Superconductivity and Low Temperature Physics II

Lecture Notes
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Chapter 2
Quantum Transport in Nanostructures
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Chapter II: Quantum Transport in Nanostructures

**Literature:**

1. Introduction to Mesoscopic Physics
   Yoseph Imry

2. Electronic Transport in Mesoscopic Systems
   Supriyoto Datta

3. Mesoscopic Electronic in Solid State Nanostructures
   Thomas Heinzel

4. Quantum Transport
   Yuli V. Nazarov, Yaroslav M. Blanter

5. Semiconductor Nanostructures
   Thomas Ihn
   Oxford University Press (2010)
II.1 Introduction

II.1.1 General Remarks

- **macroscopic solid-state systems**
  - usually consideration of thermodynamic limit $\rightarrow N \rightarrow \infty, \Omega \rightarrow \infty, N/\Omega = \text{const.}$

- **what happens if system size becomes small ?**
  - discrete spectrum of electronic levels
  - coherent motion of electrons
    $\rightarrow$ phase memory due to lack of inelastic scattering within system size:
    system size $L$ smaller than *phase coherence length* $L_\phi$
    $\rightarrow$ new *interference phenomena*
  - *validity of Boltzmann theory of electronic transport and concept of resistivity ?*
    $\rightarrow$ system size $L$ smaller than *mean free path* $\ell$: ballistic transport
  - *discreteness of electric charge and magnetic flux becomes important*
    $\rightarrow$ single electron and single flux effects
  - *concept of impurity ensemble breaks down*
    $\rightarrow$ sample properties show „fingerprint“ of detailed arrangement of impurities
II.1.2 Mesoscopic Systems

- **mesoscopic systems** (coined by Van Kampen in 1981): **mesos** *(Greek)*: between

  - *system size is between microscopic (e.g. atom, molecule) and macroscopic system (e.g. bulk solid)*

  - *system size $L$ is smaller than phase coherence length $L_\phi$* (typically in nm - µm regime)
    - $\Rightarrow$ phenomena related to phase coherence become important
    - $\Rightarrow$ statistical concepts no longer applicable due to smallness of system size
    - $\Rightarrow$ still coupling to environment/reservoir present (in contrast to microscopic objects such as atoms)

- **properties of mesoscopic systems are usually studied at low temperatures**

  - *phase coherence length $L_\phi$ decreases rapidly with increasing $T$*
    - $\Rightarrow L < L_\phi$ can usually be satisfied only at low $T$

  - *observation of level quantization effects require $k_B T < \Delta E \approx 1/L^2$*

  - **study of nanostructures at low temperature**
II.1.2 Mesoscopic Systems
Die folgende graphische Animation zeigt den Anflug auf eine Einzelektronen-Schaltung.

20 mm

Sie beginnt mit der Ansicht des gesamten Wafers und endet mit der elektronenmikroskopischen Aufnahme einer realen Struktur.
II.1.2 Mesoscopic Systems

- miniaturization of electronic devices

65 nm process
2005

45 nm process
2007

32 nm
2009

22 nm
2011

(Source: Intel Inc.)

gate length of transistors
II.1.2 Mesoscopic Systems

Transistor Fin Improvement

22 nm 1st Generation Tri-gate Transistor

14 nm 2nd Generation Tri-gate Transistor

10 nm 3rd Generation Tri-gate Transistor
II.1.3 Characteristic Length Scales

- from microscopic to macroscopic systems

**microscopic ↔ mesoscopic ↔ macroscopic**

- **Fermi wave length:** \( \lambda_F < 1 \text{ nm} \) (for metals)
  - "size" of charge carrier
- **electron mean free path:** \( \ell \approx 10 - 100 \text{ nm} \)
  - distance between (elastic) scattering events
- **phase coherence length:** \( L_\varphi \approx 1 \text{ \mu m} \)
  - loss of phase memory
- **sample size:** \( L, W \approx 0.01 - 1 \text{ \mu m} \)

**mesoscopic regime:** \( L < L_\varphi(T) \)
<table>
<thead>
<tr>
<th>Length Scale</th>
<th>Physical Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 mm</td>
<td>Mean free path in the Quantum Hall regime</td>
</tr>
<tr>
<td>100 µm</td>
<td>Mean free path / phase coherence length in high mobility semiconductors at $T &lt; 4$ K</td>
</tr>
<tr>
<td>10 µm</td>
<td>Phase coherence length in clean metal films</td>
</tr>
<tr>
<td>1 µm</td>
<td>Size of commercial semiconductor devices</td>
</tr>
<tr>
<td>100 nm</td>
<td>Fermi wave length in semiconductors</td>
</tr>
<tr>
<td>10 nm</td>
<td>Mean free path in polycrystalline metal films</td>
</tr>
<tr>
<td>1 nm</td>
<td>Fermi wave length in metals</td>
</tr>
<tr>
<td>0.1 nm</td>
<td>Distance between atoms</td>
</tr>
</tbody>
</table>
II.1.3 Characteristic Length Scales

- **electron wavelength:** \[ \lambda_F = \frac{h}{\sqrt{2m^* \varepsilon_F}} = \frac{2\pi}{(3\pi^2 n)^{1/3}} \] (Fermi wavelength)

- **mean free path:** \[ \ell = v_F \cdot \tau_m \] \[ \tau_m^{-1} = \tau_c^{-1} \cdot \alpha_m \] \( \uparrow \text{effectiveness of collision: } 0 < \alpha_m < 1 \)

- **phase relaxation length:** \[ L_\phi = v_F \tau_\phi \] \[ \tau_\phi^{-1} = \tau_c^{-1} \cdot \alpha_\phi \] \( \uparrow \text{effectiveness of collision in destroying phase coherence: } 0 < \alpha_\phi < 1 \)

- **ballistic**

- **diffusive**

\[ L_\phi = \sqrt{D \tau_\phi} = \sqrt{\frac{1}{3} v_F^2 \tau_m \tau_\phi} \]

\( \rightarrow \) **elastic impurity scattering:** \( \tau_\phi \rightarrow \infty \) or \( \alpha_\phi \rightarrow 0 \)

\( \rightarrow \) **electron-phonon scattering:** \( \tau_\phi \approx \tau_{e-\text{ph}} \)

\( \rightarrow \) **electron-electron scattering:** \( \tau_\phi \approx \tau_{e-e} \)

\( \rightarrow \) **electron-impurity scattering** (with internal degree of freedom, e.g. spin)
II.1.3 Characteristic Length Scales

- **question**: what is the effectiveness of an *inelastic scattering process* regarding destruction of phase coherence?

  - *Altshuler, Aronov, Khmelnitsky (1982):*

    if $\hbar \omega$ is the characteristic energy of an *inelastic process* (e.g. phonon energy), then the mean-squared energy spread of electron after collisions is

    \[
    \langle \Delta E \rangle^2 = (\hbar \omega)^2 \frac{\tau_\varphi}{\tau_c}
    \]

    $\tau_\varphi$ is time required to acquire a phase change of $\approx 2\pi$

    \[
    \Delta \varphi \approx \frac{\Delta E}{\hbar} \tau_\varphi \approx 2\pi \quad \Rightarrow \quad \tau_\varphi \approx \left( \frac{\tau_c}{\omega^2} \right)^{1/3}
    \]

    *low-frequency excitations are less effective in destroying phase coherence!!*

    - at low $T$: e-e scattering is dominating
II.1.4 Characteristic Energy Scales

- size quantization

- electron in a box:

- Fermi wavelength:

\[
\lambda_F = \frac{\hbar}{\sqrt{2m^*\varepsilon_F}} = \frac{2\pi}{(3\pi^2n)^{1/3}}
\]

if \( \lambda_F > L_x, L_y, L_z \) \( \rightarrow \) reduction of dimension by size quantization
3D \( \rightarrow \) 2D \( \rightarrow \) 1D \( \rightarrow \) 0D

for metals:
\[n \approx 10^{22} - 10^{23} \text{ cm}^{-3} \rightarrow \lambda_F \approx 1 \text{ nm}\]

for semiconductors:
\[n \approx 10^{16} - 10^{19} \text{ cm}^{-3} \rightarrow \lambda_F \approx 10 - 100 \text{ nm}\]

