^{1/2} dx}_{=\Gamma(3/2) \zeta(3/2)}$$

with  $x = \beta\varepsilon$  and  
fugacity  $\gamma = \exp\left(\frac{\mu}{k_B T}\right) = 1$

$$\Rightarrow n = n_0 + n_{ex} = n_0 + 2.612 \left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2} \Rightarrow \frac{n_0}{n} = 1 - \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2}$$

$n_0$ : condensate density  
 $n_{ex}$ : normal fluid density

$$T_{\text{BEC}} = \frac{2\pi\hbar^2}{mk_B} \left(\frac{n}{2.612}\right)^{2/3} \Rightarrow n = 2.612 \left(\frac{2\pi\hbar^2}{mk_B T_{\text{BEC}}}\right)^{3/2}$$

# I.2.2 Ideal Bose Gas

- Summary: temperature dependence of  $N_0$  and  $N_{ex}$

we assume constant particle number:  
 $N = N_0(T) + N_{ex}(T) = \text{const.}$

- $T > T_{\text{BEC}}$ :  $N_0 = O(1)$   $\rightarrow$  can be neglected

$$N = N_{ex} = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\sqrt{\varepsilon}}{\exp\left(\frac{\varepsilon - \mu}{k_B T}\right) - 1} d\varepsilon \quad \text{is valid}$$

- $T = T_{\text{BEC}}$ :  $\mu \rightarrow 0$ ,  $\lambda_{T,\text{BEC}}^3 \cdot n = 2.612 \rightarrow N = N_{ex} = 2.612 \cdot \frac{V}{\lambda_{T,\text{BEC}}^3}$

- $T \leq T_{\text{BEC}}$ :  $\mu = 0$ ,  $N = N_0 + N_{ex} = N_0 + 2.612 \cdot \frac{V}{\lambda_T^3} = N_0 + 2.612 \cdot \frac{V}{\lambda_T^3} \cdot \frac{\lambda_{T,\text{BEC}}^3}{\lambda_{T,\text{BEC}}^3}$

$$N = N_0 + 2.612 \cdot \frac{V}{\lambda_{T,\text{BEC}}^3} \cdot \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} = N_0 + N \cdot \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2}$$

$$\rightarrow \frac{N_0}{N} = 1 - \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} \qquad \frac{N_{ex}}{N} = \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2}$$



# I.2.2 Ideal Bose Gas

- normal fluid density ( $T < T_{\text{BEC}}$ )

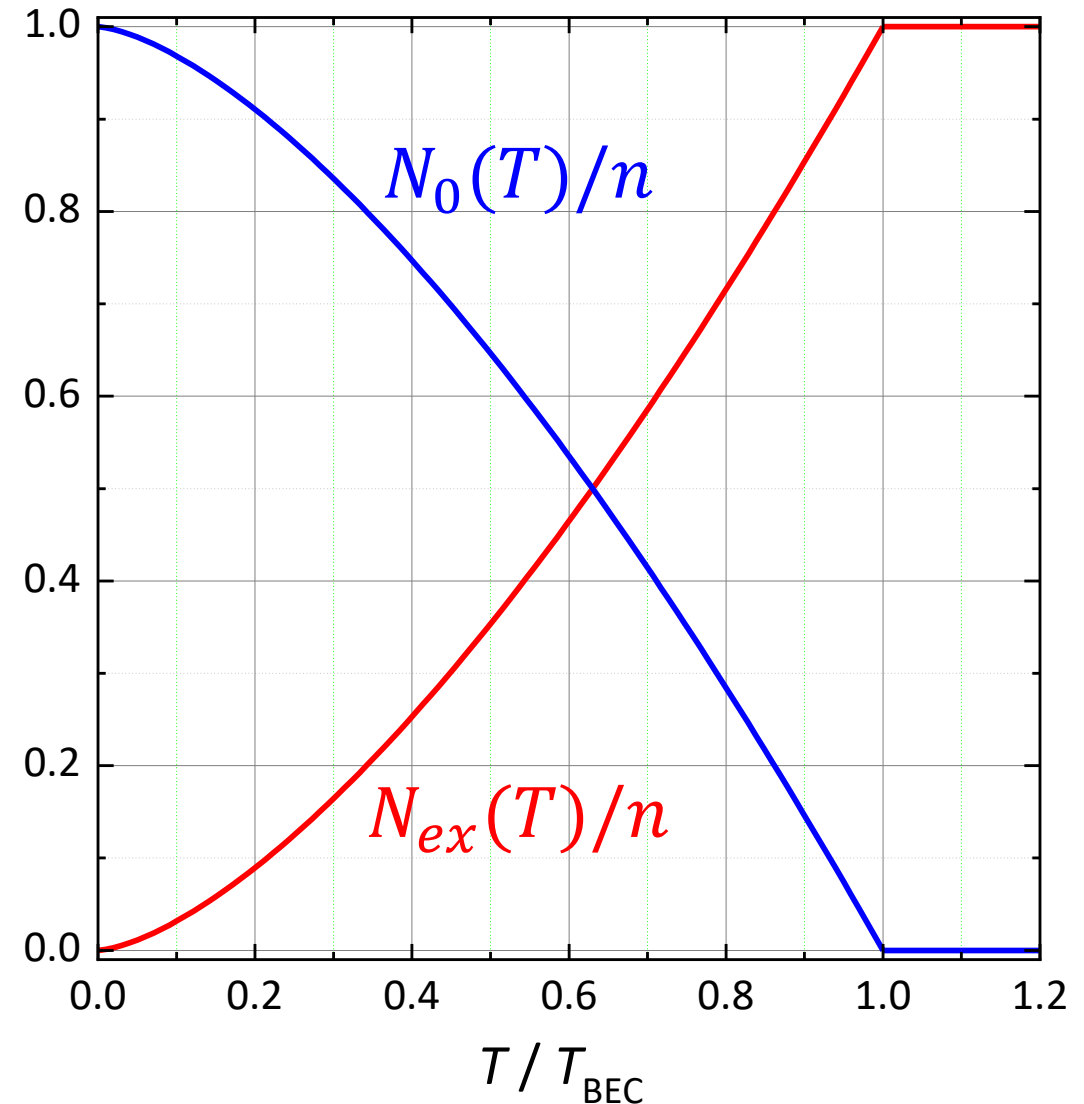
$$\frac{N_{ex}}{N} = \left( \frac{T}{T_{\text{BEC}}} \right)^{3/2}$$

- superfluid or condensate density

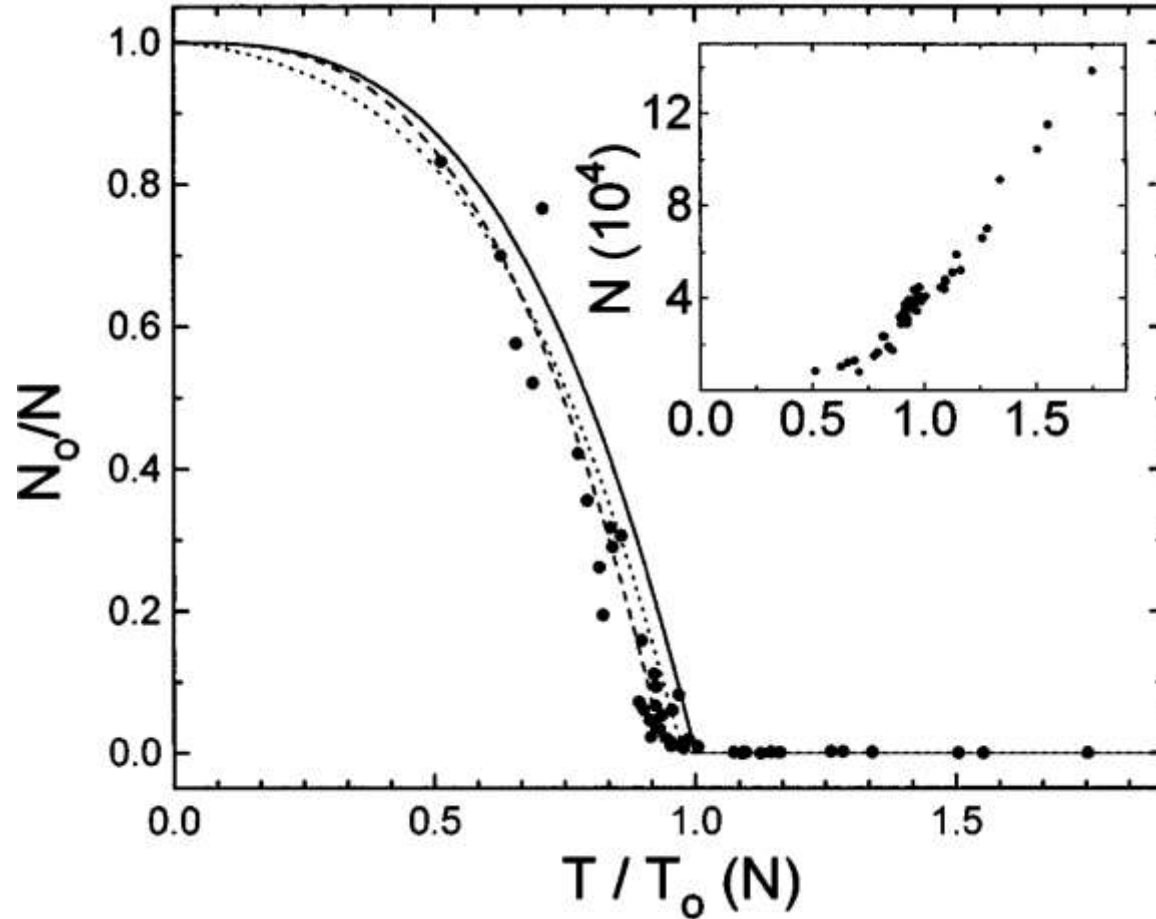
$$\frac{N_0}{N} = 1 - \left( \frac{T}{T_{\text{BEC}}} \right)^{3/2}$$

$$T_{\text{BEC}} = \frac{2\pi\hbar^2}{mk_B} \left( \frac{n}{2.612} \right)^{2/3}$$

$$\lambda_T^3 = \left( \frac{2\pi\hbar^2}{mk_B T} \right)^{3/2}$$



# I.2.2 Ideal Bose Gas



Total number  $N$  (inset) and ground-state fraction  $N_0/N$  as a function of  $T/T_{BEC}$ .

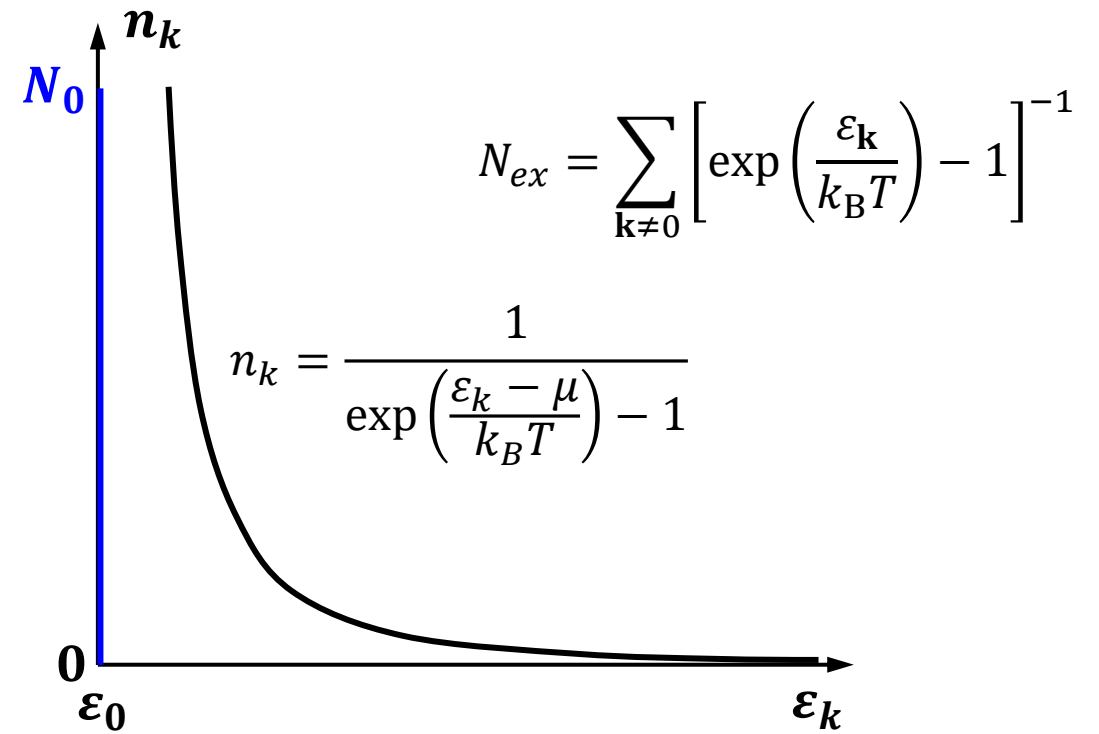
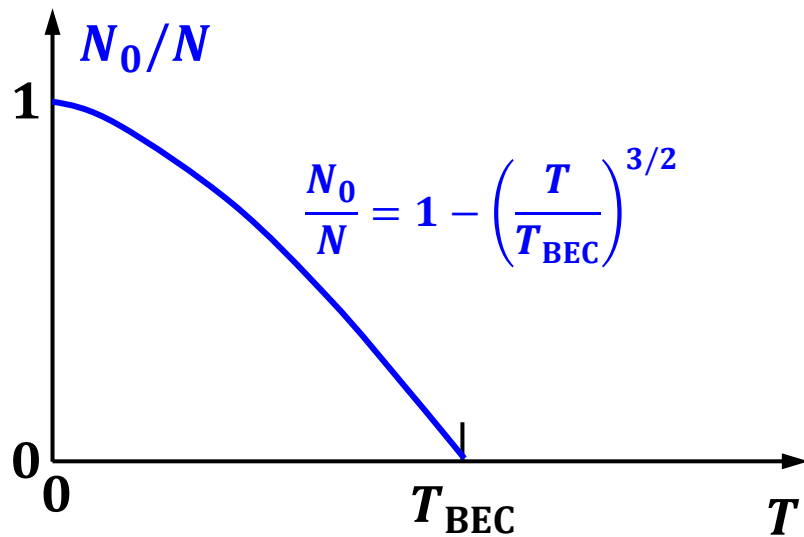
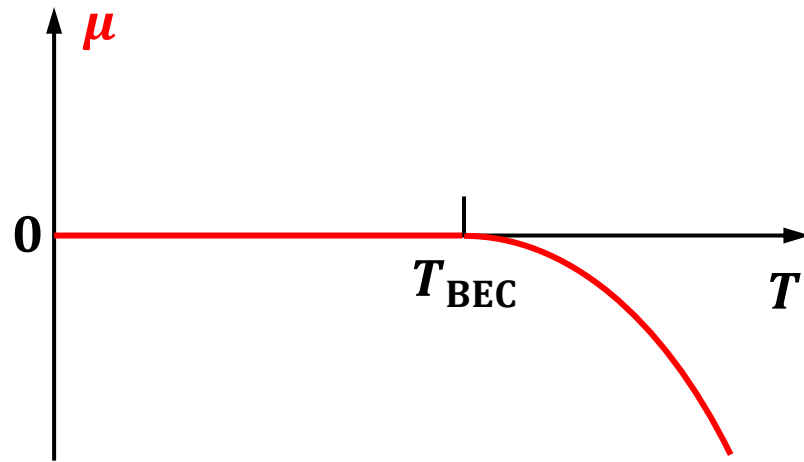
The solid (dotted) line shows the infinite (finite)  $N$  theory curves. At the transition, the cloud consists of 40 000  $^{87}\text{Rb}$  atoms at 280 nK.

## Bose-Einstein Condensation in a Dilute Gas: Measurement of Energy and Ground-State Occupation

J. R. Ensher, D. S. Jin, M. R. Matthews, C. E. Wieman, and E. A. Cornell

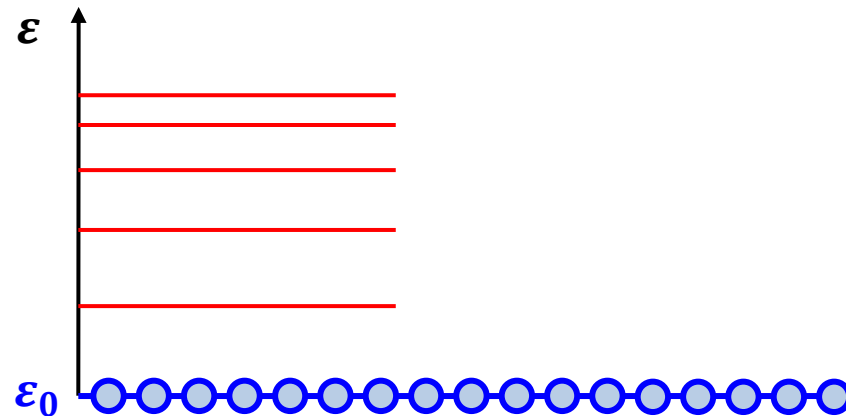
Phys. Rev. Lett. 77, 4984 (1996)

# I.2.2 Ideal Bose Gas



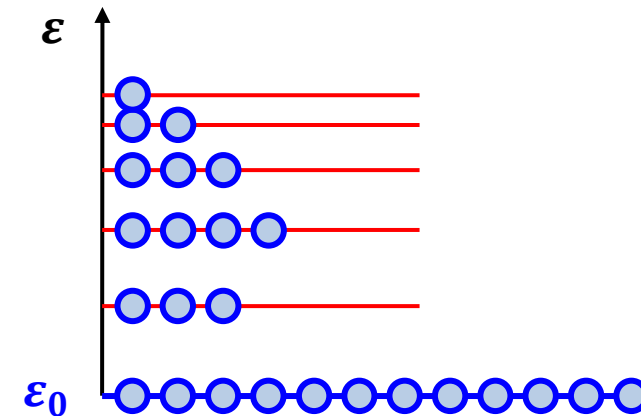
# I.2.2 Ideal Bose Gas

- Bose-Einstein condensation can be viewed as condensation of gas in momentum space  
 → *all atoms have the same momentum*
- Bose-Einstein condensation corresponds to order-disorder phase transition (first order phase transition)



$$T = 0$$

only ground state  
is occupied

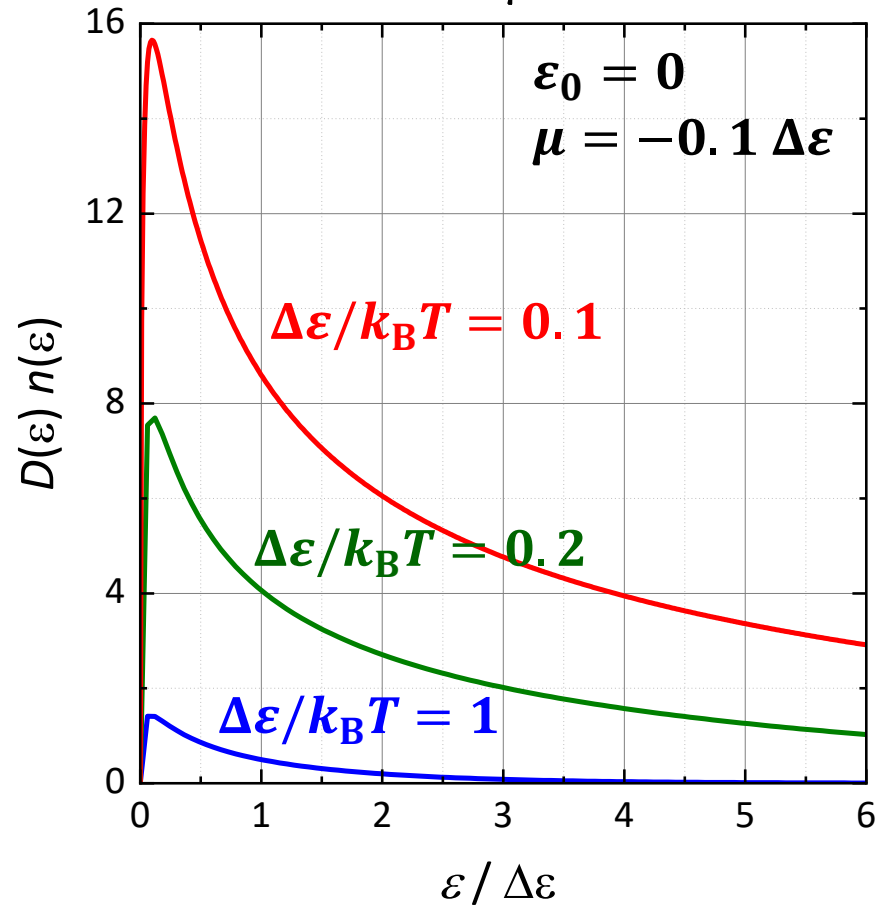


$$0 < T < T_{\text{BEC}}$$

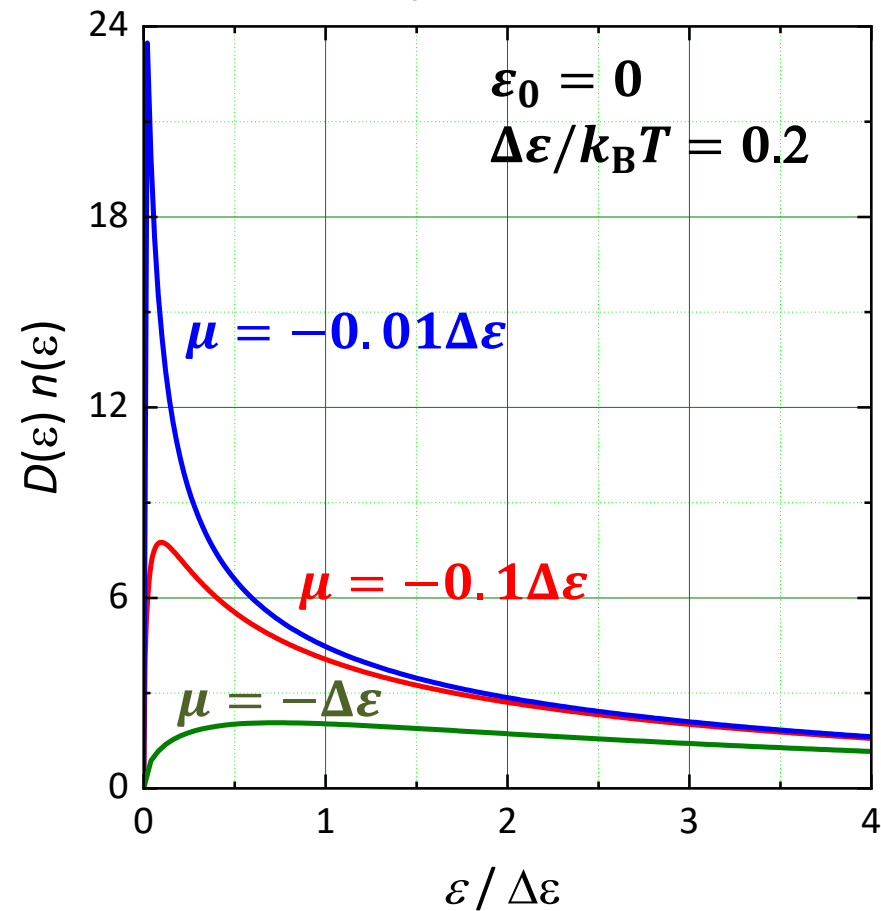
still macroscopic  
occupation of the  
ground state

# I.2.2 Ideal Bose Gas

decrease  $T$  at  $\mu = \text{const.}$



decrease  $\mu$  at  $T = \text{const.}$



we defined the Bose-Einstein temperature  $T_{\text{BEC}}$  as the temperature, where  $N_{ex}/N = 1$ :

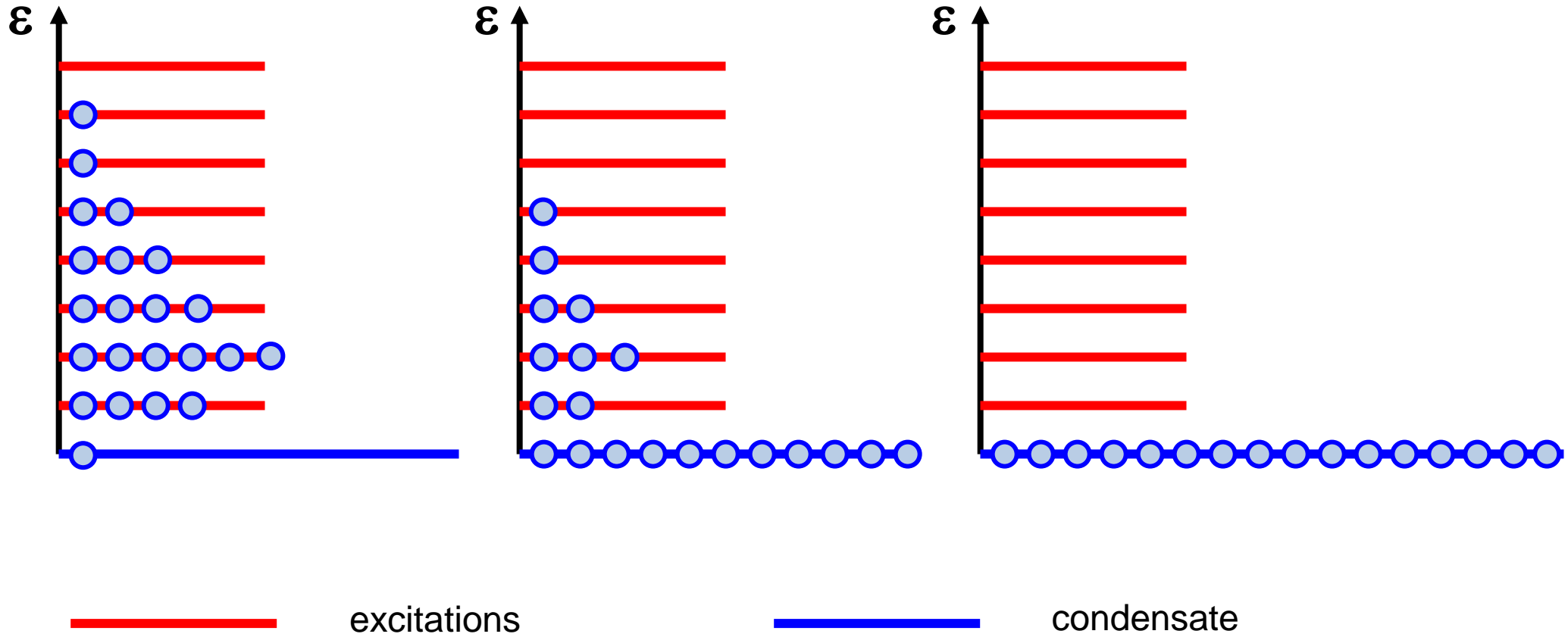
$$T_{\text{BEC}} = \frac{2\pi\hbar^2}{mk_B} \left( \frac{n}{2.612} \right)^{2/3}$$

# I.2.2 Ideal Bose Gas

$T > T_{\text{BEC}}$

$0 < T < T_{\text{BEC}}$

$T = 0$



# I.2.2 Ideal Bose Gas

free bosons:

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m}$$

$$D(\varepsilon) = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\varepsilon}$$

- thermodynamic properties of ideal Bose gas: inner energy  $U$

$$U = \sum_k \varepsilon_k n_k \quad \Rightarrow \quad U = \int D(\varepsilon) \varepsilon d\varepsilon = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int \frac{\varepsilon^{3/2}}{\exp\left(\frac{\varepsilon - \mu}{k_B T}\right) - 1} d\varepsilon$$

$$U = \frac{V}{4\pi^2} k_B T \left(\frac{2mk_B T}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\gamma e^{-x}}{1 - \gamma e^{-x}} x^{3/2} dx$$

with  $x = \varepsilon/k_B T = \beta\varepsilon$  and **fugacity**  $\gamma = \exp\left(\frac{\mu}{k_B T}\right)$

we use again  $\frac{\gamma e^{-x}}{1 - \gamma e^{-x}} = \gamma e^{-x} (1 + \gamma e^{-x} + \gamma^2 e^{-2x} + \dots) = \sum_{p=1}^\infty \gamma^p e^{-px}$  and  $\int_0^\infty e^{-px} x^{3/2} dx = \frac{1}{p^{5/2}} \underbrace{\int_0^\infty e^{-y} y^{3/2} dy}_{\Gamma(5/2) = \int_0^\infty e^{-y} y^{t-1} dy} = \frac{1}{p^{5/2}} \underbrace{\frac{3\sqrt{\pi}}{4}}_{\Gamma(5/2)}$

$$\Rightarrow \int_0^\infty \frac{\gamma e^{-x}}{1 - \gamma e^{-x}} x^{3/2} dx = \underbrace{\frac{3\sqrt{\pi}}{4}}_{\Gamma(5/2)} \underbrace{\sum_{p=1}^\infty \frac{\gamma^p}{p^{5/2}}}_{g_{5/2}(\gamma)} = \Gamma(5/2) g_{5/2}(\gamma) \quad \Rightarrow \quad U = V \frac{3}{2} k_B T \underbrace{\left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2}}_{1/\lambda_T^3} g_{5/2}(\gamma)$$

with  $V = N\lambda_T^3 / g_{3/2}(\gamma)$ , resp.  $V = N\lambda_{T,\text{BEC}}^3 / g_{3/2}(1)$  we obtain

$$U = \frac{3}{2} N k_B T \frac{g_{5/2}(\gamma)}{g_{3/2}(\gamma)}$$

$$U = \frac{3}{2} N k_B T \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} \frac{g_{5/2}(\gamma)}{g_{3/2}(1)}$$

# I.2.2 Ideal Bose Gas

- thermodynamic properties of ideal Bose gas: inner energy  $U$

i.  $T > T_{\text{BEC}}$ : 
$$U(T) = \frac{3}{2} N k_B T \frac{g_{5/2}(\gamma)}{g_{3/2}(\gamma)}$$

- ii.  $T \gg T_{\text{BEC}}$ : it is sufficient to use only the first term in the series evolution of  $g$ :

$$g_{5/2}(\gamma) = \sum_{p=1}^{\infty} \frac{\gamma^p}{p^{5/2}} \approx \gamma; \quad g_{3/2}(\gamma) = \sum_{p=1}^{\infty} \frac{\gamma^p}{p^{3/2}} \approx \gamma \quad \text{since } \gamma \rightarrow 1$$

$$U(T) = \frac{3}{2} N k_B T$$

result for gas of classical non-interacting particles  
(quantum statistics does not play any role at high  $T$ )

iii.  $T < T_{\text{BEC}}$ : 
$$U(T) = \frac{3}{2} N k_B T \frac{g_{5/2}(1)}{g_{3/2}(1)} \qquad g_{5/2}(1) = \zeta\left(\frac{5}{2}\right) = 1.341, \quad g_{3/2}(1) = \zeta\left(\frac{3}{2}\right) = 2.612$$

$$U(T) = \frac{3}{2} N_{\text{ex}} k_B T \frac{g_{5/2}(1)}{g_{3/2}(1)} = \frac{3}{2} N \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} k_B T \frac{g_{5/2}(1)}{g_{3/2}(1)} = 0.770 N \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} k_B T$$

we use 
$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} \Rightarrow N_{\text{ex}} = N - N_0 = N \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2}$$

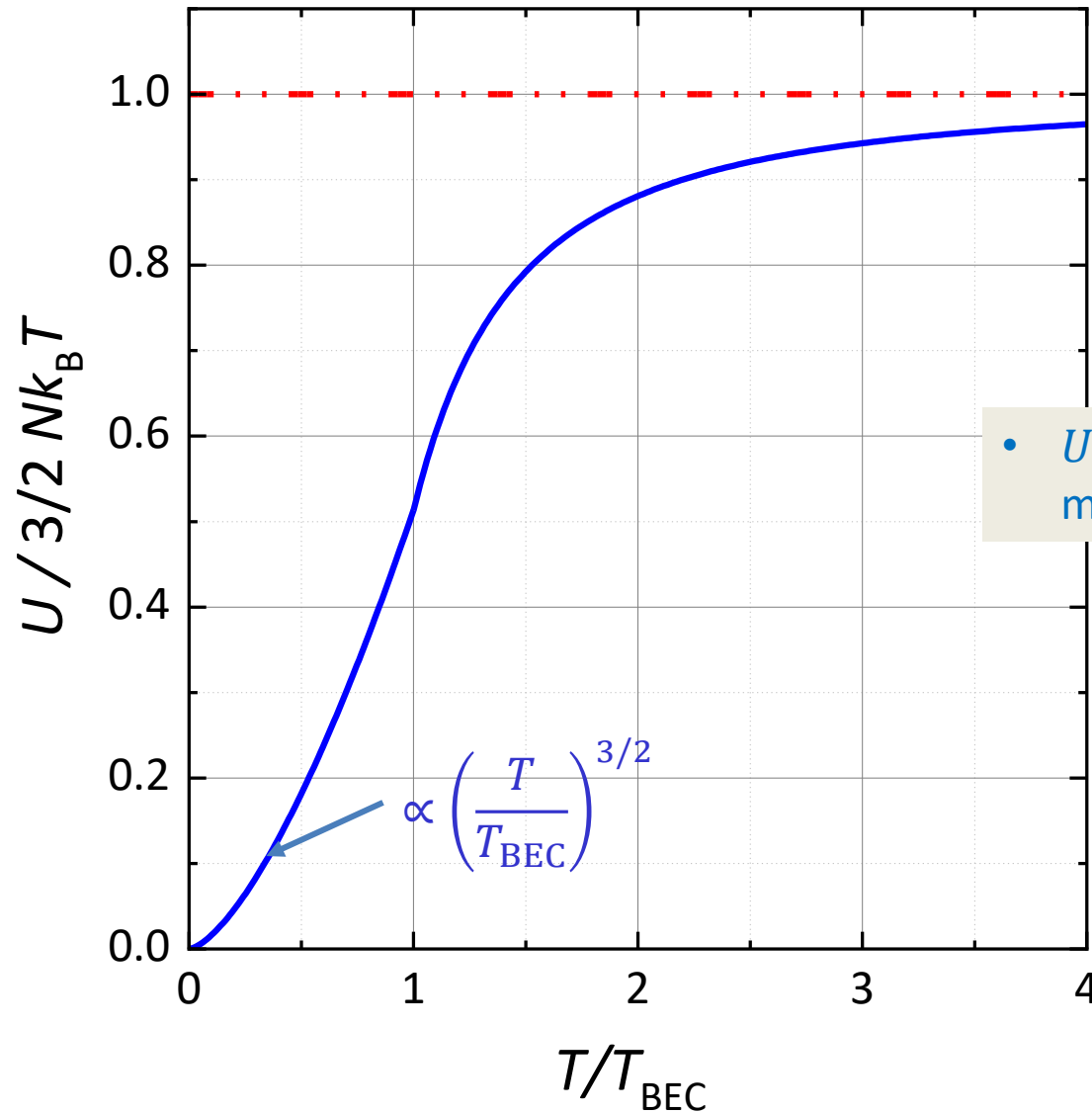
particles in ground state  $\epsilon_0 = 0$  do not contribute!

$\alpha$	$\Gamma(\alpha)$	$\zeta_\alpha(\gamma = 1)$
1	1	$\infty$
1.5	$\sqrt{\pi}/2 = 0.886$	2.612
2	1	$\pi^2/6 = 1.645$
2.5	$3\sqrt{\pi}/4 = 1.329$	1.341
3	2	1.202
3.5	$15\sqrt{\pi}/8 = 3.323$	1.127
4	6	$\pi^4/90 = 1.082$

$\Gamma$  = Gamma function  
 $\zeta$  = Riemann Zeta function



# I.2.2 Ideal Bose Gas



$$U(T) = \frac{3}{2} N k_B T \quad \text{classical result}$$

- $U(T)$  decreases towards lower  $T$  since more and more particles condense into the ground state  $\epsilon_0$

# I.2.2 Ideal Bose Gas

- thermodynamic properties of ideal Bose gas: specific heat:  $C_V(T) = \left(\frac{\partial U}{\partial T}\right)_{V,N}$

i.  $T > T_{\text{BEC}}$ :  $\mu < 0, \gamma < 1$ ,  $g_{5/2}(\gamma)$  and  $g_{3/2}(\gamma)$  are functions of  $T$  and  $n = N/V$

with  $V = N\lambda_T^3 / g_{3/2}(\gamma)$ , resp.  $V = N\lambda_{T,\text{BEC}}^3 / g_{3/2}(1)$  we obtain

$$U(T) = \frac{3}{2} Nk_B T \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} \frac{g_{5/2}(\gamma)}{g_{3/2}(1)} = \frac{3}{2} Nk_B T \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} \frac{g_{5/2}(\gamma)}{\zeta(3/2)}$$

$$C_V(T) = \left(\frac{\partial U}{\partial T}\right)_{V,N} = \frac{15}{4} Nk_B \frac{g_{5/2}(\gamma)}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} + \frac{3}{2} Nk_B \frac{T}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} \underbrace{\frac{\partial g_{5/2}(\gamma)}{\partial \gamma}}_{\gamma^{-1} g_{3/2}(\gamma)} \left(\frac{\partial \gamma}{\partial T}\right)_N$$

to derive  $\left(\frac{\partial \gamma}{\partial T}\right)_N$  we use  $\lambda_T^3 \cdot \frac{N}{V} = \lambda_T^3 \cdot n = g_{3/2}(\gamma) \Rightarrow 1 = \frac{g_{3/2}(\gamma)}{\lambda_T^3 \cdot n} = \frac{g_{3/2}(\gamma)}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2}$

temperature derivative yields:  $0 = \frac{3}{2T} \frac{g_{3/2}(\gamma)}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} + \frac{1}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} \underbrace{\frac{\partial g_{3/2}(\gamma)}{\partial \gamma}}_{\gamma^{-1} g_{1/2}(\gamma)} \left(\frac{\partial \gamma}{\partial T}\right)_N = \frac{3}{2T} g_{3/2}(\gamma) + \gamma^{-1} g_{1/2}(\gamma) \left(\frac{\partial \gamma}{\partial T}\right)_N$

$$\rightarrow \left(\frac{\partial \gamma}{\partial T}\right)_N = -\frac{3\gamma}{2T} \frac{g_{3/2}(\gamma)}{g_{1/2}(\gamma)}$$

$$U(T) = \frac{3}{2} Nk_B T \frac{g_{5/2}(\gamma)}{g_{3/2}(\gamma)}$$

$$g_{3/2}(1) = \zeta(3/2)$$

# I.2.2 Ideal Bose Gas

- thermodynamic properties of ideal Bose gas: specific heat:  $C_V(T) = \left(\frac{\partial U}{\partial T}\right)_{V,N}$

i.  $T > T_{\text{BEC}}$ :

$$C_V(T) = \left(\frac{\partial U}{\partial T}\right)_{V,N} = \frac{15}{4} Nk_B \frac{g_{5/2}(\gamma)}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} + \frac{3}{2} Nk_B \frac{T}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} \underbrace{\frac{\partial g_{5/2}(\gamma)}{\partial \gamma}}_{\gamma^{-1} g_{3/2}(\gamma)} \left(\frac{\partial \gamma}{\partial T}\right)_N$$

$$\left(\frac{\partial \gamma}{\partial T}\right)_N = -\frac{3\gamma}{2T} \frac{g_{3/2}(\gamma)}{g_{1/2}(\gamma)}$$

$$C_V(T) = \frac{15}{4} Nk_B \frac{g_{5/2}(\gamma)}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} + \frac{3}{2} Nk_B \frac{T}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} \frac{g_{3/2}(\gamma)}{\gamma} \left(-\frac{3\gamma}{2T} \frac{g_{3/2}(\gamma)}{g_{1/2}(\gamma)}\right)$$

$$C_V(T) = Nk_B \frac{g_{5/2}(\gamma)}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} \left[ \frac{15}{4} - \frac{9}{4} \frac{g_{3/2}(\gamma)}{g_{5/2}(\gamma)} \frac{g_{3/2}(\gamma)}{g_{1/2}(\gamma)} \right]$$

$$C_V(T) = Nk_B \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} \left[ \frac{15}{4} \frac{g_{5/2}(\gamma)}{\zeta(3/2)} - \frac{9}{4} \frac{g_{3/2}(\gamma)}{\underbrace{\zeta(3/2)}_{\left(\frac{T_{\text{BEC}}}{T}\right)^{3/2}}} \frac{g_{3/2}(\gamma)}{g_{1/2}(\gamma)} \right]$$

we use  $\lambda_T^3 \cdot \frac{N}{V} = \lambda_T^3 \cdot n = g_{3/2}(\gamma) \Rightarrow 1 = \frac{g_{3/2}(\gamma)}{\lambda_T^3 \cdot n} = \frac{g_{3/2}(\gamma)}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2}$

$$C_V(T) = \frac{15}{4} Nk_B \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} \frac{g_{5/2}(\gamma)}{\zeta(3/2)} - \frac{9}{4} Nk_B \frac{g_{3/2}(\gamma)}{g_{1/2}(\gamma)}$$

# I.2.2 Ideal Bose Gas

- thermodynamic properties of ideal Bose gas: specific heat:  $C_V(T) = \left(\frac{\partial U}{\partial T}\right)_{V,N}$

ii.  $T \gg T_{\text{BEC}}$ :

$$C_V(T) = \frac{3}{2} N k_B, \quad C_V^{\text{mol}}(T) = \frac{3}{2} N_A k_B = \frac{3}{2} R$$

result for classical gas of non-interacting particles

$$U(T) = \frac{3}{2} N k_B T$$

(quantum statistics does not play any role at high  $T$ )

iii.  $T < T_{\text{BEC}}$ :

$$C_V(T) = \frac{15}{4} N k_B \frac{\zeta_{5/2}(1)}{\zeta_{3/2}(1)} \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} = 1.925 \cdot N k_B \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2}$$

$$U(T) = \frac{3}{2} N \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2} k_B T \frac{g_{5/2}(1)}{g_{3/2}(1)}$$

$$g_{5/2}(1) = \zeta(5/2) \quad g_{3/2}(1) = \zeta(3/2)$$

interpretation: all  $N_{\text{ex}} = N \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2}$  non-condensed particles contribute  $O(k_B)$  to the specific heat

$$C_V(T) = \frac{5}{2} \frac{U(T)}{T}$$

$$\longleftrightarrow_{C_V = T \partial S / \partial T}$$

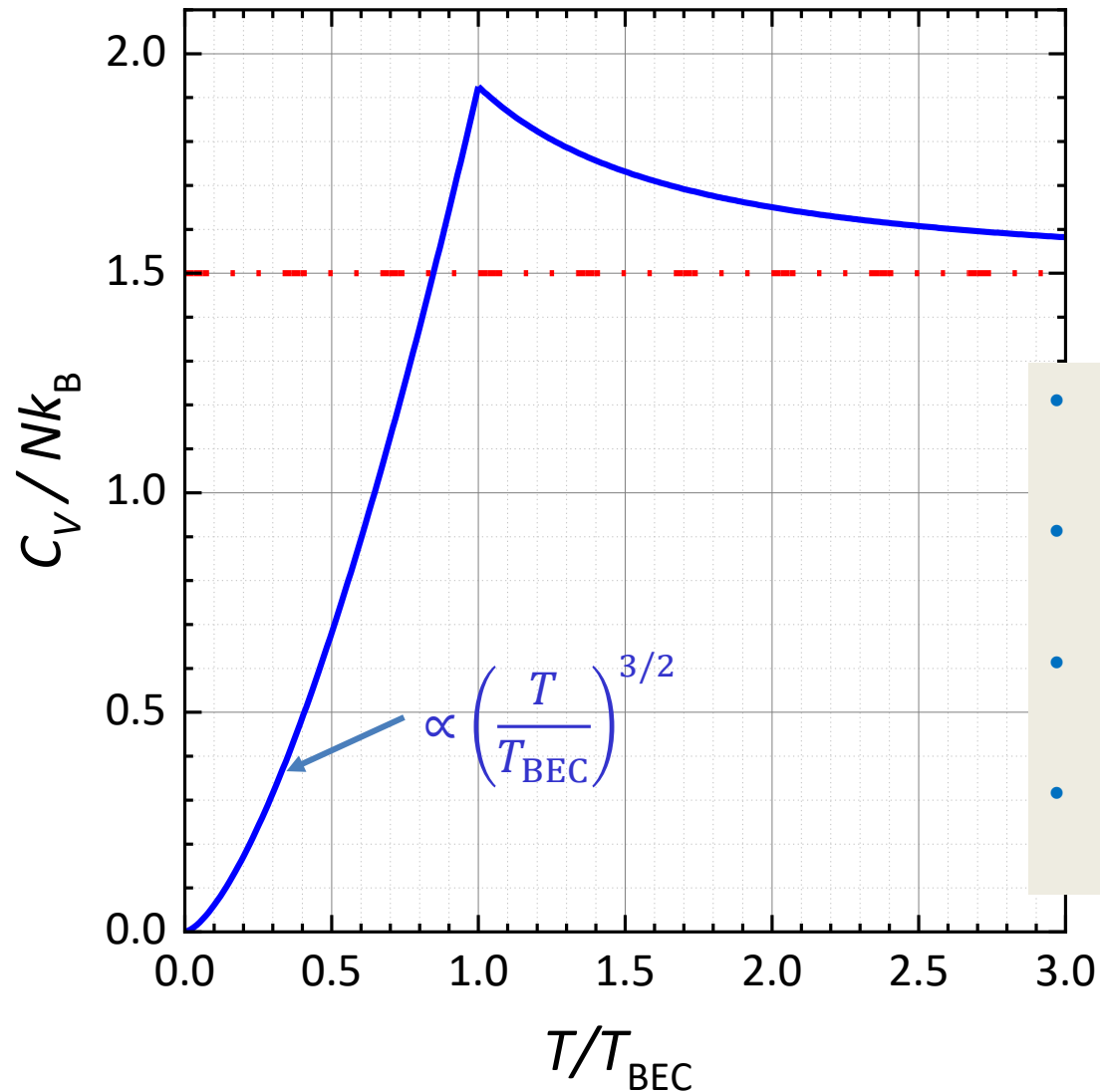
$$S(T) = \frac{5}{3} \frac{U(T)}{T}$$

$$\longleftrightarrow_{F = U - TS}$$

$$F(T) = -\frac{2}{3} U(T)$$

# I.2.2 Ideal Bose Gas

- thermodynamic properties of ideal Bose gas: specific heat:  $C_V(T) = \left(\frac{\partial U}{\partial T}\right)_{V,N}$



$$C_V(T) = \frac{3}{2} Nk_B \quad \text{classical result}$$

- $C_V(T)$  decreases towards lower  $T$  since more and more particles condense into the ground state  $\epsilon_0$
- temperature  $T_{\text{BEC}}$  represents phase transition temperature
- $C_V(T)$  shows cusp at  $T_{\text{BEC}}$ , indicating the free energy is not analytical at  $T_{\text{BEC}}$
- phase transition is usually interpreted as 1st order phase transition

# I.2.2 Ideal Bose Gas

- thermodynamic properties of ideal Bose gas: pressure  $p = \left(\frac{\partial F}{\partial V}\right)_{N,T}$
- we discuss only the case  $T < T_{\text{BEC}}$ :

$$\begin{aligned}
 F &= U - TS \\
 dF &= TdS - pdV + \mu dN - TdS - SdT \\
 dF &= -pdV + \mu dN - SdT \\
 \left(\frac{\partial F}{\partial V}\right)_{N,T} &= -p, \quad \left(\frac{\partial F}{\partial N}\right)_{V,T} = \mu, \quad \left(\frac{\partial F}{\partial T}\right)_{N,V} = -S
 \end{aligned}$$

$$p = -\left(\frac{\partial F}{\partial V}\right)_{N,T} = \frac{2}{3}\left(\frac{\partial U}{\partial V}\right)_{N,T}$$

$$\begin{aligned}
 U(T) &= \frac{3}{2} N k_B T \frac{g_{5/2}(1)}{g_{3/2}(1)} \\
 F(T) &= -\frac{2}{3} U(T)
 \end{aligned}$$

with  $V = N \lambda_{T,\text{BEC}}^3 / g_{3/2}(1)$  we obtain

$$U = \frac{3}{2} N k_B T \frac{g_{5/2}(1)}{g_{3/2}(1)} = \frac{3}{2} \frac{V}{\lambda_{T,\text{BEC}}^3} g_{3/2}(1) k_B T \frac{g_{5/2}(1)}{g_{3/2}(1)} = \frac{3}{2} \frac{V}{\lambda_{T,\text{BEC}}^3} k_B T g_{5/2}(1)$$

$$p = \frac{2}{3} \left(\frac{\partial U}{\partial V}\right)_{N,T} = \frac{k_B T}{\lambda_T^3} g_{5/2}(1)$$

pressure of the gas does not depend on  $V$  in the BEC regime  
 → compressibility of the BEC phase is infinite

this pathological behavior is removed by the inclusion of finite interactions

# I.2.2 Ideal Bose Gas

- experimental realization with ultra-cold atomic gases

- first realization in 1995 with trapped alkali metal atoms, e.g.  $^{87}\text{Rb}$  (Eric A. Cornell, Carl E. Wieman, Wolfgang Ketterle)
- typical densities:  $n \simeq 10^{11} - 10^{15} \text{ cm}^{-3} \rightarrow T_{\text{BEC}} \simeq 10 \text{ nK} - 1 \text{ }\mu\text{K}$  due to large atomic mass (compared to  $^4\text{He}$ )
- $^{87}\text{Rb}$ : electron configuration:  $(\text{Kr})5s^1 \Rightarrow J = \frac{1}{2}$ , nuclear spin  $I = \frac{3}{2} \Rightarrow$  total spin is even  $\rightarrow$  **boson**  
similar behavior:  $^7\text{Li}$ ,  $^{23}\text{Na}$
- **important:** alkali metal atoms interact (short length repulsion, attractive Van der Waals interaction)
  - atoms do not cluster since most interactions are elastic two-particle interactions:  
 $\rightarrow$  particles cannot bind since they cannot get rid of their kinetic energy
  - rate for three-particle interactions is very small
  - description of pairwise interactions:  $V(\mathbf{r}_1 - \mathbf{r}_2) \simeq g \delta(\mathbf{r}_1 - \mathbf{r}_2)$

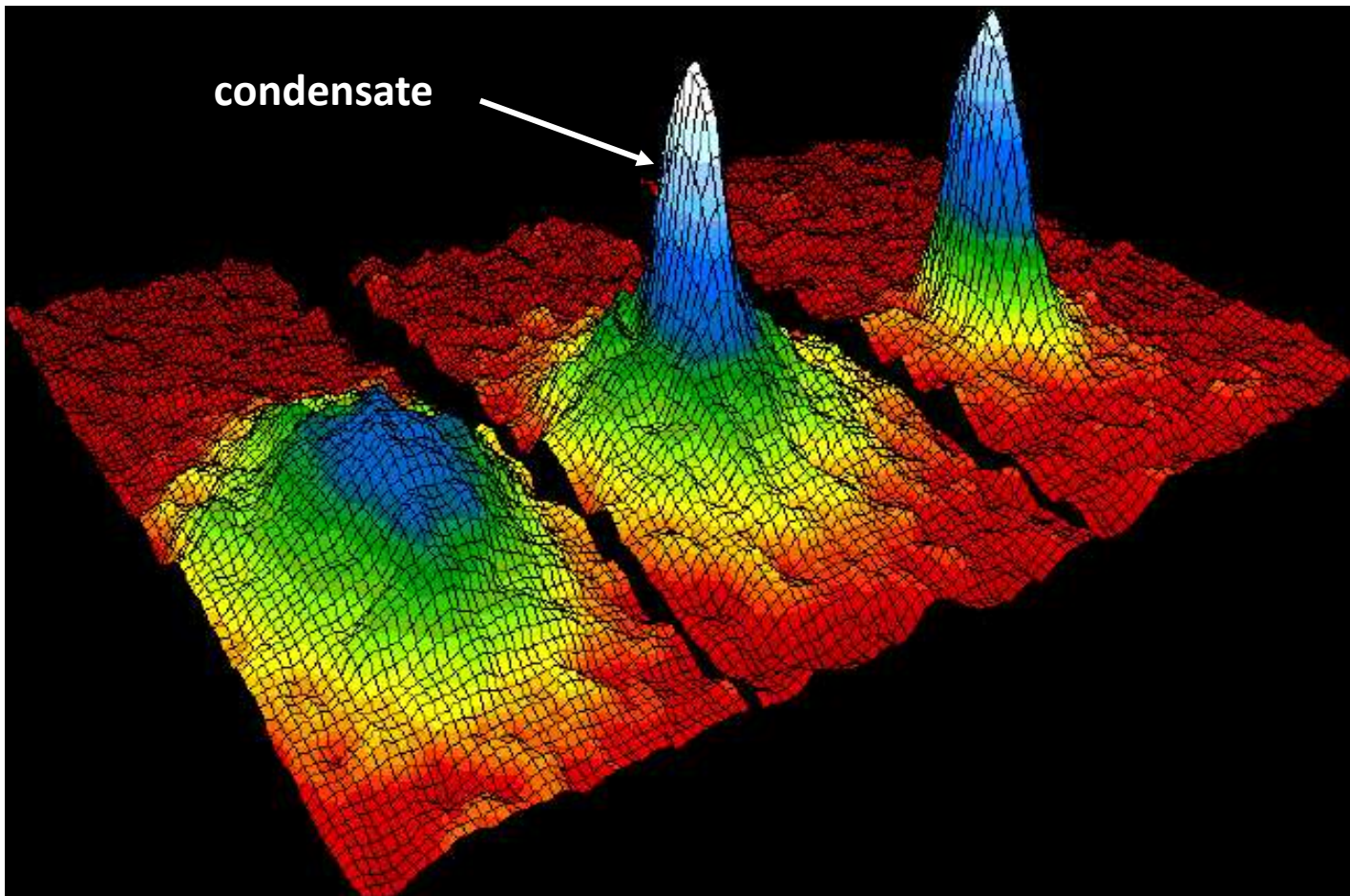
$g$  = strength of the scatterer

$$a_s = \frac{\hbar^2}{mg} = \text{s-wave scattering length}$$

(interactions and effect of trapping potential will be discussed later)



# I.2.2 Ideal Bose Gas



velocity distribution for a gas of rubidium atoms,  $n = 2.5 \times 10^{12} \text{ cm}^{-3}$

The measured distribution confirmed the existence of a Bose-Einstein condensate.

Left:  $T > T_{\text{BEC}} \approx 170 \text{ nK}$

Center:  $T < T_{\text{BEC}}$

Right: nearly pure condensate

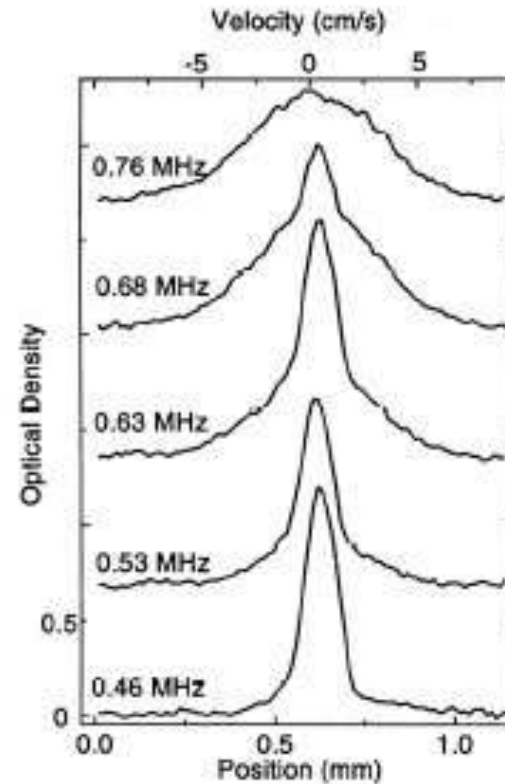
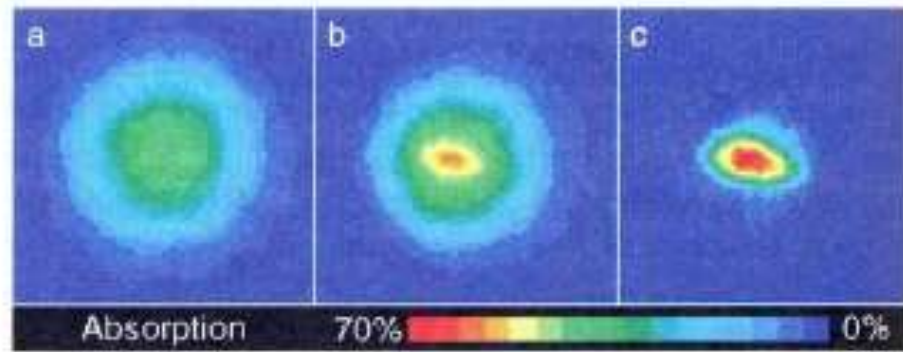
### experimental technique:

- switch-off the trap potential and let the condensate freely expand
- image the position of the atoms after a specific time by absorption of laser light by atom cloud
- measured spatial distribution is proportional to velocity distribution in condensate

Anderson, M. H.; Ensher, J. R.; Matthews, M. R.; Wieman, C. E.; Cornell, E. A.  
**Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor**  
 Science **269**, 198-201 (1995).



# I.2.2 Ideal Bose Gas



velocity distribution for a gas of sodium atoms,  $n \sim 10^{14} \text{ cm}^{-3}$

the measured distribution confirmed the existence of a Bose-Einstein condensate.

Left:  $T > T_{\text{BEC}} \approx 2 \mu\text{K}$

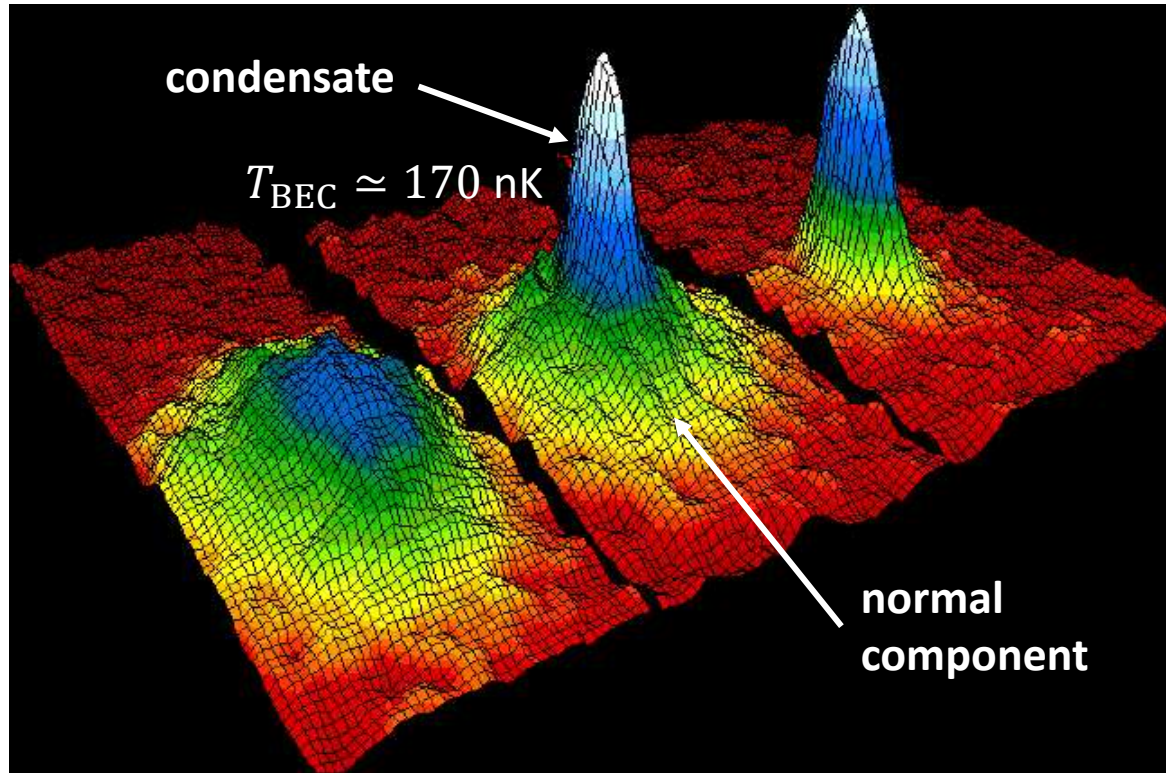
Center:  $T < T_{\text{BEC}}$

Right: after further evaporation cooling

K. B. Davis, M. -O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle  
**Bose-Einstein Condensation in a Gas of Sodium Atoms**  
 Phys. Rev. Lett. **75**, 3969 (1995)

# I.2.2 Ideal Bose Gas

- what determines the width of the central peak associated with the condensate



Anderson, M. H.; Ensher, J. R.; Matthews, M. R.; Wieman, C. E.; Cornell, E. A.  
**Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor**  
*Science* **269**, 198-201 (1995).