- single charge/flux effects:
\[
\frac{e^2}{2C} > k_B T, \quad \frac{\Phi_0^2}{2L} > k_B T
\]

level spacing:
\[
\Delta E = \frac{\hbar^2}{2m^*} \left( \frac{1}{L} \right)^2
\]

1 nm \( \leftrightarrow \) 10.000 K \( \leftrightarrow \) 800 meV
10 nm \( \leftrightarrow \) 100 K \( \leftrightarrow \) 8 meV
100 nm \( \leftrightarrow \) 1 K \( \leftrightarrow \) 0.08 meV
II.1.4 Characteristic Energy Scales

- size quantization: DOS in 3D, 2D, 1D, and 0D

<table>
<thead>
<tr>
<th>bulk</th>
<th>superlattice</th>
<th>quantum well</th>
<th>quantum wire</th>
<th>quantum dot</th>
</tr>
</thead>
</table>

- $D(\epsilon) \propto \sqrt{\epsilon}$
- $D(\epsilon) = \text{const.}$
- $D(\epsilon) \propto 1/\sqrt{\epsilon}$
- $D(\epsilon) \propto \delta(\epsilon - \epsilon_i)$
II.1.4 Characteristic Energy Scales

- **Thouless energy**
  - how long does it take for an electron to diffuse through a sample of length $L$
    \[
    L = \sqrt{D t} \quad \Rightarrow \quad t = \frac{L^2}{D} \quad D = v^2 \tau: \text{diffusion constant}
    \]
  - mean diffusion time is related to the characteristic energy (uncertainty relation)
    \[
    \varepsilon_{\text{Th}} = \frac{\hbar}{t} = \frac{\hbar D}{L^2}
    \]
    *(Thouless energy)*

- **ballistic transport regime** (see below):
  \[
  t = \frac{L}{v_F} \quad \Rightarrow \quad \varepsilon_{\text{Th}} = \frac{\hbar}{t} = \frac{\hbar v_F}{L}
  \]
  \*(v_F: Fermi velocity)*

- **macroscopic samples:** $\varepsilon_{\text{Th}} \ll k_B T$
- **mesoscopic samples:** $\varepsilon_{\text{Th}} > k_B T$

- **low $T$**
- **small $L$**
- **clean samples** (large $D$)
II.1.4 Characteristic Energy Scales

- physical meaning of the Thouless energy

\[ \epsilon_{\text{Th}} = \frac{\hbar}{t} = \frac{\hbar D}{L^2} \]

\[ \rightarrow \text{electrons in energy interval } \Delta \epsilon = \epsilon_{\text{Th}} \text{ stay phase coherent in sample of length } L \]

\[ \epsilon + \Delta \epsilon \]
\[ \epsilon \]

\[ \varphi = \varphi_i + \frac{\omega}{\hbar} \frac{\tau}{2\pi t} \]
\[ \varphi = \varphi_i + \frac{\epsilon + \Delta \epsilon}{\hbar} \frac{2\pi}{t} \]

(ballistic case)

\[ t = \frac{L}{v_F} \]
\[ t = \frac{L^2}{D} \]

(diffusive case)

\[ \Delta \varphi = 2\pi \]

after length \( L \), if \( \Delta \epsilon = \epsilon_{\text{Th}} \)

\[ \text{if } \Delta \epsilon \leq \epsilon_{\text{Th}}, \text{ the acquired phase shift is less than } 2\pi \]

example: \( D = 10^3 \text{ cm}^2/\text{s}, \ L = 1 \ \mu\text{m} \rightarrow \epsilon_{\text{Th}} / k_B \approx 1 \text{ K} \]
**II.1.5 Transport Regimes**

<table>
<thead>
<tr>
<th>macroscopic sample</th>
<th>mesoscopic sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>diffusive: $L, W \gg \ell$</td>
<td>ballistic: $L, W &lt; \ell$</td>
</tr>
<tr>
<td>incoherent: $L &gt; L_{\varphi}$</td>
<td>quasi-ballistic: $W &lt; \ell$</td>
</tr>
<tr>
<td></td>
<td>coherent: $L &lt; L_{\varphi}$</td>
</tr>
</tbody>
</table>

- @ 300 K: $\ell \sim 10$ nm due to e-ph scattering
- @ at low $T$: $\ell$ is limited by impurity and e-e scattering $\rightarrow$ sample quality matters
- $L_{\varphi}$ is limited by inelastic processes: e-ph and e-e scattering:
  - strong $T$ dependence: $L_{\varphi}$ increases with decreasing $T$
  - $L_{\varphi} \approx 1 \mu m$ @ 1K
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II.2 Description of Electron Transport by Scattering of Waves

II.2.1 Electron Waves and Waveguides (true only in vacuum)

\[ \Psi(r, t) = \frac{1}{\sqrt{V}} \exp \left( i \mathbf{k} \cdot \mathbf{r} - \frac{i}{\hbar} \varepsilon(k)t \right) \]

- \( \Psi(r, t) \): wave function
- \( |\Psi(r, t)|^2 \): probability to find electron at position \( r \) at time \( t \)
- \( V \): normalization volume
- \( k \): wave vector
- \( p = \hbar k \): momentum
- \( \varepsilon(k) = \frac{\hbar^2 k^2}{2m} \): energy
II.2.1 Electron Waves and Waveguides

• electrons as fermions:

→ *Pauli principle* (state either occupied by single electron or empty)

→ *density of states in k-space*: \( \frac{2V}{(2\pi)^3} \) (factor 2 due to spin)

→ *fraction of filled states*: \( f(\mathbf{k}, T) \)

  – important quantities:

\[
\begin{align*}
\text{density} & = \rho \\
\text{energy density} & = [\epsilon] = 2 \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{\epsilon(\mathbf{k})} \right] f(\mathbf{k}) \\
\text{current density} & = J = 2 \int \frac{d^3k}{(2\pi)^3} \left[ \frac{\epsilon(\mathbf{k})}{e\mathbf{v}(\mathbf{k})} \right] f(\mathbf{k})
\end{align*}
\]

  – \( f \) determined by statistics:

\[
f(\mathbf{k}, T) = \exp\left(\frac{\epsilon(\mathbf{k}) - \mu}{k_B T}\right) + 1
\]

Fermi statistics for electrons
II.2.1 Electron Waves and Waveguides

- **ballistic conductor as waveguide**

  - **example:** 1D free motion of charge carriers, e.g. in $x$-direction with confinement in $y,z$-direction

\[
\Psi_{k_x,n,m}(r,t) = \phi_{n,m}(y,z) \exp[i(k_x x - \omega t)]
\]

**mode index** $n,m$  \hspace{1cm} **standing wave**  \hspace{1cm} **plane wave**

\[
\varepsilon_{n,m}(k_x) = \frac{\hbar^2 k_x^2}{2m^*} + \varepsilon_{n,m} \quad \varepsilon_{n,m} = \frac{\pi^2 \hbar^2}{2m^*} \left( \frac{n_y^2}{a^2} + \frac{m_z^2}{b^2} \right)
\]

*Source: Handouts Nazarov, TU Delft*
II.2.1 Electron Waves and Waveguides

- wave guide with potential barrier

\[ \Psi(x) = 1 \cdot \exp(ik_xx) + r \cdot \exp(-ik_xx) \]

\[ \Psi(x) = A \cdot \exp(ikxx) + B \cdot \exp(-ikxx) \]

\[ \Psi(x) = t \cdot \exp(ik_xx) \]

\[ \varepsilon_{n,m}(k_x) = \frac{\hbar^2 k_x^2}{2m^*} + \varepsilon_{n,m}(0) \]

\[ \varepsilon_{n,m}(k_x) - U_0 = \frac{\hbar^2 k_x^2}{2m^*} \]