### expected:

- delta function distribution since condensate atoms occupy the same state

### observed:

- finite width  $\Delta x$  of the condensate due to trapping of condensate in trapping potential of finite width  $\Delta x$

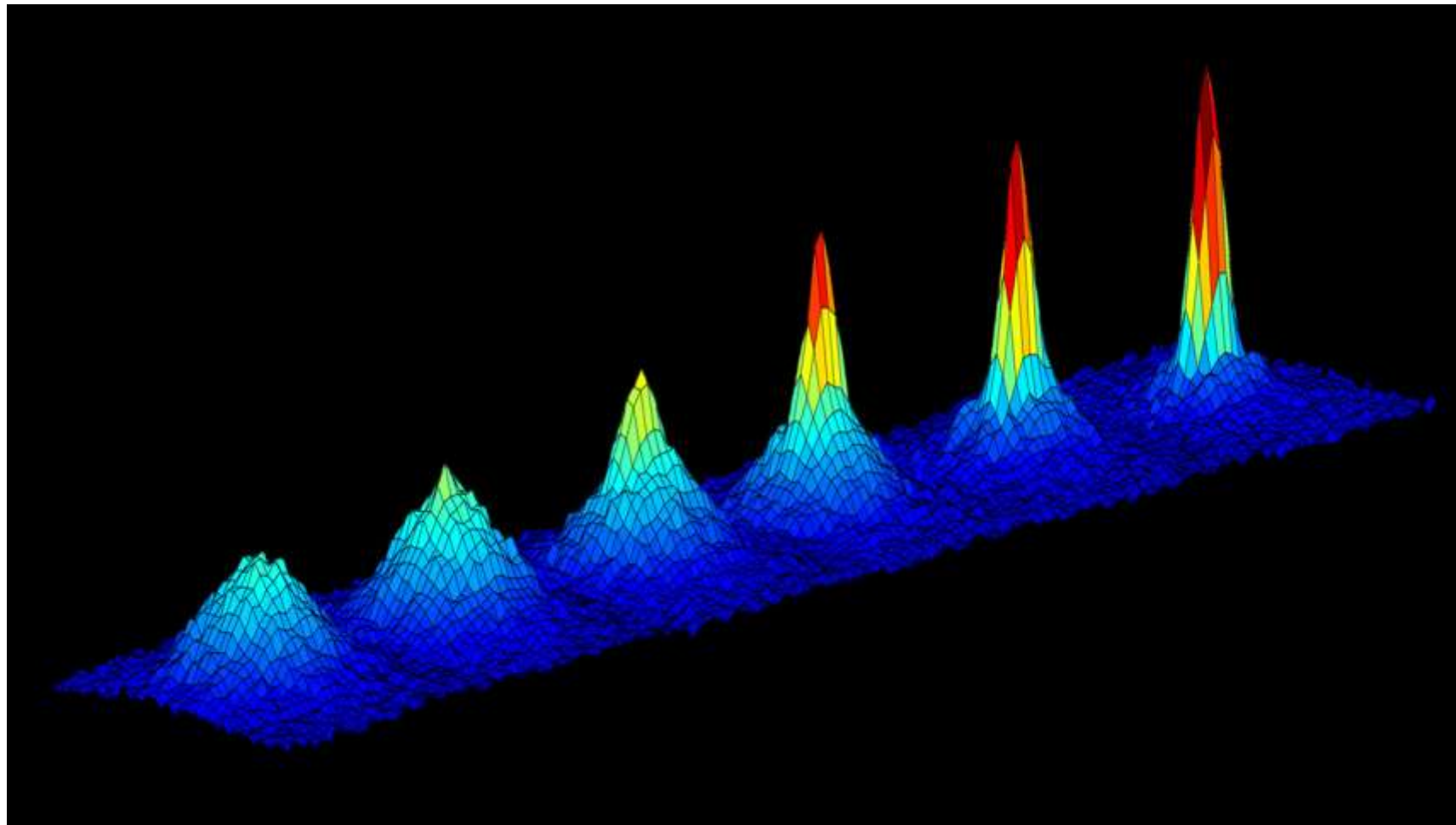
- resulting finite momentum distribution:

$$\Delta p \simeq \hbar/\Delta x \quad \Rightarrow \quad \Delta v \simeq \hbar/m\Delta x$$

- width of the normal component:

$$\frac{1}{2} m \langle v_x^2 \rangle = \frac{1}{2} k_B T \quad \Rightarrow \quad \Delta v \simeq \sqrt{k_B T/m}$$

# I.2.2 Ideal Bose Gas



## Space Station Experiment (2018)

Changing density of a cloud of rubidium atoms as it is cooled to lower and lower temperatures (going from left to right) approaching absolute zero. The emergence of a sharp peak in the later graphs confirms the formation of a Bose-Einstein condensate occurring here at about  $T_{\text{BEC}} = 130 \text{ nK}$

# I.2.2 Ideal Bose Gas

- **concluding remarks**
  - Bose-Einstein condensates have been realized with many different atoms
  - ideal model system, as most parameters can be well controlled and changed
  - it became evident that the finite interactions between the particles have an important impact
    - ideal **non-interacting Bose gas is not a superfluid** that can show frictionless flow
      - ➔ no persistent current of the condensate
      - ➔ no vortices
    - we will see later that this is due to the fact that the **Landau critical velocity** of the superfluid is zero



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Institut

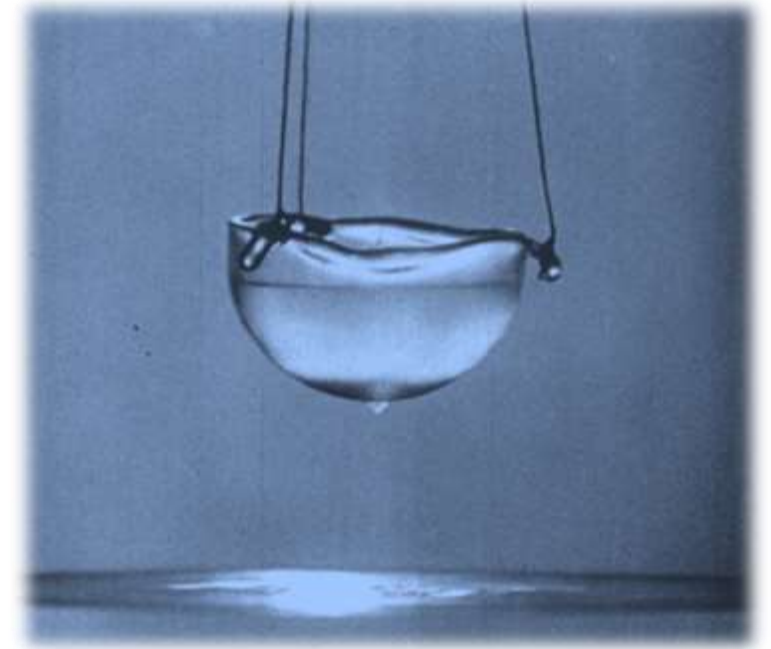


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# Superconductivity and Low Temperature Physics II



**Lecture No. 4**  
**19 May 2022**

**R. Gross**  
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# I.2.3 Bose Gas with Interactions

- treatment of the finite interaction between particles

- description by pairwise interaction:  $V(\mathbf{r}_1 - \mathbf{r}_2) \simeq g \delta(\mathbf{r}_1 - \mathbf{r}_2)$

$g$  = strength of the scatterer

$$a_s = \frac{\hbar^2}{mg} = \text{s-wave scattering length}$$

- modeling of pairwise interactions by **average „mean field“ potential** seen by every particle due to interaction with all others

$$V_{\text{eff}}(\mathbf{r}) \simeq g n(\mathbf{r})$$

- in addition, particles see finite **trapping potential**  $V_{\text{trap}}(\mathbf{r})$  of e.g. a magnetic trap

- Schrödinger equation for particles in trap:

$$\left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}}(\mathbf{r}) + V_{\text{eff}}(\mathbf{r}) \right) \psi_k(\mathbf{r}) = \varepsilon_k \psi_k(\mathbf{r})$$

**important:**

Schrödinger equation is nonlinear since  $V_{\text{eff}}(\mathbf{r})$  depends on  $n(\mathbf{r})$  and hence on wave function  $\psi_k(\mathbf{r})$  itself

→ self-consistent solution: **Gross-Pitaevskii equations**

**particle density and number:**

$$n(\mathbf{r}) = \sum_k \left[ \exp\left(\frac{(\varepsilon_k - \mu)}{k_B T}\right) - 1 \right]^{-1} |\psi_k(\mathbf{r})|^2$$

$$N = \int n(\mathbf{r}) d^3r = \sum_k \left[ \exp\left(\frac{(\varepsilon_k - \mu)}{k_B T}\right) - 1 \right]^{-1}$$

# I.2.3 Bose Gas with Interactions

- Now: more detailed discussion of finite interactions between particles

assumptions: (i) **weak interactions** (allows perturbative treatment)  
(ii) **repulsive interactions** (attractive interactions may lead to instability)

we will show: **repulsive interactions stabilize the Bose-Einstein condensate**  
(counter-intuitive, as a repulsive interaction is expected to push particles out of a single-particle ground state)

- Gedanken experiment:

*we consider two bosons and show that in the presence of a repulsive interaction it is more favorable to put them into the same eigenstate*

– bosons = particles in a box described by normalized wave functions  $\phi(\mathbf{r}) = \exp(i\mathbf{k} \cdot \mathbf{r})$ ,  $\tilde{\phi}(\mathbf{r}) = \exp(i\mathbf{k} \cdot \mathbf{r})$ ,

➤ **two bosons in the same eigenstate:**

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \phi(\mathbf{r}_1)\phi(\mathbf{r}_2) = \exp(i\mathbf{k} \cdot (\mathbf{r}_1 + \mathbf{r}_2))$$

symmetric wave function:  $\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi(\mathbf{r}_2, \mathbf{r}_1)$

$$\int |\psi(\mathbf{r}_1, \mathbf{r}_2)|^2 d^3r_1 d^3r_2 = 1$$



# I.2.3 Bose Gas with Interactions

- two bosons in different eigenstates:

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}} [\phi(\mathbf{r}_1)\tilde{\phi}(\mathbf{r}_2) + \phi(\mathbf{r}_2)\tilde{\phi}(\mathbf{r}_1)]$$

symmetric superposition of the two possibilities  
 $\phi(\mathbf{r}_1)\tilde{\phi}(\mathbf{r}_2)$  and  $\phi(\mathbf{r}_2)\tilde{\phi}(\mathbf{r}_1)$

$$\int |\psi(\mathbf{r}_1, \mathbf{r}_2)|^2 d^3r_1 d^3r_2 = 1$$

- what happens if we add repulsive potential  $V(\mathbf{r}_1 - \mathbf{r}_2)$  between the two bosons?

- energy change for the two bosons in the same eigenstate:  $\psi(\mathbf{r}_1, \mathbf{r}_2) = \phi(\mathbf{r}_1)\phi(\mathbf{r}_2) = \exp(i\mathbf{k} \cdot (\mathbf{r}_1 + \mathbf{r}_2))$

$$\delta E = \int d^3r_1 d^3r_2 |\psi(\mathbf{r}_1, \mathbf{r}_2)|^2 V(\mathbf{r}_1 - \mathbf{r}_2) = \int d^3r_1 d^3r_2 V(\mathbf{r}_1 - \mathbf{r}_2) = \int d^3r V(\mathbf{r}) = E_0 > 0$$

we use  $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$  and  $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$  and  $d^3r_1 d^3r_2 = d^3R d^3r$

note that  $\delta E = \int d^3r V(\mathbf{r}) = E_0 > 0$  for a purely repulsive potential

# I.2.3 Bose Gas with Interactions

- energy change for the two bosons in the same eigenstate:  $\psi(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}} [\phi(\mathbf{r}_1)\tilde{\phi}(\mathbf{r}_2) + \phi(\mathbf{r}_2)\tilde{\phi}(\mathbf{r}_1)]$

$$\begin{aligned} \delta E &= \int d^3r_1 d^3r_2 |\psi(\mathbf{r}_1, \mathbf{r}_2)|^2 V(\mathbf{r}_1 - \mathbf{r}_2) \\ &= \frac{1}{2} \int d^3r_1 d^3r_2 \left[ |\phi(\mathbf{r}_1)|^2 |\tilde{\phi}(\mathbf{r}_2)|^2 + |\phi(\mathbf{r}_2)|^2 |\tilde{\phi}(\mathbf{r}_1)|^2 + \phi(\mathbf{r}_1)\tilde{\phi}(\mathbf{r}_2)\phi^*(\mathbf{r}_2)\tilde{\phi}^*(\mathbf{r}_1) + c. c. \right] V(\mathbf{r}_1 - \mathbf{r}_2) \end{aligned}$$

- since  $\int d^3r_1 |\phi(\mathbf{r}_1)|^2 = 1$  and  $\int d^3r_2 |\phi(\mathbf{r}_2)|^2 = 1$ , the first two terms in [] yields the contribution  $\int d^3r V(\mathbf{r})$
- the last term in [] comes from the fact that the wave function must be symmetric with respect to the exchange of particles since they are indistinguishable → results in additional **exchange energy**
- since  $\phi(\mathbf{r}_1)\tilde{\phi}(\mathbf{r}_2)\phi^*(\mathbf{r}_2)\tilde{\phi}^*(\mathbf{r}_1) = \exp[i(\mathbf{k} - \tilde{\mathbf{k}}) \cdot (\mathbf{r}_1 - \mathbf{r}_2)]$  last term in [] results in an energy change  $V_{\mathbf{k}-\tilde{\mathbf{k}}}/2$  with

$$V_{\mathbf{k}-\tilde{\mathbf{k}}} = \int d^3r e^{i(\mathbf{k}-\tilde{\mathbf{k}})\cdot\mathbf{r}} V(\mathbf{r})$$

Fourier transform of  $V(\mathbf{r})$

(we assume  $V(\mathbf{r}) = V(-\mathbf{r})$  so that the FT is real:  $V_{\mathbf{k}}^* = V_{\mathbf{k}} \rightarrow$  the complex conjugate yields the same contribution  $V_{\mathbf{k}-\tilde{\mathbf{k}}}/2$ )

➔  $\delta E = E_0 + V_{\mathbf{k}-\tilde{\mathbf{k}}}$

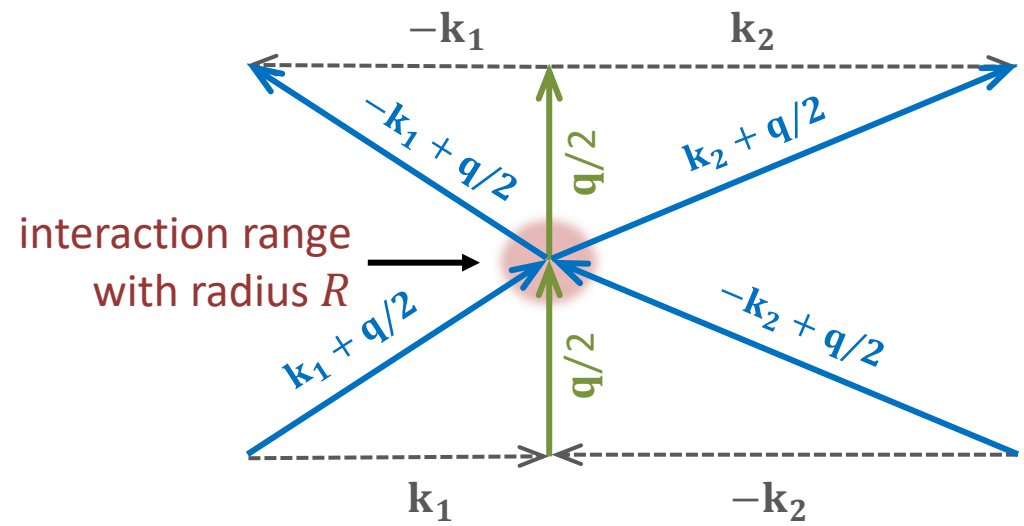
➔  $\delta E = 2E_0$  (if we assume two bosons with the same wave number)

due to the bosonic nature of the particles it is energetically favorable to put them into the same quantum state in the presence of a repulsive interaction potential

# I.2.3 Bose Gas with Interactions

- description of finite interactions between particles within second quantization formalism

$$\mathcal{H}_{\text{int}} = \frac{V_0}{2} \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{q}} \hat{b}_{\mathbf{k}_2 + \mathbf{q}/2}^\dagger \hat{b}_{-\mathbf{k}_1 + \mathbf{q}/2}^\dagger \hat{b}_{-\mathbf{k}_2 + \mathbf{q}/2} \hat{b}_{\mathbf{k}_1 + \mathbf{q}/2}$$



describes scattering process:

particle  $|\mathbf{k}_1 + \mathbf{q}/2\rangle$  is destroyed and  $|\mathbf{-k}_1 + \mathbf{q}/2\rangle$  created and particle  $|\mathbf{-k}_2 + \mathbf{q}/2\rangle$  is destroyed and  $|\mathbf{k}_2 + \mathbf{q}/2\rangle$  created

# I.2.3 Bose Gas with Interactions

- description of finite interactions between particles within second quantization formalism

$$\mathcal{H}_{\text{int}} = \frac{V_0}{2} \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{q}} \hat{b}_{\mathbf{k}_2 + \mathbf{q}/2}^\dagger \hat{b}_{-\mathbf{k}_1 + \mathbf{q}/2}^\dagger \hat{b}_{-\mathbf{k}_2 + \mathbf{q}/2} \hat{b}_{\mathbf{k}_1 + \mathbf{q}/2}$$

we have made the following simplifying assumptions:

- we assume a **local interaction**  $V(\mathbf{r}_1 - \mathbf{r}_2) \rightarrow$  Fourier transform  $V_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{q}}$  does not depend on  $\mathbf{k}_1$  and  $\mathbf{k}_2$ :  
 $V_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{q}} \simeq V_{\mathbf{q}}$
- since we are at very low  $T$ , the involved velocities and hence wavevectors  $\mathbf{q}$  are very small
- if the interaction range  $R$  of the potential is small,  $qR \ll 1$ , then the FT  $V_{\mathbf{q}}$  is essentially independent of  $\mathbf{q}$  and we can use the approximation  $V_{\mathbf{q}} = V_0$
- as above, we assume two bosons with the same wave number  $\mathbf{k}_1 = \mathbf{k}_2$  and set it equal to zero
- we describe the two states of the interacting bosons with the bosonic operators  $\hat{b}$  and  $\hat{\tilde{b}}$

$$\mathcal{H}_{\text{int}} = \frac{V_0}{2} \left[ \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + \hat{b}^\dagger \hat{\tilde{b}}^\dagger \hat{b} \hat{\tilde{b}} + \hat{\tilde{b}}^\dagger \hat{b}^\dagger \hat{b} \hat{\tilde{b}} + \hat{b}^\dagger \hat{\tilde{b}}^\dagger \hat{\tilde{b}} \hat{b} + \hat{\tilde{b}}^\dagger \hat{b}^\dagger \hat{\tilde{b}} \hat{b} \right]$$

we do not include terms like  $\hat{\tilde{b}}^\dagger \hat{\tilde{b}}^\dagger \hat{\tilde{b}} \hat{\tilde{b}}$ , as we do not consider states occupied with two  $\hat{\tilde{b}}$  bosons

# I.2.3 Bose Gas with Interactions

- **relevant relations**

- both states occupied with single boson:  $|1,1\rangle = \hat{b}^\dagger \hat{b}^\dagger |0\rangle$  with  $|0\rangle =$  normalized vacuum state
- $\langle 1,1|1,1\rangle = \langle 0|\hat{b}\hat{b} \hat{b}^\dagger \hat{b}^\dagger|0\rangle = 1,$  as  $\hat{b}$  and  $\hat{b}$  commute and  $\hat{b}\hat{b}^\dagger |0\rangle = 0$
- one state occupied with two bosons and the other is empty:  $|2,0\rangle = \hat{b}^\dagger \hat{b}^\dagger |0\rangle = (\hat{b}^\dagger)^2 |0\rangle$
- $\langle 2,0|2,0\rangle = \langle 0|b\hat{b}\hat{b}^\dagger \hat{b}^\dagger|0\rangle = \langle 0|b(1 + \hat{b}^\dagger b)\hat{b}^\dagger|0\rangle = 2,$  as  $[b, \hat{b}^\dagger] = 1$  and  $b\hat{b}^\dagger|0\rangle = 0$   
 $\Rightarrow |2,0\rangle = \frac{1}{\sqrt{2}} (\hat{b}^\dagger)^2 |0\rangle$  correctly normalized state

- **interaction energy** for the states  $|1,1\rangle$  and  $|2,0\rangle$ :

$|2,0\rangle$ : only the first term in  $\frac{V_0}{2} [\hat{b}^\dagger \hat{b}^\dagger \hat{b}\hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b}\hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b}\hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b}\hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b}\hat{b}]$  contributes

$$\frac{1}{2} \langle 0|\hat{b}\hat{b}(\hat{b}^\dagger \hat{b}^\dagger \hat{b}\hat{b})\hat{b}^\dagger \hat{b}^\dagger|0\rangle = \frac{1}{2} 2 \langle 0|\hat{b}\hat{b}(\hat{b}^\dagger \hat{b}^\dagger)|0\rangle = 2 \quad \rightarrow \langle 2,0|\mathcal{H}_{int}|2,0\rangle = V_0$$

$|1,1\rangle$ : all four last terms in  $\frac{V_0}{2} [\hat{b}^\dagger \hat{b}^\dagger \hat{b}\hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b}\hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b}\hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b}\hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b}\hat{b}]$  contribute equally, since  $\hat{b}$  and  $\hat{b}$  commute

$$4 \langle 0|\hat{b}\hat{b}(\hat{b}^\dagger \hat{b}^\dagger \hat{b}\hat{b})\hat{b}^\dagger \hat{b}^\dagger|0\rangle = 4 \langle 0|\hat{b}\hat{b}^\dagger \hat{b}\hat{b}^\dagger|0\rangle \langle 0|\hat{b}\hat{b}^\dagger \hat{b}\hat{b}^\dagger|0\rangle = 4 \quad \rightarrow \langle 1,1|\mathcal{H}_{int}|1,1\rangle = 2V_0$$

# I.2.3 Bose Gas with Interactions

- generalization to states with  $N$  bosons (without calculation)

$$\langle N, 1 | \mathcal{H}_{int} | N, 1 \rangle - \langle N + 1, 0 | \mathcal{H}_{int} | N + 1, 0 \rangle = N V_0$$

- it is energetically more favorable to put an additional boson into the same state as occupied by all the others rather than putting it into another state
- energy difference scales with the particle number  $N$  and is therefore an extensive variable

# I.2.3 Bose Gas with Interactions

- **Bogoliubov theory of weakly interacting Bose gas** (N.N Bogoliubov, J. Phys. USSR **11**, 23 (1947))
  - **starting hypothesis:** condensation of a large number of bosons into ground state also exists at finite interactions
    - since ground state  $k = 0$  is occupied by large particle number, we have to deal with large quantum numbers
      - quantum fluctuations become negligible
      - the value of the commutator  $[\hat{b}_0, \hat{b}_0^\dagger] = 1$  is small compared to these large values
        - we can set  $[\hat{b}_0, \hat{b}_0^\dagger] = 0$ , i.e. the operators commute like classical quantities
      - number of bosons in ground state,  $N_0 = \langle \hat{b}_0^\dagger \hat{b}_0 \rangle \simeq \hat{b}_0^\dagger \hat{b}_0$ , is large and also the numbers  $\hat{b}_0^\dagger$  and  $\hat{b}_0$  themselves
      - as  $\hat{b}_0^\dagger$  and  $\hat{b}_0$  are complex conjugate, we can take them real by eliminating an irrelevant phase factor:
 
$$N_0 = \hat{b}_0^\dagger \hat{b}_0 = \hat{b}_0^2 \text{ and } \hat{b}_0^\dagger = \hat{b}_0 = \sqrt{N_0}$$
  - **interaction energy:**

$$\mathcal{H}_{\text{int}}^{(0)} = \frac{V_0}{2} \left[ \underbrace{\hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b}}_{\hat{b}_0^\dagger \hat{b}_0^\dagger = N_0^2} + \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} \right]$$

dominating term is the one with all wave numbers equal and zero

$$E_{\text{int}}^{(0)} = \frac{V_0}{2} N_0^2$$

# I.2.3 Bose Gas with Interactions

– interaction energy, next order term in powers of  $\hat{b}_0$ :

→ we cannot have three  $\hat{b}_0$  factors due to momentum conservation

→ we have to consider only terms with two  $\hat{b}_0$  factors in the next order

→ momentum conservation requires that we have factors  $\hat{b}_k^\dagger \hat{b}_k$  (4 possibilities),  $\hat{b}_k^\dagger \hat{b}_{-k}^\dagger$  (1 possibility) and  $\hat{b}_k \hat{b}_{-k}$  (1 possibility)

$$\mathcal{H}_{\text{int}}^{(1)} = \frac{V_0}{2} \hat{b}_0^2 \sum_{\mathbf{k}} (4\hat{b}_k^\dagger \hat{b}_k + \hat{b}_k^\dagger \hat{b}_{-k}^\dagger + \hat{b}_k \hat{b}_{-k})$$

$\mathcal{H}_{\text{int}}^{(1)}$  contains a contribution  $\hat{b}_0^2 \sum_{\mathbf{k}} \hat{b}_k^\dagger \hat{b}_k = N_0 \sum_{\mathbf{k}} \hat{n}_{\mathbf{k}}$ , where  $\hat{n}_{\mathbf{k}}$  is the particle number operator for bosons in state  $\mathbf{k}$

→ we have to take into account the bosons that are not in the ground state  $\mathbf{k} = \mathbf{0}$  (as already done for the non-interacting Bose gas)

$$N = N_0 + \sum_{\mathbf{k}} \hat{b}_k^\dagger \hat{b}_k \Rightarrow \frac{V_0}{2} (\hat{b}_0^2)^2 = \frac{V_0}{2} N_0^2 = \frac{V_0}{2} \left( N - \sum_{\mathbf{k}} \hat{b}_k^\dagger \hat{b}_k \right)^2 \simeq \frac{V_0}{2} \left( N^2 - 2N \sum_{\mathbf{k}} \hat{b}_k^\dagger \hat{b}_k \right) \simeq \underbrace{\frac{V_0}{2} N^2}_{\simeq E_{\text{int}}^{(0)} = \frac{V_0}{2} N_0^2} - \frac{V_0}{2} 2N \sum_{\mathbf{k}} \hat{b}_k^\dagger \hat{b}_k$$

$$\Rightarrow \mathcal{H} = \underbrace{\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \hat{b}_k^\dagger \hat{b}_k}_{\text{kinetic energy}} + \underbrace{N \frac{V_0}{2} \sum_{\mathbf{k}} (2\hat{b}_k^\dagger \hat{b}_k + \hat{b}_k^\dagger \hat{b}_{-k}^\dagger + \hat{b}_k \hat{b}_{-k})}_{\text{additional ground state energy}}$$

**the last two terms do not conserve particle number:**  
additional states are excited out of the condensate or absorbed into the condensate



# I.2.3 Bose Gas with Interactions

- diagonalization of the Hamiltonian by Bogoliubov transformation
  - introduction of new operators being linear combinations of the original bosonic operators

spinors of Bogoliubov operators:

$$\mathcal{B}_{\mathbf{k}}^\dagger = (\alpha_{\mathbf{k}}^\dagger, \beta_{-\mathbf{k}}) = \underbrace{c_{\mathbf{k}}^\dagger}_{(\hat{b}_{\mathbf{k}}^\dagger, \hat{b}_{-\mathbf{k}})} \mathcal{U}_{\mathbf{k}} \quad \mathcal{B}_{\mathbf{k}} = (\alpha_{\mathbf{k}}, \beta_{-\mathbf{k}}^\dagger) = \mathcal{U}_{\mathbf{k}}^\dagger \underbrace{c_{\mathbf{k}}}_{(\hat{b}_{\mathbf{k}}, \hat{b}_{-\mathbf{k}}^\dagger)^T}$$

$$\alpha_{\mathbf{k}}^\dagger = u_{\mathbf{k}} \hat{b}_{\mathbf{k}}^\dagger - v_{\mathbf{k}} \hat{b}_{-\mathbf{k}} \quad \alpha_{\mathbf{k}} = u_{\mathbf{k}}^* \hat{b}_{\mathbf{k}} - v_{\mathbf{k}}^* \hat{b}_{-\mathbf{k}}^\dagger$$

$$\beta_{-\mathbf{k}} = v_{\mathbf{k}}^* \hat{b}_{\mathbf{k}}^\dagger + u_{\mathbf{k}} \hat{b}_{-\mathbf{k}} \quad \beta_{-\mathbf{k}}^\dagger = v_{\mathbf{k}} \hat{b}_{\mathbf{k}} + u_{\mathbf{k}} \hat{b}_{-\mathbf{k}}^\dagger$$

- appropriate unitary matrix to make transformed energy matrix  $\tilde{\mathcal{E}}_{\mathbf{k}} = \mathcal{U}_{\mathbf{k}}^\dagger \mathcal{E}_{\mathbf{k}} \mathcal{U}_{\mathbf{k}}$  diagonal:

$$\mathcal{U}_{\mathbf{k}} = \begin{pmatrix} u_{\mathbf{k}} & v_{\mathbf{k}}^* \\ -v_{\mathbf{k}} & u_{\mathbf{k}}^* \end{pmatrix}$$

$$\mathcal{U}_{\mathbf{k}}^\dagger = \begin{pmatrix} u_{\mathbf{k}}^* & -v_{\mathbf{k}}^* \\ v_{\mathbf{k}} & u_{\mathbf{k}} \end{pmatrix}$$

$$\Rightarrow \mathcal{U}_{\mathbf{k}}^\dagger \mathcal{E}_{\mathbf{k}} \mathcal{U}_{\mathbf{k}} = \tilde{\mathcal{E}}_{\mathbf{k}} = \begin{pmatrix} E_{\mathbf{k}} & D_{\mathbf{k}} \\ -D_{\mathbf{k}} & -E_{\mathbf{k}} \end{pmatrix}$$

choose  $u_{\mathbf{k}}$  and  $v_{\mathbf{k}}$  such that off-diagonal terms vanish

# I.2.3 Bose Gas with Interactions

- diagonalization of the Hamiltonian by Bogoliubov transformation (without detailed calculation)

$$u_{\mathbf{k}} = \sqrt{\frac{1}{2} \frac{(\varepsilon_{\mathbf{k}} + NV_0)}{E_{\mathbf{k}}} + 1}$$

$$v_{\mathbf{k}} = \sqrt{\frac{1}{2} \frac{(\varepsilon_{\mathbf{k}} + NV_0)}{E_{\mathbf{k}}} - 1}$$

$$u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 = 1$$

with

$$E_{\mathbf{k}} = \sqrt{(\varepsilon_{\mathbf{k}} + NV_0)^2 - (NV_0)^2} = \sqrt{\varepsilon_{\mathbf{k}} (\varepsilon_{\mathbf{k}} + 2NV_0)}$$

resulting Hamiltonian

$$\mathcal{H} = \underbrace{\mathcal{H}_0}_{\text{ground state energy}} + \underbrace{\sum_{\mathbf{k}} \{E_{\mathbf{k}} \alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + E_{\mathbf{k}} \beta_{-\mathbf{k}}^\dagger \beta_{-\mathbf{k}}\}}_{\text{elementary excitations}}$$

elementary excitations of the interacting Bose gas are non-interacting bosonic quasiparticles described by the creation and annihilation operators  $\alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}}$  and  $\beta_{-\mathbf{k}}^\dagger \beta_{-\mathbf{k}}$

# I.2.3 Bose Gas with Interactions

- nature of excitations

- large  $\mathbf{k}$ : kinetic energy dominates:  $\varepsilon_{\mathbf{k}} \gg NV_0 \Rightarrow E_{\mathbf{k}} \simeq \varepsilon_{\mathbf{k}}, u_{\mathbf{k}} \simeq 1, v_{\mathbf{k}} \simeq 0,$   
 $\alpha_{\mathbf{k}}^{\dagger} = u_{\mathbf{k}} \hat{b}_{\mathbf{k}}^{\dagger} - v_{\mathbf{k}} \hat{b}_{-\mathbf{k}} \simeq \hat{b}_{\mathbf{k}}^{\dagger}$

excitations are identical to bare bosons

- $\mathbf{k} \rightarrow \mathbf{0}$ : kinetic energy vanishes:  $\varepsilon_{\mathbf{k}} \ll NV_0 \Rightarrow E_{\mathbf{k}} \simeq \sqrt{2NV_0\varepsilon_{\mathbf{k}}} = \sqrt{NV_0/m} \hbar k = \hbar k c_s$   
 $c_s = \text{sound velocity}$

propagating sound waves are expected for interacting system, correctly described by Bogoliubov theory  $\rightarrow$  quantized sound waves: phonons

$$u_{\mathbf{k}} \simeq v_{\mathbf{k}} \text{ with } u_{\mathbf{k}}^2 \simeq v_{\mathbf{k}}^2 = 1/2$$

excitations are equal superpositions of  $\hat{b}_{\mathbf{k}}^{\dagger}$  and  $\hat{b}_{-\mathbf{k}}$  ( $\hat{b}_{-\mathbf{k}}^{\dagger}$  and  $\hat{b}_{\mathbf{k}}$ ) both adding (removing) a momentum  $\hbar\mathbf{k}$  to the system  
 $\rightarrow$  consistent with excitation of quantized sound waves

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# I.2.4 Bose-Einstein Condensation of $^4\text{He}$

- What do we already know about  $^4\text{He}$ ?
  - $^4\text{He}$  does not become solid at ambient pressure due to the weak Van der Waals interaction and the large zero-point fluctuations
  - $^4\text{He}$  is a boson and shows superfluidity  
discovered in 1937 by Pyotr Kapitsa ("*Viscosity of Liquid Helium Below the  $\lambda$ -Point*", Nature **141**, 74 (1938)) and John F. Allen, and Don Misener ("*Flow of Liquid Helium II*" Nature **142**, 643 (1938))
  - the properties of superfluid  $^4\text{He}$  often deviate from those of an ideal Bose-Einstein condensate
    - $C_V \propto T^3$  instead of  $C_V \propto T^{3/2}$  at low  $T$
    - behavior of  $C_V$  around  $T_{\text{BEC}}$  is different – sharp  $\lambda$ -shaped peak instead of "cusp":  $^4\text{He}$  shows critical behavior consistent with 3D-XY universality class
  - description of superfluid  $^4\text{He}$  by macroscopic wave function  $\psi(\mathbf{r})$

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r})e^{i\theta(\mathbf{r})}, \quad |\psi_0(\mathbf{r})|^2 = n_0(\mathbf{r}), \quad N_0 = \int |\psi_0(\mathbf{r})|^2 d^3r$$

$$n_0 = \frac{N_0}{V} = \text{particle density in condensate}$$

# I.2.4 Bose-Einstein Condensation of $^4\text{He}$

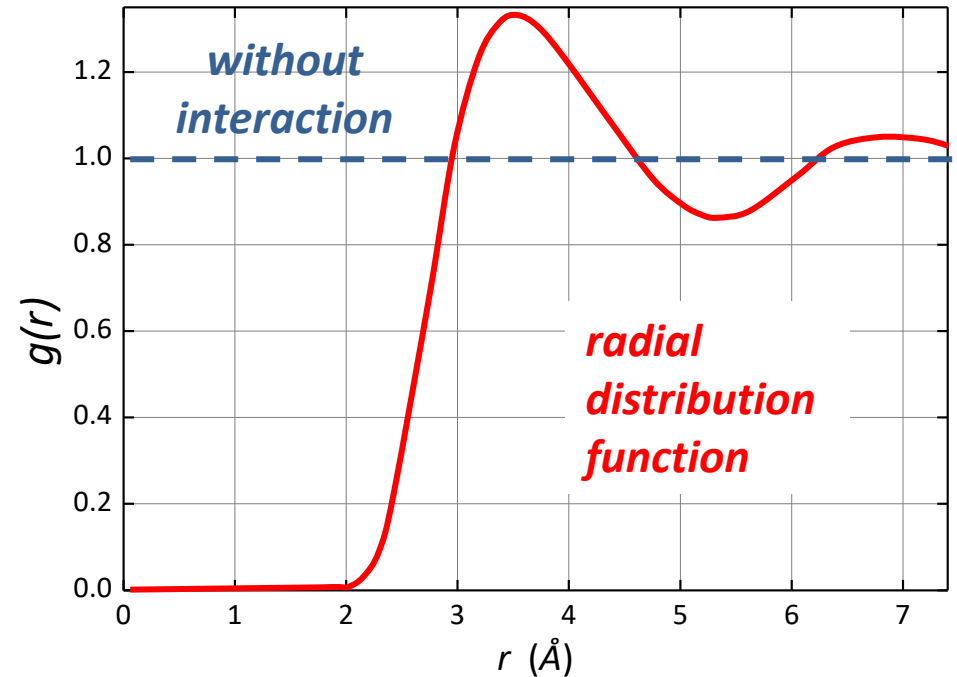
- How well does the ideal Bose gas describe the normal-superfluid transition temperature?

$$T_{\text{BEC}} = \frac{2\pi\hbar^2}{mk_B} \left( \frac{n}{2.612} \right)^{2/3}$$

- $T_{\text{BEC}} \approx 0.5 \text{ K}$  by using data of gaseous  $^4\text{He}$  (is well below normal-superfluid transition temperature)
- $T_{\text{BEC}} \approx 3.1 \text{ K}$  by using data of liquid  $^4\text{He}$  (agrees only qualitatively with measured  $T_\lambda = 2.17 \text{ K}$ )

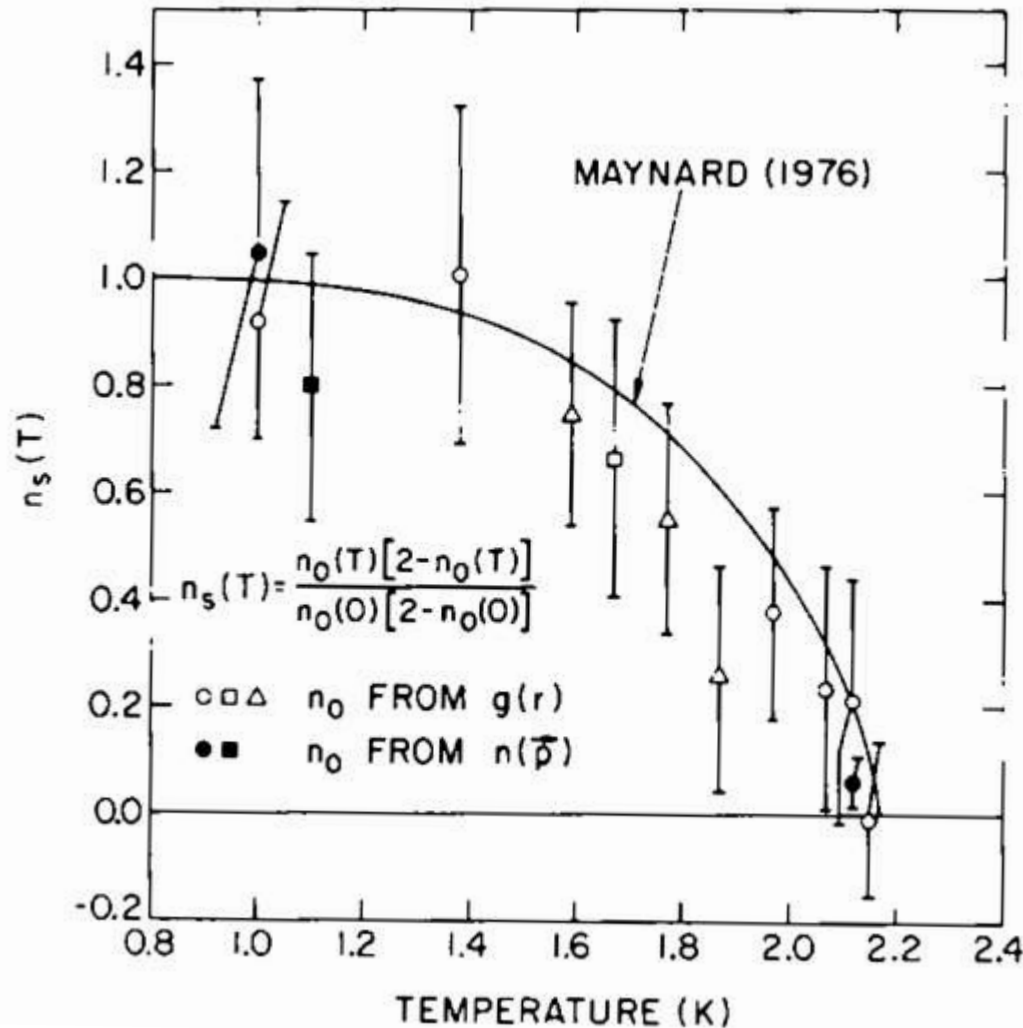
$^4\text{He}$  is not an ideal Bose gas:  
*interaction of He atoms*

- hard core repulsion
- admixture of higher  $\mathbf{k}$ -states (is equivalent to larger  $k$ -uncertainty)
- reduction of particle number in ground state



# I.2.4 Bose-Einstein Condensation of $^4\text{He}$

- superfluid fraction of  $^4\text{He}$  is only about 14% for  $T \rightarrow 0$ :  $n_0(T = 0) \simeq 0.14 n$



**experimental observations**

$$T \ll T_\lambda: n_s(T) \simeq n_s(0) - AT^\alpha \text{ with } \alpha \simeq 4$$

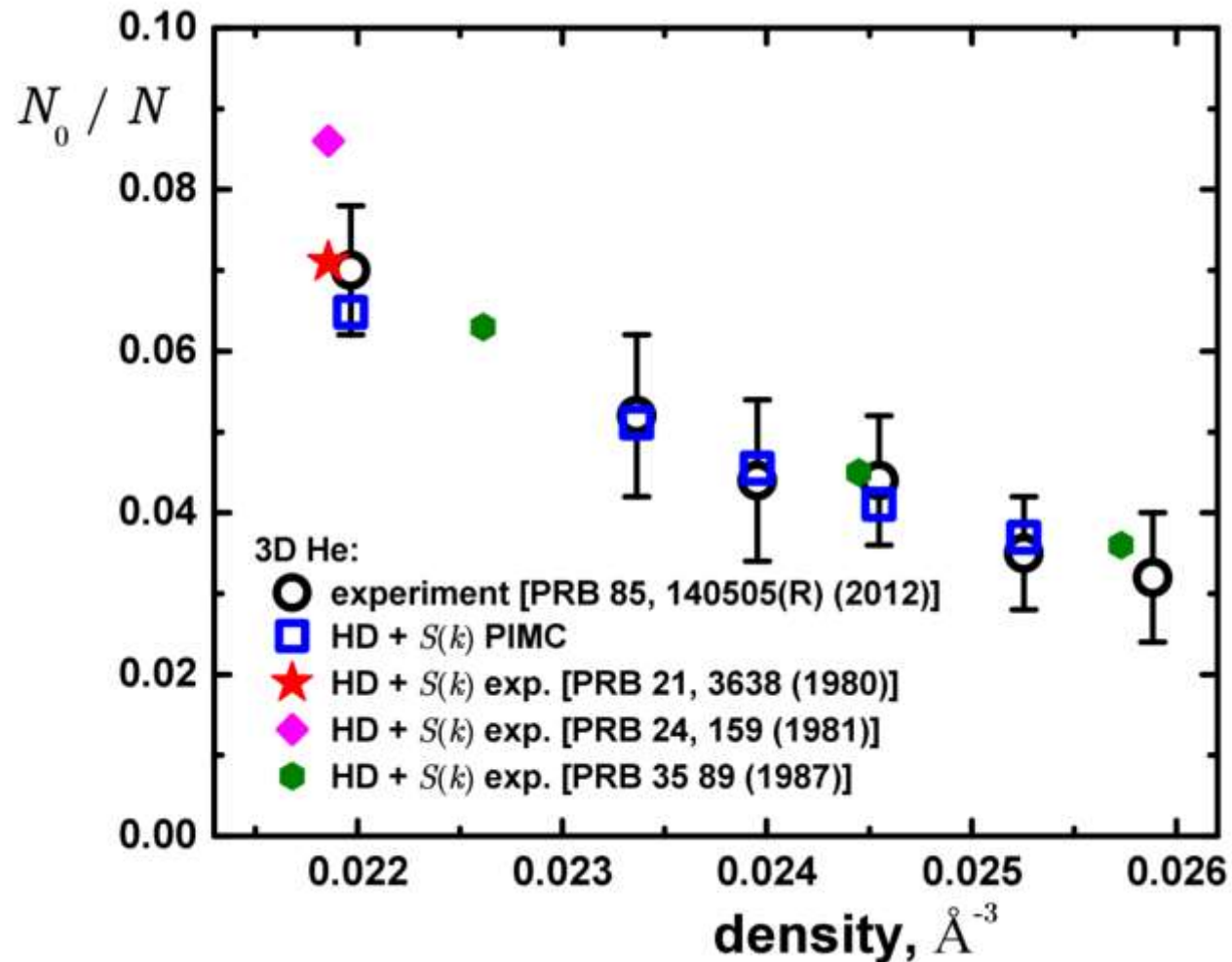
$$T \sim T_{\text{BEC}}: n_s(T) \simeq \begin{cases} B(T_\lambda - T)^\beta & \text{for } T < T_\lambda \\ 0 & \text{for } T > T_\lambda \end{cases}$$

**data obtained from x-ray scattering and neutron scattering**

J.S. Brooks and R.J. Donnelly, J. Phys. Chem. Ref. Data 6, 51 (1977).

# I.2.4 Bose-Einstein Condensation of $^4\text{He}$

- superfluid fraction of  $^4\text{He}$



condensate fraction of superfluid  $^4\text{He}$  as a function of the density. Open circles are experimental results obtained from deep inelastic neutron scattering, squares are HD predictions at the same densities. The star diamond, and hexagonal points correspond to quantum-field hydrodynamics (HD) results using the experimental measures of the static structure factor  $S(\mathbf{p})$

Lozovik, Yurii & Kurbakov, I. & Astrakharchik, G. & Boronat, J. (2020). *Universal relation between the static structure factor and the condensate fraction.*



# I.2.4 Bose-Einstein Condensation of $^4\text{He}$

- superfluid fraction of  $^4\text{He}$  – more detailed treatment (1)

- reduced superfluid fraction is caused by the fact that superfluid  $^4\text{He}$  is a strongly interacting Bose gas described by

$$\mathcal{H} = \sum_{i=1}^N \frac{\hbar^2}{2m} \nabla_i^2 + \frac{1}{2} \sum_{i \neq j} V(\mathbf{r}_i - \mathbf{r}_j)$$

- solutions are symmetric  $N$ -particle wave functions

$$\psi(\mathbf{r}_1, \dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots, \mathbf{r}_N) = \psi(\mathbf{r}_1, \dots, \mathbf{r}_j, \dots, \mathbf{r}_i, \dots, \mathbf{r}_N)$$

$$\mathcal{H}\psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = E_n \psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$$

- occupation probability of eigenstate with energy  $E_n$

$$P_n = \frac{1}{Z_N} \exp(-E_n/k_B T) \quad \text{with partition function} \quad Z_N = \sum_n \exp(-E_n/k_B T)$$

- if particle number is not fixed (grand canonical ensemble):  $Z_N = \sum_{n,N} \exp\left(-\frac{E_n^{(N)}}{k_B T} - \mu N\right)$

# I.2.4 Bose-Einstein Condensation of $^4\text{He}$

- superfluid fraction of  $^4\text{He}$  – more detailed treatment (2)

- for grand canonical ensemble:

$$\Omega(T, \mu) = -k_B T \ln Z$$

grand canonical potential

$$\langle N \rangle = k_B T \frac{\partial \ln Z}{\partial \mu}$$

average particle number

$$U = \langle \mathcal{H} \rangle = \mu \langle N \rangle - k_B T \frac{\partial \ln Z}{\partial \beta}$$

inner energy (with  $\beta = 1/k_B T$ )

- calculations have to be performed numerically, e.g. by perturbation theory (if interactions are weak) or quantum Monte Carlo simulations in the case of strong interactions (as for superfluid  $^4\text{He}$ )

**key result: superfluid  $^4\text{He}$  forms a condensate of He atoms in the ground state, however, not all atoms occupy ground state at  $T = 0$**

see e.g. D.M. Ceperley, Rev. Mod. Phys. 67, 279-355 (1995)

# I.2.4 Bose-Einstein Condensation of $^4\text{He}$

- superfluid fraction of  $^4\text{He}$  – more detailed treatment (3)

- single particle density matrix:

$$\rho(\mathbf{r}_1 - \mathbf{r}'_1) = N \int \psi_0^*(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) \psi_0(\mathbf{r}'_1, \mathbf{r}_2, \dots, \mathbf{r}_N) d^3r_2 \dots d^3r_N$$

- corresponds to correlation function between the many-body wave functions at the coordinates  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N$  and  $\mathbf{r}'_1, \mathbf{r}_2, \dots, \mathbf{r}_N$

- by integrating of all coordinates despite  $\mathbf{r}_1$  and  $\mathbf{r}'_1$ , we average over all configurations of the particles despite the first ones

- properties if the single particle density matrix:

- depends only on relative coordinate  $\mathbf{r}_1 - \mathbf{r}'_1$  due to translational invariance

- for  $\mathbf{r}_1 = \mathbf{r}'_1$  we obtain:  $\int \rho_1(0) d^3r_1 = \frac{N}{V} = n$

- for ideal non-interacting Bose gas we have  $\psi_0(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \psi_0(\mathbf{r}_1) \psi_0(\mathbf{r}_2) \dots \psi_0(\mathbf{r}_N)$

$$\rho(\mathbf{r}_1 - \mathbf{r}'_1) = N \psi_0^*(\mathbf{r}_1) \psi_0(\mathbf{r}'_1) \int |\psi_0(\mathbf{r}_1)|^2 |\psi_0(\mathbf{r}_2)|^2 \dots |\psi_0(\mathbf{r}_N)|^2 d^3r_2 \dots d^3r_N = N \psi_0^*(\mathbf{r}_1) \psi_0(\mathbf{r}'_1)$$

$$\rho(\mathbf{r}_1 - \mathbf{r}'_1) = \frac{N}{V} = n \quad \text{since } \psi_0(\mathbf{r}) = \frac{1}{\sqrt{V}} \exp(i\mathbf{k} \cdot \mathbf{r})$$

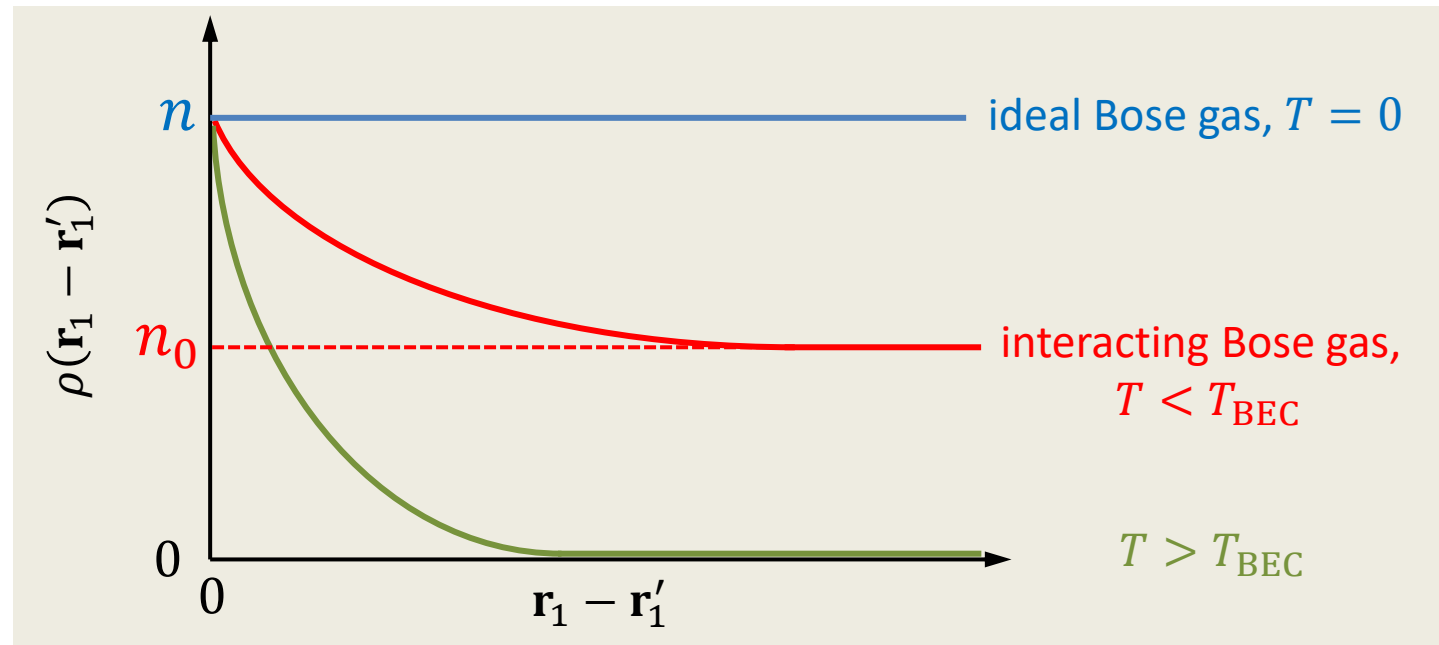
# I.2.4 Bose-Einstein Condensation of $^4\text{He}$

- superfluid fraction of  $^4\text{He}$  – more detailed treatment (4)
  - for interacting Bose gas we obtain more complicated result
    - $\rho(\mathbf{r}_1 - \mathbf{r}'_1)$  is no longer independent of  $\mathbf{r}_1 - \mathbf{r}'_1$
    - $\rho(\mathbf{r}_1 - \mathbf{r}'_1)$  decreases with increasing  $\mathbf{r}_1 - \mathbf{r}'_1$  and converges towards a constant value at large  $\mathbf{r}_1 - \mathbf{r}'_1$
  - condensate density:

$$n_0 = \lim_{|\mathbf{r}_1 - \mathbf{r}'_1| \rightarrow \infty} \rho(\mathbf{r}_1 - \mathbf{r}'_1)$$

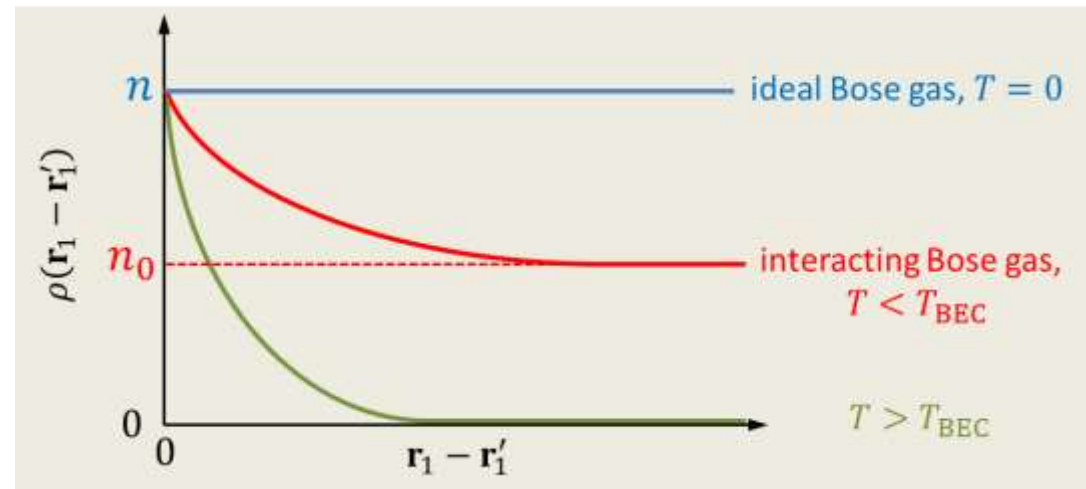
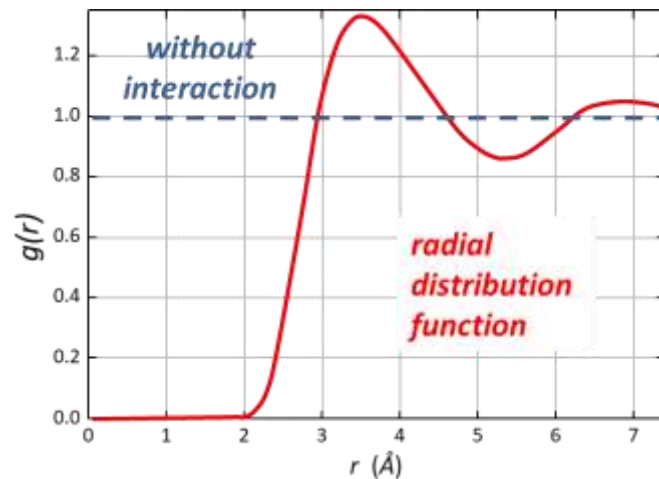
for superfluid  $^4\text{He}$ :

$$n_0(T = 0) \simeq 0.14 n$$



# I.2.4 Bose-Einstein Condensation of $^4\text{He}$

- superfluid fraction of  $^4\text{He}$  – more detailed treatment (5)
  - plausibility consideration by considering spatial distribution function



- without interactions particles can be described by plane waves with  $k = k_0$ 
  - $g(\mathbf{r}_1 - \mathbf{r}_2) = 1$ , all particles in ground state:  $n_0 = n$
- in the presence of interactions,  $g(\mathbf{r}_1 - \mathbf{r}_2) \neq 1$  for short distances
  - we have to admix states with higher  $k$  to describe the situation: spatial localization requires finite  $\Delta k$
  - some states with  $k > k_0$  are occupied:  $n_0 < n$
  - $n_0$  becomes smaller, if  $g(\mathbf{r}_1 - \mathbf{r}_2)$  deviates more strongly from  $g(\mathbf{r}_1 - \mathbf{r}_2) = 1$ , i.e. with increasing interactions

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- I.2.2 Ideal Bose Gas
- I.2.3 Bose Gas with Interactions
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### I.3 Superfluid $^4\text{He}$

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### I.5 $^3\text{He}$

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- I.5.3 superfluid  $^3\text{He}$

### I.6 $^3\text{He}$ / $^4\text{He}$ mixtures

# I.3 Superfluid $^4\text{He}$

- nomenclature and history

- superfluid  $^4\text{He}$  can flow through tiny capillaries without any friction → nomenclature *superfluidity*

- **discovery of superfluidity** in 1937 by two groups:
    - (i) *P. Kapitza, Nature 141, 74 (1938)*
    - (ii) *J.F. Allen, A.D. Misener, Nature 141, 75 (1938)*

- **theory of superfluidity:**

- **F. London** (1938): superfluidity caused by ordered state in momentum space  
*F. London, Nature 141, 643 (1938); Phys. Rev. 54, 947 (1938).*

- **L. Tisza** (1938): two-fluid model: prediction of second sound  
*L. Tisza, Nature 141, 913 (1938).*

- **L.D. Landau** (1941-47): two-fluid hydrodynamics of superfluid  $^4\text{He}$   
→ *superfluidity as a consequence of the excitation spectrum*  
*L.D. Landau, J. Phys. USSR 5, 71 (1941); ibid. 8, 1 (1944); ibid. 11, 91 (1947).*

- **R.P. Feynman** (1953): excitation spectrum postulated by Landau follows from quantum mechanics  
*R.P. Feynman, Phys. Rev. 90, 1116 (1953).*

# I.3.1 Two-Fluid Model

- *two-fluid model allows phenomenological description of superfluid  $^4\text{He}$*   
 microscopic description is difficult  $\rightarrow$  treatment using hydrodynamics

- already in 1938: anomalous properties of superfluid  $^4\text{He}$  can be well described by two-fluid model:  
*mixture of two non-interacting fluid components* *(l. Tisza, J. de Phys. et Radium 1, 164 (1938))*
- formal description of superfluid  $^4\text{He}$  as the sum of a normal and a superfluid component

$$\rho = \rho_s + \rho_n \quad \rho_s, \rho_n = \text{superfluid and normal fluid densities}$$

- $T = 0$ :  $\rho_s = \rho$  and  $\rho_n = 0$
- $T = T_\lambda$ :  $\rho_s = 0$  and  $\rho_n = \rho$

- superfluid component: **no entropy:  $S_s = 0$**       **zero viscosity:  $\eta_s = 0$**
- normal fluid component: **carries total entropy:  $S_n = S$ ,**      **finite viscosity:  $\eta_n = \eta$**



# I.3.1 Two-Fluid Model

- **superfluid component**

- description of superfluid component by macroscopic wave function

as done in superconductivity for electronic system: charged superfluid

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r})e^{i\theta(\mathbf{r})}, \quad |\psi_0(\mathbf{r})|^2 = n_0(\mathbf{r}), \quad N_0 = \int |\psi_0(\mathbf{r})|^2 d^3r \quad n_0 = \frac{N_0}{V} = \text{particle density in condensate}$$

- use quantum mechanics to derive expression for the particle current density (probability current density in 3D)

$$J_\rho = \frac{\hbar}{2mi} [\psi^*(\mathbf{r})\nabla\psi(\mathbf{r}) - \psi(\mathbf{r})\nabla\psi^*(\mathbf{r})]$$

$$J_\rho = \frac{\hbar}{2mi} \left[ \underbrace{\sqrt{n_0} e^{-i\theta(\mathbf{r})}}_{\psi^*(\mathbf{r})} \underbrace{\left( e^{i\theta(\mathbf{r})}\nabla\sqrt{n_0} + i\sqrt{n_0}e^{i\theta(\mathbf{r})}\nabla\theta(\mathbf{r}) \right)}_{\nabla\psi(\mathbf{r})} - \underbrace{\sqrt{n_0} e^{i\theta(\mathbf{r})}}_{\psi(\mathbf{r})} \underbrace{\left( e^{-i\theta(\mathbf{r})}\nabla\sqrt{n_0} - i\sqrt{n_0}e^{-i\theta(\mathbf{r})}\nabla\theta(\mathbf{r}) \right)}_{\nabla\psi^*(\mathbf{r})} \right]$$

$$J_\rho(\mathbf{r}) = n_0 \frac{\hbar}{m} \underbrace{\nabla\theta(\mathbf{r})}_{v_s} = n_0 v_s(\mathbf{r})$$

**current density – phase relation** (as in superconductivity)

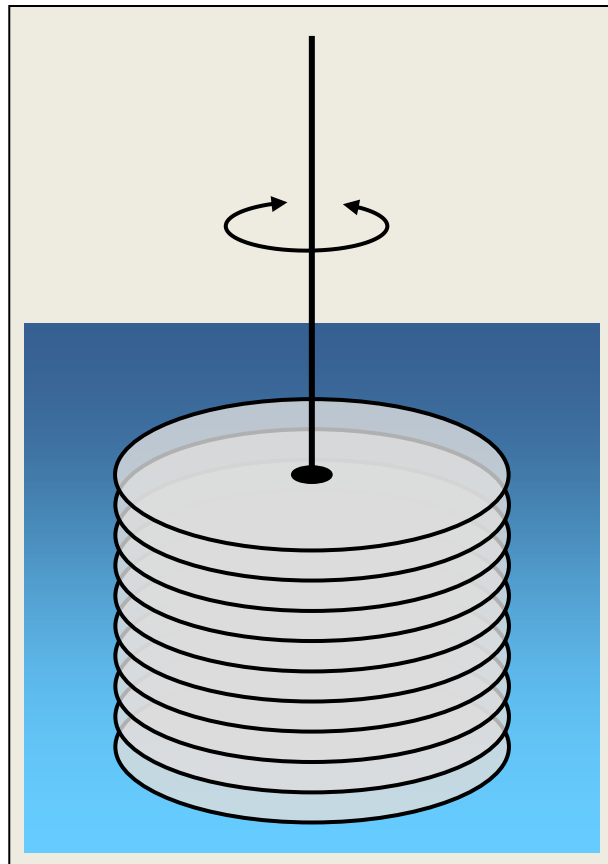
$v_s$  = superfluid velocity

# I.3.1 Two-Fluid Model

- what is the ratio of the normal und superfluid component? How does it depend on temperature?

- torsional pendulum experiment: 50 Al disks, thickness: 13  $\mu\text{m}$ , diameter: 3.5 cm, disk separation: 210  $\mu\text{m}$

(E.L. Andronikashvili, Zh. Eksperim. i. Teor. Fiz. 18, 424 (1948))



- only normal component is moved during rotation

- skin depth of viscous wave:

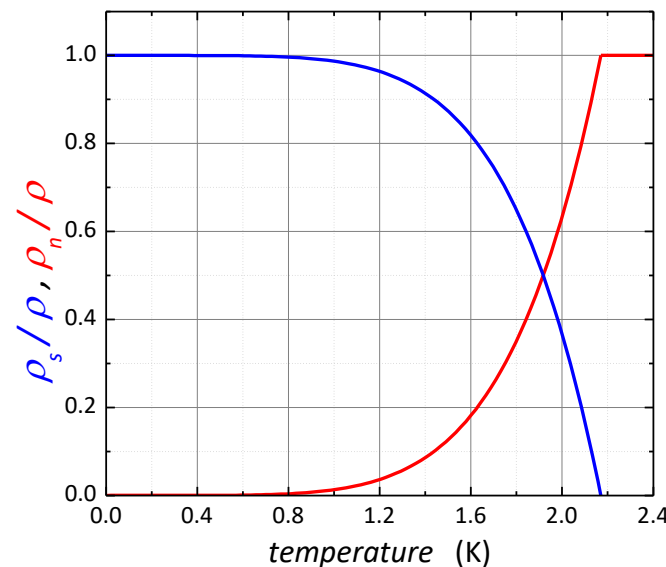
$$\sqrt{2\eta_n/\rho_n\omega}$$

→ large compared to disk separation below  $T_\lambda$

→ complete normal component is moved

- moved mass of normal component results in change of moment of inertia

→ change of oscillation frequency:  $\omega \propto 1/\text{mass}$



empirical relation by fitting data:

$$\frac{\rho_n}{\rho(T_\lambda)} = \left(\frac{T}{T_\lambda}\right)^{5.6}$$

cf. normal fluid density of ideal BEC:

$$\frac{\rho_n}{\rho} = \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2}$$

# I.3.1 Two-Fluid Model

- physical consequence of macroscopic wave function description

$$J_{\rho}(\mathbf{r}) = n_0 \frac{\hbar}{m} \nabla \theta(\mathbf{r}) = n_0 v_s(\mathbf{r}) \Rightarrow v_s(\mathbf{r}) = \frac{\hbar}{m} \nabla \theta(\mathbf{r})$$

- since the curl of the gradient of a scalar function vanishes, we obtain

$$\nabla \times v_s(\mathbf{r}) = 0$$

$v_s(\mathbf{r})$  cannot take the form  $\Omega \times \mathbf{r}$  corresponding to a rotating liquid

➔ it is impossible to set the superfluid into rotation

**note:**

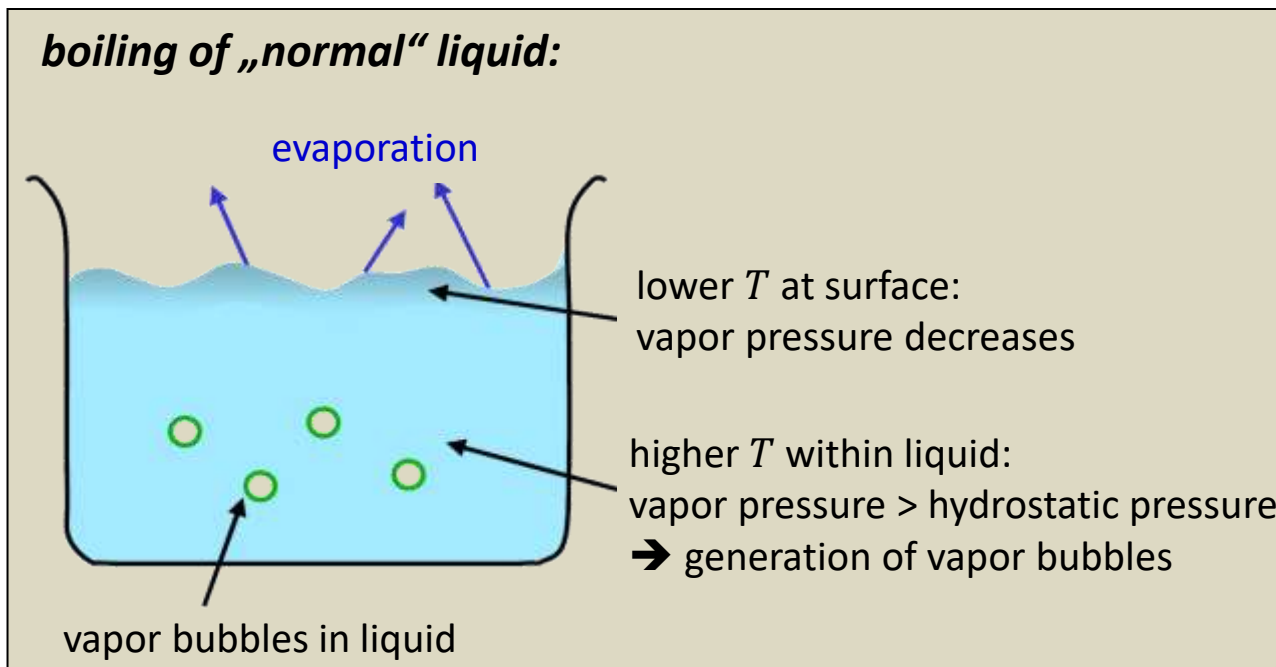
we will see later that this is true only for slow enough rotations of the vessel containing the superfluid. At large rotation speed, we are generating vortices in the superfluid

➔ the critical rotation speed corresponds to the lower critical field in superconductors!!