4 unknown variables: 
- \( A, B, r, t \)
- \( t \): transmission amplitude
- \( r \): reflection amplitude

4 equations
- (wave function matching at interfaces)
II.2.1 Electron Waves and Waveguides

- wave guide with potential barrier \( \rightarrow \) example: rectangular barrier

\[ \sinh^2(\kappa d) = \left[ \exp(\kappa d) - \exp(-\kappa d) \right]^2 \approx \exp(2\kappa d) \]

Transmission probability/coefficient:

\[ T(\varepsilon) \equiv |t^2| = \frac{1}{1 + \left( \frac{k_x^2 - \kappa^2}{2k_x\kappa} \right)^2 \sinh^2 \kappa d} \]

\( \kappa d \gg 1 \):
II.2.1 Electron Waves and Waveguides

- quantum tunneling through a thin potential barrier

  - total reflection at boundary (barrier with infinite thickness)

  \[ R = 1 \quad T = 0 \]

  \[ R < 1 \quad T > 0 \]

  - partial reflection/tunneling at barrier of finite thickness
II.2.1 Electron Waves and Waveguides

- quantum tunneling through a thin potential barrier: a rectangular barrier

\[ \Psi(x) = 1 \cdot \exp(ik_x x) \quad \Psi(x) = A \cdot \exp(ikx) \quad \Psi(x) = t \cdot \exp(ik_x x) \]

\[ \Psi(x) = r \cdot \exp(-ik_x x) \quad \Psi(x) = B \cdot \exp(ikx) \]

- in regions 1 and 3:
  \[ -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} = \varepsilon \Psi(x) \]
  \[ k_x^2 = \frac{2m\varepsilon}{\hbar^2} \]

- in region 2:
  \[ -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} = (\varepsilon - U_0) \Psi(x) \]
  \[ \kappa^2 = \frac{2m(\varepsilon - U_0)}{\hbar^2} \]
II.2.1 Electron Waves and Waveguides

- quantum tunneling through a thin potential barrier: a rectangular barrier

\[ T(\varepsilon) \equiv |t|^2 = \frac{1}{1 + \left( \frac{k_x^2 - \kappa^2}{2k_x\kappa} \right)^2 \sinh^2 \kappa d} \]

\[ T(\varepsilon) \equiv |t|^2 = \frac{1}{1 + \frac{U_0^2}{4\varepsilon(U_0 - \varepsilon)} \sinh^2 \kappa d} \]

for \( \kappa d \gg 1 \):
\[ \sinh^2(\kappa d) = [\exp(\kappa d) - \exp(-\kappa d)]^2 \approx \exp(2\kappa d) \]

\[ T(\varepsilon) \equiv |t|^2 = \frac{1}{1 + \frac{U_0^2}{4\varepsilon(U_0 - \varepsilon)} \exp(2\kappa d)} \]
II.2.1 Electron Waves and Waveguides

- modelling of nanostructures as complex waveguides

→ transport channels + potential barrier

- description of transport by a set of transmission coefficients $T_n$

sufficient to describe transport !!

examples:
(i) adiabatic quantum transport
(ii) quantum point contact
II.2.1 Electron Waves and Waveguides

- modelling of nanostructures as complex waveguides
  
  - example: adiabatic quantum transport $\rightarrow$ constriction as a potential barrier

\[
\varepsilon_{n,m}(k_x, x) = \frac{\hbar^2 k_x^2}{2m^*} + \frac{\pi^2 \hbar^2}{2m^*} \left( \frac{n_y^2}{a^2(x)} + \frac{m_z^2}{b^2(x)} \right)
\]

adiabatic waveguide:
variation of dimensions occurs on length scale large compared to width
$\rightarrow$ waveguide walls can be assumed parallel locally
II.2.1 Electron Waves and Waveguides

- modelling of nanostructures as complex waveguides
  - example: adiabatic quantum transport $\rightarrow$ constriction as a potential barrier

\[ a = a(x) \]

\[ \varepsilon_{n,m}(k_x, x) = \frac{\hbar^2 k_x^2}{2m^*} + \frac{\pi^2 \hbar^2}{2m^*} \left( \frac{n_y^2}{a^2(x)} + \frac{m_z^2}{b^2(x)} \right) \]

- parabolic approximation of potential step

\[ V(x) \approx -\frac{1}{2} m \Omega^2 x^2 \]

**transmission probability:**

\[ T(\varepsilon_F) = \frac{1}{\exp \left( -\frac{2\pi \varepsilon_F}{\hbar \Omega} \right)} + 1 \]

E.C. Kemble, 1935
II.2.1 Electron Waves and Waveguides

- modelling of nanostructures as complex waveguides
  - example: *quantum point contact*

\[ I = I_l - I_r \]

net current:

\[ I_l = T \frac{2}{2\pi} \int dk_x ev_x f_l(k_x) \]

\[ I_r = T \frac{2}{2\pi} \int dk_x ev_x f_r(k_x) \]

\[ v_x = \frac{1}{\hbar} \frac{\partial \varepsilon}{\partial k_x} \]

open channel: \( T = 1 \)
closed channel: \( T = 0 \)

quantized conductance !!
II.2.1 Electron Waves and Waveguides

• what is the meaning of the quantity

\[ G = \frac{I}{V} = 2 \frac{e^2}{\hbar} N_{\text{open}} = 2 G_Q N_{\text{open}} \]

➢ for ballistic transport and reflectionless contacts \((T = 1)\) there should not be any resistance!

➢ where does the resistance come from?

→ contact resistance from the interface between the ballistic conductor and the contact pads

→ resistance is denoted as contact resistance

\[ G_{c}^{-1} = \frac{h}{e^2} \frac{1}{2N_{\text{open}}} = \frac{G_Q^{-1}}{2N_{\text{open}}} \]

\(G_Q\) determined by fundamental constants, does not depend on materials properties, geometry or size of nanostructure

\[ 25 \, 812.807 \, \Omega = 1 \text{ Klitzing} \]
II.2.1 Electron Waves and Waveguides

Electron waves in a ballistic conductor are characterized by their energy momentum relations, which are reflected in the dispersion relation. The states near the Fermi level are important for understanding the behavior of electrons in such systems. The Fermi level, which separates occupied and unoccupied states, acts as a reference point for the energy levels.

In a ballistic conductor, the electrons move without scattering, leading to a smooth transition between the reservoirs. The electrons are transferred from one reservoir to the other through a region where their energy is conserved. The voltage drop at interfaces (contact resistance) is a significant factor in determining the overall electrical properties of the system.

The figure illustrates the transition of electrons from one reservoir to another, with the Fermi level serving as a reference. The states near the Fermi level are important for the electrical behavior of the system, and the voltage drop at interfaces plays a crucial role in the overall electrical properties.
II.2.1 Electron Waves and Waveguides

- quantum point contact: experimental results

first experiment by

increasing gate voltage narrows channel → reduction of $N_{\text{open}}$

II.2.1 Electron Waves and Waveguides

**conduction through a single atom!**

*(Elke Scheer, Univ. Konstanz)*
II.2.2 Landauer Formalism

• considered examples have been too simple: $T$ only 1 (open) or 0 (closed)

• more complicated situation: ideal sample + scattering sites

Transmission probability of the different modes will no longer be only 0 or 1

⇒ „dusty waveguide“

$0 \leq T \leq 1$

• $T$ represents the average probability that an electron injected at one end will be transmitted to the other end

• treatment of the situation by a scattering matrix
II.2.2 Landauer Formalism

**Scattering Region**

- Left reservoir $N_l$
  - $r_{12}$
  - $r_{22}$
  - $r_{32}$

- Right reservoir $N_r$
  - $t_{12}$
  - $t_{22}$
  - $t_{32}$

Incoming amplitudes $a_l$, $a_r$

Outgoing amplitudes $b_l$, $b_r$

**Reflection and Transmission Matrices**

- Reflection matrix $\hat{S}$
  - $\hat{S}_{ll}$
  - $\hat{S}_{lr}$
  - $\hat{S}_{rl}$
  - $\hat{S}_{rr}$