# I.3.2 Experimental Observations

- boiling of liquid helium

- $T > T_\lambda$ :
  - liquid He is boiling
  - latent heat of evaporation is removed from bath
  - temperature is decreasing
- $T \simeq T_\lambda$ :
  - temperature is decreasing only slowly → large specific heat
  - temperature does not stay constant → no latent heat, 2. order phase transition
  - **boiling suddenly stops !!!**



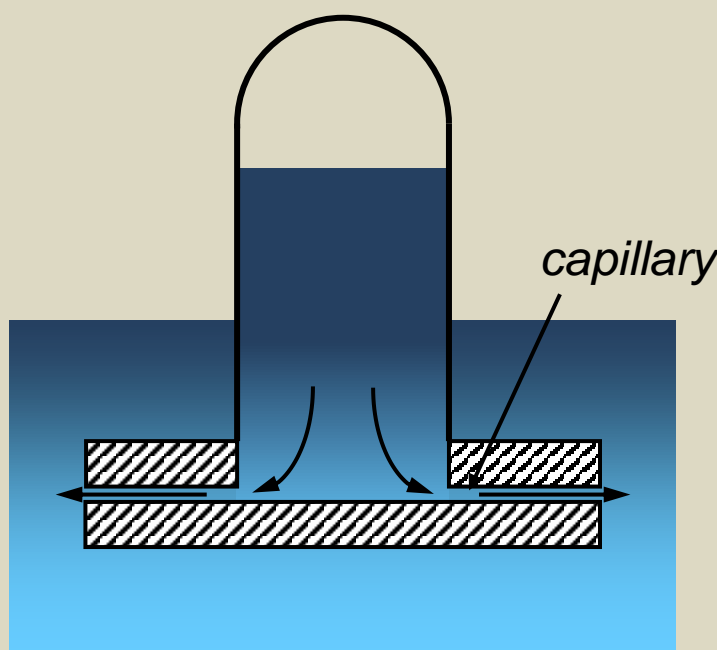
**boiling of superfluid helium:**

- **very high thermal conductivity**
  - low  $T$  not only at surface, but also within liquid
  - no vapor bubbles in liquid
- **evaporation only from surface, no boiling**

# I.3.2 Experimental Observations

- viscosity  $\eta$  of superfluid  $^4\text{He}$ : *two different measuring methods*

**flow through narrow capillary**



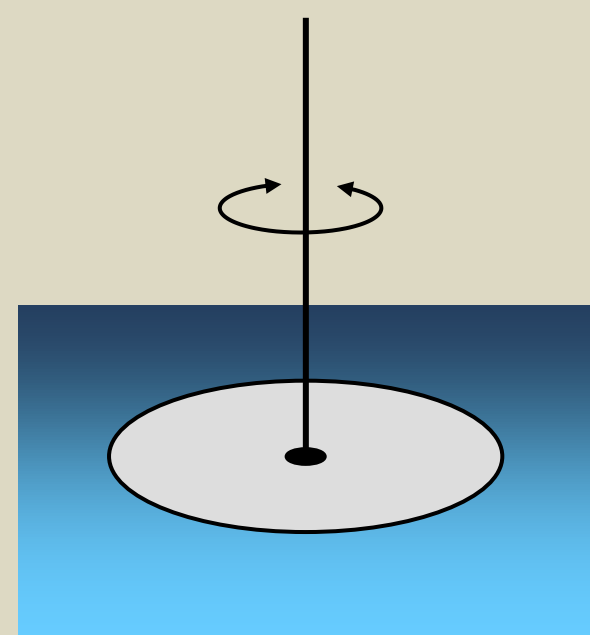
$\eta < 10^{-11}$  poise

(cf. water:  $8.90 \times 10^{-5}$  poise @  $25^\circ\text{C}$ )

- flow without measurable friction
- viscosity  $\eta \approx 0$
- *superfluid component flows out*

(1 Poise =  $1 \text{ g/cm s} = 0.1 \text{ Pa} \cdot \text{s}$ )

**torsional pendulum**



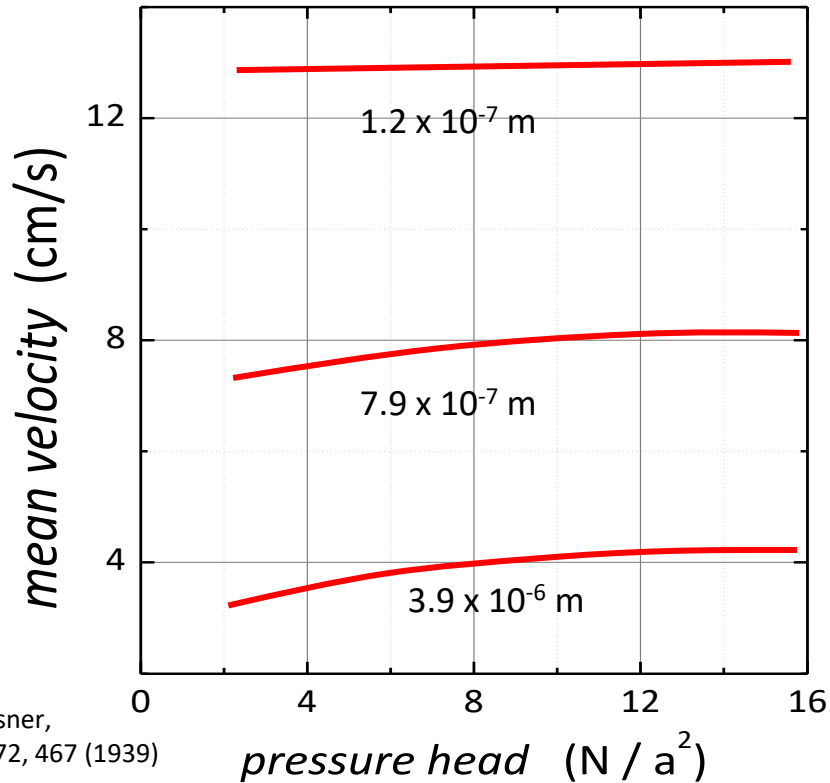
$\eta \approx 10^{-5}$  poise

- damping of oscillation:  $\eta > 0$
- scattering of **phonons** and **rotons**
- *normal component picks up momentum*

# I.3.2 Experimental Observations

- viscosity  $\eta$  of superfluid  $^4\text{He}$ : *experimental results*

## ➤ flow through narrow capillary



J.F. Allen, A.D. Miesner,  
Proc. Roy. Soc. A 172, 467 (1939)

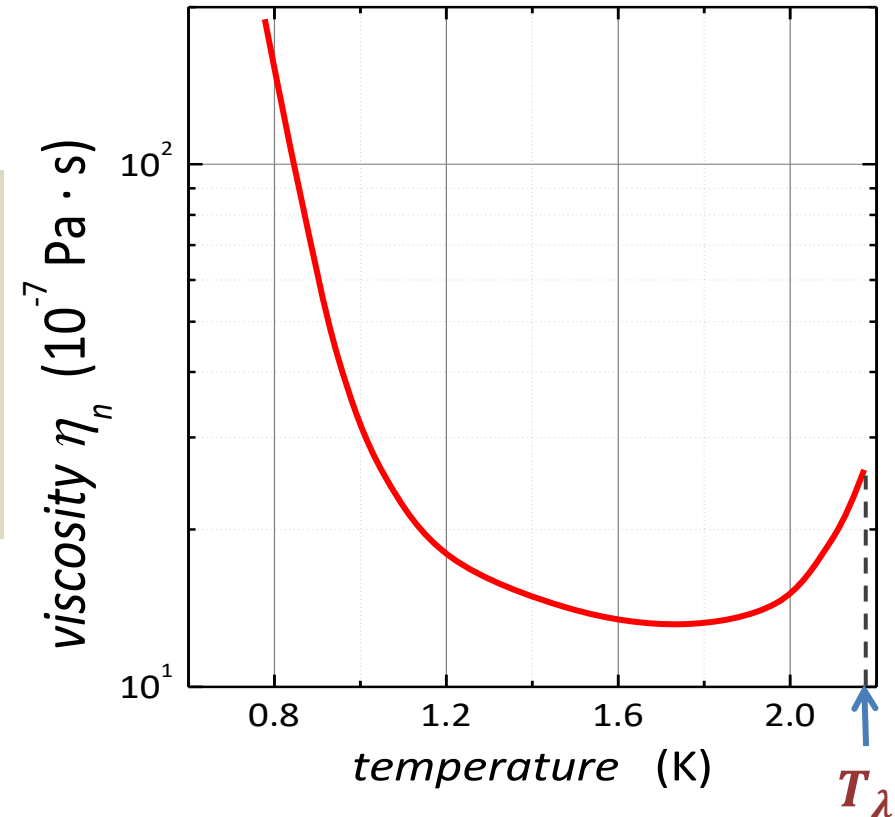
- flow velocity  $v$  about independent of pressure
- flow velocity increases  $v$  with decreasing diameter  $a$  of capillary

*experimental result depends on measuring method !!*



*explanation by two-fluid model*

## ➤ torsional pendulum

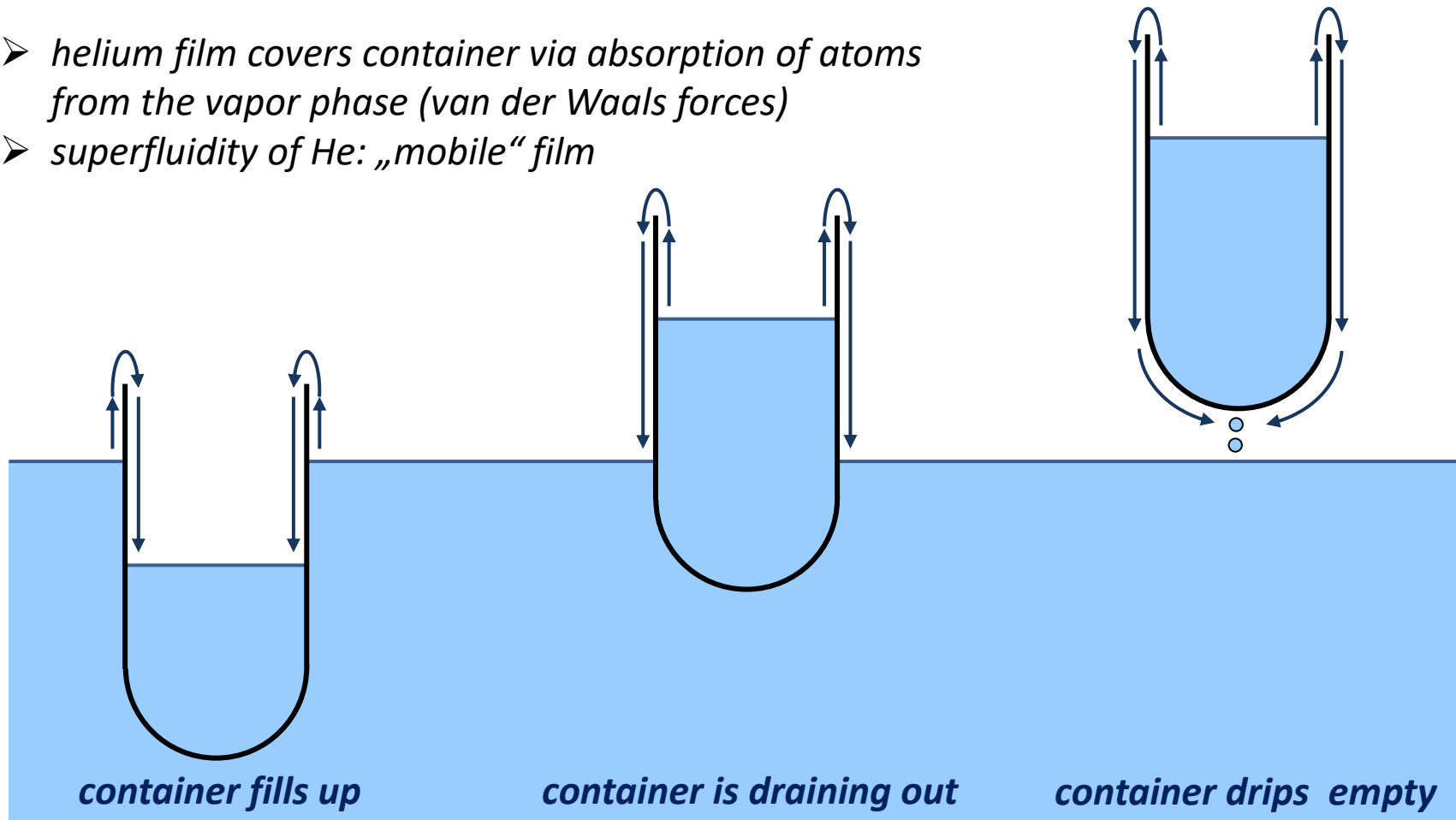


expected for viscous flow through tube of length  $L$  and diameter  $a$ :  $\Delta p/L \sim \eta(v/a^4)$  (laminar flow, Hagen-Poiseuille law)  
 experimental result of Kapitza:  $\Delta p \simeq 0$  independent of flow velocity  $v \rightarrow \eta \simeq 0$

# I.3.2 Experimental Observations

- superfluid film flow

- helium film covers container via absorption of atoms from the vapor phase (van der Waals forces)
- superfluidity of He: „mobile“ film



overshooting during fill up and draining out → oscillation of filling height

# I.3.2 Experimental Observations

- **superfluid film flow: estimate of superfluid film thickness**

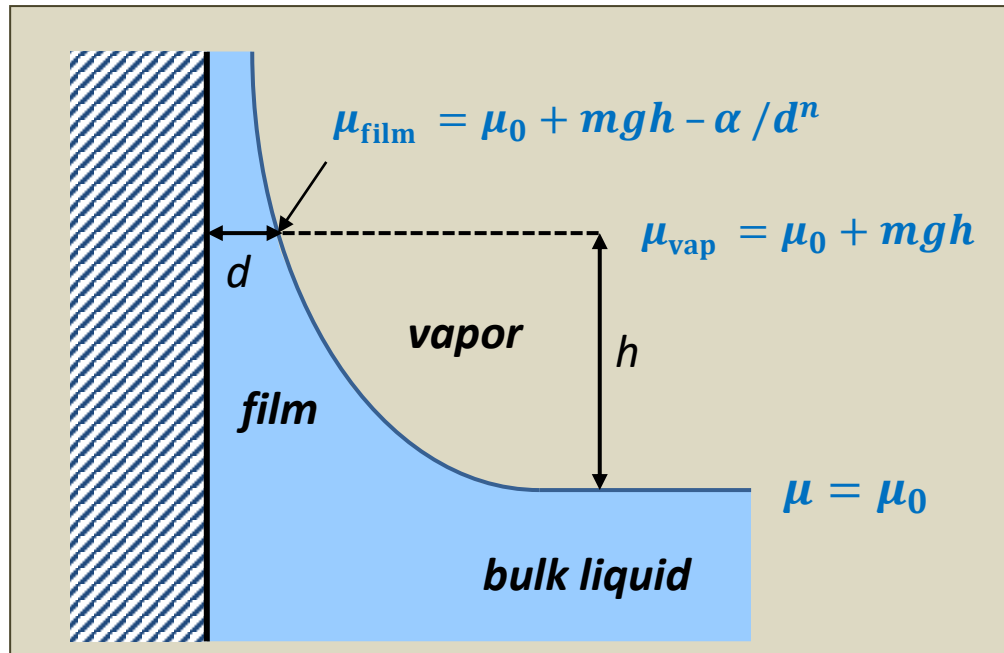
- in equilibrium **chemical potentials** are identical:  $\mu_{\text{film}} = \mu_{\text{vap}} = \mu_0$  (for film in saturated vapor)

- for film: 
$$\mu_{\text{film}} = \mu_0 + \underbrace{mgh - \alpha / d^n}_{= 0}$$

$\mu_0$  ← bulk liquid
gravitation
van der Waals potential

( $n \approx 3$  for  $d < 5$  nm,  $\alpha$  = Hamaker constant,  $\alpha$  is determined by dielectric properties of wall and He atoms)

- for vapor: 
$$\mu_{\text{vap}} = \mu_0 + mgh$$



**equilibrium condition:**  $\mu_{\text{film}} = \mu_0$

$$d = \left( \frac{\alpha}{mgh} \right)^{1/3}$$

typical values:

$$d \text{ (nm)} \approx 30 / [h \text{ (cm)}]^{1/3}$$

$$h = 5 \text{ cm} \rightarrow d = 20 \text{ nm}$$

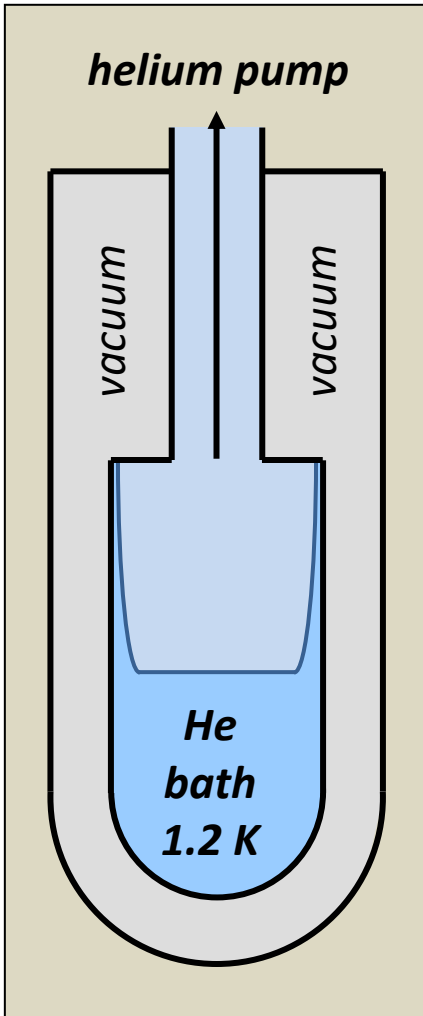
normal fluid blocked in thin film

➔ **superleak**



# I.3.2 Experimental Observations

- superfluid film flow: *acting as a heat leak*



- film creeps up along the container walls and pumping line
- film creeps up to the position, where  $T = 2.18$  K
- evaporation from film dominates, since vapor pressure at height  $h$  is much lower
- estimate of film thickness for „*unsaturated film*“  
 →  $d$  depends on gas pressure  $p$  (kinetic gas theory)

$$p(h) = p_{\text{sat}} \exp\left(-\frac{mgh}{k_B T}\right) \Rightarrow h = -\frac{k_B T}{mg} \ln\left(\frac{p(h)}{p_{\text{sat}}}\right)$$

$$d = \left(\frac{\alpha}{mgh}\right)^{1/3} \Rightarrow d = \left(\frac{-\alpha}{k_B T \ln(p/p_{\text{sat}})}\right)^{1/3}$$

- estimate of the flow of liquid for a critical superfluid velocity of  $v_s = 30$  cm/s:

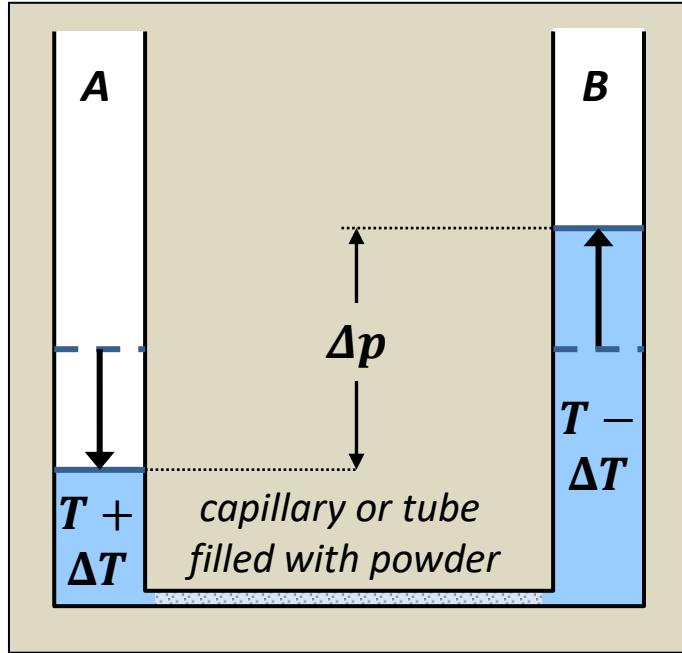
$$\dot{V}_{\text{liq}} = 2\pi R d v_s \approx 1 \text{ cm}^3/\text{h} \text{ for } R = 5 \text{ mm} \Rightarrow \dot{V}_{\text{gas}} \approx 1 \text{ l/h}$$

- min. pressure achieved by vacuum pump with  $\dot{V}_{\text{pump}} \approx 10^4$  l/h

$$p_{\text{min}} = p \dot{V}_{\text{gas}} / \dot{V}_{\text{pump}} \approx 10^{-4} \text{ bar} \Rightarrow T_{\text{min}} \approx 1 \text{ K}$$

# I.3.2 Experimental Observations

- thermomechanical effect: *generate  $\Delta T$  by applying  $\Delta p$*



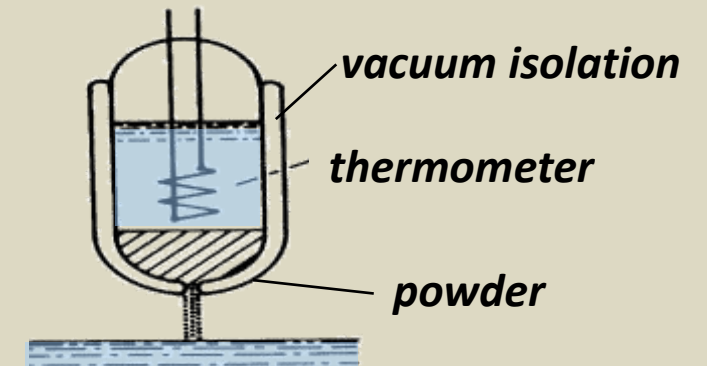
- starting point: same filling level and temperature  $T$  of container A and B
- apply  $\Delta p \rightarrow$  only superfluid component flows through capillary from A to B, but cannot transport heat from A to B
- *temperature decreases in B and increases in A, since normal fluid fraction increases in A and decreases in B*
- remove  $\Delta p \rightarrow$  original state is recovered
- note: mass flow is not accompanied by heat flow since only superfluid component can flow

*only superfluid can flow from A to B through narrow capillary*

increase/decrease of normal fluid concentration in A/B

*similar experiment:*

- only superfluid flows through powder
- increase of normal fluid concentration
- increase of  $T$  in container



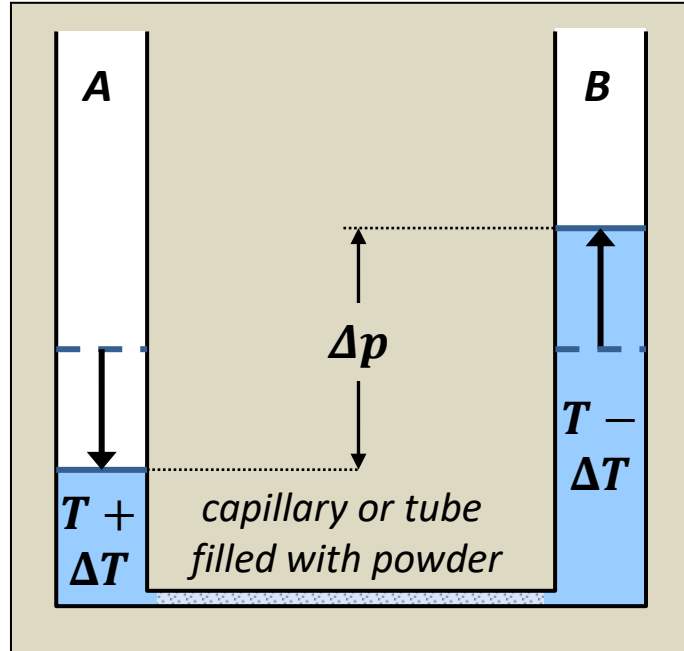
# I.3.2 Experimental Observations

- thermomechanical effect: *generate*  $\Delta T$  by applying  $\Delta p$

$$G = U - TS + pV$$

$$dG = TdS - pdV + \mu dN - TdS - SdT + pdV + Vdp$$

$$dG = -SdT + Vdp + \mu dN$$



*only superfluid can flow from A to B through narrow capillary*

➤ since  $S = S_n$

$$\Delta p = \frac{S_n}{V} \Delta T = \frac{S_n M_n}{M_n V} \Delta T = \sigma_n \rho_n \Delta T$$

$$\Delta p = \sigma_n \rho_n \Delta T \quad \rightarrow \quad \Delta p \rightarrow 0 \quad \text{for} \quad \rho_n \rightarrow 0$$

- the configuration can be considered as two systems A and B in equilibrium:  
 $\Rightarrow \mu_A = \mu_B = \mu$  and particle number is conserved:  $N = N_A + N_B$

– **Gibbs-Duhem relation:**

$$G(T, p, \mu_A, \mu_B) = N_A \mu_A + N_B \mu_B = N \mu$$

$$\Rightarrow dG = \mu dN + N d\mu$$

- comparison with  $dG = -SdT + Vdp + \mu dN$  yields

$$-SdT + Vdp - Nd\mu = 0$$

$$-S(T_B - T_A) + V(p_B - p_A) - N \underbrace{(\mu_B - \mu_A)}_{=0} = 0$$

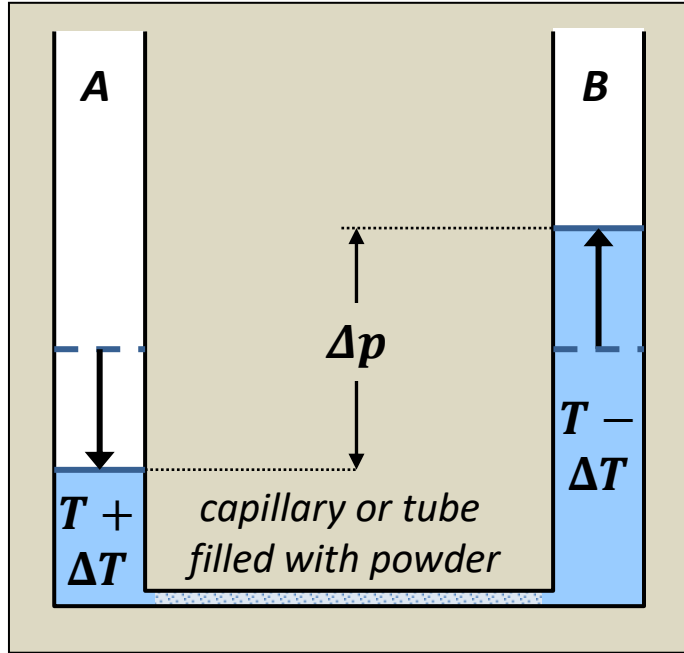


$$\underbrace{V(p_B - p_A)}_{\text{work associated with mass flow}} = \underbrace{S(T_B - T_A)}_{\text{compensating heat change}}$$

- since superfluid flow does not carry heat, there is no mechanism to equilibrate  $T_A$  and  $T_B$
- nevertheless, the particle flow results in  $\mu_A = \mu_B = \mu$

# I.3.2 Experimental Observations

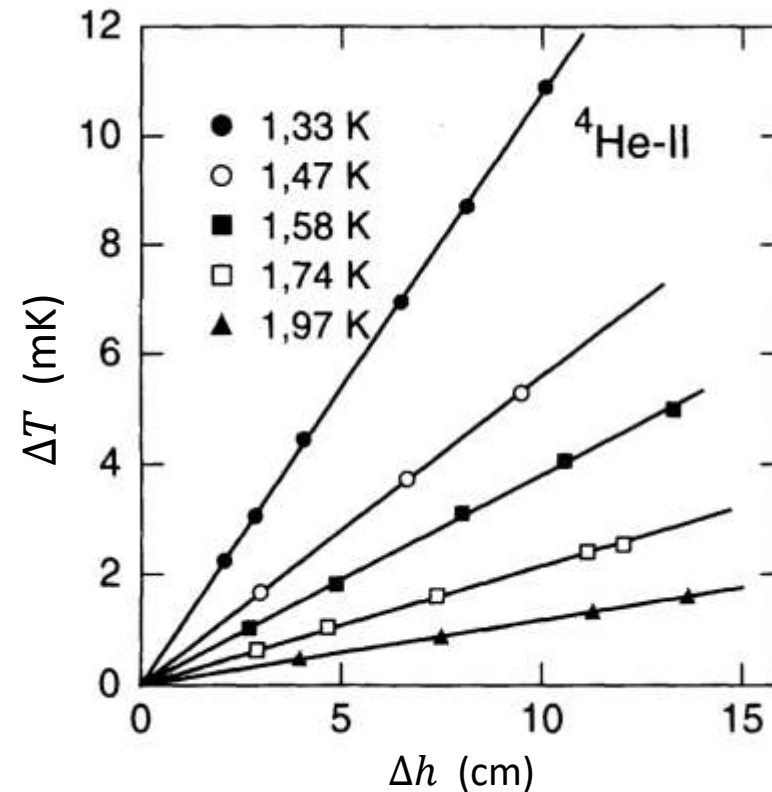
- thermomechanical effect: *generate  $\Delta T$  by applying  $\Delta p$*



➤ since  $S = S_n$

$$\Delta p = \frac{S_n}{V} \Delta T = \frac{S_n M_n}{M_n V} \Delta T = \sigma_n \rho_n \Delta T$$

$$\Delta p = \sigma_n \rho_n \Delta T \quad \rightarrow \Delta p \rightarrow 0 \text{ for } \rho_n \rightarrow 0$$

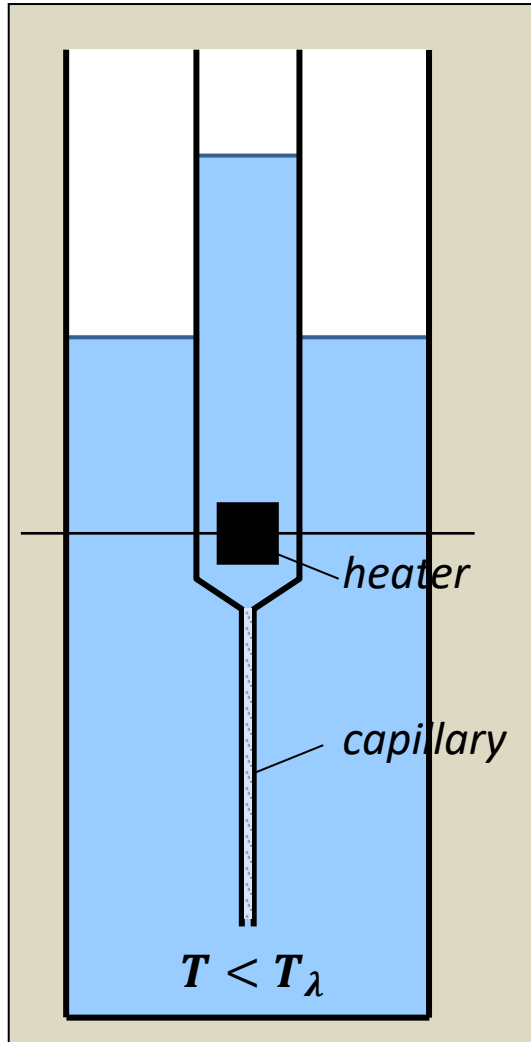


Enns, Hunklinger,  
Low Temperature Physics, Springer (2000)

# I.3.2 Experimental Observations

- inverse thermomechanical effect: *generate  $\Delta p$  by applying  $\Delta T$*

$$\Delta p = \sigma_n \rho_n \Delta T$$

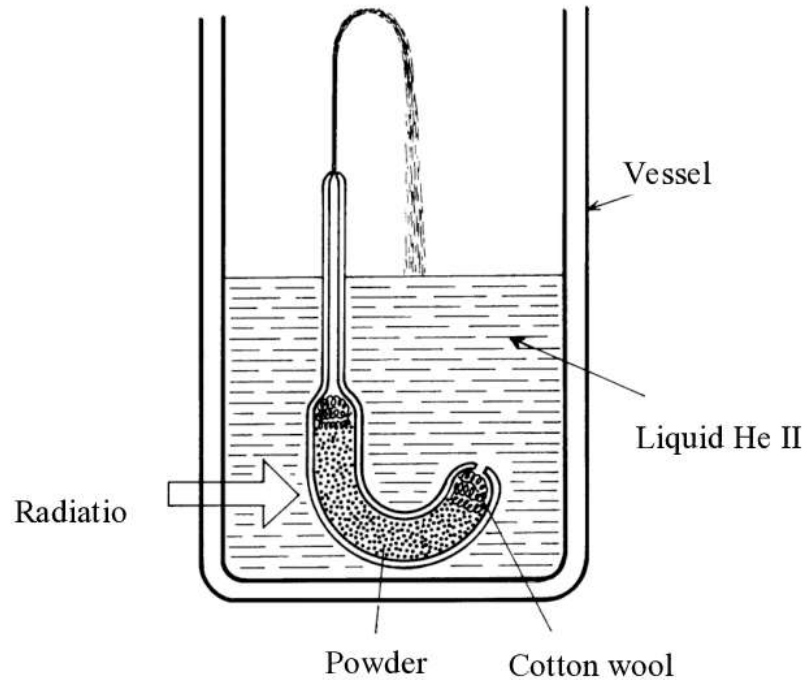


- heating generates increase of normal fluid concentration
- superfluid components flows through capillary to establish balance
- analogous to osmotic pressure
- superfluid flows very rapidly through capillary so that a fountain pressure can be generated

→ ***fountain effect***

# I.3.2 Experimental Observations

- inverse thermomechanical effect: *fountain effect*



- light is absorbed by emery powder → heating
- lower opening is closed by cotton wool → only superfluid can enter
- a He fountain is generated by the thermomechanical effect (typical height up to 30 cm)



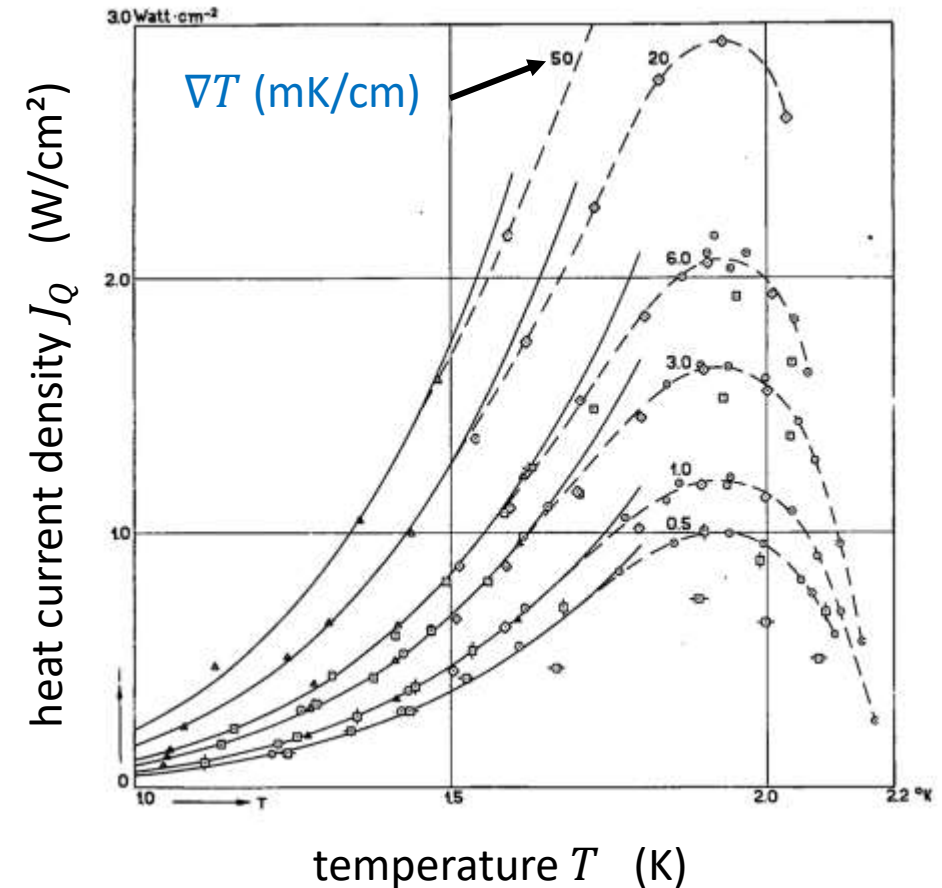
# I.3.2 Experimental Observations

- thermal conductivity

- thermal conductivity of superfluid  $^4\text{He}$  is by more than five orders of magnitude larger than that of normal liquid  $^4\text{He}$
- explains disappearance of boiling below  $T_\lambda$
- heat conduction through narrow capillaries has maximum at  $\approx 1.8$  K
- heat current density varies as  $J_Q \propto \nabla T^{1/3}$   
(if we would use  $J_Q \simeq -\lambda \nabla T$ , then  $\lambda \propto \nabla T^{-2/3}$  and would diverge for small  $\nabla T$ )
- linear  $J_Q \propto \nabla T$  dependence only for small heat current densities

➔ **no Umklapp (U) processes possible in superfluid  $^4\text{He}$**   
(no periodic lattice)

➔ **excitations (phonons, rotons) move ballistically, only surface scattering**



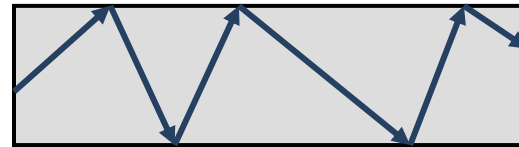
W.H. Keesom, B.F. Saris, L. Meyer, Physica 7, 817 (1940)



# I.3.2 Experimental Observations

- thermal conductivity: comparison to thermal conductivity of solids

solids at low  $T$ :



- ballistic phonons
- $N$  processes do not cause momentum relaxation
- only surface scattering

kinetic gas theory:

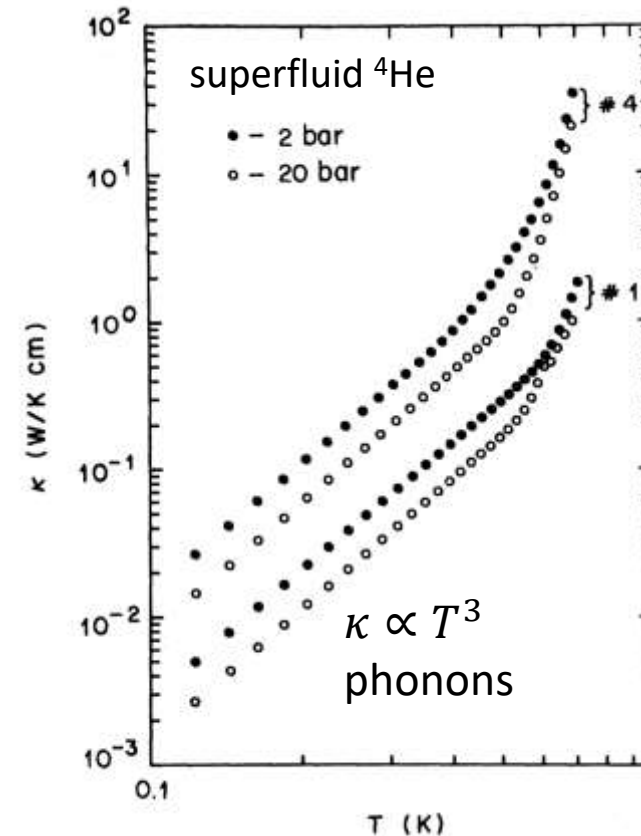
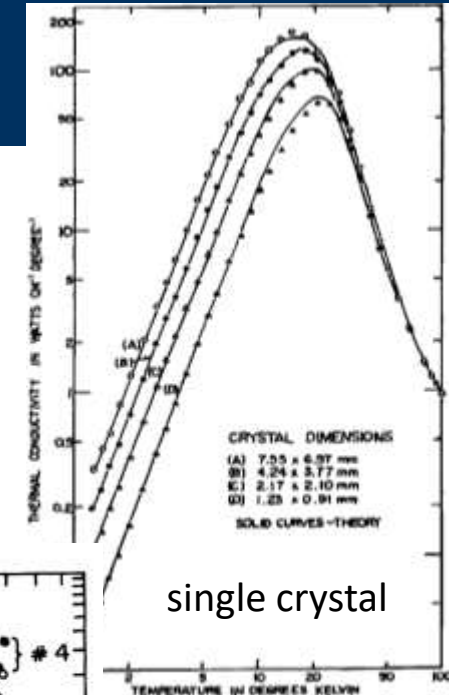
$$\kappa = \frac{1}{3} \langle v \rangle \ell C_V$$

function of  $d$   
(e.g. Hagen-Poiseuille)

$n \cdot c_V^*$

thermal conductivity  $\kappa$  is proportional to

- mean velocity of the particles
- mean free path
- density of the particles
- specific heat carried by particles



Dennis S. Greywall  
Phys. Rev. B 23, 2152 (1981)





Walther  
Meißner  
Institut

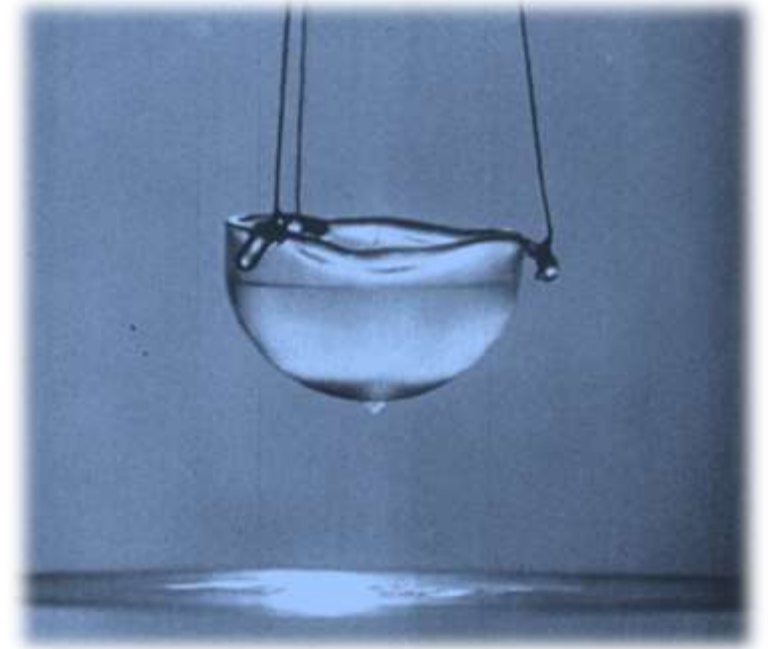
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# Superconductivity and Low Temperature Physics II



**Lecture No. 5  
02 June 2022**

**R. Gross  
© Walther-Meißner-Institut**

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# I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics

- starting point for hydrodynamic description are **Navier-Stokes equations**:

- **normal fluid**:

$$\underbrace{\rho_n \frac{D\mathbf{v}_n}{Dt}}_{\text{inertia}} = - \underbrace{\frac{\rho_n}{\rho} \nabla p}_{\text{pressure gradient}} - \underbrace{\frac{\rho_n \sigma_n \nabla T}{\rho}}_{\substack{\text{temperature} \\ \text{gradient}}} - \underbrace{\frac{\rho_s \rho_n}{2\rho} \nabla (\mathbf{v}_n - \mathbf{v}_s)^2}_{\text{add. term due to compressibility}} + \underbrace{\eta_n \nabla^2 \mathbf{v}_n}_{\text{viscosity}}$$

entropy per mass

$\rho_s, \rho_n$  = superfluid and normal fluid densities  
 $\mathbf{v}_s, \mathbf{v}_n$  = superfluid and normal fluid velocities  
 $\eta_n$  = normal fluid viscosity

terms have unit energy/volume

with the definition of the material derivative  $D\mathbf{v}/Dt$ :

$$\frac{D\mathbf{v}_n}{Dt} = \frac{\partial \mathbf{v}_n}{\partial t} + (\nabla \cdot \mathbf{v}_n) \mathbf{v}_n \quad \text{and} \quad \nabla \cdot \mathbf{v}_n = \nabla \left( \frac{\|\mathbf{v}_n\|^2}{2} \right) + \underbrace{(\nabla \times \mathbf{v}_n) \times \mathbf{v}_n}$$

the tensor derivative of the velocity vector

curl of the velocity (called vorticity)  
 disappears for irrotational flow  
 (no vortices)

# I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics

➤ **superfluid (no viscous friction):**

$$\underbrace{\rho_s \frac{D\mathbf{v}_s}{Dt}}_{\text{inertia}} = - \underbrace{\frac{\rho_s}{\rho} \nabla p}_{\text{pressure gradient}} - \underbrace{\rho_s \sigma_s \nabla T}_{=0 \text{ as superfluid carries no entropy}} - \underbrace{\frac{\rho_s \rho_n}{2\rho} \nabla(\mathbf{v}_n - \mathbf{v}_s)^2}_{\text{add. term due to compressibility}} + \underbrace{\eta_s \nabla^2 \mathbf{v}_s}_{=0 \text{ as } \eta_s=0}$$

$\rho_s, \rho_n$  = superfluid and normal fluid densities  
 $\mathbf{v}_s, \mathbf{v}_n$  = superfluid and normal fluid velocities  
 $\eta_n$  = normal fluid viscosity

terms have unit energy/volume

with the definition of the material derivative  $D\mathbf{v}/Dt$ :

$$\frac{D\mathbf{v}_s}{Dt} = \frac{\partial \mathbf{v}_s}{\partial t} + (\nabla \cdot \mathbf{v}_s) \mathbf{v}_s \quad \text{and} \quad \nabla \cdot \mathbf{v}_s = \nabla \left( \frac{\|\mathbf{v}_s\|^2}{2} \right) + \underbrace{(\nabla \times \mathbf{v}_s) \times \mathbf{v}_s}_{\text{curl of the velocity (called vorticity) disappears for irrotational flow (no vortices)}}$$

the tensor derivative of the velocity vector

- ➔ **Euler type equation for superfluid** (if there are no vortices)
- ➔ detailed treatment including vortices: **Gross-Pitaevskii equation**

# I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics (1) - a simplified description

- mass flow density: 
$$\mathbf{J} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s$$

$\nearrow$ 
 $\uparrow$   
*normal*    *superfluid component*

- conservation of mass: 
$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$$

- viscosity of normal component is very small and can be neglected in most situations
  - in first order approximation we treat superfluid He as ideal liquid
  - description by **Euler equation** (*motion of a compressible, inviscid fluid*)

$$\frac{\partial \mathbf{J}}{\partial t} + \underbrace{\rho \mathbf{v} \cdot \nabla \mathbf{v}}_{\approx 0} = -\nabla p \quad (\mathbf{p} = \text{pressure})$$

→ if  $\mathbf{v}$  is small, quadratic terms in  $\mathbf{v}$  can be neglected

- since we are neglecting dissipative effects (no viscous friction), entropy is conserved:

$$\frac{\partial S}{\partial t} = \frac{\partial(\rho\sigma)}{\partial t} = -\nabla \cdot (\rho\sigma) \mathbf{v}_n \quad \sigma = \text{entropy per mass}$$

# I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics (2)

- exact derivation of equation of motion of normal and superfluid phase is difficult  
 → we use approach following „Gedanken“ experiment of Landau

- we are adding superfluid component at constant volume and consider change of inner energy

→ the resulting operation only changes the mass of the superfluid component, no entropy and volume change)

$$dU = TdS - pdV + \underbrace{\mu_m}_{\substack{\text{potential energy} \\ \text{per mass}}} dm \Rightarrow -\nabla(\mu_m dm) = \mathbf{F} = dm \frac{d\mathbf{v}_s}{dt}$$

- since adding particles to the superfluid does not change the energy, we have

$$0 = TdS - pdV + \mu_m dm \Rightarrow \frac{d\mathbf{v}_s}{dt} = \frac{\Delta S}{\Delta m} \nabla T - \frac{\Delta V}{\Delta m} \nabla p = \sigma \nabla T - \frac{1}{\rho} \nabla p$$

- with  $\frac{d\mathbf{v}_s}{dt} = \frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{v}_s \approx \frac{\partial \mathbf{v}_s}{\partial t}$  (neglecting quadratic terms in  $\mathbf{v}_s$ ) we obtain

$$\frac{\partial \mathbf{v}_s}{\partial t} = \sigma \nabla T - \frac{1}{\rho} \nabla p$$

# I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics (3)

- for the normal fluid component we obtain

$$\frac{\partial \mathbf{J}}{\partial t} = \frac{\partial(\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s)}{\partial t} = -\nabla \mathbf{p} \quad \Rightarrow \quad \frac{\partial \mathbf{v}_n}{\partial t} = -\frac{1}{\rho_n} \nabla \mathbf{p} - \frac{\rho_s}{\rho_n} \frac{\partial \mathbf{v}_s}{\partial t}$$

with  $\frac{\partial \mathbf{v}_s}{\partial t} = \sigma \nabla T - \frac{1}{\rho} \nabla p$  we then obtain

$$\frac{\partial \mathbf{v}_n}{\partial t} = -\frac{1}{\rho_n} \nabla \mathbf{p} - \frac{\rho_s}{\rho_n} \sigma \nabla T + \frac{\rho_s}{\rho_n} \frac{1}{\rho} \nabla p = -\frac{\rho_s}{\rho_n} \sigma \nabla T + \frac{\rho_s - \rho}{\rho_n \rho} \nabla p = -\frac{\rho_s}{\rho_n} \sigma \nabla T - \frac{1}{\rho} \nabla p$$

$$\frac{\partial \mathbf{v}_n}{\partial t} = -\frac{\rho_s}{\rho_n} \sigma \nabla T - \frac{1}{\rho} \nabla p$$

- equations of motion

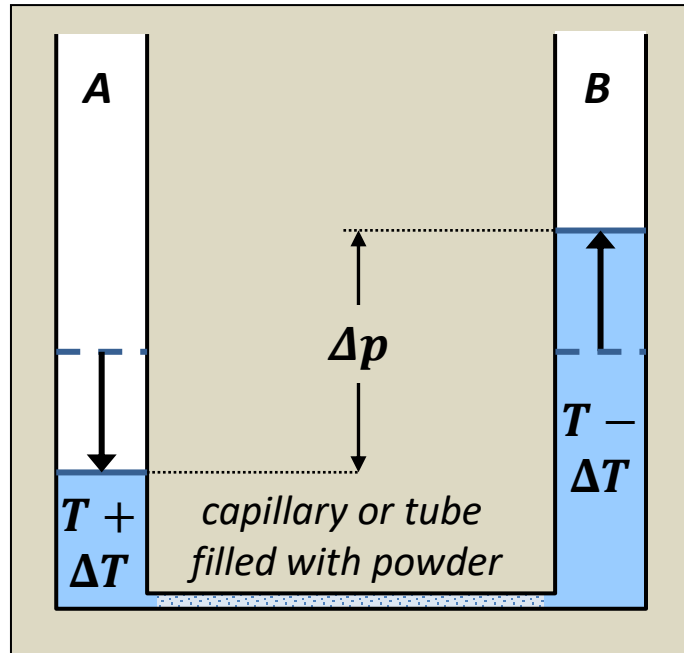
$$\frac{\partial \mathbf{v}_s}{\partial t} = \sigma \nabla T - \frac{1}{\rho} \nabla p$$

$$\frac{\partial \mathbf{v}_n}{\partial t} = -\frac{\rho_s}{\rho_n} \sigma \nabla T - \frac{1}{\rho} \nabla p$$

**note:**  
the equations of motion are valid only for the linear regime !

# I.3.3 Two-Fluid Hydrodynamics

- example: explanation of thermomechanical effect:



$$\frac{\partial \mathbf{v}_s}{\partial t} = \sigma \nabla T - \frac{1}{\rho} \nabla p$$

– in steady state:  $\frac{\partial \mathbf{v}_s}{\partial t} = 0$

$$0 = \sigma \nabla T - \frac{1}{\rho} \nabla p$$

$$\Delta p = \sigma \rho \Delta T$$



# I.3.3 Two-Fluid Model

- two-fluid hydrodynamics can be used to understand sound propagation in superfluid helium

- *first sound:*

normal and superfluid component move *in-phase*:

$$v_n = v_s$$

$$v_1 = 238 \text{ m/s for } T \rightarrow 0$$

- *second sound:*

normal and superfluid component move *out-of-phase*:  
(Peshkov 1944)

$$\rho_n v_n + \rho_s v_s = 0$$

$$v_2 = \frac{v_1}{\sqrt{3}} \simeq 137 \text{ m/s for } T \rightarrow 0$$

- *third sound: waves propagating in thin He films*

- *fourth sound: compression wave propagating in a super leak*

use two-fluid hydrodynamics to understand this behavior

(super leak: very small opening or capillary, in which normal fluid cannot move because of its finite viscosity)

# I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics – sound propagation (1)

- take time derivative of  $\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$  and insert into  $\frac{\partial \mathbf{J}}{\partial t} + \underbrace{\rho \mathbf{v} \cdot \nabla \mathbf{v}}_{\approx 0} = -\nabla p$

$$\frac{\partial^2 \rho}{\partial t^2} = -\nabla \cdot \frac{\partial \mathbf{J}}{\partial t} = -\nabla \cdot (-\nabla p) \Rightarrow \frac{\partial^2 \rho}{\partial t^2} = \nabla^2 p$$

- elimination of terms with  $\mathbf{v}_n$  and  $\mathbf{v}_s$  in  $\frac{\partial S}{\partial t} = \frac{\partial(\rho\sigma)}{\partial t} = -\nabla \cdot (\rho\sigma) \mathbf{v}_n$  and  $\frac{\partial \mathbf{v}_s}{\partial t} = \sigma \nabla T - \frac{1}{\rho} \nabla p$  (cannot be observed experimentally)

→ after some steps and neglecting higher order terms one obtains

$$\frac{\partial^2 \sigma}{\partial t^2} = \frac{\rho_s}{\rho_n} \sigma^2 \nabla^2 T$$

$$\frac{\partial \mathbf{v}_s}{\partial t} = \sigma \nabla T - \frac{1}{\rho} \nabla p$$

$$\frac{\partial \mathbf{v}_n}{\partial t} = -\frac{\rho_s}{\rho_n} \sigma \nabla T - \frac{1}{\rho} \nabla p$$

- in the two equations we have four variables:  $\rho, \sigma, p$  and  $T$  → only two of them are independent, we choose  $\rho$  and  $\sigma$

- dependence of  $p$  and  $T$  on  $\rho$  and  $\sigma$  can be expressed as

$$\delta p = \left( \frac{\partial p}{\partial \rho} \right)_\sigma \delta \rho + \left( \frac{\partial p}{\partial \sigma} \right)_\rho \delta \sigma \quad \delta T = \left( \frac{\partial T}{\partial \rho} \right)_\sigma \delta \rho + \left( \frac{\partial T}{\partial \sigma} \right)_\rho \delta \sigma$$

we insert relations into expression for  $\frac{\partial^2 \rho}{\partial t^2}$  and  $\frac{\partial^2 \sigma}{\partial t^2}$

# I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics – sound propagation (2)

- insertion yields

$$\frac{\partial^2 \rho}{\partial t^2} = \left( \frac{\partial p}{\partial \rho} \right)_\sigma \nabla^2 \rho + \left( \frac{\partial p}{\partial \sigma} \right)_\rho \nabla^2 \sigma$$

$$\frac{\partial^2 \sigma}{\partial t^2} = \frac{\rho_s}{\rho_n} \sigma^2 \left[ \left( \frac{\partial T}{\partial \rho} \right)_\sigma \nabla^2 \rho + \left( \frac{\partial T}{\partial \sigma} \right)_\rho \nabla^2 \sigma \right]$$

2<sup>nd</sup> order partial differential equations

- to solve these equations we use the ansatz (1D for simplicity)

$$\rho = \rho_0 + \rho_1 \exp \left[ i\omega \left( t - \frac{x}{v} \right) \right] \quad \sigma = \sigma_0 + \sigma_1 \exp \left[ i\omega \left( t - \frac{x}{v} \right) \right]$$

- inserting the ansatz yields

*linear system  
of equations*

$$\left[ \left( \frac{v}{v_1} \right)^2 - 1 \right] \rho_1 + \left( \frac{\partial p}{\partial \sigma} \right)_\rho \left( \frac{\partial \rho}{\partial p} \right)_\sigma \sigma_1 = 0$$

$$\left( \frac{\partial T}{\partial \rho} \right)_\sigma \left( \frac{\partial \sigma}{\partial T} \right)_\rho \rho_1 + \left[ \left( \frac{v}{v_2} \right)^2 - 1 \right] \sigma_1 = 0$$

with abbreviations

$$v_1^2 = \left( \frac{\partial p}{\partial \rho} \right)_\sigma$$

$$v_2^2 = \frac{\rho_s}{\rho_n} \sigma^2 \left( \frac{\partial T}{\partial \sigma} \right)_\rho$$

# I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics – sound propagation (3)

- constraint equation for coefficients

$$\left[ \left( \frac{v}{v_1} \right)^2 - 1 \right] \cdot \left[ \left( \frac{v}{v_2} \right)^2 - 1 \right] = \left( \frac{\partial p}{\partial \sigma} \right)_\rho \left( \frac{\partial \rho}{\partial p} \right)_\sigma \left( \frac{\partial T}{\partial \rho} \right)_\sigma \left( \frac{\partial \sigma}{\partial T} \right)_\rho$$

- the r.h.s. can be rewritten by using known thermodynamic relations

$$\left[ \left( \frac{v}{v_1} \right)^2 - 1 \right] \cdot \left[ \left( \frac{v}{v_2} \right)^2 - 1 \right] = \frac{C_p - C_V}{C_p} \quad C_p, C_V: \text{specific heat at constant pressure/volume}$$

- for superfluid helium the difference  $C_p - C_V$  is very small  $\rightarrow$  we can use the approximation

$$\left[ \left( \frac{v}{v_1} \right)^2 - 1 \right] \cdot \left[ \left( \frac{v}{v_2} \right)^2 - 1 \right] \approx 0 \quad v_1^2 = \left( \frac{\partial p}{\partial \rho} \right)_\sigma \quad v_2^2 = \frac{\rho_s}{\rho_n} \sigma^2 \left( \frac{\partial T}{\partial \sigma} \right)_\rho$$

# I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics: **first sound**

- propagation of usual sound waves with velocity  $v = v_1$  is given by

$$\left[ \left( \frac{v}{v_1} \right)^2 - 1 \right] \rho_1 + \left( \frac{\partial p}{\partial \sigma} \right)_\rho \left( \frac{\partial \rho}{\partial p} \right)_\sigma \sigma_1 = 0 \quad \text{if we set } \rho_1 \neq 0 \text{ and } \sigma_1 = 0$$

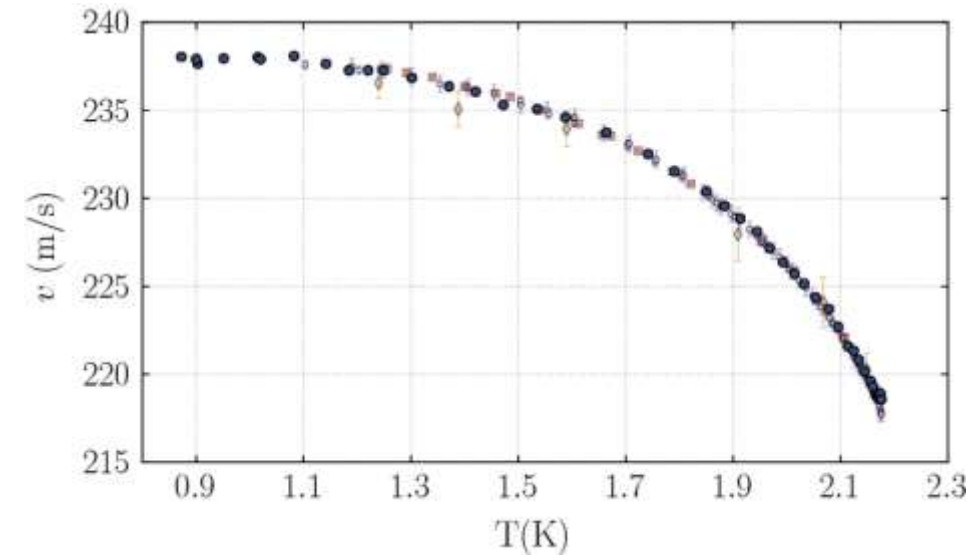
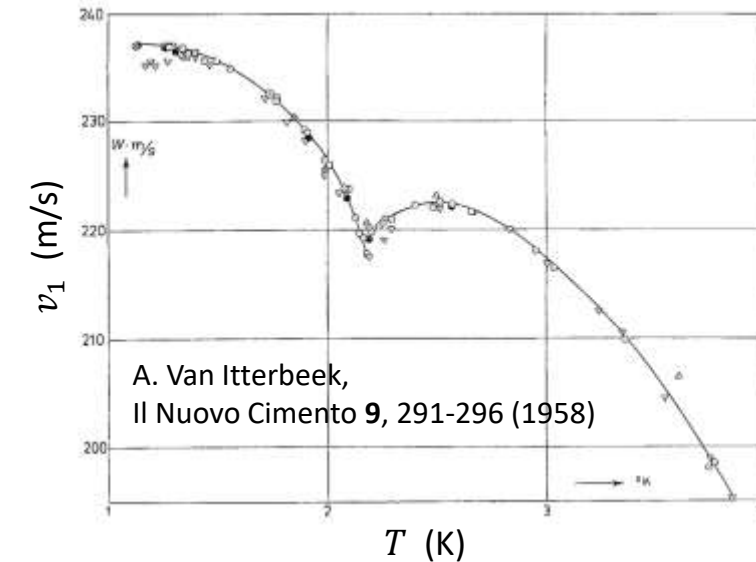
under these conditions we have  $\nabla T = 0$  as for a usual sound waves (**first sound**)

- from  $\frac{\partial \mathbf{J}}{\partial t} = -\nabla \mathbf{p}$  and  $\frac{\partial \mathbf{v}_s}{\partial t} = \sigma \nabla T - \frac{1}{\rho} \nabla p$  we obtain

$$v_n = v_s \quad (\text{acoustic mode})$$

→ **normal and superfluid component move in phase (first sound)**

- density changes are adiabatic
- superfluid He behaves as usual liquid
- sound velocity  $v = v_1 = 238 \text{ m/s}$  for  $T \rightarrow 0$
- $v_1(T)$  shows anomaly at  $T = T_\lambda$



L. Djadaojee, A. Douillet, J. Grucker, Jules, J. Low Temp. Phys. 203 (2021)

# I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics: **second sound**

- if we set  $\rho_1 = 0$  and  $\sigma_1 \neq 0$  we obtain temperature wave propagating with velocity  $v = v_2$

$$\left(\frac{\partial T}{\partial \rho}\right)_\sigma \left(\frac{\partial \sigma}{\partial T}\right)_\rho \rho_1 + \left[\left(\frac{v}{v_2}\right)^2 - 1\right] \sigma_1 = 0$$

- from  $\frac{\partial^2 \rho}{\partial t^2} = \nabla^2 p$  we obtain  $\nabla p = 0$

- together with Euler's equation  $\frac{\partial \mathbf{J}}{\partial t} + \underbrace{\rho \mathbf{v} \cdot \nabla \mathbf{v}}_{\approx 0} = -\nabla p$  we obtain:  $\frac{\partial \mathbf{J}}{\partial t} = \frac{\partial(\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s)}{\partial t} = 0$

- since  $\mathbf{J}$  cannot be constant in a closed container  $\rightarrow \mathbf{J} = 0$

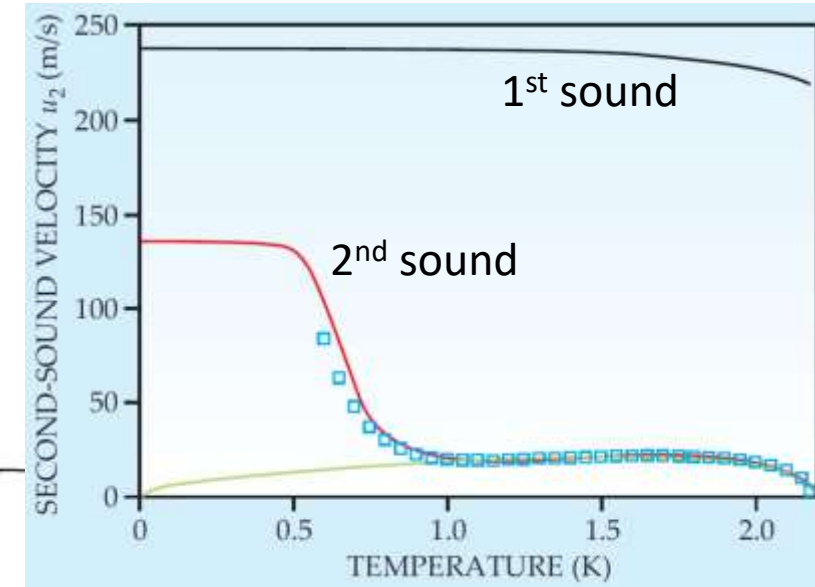
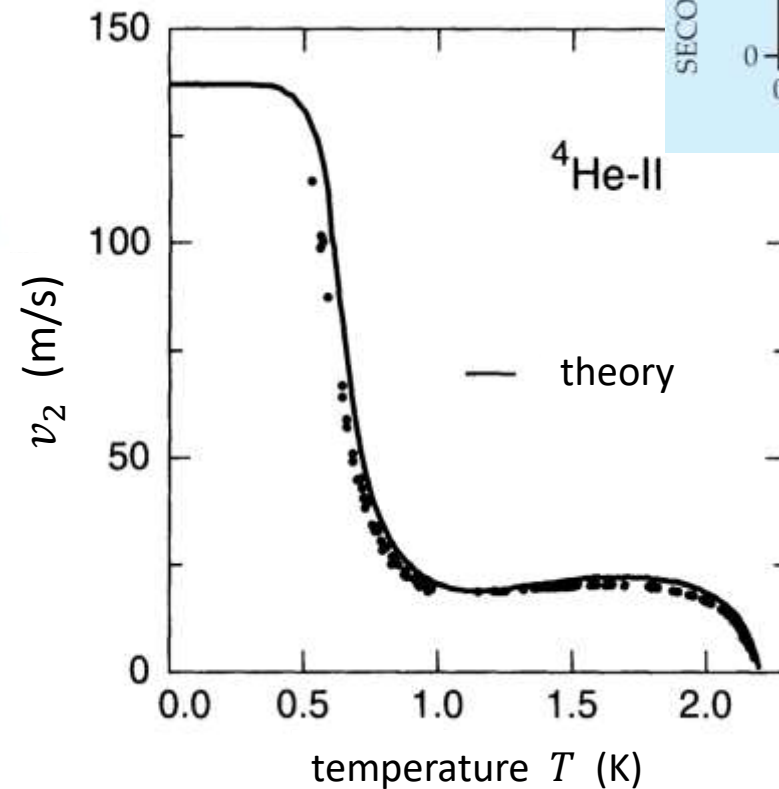
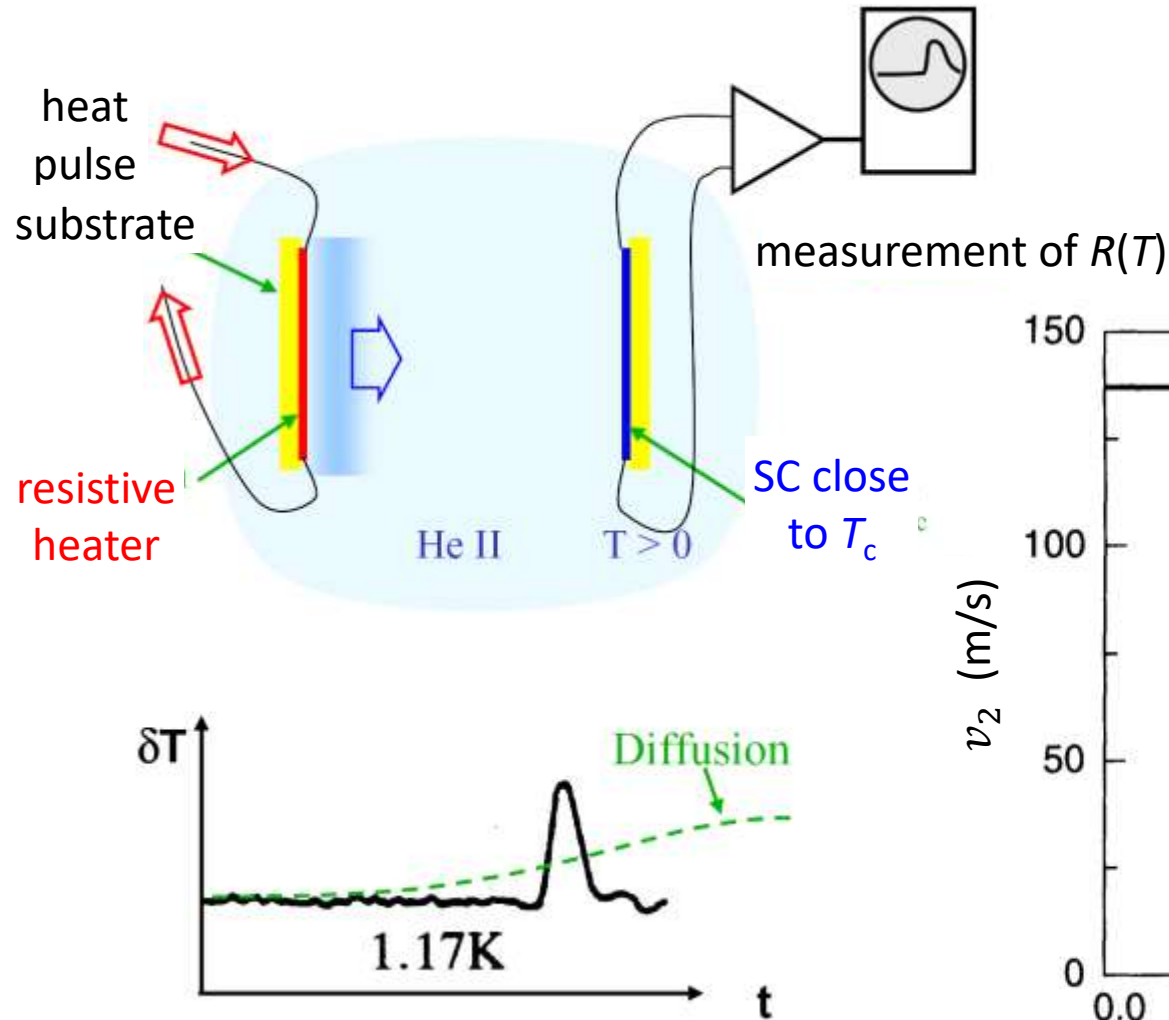
$$\rho_n v_n + \rho_s v_s = 0 \quad (\text{optical mode})$$

- velocity of the temperature wave

$$v_2^2 = \frac{\rho_s}{\rho_n} \sigma^2 \left(\frac{\partial T}{\partial \sigma}\right)_\rho = \frac{\rho_s}{\rho_n} \sigma^2 \frac{T}{C_p}, \quad v_2 = \frac{v_1}{\sqrt{3}} \simeq 137 \text{ m/s for } T \rightarrow 0$$

# I.3.3 Two-Fluid Hydrodynamics

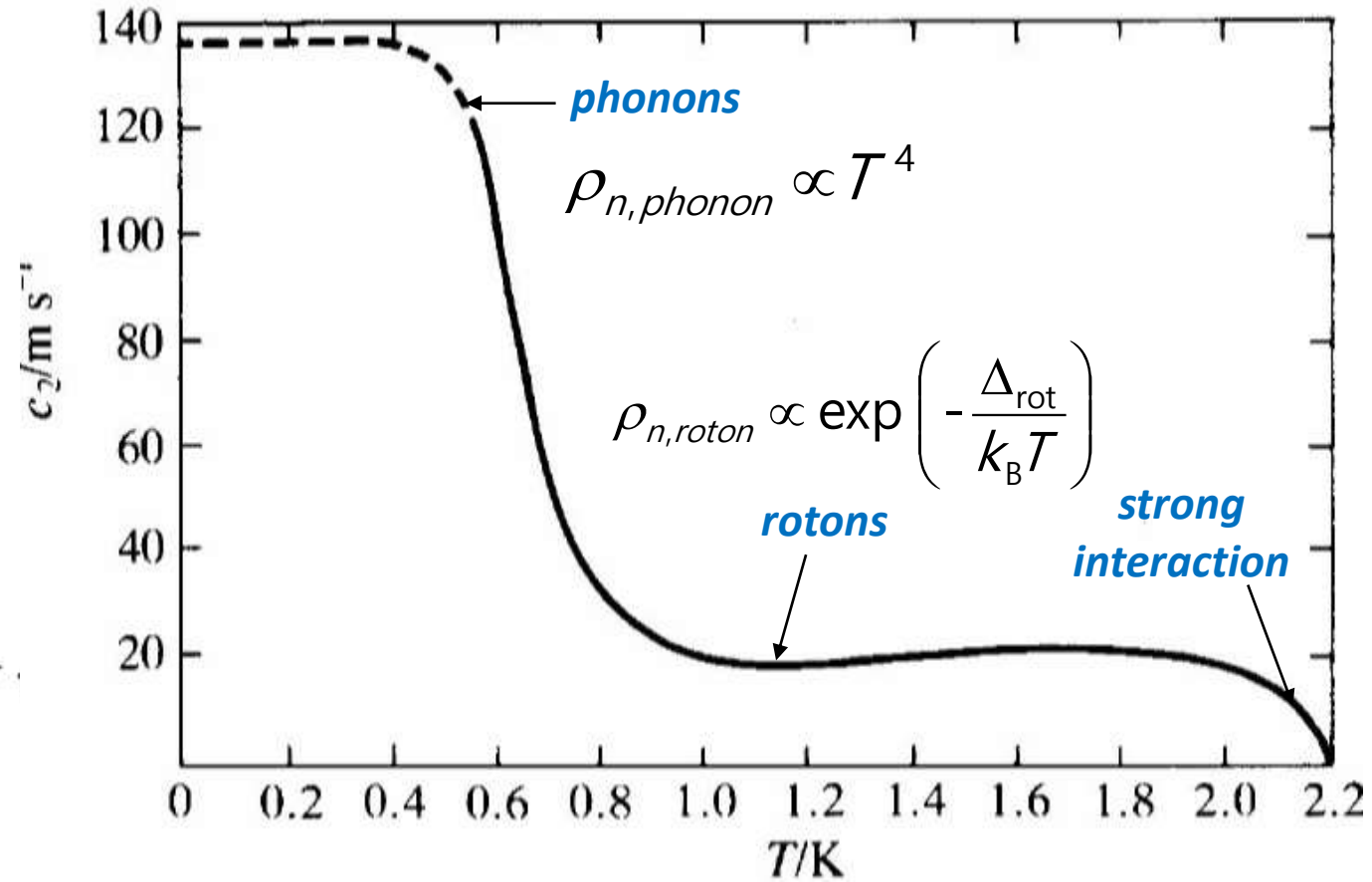
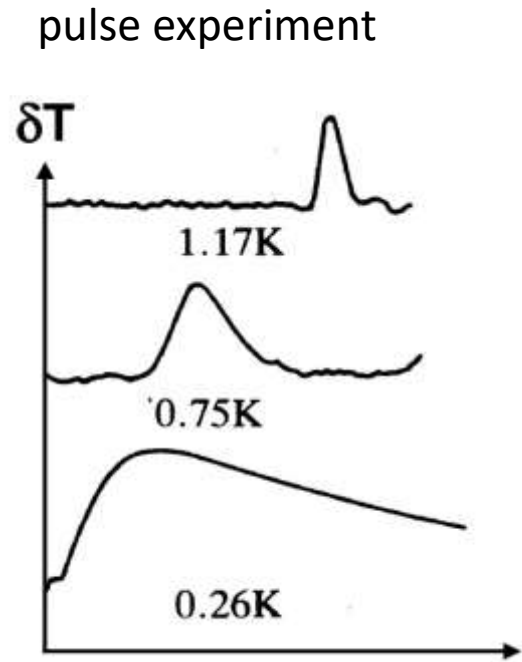
- two-fluid hydrodynamics: **second sound** – experiment



Enns, Hunklinger,  
Low Temperature Physics, Springer (2000)

# I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics: **second sound – experiment**



$T \rightarrow T_\lambda:$

- collision probability increases
- picture of elementary excitations no longer valid
- vortex rings



# I.3.3 Two-Fluid Hydrodynamics

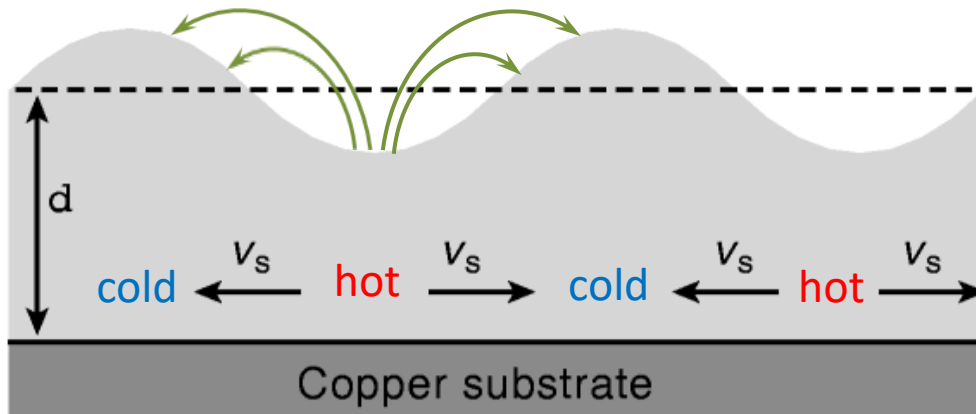
- two-fluid hydrodynamics: **third sound – thickness waves in thin helium films**

- we already discussed the thickness of helium films on surfaces

$$d = \left( \frac{-\alpha}{k_B T \ln(p/p_{\text{sat}})} \right)^{1/3} = \left( \frac{\alpha}{k_B T (\ln p_{\text{sat}} - \ln p)} \right)^{1/3}$$

- ➔ film thickness depends on pressure and temperature

- ➔ local periodic heating causes thickness variation propagating along surface

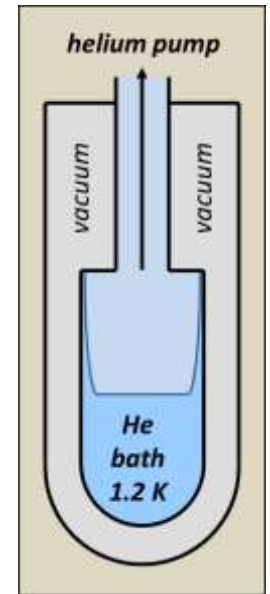


- viscous normal component is fixed at surface
- superfluid component oscillates freely parallel to surface

wave crest: surplus of superfluid → **cold**

wave valley: deficit of superfluid → **hot**

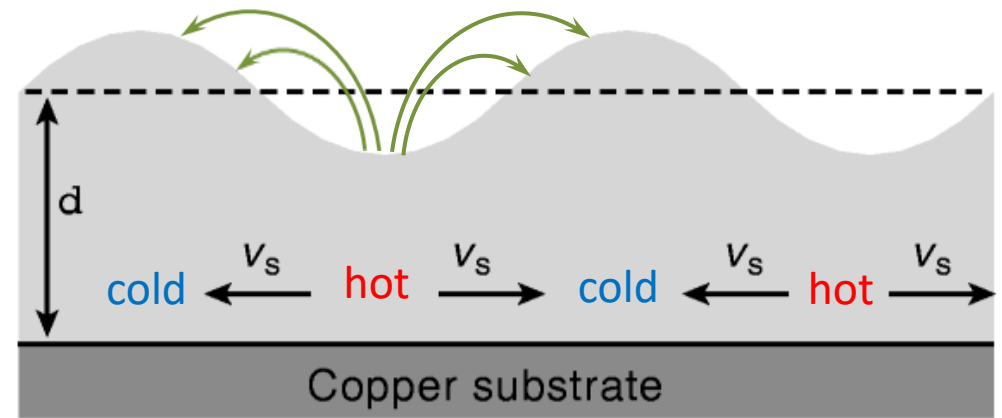
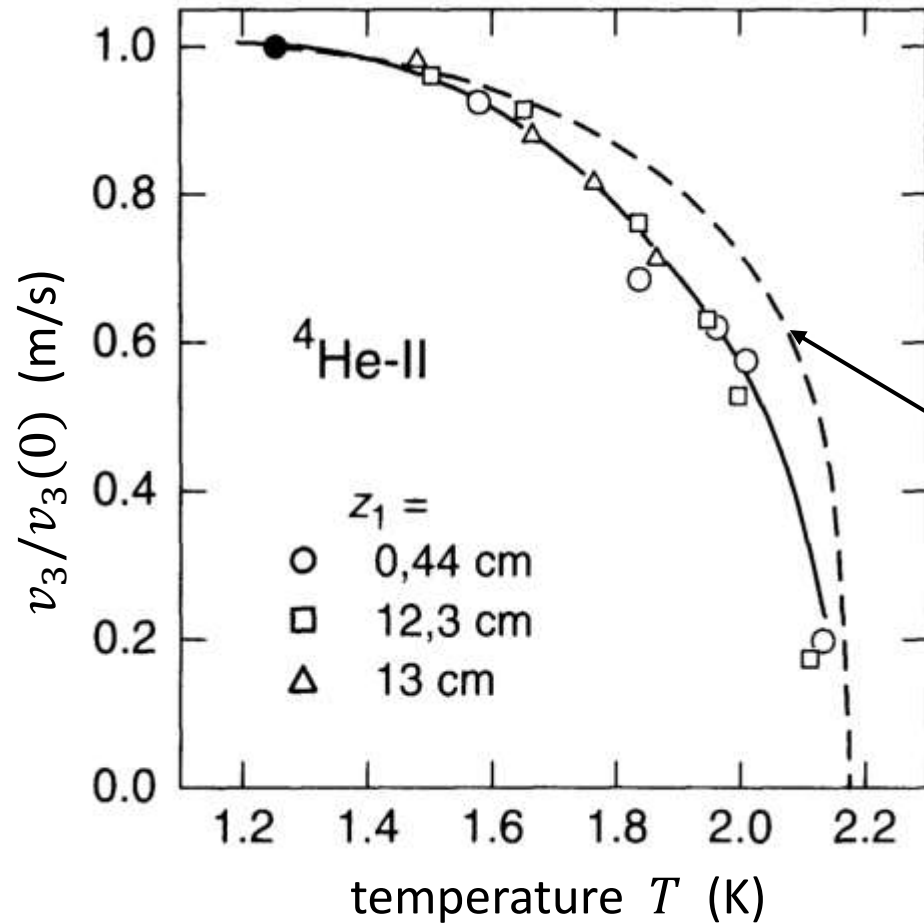
sound velocity:  $v_3 \sim 10 \dots 100 \text{ m/s}$



**detailed discussion complicated due to additional evaporation and condensation at surface**

# I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics: **third sound** – thickness waves in thin helium films



$$v_3^2 = \frac{3\alpha}{d^3} \frac{\rho_s}{\rho} \left( 1 + \frac{\sigma T}{L} \right)$$

$L = \text{latent heat}$

Enns, Hunklinger,  
Low Temperature Physics, Springer (2000)

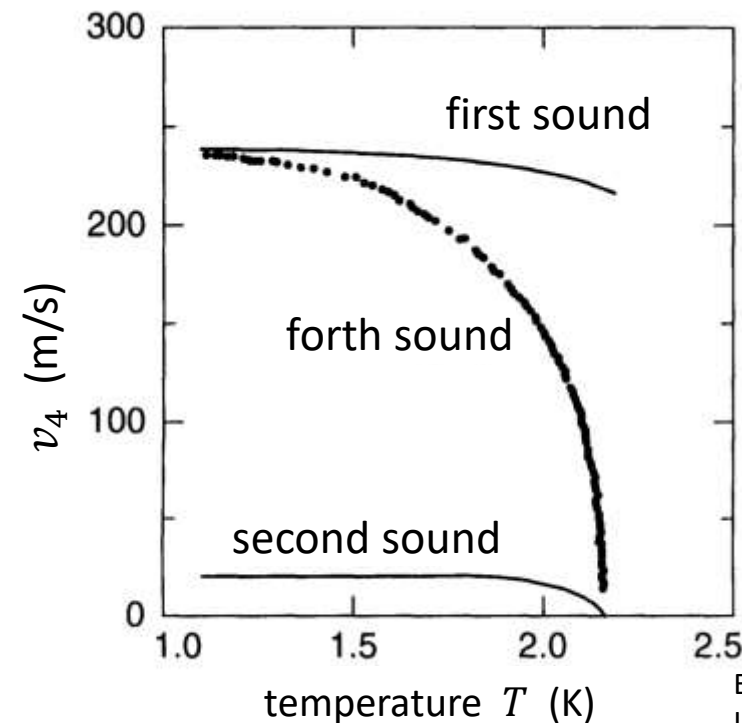
# I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics: **forth sound – compression waves in superleaks**

(super leak = very small opening or capillary, in which normal fluid cannot move because of its finite viscosity)

- only superfluid can move, whereas normal fluid is blocked due to its finite viscosity
- compression waves are not only density oscillations, but also oscillations of the superfluid density  $\rho_s/\rho_n$ , the pressure  $p$ , the temperature  $T$ , and the entropy  $\sigma$
- sound velocity (without derivation)

$$v_4^2 \simeq \frac{\rho_s}{\rho} v_1^2 + \frac{\rho_n}{\rho} v_2^2$$



Enns, Hunklinger,  
Low Temperature Physics, Springer (2000)

# I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics: **heat transport**

- example: mass flow  $\dot{V}_n$  through capillary of diameter  $d$ :
  - the normal fluid component flows from hot to cold side
  - this mass flow is compensated by opposite flow of superfluid component
  - due to different entropies of both components this results in a finite heat flow (limited by  $\eta_n$ )
- mass flow  $\dot{V}_n$  of normal component according to classical hydrodynamics

$$\dot{V}_n = \frac{\beta}{\eta_n} \frac{\Delta p}{L}$$

$\eta_n$ : viscosity of normal component

$L$ : length of capillary

$\Delta p$ : pressure difference

- the parameter  $\beta$  depends on the geometry of the capillary: e.g.  $\beta \propto r^4$  according to Hagen-Poiseuille law

- heat flow = mass flow times heat/mass:  $\dot{Q} = \rho_n \dot{V}_n \cdot \sigma_n T = \dot{V}_n S_n T$

$$\dot{Q}_n = \rho_n \underbrace{\dot{V}_n}_{\frac{\beta}{\eta_n} \frac{\Delta p}{L}} \cdot \sigma_n T = \rho_n \frac{\beta}{\eta_n} \frac{\Delta p}{L} \sigma_n T \quad \text{with } \Delta p = \rho_n \sigma_n \Delta T \Rightarrow$$

$$\dot{Q}_n = \frac{\beta T}{\eta_n L} (\rho_n \sigma_n)^2 \Delta T$$

# I.3.3 Two-Fluid Hydrodynamics

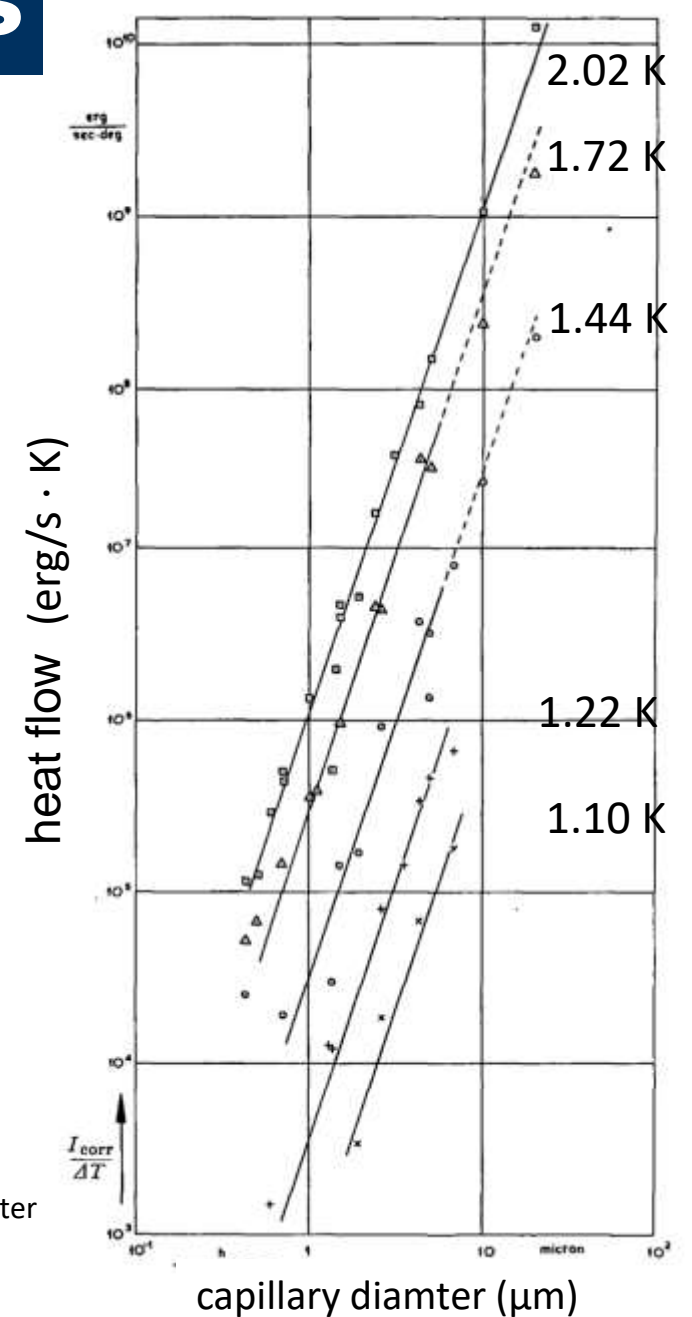
- two-fluid hydrodynamics: **heat transport**

$$\dot{Q}_n = \frac{\beta T}{\eta_n L} (\rho_n \sigma_n)^2 \Delta T$$

- experiment:

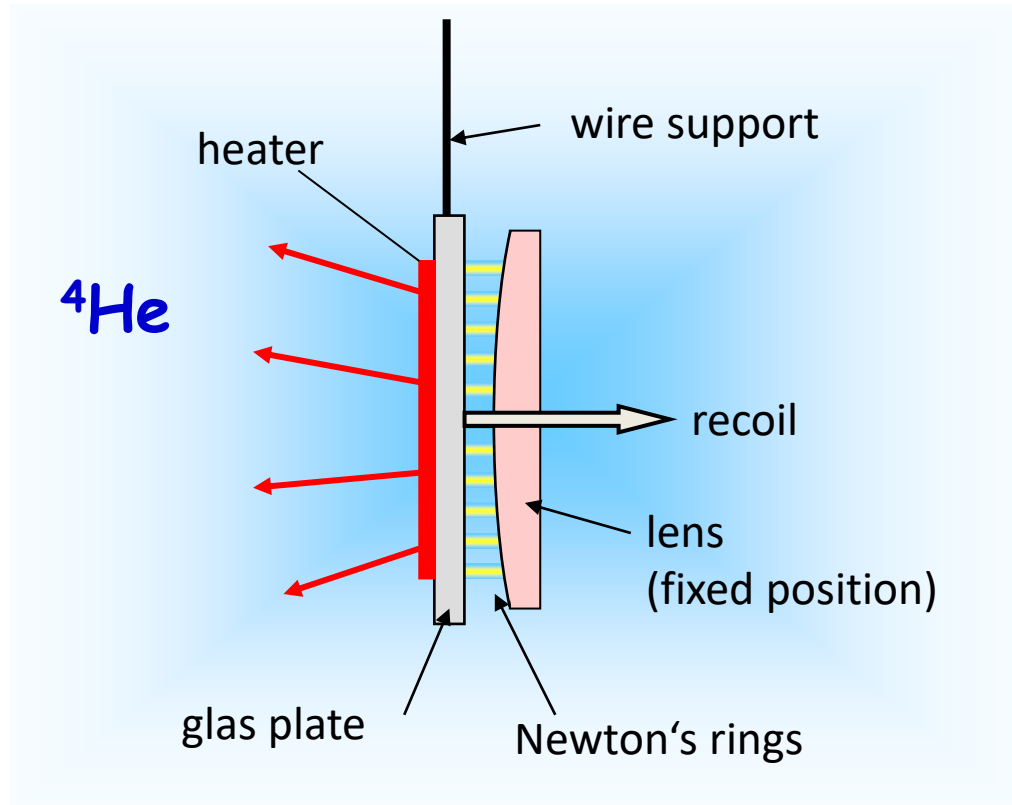
heat flow is found to be proportional to  $\beta \propto d^3$  (for slit of width  $d$ )

A. Broese Van Groenou, J.D. Poll, A.M.G. Delsing, C.J. Gorter  
 Physica 22, 905-910 (1956)



# I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics: **momentum flow associated with heat flow**



- heater generates normal fluid: phonons and rotons
- radiation to the left
- recoil to the right
- measurement of displacement of the glass plate via Newton's rings between glass plate and lens

– momentum flow (momentum / area and time):  $\rho \mathbf{v} \cdot \mathbf{v}$

– resulting pressure on heat source:

$$p = \rho_n v_n^2 + \rho_s v_s^2$$

– with heat flow per area:

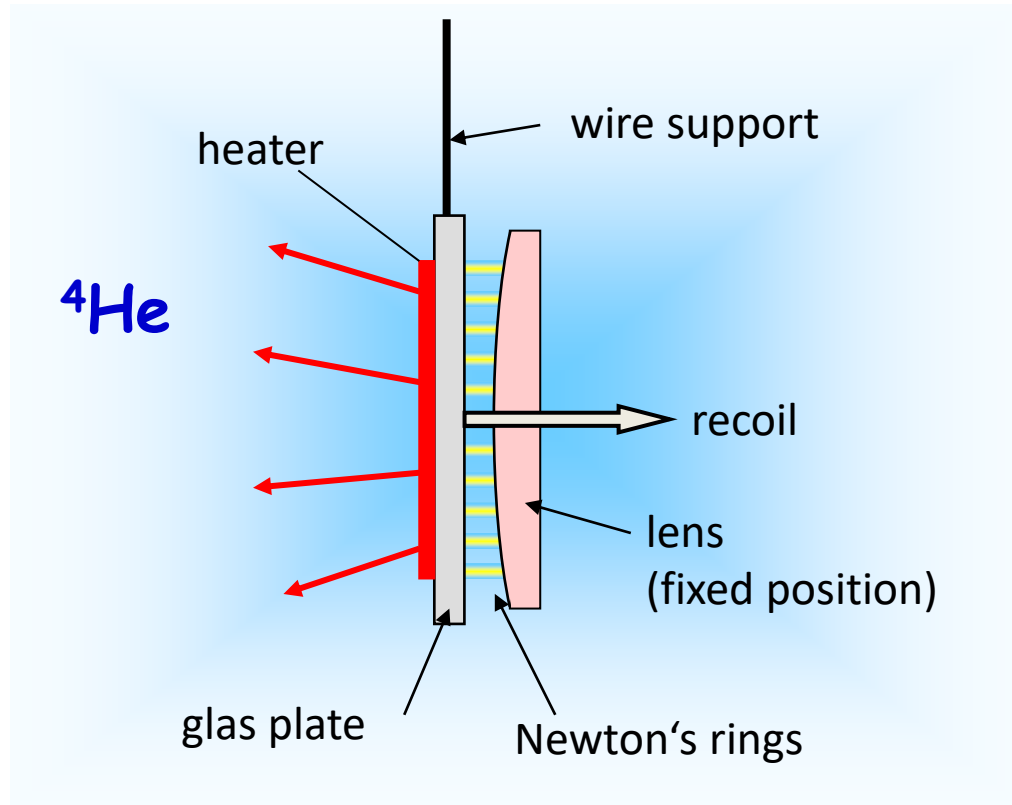
$$J_h = \rho \sigma T v_n$$

– and the fact that there is no mass transport,  
 $\rho_n v_n + \rho_s v_s = 0$ ,  
 we obtain pressure

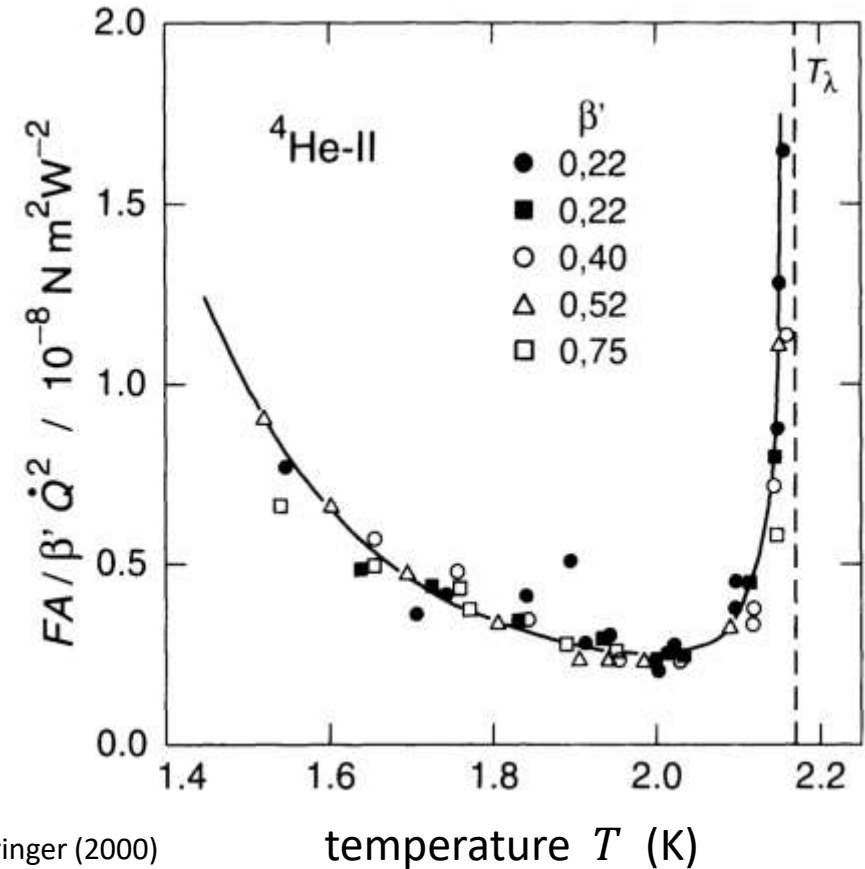
$$p = \frac{\rho_n J_H^2}{\rho_s \rho T^2 \sigma^2} = \frac{J_h^2}{v_2^2 \rho C_p T}$$

# I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics: momentum flow associated with heat flow



$$p = \frac{\rho_n J_H^2}{\rho_s \rho T^2 \sigma^2} = \frac{J_h^2}{v_2^2 \rho C_p T}$$



Enns, Hunklinger,  
Low Temperature Physics, Springer (2000)

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# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

- **important question:** **does the formation of a condensate automatically result in superfluidity? Does an ideal Bose-Einstein condensate show superfluidity as  $^4\text{He-II}$ ?**

**Answer:** *Structure of the excitation spectrum is crucial for observability of superfluidity*  
(Lev Landau, 1947)

– **note:**

- we have used a phenomenological two-fluid model to describe superfluid current density in  $^4\text{He-II}$

$$\mathbf{J}_\rho(\mathbf{r}) = n_0 \frac{\hbar}{m} \underbrace{\nabla\theta(\mathbf{r})}_{v_s} = n_0 \mathbf{v}_s(\mathbf{r})$$

– **still unknown:**

- is there a maximum value for the superfluid velocity  $\mathbf{v}_s$  ?
- if yes, what determines the value of  $\mathbf{v}_s$  ?
- is there a qualitative difference between a condensate of a non-interacting Bose gas and an interacting Bose gas ?

# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

- Gedanken experiment by Lev Landau

- what happens, if we move an object (e.g. sphere) of mass  $M$  at velocity  $\mathbf{v}_i$  through the condensate/superfluid?
  - at  $T = 0$ , the deceleration of the object is possible only by the generation of excitations
  - at which velocity can the moving object generate an **excitation** of energy  $E_p$  and momentum  $\mathbf{p} = \hbar\mathbf{k}$ ?

- with velocity difference  $v_i - v_f$  of object we obtain (*energy and momentum conservation*):

$$E_p = \frac{1}{2} M (v_i^2 - v_f^2) \quad \mathbf{p} = M (\mathbf{v}_i - \mathbf{v}_f)$$

- elimination of  $v_f$  yields

$$E_p = \mathbf{p} \cdot \mathbf{v}_i - \frac{1}{2M} p^2 = \mathbf{p} \cdot \mathbf{v}_i - \mathcal{O}\left(\frac{1}{M}\right) \quad \Rightarrow$$

- there may be a minimum value of  $v_i$  for which the equation can be satisfied
- mass of object can be large → we can neglect 2nd term on rhs

$$v_L = \min \left\{ \frac{E_p}{|\mathbf{p}|} \right\} \quad \text{Landau critical velocity}$$

$v_L =$  smallest  $v_i$  for momentum  $\mathbf{p}$  of the excitation parallel to  $\mathbf{v}_i$

# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

- Gedanken experiment by Lev Landau

$$E_p = \mathbf{p} \cdot \mathbf{v}_i - \frac{1}{2M} p^2 = \mathbf{p} \cdot \mathbf{v}_i - \mathcal{O}\left(\frac{1}{M}\right)$$

- for free bosons:  $E_p = p^2/2m$  ( $m = \text{mass of the boson}$ )

$$v_L = \min \left\{ \frac{E_p}{|\mathbf{p}|} \right\}$$

$$\frac{p^2}{2m} = \mathbf{p} \cdot \mathbf{v}_i \quad \text{equation always satisfied for cone of momentum vectors } |p| = 2mv_i \cos \vartheta, \text{ where } \vartheta = \angle(\mathbf{p}, \mathbf{v}_i)$$

- object (e.g. a sphere) moving in superfluid can always transfer momentum to superfluid and is thereby decelerated →  $v_L = 0$

ideal Bose-Einstein condensate of free (non-interacting) bosons with dispersion  $E_p = p^2/2m$  does not show superfluidity !!

# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

- motivation for Landau critical velocity

$$E_p = \mathbf{p} \cdot \mathbf{v}_i - \frac{1}{2M} p^2 \approx \mathbf{p} \cdot \mathbf{v}_i$$

- deceleration of moving object by generation of excitations with energy  $E_p$  only possible if  $\frac{p^2}{2m} = \mathbf{p} \cdot \mathbf{v}_i$  can be satisfied

$$v_L = \min \left\{ \frac{E_p}{|\mathbf{p}|} \right\}$$

$$\Rightarrow E_p = \hbar\omega = \underbrace{\frac{\partial E_p}{\partial p}}_{\mathbf{v}_i} \cdot \mathbf{p} = \hbar \mathbf{v}_i \cdot \mathbf{k} = \hbar v_i k \cos \vartheta \leq \hbar v_i k$$

$$v_i \geq \frac{\omega}{k} = v_{\text{ph}}$$

$v_{\text{ph}}$  = Phasengeschwindigkeit

$$v_L = \min(v_{\text{ph}})$$

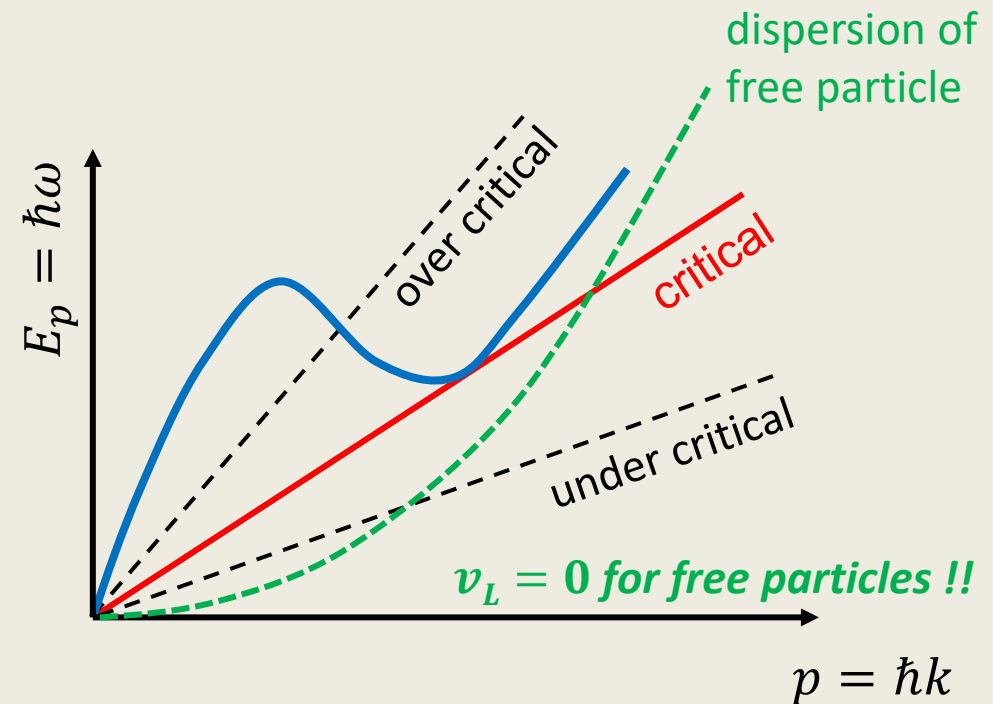
$v_L$  = **Landau critical velocity**  
 $\min(v_{\text{ph}}) = 0$  for free bosons

no deceleration of object, if  $v_i < v_L$

**Laudau:**

**in order to have a superfluid, we need a dispersion relation differing significantly from that of free bosons !**

how to obtain the blue dispersion curve ??



# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

dispersion relation must deviate from that of free bosons,  $E_p = p^2/2m$ , as in this case  $v_L = 0$  !!

- excitation spectrum of superfluid  $^4\text{He}$ : historical aspects (1)



**Nikolai Bogoliubov** (1947):

- interaction effects drastically change the nature of elementary excitations **collective excitations** (many particles move at the same time)
- even for weak repulsive interaction:  $E_p = c |\mathbf{p}|$  (phonon-like dispersion)



**Lev Landau** (1947) and **Richard Feynman** (1953):

- discussion of situation for superfluid  $^4\text{He}$  (stronger interaction)
- postulation of the following excitation spectrum (phonons and rotons):

(i) small  $p$ :

$$\lim_{p \rightarrow 0} E_p = c_s |\mathbf{p}|$$

**phonon-like collective excitations**



(ii) higher  $p$ :

$$\lim_{|p| \rightarrow p_0} E_p = \Delta_{\text{rot}} + \frac{(|\mathbf{p}| - p_0)^2}{2m_{\text{rot}}}$$

**roton-like collective excitations**  
(finite energy gap  $\Delta_{\text{rot}}$  for excitation)

expression „**roton**“ used by Feynman due to analogy with „**smoke ring**“, since it is connected with a forward motion of a particle accompanied by a ring of back-flowing particles

→ **however: detailed nature of rotons still not clarified**

# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

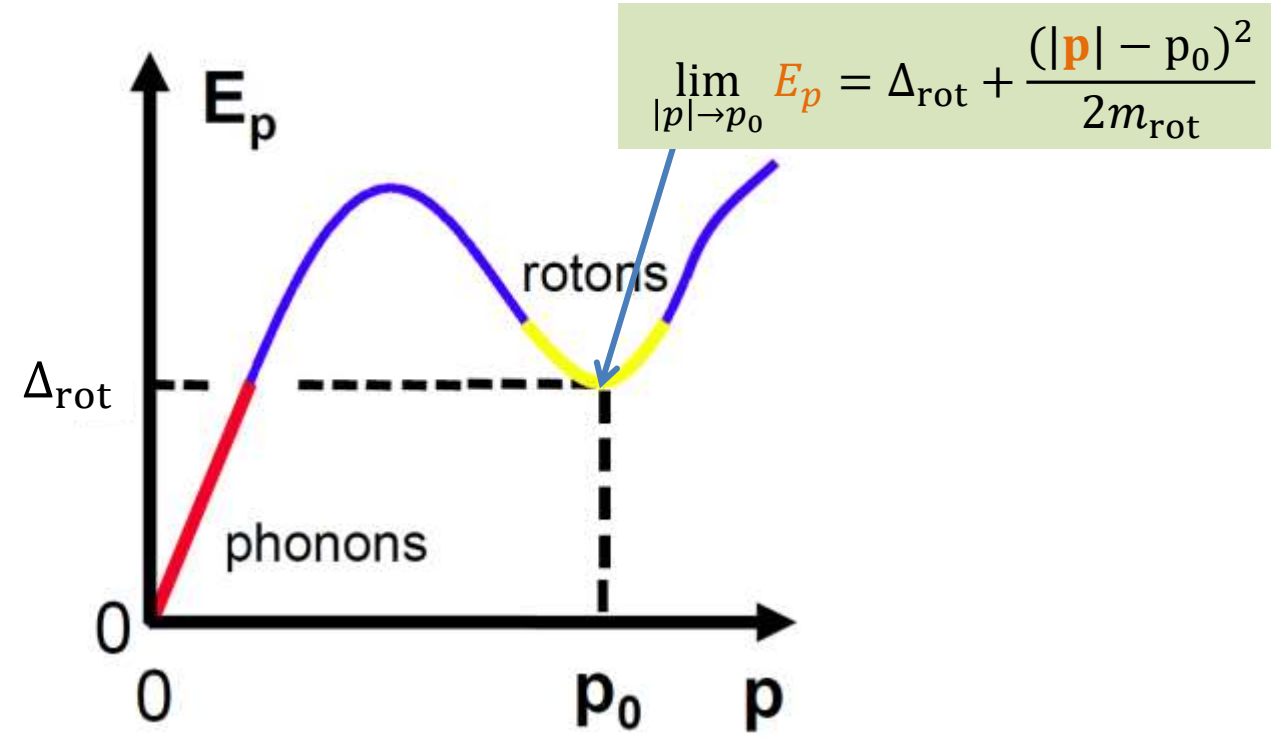
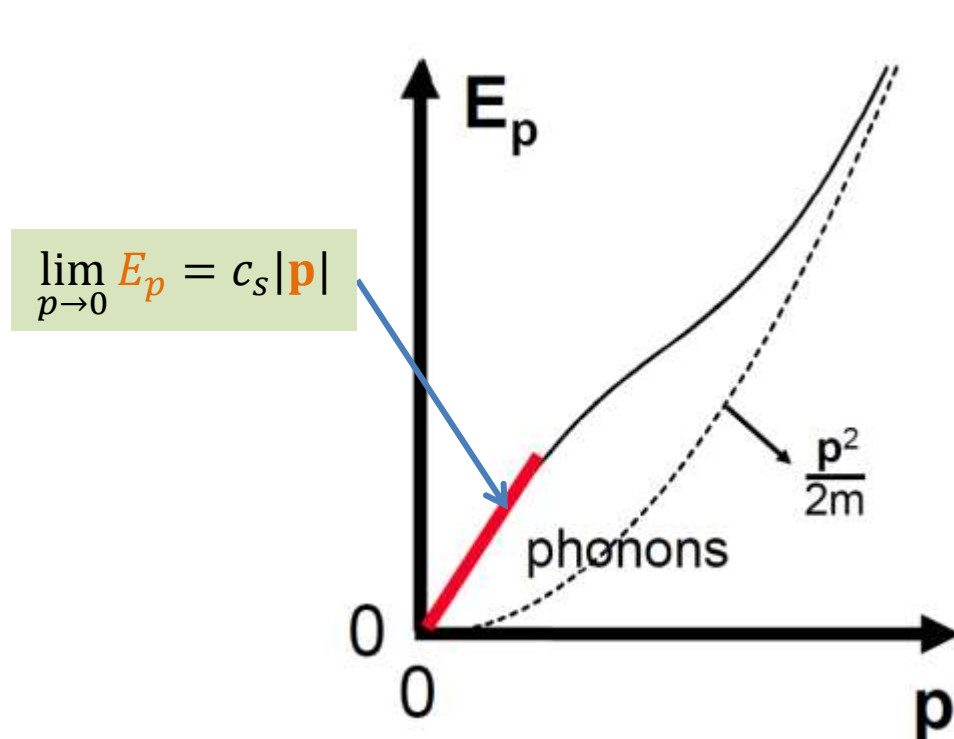
- excitation spectrum of superfluid  $^4\text{He}$ : historical aspects (2)

interacting Bose gas  
 → finite stiffness  
 → solid like excitations

weak interactions:  
*phonons*  
 [Bogoliubov, 1947]

strong interactions:  
*phonons and rotons*

[Landau, 1947, Feynman & Cohen, 1957]



# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

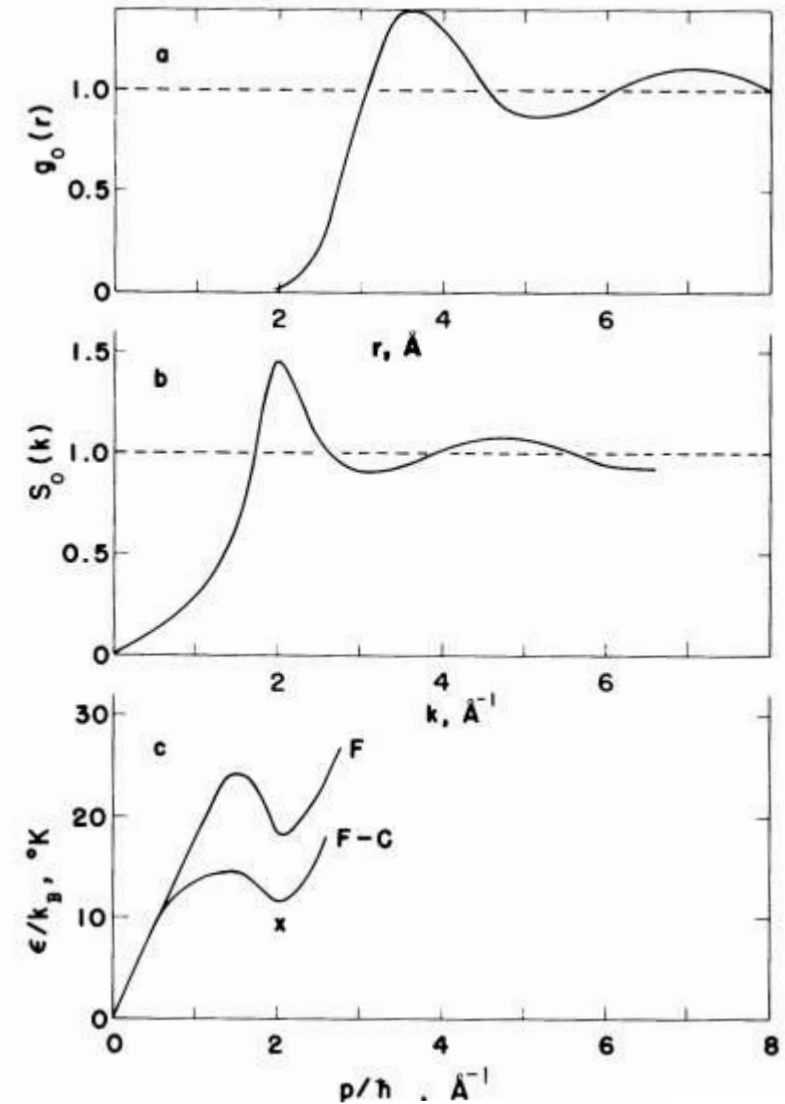
- excitation spectrum of superfluid  $^4\text{He}$ : historical aspects (3)

**Richard P. Feynman:**

(from: „The beat of a different drum“  
Jagdish Mehra, 1994)

„I cannot remember, how it happened.  
I was walking along the street... and  
zing! I understood it!“ ...

... „But of course, that is just what liquids  
must do. If you measure X-ray diffraction,  
then because of the spatial structure of  
the liquid, *which is almost like a solid*,  
there will be a maximum corresponding  
to the first diffraction ring of the X-ray  
pattern! That was a terrific moment!“

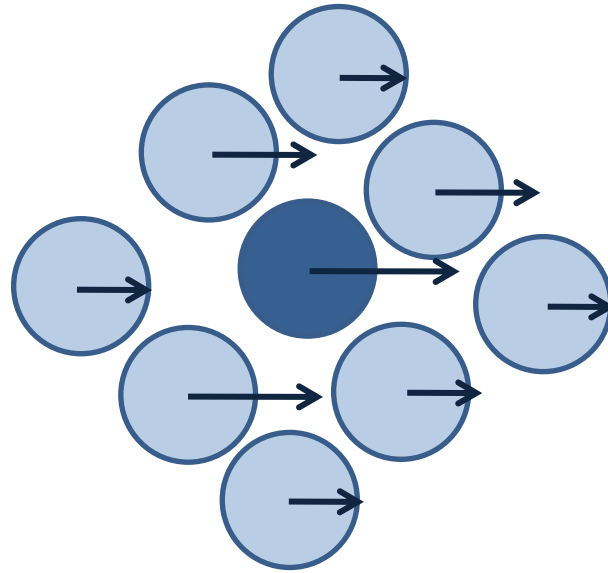


# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

- excitation spectrum of superfluid  $^4\text{He}$ : plausibility considerations

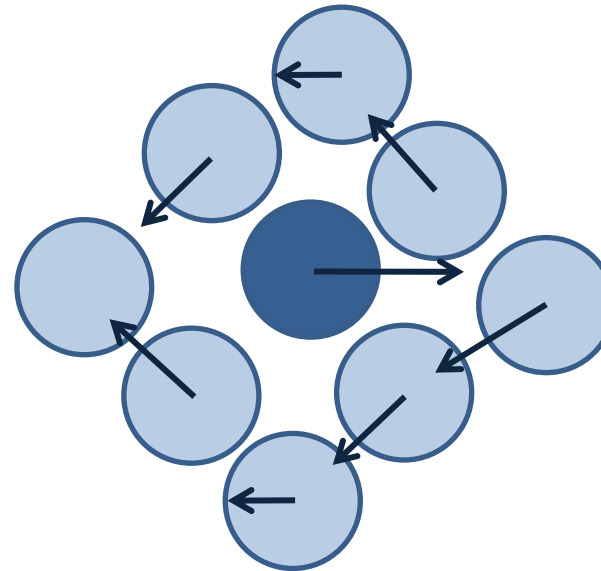
physical interpretation of the phonon and roton part of the excitation spectrum

phonon



- wavelength  $\lambda = 2\pi/k \gg$  atomic distance
- coupled motion of group of atoms (like for acoustic phonon in solid)

roton



- wavelength  $\lambda = 2\pi/k \simeq$  interatomic distance
- central atom moves forward while closely packed neighbors move (backward) out of the way in circular motion (like smoke ring)

in both cases we have collective excitations with dispersion  $E_p(k) \rightarrow$  **quasiparticles**



# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

- excitation spectrum of superfluid  $^4\text{He}$ : measurement by inelastic scattering experiments

- experimental study:  
inelastic neutron scattering  
(coupling to density)

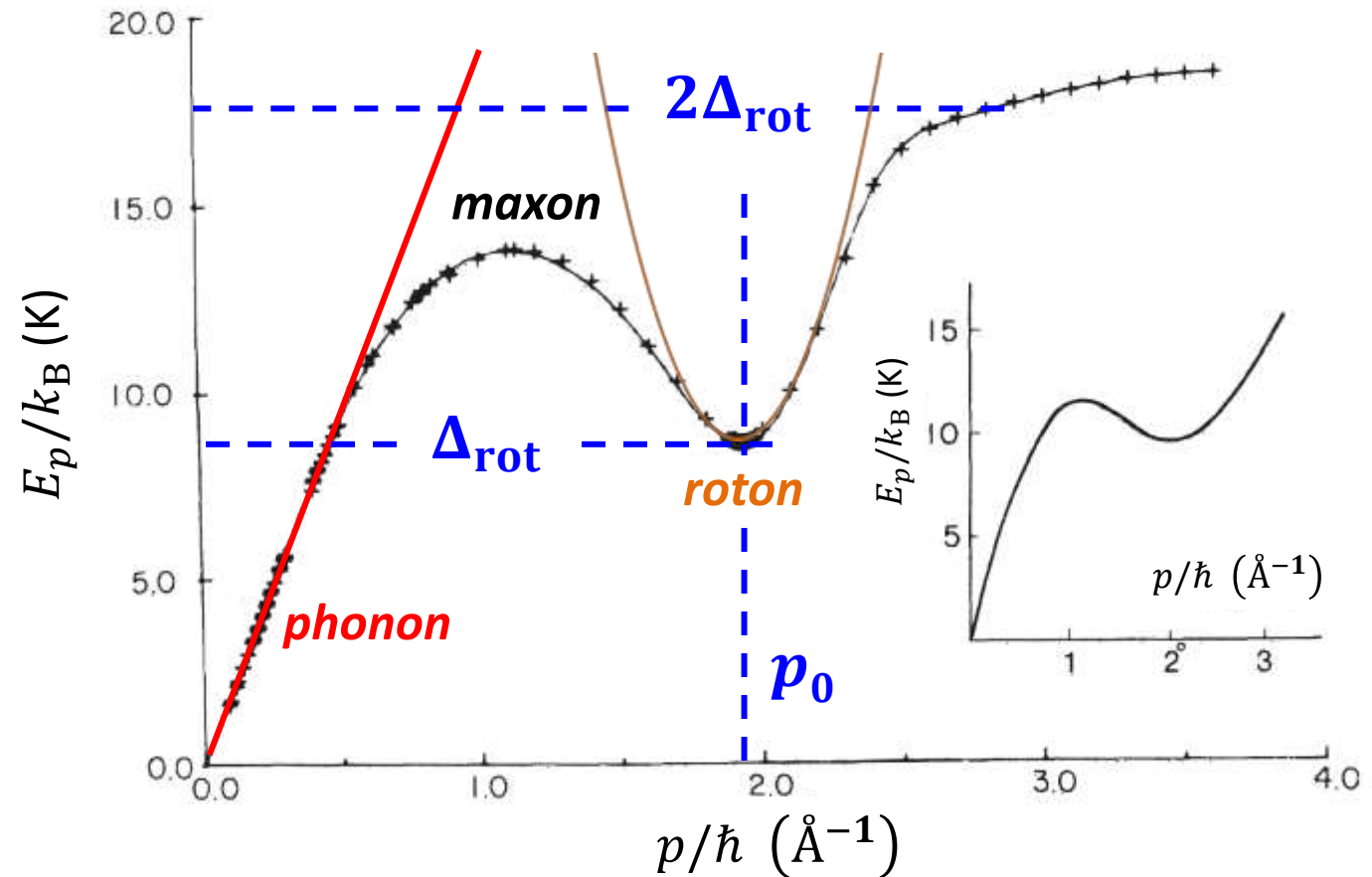
- experimental parameter:

$$\frac{\Delta_{\text{rot}}}{k_B} = 8.61 \text{ K}, \frac{p_0}{\hbar} = 1.92 \text{ \AA}^{-1}, m_{\text{rot}} = 0.14 m_{\text{He}}$$

TABLE IV. Zero pressure roton parameters (this work) compared to previous results: Woods *et al.* [58], Stirling [14,62,64], Andersen *et al.* [68–72], Gibbs *et al.* [73,74], and Pearce *et al.* [76].  $\Delta_R$  at  $P = 0$  (shown below in italics) is taken from Refs. [62,64] in our instrument calibration procedure.