- Transmission matrix $\hat{T}$
  - $\hat{T}$
  - $\hat{T}'$

**Transfer Matrix $\hat{M}$**

\[
\begin{bmatrix}
    b_l \\
    b_r
\end{bmatrix} = \hat{M}
\begin{bmatrix}
    a_l \\
    a_r
\end{bmatrix} = \begin{bmatrix}
    \hat{r} & \hat{r}' \\
    \hat{t} & \hat{t}'
\end{bmatrix}
\begin{bmatrix}
    a_l \\
    a_r
\end{bmatrix}
\]

- Relates amplitudes of waves right of the scatterer with those left of the scatterer
- "Transfers" states across the scatterer
II.2.2 Landauer Formalism

• properties of the scattering matrix

- for given time reversal symmetry: \( \hat{t}^T = \hat{t}^\prime \) \( \Rightarrow \hat{S}^T = \hat{S} \) \( \text{symmetric matrix} \)

- electrons do not disappear:

\[
\sum_{n'} |r_{nn'}|^2 + \sum_m |t_{mn}|^2 = (\hat{S}^\dagger \hat{S})_{nn} = 1
\]

\[
R_n = (\hat{r}^\dagger \hat{r})_{nn} \quad T_n = (\hat{t}^\dagger \hat{t})_{nn}
\]

\( \hat{S}^\dagger \hat{S} = \hat{1} \) \( \text{unitary matrix} \)
II.2.2 Landauer Formalism

- properties of the scattering matrix

  - example: one channel scatterer

\[
\begin{bmatrix}
    r & t \\
    t' & r'
\end{bmatrix} =
\begin{bmatrix}
    b_l \\
    b_r
\end{bmatrix} =
\begin{bmatrix}
    a_l \\
    a_r
\end{bmatrix}
\]

\[r, t, r', t'\] are complex numbers

condition of unitarity

\[R = |r|^2 = 1 - |t|^2 = 1 - T\]

follows from condition of unitarity

- the phases \(\theta\) and \(\eta\) do not manifest themselves in transport across a single scatterer

\[\Rightarrow\text{ lead to quantum interference effects in multi-scatterer configurations}\]
• properties of the scattering matrix: condition of unitarity: $\hat{s}^\dagger \hat{s} = \hat{1}$

\[
\begin{bmatrix}
\hat{r}^* & \hat{t}^* \\
\hat{t}^* & \hat{r}^*
\end{bmatrix} \cdot \begin{bmatrix}
\hat{r} & \hat{t} \\
\hat{t} & \hat{r}
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

(i) $r^* t + t^* r' = 0$

\[
\sqrt{R} e^{-i\eta} \sqrt{T} e^{i\eta} - \sqrt{T} e^{-i\eta} \sqrt{R} e^{i(2\eta-\theta)} = 0
\]

(ii) $t^* r + r^* t' = 0$

\[
\sqrt{T} e^{-i\eta} \cdot \sqrt{R} e^{i\theta} - \sqrt{R} e^{-i(2\eta-\theta)} \cdot \sqrt{T} e^{i\eta} = 0
\]
II.2.2 Landauer Formalism

- description of transport properties by scattering matrix

  - expression for the current:

    \[ I = 2e \sum_n \sum_{k_x} v_x(k_x) f_n(k_x) = 2e \sum_n \int_0^\infty \frac{dk_x}{2\pi} v_x(k_x) f_n(k_x) \]

  - occupation probabilities for right- and left-moving electrons (for current in the left waveguide):

    i. \( k_x > 0 \): \( f_l(\varepsilon) \) (electrons moving to the right)
    
    ii. \( k_x < 0 \): \( R_n f_l(\varepsilon) + (1 - R_n) f_r(\varepsilon) \) (electrons moving to the left)
II.2.2 Landauer Formalism

- description of transport properties by scattering matrix

- \( \text{Tr} \left[ \hat{\epsilon} \dagger \hat{\epsilon} \right] \) can be represented by sum of 'transmission' eigenvalues \( T_p \) of Hermitian matrix \( \hat{\epsilon} \dagger \hat{\epsilon} \) (for each energy \( \epsilon \))

- expression for the current:
  \[
  I = \frac{2e}{2\pi \hbar} \sum_p \int d\epsilon \ T_p(\epsilon) \cdot \left[ f_l(\epsilon) - f_r(\epsilon) \right] = 2G_Q \sum_p T_p \cdot V
  \]

Rolf Wilhelm (William) Landauer
born 4. February 1927 in Stuttgart
† 27. April 1999 in Briarcliff Manor, N.Y.

Landauer formula $\rightarrow$ ‘mesoscopic version’ of Einstein relation

\[
\sigma = 2e^2 N(\epsilon_F)D \quad \iff \quad G = 2 \frac{e^2}{\hbar} N T
\]

Einstein relation $\leftrightarrow$ Landauer formula

- can usually assumed to be independent of \( \epsilon \)
- this gives just the number of open channels, if \( T_p \) is either 0 or 1

Landauer Formalism

- description of transport properties by scattering matrix

- \( \text{Tr} \left[ \hat{\epsilon} \dagger \hat{\epsilon} \right] \) can be represented by sum of 'transmission' eigenvalues \( T_p \) of Hermitian matrix \( \hat{\epsilon} \dagger \hat{\epsilon} \) (for each energy \( \epsilon \))

- expression for the current:
  \[
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  \]

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II.2.2 Landauer Formalism

- description of transport properties by scattering matrix: plausibility consideration

  - consider a conductor with a single conduction channel

  - reservoir biased at $V$ sends out the following number of electrons:

    $$ N(t) = \frac{Z(k)\Delta k}{\text{number}} \cdot \frac{1}{\hbar} \frac{\Delta \varepsilon}{\Delta k} \cdot \frac{t}{\text{time}} = \frac{2}{2\pi} \frac{\Delta k}{\hbar\Delta k} \cdot \frac{eV}{h} \cdot t = \frac{2eV}{h} \cdot t $$

    *emission frequency*

  - the chance to pass is $T_0$, then the passed charge is just $Q(t) = eT_0 N(t)$

  - the average current is charge per time:

    $$ I = \frac{Q}{t} = 2\frac{e^2}{h} T_0 V $$

  - many channels: just sum up to obtain

    $$ I = 2GQ \sum_p T_p V $$
II.2.2 Landauer Formalism

- description of transport properties by scattering matrix: limitations and restrictions

  - restrictions:
    - only *elastic* scattering (electrons pass the conductor at constant energy)
    - *no interactions* between electrons

  - limitations:
    - low temperatures and low voltages
    - short conductors (shorter than inelastic scattering length)
II.2.3 Multi-terminal Conductors

- Landauer formalism: multi-terminal conductors

  - so far discussion of two-terminal systems, extension to multi-terminal conductors?

  \[ V_1 \quad V_2 \quad V_3 \]
  \[ I_1 \quad I_2 \quad I_3 \]

  \textit{ideal conductor} \quad \textit{reservoir} \quad \textit{scattering region} \quad \textit{gate}

  \textit{how to express currents in terms of voltages using the Landauer formalism?}
II.2.3 Multi-terminal Conductors

- Landauer formalism: multi-terminal conductors

- Conduction matrix $G_{kl}$

\[
\begin{pmatrix}
I_1 \\
\vdots \\
I_n
\end{pmatrix} =
\begin{pmatrix}
G_{11} & \ldots & G_{1n} \\
\vdots & \ddots & \vdots \\
G_{n1} & \ldots & G_{nn}
\end{pmatrix} \begin{pmatrix}
V_1 \\
\vdots \\
V_n
\end{pmatrix}
\]