	$\Delta_R$ (meV)	$k_R$ ( $\text{\AA}^{-1}$ )	$\mu_R$
This work	<i>0.7418(10)</i>	1.918(2)	0.141(2)
Woods 1977	0.7426(10)	1.926(5)	0.126(30)
Stirling 1991	0.7418(10)	1.920(2)	0.136(5)
Andersen 1992-1994	0.743(1)	1.931(3)	0.144(3)
Gibbs 1999	0.7426(21)	1.929(2)	0.161(4)
Pearce 2001	0.7440(20)	1.926(-)	0.166(10)

H. Godfrin *et al.*, Phys. Rev. B **103**, 104516 (2021)

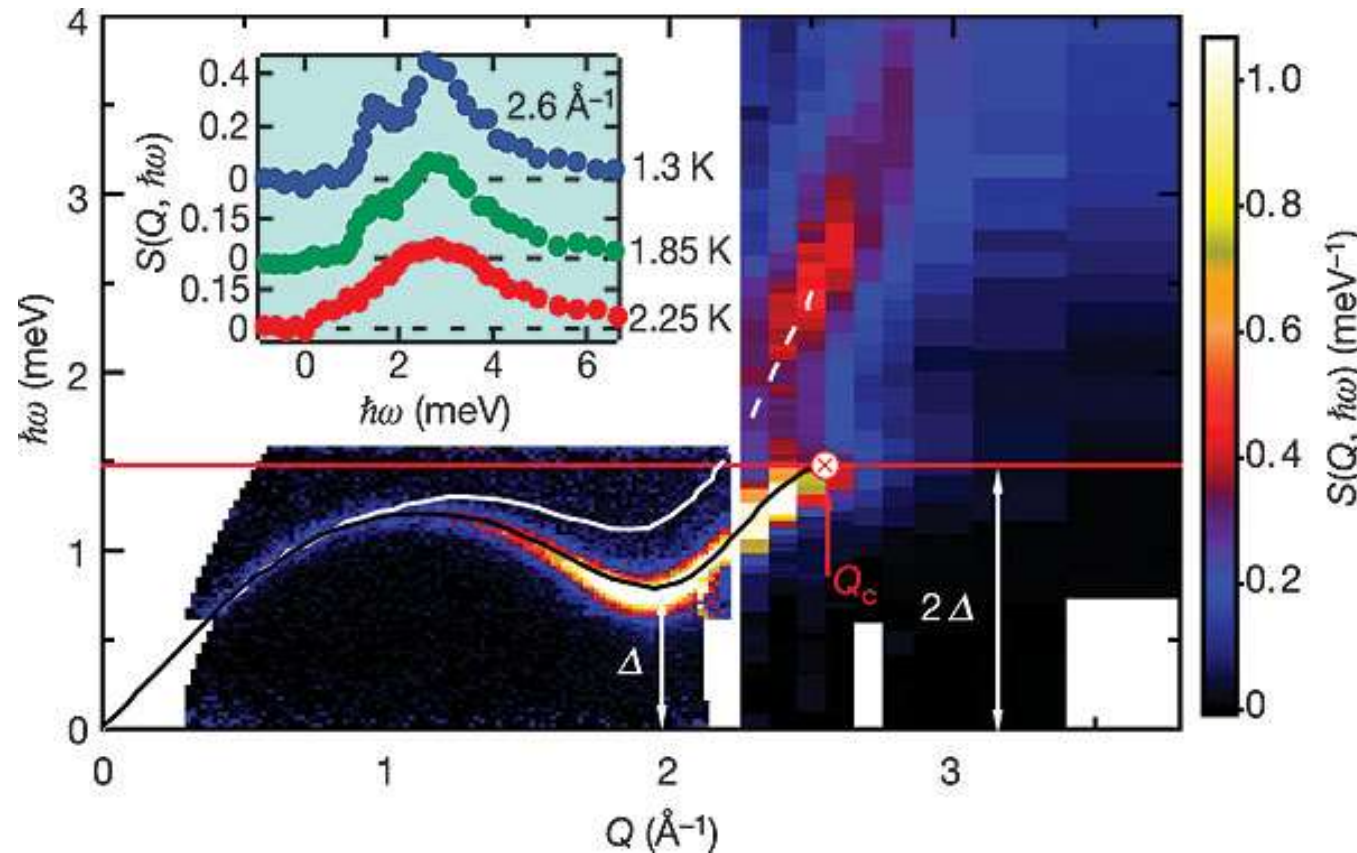


R. J. Donnelly, J. A. Donnelly, R. N. Hills,  
J. Low Temp. Phys. **44**, 471 (1981).

Inset:  
L. D. Landau, J. Phys. U.S.S.R. **11**, 91 (1947).

# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

- excitation spectrum of superfluid  $^4\text{He}$ : measurement by inelastic scattering experiments



**Liquid helium excitation spectrum  $S(Q, \hbar\omega)$  from inelastic neutron scattering measurements.**

Main panel, excitation spectrum in  $^4\text{He}$  for  $1.5 < T < 1.8$  K. Data for wave vector  $Q \geq 2.3 \text{ \AA}^{-1}$  are reproduced from ref. 13, data at smaller  $Q$  are from C.L.B. and S.-H. Lee, unpublished results. Solid black line, dispersion from ref. 13; red circle with cross, spectrum termination point at  $Q = Q_c$  and  $\hbar\omega = 2\Delta$ . White line, Feynman–Cohen bare dispersion in absence of decays; horizontal red line at  $\hbar\omega = 2\Delta$ , onset of two-roton states for  $\hbar\omega \geq 2\Delta$ . Inset, excitations near termination point, at  $Q = 2.6 \text{ \AA}^{-1} \approx Q_c$ , for several temperatures.

[Quasiparticle breakdown in a quantum spin liquid](#)

Matthew B. Stone, Igor A. Zaliznyak, Tao Hong, Collin L. Broholm and Daniel H. Reich  
 Nature **440**, 187-190 (9 March 2006)

# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

- excitation spectrum of superfluid  $^4\text{He}$ : qualitative understanding

*periodic crystal:*

$\mathbf{k}$  and  $\mathbf{k} + \mathbf{G}$  are equivalent

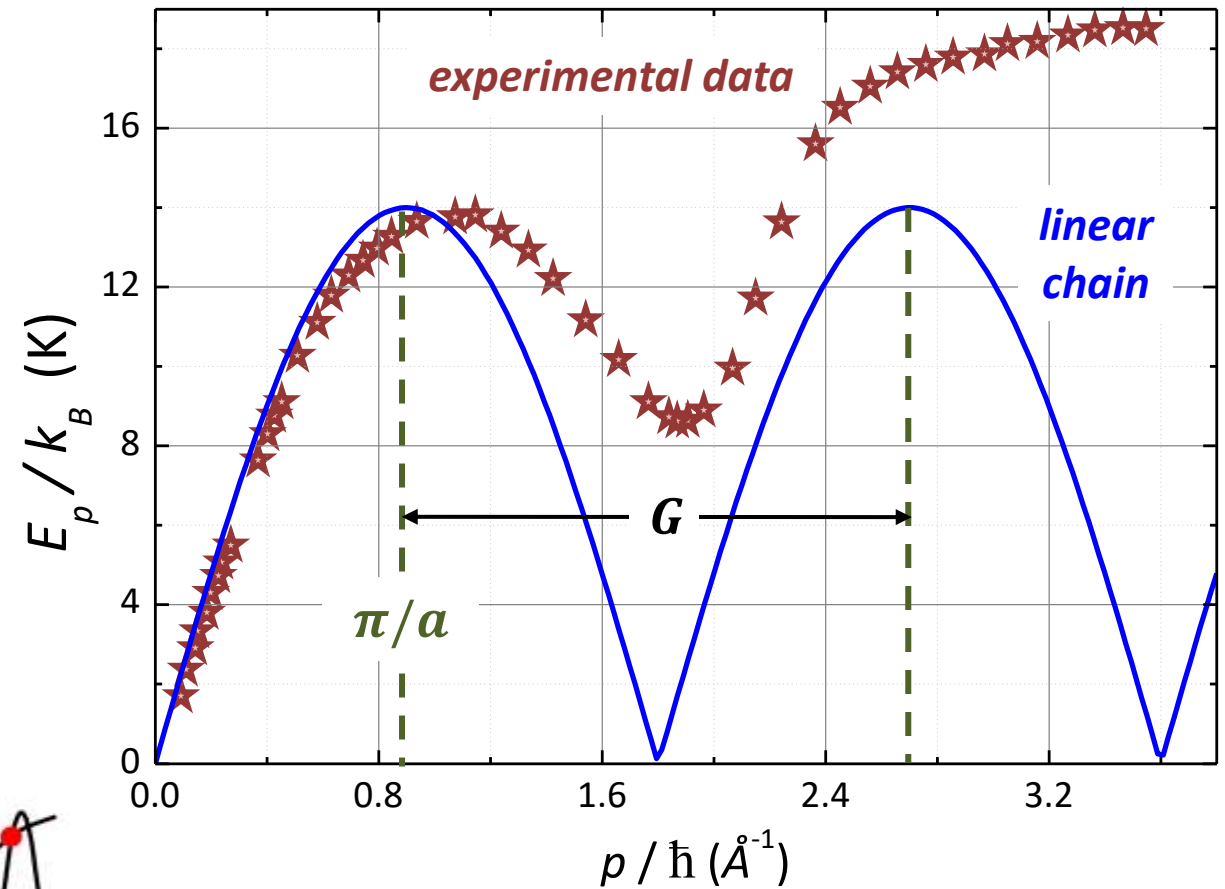
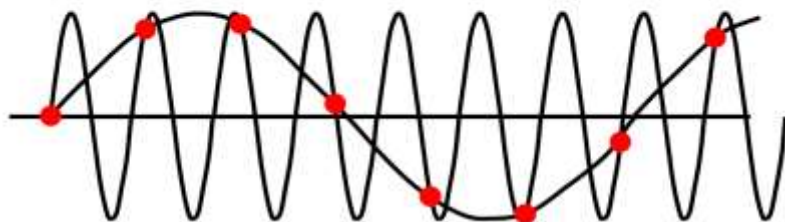
*liquid:*

only short distance ordering

→  $a$  becomes smeared out

→ also  $G = 2\pi/a$  is smeared out

$a = 3.6 \text{ \AA} \rightarrow 2\pi/a \approx 1.8 \text{ \AA}^{-1}$   
(reasonable agreement)



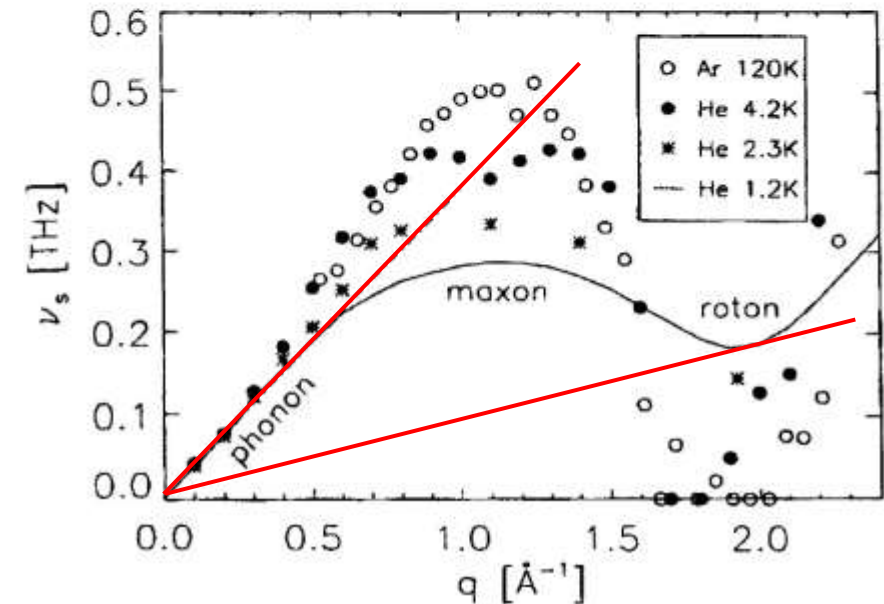
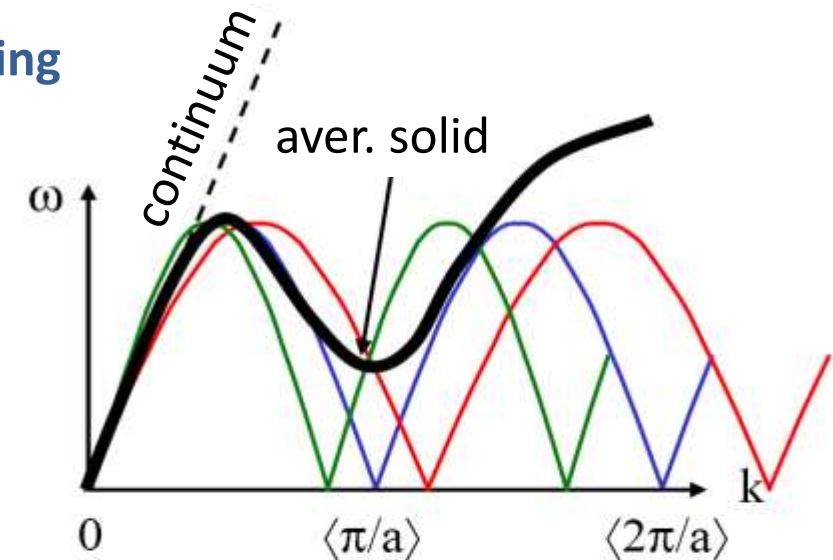
# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

- excitation spectrum of superfluid  $^4\text{He}$ : qualitative understanding

- smearing of  $G = 2\pi/a$**   
 → modelling by average solid + continuum
- other liquids:**  
 → rotons strongly damped
- superfluid He:**  
 → no damping, since there are no single particle excitations
- critical velocities:**  
 →  $v_L = 238 \text{ m/s}$  for phonons  
 →  $v_L \approx 60 \text{ m/s}$  for rotons

measured  $v_L$  usually smaller

→ **excitation of vortices**



# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

- measurement of Landau critical velocity

*measurement of drag  
on ions in  $^4\text{He-II}$*

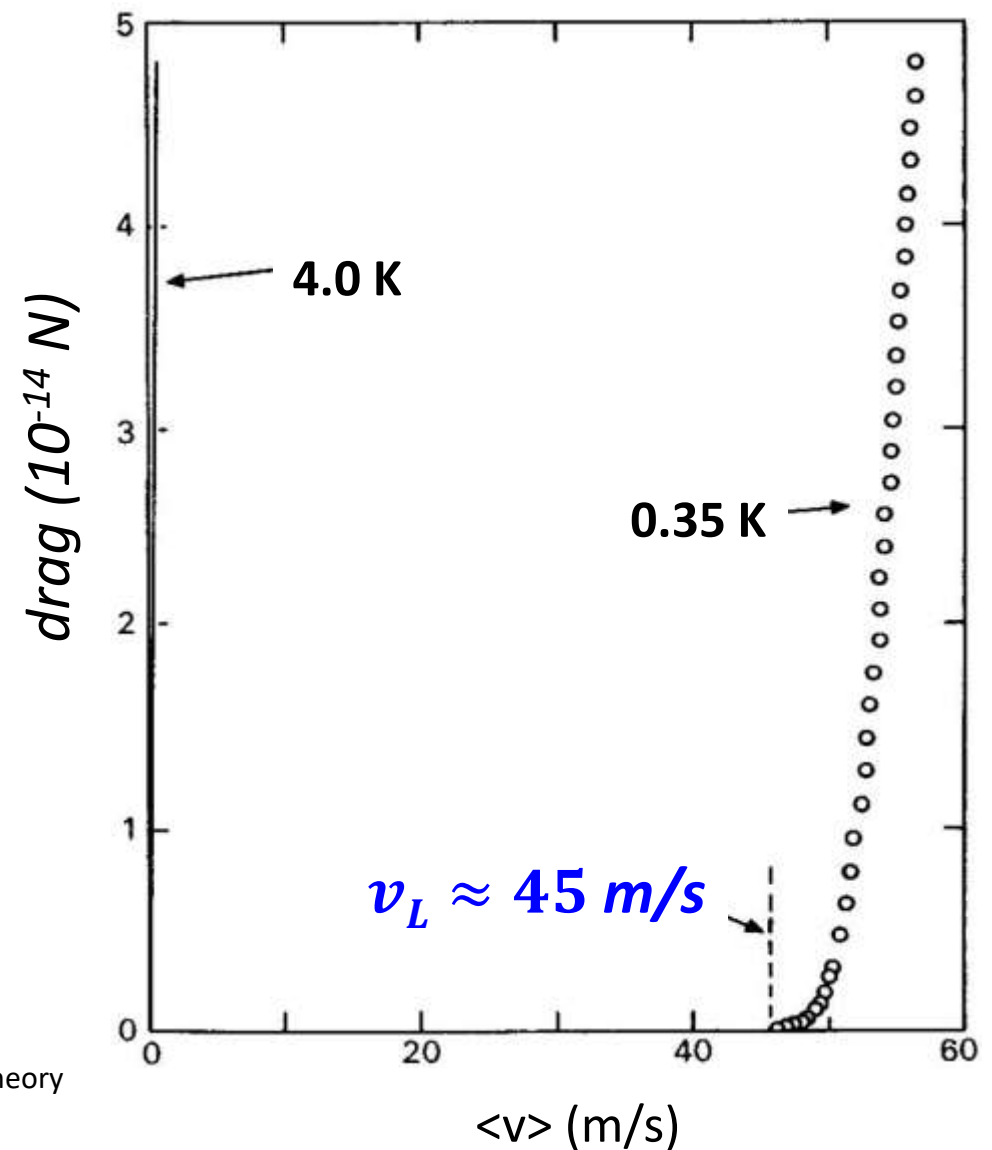
→ *no generation of excitations  
for  $v < v_L$*

→ *no friction*

**key result:**

*there is a finite Landau critical  
velocity in  $^4\text{He-II}$ !*

The breakdown of superfluidity in liquid  $^4\text{He}$  : an experimental test of Landau's theory  
D. R. Allum, Peter V. E. McClintock, A. Phillips, R. M. Bowley  
Philosophical Transactions of the Royal Society of London A **284**, 179-224 (1977).

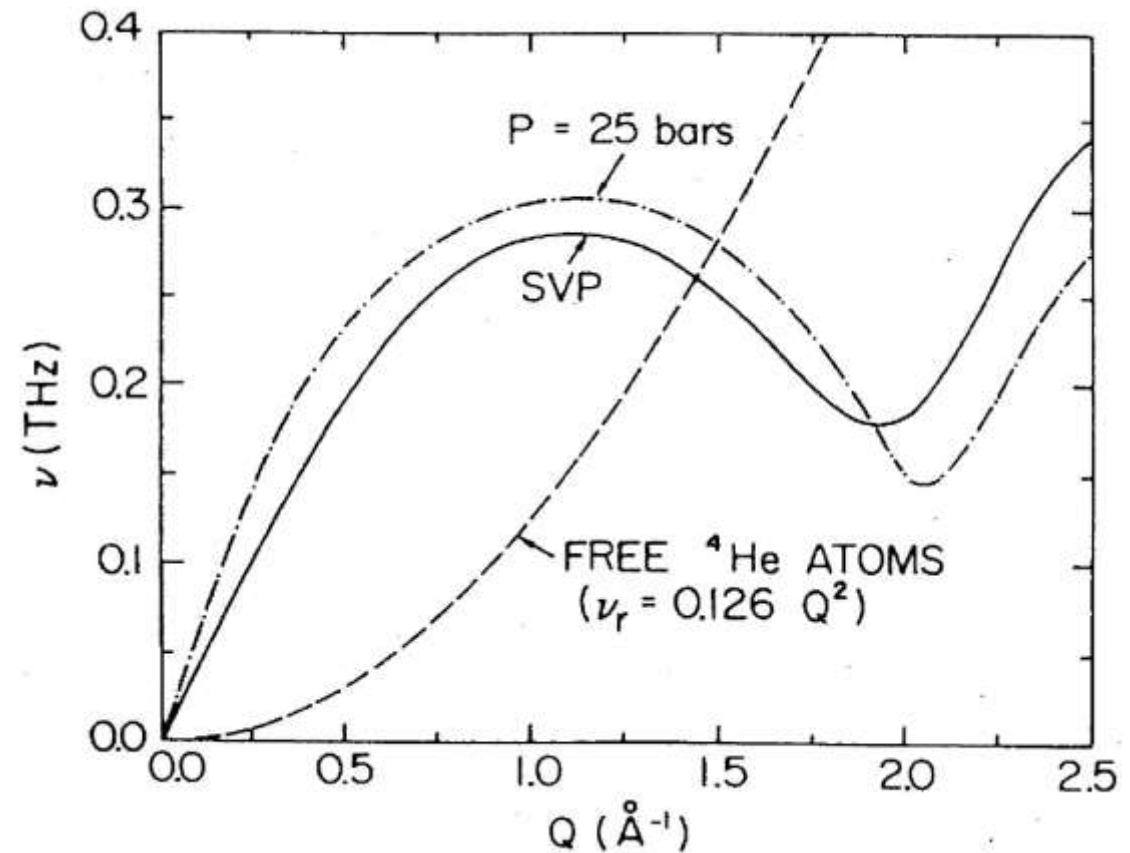




# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

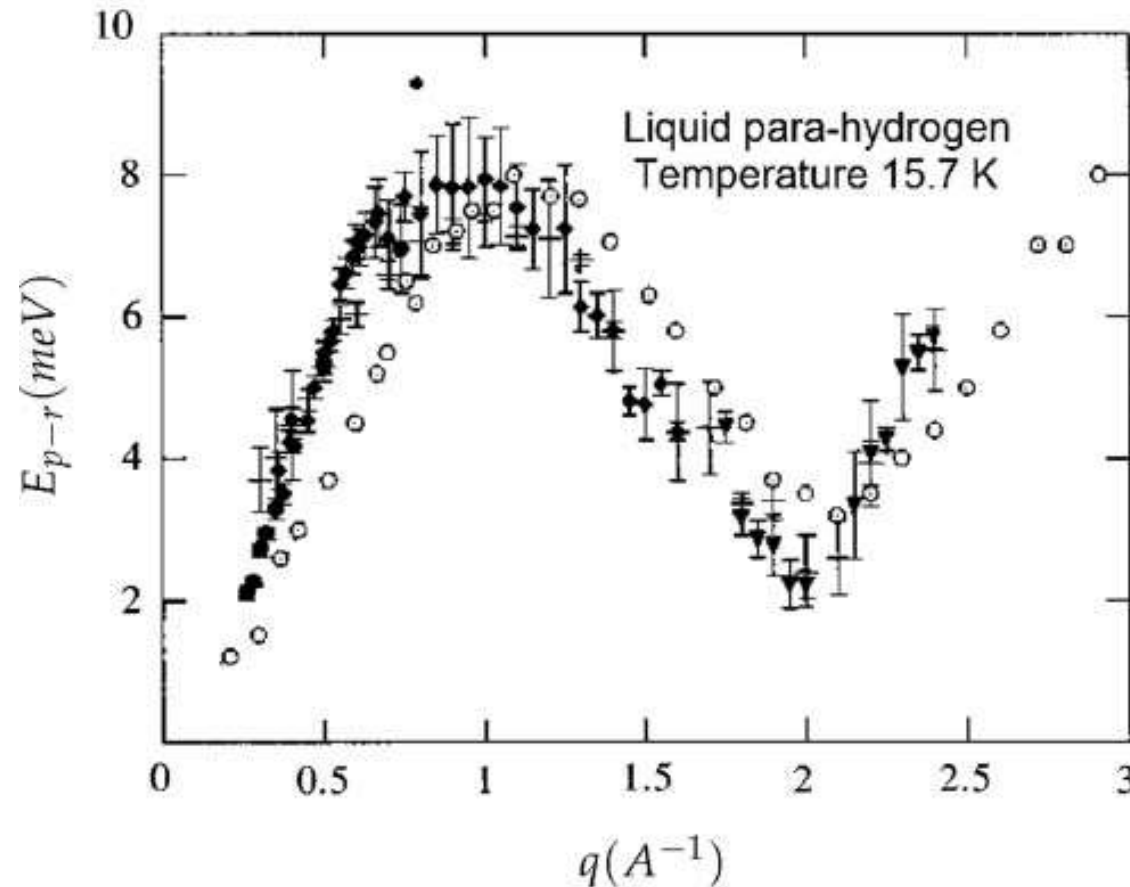
- excitation spectrum of superfluid  $^4\text{He}$ : pressure dependence

- average distance  $a$  becomes smaller
- He becomes harder  
 → *sound velocity increases*
- ordering increases  
 → *roton gap decreases*



# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

- excitation spectrum of superfluid  $^4\text{He}$ : pressure dependence



The experimental INS phonon–roton curve for molecular para-hydrogen at  $T = 15.7$  K. The filled symbols (inverted triangles, diamonds and crosses) depict data measured at different spectrometers and the open circles with a dot show the simulation results.

J. Dawidowski, F. J. Bermejo, M. L. Ristig, B. Fak, C. Cabrillo, et al., Phys. Rev. **B 60**, 15154 (1999).

# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

- excitation spectrum of superfluid  $^4\text{He}$ : pressure dependence

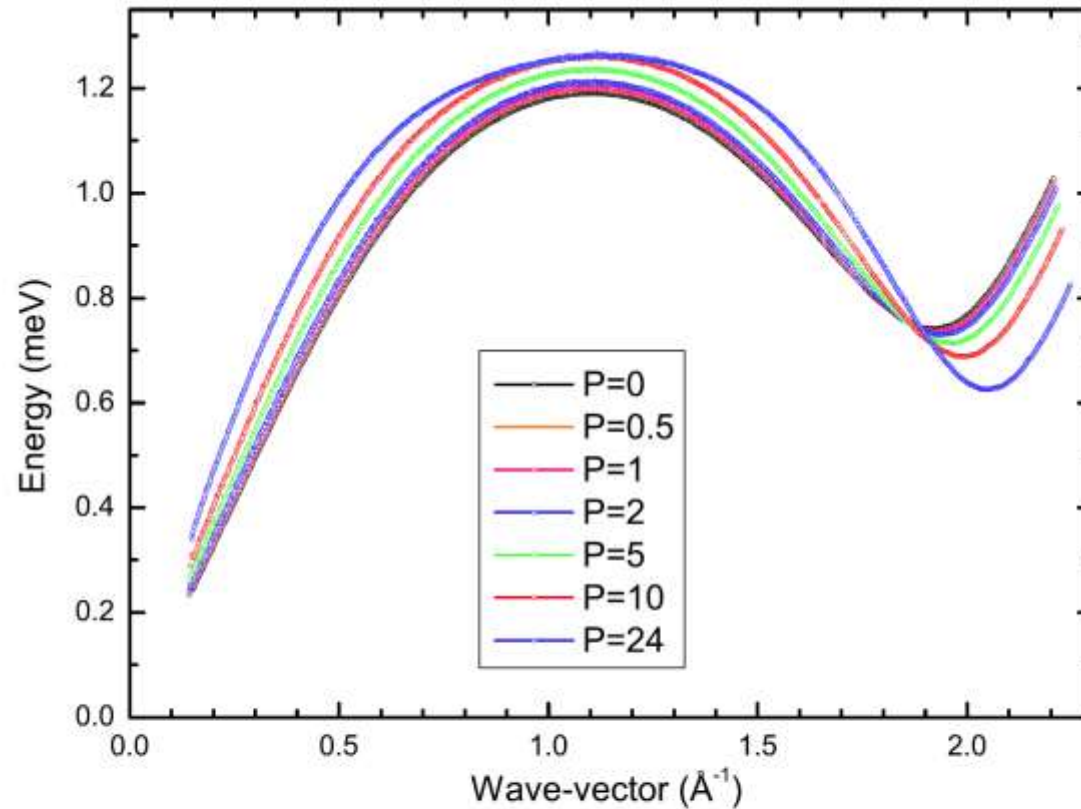


FIG. 17. Dispersion curves  $\epsilon(k)$  measured for several pressures in the 0 to 24 bar range. The individual data points are represented by small circles (best seen on-line). Accurate values for the pressures are given in Table I.

H. Godfrin, K. Beauvois, A. Sultan, E. Krotscheck, J. Dawidowski, B. Fåk, and J. Ollivier  
 Phys. Rev. B **103**, 104516 (2021)





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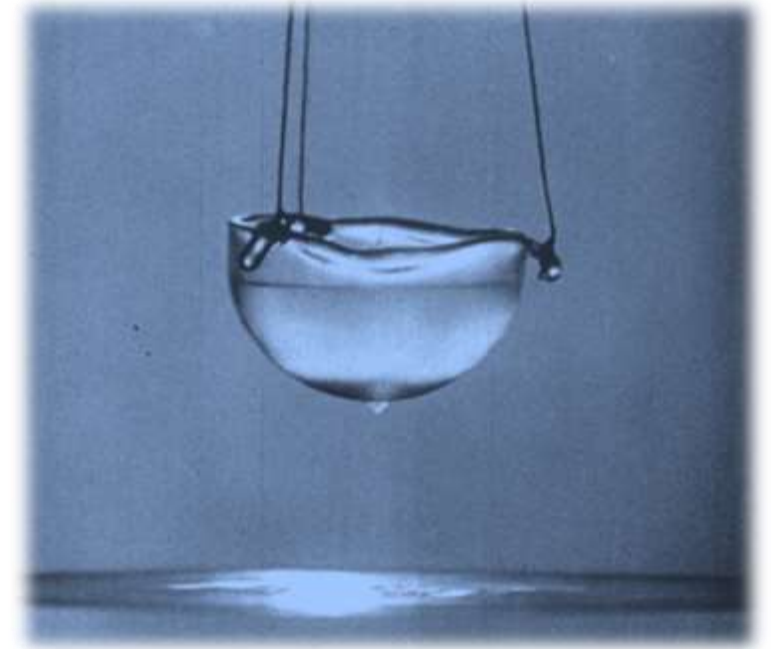
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München

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# Superconductivity and Low Temperature Physics II



**Lecture No. 6**  
**09 June 2022**

**R. Gross**  
**© Walther-Meißner-Institut**

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### I.1 Foundations and General Properties

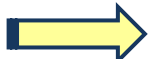
- I.1.1 Quantum Fluids
- I.1.2 Helium
- I.1.3 Van der Waals Bonding
- I.1.4 Zero-Point Fluctuations
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- I.1.6 pT-Phase Diagram of  $^4\text{He}$  and  $^3\text{He}$
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- I.2.1 Bose-Einstein Condensation
- I.2.2 Ideal Bose Gas
- I.2.3 Bose Gas with Interactions
- I.2.4 Bose-Einstein Condensation of  $^4\text{He}$

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- I.3.2 Experimental Observations
- I.3.3 Two-Fluid Hydrodynamics
- I.3.4 Excitation Spectrum of  $^4\text{He}$



### I.4 Vortices

- I.4.1 Quantization of Circulation
- I.4.2 Experimental Study of Vortices

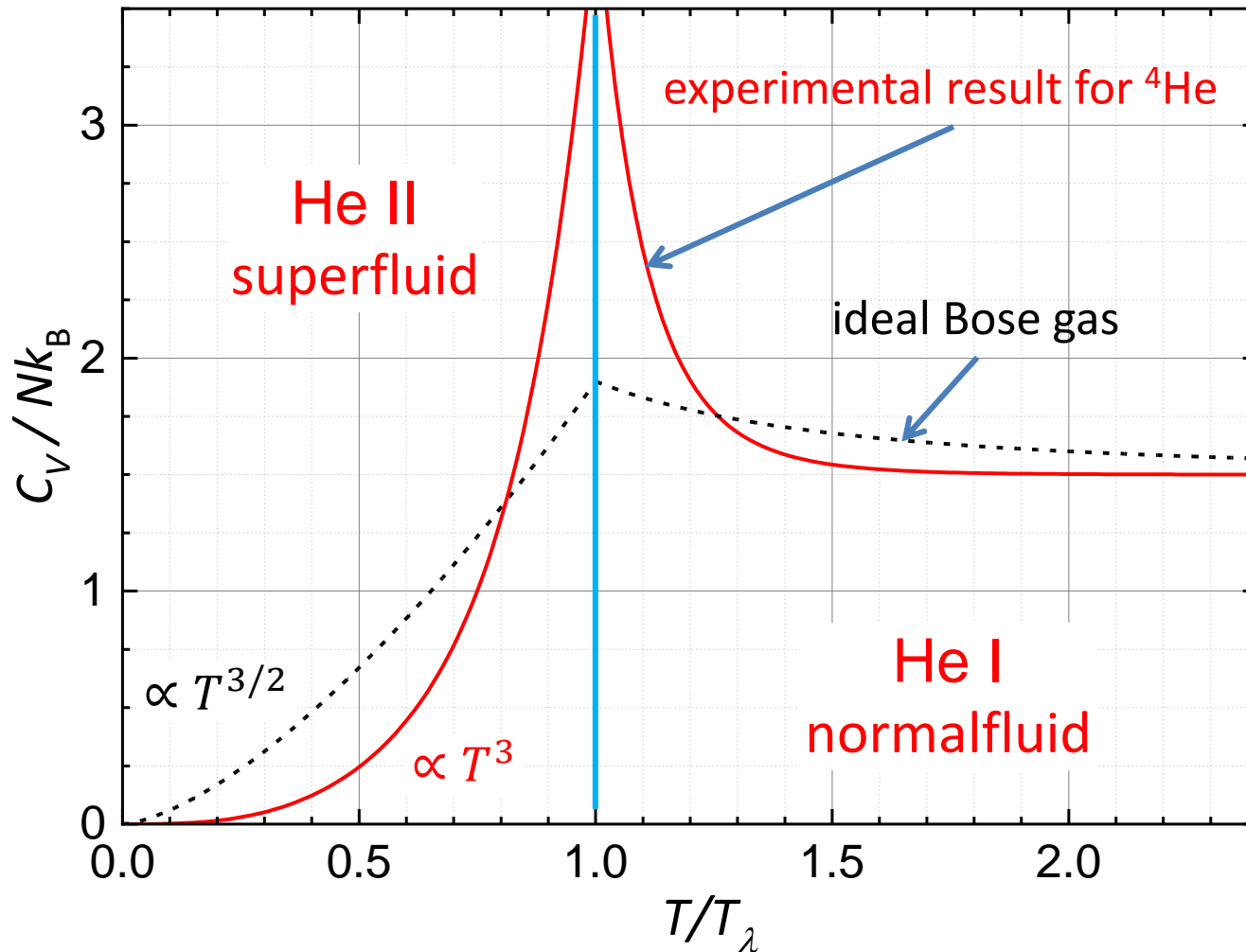
### I.5 $^3\text{He}$

- I.5.1 normal fluid  $^3\text{He}$
- I.5.2 solid  $^3\text{He}$  and Pomeranchuk effect
- I.5.3 superfluid  $^3\text{He}$

### I.6 $^3\text{He}$ / $^4\text{He}$ mixtures

# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

- specific heat of  $^4\text{He}$  revisited



deviations from ideal Bose gas are expected due to finite interactions  
 → *collective excitations: phonons, rotons*

$C_V(T)$  determined by freeze out of phonons and rotons at  $T < T_\lambda$

2<sup>nd</sup> order phase transition is expected for interacting Bose gas

→ however, no jump of  $C_V$  at  $T = T_\lambda$  is observed, as expected for 2<sup>nd</sup> order phase transition

# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

- specific heat of  $^4\text{He}$  revisited: temperature dependence below  $T_\lambda$

## I. $T < 0.6 \text{ K}$ :

in this temperature regime the dominant excitations are long wave length phonons

$$C_{V,ph} \propto T^3 \quad (\text{corresponds to Debye model in solid state physics})$$

$$\frac{C_V}{M} = \frac{2}{15} \frac{\pi^2 k_B^4}{\hbar^3 v_s^3} \frac{V}{M} T^3 \quad \text{single crystal, for single acoustic phonon branch}$$

$$\frac{C_{V,ph}}{M} = \frac{2}{15} \frac{\pi^2 k_B^4}{\hbar^3 v_s^3} \frac{V}{M} T^3 \quad \text{phononic specific heat of } ^4\text{He-II}$$

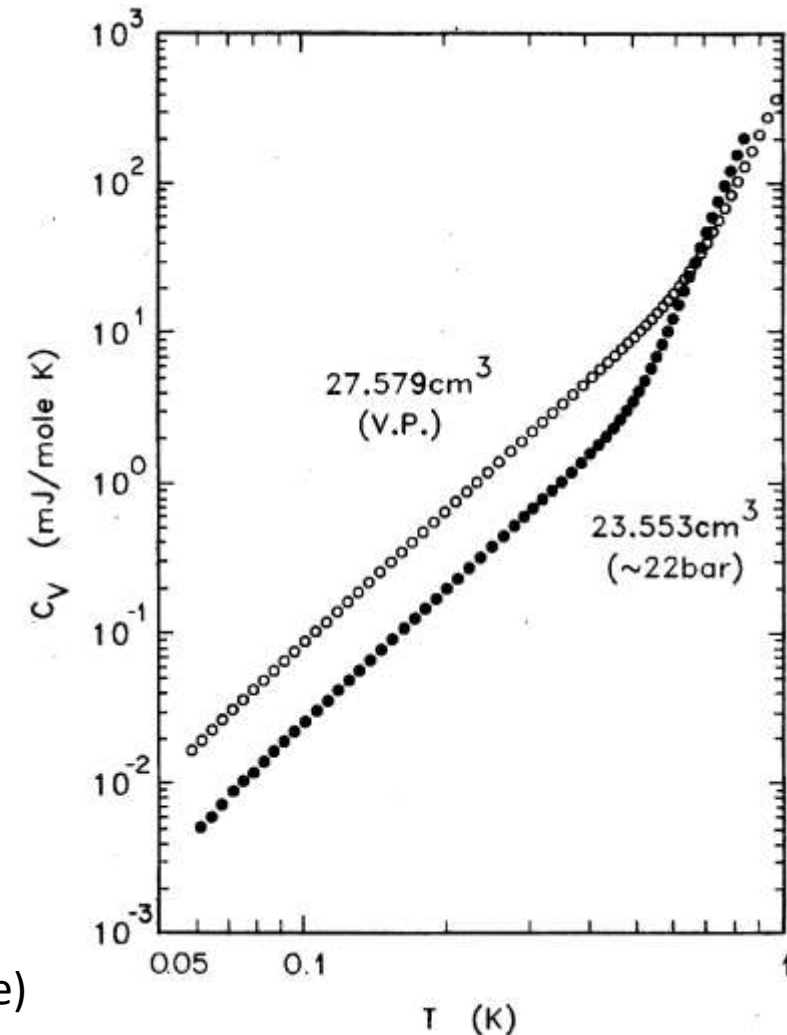
thermal conductivity:

$$\kappa = \frac{1}{3} C_V v \ell \approx \frac{1}{3} C_V v f(d) \propto T^3$$

if  $\ell \simeq d$  (sample dimension, Casimir regime)

absolute value of  $\kappa$  depends on details:

e.g.,  $\kappa \propto d^3$  for flow through narrow slit of width  $d$



Specific Heat and Phonon Dispersion of liquid  $^4\text{He}$   
 D. S. Greywall, Phys. Rev. B **18**, 2127 (1978);  
 see also Phys. Rev. B **21**, 1329 (1979)

# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

- specific heat of  $^4\text{He}$  revisited: temperature dependence below  $T_\lambda$

## II. $0.6 \text{ K} < T < 1.2 \text{ K}$ :

in this temperature regime the dominant excitations are rotons, number  $N_R$  of rotons increases with  $T \rightarrow$  thermal activation

$$C_{V,R} \propto N_R \propto \frac{1}{\sqrt{T}} \exp\left(-\frac{\Delta_{\text{rot}}}{k_B T}\right)$$

## III. $1.2 \text{ K} < T < T_\lambda$ :

- lifetime broadening of roton states comparable to  $\Delta_{\text{rot}}$
- additional excitations
- more complicated behavior

# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

- normal fluid density

- within two-fluid model, the normal component can be considered as formed by the excitations out of the ground state (quasiparticles)

→ normal fluid density is determined by phonon and roton density

- temperature dependence of phonon and roton density is given by

$$\rho_{n,\text{ph}} = \frac{1}{V} \int D_{\text{ph}}(E) n_{\text{ph}}(E, T) dE$$

$n_{\text{ph}}, n_{\text{rot}}$  = occupation probability

$$\rho_{n,\text{rot}} = \frac{1}{V} \int D_{\text{rot}}(E) n_{\text{rot}}(E, T) dE$$

$D_{\text{ph}}, D_{\text{rot}}$  = density of states

- after some math

$$\rho_{n,\text{ph}} = \frac{2\pi^2 k_B^4}{45 \hbar^3 v_{\text{ph}}^5} T^4$$

$$\rho_{n,\text{rot}} = \frac{2p_0^4}{3\hbar^3} \sqrt{\frac{m_{\text{rot}}}{(2\pi)^3 k_B T}} \exp\left(-\frac{\Delta_{\text{rot}}}{k_B T}\right)$$

experimental parameter:

$$\frac{\Delta_{\text{rot}}}{k_B} = 8.61 \text{ K}$$

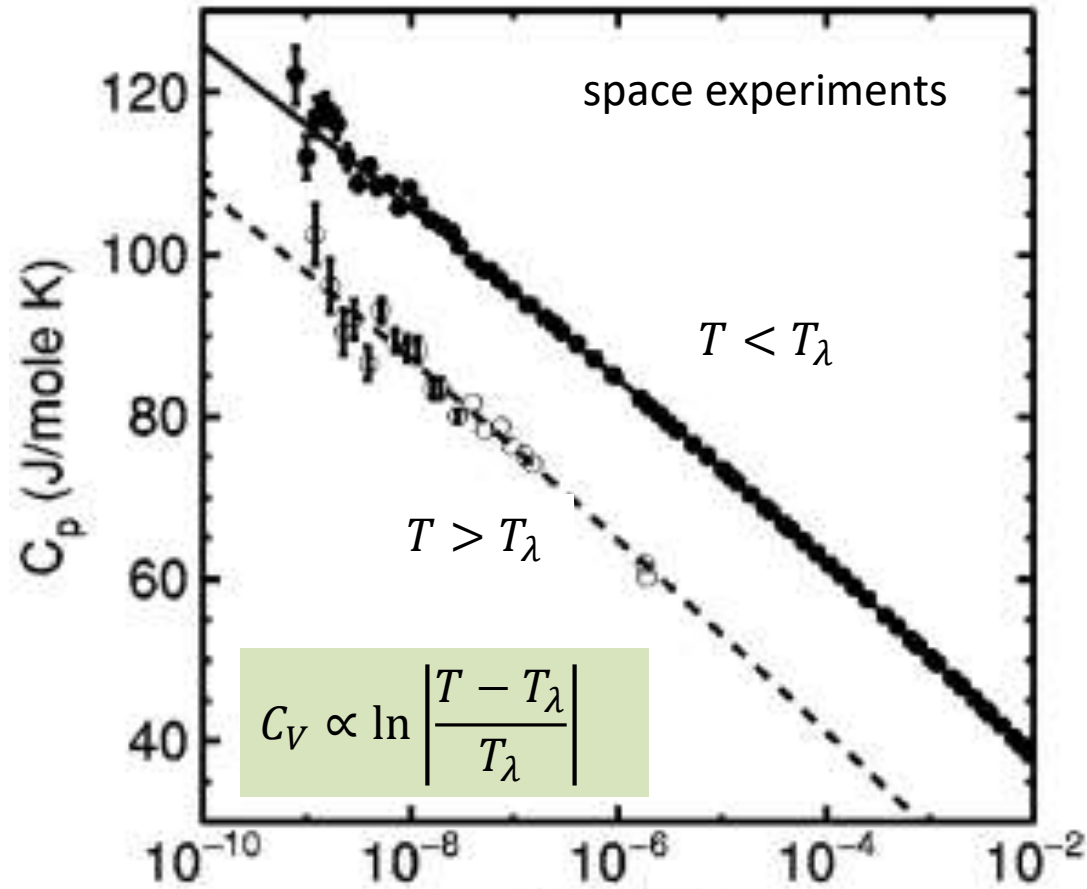
$$\frac{p_0}{\hbar} = 1.92 \text{ \AA}^{-1}$$

$$m_{\text{rot}} = 0.15 m_{\text{He}}$$

$$v_{\text{ph}} = 238 \text{ m/s}$$

# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

- specific heat of  $^4\text{He}$  revisited: temperature dependence close to  $T_\lambda$



experiment:

*second order phase transition with logarithmic singularity*

theory of phase transitions:

$$C_p = a^\pm + b^\pm \frac{t^{-\alpha}}{\alpha} (1 - c^\pm t^\delta + \dots) \quad \text{with } t = \frac{T}{T_\lambda} - 1$$

$\alpha \simeq -0.0127 \pm 0.0003$  (critical exponent),  $\delta = 0.529$

$a^\pm, b^\pm, c^\pm = \text{constants}$  (agree well with 3D XY universality class)

note that  $t^{-\alpha} = \exp(-\alpha \ln t) \simeq 1 - \alpha \ln t$ , since  $\alpha \ln t \ll 1$

$$\Rightarrow C_p \propto b^\pm \frac{t^{-\alpha}}{\alpha} = \frac{b^\pm}{\alpha} - b^\pm \ln t$$

- $T_\lambda$  depends on pressure, i.e. on the filling height of the He container  
→ results in an experimental broadening of the cusp

→ *precision experiment in space shuttle*

- deviations also appear, if the size of the container becomes smaller than the coherence length, which diverges at  $T_\lambda$

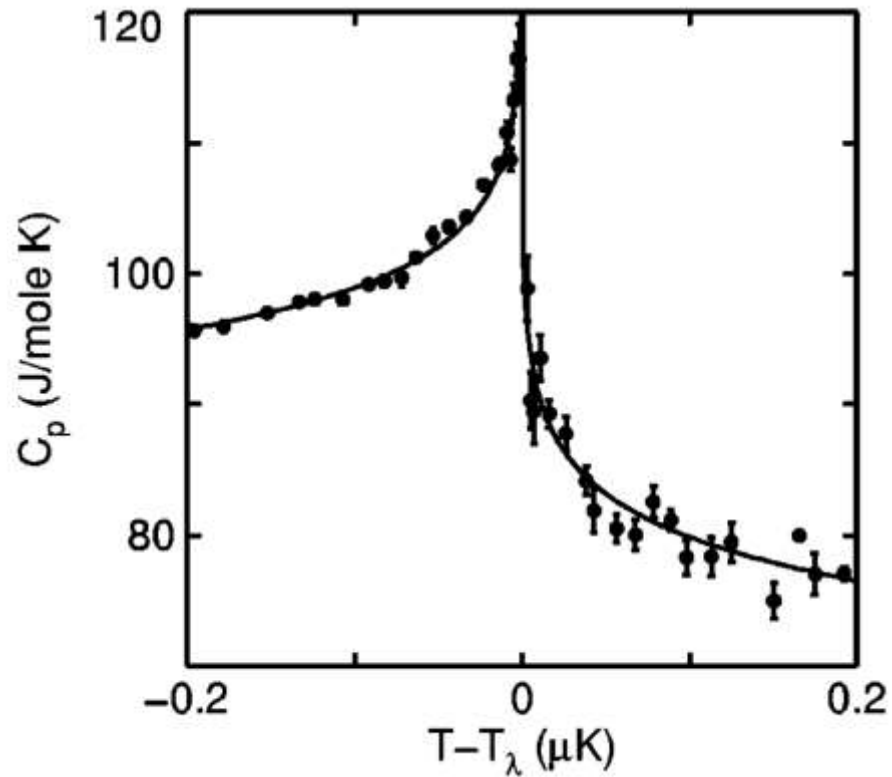
Specific heat of liquid helium in zero gravity very near the lambda point

J. A. Lipa et al., Phys. Rev. B 68, 174518 (2003)

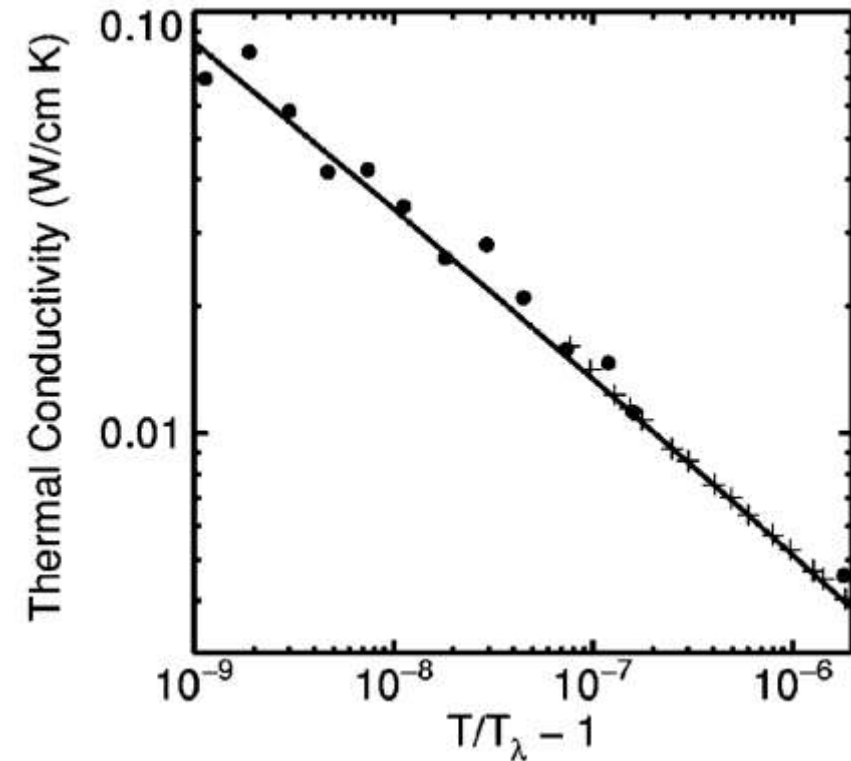
$$\left| \frac{T}{T_\lambda} - 1 \right|$$

# I.3.4 Excitation Spectrum of Superfluid $^4\text{He}$

- specific heat and thermal conductivity of  $^4\text{He}$  revisited: temperature dependence close to  $T_\lambda$



Averaged data close to the transition. Line shows the best-fit function.



Log-log plot of thermal conductivity vs reduced temperature above the lambda point. Filled circles: data, curve: theory model.

Specific heat of liquid helium in zero gravity very near the lambda point  
 J. A. Lipa et al., Phys. Rev. B 68, 174518 (2003)



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### I.6 $^3\text{He}$ / $^4\text{He}$ mixtures



- vortices in superfluid  $^4\text{He}$

- in the discussion of the two-fluid model, we assumed the absence of turbulence in  $\mathbf{J}_s$ :  $\Rightarrow \nabla \times \mathbf{v}_s = 0$
- 1941: **Landau** proposed to check this assumption in experiments with superfluid He in rotating containers
- 1949: **Onsager** predicted the appearance of vortices in rotating superfluid He
- 1953: **Feynman** noted that circulation in superfluid  $^4\text{He}$  should be quantized
- 1961: first experimental proof of quantization by **Vinen**
  - ➔ *quantized vortices in superfluid  $^4\text{He}$  as in superconductors*
  - ➔ *difference: superfluid  $^4\text{He}$  is not charged, neutral circulation*

# I.4.1 Quantization of Circulation

- vortices – nomenclature and some definitions

- ideal vortex in z-direction at position  $\mathbf{r} = 0$ :  $\nabla \times \mathbf{v}(\mathbf{r}) = \Gamma \cdot \delta(x) \cdot \delta(y)$

- circulation  $\Gamma$ :

$$\int_A \nabla \times \mathbf{v}(\mathbf{r}) dA = \oint_S \mathbf{v}(\mathbf{r}) \cdot d\mathbf{r} = \Gamma = \text{const.}$$

Stokes

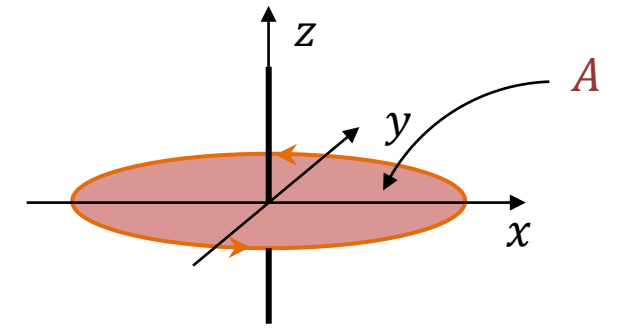
Thomson circulation theorem

- if  $A$  is a circle, then  $\Gamma = 2\pi r \cdot v_\phi$

and  $v_\phi = \frac{\Gamma}{2\pi r} \hat{e}_\phi$

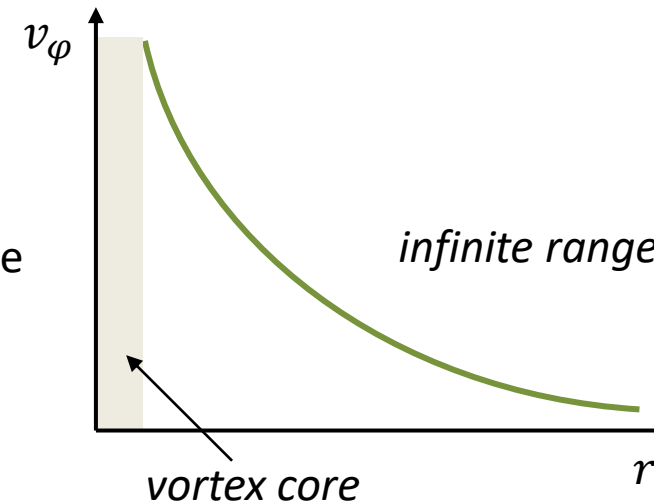
note that  $\nabla \times \mathbf{v}_\phi = \mathbf{0}$  apart from the vortex core

→ for superfluid He: circulation  $\Gamma$  is quantized



$$\nabla \times \mathbf{v} = \begin{pmatrix} \left[ \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right] \hat{x} \\ \left[ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right] \hat{y} \\ \left[ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right] \hat{z} \end{pmatrix}$$

$$\nabla \times \mathbf{v} = \begin{pmatrix} \left[ \frac{1}{r} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{e}_r \\ \left[ \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right] \hat{e}_\phi \\ \frac{1}{r} \left[ \frac{\partial}{\partial r} (r v_\phi) - \frac{\partial v_r}{\partial \phi} \right] \hat{e}_z \end{pmatrix}$$



# I.4.1 Quantization of Circulation

- description of superfluid component by macroscopic wave function

as done in superconductivity for electronic system: charged superfluid

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r})e^{i\theta(\mathbf{r})}, \quad |\psi_0(\mathbf{r})|^2 = n_0(\mathbf{r}), \quad N_0 = \int |\psi_0(\mathbf{r})|^2 d^3r \quad n_0 = \frac{N_0}{V} = \text{particle density in condensate}$$

- phase is related to velocity  $\mathbf{v}_s(\mathbf{r})$  of superfluid:

$$\mathbf{J}_\rho(\mathbf{r}) = n_0 \underbrace{\frac{\hbar}{m_{\text{He}}} \nabla\theta(\mathbf{r})}_{\mathbf{v}_s(\mathbf{r})} = n_0 \mathbf{v}_s(\mathbf{r}) \quad \text{current density – phase relation (as in superconductivity)}$$

$\mathbf{v}_s = \text{superfluid velocity, } n_0 = \text{superfluid density}$

- analogy to superconductors

$$\mathbf{J}_s(\mathbf{r}) = n_s q_s \underbrace{\left[ \frac{\hbar}{m_s} \nabla\theta(\mathbf{r}) - \frac{q_s}{m_s} \mathbf{A}(\mathbf{r}) \right]}_{\mathbf{v}_s} \quad \mathbf{v}_s(\mathbf{r}) = \frac{\hbar}{m_s} \nabla\theta(\mathbf{r}) - \frac{q_s}{m_s} \mathbf{A}(\mathbf{r})$$

superfluid He is uncharged:  $q_s = 0$

$$\mathbf{v}_s(\mathbf{r}) = \frac{\hbar}{m_{\text{He}}} \nabla\theta(\mathbf{r})$$



# I.4.1 Quantization of Circulation

- circulation of superfluid component

$$\Gamma = \oint_S \mathbf{v}_s(\mathbf{r}) \cdot d\mathbf{r} = \frac{\hbar}{m_{\text{He}}} \oint_S \underbrace{\nabla\theta(\mathbf{r}) \cdot d\mathbf{r}}_{=2\pi n} = \frac{\hbar}{m_{\text{He}}} \cdot 2\pi n = \frac{h}{\underbrace{m_{\text{He}}}_{\kappa}} \cdot n$$

→ circulation is quantized in units of  $h/m_{\text{He}} = \kappa$  (*vorticity*)

→ quantum number  $n$  corresponds to **topological winding number**

(counts the  $2\pi$  windings along a closed contour)

- generation of vortices by rotation:

$$\text{angular frequency } \Omega = \Gamma/A \Leftrightarrow \text{magnetic field } B = \Phi/A$$

$$\text{Coriolis force } \mathbf{F}_{\text{cor}} = 2m_{\text{He}}\mathbf{v}_s \times \boldsymbol{\Omega} \Leftrightarrow \text{Lorentz force } \mathbf{F}_L = q_s\mathbf{v}_s \times \mathbf{B}$$

$$2m_{\text{He}}\Omega \Leftrightarrow q_s B$$

# I.4.1 Quantization of Circulation

- consequences of macroscopic quantum description of superfluid

– superfluid velocity:  $\mathbf{v}_s(\mathbf{r}) = \frac{\hbar}{m_{\text{He}}} \nabla\theta(\mathbf{r}) \Rightarrow \nabla \times \mathbf{v}_s(\mathbf{r}) = \frac{\hbar}{m_{\text{He}}} \underbrace{\nabla \times \nabla\theta(\mathbf{r})}_{=0} = \mathbf{0}$

$\Rightarrow \oint \mathbf{v}(\mathbf{r}) \cdot d\mathbf{r} = \Gamma = 0$  *an uncharged superfluid cannot rotate homogeneously*

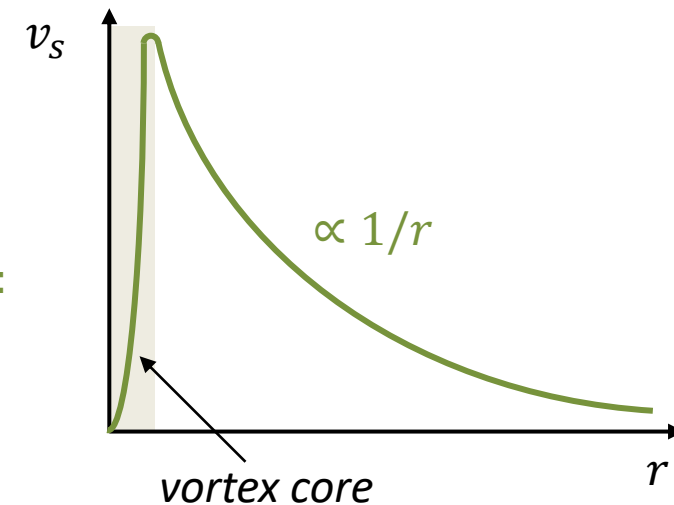
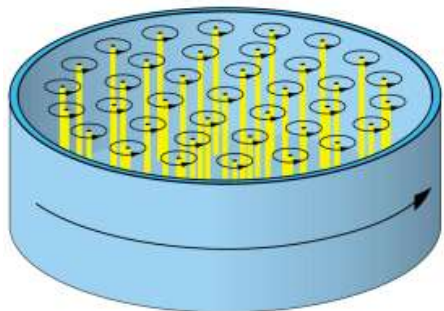
- experimental observation:

- if we rotate a cylinder filled with superfluid  $^4\text{He}$  at angular frequency  $\omega$ , the surface of the superfluid does not stay flat but shows curvature due to centrifugal force

parabolic radial height profile:  $z = \frac{\omega^2}{2g} r^2$

- apparent contradiction is caused by the presence of vortices

- $\nabla \times \mathbf{v}_s(\mathbf{r}) = \mathbf{0}$  is satisfied apart inside the vortex core at  $r = 0$ : macroscopic wave function  $|\psi_0(0)|^2 = 0$ , phase  $\theta$  is not defined, superfluid density  $J_s = n_0 v_s = 0$



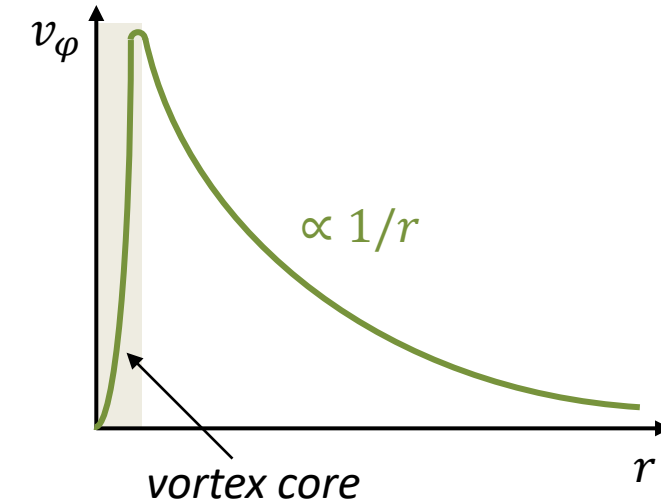
# I.4.1 Quantization of Circulation

- some estimates

- magnitude of the superfluid velocity (in circular vortex)

$$\Gamma = \frac{\hbar}{m_{\text{He}}} n = 2\pi r v_{\varphi}$$

$$\Rightarrow v_{\varphi} = \frac{\hbar}{m_{\text{He}}} n \frac{1}{2\pi r} = \frac{\kappa}{2\pi r} n \simeq \frac{4 \times 10^{-8}}{r} n \left[ \frac{\text{m}}{\text{s}} \right]$$



- diameter  $d_{\text{VC}}$  of vortex core

$$\Rightarrow d_{\text{VC}} = \frac{\hbar}{m_{\text{He}}} n \frac{1}{2\pi v_{\varphi}} = \frac{\kappa}{2\pi v_{\varphi}} n \sim 1 \text{ \AA}$$

for  $n = 1$  and  $v_{\varphi} =$  critical velocity for roton formation

- energy of vortex is proportional to  $v_{\varphi}^2$

$$E_{\text{V}} \approx E_{\text{kin}} \propto \kappa^2 \propto n^2$$

→ it is more favorable to generate two vortices with  $n = 1$  instead of a single vortex with  $n = 2$



# I.4.1 Quantization of Circulation

- are there vortices in thermal equilibrium ?

- excitation energy  $\approx E_{\text{kin}}$  should be as small as possible:

- minimum  $v_s$

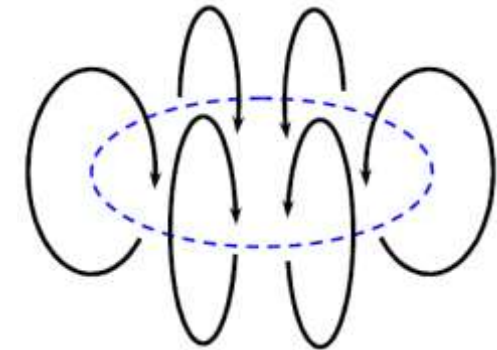
- $n = 1$

- minimum length

- vortex rings

- smallest ring

- atomic dimension



- vortex still macroscopic object

- $E_V \gg k_B T$  except for  $T \sim T_\lambda$

- momentum is large →  $v_L = E / p$  is small for vortex rings

- critical velocity  $v_L$  is determined by excitation of vortex rings

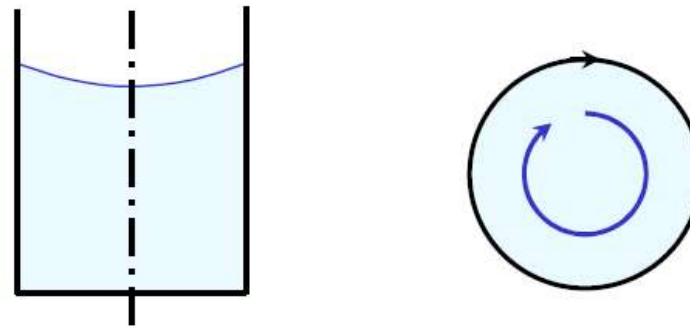
# I.4.2 Experimental Study of Vortices

- cooling down a rotating  $^4\text{He}$  container below  $T_\lambda$

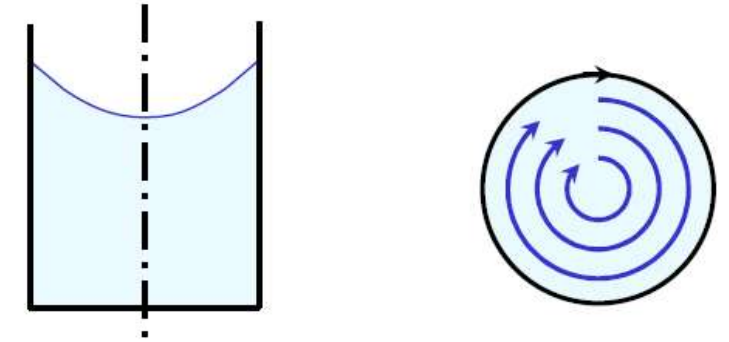
first experiments by Vinen in 1961

The detection of a single quantum of circulation in liquid helium II, W.F. Vinen, *Proc. Roy. Soc. (Lond.) A* **260**,218 (1961).

- above  $T_\lambda$ : rigid rotation

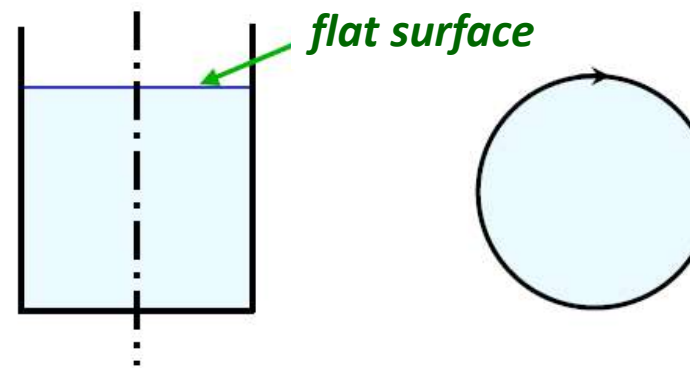


*weak circulation*

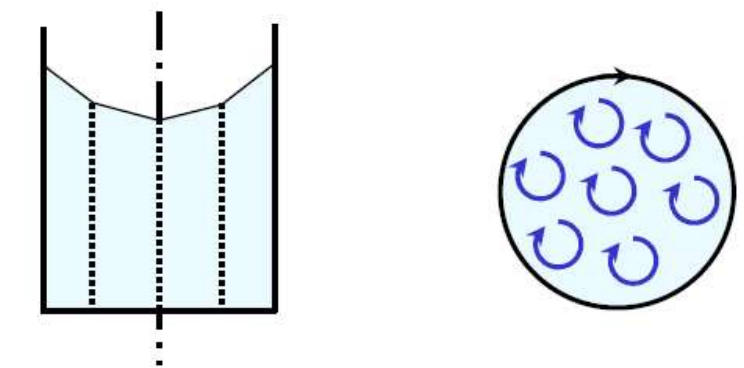


*strong circulation*

- below  $T_\lambda$ :



*screening of circulation  
„Meißner State“*

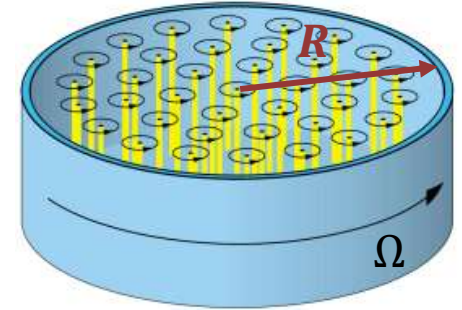


*penetration of circulation  
„Mixed State“*

# I.4.2 Experimental Study of Vortices

- what is the density of vortices inside the container ?
  - circulation at the edge of the container with radius  $R$ :

$$\Gamma = \oint_S \mathbf{v}(\mathbf{r}) \cdot d\mathbf{r} = 2\pi R v(R) = 2\pi R \Omega R = 2\pi R^2 \Omega$$



- circulation must be equal to number of vortices  $N_V$   $\times$  vorticity  $\kappa$   $\times$  area  $\pi R^2$

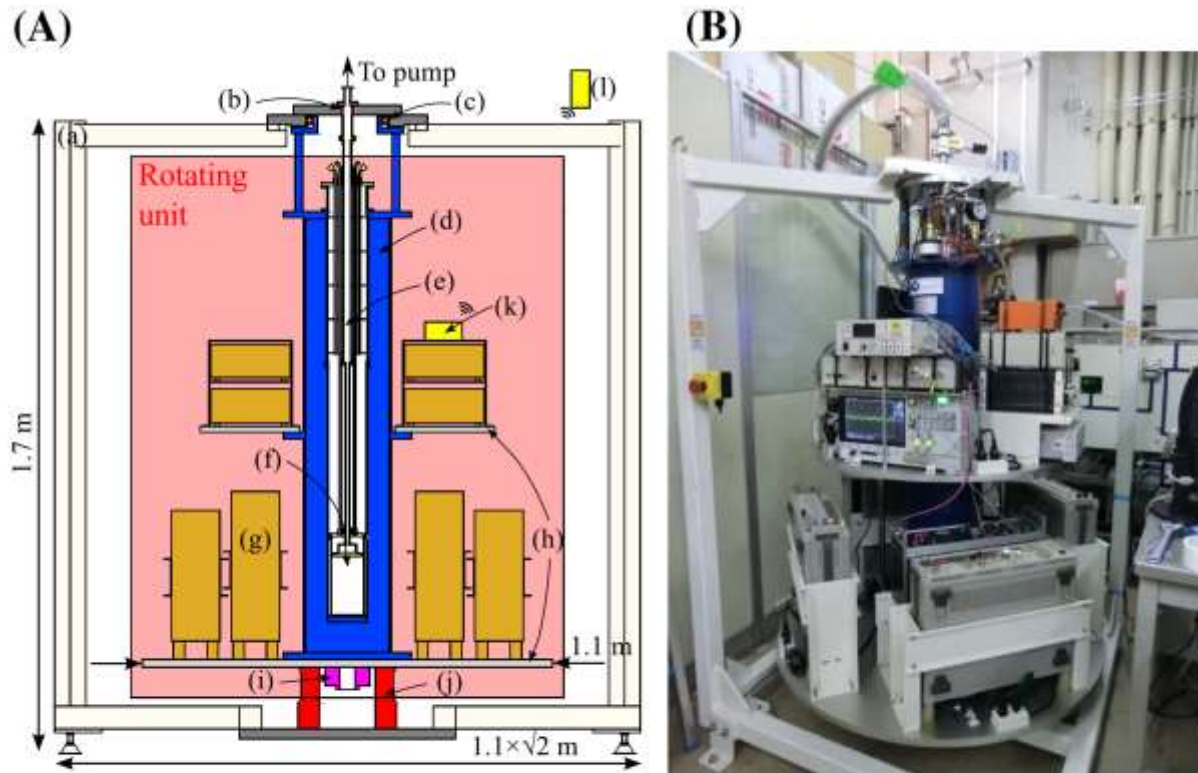
$$\Gamma = N_V \kappa \pi R^2 = 2\pi R^2 \Omega$$

$$N_V = \frac{2\Omega}{\kappa} = \frac{2m_{\text{He}}\Omega}{h} = \frac{m_{\text{He}}\Omega}{\pi\hbar}$$

$$N_V \simeq \frac{20}{\text{mm}^2} \times \Omega [\text{Hz}]$$

# I.4.2 Experimental Study of Vortices

- rotating cryostats



**Fig. 1** **a** Schematic vertical section of the rotating cryostat: (a) Steel frame, (b) rotating o-ring seal, (c) ball bearing, (d) aluminum/FRP dewar, (e) 1K cryostat insert, (f) sample space, (g) instruments, (h) duralumin round tables, (i) slip ring for power supply, (j) AC servomotor, (k) GPIB-USB interface and wireless USB device server and (l) wireless LAN access point. **b** Photograph of the rotating cryostat (Color figure online)

A Compact Rotating 1K Cryostat for  $^4\text{He}$  Studies  
 Takahiko Makiuchi, Satoshi Murakawa & Keiya Shirahama  
 J. Low Temp. Phys. **187**, 633-638 (2017)



Helsinki university

<http://itl.tkk.fi/research/applied/rotating3he.html>



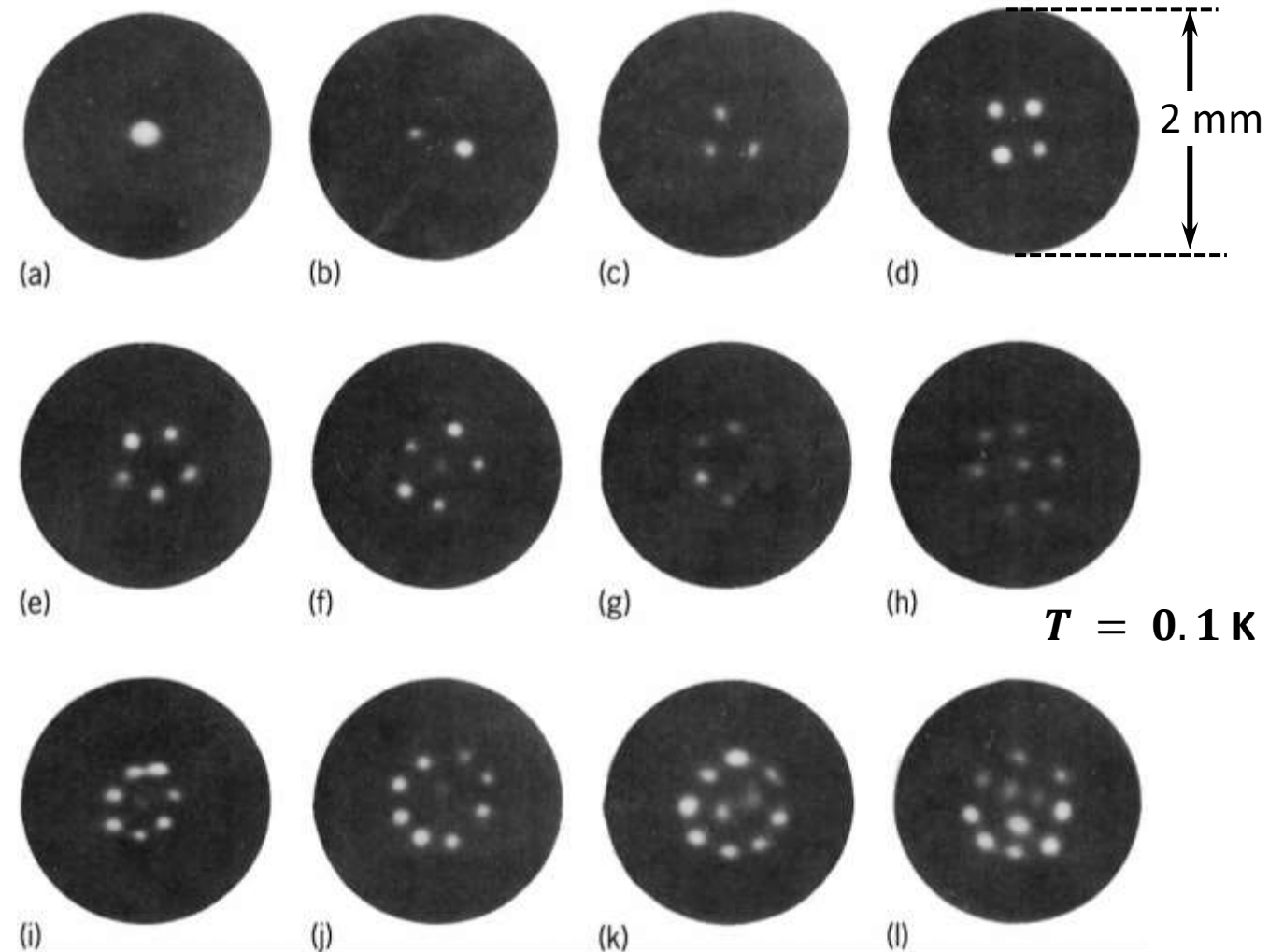
# I.4.2 Experimental Study of Vortices

- imaging of vortices in superfluid  $^4\text{He}$

- *He is covered by electrons*
- *electrons are trapped in vortex cores*
- *on applying an electric field, electrons are extracted to fluorescent screen*

*pattern and camera are rotating*

*pinning of vortices at bottom of container*



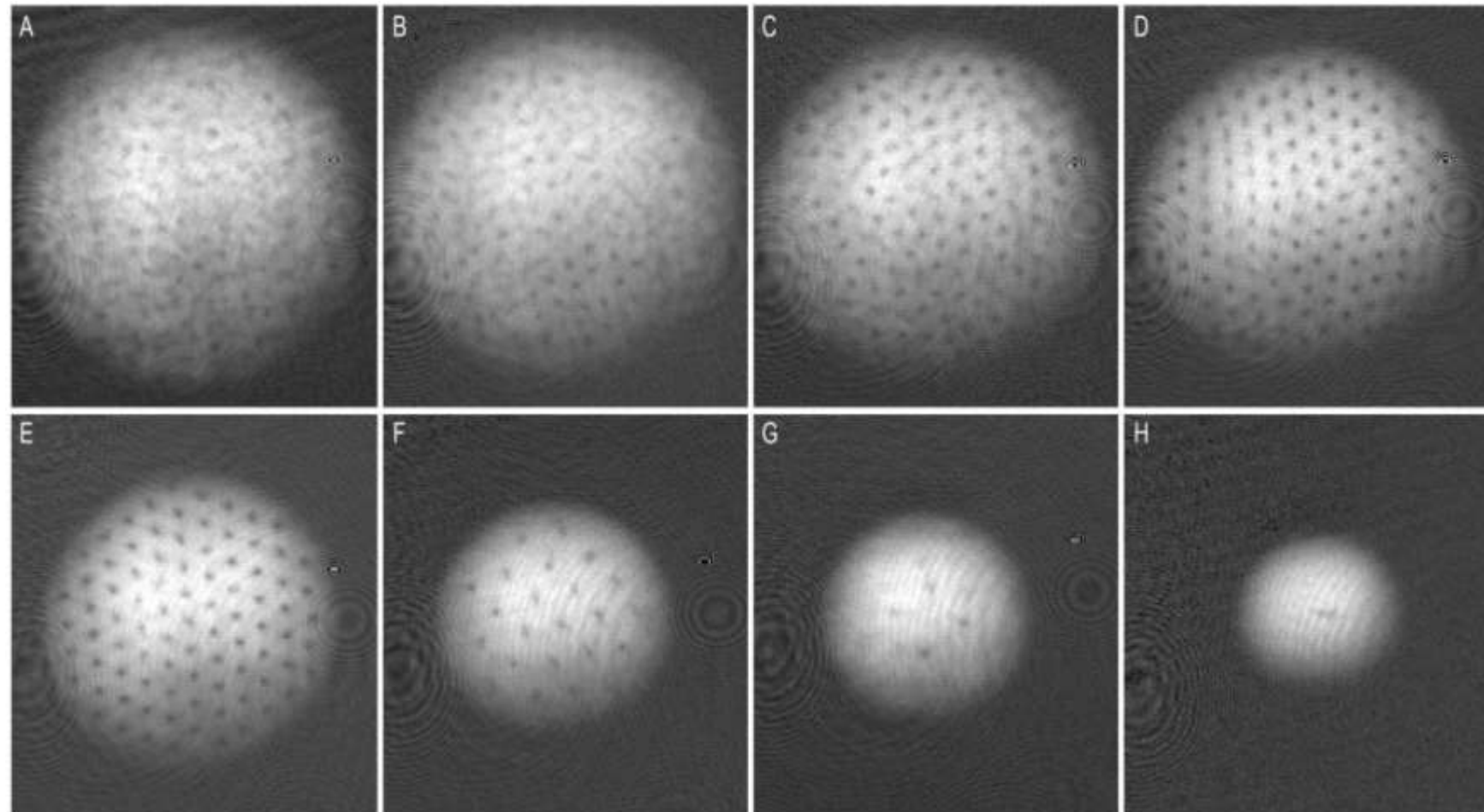
E.J. Yarmchuk, M.J.V. Gordon, R.E. Packard, Phys. Rev. Lett. **43**, 214 (1979)

E.J. Yarmchuk, R.E. Packard, J. Low Temp. Phys. **46**, 479 (1982)

# I.4.2 Experimental Study of Vortices

- imaging of vortices in a Bose-Einstein condensate of Na atoms

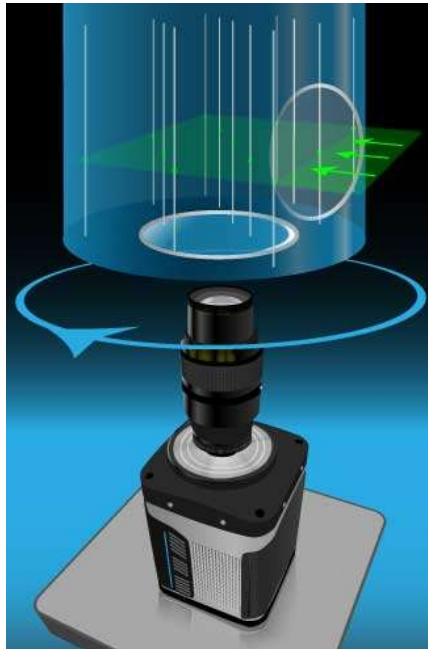
Formation and decay of a vortex lattice. The condensate was rotated for 400 ms and then equilibrated in the stationary magnetic trap for various hold times. **(A)** 25 ms, **(B)** 100 ms, **(C)** 200 ms, **(D)** 500 ms, **(E)** 1 s, **(F)** 5 s, **(G)** 10 s, and **(H)** 40 s. The decreasing size of the cloud in (E) to (H) reflects a decrease in atom number due to inelastic collisions. The field of view is about 1 mm by 1.15 mm.



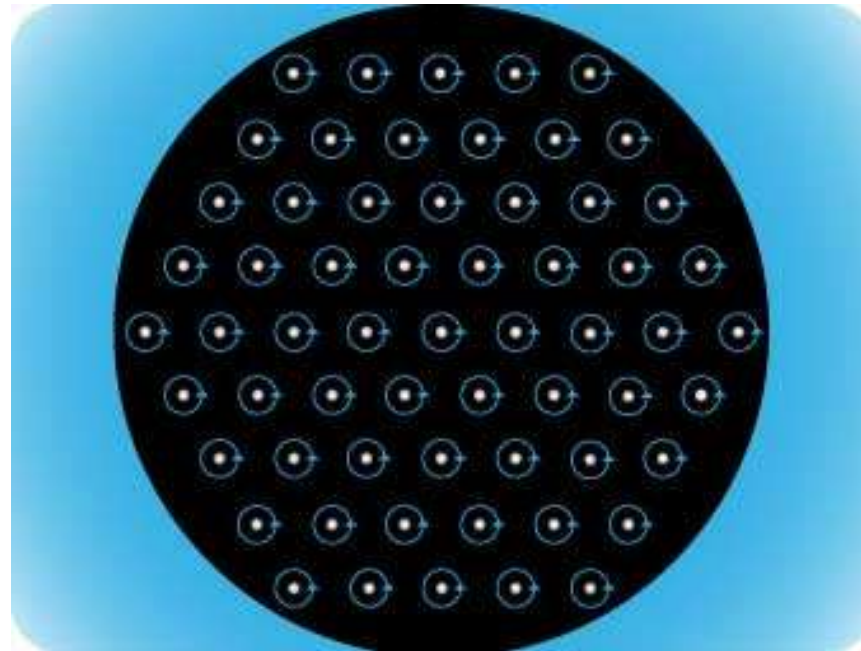
Observation of Vortex Lattices in Bose-Einstein Condensates  
 W. Ketterle et al., Science **292**, 476 (2001)

# I.4.2 Experimental Study of Vortices

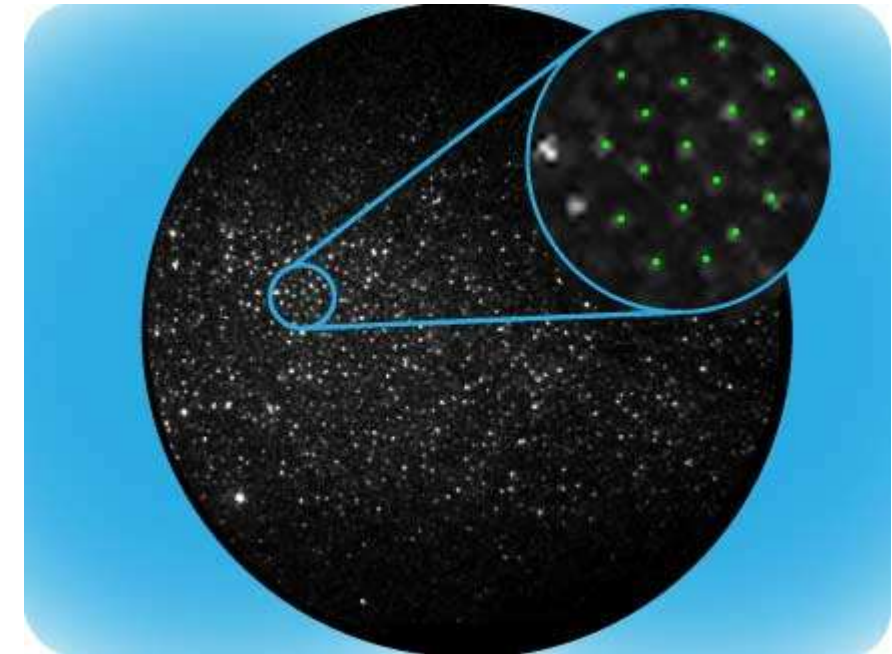
- imaging of vortices



A sheet of laser light illuminates frozen hydrogen particles trapped on vortex cores, which are then captured on video.



Quantized vortices arrange themselves in a triangular lattice pattern.



A still image captures the vortex lattice along the axis of rotation in a vat of superfluid helium. A perfect lattice is not observed due to waves and other boundary effects.

<https://engage.aps.org/dfd/resources/virtual-presroom/2009-presroom/language-papers-2009/tornadoes>

# I.4.2 Experimental Study of Vortices

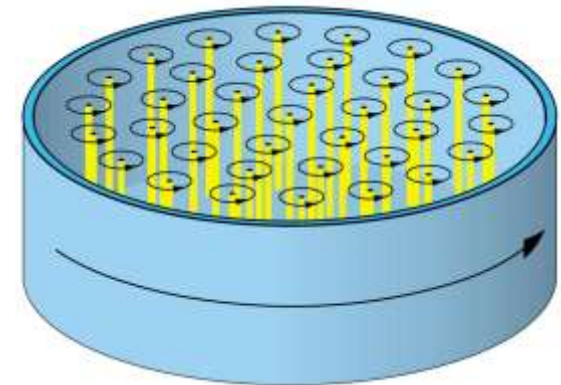
- generation of persistent current

- rotate a normal fluid at  $T > T_\lambda$
- cool down the rotating normal fluid to  $T < T_\lambda \rightarrow$  the rotation is frozen in and does not decay

- generation of circulation jumps or phase slip events

- start with a non-rotating fluid at  $T < T_\lambda$ :  $\kappa = 0, n = 0$
- rotate the cylindrical vessel containing the fluid:
  - $\rightarrow$  normal fluid rotates at the speed of the vessel
  - $\rightarrow$  superfluid stays in rest
- reduce the temperature:

- $\rightarrow$  more and more particles of the normal fluid enter the superfluid
- $\rightarrow$  the lost angular momentum is transferred to the superfluid, which starts to rotate
- $\rightarrow$  the circulations and the vorticity  $\kappa$  do not change continuously but in jumps of  $h/m_{\text{He}}$
- $\rightarrow$  jumps are associated with **additional vortex** corresponding to phase change of  $2\pi$  called **phase slip**
- $\rightarrow$  phase slips correspond to **transition between two quantum states** with different quantum number  $n$





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- I.3.1 Two-Fluid Model
- I.3.2 Experimental Observations
- I.3.3 Two-Fluid Hydrodynamics
- I.3.4 Excitation Spectrum of  $^4\text{He}$

- basic properties of <sup>3</sup>He

<sup>4</sup> He	p <sup>+</sup> ↑ p <sup>+</sup> ↓	n↑ n↓	e <sup>-</sup> ↑ e <sup>-</sup> ↓	<b>boson</b>
<sup>3</sup> He	p <sup>+</sup> ↑ p <sup>+</sup> ↓	n↑	e <sup>-</sup> ↑ e <sup>-</sup> ↓	<b>fermion</b>

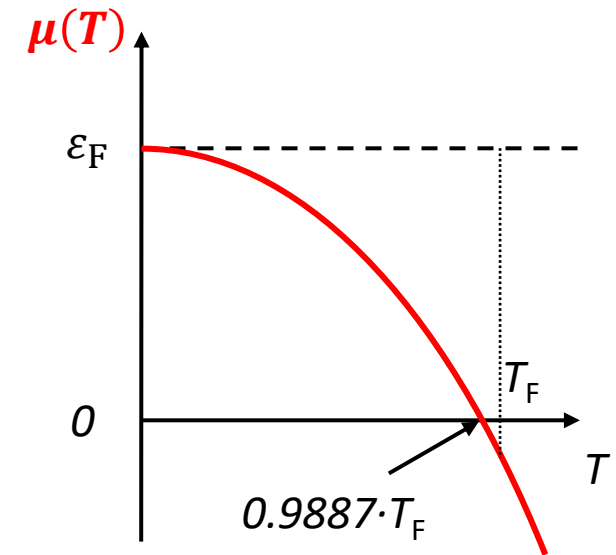
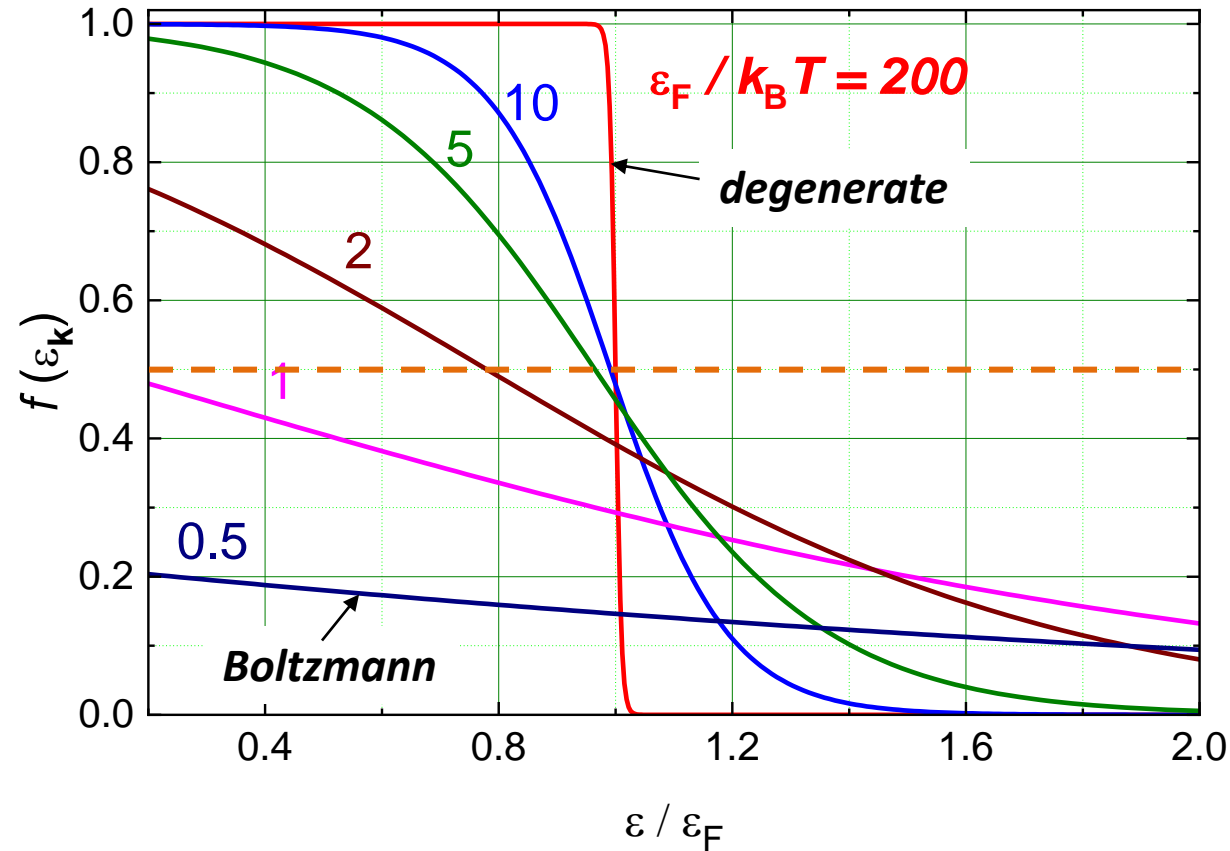
- <sup>3</sup>He has spin ½ → **Fermion**
  - fluid <sup>3</sup>He: **interacting Fermi gas** → Fermi liquid (*ideal non-interacting Fermi gas with finite interaction*)  
→ theoretical description by Landau (1956-1958)  
L.D. Landau, Soviet. Phys. JETP 3, 920 (1957)  
L.D. Landau, Soviet. Phys. JETP 5, 101 (1957)
  - solid <sup>3</sup>He shows nuclear magnetism
- due to different quantum statistics:  
*liquid <sup>3</sup>He has completely different properties than liquid <sup>4</sup>He*

# I.5.1 Normal Fluid <sup>3</sup>Helium

- first approximation: description by an *ideal Fermi gas* (*non-interacting Fermions*)

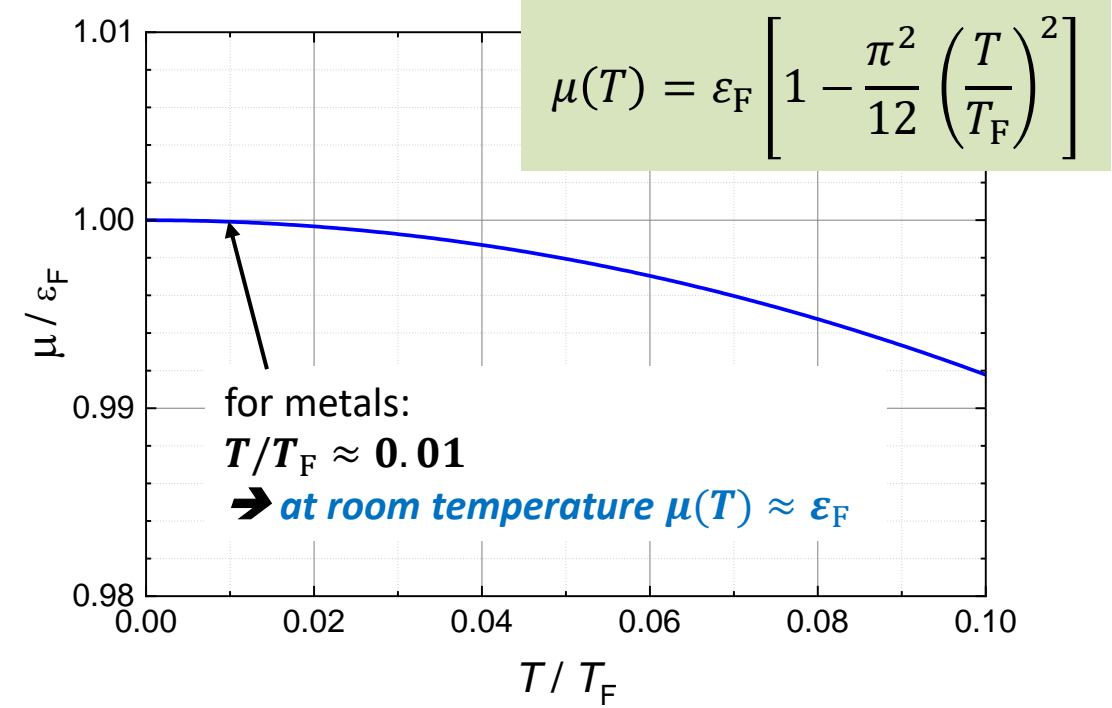
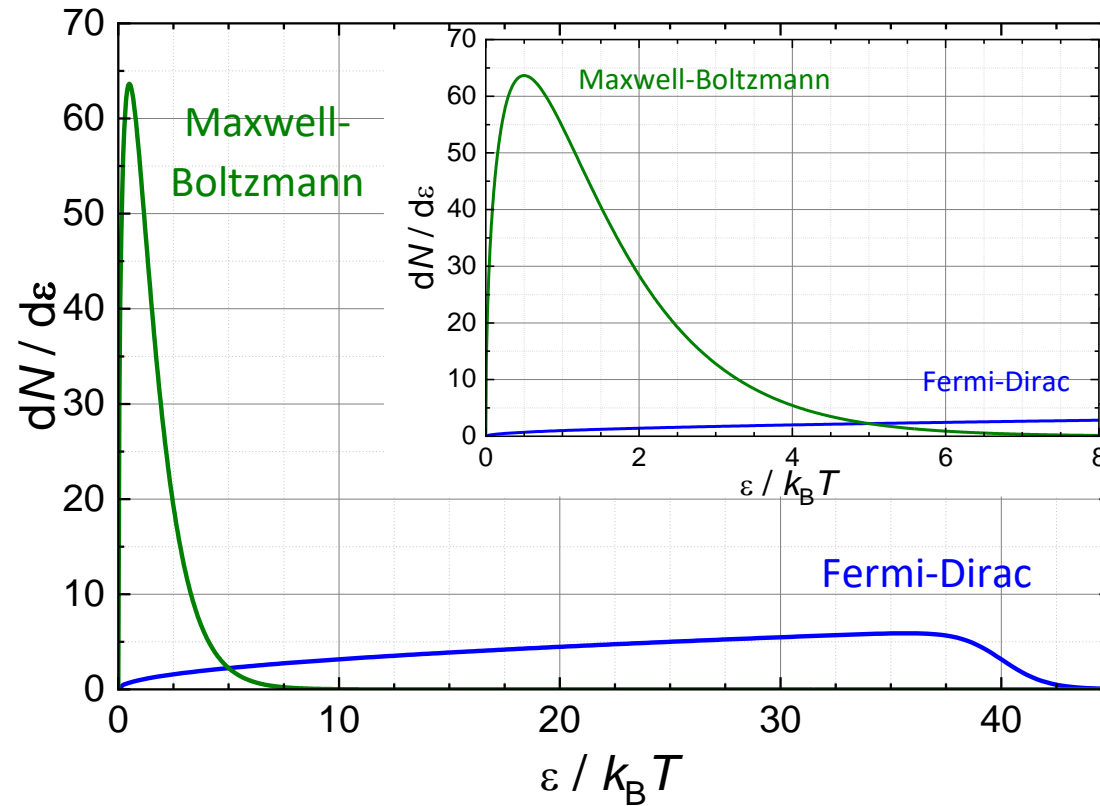
– Fermi-Dirac distribution:

$$f(\epsilon_{\mathbf{k}}) = \frac{1}{\exp\left(\frac{\epsilon_{\mathbf{k}} - \mu}{k_B T}\right) + 1}$$



# I.5.1 Normal Fluid $^3\text{He}$

- ideal (non-interacting) Fermi gas



$$N = \int D(\varepsilon) f(\varepsilon) d\varepsilon$$

$$\frac{dN}{d\varepsilon} = D(\varepsilon) f(\varepsilon)$$

# I.5.1 Normal Fluid $^3\text{He}$

- ideal (non-interacting) Fermi gas

$$\varepsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m} (3\pi^3 n)^{2/3} \quad \textit{Fermi energy}$$

$$T_F = \varepsilon_F / k_B \quad \textit{Fermi temperature}$$

$$\lambda_F = 2\pi / k_F \quad \textit{Fermi wavelength}$$

$$v_F = \frac{p_F}{m} = \frac{\hbar k_F}{m} \quad \textit{Fermi velocity}$$

$N$  = particle number  
 $n$  = particle density [ $1/\text{m}^3$ ]

$m = m_{\text{He}}$  is large  $\rightarrow$  expected Fermi temperature (energy) is small:  $T_F \simeq 4.2 \text{ K}$

comparison to electron gas in metal:

**electrons in metals:**

$n \approx 10^{23} \text{ cm}^{-3}$  is large,  $m = m_e$  is small  
 $\rightarrow T_F \approx 10^5 \text{ K}$  is large

**$^3\text{He}$ :**

$n \approx 10^{22} \text{ cm}^{-3}$  is smaller,  $m = m_{\text{He}}$  is large  
 $\rightarrow T_F \approx \text{few K}$  is small

$$T_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2mk_B} (3\pi^3 n)^{2/3}$$

# I.5.1 Normal Fluid <sup>3</sup>Helium

- ideal (non-interacting) Fermi gas: specific heat

– classical:  $C_V^{\text{class}} = \left. \frac{\partial \langle U \rangle}{\partial T} \right|_V = N \cdot 3 \cdot \frac{1}{2} k_B = \frac{3}{2} N k_B$  (Dulong-Petit)

kinetic degrees of freedom  
contribution per degree of freedom (equipartition theorem)

– quantum mechanical:  $U = \int_0^\infty \varepsilon D(\varepsilon) f(\varepsilon) d\varepsilon$

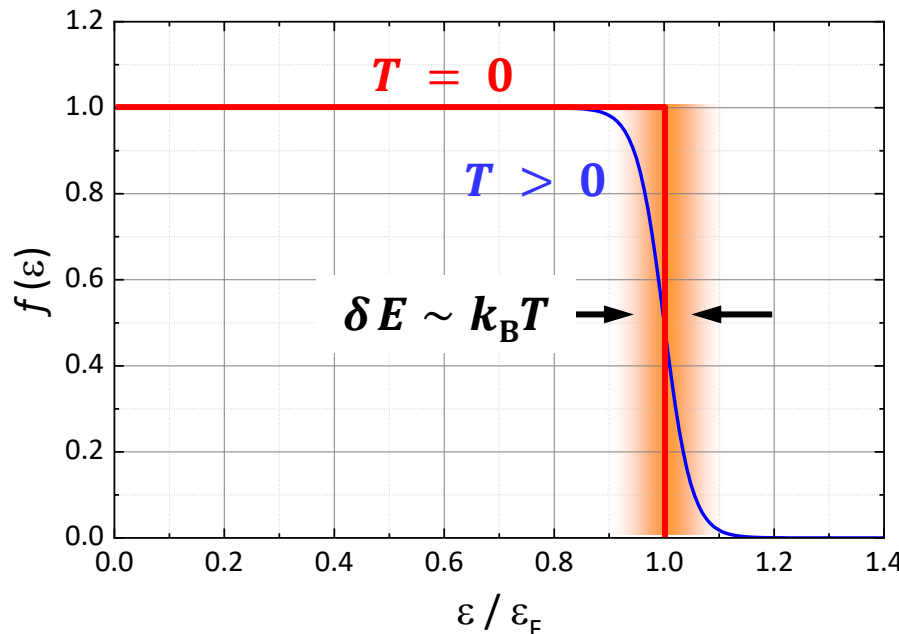
Sommerfeld expansion  $U = U(T=0) + (k_B T)^2 \frac{\pi^2}{6} D(\varepsilon_F)$

$$\frac{D(\varepsilon_F)}{V} = \frac{3n}{2\varepsilon_F}$$

$$\frac{C_V^{\text{qm}}}{V} = \frac{\pi^2}{2} n k_B^2 \frac{T}{\varepsilon_F} = \gamma \cdot T$$

Sommerfeld coefficient

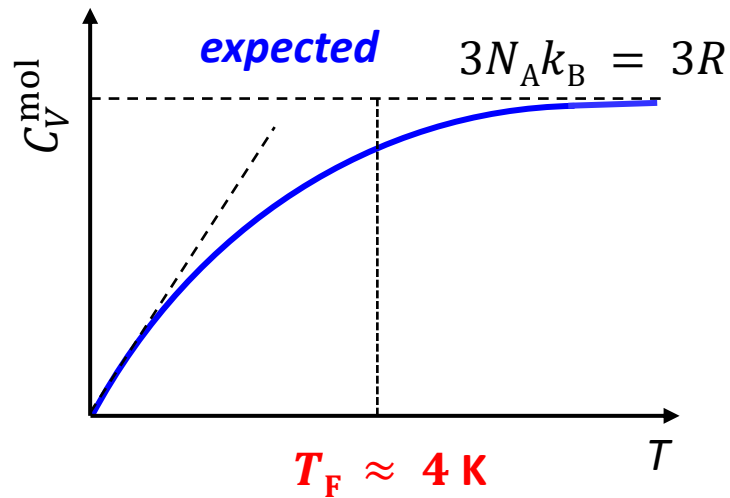
$$\gamma = \frac{\pi^2}{3} k_B^2 \frac{D(\varepsilon_F)}{V} = \frac{\pi^2}{2} \frac{n k_B^2}{\varepsilon_F}$$



- only small fraction  $\approx T/T_F$  of particles in energy interval  $k_B T$  around  $E_F$  contributes to specific heat
- energy change per particle  $\approx k_B T$
- total energy change  $\approx k_B T \cdot N k_B \left( \frac{T}{T_F} \right) \propto D(E_F) T^2$
- $C_V \propto D(E_F) T$

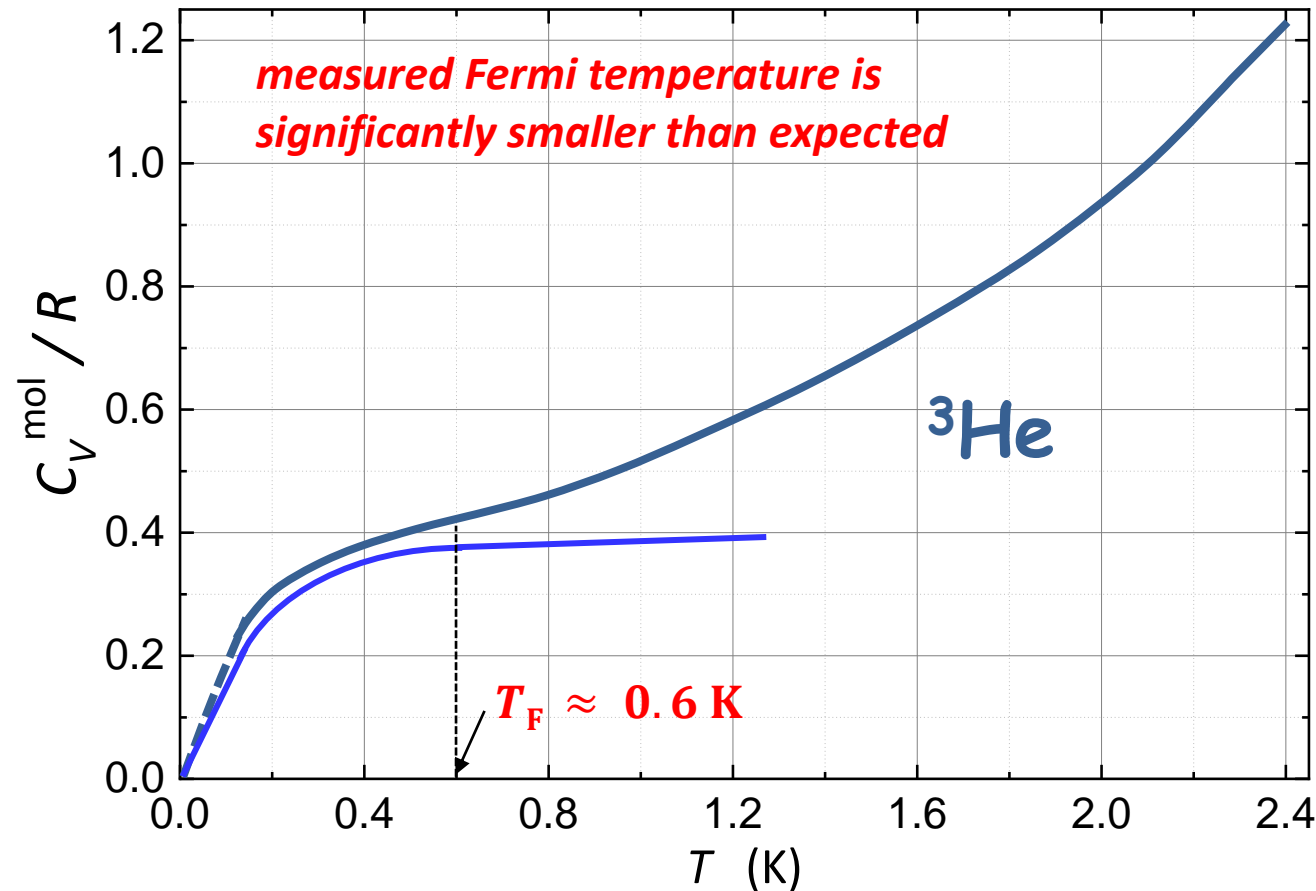
# I.5.1 Normal Fluid $^3\text{He}$

- ideal (non-interacting) Fermi gas: specific heat



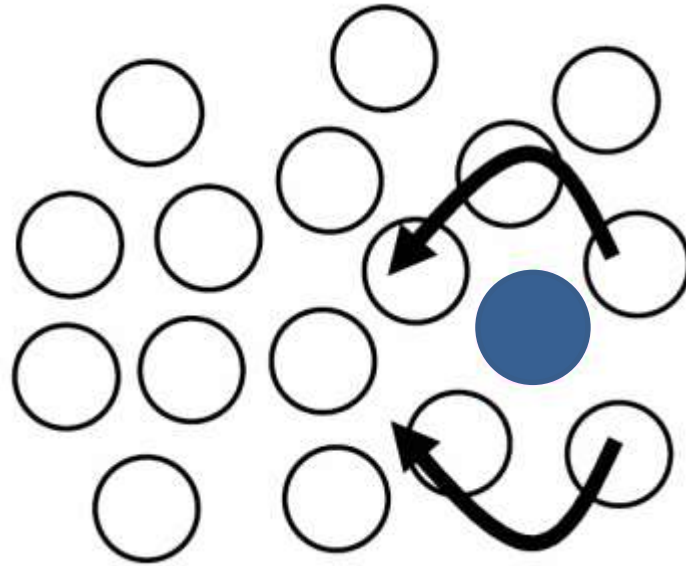
- interacting Fermi gas → **Fermi liquid** (Landau)
- increased effective mass  $m^*$ 
  - smaller Fermi temperature  $T_F \propto 1/m^*$
  - $m^*/m_{\text{He}} \approx 2.8$  (@  $p = 1 \text{ bar}$ )
  - $\approx 5.5$  (@  $p = 30 \text{ bar}$ )

- additional contribution due to **spin fluctuations (paramagnons)**
  - strong repulsive interaction at short distance → antisymmetric orbital wavefunction
  - symmetric spin wavefunction → parallel orientation of neighboring spins



# I.5.1 Normal Fluid $^3\text{He}$

- intuitive argument for increased effective mass:



interaction causes „backflow“ of particles

→ increased effective mass  $m^*$

→ reduced Fermi temperature  $T_F \propto 1/m^*$



# I.5.1 Normal Fluid <sup>3</sup>Helium

- magnetic susceptibility: nuclear spin  $I = 1/2$  determines magnetic properties

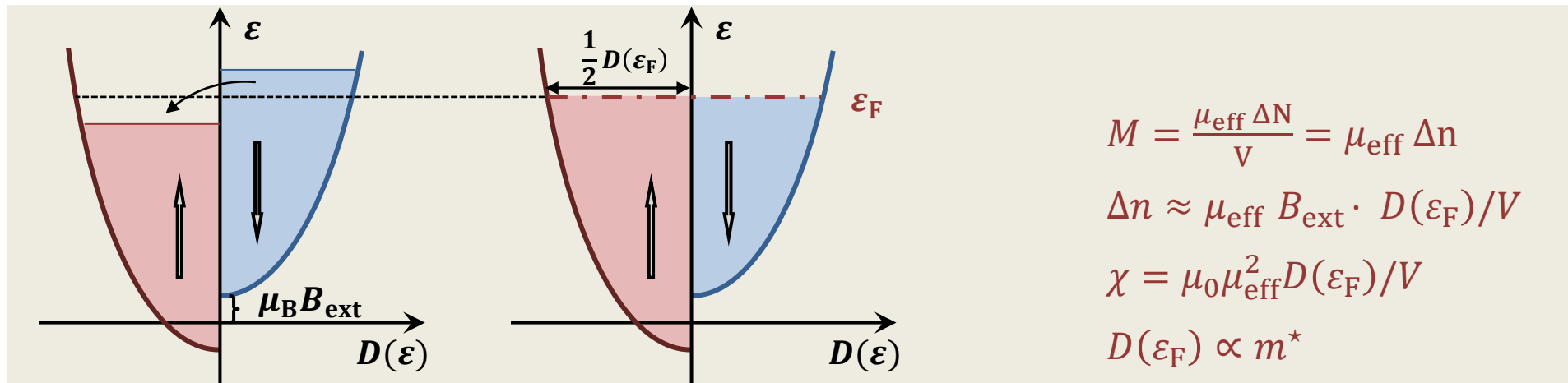
- $T > T_F$ : **non-degenerate Fermi gas** → magnetic susceptibility follows **Curie law**:  $\chi \propto C/T$

$$\chi = \mu_0 \left( \frac{\partial M}{\partial B_{\text{ext}}} \right)_{T,V} = \frac{\mu_0 n \mu_{\text{eff}}^2}{k_B T} = \frac{C}{T} \quad \mu_{\text{eff}} = g_n \mu_n \quad \begin{array}{l} g_n = \text{Landé factor} \\ \mu_n = \text{magnetic moment of } ^3\text{He} \end{array}$$

- $T < T_F$ : **degenerate Fermi gas** → temperature independent **Pauli spin susceptibility** (only fraction  $T/T_F$  contributes)

for electron liquid:  $\chi_P = \mu_0 \left( \frac{\partial M}{\partial B_{\text{ext}}} \right)_{T,V} = \mu_0 \mu_B^2 \frac{D(\epsilon_F)}{V} = \frac{3 \mu_0 n \mu_B^2}{2 k_B T_F} \approx \text{const.}$

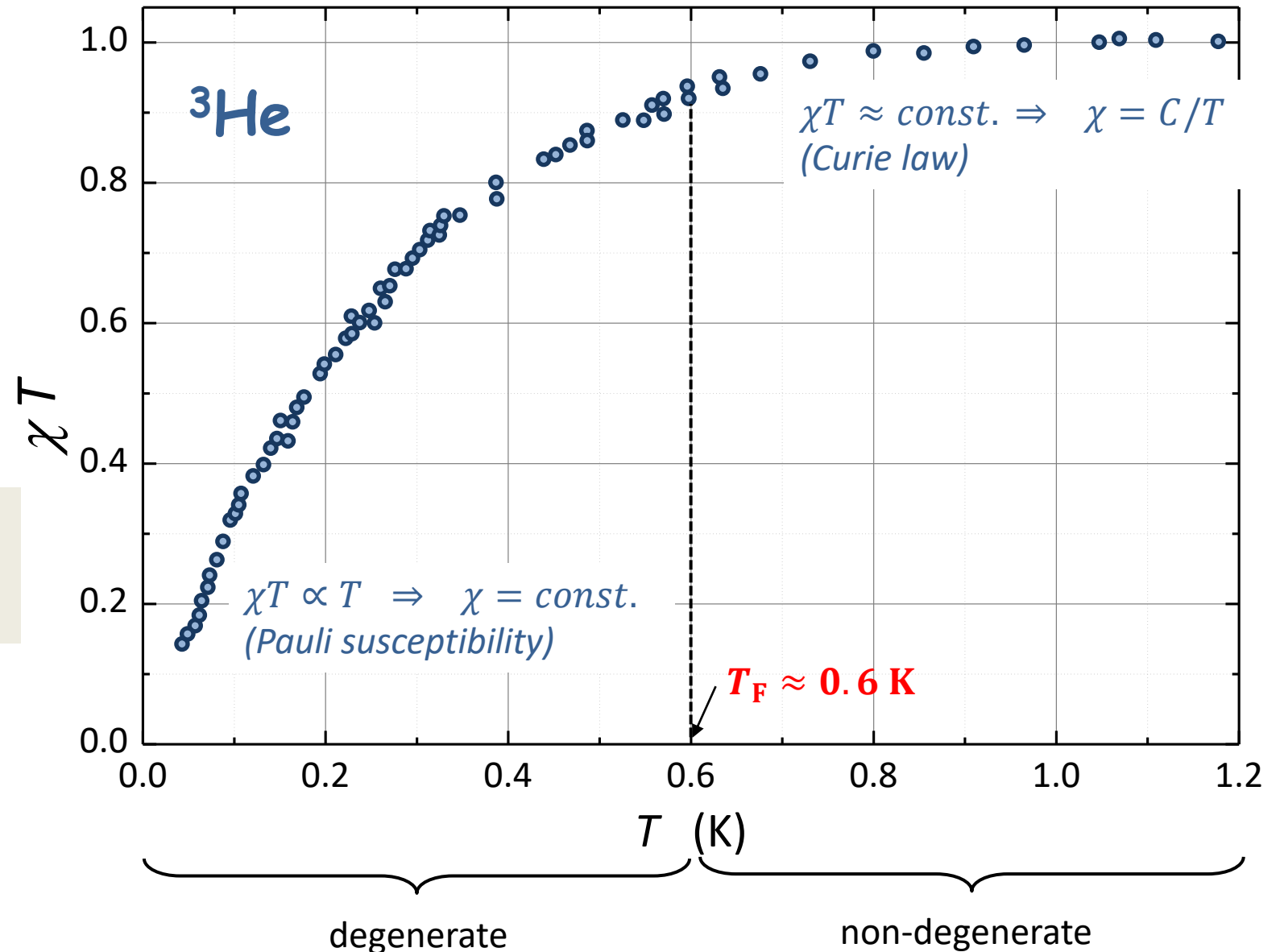
for <sup>3</sup>He liquid:  $\chi_P = \mu_0 \mu_{\text{eff}}^2 \frac{D(\epsilon_F)}{V}$  with  $\mu_{\text{eff}} = g_n \mu_n$



# I.5.1 Normal Fluid $^3\text{He}$

- magnetic susceptibility

cannot be measured for an electron gas system in a metal since  $T_F \gg$  melting temperature



# I.5.1 Normal Fluid $^3\text{He}$

- transport properties

- description by *kinetic gas theory*:

→ viscosity:  $\eta = \frac{1}{3} \rho v \ell$  ( $\ell$  = mean free path)

→ diffusion coefficient:  $D = \frac{1}{3} v \ell$  (self-diffusion)

→ thermal conductivity:  $\kappa = \frac{1}{3} C_V v \ell$

- classical gas:  $v_{\text{th}}$  = thermal velocity  $v_{\text{th}} = \sqrt{2k_B T/m}$  (Maxwell-Boltzmann statistics)

- Fermi gas:  $v_F$  = Fermi velocity  $v_F = \frac{\hbar}{m} (3\pi^3 n)^{1/3}$  (Fermi statistics)

- Fermi gas, with  $\ell = v_F \tau$

$$\eta = \frac{1}{3} \rho \tau v_F^2 \propto \frac{1}{T^2}$$

$$D = \frac{1}{3} \tau v_F^2 \propto \frac{1}{T^2}$$

$$\kappa = \frac{1}{3} C_V \tau v_F^2 \propto \frac{1}{T}$$

$\tau^{-1} \propto \left(\frac{T}{T_F}\right)^2$   
particle-particle scattering rate



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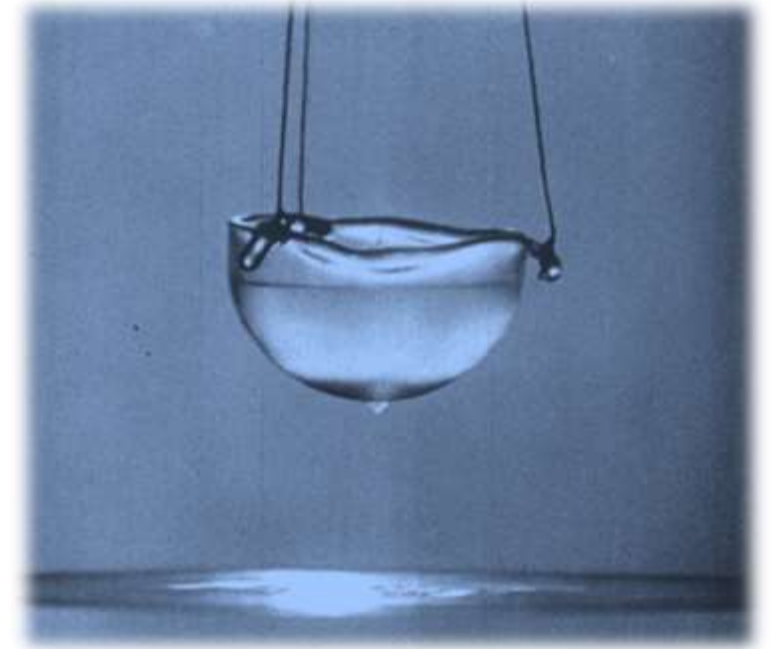
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Technische  
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**TUM**

# Superconductivity and Low Temperature Physics II



**Lecture No. 7**  
**23 June 2022**

**R. Gross**  
**© Walther-Meißner-Institut**

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- I.5.3 superfluid  $^3\text{He}$



### I.6 $^3\text{He}$ / $^4\text{He}$ mixtures

# I.5.1 Normal Fluid $^3\text{He}$

- comparison of properties of liquid  $^3\text{He}$  with those of ideal (non-interacting) Fermi gas

	$^3\text{He}$	Fermi Gas	ratio
$C_V^{\text{mol}} / N_A k_B T$	<b>2.78</b>	<b>1.0</b>	<b>2.78</b>
$v_F$ (m/s)	<b>56</b>	<b>157</b>	<b>2.80</b>
$\chi / \mu_0 \mu_{\text{eff}}^2$ (Jm <sup>3</sup> ) <sup>-1</sup>	<b><math>3.3 \cdot 10^{51}</math></b>	<b><math>3.6 \cdot 10^{50}</math></b>	<b>9.2</b>

*significant deviations due to finite interactions* → Landau Fermi liquid  
 → effective mass

mole volume:  $36.84 \text{ cm}^3$  @  $p = 0 \text{ bar}$ ,  $T = 0 \text{ K}$  →  $n = 1.63 \times 10^{28} \text{ m}^{-3}$   
 mass:  $m_{^3\text{He}} = 3.016 \text{ amu} = 5.006 \times 10^{-27} \text{ kg}$

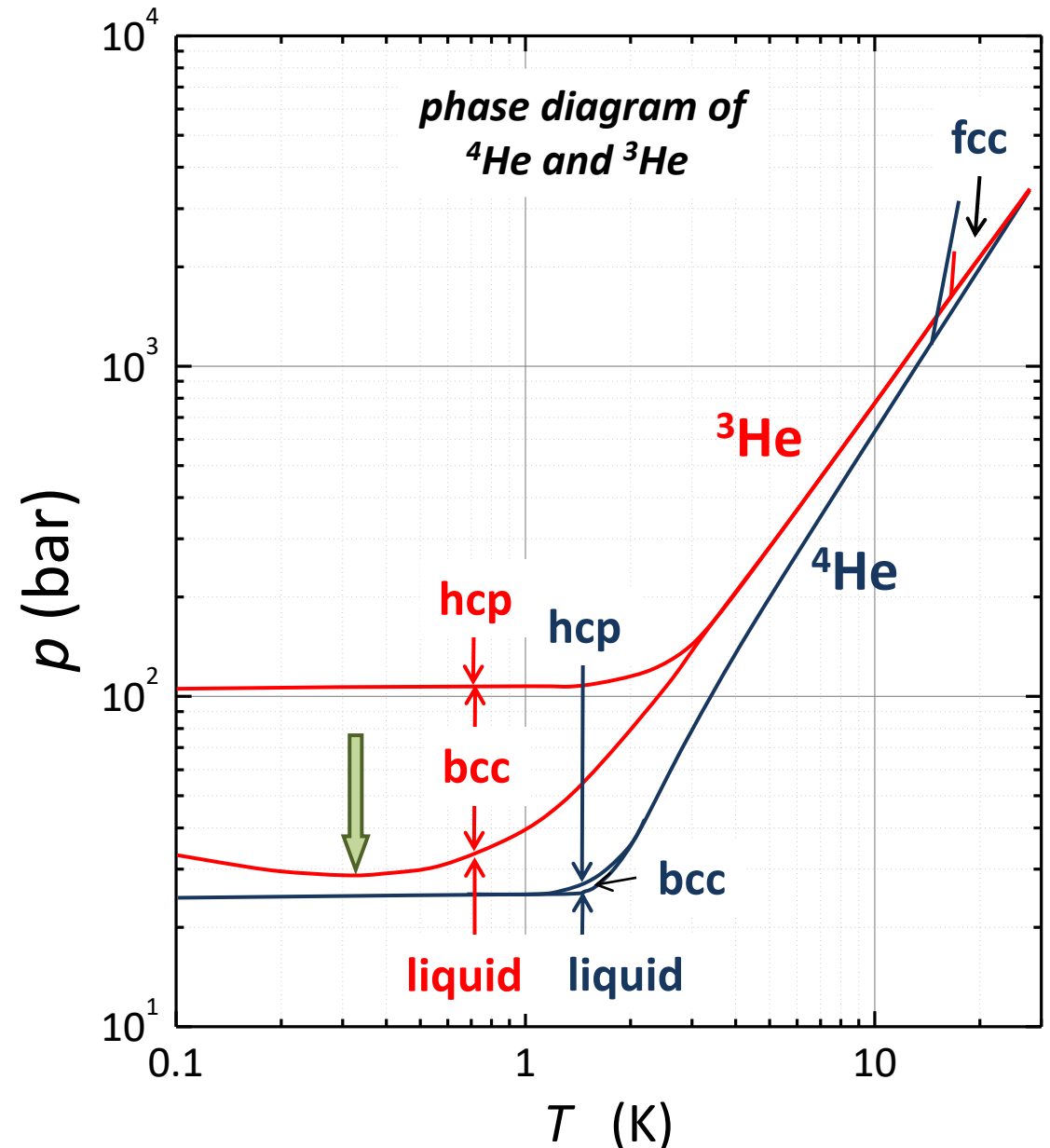
# I.5.2 Solid $^3\text{He}$ and Pomerantchuk Effect

- comparison of phase diagrams of  $^3\text{He}$  and  $^4\text{He}$

- $^4\text{He}$  mostly hcp, small region with bcc, at high  $T$ : fcc
- $^3\text{He}$  shows bcc-hcp transition at 100 bar, at high  $T$ : fcc
- $^3\text{He}$  shows **minimum in melting curve** at  **$T = 0.32 \text{ K}$**

→ can be used for cooling of  $^3\text{He}$   
**Pomerantchuk effect**

**Isaak Jakowlewitsch Pomerantschuk**  
(Исаак Яковлевич Померанчук;  
geb: 20. Mai 1913 in Warschau;  
gest: 14. Juli 1966 in Moskau)  
russischer Physiker.