- Properties of conduction matrix:

  - Current conservation (Kirchhoff’s law):

  \[\sum_{k=1}^{n} I_k = 0 \Rightarrow \sum_{k=1}^{n} G_{kl} = 0\]

  - No current, if potential is shifted by the same amount in all leads

  \[\sum_{l=1}^{n} G_{kl} = 0\]

- Consequence of the sum rules: currents $I_k$ and voltage differences
II.2.3 Multi-terminal Conductors

- Landauer formalism: multi-terminal conductors
  - simplest case: two-terminal conductor

- the conduction matrix only has a *single independent element*:

\[
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix} =
\begin{bmatrix}
-G & G \\
G & -G
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
\]

\[I_1 = G(V_2 - V_1)\]
\[I_2 = G(V_1 - V_2)\]
II.2.3 Multi-terminal Conductors

- Landauer formalism: multi-terminal conductors
  - scattering matrix for *multi-terminal* conductors
  - number of modes: \( N = N_1 + N_2 + N_3 + \cdots \)
    \[ \rightarrow \text{scattering matrix is } N \times N \text{ matrix} \]
  - meaning of \( s_{\beta m, \alpha n} \):
    \[ b_{\beta m} = s_{\beta m, \alpha n} a_{\alpha n} \]
    \[ \rightarrow \text{propagation amplitude from terminal } \alpha, \text{ transport channel } n, \text{ to the terminal } \beta, \text{ transport channel } m \]
  - transmission probability from lead \( \alpha \) to \( \beta \):
    \[ T_{\alpha \beta} = \sum_{n=1}^{N_\alpha} \sum_{m=1}^{N_\beta} |s_{\alpha n, \beta m}|^2 \]
  - reflection probability from lead \( \alpha \) into \( \alpha \):
    \[ R_\alpha = \sum_{n=1}^{N_\alpha} \sum_{m=1}^{N_\alpha} |s_{\alpha n, \alpha m}|^2 \]

\[ R_\alpha + \sum_{\beta, \beta \neq \alpha} T_{\alpha \beta} = N_\alpha \]

# of transport channels
II.2.3 Multi-terminal Conductors

- **Landauer formalism: multi-terminal conductors**
  
  \[ \begin{align*}
  & \rightarrow \text{reflection back into same lead } \alpha: \quad s_{\alpha n, \alpha m} \\
  & \rightarrow \text{transmission from lead } \beta \text{ to lead } \alpha: \quad s_{\alpha n, \beta m}
  \end{align*} \]

- current conservation requires
  
  \[ \hat{S}^\dagger \hat{S} = \hat{1} \quad \text{(unitary matrix)} \]

  \[ \sum_{\alpha n,\gamma l} s_{\alpha n,\gamma l}^* s_{\alpha n,\beta m} = \delta_{\gamma \beta} \delta_{lm} \]

- time reversibility relation
  
  ➢ we know: if \( \Psi(r, B) \) solves Schrödinger equation then also \( \Psi^*(r, -B) \)
  
  ➢ application to asymptotic scattering states: taking complex conjugate of scattering state \( b \), then the incoming state \( a \) becomes the complex conjugate of \( b^* \) ➔ corresponds to reversal of time direction

  \[
  \begin{align*}
  b &= \hat{s(B)}a \quad \Rightarrow \quad b^* = s^*(B) a^* \\
  a^* &= s(-B)b^* \quad \Rightarrow \quad s^{-1}(-B)a^* = s^{-1}(-B)s(-B)b^* \quad \Rightarrow \quad s^{-1}(-B)a^* = b^* \\
  \end{align*}
  \]

  \[ \Rightarrow \quad s^{-1}(-B) = s^*(B) = s^{\dagger}(B) \]

  due to unitarity

  \[ s_{\alpha n, \beta m}(B) = s_{\beta m, \alpha n}(-B) \]
### II.2.3 Multi-terminal Conductors

- **Landauer formalism: multi-terminal conductors**
  - sum rules:
    
    \[
    R_\alpha + \sum_{\beta, \beta \neq \alpha} T_{\alpha \beta} = N_\alpha \quad \text{# of transport channels in lead } \alpha
    \]

    \[
    R_\beta + \sum_{\alpha, \alpha \neq \beta} T_{\beta \alpha} = N_\beta \quad \text{# of transport channels in lead } \beta
    \]

  - example: *two-terminal conductor*

<table>
<thead>
<tr>
<th></th>
<th>$\beta = 1$</th>
<th>$\beta = 2$</th>
<th>$\sum = \ N_1$</th>
<th>$N_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 1$</td>
<td>$R_1$</td>
<td>$T_{12}$</td>
<td>$N_1$</td>
<td></td>
</tr>
<tr>
<td>$\alpha = 2$</td>
<td>$T_{21}$</td>
<td>$R_2$</td>
<td>$N_2$</td>
<td></td>
</tr>
<tr>
<td>$\sum = \ $</td>
<td>$N_1$</td>
<td>$N_2$</td>
<td>\</td>
<td></td>
</tr>
</tbody>
</table>

\[
R_1 + T_{12} = R_1 + T_{21} \quad \Rightarrow \quad T_{12} = T_{21}
\]

*transmission function is reciprocal!*  
\[\Rightarrow \text{time reversal symmetry}\]
II.2.3 Multi-terminal Conductors

- Landauer formalism: multi-terminal conductors

  - multi-terminal expression of Landauer formula relates currents to voltages via a scattering matrix (cf. page 42)

  \[
  I_\alpha = 2e \sum_n \left\{ \int_0^{\infty} \frac{dk_x}{2\pi} v_x(k_x) f_\alpha(\varepsilon) + \int_{-\infty}^{0} \frac{dk_x}{2\pi} v_x(k_x) \sum_{\beta m} |s_{\alpha n,\beta m}|^2 f_\beta(\varepsilon) \right\}
  \]

  \[
  I_\alpha = 2e \sum_n \int_0^{\infty} \frac{dk_x}{2\pi} v_x(k_x) \sum_{\beta m} \left\{ |s_{\alpha n,\beta m}|^2 - \delta_{\alpha \beta} \delta_{mn} \right\} f_\beta(\varepsilon) \]

  - probability for transmission from \( \alpha \) to \( \beta \):

    \[
    I_\alpha = -\frac{G_Q}{e} \int_0^{\infty} d\varepsilon \sum_\beta \text{Tr} \left\{ \delta_{\alpha \beta} \delta_{mn} - s^\dagger_{\alpha \beta} s_{\alpha \beta} \right\} f_\beta(\varepsilon)
    \]

    - trace includes all possible transport channels
    - if all \( f_\beta(\varepsilon) \) are the same, e.g. in thermal equilibrium and no voltages applied, then \( \sum_\beta = 0 \) (current conservation, follows from unitarity)

  - we apply voltage \( V_\gamma \) to terminal \( \gamma \) and keep all other at \( \varepsilon_F \) \( \Rightarrow \) the only surviving term in \( \sum_\beta \) is the one for \( \beta = \gamma \) and the integral yields \( eV_\gamma \)

  \[
  I_\alpha = -\frac{G_Q}{e} \text{Tr} \left\{ \delta_{\alpha \gamma} \delta_{mn} - s^\dagger_{\alpha \gamma} s_{\alpha \gamma} \right\} eV_\gamma = G_{\alpha \gamma} V_\gamma
  \]

  \[
  G_{\alpha \gamma} = -G_Q \text{ Tr} \left\{ \delta_{\alpha \gamma} \delta_{mn} - s^\dagger_{\alpha \gamma} s_{\alpha \gamma} \right\}
  \]
II.2.3 Multi-terminal Conductors

- **Summary: Landauer formalism: multi-terminal conductors**

  - linear transport regime:
    \[ G_{\alpha\gamma} = -G_Q \text{ Tr} \{ \delta_{\alpha\gamma}\delta_{mn} - \hat{s}^\dagger_{\alpha\gamma}\hat{s}_{\alpha\gamma} \} \]

  - relation to two-terminal expression: \( \alpha, \gamma = l, r \)
    \[ G_{lr} = G_Q \text{ Tr} \{ \hat{s}^{\dagger}_{lr}\hat{s}_{lr} \} = G_Q \text{ Tr} \{ t^{\dagger} \hat{t} \} \]

  - time reversal symmetry:
    \[ G_{\alpha\gamma}(B) = G_{\gamma\alpha}(-B) \]

  *this is in agreement with Onsager symmetry relations!*
II.2.3 Multi-terminal Conductors

- Landauer formalism: multi-terminal conductors
  - example: three-terminal scattering element

scattering matrix for fully symmetric beam splitter:

\[
\hat{S}_{BS} = \frac{1}{3} \begin{pmatrix} 1 + 2e^{i\varphi} & 1 - e^{i\varphi} & 1 - e^{i\varphi} \\ 1 - e^{i\varphi} & 1 + 2e^{i\varphi} & 1 - e^{i\varphi} \\ 1 - e^{i\varphi} & 1 - e^{i\varphi} & 1 + 2e^{i\varphi} \end{pmatrix}
\]

diagonal elements:

- \[ R = |1 + 2e^{i\varphi}|^2 = \frac{[5 + 4 \cos \varphi]}{9} \]
  - \( R = 1 \) for \( \varphi = 0 \) (total reflection)
  - \( R = 1/3 \) for \( \varphi = \pi \) (equal division)

fully symmetric ideal beam splitter
II.2.3 Multi-terminal Conductors

- Landauer formalism: multi-terminal conductors

- example: three-terminal scattering element

\[
\hat{s}_{BS} = \begin{pmatrix}
-\sin^2\left(\frac{\varphi}{2}\right) & \cos^2\left(\frac{\varphi}{2}\right) & \sin\left(\frac{\varphi}{2}\right) \\
\cos^2\left(\frac{\varphi}{2}\right) & -\sin^2\left(\frac{\varphi}{2}\right) & \sin\left(\frac{\varphi}{2}\right) \\
\sin\left(\frac{\varphi}{2}\right) & \sin\left(\frac{\varphi}{2}\right) & -\cos\varphi
\end{pmatrix}
\]

scattering matrix:

for \( \varphi = \pi/2 \):

\[
\hat{s}_{BS} = \begin{pmatrix}
-1/2 & 1/2 & 1/\sqrt{2} \\
1/2 & -1/2 & 1/\sqrt{2} \\
1/\sqrt{2} & 1/\sqrt{2} & 0
\end{pmatrix}
\]

conductance matrix:

(for \( \varphi = \pi/2 \))

\[
G_{\alpha\beta} = G_Q \begin{pmatrix}
-3/4 & 1/4 & 1/2 \\
1/4 & -3/4 & 1/2 \\
1/2 & 1/2 & 1
\end{pmatrix}
\]

example: \( I_3 = G_Q V, \ I_1 = I_2 = -G_Q V / 2 \)
II.2.4 Statistics of Charge Transport

- **Landauer formalism: counting electrons**
  
  - electron transfer is stochastic process
  
  ➔ *measured number of electrons transferred in time interval $\Delta t$ is random*

- **important aspects:**
  
  i. averaging allows to get rid of fluctuations of individual measurements
  ii. study of statistics provides additional information on nanostructure

- probability $P_N$ to count $N$ electrons:
  
  $$\sum_N P_N = 1$$

  **normalization of distribution**

  $$\langle N \rangle = \sum_N N P_N$$

  **average number (1st cumulant)**

  $$\langle (N - \langle N \rangle)^2 \rangle = \sum_N N^2 P_N - \left(\sum_N N P_N\right)^2$$

  **variance (2nd cumulant)**
II.2.4 Statistics of Charge Transport

- cumulant generation function

\[ K(t) = \ln\langle e^{tN} \rangle = \sum_{n=1}^{\infty} \kappa_n \frac{t^n}{n!} = \mu t + \sigma^2 \frac{t^2}{2} + \cdots \]

\( \kappa_1 = \mu = \langle N \rangle \) average value

\( \kappa_2 = \sigma^2 = \langle (N - \langle N \rangle)^2 \rangle \) variance

\[ \langle e^{itN} \rangle = \sum_N P_N e^{itN} \] (Fourier transform of the probability density function)

- characteristic function

\[ H(t) = \ln\langle e^{itN} \rangle = \sum_{n=1}^{\infty} \kappa_n \frac{(it)^n}{n!} = \mu it - \sigma^2 \frac{t^2}{2} + \cdots \]

\( \kappa_1 = \mu = \langle N \rangle \) average value

\( \kappa_2 = \sigma^2 = \langle (N - \langle N \rangle)^2 \rangle \) variance

\[ \langle e^{itN} \rangle = \sum_N P_N e^{itN} \] (Fourier transform of the probability density function)

\( k^{\text{th}} \) cumulant: differentiate expansion \( k \)-times with respect to \( t \) and evaluate result at \( t = 0 \)

\[ \kappa_n = K^{(n)}(0) \]

example: 1st cumulant

\[ \frac{\partial}{\partial (it)} \ln\left(\sum_N P_N e^{itN}\right) \bigg|_{t=0} = \frac{1}{\sum_N P_N e^{itN}} \sum_N P_N Ne^{itN} \equiv \sum_N P_N N = \langle N \rangle \]

(Fourier transform of the probability density function)
II.2.4 Statistics of Charge Transport

- characteristic function (1)

  - use of characteristic function \( H(t) = \ln(e^{itN}) = \ln \sum_N P_N e^{itN} \)  
    \((e^{itN}) = \text{Fourier transform of the probability density function}\)

  - application to statistics of electron transfer:
    - we assume large measurement time \( \Delta t \) so that \( \langle Q \rangle = \langle I \rangle \Delta t \gg e \)
    - we divide \( \Delta t \) into very small intervals \( dt \) so that \( \langle Q \rangle = \langle I \rangle dt \ll e \)
      - probability to transfer one electron within \( dt \): \( \Gamma dt \ll 1 \)  
      - probability to transfer no electron within \( dt \): \( 1 - \Gamma dt \)
    - we assume that all electrons move in the same direction
    - we neglect probability to transfer two (or more) electrons within \( dt \) \((\mathcal{O}(\Gamma dt)^2))\)

\[
\langle e^{itN} \rangle_{N, dt} = \sum_N P_N e^{itN} = \left(1 - \Gamma dt\right) + \left(\Gamma dt\right)e^{it} + \cdots = 1 + \Gamma dt(e^{it} - 1) + \cdots \approx \exp[\Gamma dt(e^{it} - 1)]
\]

\[
\langle e^{itN} \rangle_{N, \Delta t} \equiv \left[\Pi_{N, dt}(t)\right]^{\Delta t/dt} = \{\exp[\Gamma dt(e^{it} - 1)]\}^{\Delta t/dt} = \exp\left[\frac{\Gamma \Delta t (e^{it} - 1)}{e^{\Delta t}}\right] = \exp[\overline{N}(e^{it} - 1)]
\]

\[
\begin{align*}
P_N &= \int_0^{2\pi} \frac{dt}{2\pi} \langle e^{itN} \rangle_{N, \Delta t} e^{-iNt} = \int_0^{2\pi} \frac{dt}{2\pi} e^{i[N(e^{it} - 1)]} e^{-iNt} = \frac{\overline{N}^N}{N!} e^{-N\Delta t}
\end{align*}
\]

\(k\text{th cumulant: differentiate } H(t) \text{ } k\text{-times with respect to } it \text{ and set } t = 0 \text{ afterwards}\)

example: 1st cumulant

\[
\frac{\partial}{\partial (it)} \ln \left(\sum_N P_N e^{itN}\right) \bigg|_{t=0} = \frac{1}{\sum_N P_N e^{itN}} \sum_N P_N e^{itN} \equiv \sum_N P_N N = \langle N \rangle
\]
II.2.4  Statistics of Charge Transport