# I.5.2 Solid $^3\text{He}$ and Pomerantchuk Effect

– Clausius-Clapeyron equation:  $\left. \frac{\partial p_m}{\partial T} \right|_{\text{melting}} = \frac{S_{\text{liq}} - S_{\text{sol}}}{V_{\text{liq}} - V_{\text{sol}}} = \frac{\Delta S_m}{\Delta V_m} = \frac{\Delta Q_m}{\Delta V_m T}$  and  $V_{\text{liq}} > V_{\text{sol}}$

for  $T < 0.32 \text{ K}$ :  $\left. \frac{\partial p_m}{\partial T} \right|_{\text{melting}} < 0 \Rightarrow S_{\text{liq}} < S_{\text{sol}} \rightarrow$  *disorder larger in solid than in liquid phase ??*

– **explanation:**

→ **solid phase:** atoms are ordered,

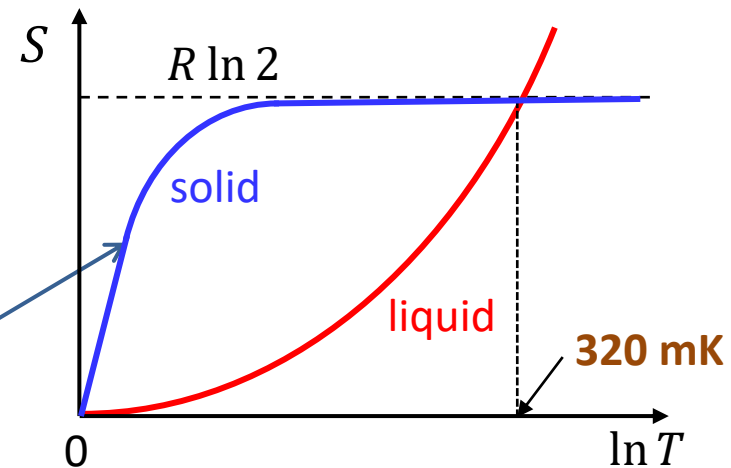
→ spins are disordered and determine entropy:  $S_{\text{solid}} = R \ln 2$   
 at low  $T$ : antiparallel ordering of spins,  $S$  decreases towards zero

→ **liquid phase:** atoms are spatially disordered, but ordering in  $k$ -space (Fermi liquid)

→ entropy of Fermi liquid:

$$S_{\text{liquid}} = \int_0^T \frac{C_V}{T'} dT' \propto \frac{T}{T_F}$$

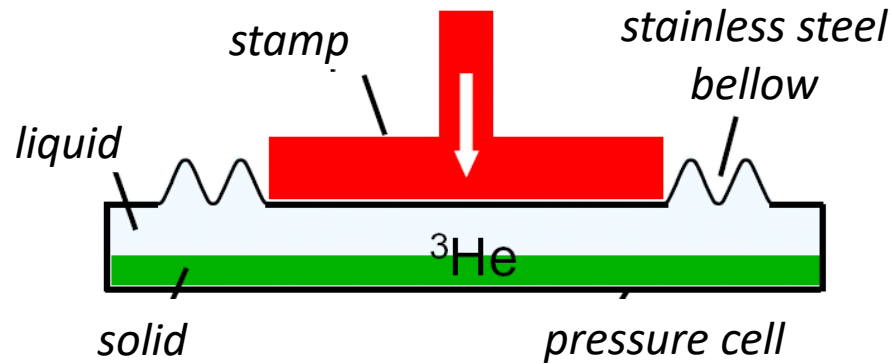
*drop due to interactions*



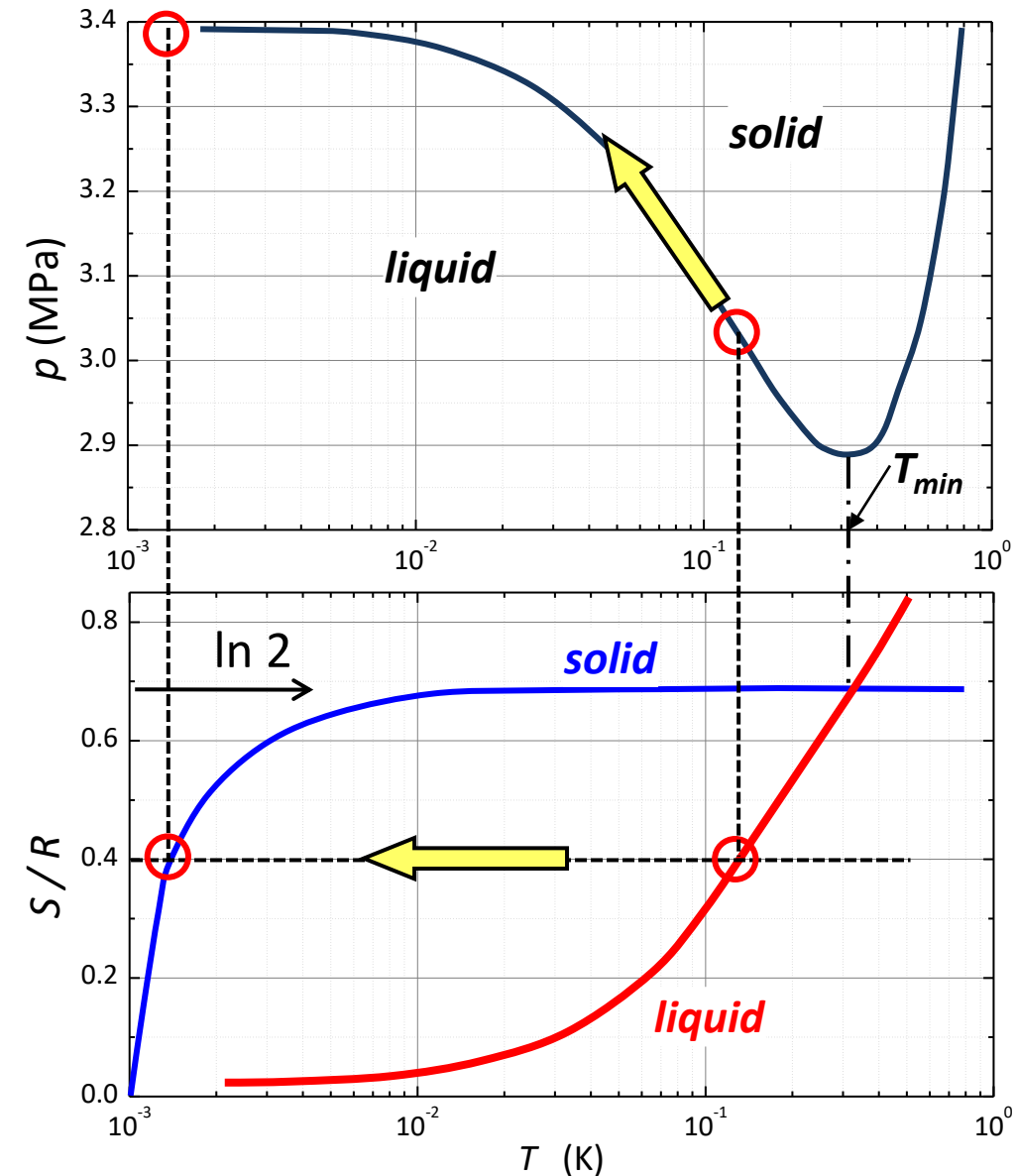


# I.5.2 Solid $^3\text{He}$ and Pomerantchuk Effect

- Pomerantchuk cooling

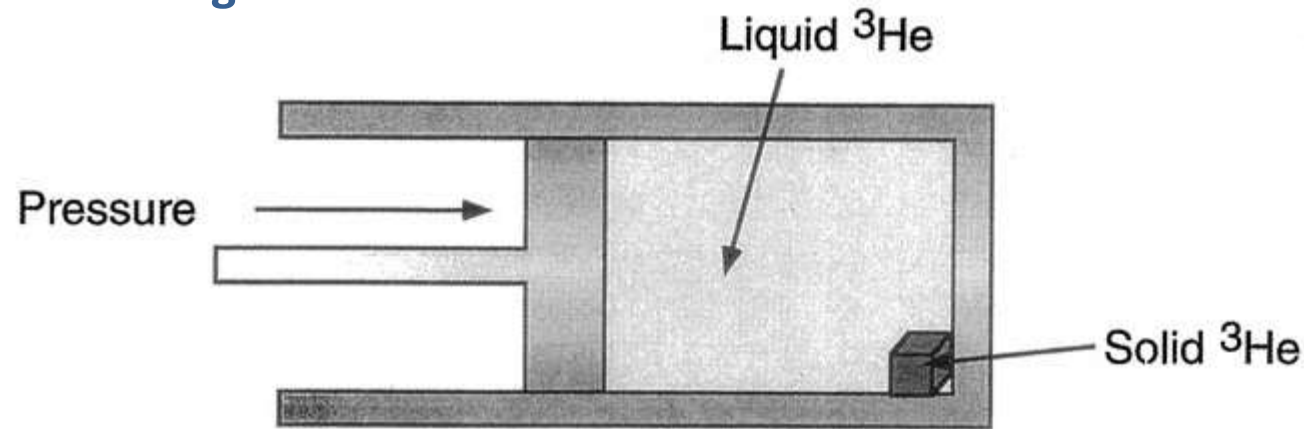


- precooling to  $T < T_{\min}$
- adiabatic compression  
→ *solidification and cooling*
- lowest  $T$ :  $\approx 1.5$  mK
- *limitation due to antiparallel spin ordering in solid  $^3\text{He}$*



# I.5.2 Solid $^3\text{He}$ and Pomerantchuk Effect

- Pomerantchuk cooling



Richardson,  
Nobel Prize Lecture (1996)

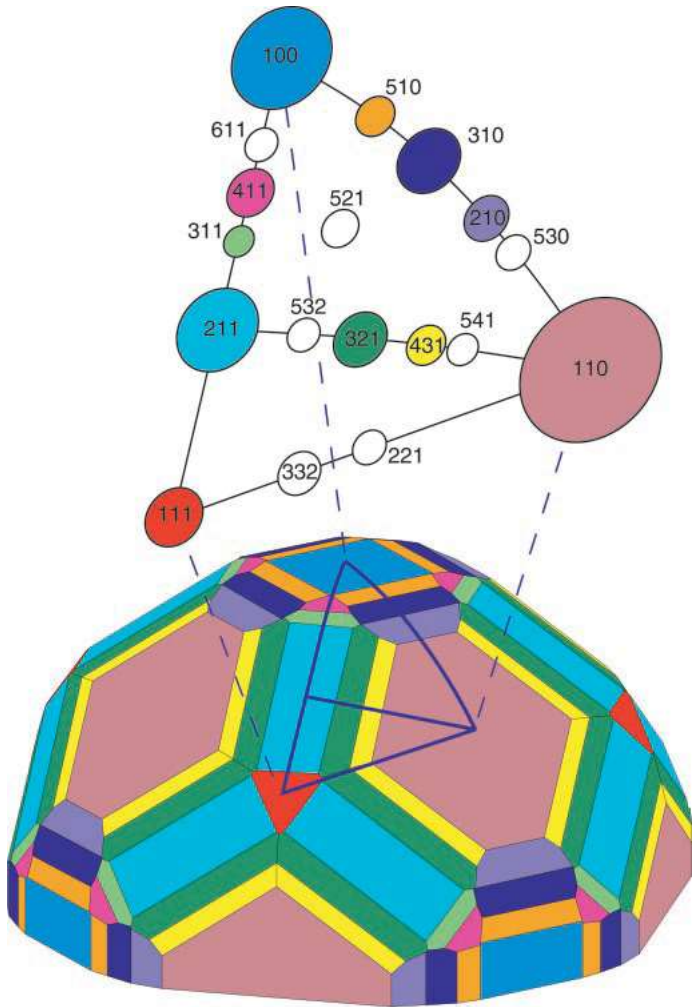
$$\frac{\Delta V}{V} \approx 5\% \quad \text{volume change for complete conversion of liquid to solid}$$

## Pomeranchuk's suggestion for cooling a melting mixture of $^3\text{He}$ :

The solid phase has a higher entropy than the liquid at low temperatures. As the liquid-solid mixture is compressed, heat is removed from the liquid phase as solid crystallites form. The fractional change of volume required to completely convert liquid into solid is approximately 5%. Unlike melting water, the solid phase forms at the hottest part of the container.

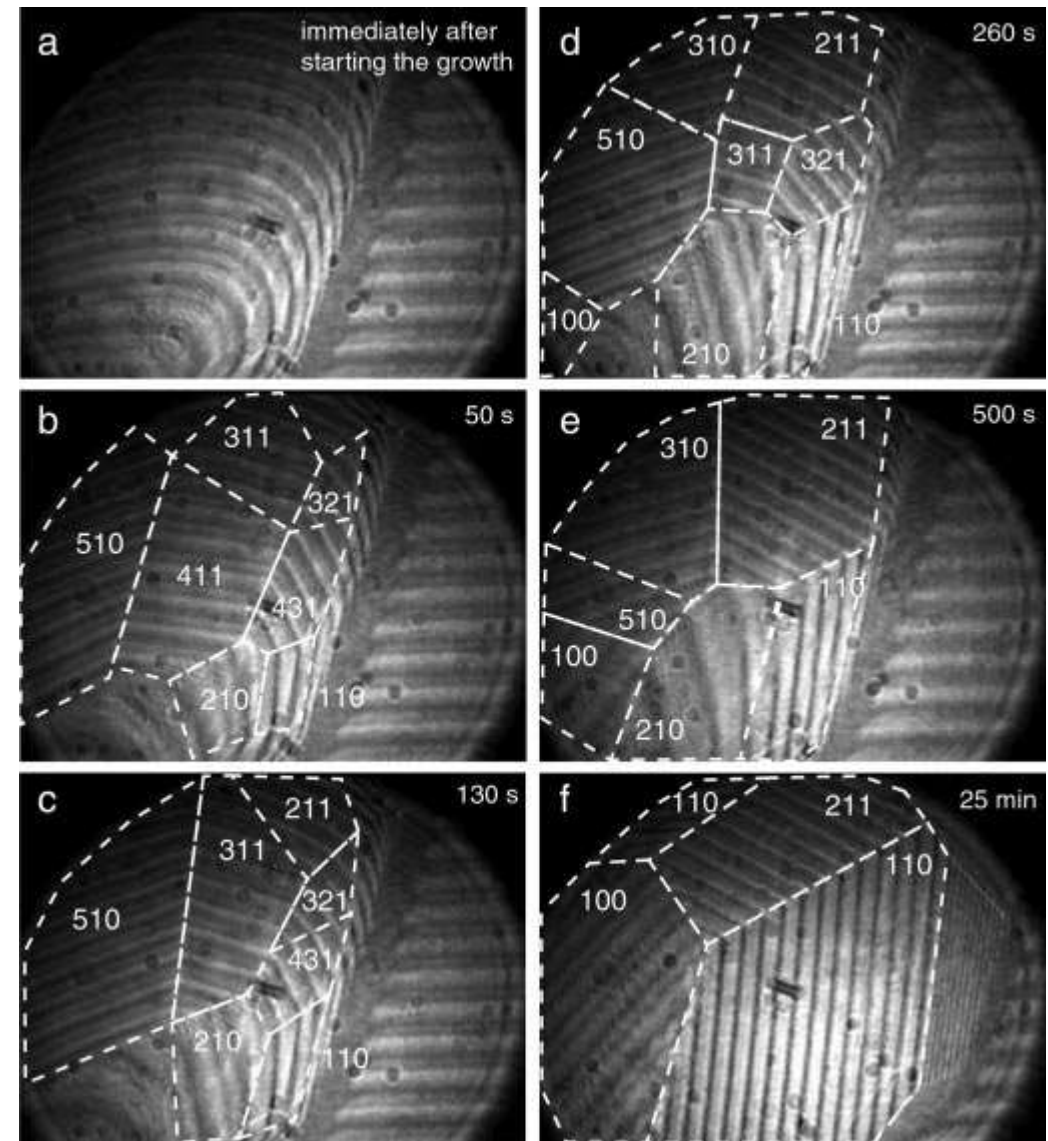
# I.5.2 Solid $^3\text{He}$ and Pomerantchuk Effect

- solid  $^3\text{He}$ : single crystals



computer-generated shape of a bcc crystal presented together with an elementary patch of a crystal habit where the facets are labeled with Miller indices.

H. Alles et al., PNAS **99**, 1796-1800 (2002)



Sequence of interferograms of a growing  $^3\text{He}$  crystal at  $T = 0.55$  mK. The dashed white lines outline the identified facets marked with Miller indices.

# I.5.3 Superfluid $^3\text{He}$

- **history of superfluid  $^3\text{He}$** 
  - **initial believe:** no superfluid  $^3\text{He}$  due to Pauli's exclusion principle for fermions  
→ no Bose-Einstein condensation of  $^3\text{He}$
  - **after development of BCS theory: pairing of  $^3\text{He}$  atoms to Cooper pairs ??**  
→ problem: liquid  $^3\text{He}$  is system of strongly interacting particles  
→ formation of Cooper pairs out of „naked“  $^3\text{He}$  atoms not possible (nature of exchange boson ??)  
→ Landau: weakly interacting quasiparticles in Fermi liquid
  - **1963:** critical pairing temperature of about 100 mK was expected
  - **1971:** discovery of superfluid  $^3\text{He}$  at  $T < 2.7$  mK by **Osheroff, Richardson** and **Lee** (used Pomerantschuk cooling)  
*(Nobel Prize in Physics 1996 „for their discovery of superfluid  $^3\text{He}$ “)*  
D.D. Osheroff, R.C. Richardson, D.M. Lee, Phys. Rev. Lett. **28**, 855 (1972),  
D.D. Osheroff, W.J. Gully, R.C. Richardson, D.M. Lee, Phys. Rev. Lett. **29**, 920 (1972)
  - **initially:** two superfluid phases **A** and **B**  
**later:** three phases (in magnetic field additional **A<sub>1</sub>** phase)
  - experimental result: pairs have spin  **$S = 1$** , **spin triplet pairs**  
→ pairs with  **$S = 1$**  must have  **$L = 1$**  (see below)  
→ occupation probability for same position is negligible (*reduction of short length repulsion*)

# I.5.3 Superfluid $^3\text{He}$

- discovery of superfluid  $^3\text{He}$  in 1971



**Douglas D. Osheroff,**  
Stanford University,  
Stanford, California, USA



**David M. Lee,**  
Cornell University, Ithaca,  
New York, USA



**Robert C. Richardson,**  
Cornell University, Ithaca,  
New York, USA

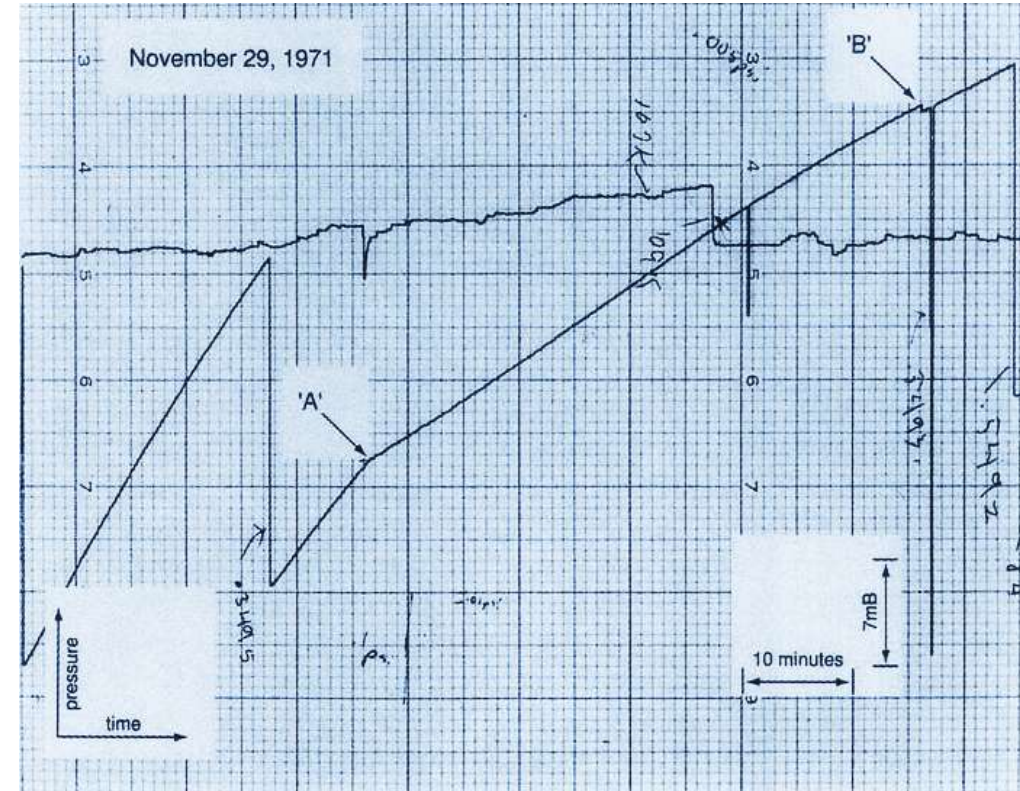
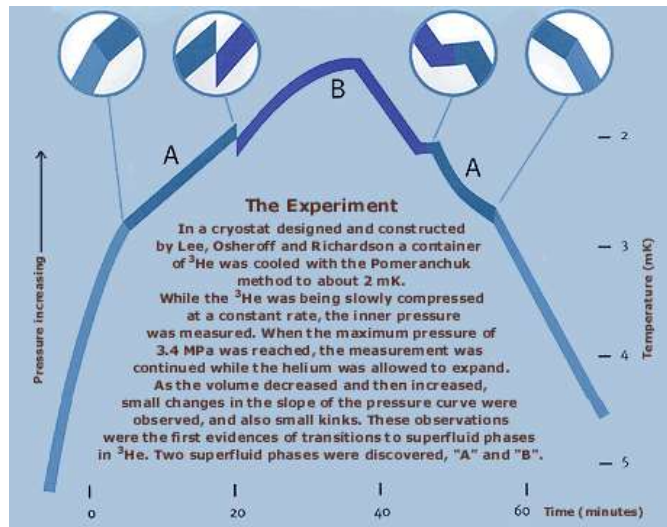
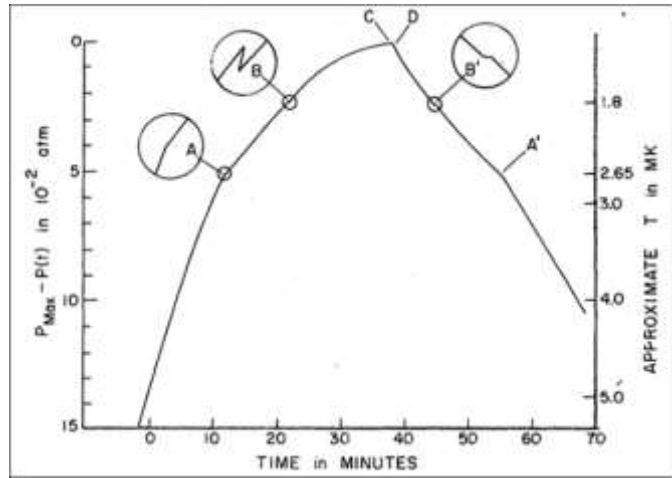
**The Nobel Prize in Physics 1996**

*"for their discovery of superfluidity in helium-3"*



# I.5.3 Superfluid $^3\text{He}$

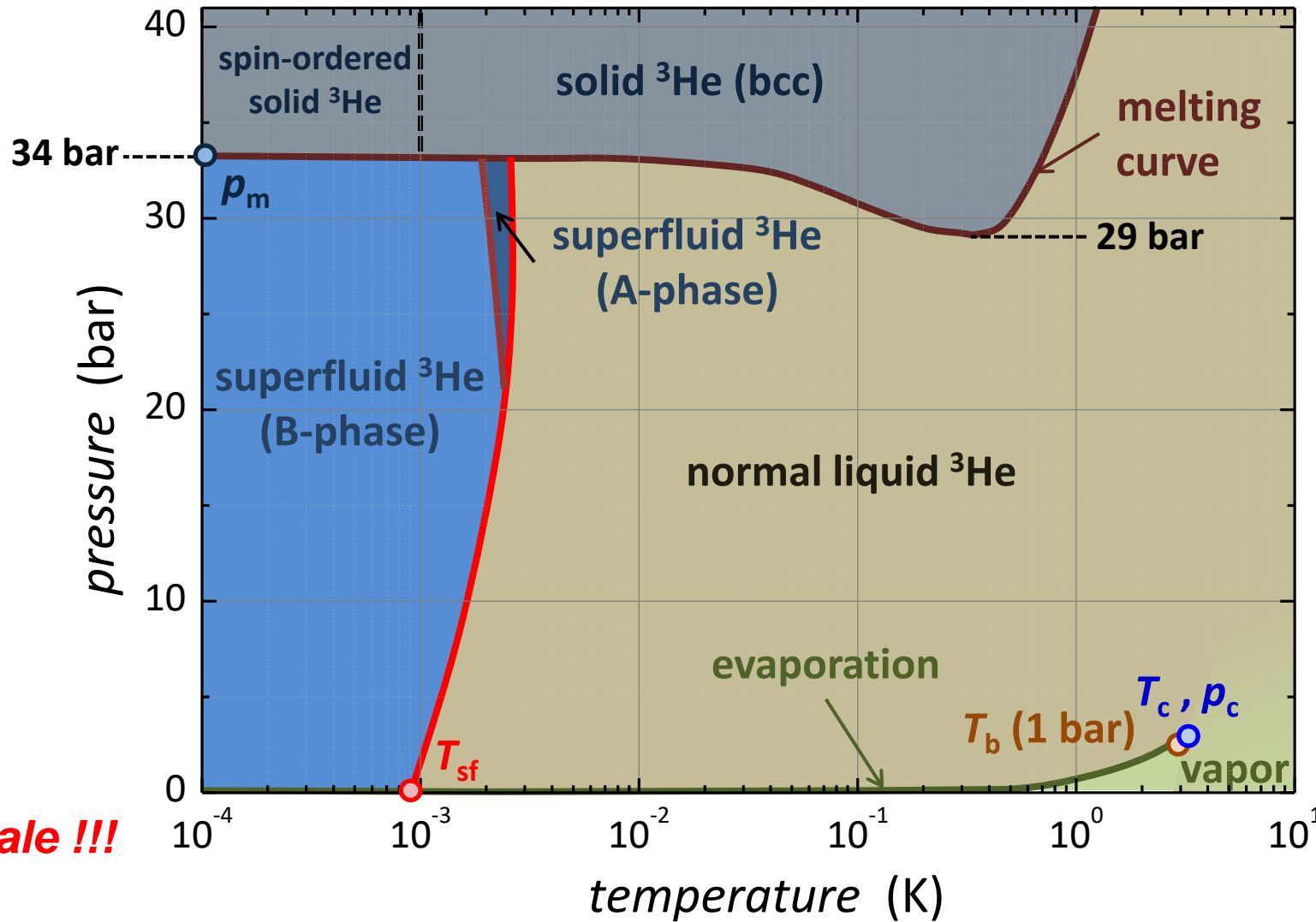
- discovery of superfluid  $^3\text{He}$  in 1971



- cool down of  $^3\text{He}$  via Pomeranchuk effect
- observation of melting curve
  - *anomalies give hint to phase transitions*

source: Nobel Prize Lecture Lee/Osheroff (1996)

# I.5.3 Superfluid $^3\text{He}$ : phase diagram for $B = 0$



big difference in value of  $T_{sf}$  compared to  $^4\text{He}$

- normal/superfluid transition is dominated by quantum effects
- quantum statistics matters!! (bosons vs. fermions)
- nature of superfluid phase in  $^4\text{He}$  and  $^3\text{He}$  is different

**log T scale !!!**

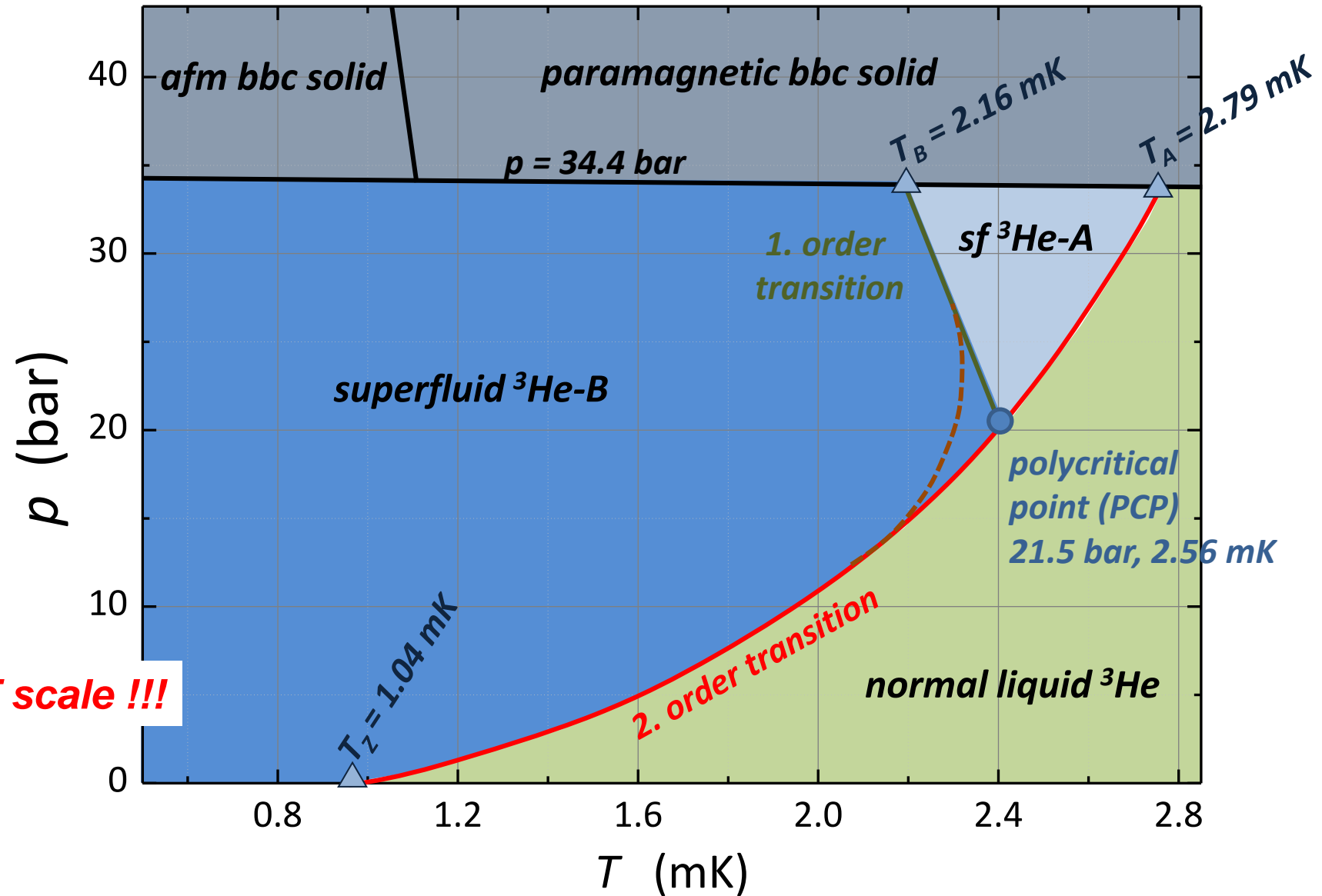
- phase diagram of  $^3\text{He}$ : note the logarithmic temperature scale. There are **two superfluid phases of  $^3\text{He}$ , A and B.**
- the line within the solid phase indicates a transition between spin-ordered and spin disordered structures (at low and high temperatures, respectively).

# I.5.3 Superfluid $^3\text{He}$ : phase diagram for $B = 0$

R. Gross © Walther-Meißner-Institut (2004 - 2022)

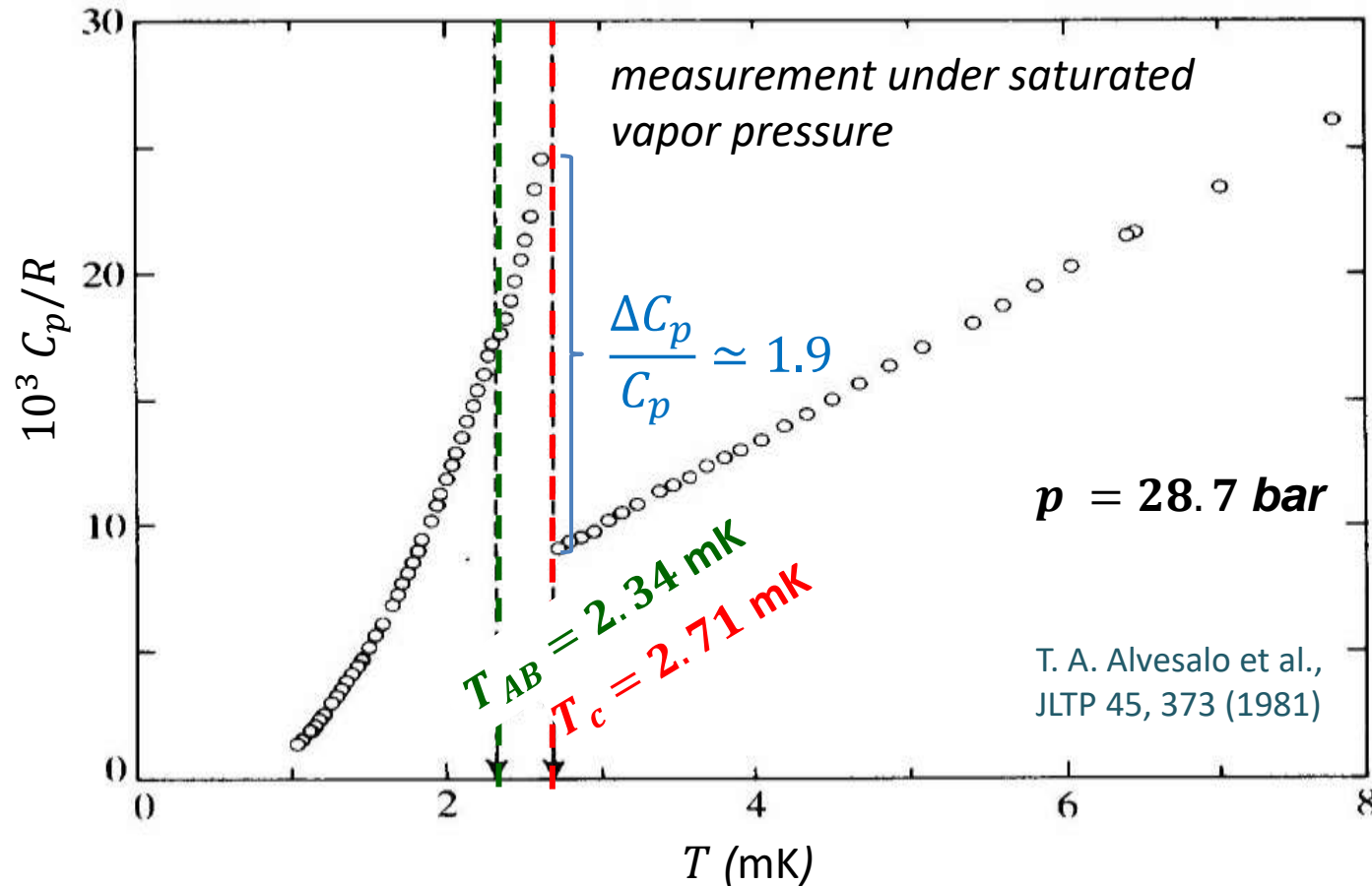
pressure increases  
pairing interaction

linear  $T$  scale !!!





# I.5.3 Superfluid $^3\text{He}$ : specific heat



$\Delta C_p / C_p \approx 2$   
@  $p = 30 \text{ bar}$

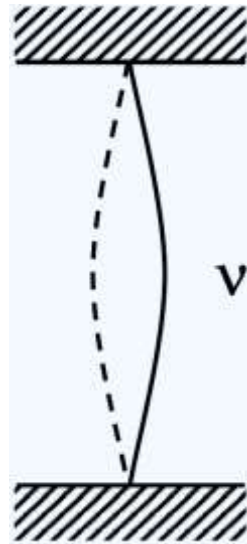
$\Delta C_p / C_p \approx 1.4$   
@  $p = 1 \text{ bar}$

weak coupling  
BCS theory:  
 $\Delta C_p / C_p = 1.43$

- $T_c$ : 2. order phase transition → jump of  $C_p$  as for superconductor
- $T_{AB}$ : 1. order phase transition → small change of temperature dependence of  $C_p$   
→ hysteresis, latent heat
- PCP: first and second order phase transition merge

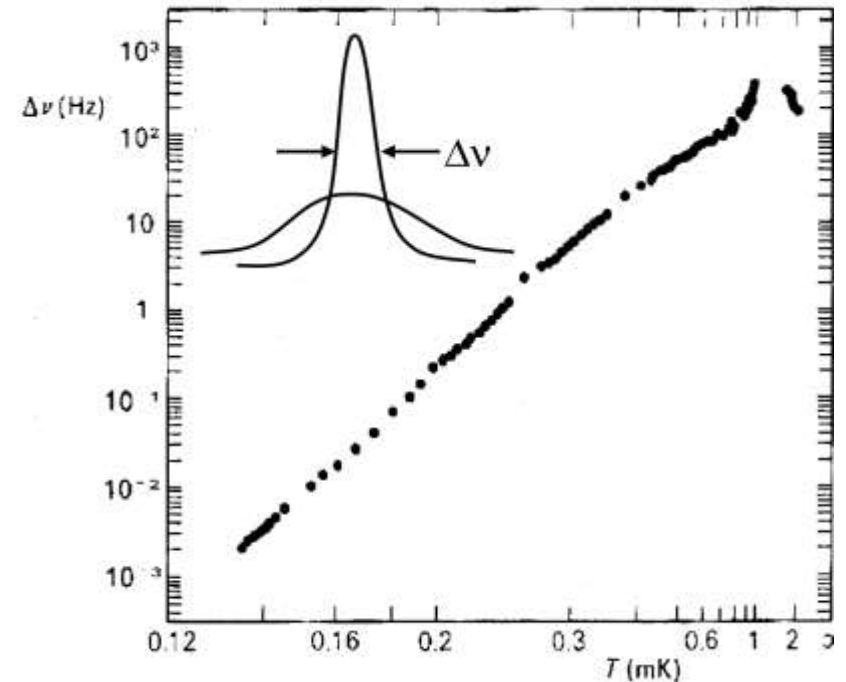
# I.5.3 Superfluid $^3\text{He}$ : proof of superfluidity

- use of *vibrating wire*, measurement of quality factor  $\nu / \Delta\nu$



*viscosity changes  
by 5 orders of  
magnitude*

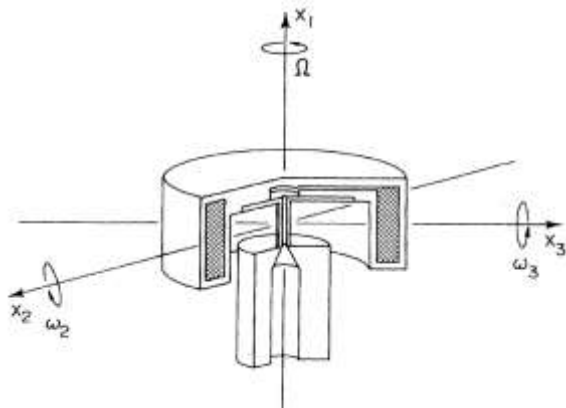
*normal fluid  
freezes out*



- *other experiments:*

→ *persistent flow experiments:*

- viscosity of B phase changes at least by 12 orders of magnitude
- persistent flow in A phase decays slowly (appearance of textures)
- small critical velocity: 1 to 100 mm/s



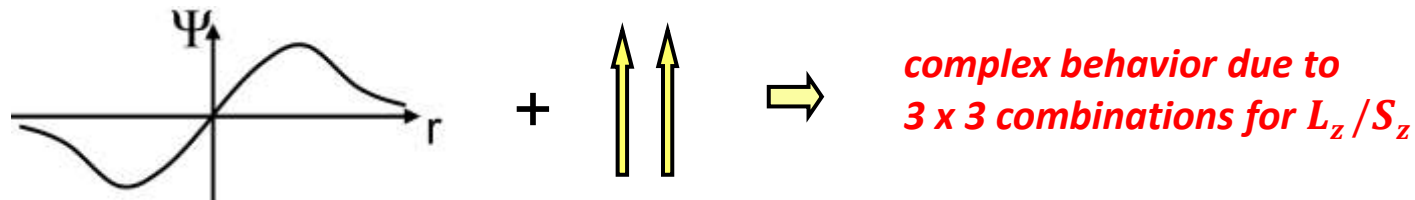
Persistent-Current Experiments on Superfluid He 3 -B and He 3 -A  
J. P. Pekola, J. T. Simola, K. K. Nummilla, O. V. Lounasmaa, R. E. Packard,  
Phys. Rev. Lett. **53**, 70 (1984)

# I.5.3 Superfluid $^3\text{He}$ : pairing

- $^3\text{He}$  is **strongly interacting Fermi liquid** (different to electrons in metals)
  - exchange interaction favors parallel spin alignment
  - Fermi statistics favors anti-parallel spin orientation
- $^3\text{He}$  forms Cooper pairs with spin  $S = 1$  (**spin triplet pairs**)  
 in contrast to electrons in metals:  $S = 0$  (**spin singlet pairs pairs**)
- note the general requirement: **antisymmetric pair wave function** for fermions

**symmetric** spin wave function → **antisymmetric** orbital wave function ( $L = 1, 3, \dots$ )  
**antisymmetric** spin wave function → **symmetric** orbital wave function ( $L = 0, 2, \dots$ )

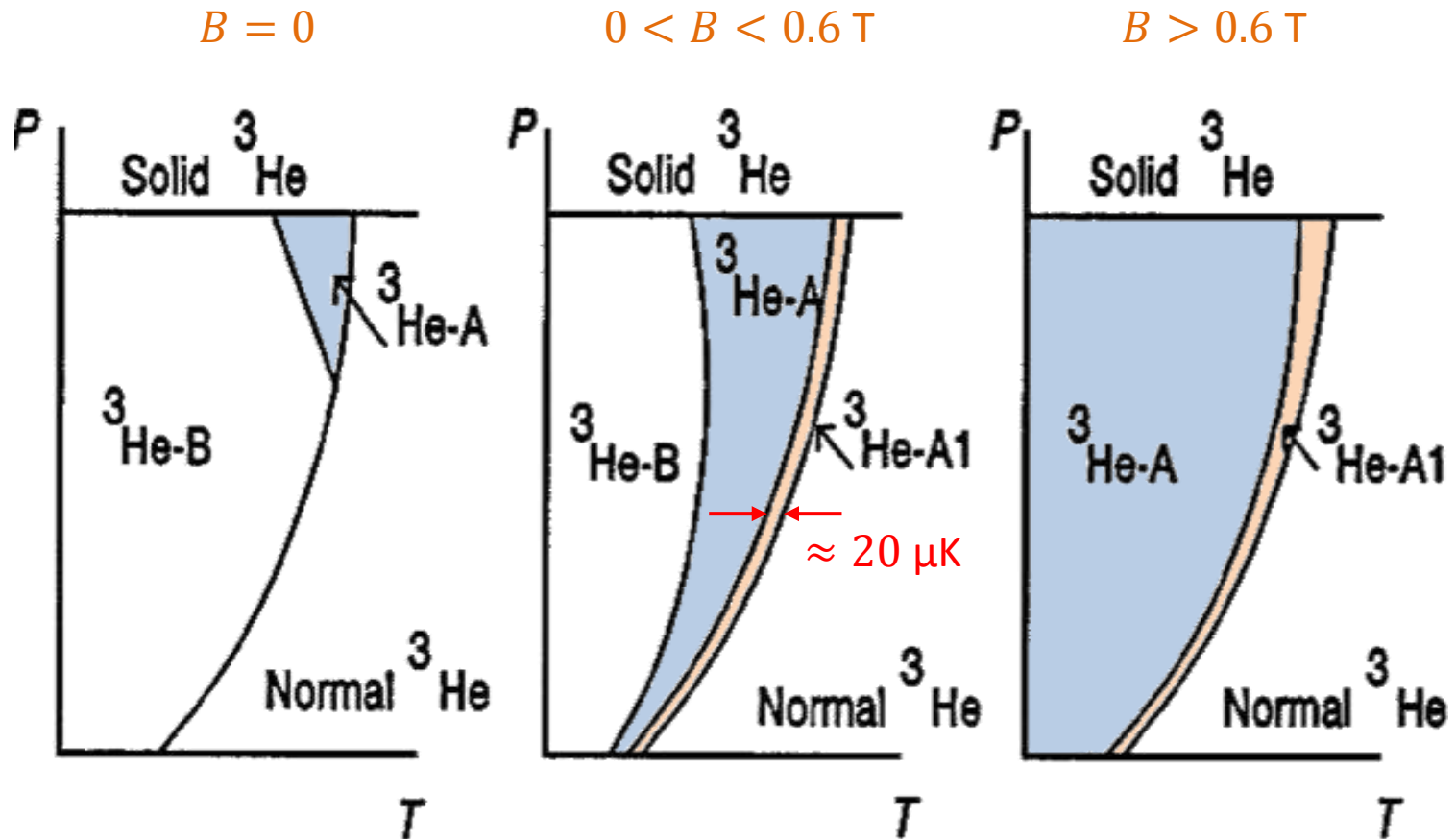
- examples:
- (i) Cooper pairs in metallic superconductors ( $L = 0, S = 0$ ) → **s-wave, spin-singlet pairs**
  - (ii) Cooper pairs in high  $T_c$  superconductors ( $L = 2, S = 0$ ) → **d-wave, spin-singlet pairs**
  - (iii) Cooper pairs in superfluid  $^3\text{He}$  ( $L = 1, S = 1$ ) → **p-wave, spin-triplet pairs**



**p-type orbital wave function minimizes hard core repulsion**

# I.5.3 Superfluid $^3\text{He}$ : in $B \neq 0$

spin triplet pairs have magnetic moment  $\rightarrow$  phase diagram depends on magnetic field

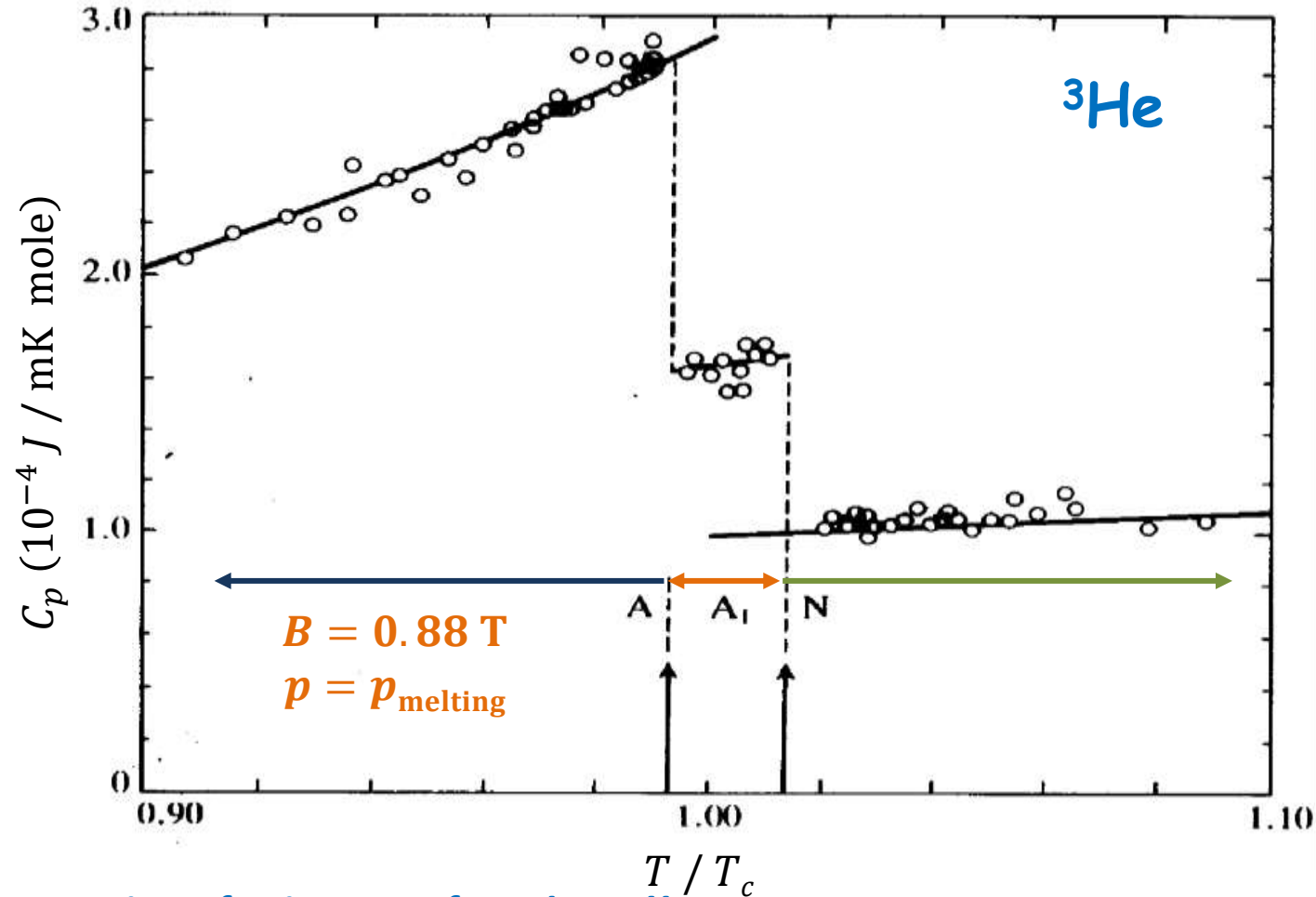


-new phase  $A_1$   
 -phase A gains,  
 -phase B loses

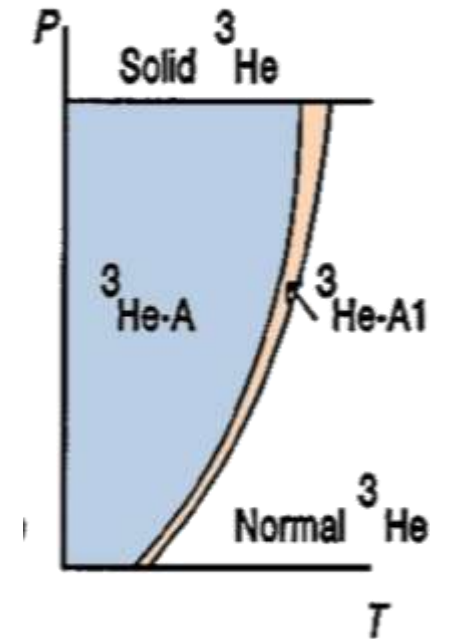
-phase B disappears

# I.5.3 Superfluid $^3\text{He}$ : in $B \neq 0$

- specific heat



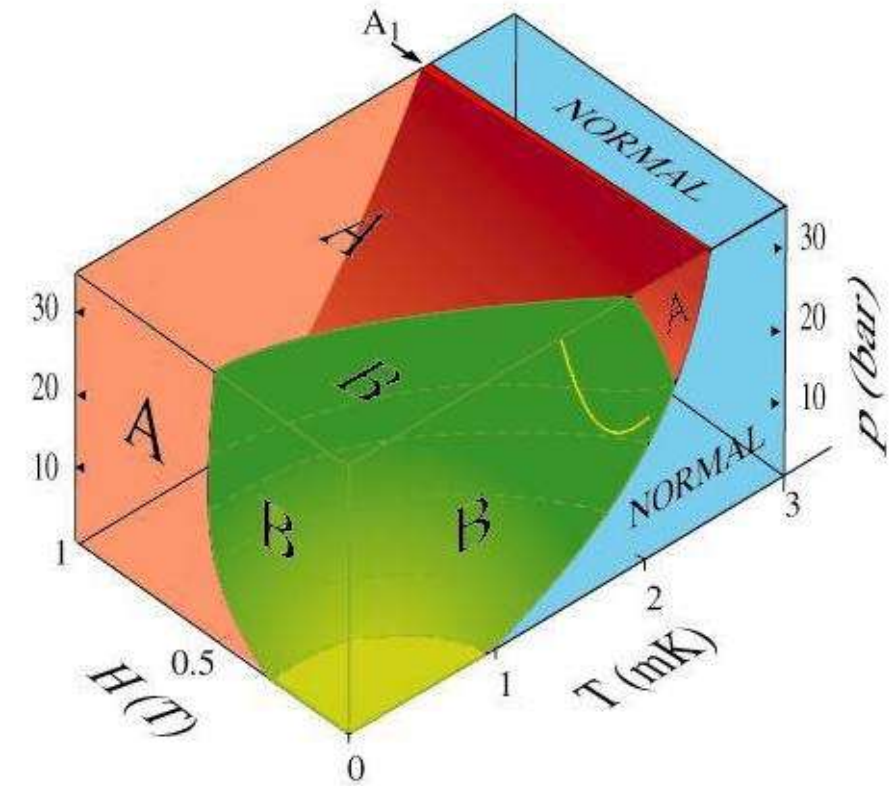
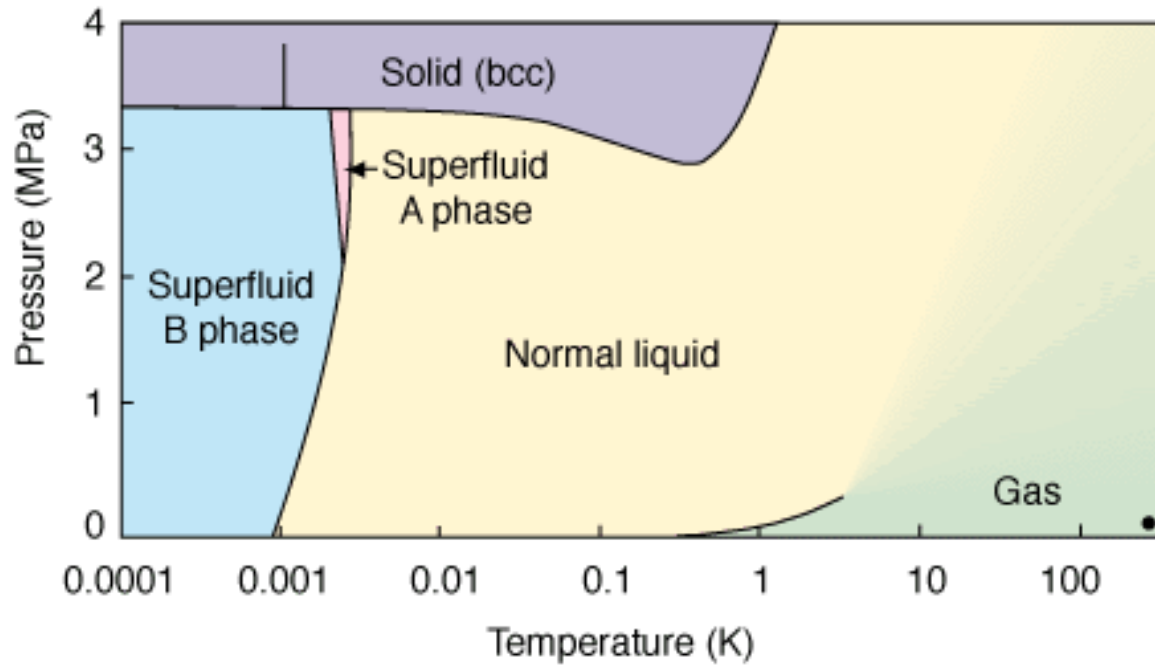
*clear demonstration of existence of  $A_1$  phase !!*



W.P. Halperin et al.,  
Phys. Rev. B13, 2124 (1974)

# I.5.3 Superfluid $^3\text{He}$ : in $B \neq 0$

- phase diagram of  $^3\text{He}$



source: <http://tl.tkk.fi/research/theory/helium.html>

# I.5.3 Superfluid $^3\text{He}$ : Fermi liquid description

- description of liquid  $^3\text{He}$  by a Fermi liquid (interacting Fermi gas)

**Landau:** what happens, if we increase the interaction strength in a non-interacting systems continuously?

- non-interacting system:** addition/removal of one particle with momentum  $\mathbf{k}$  and spin  $\sigma$  to ground state  $|\tilde{\Psi}_0\rangle$

$$|\tilde{\Psi}_{\mathbf{k}\sigma}\rangle = c_{\mathbf{k}\sigma}^\dagger |\tilde{\Psi}_0\rangle \quad \tilde{\varepsilon}_{\mathbf{k}} = \frac{\hbar^2(k^2 - k_F^2)}{2m} \quad f_{\mathbf{k}\sigma}(T=0) = \begin{cases} 0 & \text{for } k < k_F \\ 1 & \text{for } k > k_F \end{cases}$$

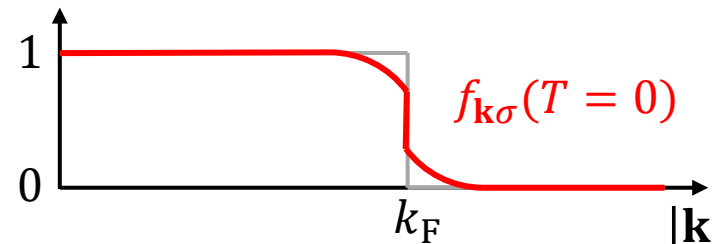
- interacting system:** addition/removal of one particle changes energies of all other particles due to interaction

Landau's argument of *adiabatic continuity*:

quasiparticle energy has similar  $k$ -dependence:  $\varepsilon_{\mathbf{k}} = \frac{\hbar^2(k^2 - k_F^2)}{2m^*}$   $m^*$  = effective mass

for  $k$  close to  $k_F$ :  $\varepsilon_{\mathbf{k}} = \left. \frac{\partial \varepsilon_{\mathbf{k}}}{\partial k} \right|_{k=k_F} (|\mathbf{k}| - k_F) = \frac{\hbar k_F}{m^*} (|\mathbf{k}| - k_F)$

$f_{\mathbf{k}\sigma}(T=0)$  has shows discontinuity, but height of the discontinuity is reduced to smaller value  $Z < 1$





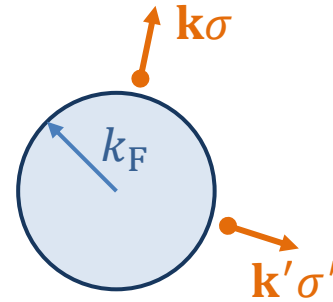
# I.5.3 Superfluid $^3\text{He}$ : Fermi liquid description

- description of liquid  $^3\text{He}$  by a Fermi liquid (interacting Fermi gas)

➤ total energy: 
$$E = \underbrace{E_0}_{\text{ground state}} + \underbrace{\sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}} n_{\mathbf{k}\sigma}}_{\text{quasiparticle excitations}} + \underbrace{\frac{1}{2} \sum_{\mathbf{k}\sigma, \mathbf{k}'\sigma'} \eta(\mathbf{k}\sigma, \mathbf{k}'\sigma') n_{\mathbf{k}\sigma} n_{\mathbf{k}'\sigma'}}_{\text{interaction energy}}$$

➤ two particle energy:  $\varepsilon_{\mathbf{k}} + \varepsilon_{\mathbf{k}'} + \underbrace{\eta(\mathbf{k}\sigma, \mathbf{k}'\sigma')}_{\text{interaction}} \neq \text{sum of single particle energies } \varepsilon_{\mathbf{k}}, \varepsilon_{\mathbf{k}'}$

describes effective interaction between two quasiparticles:  
is usually split in parts  $\eta_1(\mathbf{k}, \mathbf{k}')$  and  $\eta_2(\mathbf{k}, \mathbf{k}')$  for singlet and triplet contribution



- at low  $T$ :  $|\mathbf{k}|, |\mathbf{k}'| \simeq k_F$  and due to spherical symmetry  $\eta(\mathbf{k}, \mathbf{k}') \propto \mathbf{k} \cdot \mathbf{k}' = k_F^2 \cos \theta$   
 ➔ we can express  $\eta_1(\mathbf{k}, \mathbf{k}')$  and  $\eta_2(\mathbf{k}, \mathbf{k}')$  by Legendre polynoms  $P_\ell(\cos \theta)$

$$g(\varepsilon_F) \eta_1(\mathbf{k}, \mathbf{k}') = \sum_{\ell} F_{\ell} P_{\ell}(\cos \theta) \qquad g(\varepsilon_F) \eta_2(\mathbf{k}, \mathbf{k}') = \sum_{\ell} Z_{\ell} P_{\ell}(\cos \theta)$$

$g(\varepsilon_F)$  = density of states (states per energy interval) ➔  $F_{\ell}, Z_{\ell}$  are dimensionless

**Theory by Leggett:** ➤  $F_0, F_1$  are large and positive ➔  $m^* = m(1 + F_1/3)$  with  $F_1 \simeq 5.4$  (increases with pressure)

➤  $Z_0 \simeq -3$  and  $Z_1$  is small ➔ magnetic susceptibility  $\chi = \frac{m^*}{m} \frac{\chi_0}{1 + (Z_0/4)}$  with  $Z_0 \simeq -2.78$   
 ➔  $\chi \simeq 9.2 \chi_0$ : He is close to ferromagnetic instability



# I.5.3 Superfluid $^3\text{He}$ : pairing interaction

- interaction of  $^3\text{He}$  atoms due to van der Waals potential  $V(\mathbf{r}_i - \mathbf{r}_j) = V(\mathbf{r})$

$$V(\mathbf{k}, \mathbf{k}') = \int e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} V(\mathbf{r}) d^3r \quad \text{pairing potential in } k\text{-space}$$

- at low  $T$ :  $|\mathbf{k}|, |\mathbf{k}'| \simeq k_F$  and due to spherical symmetry  $V(\mathbf{k}, \mathbf{k}') \propto \mathbf{k} \cdot \mathbf{k}' = k_F^2 \cos \theta$   
 ➔ we can express  $V(\mathbf{k}, \mathbf{k}')$  by Legendre polynoms  $P_\ell(\cos \theta)$

$$V(\mathbf{k}, \mathbf{k}') = \sum_{\ell} \frac{2\ell + 1}{2} V_{\ell} P_{\ell}(\cos \theta)$$

$$\text{with } V_{\ell} = \frac{1}{4\pi} \int V(\hat{\mathbf{n}}(\theta, \varphi), \hat{e}_z) P_{\ell}(\cos \theta) \sin \theta d\theta d\varphi$$

$$\frac{2\ell+1}{2} = \int P_{\ell}^2(\cos \theta) \sin \theta d\theta \quad (\text{normalization})$$

- calculation of  $V_{\ell}$  for **van der Waals potential** (e.g. by Leggett 1975):  
 $V_0 =$  strongly repulsive,  $V_1, V_2 =$  weakly attractive ➔ **cannot explain pairing in superfluid  $^3\text{He}$**
- attractive interaction results from exchange of **virtual spin fluctuations**

one  $^3\text{He}$  atom polarizes neighboring nuclear spins which then result in an effective attractive interaction for 2<sup>nd</sup>  $^3\text{He}$  atom

$$V(\mathbf{k}, \mathbf{k}') \propto \frac{1}{g(\varepsilon_F)} \frac{\chi_0}{1 + (Z_0/4)} \hat{\mathbf{S}}_{\mathbf{k}} \cdot \hat{\mathbf{S}}_{\mathbf{k}'}$$

(Leggett 1975)

$$\hat{\mathbf{S}}_{\mathbf{k}} \cdot \hat{\mathbf{S}}_{\mathbf{k}'} = +1/4 \text{ for triplet}$$

$$\hat{\mathbf{S}}_{\mathbf{k}} \cdot \hat{\mathbf{S}}_{\mathbf{k}'} = -3/4 \text{ for singlet}$$

# I.5.3 Superfluid $^3\text{He}$ : superfluid phases

- starting point: BCS Hamiltonian

formalism of second quantization is used (>1927, Dirac, Fock, Jordan et al.)