• characteristic function (2)
  
  – opposite example: ideally transmitting channel
    ➢ since there is no scattering, the total momentum of all electrons does not change ➔ current does not fluctuate
    \[ P_N = \delta(N - \bar{N}) \] ➔ \[ \langle e^{itN} \rangle = \sum_N P_N e^{it(N-\bar{N})} \]
  
  – intermediate case: \( 0 < T_p < 1 \):
    ➢ the transmitted electrons are correlated, but not fully
    ➢ characteristic function is given by Levitov formula
    \[
    \ln\langle e^{itN} \rangle = 2 \Delta t \int \frac{d\epsilon}{2\pi\hbar} \sum_p \ln\{1 + T_p (e^{it} - 1) f_l(\epsilon)[1 - f_r(\epsilon)] + T_p (e^{-it} - 1)f_r(\epsilon)[1 - f_l(\epsilon)]\}
    \]
  
  note:
    ➢ the transfer processes from left to right and vice versa are correlated
    ➢ for \( f_l(\epsilon) = f_r(\epsilon) = 1 \), the total current is zero
    ➢ if there are no correlations, there would be current fluctuations
    ➢ electrons moving left are blocked by electrons filling the state and vice versa
  
  – limiting case: \( k_B T \ll eV \): integral over energy gives \( eV \)
    \[
    \ln\langle e^{itN} \rangle = \pm \frac{2eV\Delta t}{2\pi\hbar} \sum_p \ln\{1 + T_p (e^{\pm it} - 1)\} \]
    ± for different sign of voltage

L. S. Levitov and G. B. Lesovik, JETP Lett. 58, 230 (1993)
II.2.4 Statistics of Charge Transport

- calculation of cumulants
  - starting point is Levitov formula

\[
\ln\langle e^{i t N} \rangle = 2 \Delta t \int \frac{d \varepsilon}{2 \pi \hbar} \sum_p \ln\{1 + T_p(e^i - 1) f_l(\varepsilon)[1 - f_r(\varepsilon)] + T_p(e^{-i} - 1)f_r(\varepsilon)[1 - f_l(\varepsilon)]\}
\]

- 1st cumulant: 
  \[
  \langle N \rangle = \left. \frac{\partial \ln\langle e^{i t N} \rangle}{\partial (it)} \right|_{t=0} = \frac{2 e V \Delta t}{2 \pi \hbar} \sum_p \int d \varepsilon T_p(\varepsilon)[f_l(\varepsilon) - f_r(\varepsilon)]
  \]

\[
I = e \frac{\langle N \rangle}{\Delta t} = \frac{2 e V}{2 \pi \hbar} \sum_p \int d \varepsilon T_p(\varepsilon)[f_l(\varepsilon) - f_r(\varepsilon)] 
\]

  Landauer formula

- 2nd cumulant: 
  \[
  \langle (N - \langle N \rangle)^2 \rangle = \left. \frac{\partial^2 \ln\langle e^{i t N} \rangle}{\partial (it)^2} \right|_{t=0}
  \]

\[
\langle (N - \langle N \rangle)^2 \rangle = \frac{2 e V \Delta t}{2 \pi \hbar} \sum_p \int d \varepsilon \left\{T_p(\varepsilon)[f_l(\varepsilon)(1 - f_l(\varepsilon)) + f_r(\varepsilon)(1 - f_r(\varepsilon))] + T_p(\varepsilon)\left(1 - T_p(\varepsilon)\right)(f_l(\varepsilon) - f_r(\varepsilon))^2\right\}
\]

Case 1: equilibrium: \( V = 0 \) \((f_l(\varepsilon) = f_r(\varepsilon))\):

\[
\langle (Q - \langle Q \rangle)^2 \rangle_{eq} = \frac{2 e^2 \Delta t}{2 \pi \hbar} k_B T \sum_{p=1}^{G} T_p = 2G_k B \Delta t
\]
II.2.4 Statistics of Charge Transport

- **Nyquist-Johnson noise**

  - interpretation of result

\[
\langle (Q - \langle Q \rangle)^2 \rangle_{eq} = 2G_Q k_B T \Delta t
\]

  - if \( \Delta t \) is large enough, variance of the transmitted charge can be interpreted as zero-frequency current noise

    with \( \Delta I = \Delta Q / \Delta t \) we obtain the current fluctuation

\[
\Delta I^2 = \langle (Q - \langle Q \rangle)^2 \rangle_{eq} / \Delta t^2 = 2G_Q k_B T / \Delta t
\]

  - with the current noise power spectral density \( S_I(0) = \Delta I^2 2\Delta t = \frac{\Delta I^2}{BW} \), we obtain

\[
S_I(0) = 4G_Q k_B T
\]

- **Wiener–Khinchin theorem**: relates the autocorrelation function \( AC_I(\tau) \) to the power spectral density \( S_I(\omega) \)

\[
AC_I(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_I(\omega) e^{i\omega \tau} d\omega
\]

\[
S_I(\omega) = \int_{-\infty}^{\infty} AC_I(\tau) e^{-i\omega \tau} d\tau
\]

\[
AC_I(\tau) = \langle I(t)\hat{I}^*(t + \tau) \rangle
\]

\[
= \lim_{\Delta t \to \infty} \int_{-\Delta t}^{+\Delta t} dt I(t)\hat{I}^*(t + \tau)
\]
II.2.4 Statistics of Charge Transport

- shot noise

\[
\langle (N - \langle N \rangle)^2 \rangle = \frac{2e\Delta t}{2\pi\hbar} \sum_p \int d\varepsilon \left\{ T_p(\varepsilon)[f_i(\varepsilon)(1 - f_i(\varepsilon)) + f_r(\varepsilon)(1 - f_r(\varepsilon))] + T_p(\varepsilon)\left(1 - T_p(\varepsilon)\right)(f_i(\varepsilon) - f_r(\varepsilon))^2 \right\}
\]

**Case 2:** \(eV \gg k_B T\) \(\Rightarrow\) only 2nd term on rhs survives (we assume \(T_p(\varepsilon) = \text{const.}\))

\[
\langle (Q - \langle Q \rangle)^2 \rangle_{eV \gg k_B T} = 4eG_QV\Delta t \sum_p T_p(1 - T_p)
\]

with \(S_I = \langle (Q - \langle Q \rangle)^2 \rangle_{\text{eq}} / \Delta t^2\)

\[
S_I(\omega) = 4eG_QV \sum_p T_p(1 - T_p)
\]

with \(\langle I \rangle = 2G_QV \sum_p T_p\)

\[
S_I(\omega) = 2e\langle I \rangle \left[ \frac{\sum_p T_p(1 - T_p)}{\sum_p T_p} \right]
\]

**Schottky expression**

\(F = 1\)

no correlations in transmission: Poisson process

**Fano factor**

\(0 \leq F \leq 1\)

takes into account correlations in the transmission processes

ideal quantum point contact:
only open \((T_p = 1)\) or closed \((T_p = 0)\) channels
\(\Rightarrow\) no shot noise!
Superconductivity and Low Temperature Physics II

Lecture No. 10
14 July 2022

R. Gross
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   II.1.4 Characteristic Energy Scales
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II.2 Description of Electron Transport by Scattering of Waves
   II.2.1 Electron Waves and Waveguides
   II.2.2 Landauer Formalism
   II.2.3 Multi-terminal Conductors
   II.2.4 Statistics of Charge Transport

II.3 Quantum Interference Effects
   II.3.1 Double Slit Experiment
   II.3.2 Two Barriers – Resonant Tunneling
   II.3.3 Aharonov-Bohm Effect
   II.3.4 Weak Localization
   II.3.5 Universal Conductance Fluctuations

II.4 From Quantum Mechanics to Ohm’s Law

II.5 Coulomb Blockade
II.3 Quantum Interference Effects

charge carriers are phase coherent if \( L_\phi > L \)