BCS Hamiltonian for  $N$  interacting electrons

$$\mathcal{H}_{\text{BCS}} = \sum_{\sigma} \sum_{i=1}^N \left( \underbrace{-\frac{\hbar^2}{2m} \nabla_i^2}_{\text{kinetic energy}} + \underbrace{V_{\text{ext}}(\mathbf{r})}_{\text{potential energy}} \right) + \frac{1}{2} \sum_{\sigma} \sum_{i,j=1}^N \underbrace{V_{\text{int}}(\mathbf{r}_i - \mathbf{r}_j)}_{\text{interaction energy}}$$



insertion of field operators and integration over volume  $\rightarrow$  FT of  $\mathcal{H}_{\text{BCS}}$  into  $k$ -space (see R. Gross, A. Marx, „Festkörperphysik“, 3. Auflage, appendix H.2)

$$\mathcal{H}_{\text{BCS}} = \underbrace{\sum_{\sigma} \sum_{\mathbf{k}} (\varepsilon_{\mathbf{k}} - \mu) \hat{c}_{\mathbf{k}\sigma}^{\dagger} \hat{c}_{\mathbf{k}\sigma}}_{\text{energy of non-interacting free electron gas}} + \underbrace{\frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} \sum_{\mathbf{k},\mathbf{k}'} V_{\alpha\beta\gamma\delta}(\mathbf{k}, \mathbf{k}') \hat{c}_{\mathbf{k}'\alpha}^{\dagger} \hat{c}_{-\mathbf{k}'\beta}^{\dagger} \hat{c}_{-\mathbf{k}\gamma} \hat{c}_{\mathbf{k}\delta}}_{\text{interaction energy}}$$

$$\varepsilon_{\mathbf{k}} - \mu = \frac{\hbar^2(k - k_{\text{F}})^2}{2m}$$

$$V(\mathbf{k}, \mathbf{k}') = \frac{1}{\Omega} \int_{\Omega} V(\mathbf{r}) e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} dV$$

factor  $\frac{1}{2}$  avoids double counting

operator describes scattering from state  $(-\mathbf{k}\gamma, \mathbf{k}\delta)$  into  $(\mathbf{k}'\alpha, -\mathbf{k}'\beta)$  by exchange of exchange boson with wave vector  $(\mathbf{k} - \mathbf{k}')$

# I.5.3 Superfluid $^3\text{He}$ : superfluid phases

- symmetry of the pair wavefunction

- **important:** pair consists of two fermions  $\rightarrow$  **total wavefunction must be antisymmetric: minus sign for particle exchange**

$$\Psi(\mathbf{r}_1, \boldsymbol{\sigma}_1, \mathbf{r}_2, \boldsymbol{\sigma}_2) = \underbrace{\frac{1}{\sqrt{V}} e^{i \mathbf{K}_s \cdot \mathbf{R}_s}}_{\substack{\text{center of mass motion} \\ \text{we assume } \mathbf{K}_s = 0}} \underbrace{f(\mathbf{q}, \mathbf{r})}_{\substack{\text{orbital} \\ \text{part}}} \underbrace{\chi(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)}_{\substack{\text{spin} \\ \text{part}}} = -\Psi(\mathbf{r}_2, \boldsymbol{\sigma}_2, \mathbf{r}_1, \boldsymbol{\sigma}_1)$$

$$\mathbf{R}_s = (\mathbf{r}_1 + \mathbf{r}_2)/2$$

$$\mathbf{r} = (\mathbf{r}_1 - \mathbf{r}_2)$$

$$\mathbf{K}_s = (\mathbf{k} + \mathbf{k}')/2$$

$$\mathbf{q} = (\mathbf{k} - \mathbf{k}')$$

- possible **spin wavefunctions**  $\chi(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$  for electron pairs

$$S = \begin{cases} 0 & m_s = 0 & \chi^a = \frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow) & \rightarrow \text{singlet pairing, antisymmetric spin wavefunction} \\ & & & \text{symmetric orbital function: } L = 0, 2, \dots (s, d, \dots) \\ 1 & m_s = \begin{cases} -1 & \chi^s = \downarrow\downarrow \\ 0 & \chi^s = \frac{1}{\sqrt{2}} (\uparrow\downarrow + \downarrow\uparrow) \\ +1 & \chi^s = \uparrow\uparrow \end{cases} & \rightarrow \text{triplet pairing, symmetric spin wavefunction} \\ & & & \text{antisymmetric orbital function: } L = 1, 3, \dots (p, f, \dots) \end{cases}$$

# I.5.3 Superfluid $^3\text{He}$ : superfluid phases

- BCS mean-field approximation

pairing amplitude:

$$\hat{c}_{-\mathbf{k}\alpha}\hat{c}_{\mathbf{k}\beta} = \underbrace{\langle \hat{c}_{-\mathbf{k}\alpha}\hat{c}_{\mathbf{k}\beta} \rangle}_{g_{\alpha\beta}(\mathbf{k})} + \underbrace{\hat{c}_{-\mathbf{k}\alpha}\hat{c}_{\mathbf{k}\beta} - \langle \hat{c}_{-\mathbf{k}\alpha}\hat{c}_{\mathbf{k}\beta} \rangle}_{\delta g_{\alpha\beta}(\mathbf{k})} \quad \hat{c}_{\mathbf{k}\alpha}^\dagger\hat{c}_{-\mathbf{k}\beta}^\dagger = \underbrace{\langle \hat{c}_{\mathbf{k}\alpha}^\dagger\hat{c}_{-\mathbf{k}\beta}^\dagger \rangle}_{g_{\alpha\beta}^*(\mathbf{k})} + \underbrace{\hat{c}_{\mathbf{k}\alpha}^\dagger\hat{c}_{-\mathbf{k}\beta}^\dagger - \langle \hat{c}_{\mathbf{k}\alpha}^\dagger\hat{c}_{-\mathbf{k}\beta}^\dagger \rangle}_{\delta g_{\alpha\beta}^*(\mathbf{k})}$$

$$g_{\alpha\beta}(\mathbf{k}) \equiv \langle \hat{c}_{-\mathbf{k}\alpha}\hat{c}_{\mathbf{k}\beta} \rangle \neq 0$$

$$g_{\alpha\beta}^*(\mathbf{k}) \equiv \langle \hat{c}_{\mathbf{k}\alpha}^\dagger\hat{c}_{-\mathbf{k}\beta}^\dagger \rangle \neq 0$$

- make use of **pairing potential**  $\Delta_{\alpha\beta}(\mathbf{k}) \equiv - \sum_{\mathbf{k}',\gamma,\delta} V_{\alpha\beta\gamma\delta}(\mathbf{k},\mathbf{k}')g_{\alpha\beta}(\mathbf{k}')$ 

$\Delta_{\alpha\beta}^*(\mathbf{k}) \equiv - \sum_{\mathbf{k}',\gamma,\delta} V_{\alpha\beta\gamma\delta}(\mathbf{k},\mathbf{k}')g_{\alpha\beta}^*(\mathbf{k}')$

four component function due to two values  $\alpha, \beta$  of spin spin structure orbital structure



$$\mathcal{H}_{\text{BCS}} - \mathcal{N}\mu = \sum_{\mathbf{k},\sigma} (\varepsilon_{\mathbf{k}} - \mu) c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} - \sum_{\mathbf{k},\alpha,\beta} \left[ \Delta_{\alpha\beta}(\mathbf{k}) \hat{c}_{\mathbf{k}\alpha}^\dagger \hat{c}_{-\mathbf{k}\beta}^\dagger + \Delta_{\alpha\beta}^*(\mathbf{k}) \hat{c}_{\mathbf{k}\alpha} \hat{c}_{-\mathbf{k}\beta} \right]$$

- with  $\alpha, \beta = \uparrow, \downarrow$ : two fermions with  $\mathbf{s} = 1/2$  and  $\alpha, \beta = m_s = \uparrow, \downarrow$

$$g(\mathbf{k}) \equiv \begin{pmatrix} \langle \hat{c}_{-\mathbf{k}\uparrow}\hat{c}_{\mathbf{k}\uparrow} \rangle & \langle \hat{c}_{-\mathbf{k}\uparrow}\hat{c}_{\mathbf{k}\downarrow} \rangle \\ \langle \hat{c}_{-\mathbf{k}\downarrow}\hat{c}_{\mathbf{k}\uparrow} \rangle & \langle \hat{c}_{-\mathbf{k}\downarrow}\hat{c}_{\mathbf{k}\downarrow} \rangle \end{pmatrix} \quad g^*(\mathbf{k}) \equiv \begin{pmatrix} \langle \hat{c}_{-\mathbf{k}\uparrow}^\dagger\hat{c}_{\mathbf{k}\uparrow}^\dagger \rangle & \langle \hat{c}_{-\mathbf{k}\uparrow}^\dagger\hat{c}_{\mathbf{k}\downarrow}^\dagger \rangle \\ \langle \hat{c}_{-\mathbf{k}\downarrow}^\dagger\hat{c}_{\mathbf{k}\uparrow}^\dagger \rangle & \langle \hat{c}_{-\mathbf{k}\downarrow}^\dagger\hat{c}_{\mathbf{k}\downarrow}^\dagger \rangle \end{pmatrix} \quad \Delta(\mathbf{k}) = \begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} \quad \Delta^*(\mathbf{k}) = \begin{pmatrix} \Delta_{\uparrow\uparrow}^*(\mathbf{k}) & \Delta_{\uparrow\downarrow}^*(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}^*(\mathbf{k}) & \Delta_{\downarrow\downarrow}^*(\mathbf{k}) \end{pmatrix}$$

4 components covering all spin orientations before and after interaction

# I.5.3 Superfluid $^3\text{He}$ : superfluid phases

- diagonalization of mean-field Hamiltonian by Bogoliubov transformation

- BCS energy gap equation:  
(Nambu, 1962)

$$\begin{array}{c}
 \begin{array}{cc} \text{particles} & \text{pairs} \end{array} \\
 \left( \begin{array}{cccc}
 \varepsilon_{\mathbf{k}} - \mu & 0 & \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\
 0 & \varepsilon_{\mathbf{k}} - \mu & \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \\
 \Delta_{\uparrow\uparrow}^*(\mathbf{k}) & \Delta_{\uparrow\downarrow}^*(\mathbf{k}) & -(\varepsilon_{\mathbf{k}} - \mu) & 0 \\
 \Delta_{\downarrow\uparrow}^*(\mathbf{k}) & \Delta_{\downarrow\downarrow}^*(\mathbf{k}) & 0 & -(\varepsilon_{\mathbf{k}} - \mu)
 \end{array} \right)
 \begin{pmatrix} u_{\mathbf{k}\uparrow n} \\ u_{\mathbf{k}\downarrow n} \\ v_{\mathbf{k}\uparrow n} \\ v_{\mathbf{k}\downarrow n} \end{pmatrix} = E_{\mathbf{k}n}
 \begin{pmatrix} u_{\mathbf{k}\uparrow n} \\ u_{\mathbf{k}\downarrow n} \\ v_{\mathbf{k}\uparrow n} \\ v_{\mathbf{k}\downarrow n} \end{pmatrix}
 \end{array}$$

particles
pairs
pairs
holes

- 4 x 4 matrix equation instead of 2 x 2 matrix equation for spin singlet superconductor
  - diagonalization by suitable Bogoliubov transformation
  - two positive eigenvalues  $E_{\mathbf{k}n}$  ( $n = 1, 2$ ) and two negative eigenvalues  $-E_{\mathbf{k}n}$
  - general gap equation allows for both spin singlet and spin triplet pairing
- symmetry properties of pairing amplitude and pairing potential (follows from anti-commutation relation of fermionic operators)

$$g_{\alpha\beta}(\mathbf{k}) = \langle \hat{c}_{-\mathbf{k}\alpha} \hat{c}_{\mathbf{k}\beta} \rangle = -\langle \hat{c}_{\mathbf{k}\beta} \hat{c}_{-\mathbf{k}\alpha} \rangle = -g_{\beta\alpha}(-\mathbf{k})$$

$$\Delta_{\alpha\beta}(\mathbf{k}) = -\Delta_{\beta\alpha}(-\mathbf{k})$$

# I.5.3 Superfluid $^3\text{He}$ : superfluid phases

- representation of the energy gap matrix  $\Delta_{\alpha\beta}(\mathbf{k})$  by a scalar gap function  $\Delta(\mathbf{k})$  and a vector  $\mathbf{d}(\mathbf{k})$

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = i [\Delta(\mathbf{k}) \mathbf{I} + \mathbf{d}(\mathbf{k}) \cdot \boldsymbol{\sigma}] \sigma_y$$

scalar gap function

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

d-vector

Pauli spin matrix  $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$  = vector of Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -d_x(\mathbf{k}) + id_y(\mathbf{k}) & \Delta(\mathbf{k}) + d_z(\mathbf{k}) \\ -\Delta(\mathbf{k}) + d_z(\mathbf{k}) & d_x(\mathbf{k}) + id_y(\mathbf{k}) \end{pmatrix}$$

$$\mathbf{d} = \hat{\mathbf{d}} |\mathbf{d}| e^{i\phi}$$

direction of spin wave function

superfluid density

phase factor

➤ amplitude  $|\mathbf{d}|$  ( $\mathbf{d}$  is vector with complex components):  
pair density  $|d|^2 = d_x^2 + d_y^2 + d_z^2 = n_s$

➤ direction  $\hat{\mathbf{d}} = \mathbf{d}/|\mathbf{d}|$ :  
spatial orientation of spin wave function:  $\hat{\mathbf{d}} \cdot \mathbf{S} = 0$

# I.5.3 Superfluid $^3\text{He}$ : superfluid phases

- parity consideration

- with  $g_{\alpha\beta}(\mathbf{k}) = -g_{\beta\alpha}(-\mathbf{k})$  and  $\Delta_{\alpha\beta}(\mathbf{k}) = -\Delta_{\beta\alpha}(-\mathbf{k})$  we obtain

$$\Delta(\mathbf{k}) = \Delta(-\mathbf{k})$$

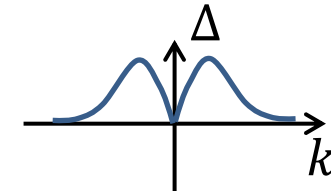
scalar component is **even** with respect to inversion  $\mathbf{k} \rightarrow -\mathbf{k}$

$$\mathbf{d}(\mathbf{k}) = -\mathbf{d}(-\mathbf{k})$$

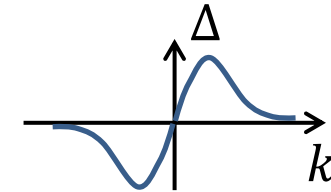
vector component is **odd** with respect to inversion  $\mathbf{k} \rightarrow -\mathbf{k}$

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -d_x(\mathbf{k}) + id_y(\mathbf{k}) & \Delta(\mathbf{k}) + d_z(\mathbf{k}) \\ -\Delta(\mathbf{k}) + d_z(\mathbf{k}) & d_x(\mathbf{k}) + id_y(\mathbf{k}) \end{pmatrix}$$

*orbital structure of the pairing potential*



$L = 0, 2, 4, \dots$



$L = 1, 3, 5, \dots$

- anti-symmetry of pair wave function requires

- spin singlet ( $S = 0$ ):

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} 0 & \Delta(\mathbf{k}) \\ -\Delta(\mathbf{k}) & 0 \end{pmatrix}$$

- spin triplet ( $S = 1$ ):

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -d_x(\mathbf{k}) + id_y(\mathbf{k}) & +d_z(\mathbf{k}) \\ +d_z(\mathbf{k}) & d_x(\mathbf{k}) + id_y(\mathbf{k}) \end{pmatrix}$$

# I.5.3 Superfluid $^3\text{He}$ : superfluid phases

- representation of the scalar function  $\Delta(\mathbf{k})$  and vector  $\mathbf{d}(\mathbf{k})$ 
  - we can restrict ourselves to wave vectors near the Fermi level
    - ➔ due to spherical symmetry, we can use spherical harmonics  $Y_{lm}$  to represent

$$d_i(\mathbf{k}) = \sum h_{ilm} Y_{lm}(\theta_{\mathbf{k}}, \varphi_{\mathbf{k}}) \quad i = x, y, z$$

- spin singlet:  $l = 0, 2, 4, \dots$
- spin triplet:  $l = 1, 3, 5, \dots$

- in general, one has to sum up over all angular momentum quantum numbers  $l$
- practically, one can restrict oneself to a particular quantum number (dominating pairing channel, all other channels have much smaller  $T_c$ )
- for superfluid  $^3\text{He}$ , the **dominating pairing channel is the  $l = 1$  channel** mediated by spin fluctuations

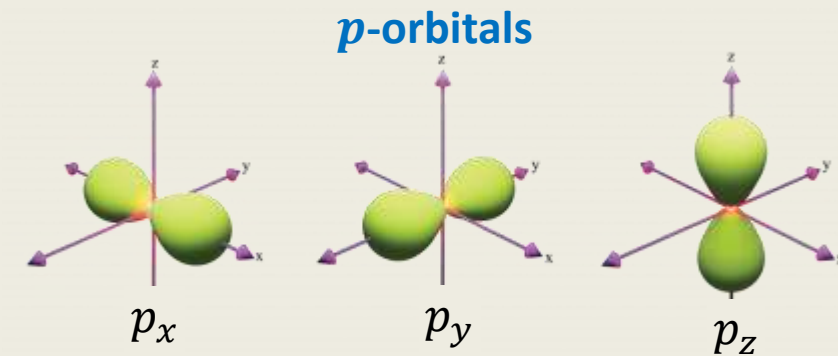


$$d_i(\mathbf{k}) = \sum h_{im} p_j(\theta_{\mathbf{k}}, \varphi_{\mathbf{k}})$$

$$p_x(\theta_{\mathbf{k}}, \varphi_{\mathbf{k}}) = \sqrt{3/4\pi} \cos \varphi_{\mathbf{k}} \sin \theta_{\mathbf{k}}$$

$$p_y(\theta_{\mathbf{k}}, \varphi_{\mathbf{k}}) = \sqrt{3/4\pi} \sin \varphi_{\mathbf{k}} \cos \theta_{\mathbf{k}}$$

$$p_z(\theta_{\mathbf{k}}, \varphi_{\mathbf{k}}) = \sqrt{3/4\pi} \cos \theta_{\mathbf{k}}$$



$h_{im} = 3 \times 3$  matrix as  $i = x, y, z$  and  $m = -1, 0, +1$

**complex matrix elements have to be determined by solving BCS gap equation**



# I.5.3 Superfluid $^3\text{He}$

- superfluid phases of  $^3\text{He}$

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -d_x(\mathbf{k}) + id_y(\mathbf{k}) & +d_z(\mathbf{k}) \\ +d_z(\mathbf{k}) & d_x(\mathbf{k}) + id_y(\mathbf{k}) \end{pmatrix}$$

$$d_i(\mathbf{k}) = \sum h_{im} p_j(\theta_{\mathbf{k}}, \varphi_{\mathbf{k}})$$

$$h_{im} = \begin{pmatrix} h_{xx} & h_{xy} & h_{xz} \\ h_{yx} & h_{yy} & h_{yz} \\ h_{zx} & h_{zy} & h_{zz} \end{pmatrix}$$

- Anderson-Brinkman-Morrel (ABM or A) phase

$$h_{im} = \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow d_x(\mathbf{k}) = p_x + ip_y$$

$$|\psi\rangle_{\text{ABM}} = (p_x + ip_y) |\downarrow\downarrow\rangle - (p_x - ip_y) |\uparrow\uparrow\rangle$$

linear combination of  $S_z = \pm 1$

(i)  $B = 0: |h_{xx}| = |h_{xy}| \rightarrow \langle S_z \rangle = 0$

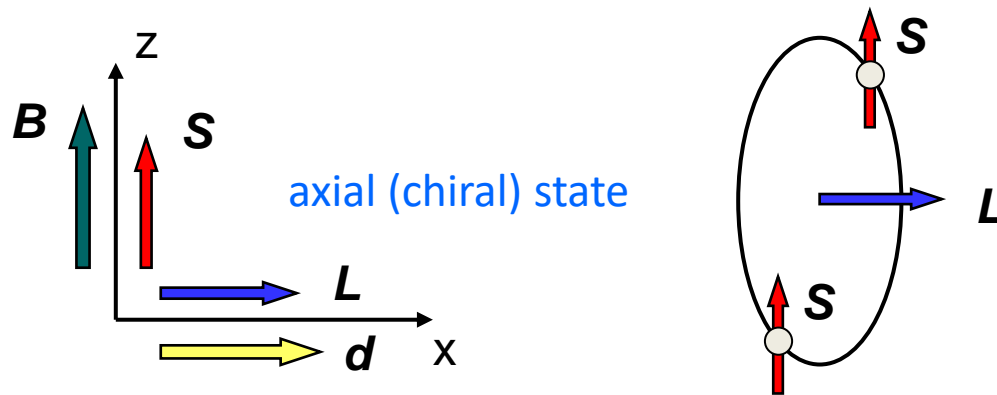
(ii)  $B \neq 0: |h_{xx}| \neq |h_{xy}| \rightarrow \langle S_z \rangle \neq 0$  (anisotropic, paramagnetic)

# I.5.3 Superfluid $^3\text{He}$

## – Anderson-Brinkman-Morrel (ABM or A) phase

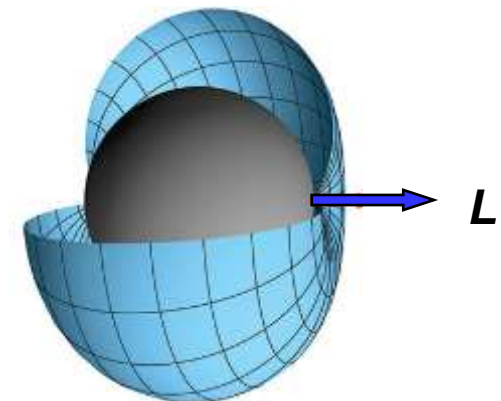
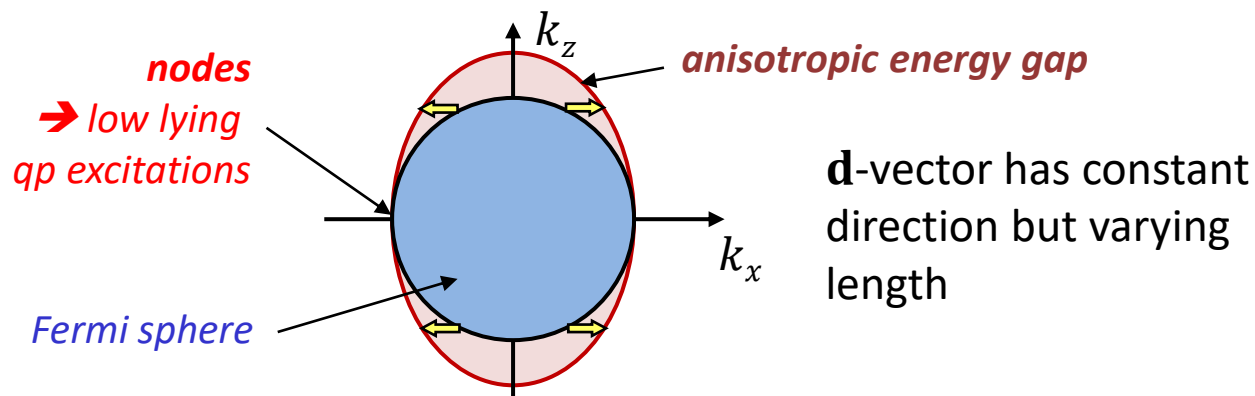
$$|\psi\rangle_{\text{ABM}} = (p_x + ip_y) |\downarrow\downarrow\rangle - (p_x - ip_y) |\uparrow\uparrow\rangle$$

- both spins are always parallel
- all paired atoms are moving in plane perpendicular to  $\mathbf{L}$



*note that  $\mathbf{L} \parallel \mathbf{S}$  would result in repulsive spin-orbit interaction*

- unpaired atoms moving in  $\mathbf{L}$  direction do not perturb  $\rightarrow$  no energy gap in  $\mathbf{L}$ -direction



# I.5.3 Superfluid $^3\text{He}$

- superfluid phases of  $^3\text{He}$

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -d_x(\mathbf{k}) + id_y(\mathbf{k}) & +d_z(\mathbf{k}) \\ +d_z(\mathbf{k}) & d_x(\mathbf{k}) + id_y(\mathbf{k}) \end{pmatrix}$$

$$d_i(\mathbf{k}) = \sum h_{im} p_j(\theta_{\mathbf{k}}, \varphi_{\mathbf{k}})$$

$$h_{im} = \begin{pmatrix} h_{xx} & h_{xy} & h_{xz} \\ h_{yx} & h_{yy} & h_{yz} \\ h_{zx} & h_{zy} & h_{zz} \end{pmatrix}$$

- Balian-Werthamer (BW or B) phase

$$h_{im} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rightarrow d_x(\mathbf{k}) = p_x, \quad d_y(\mathbf{k}) = p_y, \quad d_z(\mathbf{k}) = p_z,$$

$$|\psi\rangle_{\text{BW}} = (p_x + ip_y) |\downarrow\downarrow\rangle - (p_x - ip_y) |\uparrow\uparrow\rangle + p_z (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

linear combination of  $S_z = -1, 0, +1$  (all spin states are represented equally)

$$|h_{xx}| = |h_{yy}| = |h_{zz}| \neq 0 \quad \rightarrow \quad \langle S_z \rangle = 0 \quad (\text{isotropic, non-magnetic})$$

$$B \neq 0: \quad S_z = 0 \text{ component is non-magnetic} \rightarrow \chi_{\text{BW}} = \frac{2}{3} \chi_{\text{ABM}}$$

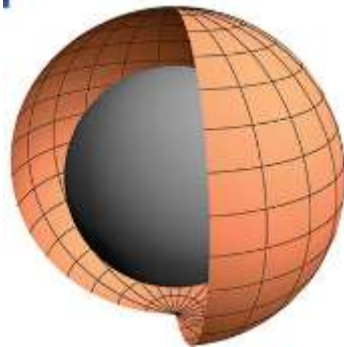
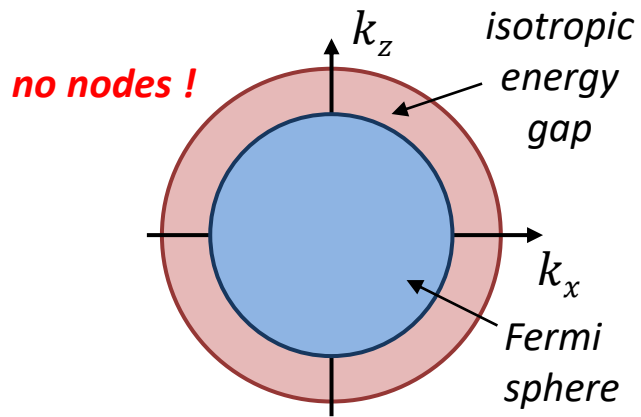
**BW phase energetically less favorable than ABM phase in applied magnetic field !!**

# I.5.3 Superfluid $^3\text{He}$

## – Balian-Werthamer (BW or B) phase

$$|\psi\rangle_{\text{BW}} = (p_x + ip_y) |\downarrow\downarrow\rangle - (p_x - ip_y) |\uparrow\uparrow\rangle + p_z (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

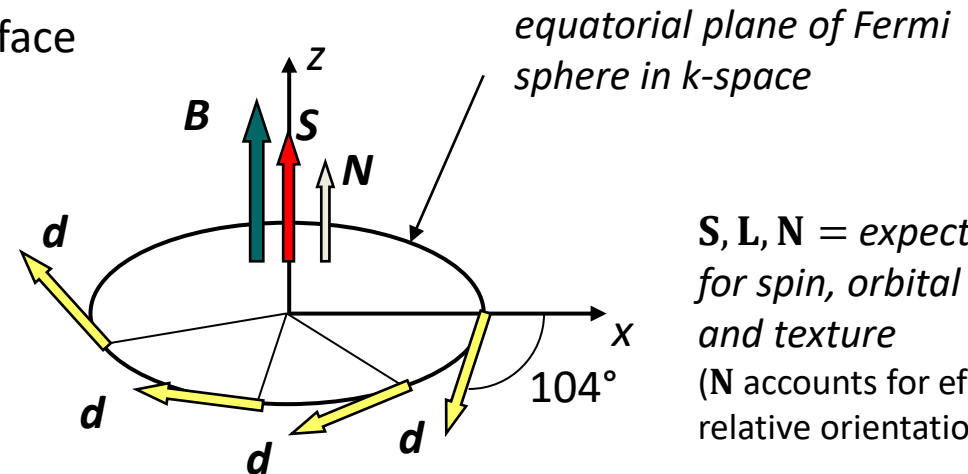
- **d**-vector has constant length all over the Fermi surface  
 ➔ constant magnitude of the energy gap



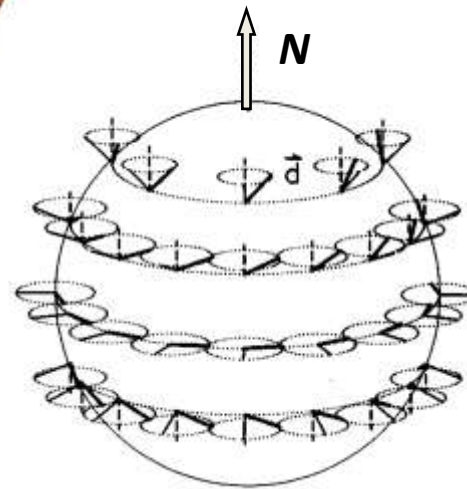
pseudo-isotropic state

(constant gap as s-wave superconductor)

$$C_V(T) \propto \exp(-\Delta(T)/k_B T)$$



**S, L, N** = expectation values for spin, orbital momentum and texture  
 (N accounts for effects related to relative orientation of **d** and **L**)



**d**-vectors (thick lines) are rotated by  $104^\circ$  about a vector defining the radial directions (thin lines) for all points on the Fermi sphere.  
 D.M. Lee, Reviews of Modern Physics **69**, 645 (1997)

# I.5.3 Superfluid $^3\text{He}$

- superfluid phases of  $^3\text{He}$

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -d_x(\mathbf{k}) + \imath d_y(\mathbf{k}) & +d_z(\mathbf{k}) \\ +d_z(\mathbf{k}) & d_x(\mathbf{k}) + \imath d_y(\mathbf{k}) \end{pmatrix}$$

$$d_i(\mathbf{k}) = \sum h_{im} p_j(\theta_{\mathbf{k}}, \varphi_{\mathbf{k}})$$

$$h_{im} = \begin{pmatrix} h_{xx} & h_{xy} & h_{xz} \\ h_{yx} & h_{yy} & h_{yz} \\ h_{zx} & h_{zy} & h_{zz} \end{pmatrix}$$

- $A_1$  phase (only for  $B > 0$ )

$$h_{im} = \begin{pmatrix} 1 & \imath & 0 \\ \imath & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow d_x(\mathbf{k}) = p_x + \imath p_y$$

$$\rightarrow d_y(\mathbf{k}) = \imath p_x + p_y$$

$$|\psi\rangle_{A_1} = 2(p_x + \imath p_y) |\uparrow\uparrow\rangle$$

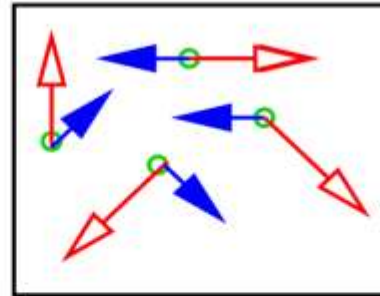
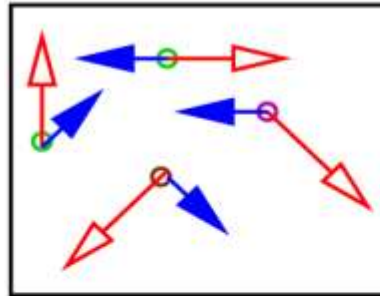
only state  $|\uparrow\uparrow\rangle$  contributes:  $\langle S_z \rangle = 1$  (“ferro”magnetic)

**→ long-range ordered magnetic liquid**

# I.5.3 Superfluid $^3\text{He}$ : broken symmetries

## disordered state

(isotropic with respect to orientation of both degrees of freedom)

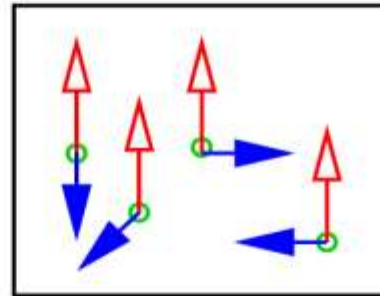
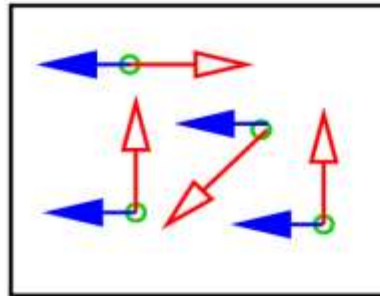


broken gauge symmetry  
"superfluid"

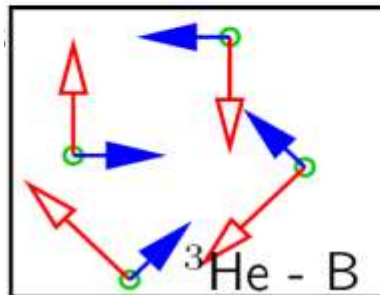
↑ spin moment  
↑ orbital

## broken gauge and rotational symmetry in spin space

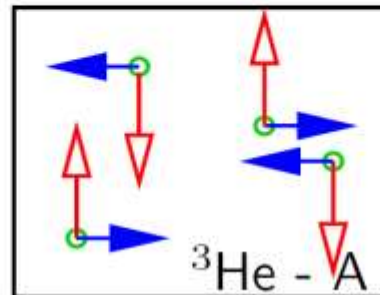
"ferromagnetic superfluid"



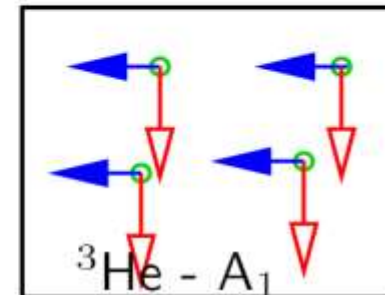
broken gauge and rotational symmetry in orbital space  
"ferroorbital superfluid"



only symmetry related to relative orientation is broken



rotational symmetry is broken separately in spin and orbital space



rotational symmetry is broken separately in spin and orbital space



Walther  
Meißner  
Institut

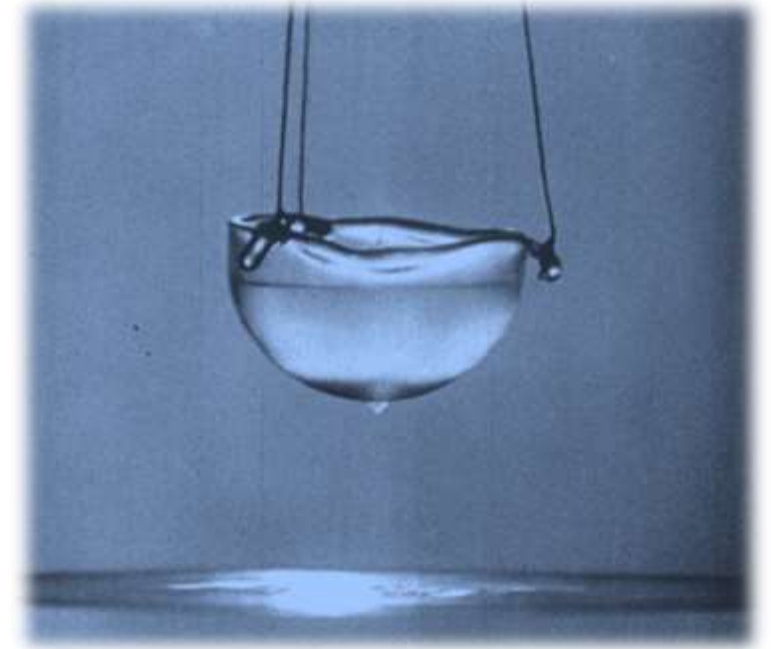
**BAW**

BAYERISCHE  
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DER  
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Technische  
Universität  
München

**TUM**

# Superconductivity and Low Temperature Physics II



**Lecture No. 8**  
**30 June 2022**

**R. Gross**  
**© Walther-Meißner-Institut**

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### I.6 $^3\text{He}$ / $^4\text{He}$ mixtures



# I.6 $^3\text{He}/^4\text{He}$ mixtures

- **why interesting:**

- model system for testing Fermi liquid theories

- ( $^3\text{He}$  is fermion, can be diluted in passive  $^4\text{He}$  background → change of interaction strength)

- technical application in cooling machines

- H.E. Hall, P.J. Ford and K. Thomson, *Cryogenic* **6** (1966) 80

- B.S. Neganov, N.S. Borisov and M.Yu. Liburg, *Zh. Exp. Teor. Fiz.* **50** (1966) 1445.

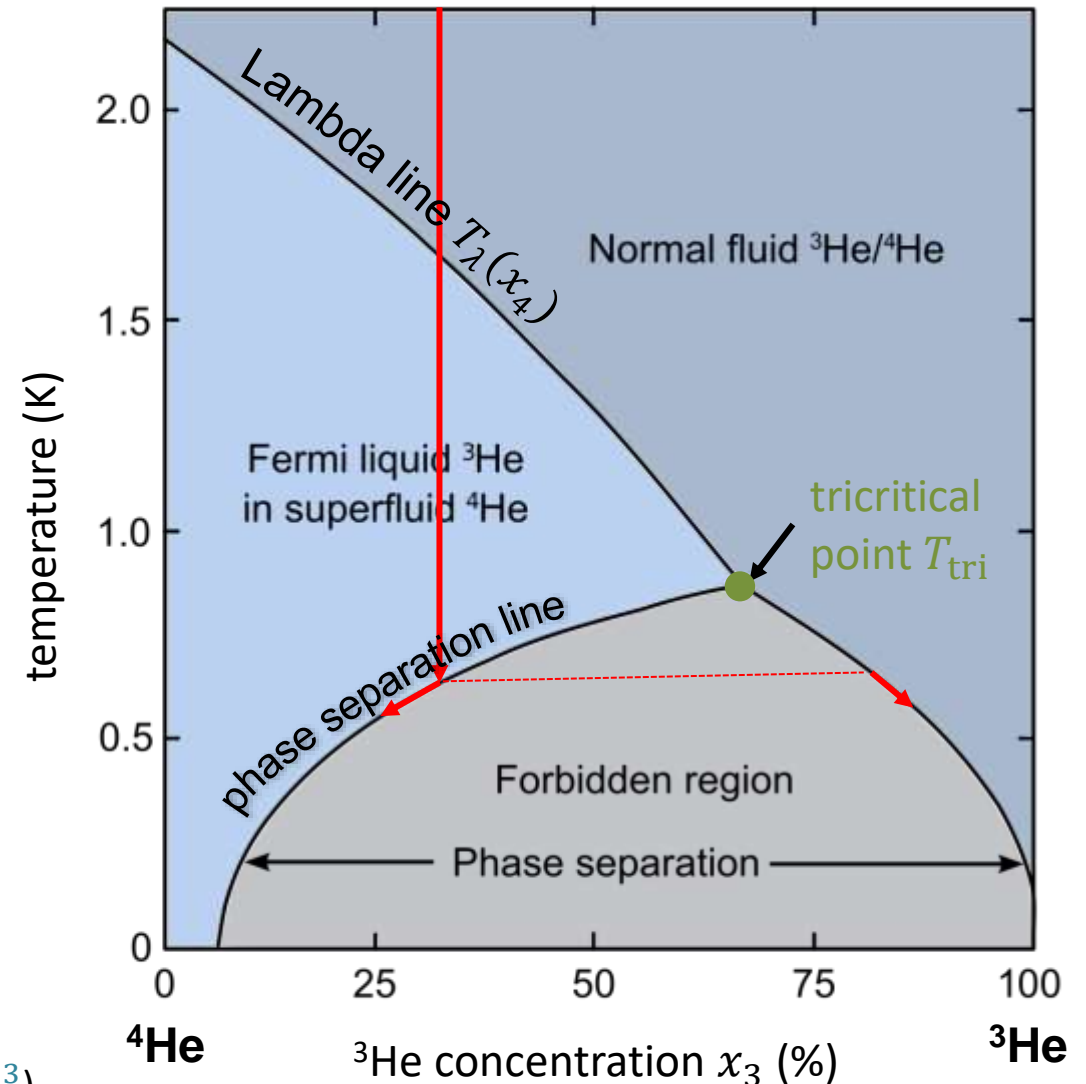
- **experimental observations:**

- specific heat vs.  $T$  curve of  $^4\text{He}$  is shifted to lower  $T$  on adding  $^3\text{He}$ , but does not change its shape

- phase separation of  $^4\text{He}$  and  $^3\text{He}$  at low  $T$

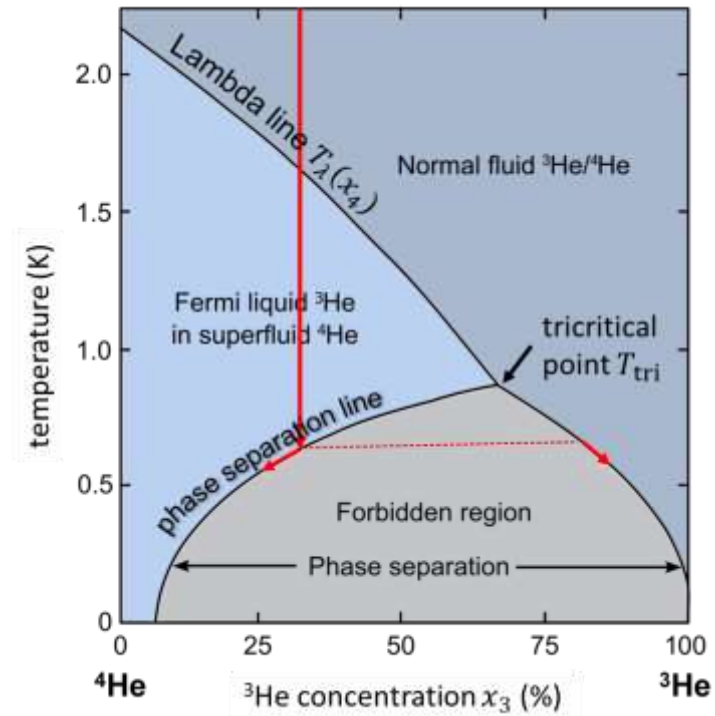
# I.6 $^3\text{He}/^4\text{He}$ mixtures: phase diagram

- high  $T$ :  
 $^3\text{He}$  reduces  $T_\lambda \propto n_4$  ( $^4\text{He}$  concentration)
- tricritical point  $T_{\text{tri}} = 0.87 \text{ K}$
- $T < T_{\text{tri}} = 0.87 \text{ K}$ :  
 → **phase separation**  
 $^3\text{He}$  rich phase: lighter  
 $^3\text{He}$  poor phase: heavier
- $T \rightarrow 0$ :  
 → **incomplete phase separation**  
 6.5% of  $^3\text{He}$  still dissolved in  $^4\text{He}$
- concentrations:  $x_3 = \frac{n_3}{n_3+n_4}$ ;  $x_4 = 1 - x_3 = \frac{n_4}{n_3+n_4}$
- limiting concentrations:  
 $x_4 = aT^{3/2}e^{-b/T}$  ( $a = 0.85 \text{ K}^{-3/2}$ ,  $b = 0.56 \text{ K}^{-1}$ )  
 $x_3 = 0.0648 (1 + cT^2 + dT^3)$  ( $c = 8.4 \text{ K}^{-2}$ ,  $d = 9.4 \text{ K}^{-3}$ )



# I.6 $^3\text{He}/^4\text{He}$ mixture as diluted Fermi liquid

- at  $T_{\text{phase sep}} < T < T_\lambda(x_4)$  :
  - $^4\text{He}$  forms superfluid with negligible amounts of excitations (phonons, rotons)
  - viscosity, entropy and specific heat go to zero:
    - $^4\text{He} = \textit{inert superfluid background}$
- $^3\text{He}$  obeys Fermi statistics
  - Fermi temperature is about 1 K for pure  $^3\text{He}$
  - dissolving  $^3\text{He}$  in  $^4\text{He}$  reduces the Fermi temperature

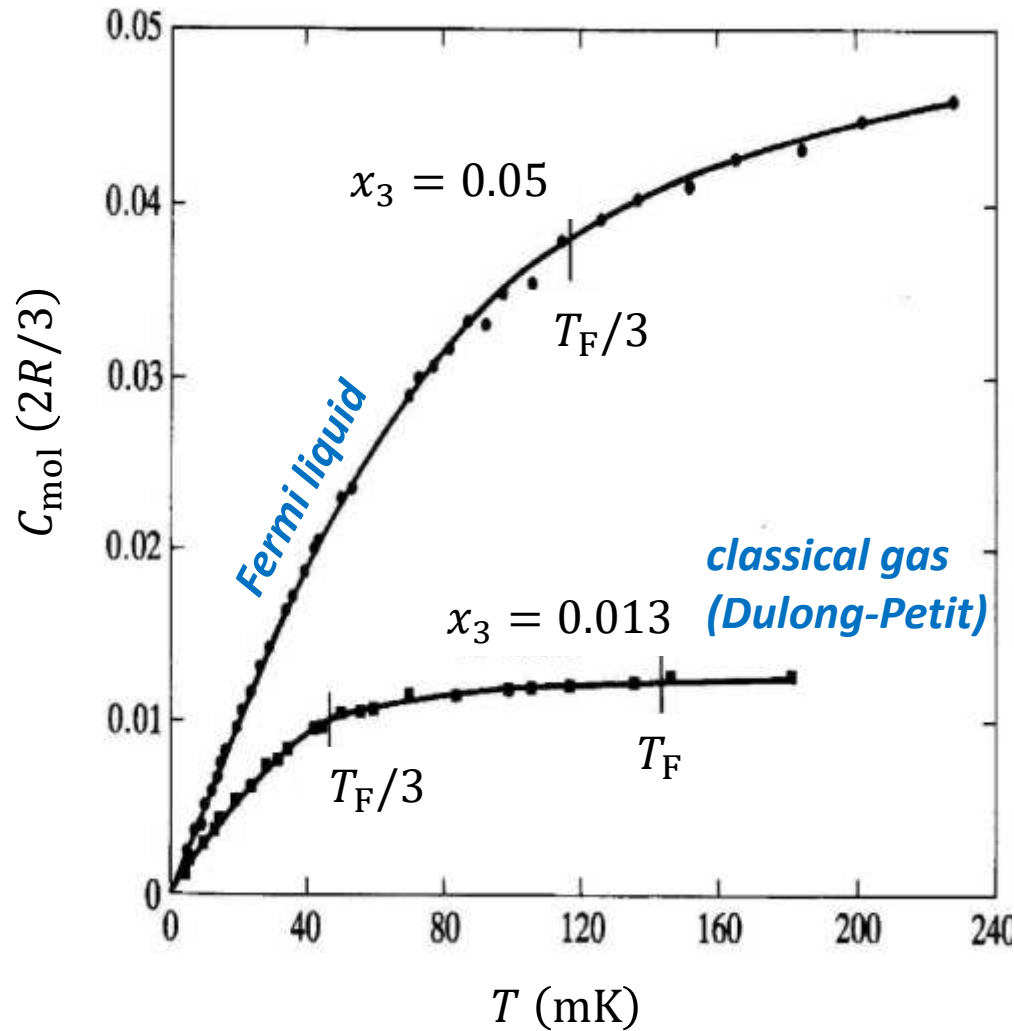


$$T_F = \frac{\hbar^2}{2m_{^3\text{He}}^*} (3\pi^3 n_{^3\text{He}})^{2/3} \propto x_3^{2/3}$$

$m_{^3\text{He}}^*$ : effective mass of  $^3\text{He}$  dissolved in  $^4\text{He}$   
 $n_{^3\text{He}} = \frac{N_{^3\text{He}}}{V} = \frac{x_3 N}{V}$ : density of  $^3\text{He}$  atoms

# I.6 $^3\text{He}/^4\text{He}$ mixture as diluted Fermi liquid

- specific heat



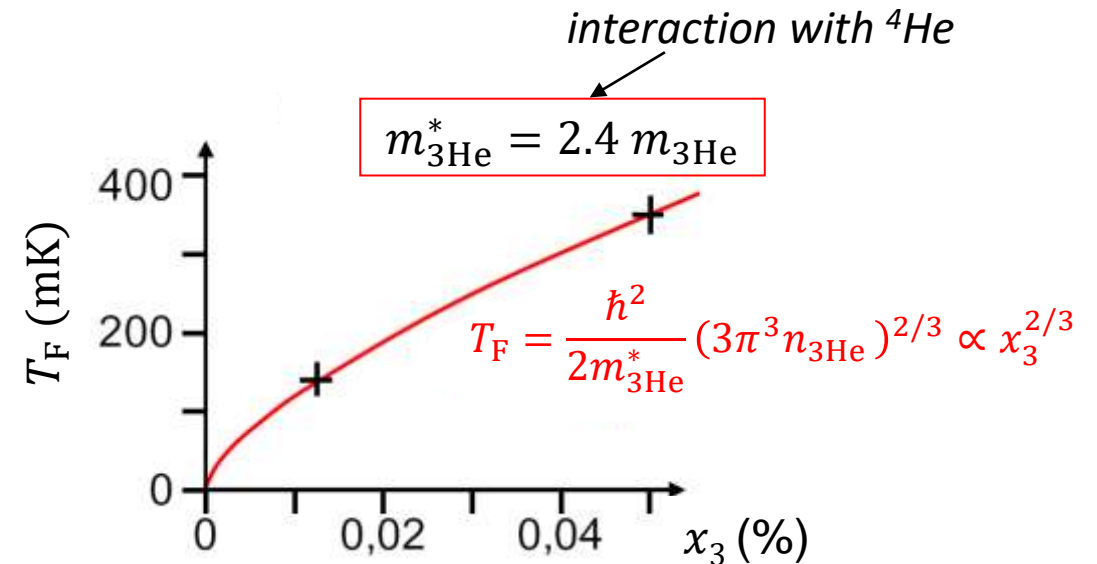
- Fermi gas ( $T \ll T_F$ )

$$C_{\text{mol}} = \frac{\pi^2}{2} R \frac{T}{T_F}$$

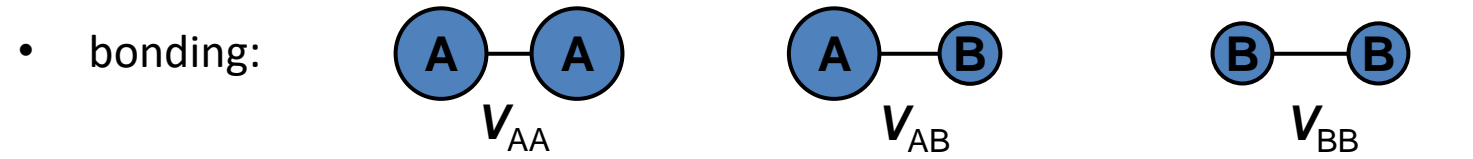
- classical gas ( $T > T_F$ )

$$C_{\text{mol}} = \frac{3}{2} N_A k_B = \frac{3}{2} R$$

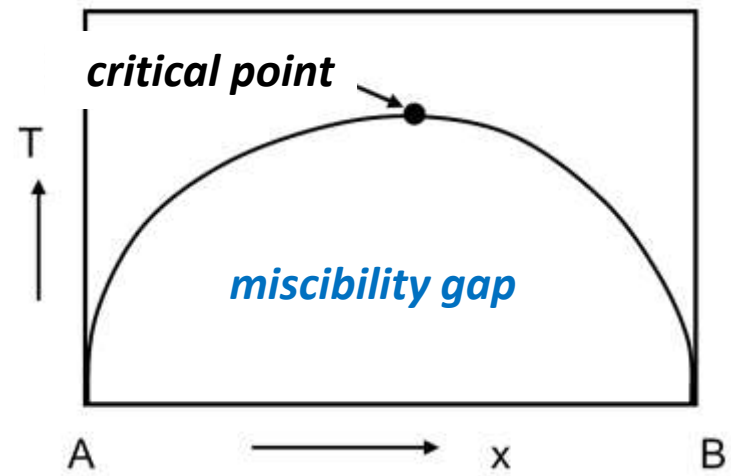
- $T_F$  increases with  $x_3$



# I.6 $^3\text{He}/^4\text{He}$ mixtures: miscibility gap in binary systems



- $V_{AB} > \frac{1}{2}(V_{AA} + V_{BB})$     **complete miscibility** (e.g. water and alcohol)
- $\frac{1}{2}(V_{AA} + V_{BB}) > V_{AB}$     **phase separation** (e.g. water and petrol)     $\rightarrow$  **optimization of binding total energy  $U$**   
complete phase separation @  $T = 0$



increasing mixing with increasing  $T$ :

$$F = U - TS \rightarrow \min$$

$\nearrow$

binding energy

$\nwarrow$

thermal motion

$\rightarrow$  **minimization of free energy**

# I.6 $^3\text{He}/^4\text{He}$ mixtures: finite solubility of $^3\text{He}$ in $^4\text{He}$

- binding energy of  $^3\text{He}$  in  $^3\text{He}$  ( $V_{33}$ ) and  $^4\text{He}$  ( $V_{34}$ ):

$$L_3 = U_{\text{vap}} - U_{\text{liq}} + p\Delta V$$

$p\Delta V = \text{work done against ambient pressure}$

- single  $^3\text{He}$  atom in liquid  $^3\text{He}$ :** average binding energy is given by the latent heat of evaporation  $L_3$ :

binding energy for single  $^3\text{He}$  atom:  $\epsilon_{3,c} = -\frac{L_3}{N_A} = \mu_{3,c}$   $\mu_{3,c} = \text{chemical potential of pure (concentrated) liquid}$

- single  $^3\text{He}$  atom in liquid  $^4\text{He}$ :**

binding energy for single  $^3\text{He}$  atom:  $\epsilon_{3,d}(x_3 \rightarrow 0) = \mu_{3,d}(x_3 \rightarrow 0)$   $\mu_{3,d} = \text{chemical potential of dilute phase}$

–  $|\epsilon_{3,d}(0)| > |\epsilon_{3,c}|$  or vice versa ?

$^3\text{He}$  has smaller mass  $\rightarrow$  larger zero point fluctuations  $\rightarrow$  occupies larger volume

$\rightarrow$  binding of  $^3\text{He}$  is larger in  $^4\text{He}$  than in  $^3\text{He}$  due to larger density of  $^4\text{He}$

$\rightarrow |\epsilon_{3,d}(0)| > |\epsilon_{3,c}|$  (it is energetically more favorable to add a  $^3\text{He}$  atom to the  $^4\text{He}$  superfluid than to pure  $^3\text{He}$ )

$\rightarrow$  corresponds to case  $V_{AB} > \frac{1}{2}(V_{AA} + V_{BB}) \rightarrow$  **complete miscibility expected,**  
**why miscibility only up to 6.5% ??**

# I.6 $^3\text{He}/^4\text{He}$ mixtures: finite solubility of $^3\text{He}$ in $^4\text{He}$

- questions

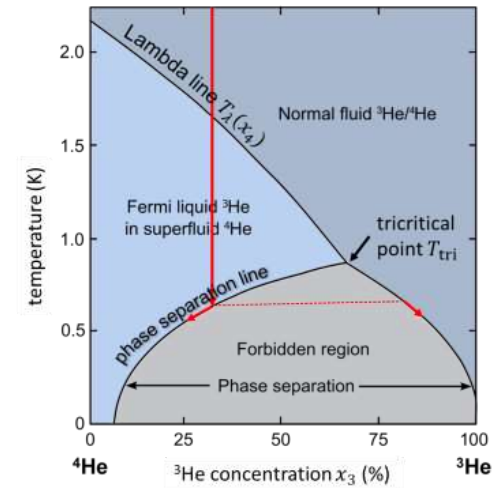
- why can't we dissolve more than 6.5% of  $^3\text{He}$  in  $^4\text{He}$  at  $T = 0$  ?

*we have to take into account two effects:*

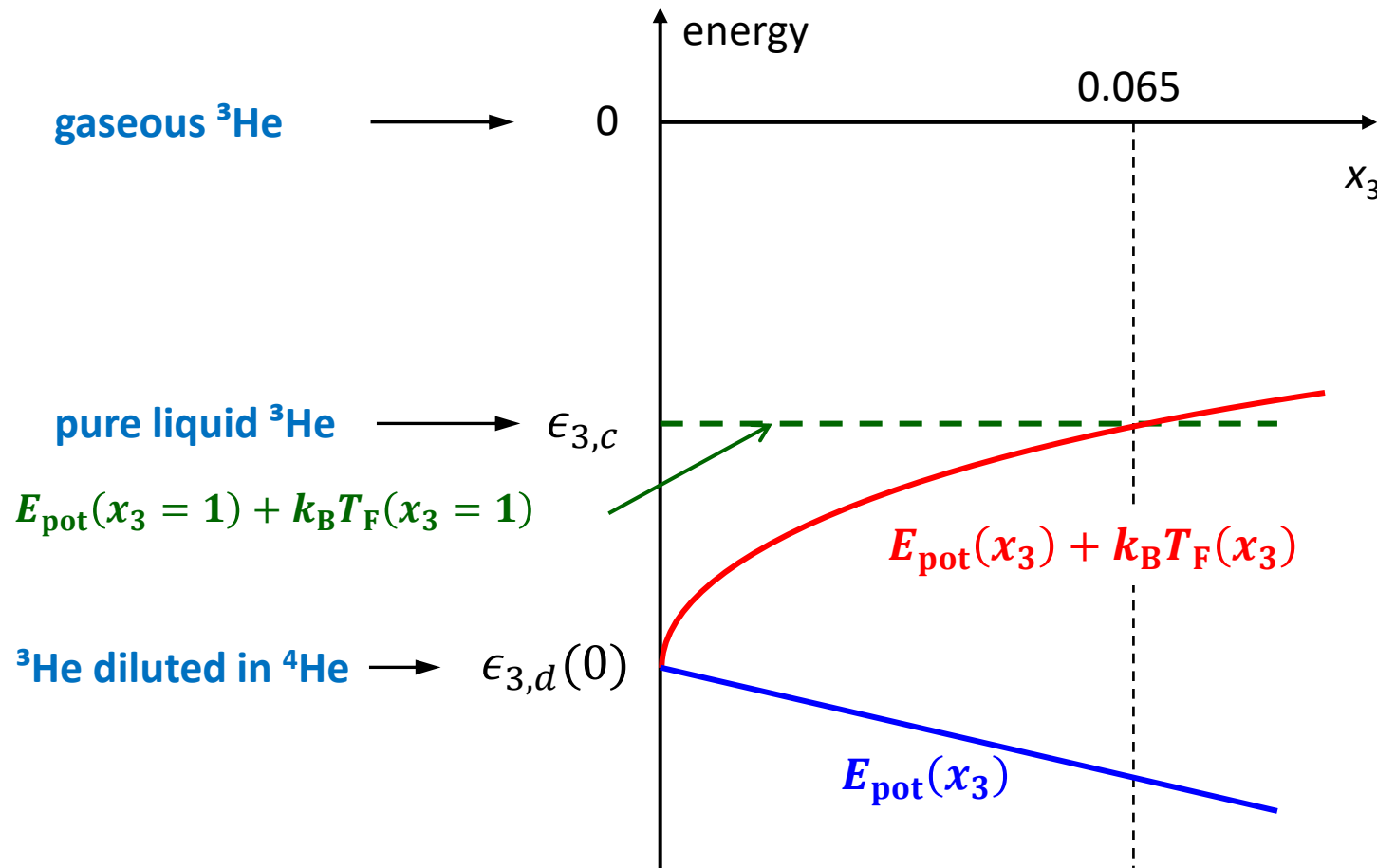
- $^3\text{He}$  forms degenerate Fermi liquid
  - $T_F$  increases with  $x_3$
  - for  $x_3 > 6.5\%$ , the increase of Fermi energy exceeds the gain in binding energy
- effective attraction between two  $^3\text{He}$  atoms (*magnetic and volume effect*)
  - $|\epsilon_{3,d}(x_3)| > |\epsilon_{3,d}(x_3 = 0)|$

- why don't we have a complete phase separation into pure  $^3\text{He}$  and pure  $^4\text{He}$  at  $T = 0$  ?

- incomplete phase separation results in finite disorder, violation of 3<sup>rd</sup> law of thermodynamic ?
- no: we have **degenerate Fermi gas, ordering in k-space**



# I.6 $^3\text{He}/^4\text{He}$ mixtures: finite solubility of $^3\text{He}$ in $^4\text{He}$



for  $x_3 > 6.5\%$ :

$$|E_{\text{pot}}(x_3) + k_B T_F(x_3)| < |\epsilon_{3,c}| = |E_{\text{pot}}(x_3 = 1) + k_B T_F(x_3 = 1)|$$

→ separation of pure  $^3\text{He}$



# I.6 $^3\text{He}/^4\text{He}$ mixtures: cooling effect

- $^3\text{He}/^4\text{He}$  dilution refrigeration

- operation principle:

- remove  $^3\text{He}$  atoms from the dilute phase below  $T_{\text{tri}} = 0.87 \text{ K}$
- transport of  $^3\text{He}$  atoms across phase boundary to maintain equilibrium concentration
- corresponds to evaporation of  $^3\text{He}$  from concentrated phase
- **cooling effect as the latent heat of evaporation is removed**

- for Fermi liquid:

$$\rightarrow C_{V,\text{concentrated}} < C_{V,\text{diluted}} (x_3 = 0.065) \quad (C_V \propto T/T_F, \quad T_F \propto n_{^3\text{He}}^{2/3})$$

- with  $U = Q = \int_0^T C_V(T') dT' = \frac{\gamma}{2} T^2$  we therefore obtain ( $C_V = \gamma T$ ,  $\gamma =$  Sommerfeld coefficient):

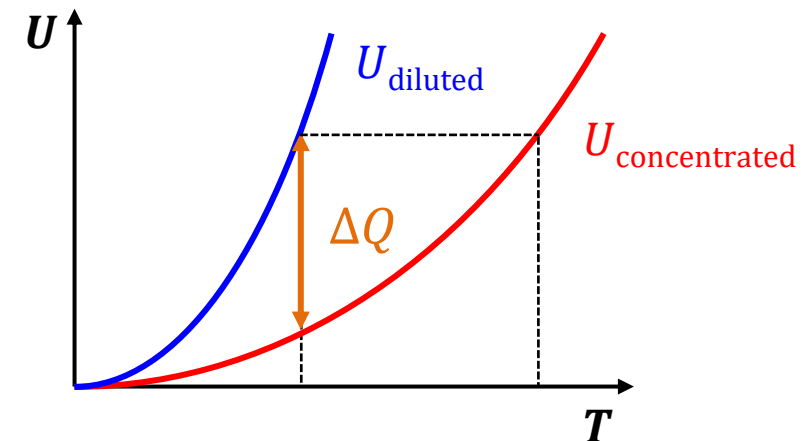
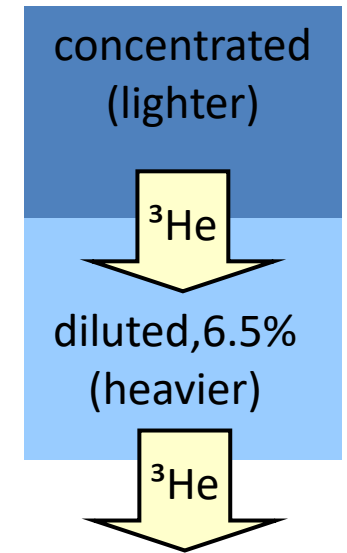
$$U_{\text{concentrated}}(T) < U_{\text{diluted}}(T)$$

- on transition across phase boundary:

$$dG = 0 = dU - TdS \Rightarrow dU = TdS = dQ$$

- removal of heat  $\Delta Q = T\Delta S = T [S_{\text{dil}}(T) - S_{\text{con}}(T)]$
- **cooling effect**

- $^3\text{He}/^4\text{He}$  dilution refrigerator



# I.6 $^3\text{He}/^4\text{He}$ mixtures: cooling effect

- $^3\text{He}/^4\text{He}$  dilution refrigeration: cooling power

- assumption: *one mole of  $^3\text{He}$  crosses boundary between diluted and concentrated phase*

- removed heat:  $\Delta Q = T\Delta S = T [S_{\text{dil}}(T) - S_{\text{con}}(T)]$

- cooling power:  $\Delta\dot{Q} = \left(\frac{\Delta Q}{n_{^3\text{He}}}\right) \cdot \dot{n}_{^3\text{He}} \Rightarrow \Delta\dot{Q} = \dot{n}_{^3\text{He}} T [s_{\text{dil}}(T) - s_{\text{con}}(T)]$   $s_{\text{dil}} = \frac{S_{\text{dil}}}{n_{^3\text{He}}}, s_{\text{con}} = \frac{S_{\text{con}}}{n_{^3\text{He}}}$

- since there is no volume change:  $dQ = dU = TdS = C_V dT, \quad U(T) = \int_0^T C_V dT', \quad S(T) = \int_0^T \frac{C_V}{T'} dT'$

➤ with  $C_{\text{mol}} = \frac{\pi^2 N_A k_B^2 T}{2\varepsilon_F}$  and  $\varepsilon_F = \frac{\hbar^2}{2m_{^3\text{He}}^*} \left(\frac{3\pi^2 N_{^3\text{He}}}{V}\right)^{2/3} = \frac{\hbar^2}{2m_{^3\text{He}}^*} \left(\frac{3\pi^2 N_A}{V_{\text{mol}}}\right)^{2/3}$  (standard expressions for Fermi liquid)

we obtain the entropy  $S_{\text{mol}} = \frac{\pi^2 N_A k_B^2 m_{^3\text{He}}^*}{\hbar^2} \left(\frac{V}{3\pi^2 N_{^3\text{He}}}\right)^{2/3} T \propto \frac{T}{n_{^3\text{He}}}$

⇒  $S_{\text{con}} < S_{\text{dil}}$  (as  $n_{^3\text{He,con}} > n_{^3\text{He,dil}}$ )

⇒  $Q(\dot{T}) = 84 \dot{n}_{^3\text{He}} T^2 [\text{W}]$  (cooling power  $\propto$   $^3\text{He}$  throughput  $\dot{n}_3$  [mol/s])

# I.6 $^3\text{He}/^4\text{He}$ mixtures: cooling effect

- $^3\text{He}/^4\text{He}$  dilution refrigeration: plausibility consideration