- low temperatures (\( \rightarrow L_\phi \) gets large), nanoscale samples (\( L \) gets small)
  - interference of multiply scattered charge carriers
  - \textit{corrections to the classical conductance}

- \textit{macroscopic and mesoscopic samples}:
  - \textit{weak localization (WL)}

- \textit{mesoscopic samples}:
  - \textit{Aharonov-Bohm (AB) oscillations}
  - \textit{Universal Conductance Fluctuations (UCFs)}
II.3.1 Double Slit Experiment

- effect of quantum coherence: transmission through double slit

- basic quantum mechanics: double slit experiment

- probability of propagation from point A to point B:

\[ P_{AB} = |A_1 + A_2|^2 = |A_1|^2 + |A_2|^2 + A_1A_2^* + A_1^*A_2 \]

\[
\begin{align*}
&= P_1 + P_2 + 2\text{Re}\{A_1A_2^*\} \\
&= P_1 + P_2 + \text{classical result} + \text{interference term: quantum mechanical}
\end{align*}
\]
II.3.1 Double Slit Experiment

- effect of quantum coherence: transmission through double slit

\[ P_{AB} = P_{\text{classical}} + 2 \sqrt{P_1 P_2} \cos \varphi \]

interference terms may be destructive or constructive

→ depends on phase shift \( \varphi \)

**problem:**
calculate phase shift \( \varphi \) as a function of geometry, electric potential, magnetic field, ...
II.3.1 Double Slit Experiment

- **phase shifts**
  
  - **geometric phase:**
    \[
    \psi(x) = \exp[i\varphi(x)] = \exp[i k(x)x]
    \]
    \[
    \frac{d\varphi}{dx} = k(x) = \frac{\sqrt{2m[\varepsilon - V(x)]}}{\hbar}
    \]
    
    \[
    \Delta \varphi = \int_A^B \frac{d\varphi}{dx} \, dx = \int_A^B k(x) \, dx = \varphi(B) - \varphi(A) = \frac{kL}{V(x) = \text{const}}
    \]
    
    ➢ usually, absolute value of phase is not interesting, but the relative phase shift between different paths

  - **dynamical phase:**
    
    \[
    \frac{d\varphi}{d\varepsilon} = \frac{d\varphi}{dk} \frac{dk}{d\varepsilon} = \frac{d\varphi}{dk} \frac{1}{\hbar v(x)} = \int_A^B \frac{dx}{\hbar v(x)} = \int_{t_A}^{t_B} \frac{dt}{\hbar} \quad \frac{\tau}{\hbar}
    \]
    
    time of flight between points \( A \) and \( B \) at energy \( \varepsilon \)

    \[
    \Delta \varphi = \frac{d\varphi}{d\varepsilon} \Delta \varepsilon = \int_A^B eV(x) \frac{dx}{\hbar v(x)} = \int_{t_A}^{t_B} eV(x(t)) \frac{dt}{\hbar} \quad \frac{eV}{\hbar} \tau
    \]
    
    ➢ same phase shift for time-reversed path
    
    **dynamical phase**
    
    e.g. by potential \( V \) along path
II.3.1 Double Slit Experiment

- **phase shifts**
  - **Aharonov-Bohm phase** (charged particle in magnetic field)
    - canonical momentum: \( p = m v + qA \)
    - \( k(x) \rightarrow k(x) - \frac{q}{\hbar} A(x) \)
    - results in phase shift \( \varphi_{\text{mag}} \) due to vector potential \( A(x) \)
      \[
      \varphi_{\text{mag}} = \frac{e}{\hbar} \int_{A}^{B} A \cdot dx = \frac{e}{\hbar} \int_{t_A}^{t_B} A \cdot v(t) \, dt \quad (q = -e)
      
      \text{opposite phase shift for time-reversed path}
      \]
    - **note:** \( \varphi_{\text{mag}} \) depends on gauge \( A \rightarrow A + \nabla \chi(x) \) and is therefore unphysical and not observable
    - \( \varphi_{\text{AB}} \) due to opposite sign of phase shift on time-reversed path

\[
\Phi_0 = \frac{\hbar}{e} \quad (\text{"normal" flux quantum})
\]

\[
\Phi = \frac{\Phi_0}{2} = \frac{\hbar}{2e} \quad (\text{in superconductors we have } q_s = -2e \text{ and therefore } \Phi_0 = \frac{\hbar}{2e})
\]

phase difference \( \Delta \varphi = \varphi_1 - \varphi_2 \) between 1 and 2 corresponds to \( \varphi_{\text{AB}} \) due to opposite sign of phase shift on time-reversed path

II.3.2 Double Tunnel Junction

- quantum interference effect in double tunnel junction
  
  - we consider only a single conductance channel
  - no magnetic field

- "classical" expectation:
  (tunneling) resistances are added
  multiplication of transmission probabilities $T_L \cdot T_R$

- what is the role of quantum interference?
- how do individual scattering matrices have to be combined?

Ohm’s law:

$R = R_1 + R_2$

$G = \frac{G_1 G_2}{G_1 + G_2}$
II.3.2 Double Tunnel Junction

- quantum interference effect in double tunnel junction

\[ \varphi = k \cdot s \]

scattering matrix of left barrier

propagation between barriers

scattering matrix of right barrier

acquired phase during propagation between barriers
II.3.2 Double Tunnel Junction

- quantum interference effect in double tunnel junction

\[
\begin{bmatrix}
    r_{\text{tot}} \\
    a
\end{bmatrix} = \begin{bmatrix}
    \hat{r}_L & \hat{t}'_L \\
    \hat{t}_L & \hat{r}'_L
\end{bmatrix} \begin{bmatrix}
    1 \\
    de^{i\phi}
\end{bmatrix}
\]

\[
\begin{bmatrix}
    d \\
    t_{\text{tot}}
\end{bmatrix} = \begin{bmatrix}
    \hat{r}_R & \hat{t}'_R \\
    \hat{t}_R & \hat{r}'_R
\end{bmatrix} \begin{bmatrix}
    ae^{i\phi} \\
    0
\end{bmatrix}
\]
II.3.2 Double Tunnel Junction

- quantum interference effect in double tunnel junction

**Process**

- Amplitude:
  \[ t_L t_R e^{i\varphi} \]

- Probability:
  \[ T_L T_R \]

- Sum of all amplitudes:
  \[ t_L t_R r'_L r'_R e^{i3\varphi} \]

- Sum of all probabilities:
  \[ T_L T_R R_L R_R \]

Path can be viewed as Feynman path

\[ t_{\text{tot}} = \frac{t_L t_R}{1 - r'_L r'_R e^{i2\varphi}} \]

\[ T_{\text{tot}} = |t_{\text{tot}}|^2 \]

- Coherent:
  \[ T_{\text{tot}} = |t_{\text{tot}}|^2 = \frac{T_L T_R}{1 + R_L R_R - 2 \sqrt{R_L R_R} \cos \chi} \]

- Incoherent:
  \[ T_{\text{classical}} = \frac{T_L T_R}{1 - R_L R_R} \]

For \( T_{LR} \ll 1 \):

\[ G_{\text{class}} = G_Q T_{\text{class}} = \frac{G_L G_R}{G_Q [1 - (1 - T_L)(1 - T_R)]} \]

\[ G_{\text{class}} \approx \frac{G_L G_R}{G_Q |T_L + T_R|} = \frac{G_L G_R}{G_L + G_R} \]
II.3.2 Double Tunnel Junction

- quantum interference effect in double tunnel junction

\[ T_{\text{tot}}(\varepsilon) = |t_{\text{tot}}|^2 = \frac{T_LT_R}{1 + R_LR_R - 2\sqrt{R_LR_R}\cos \chi(\varepsilon)} \]

\[ t_{\text{tot}} = \frac{t_Lt_R}{1 - r'L'r_Re^{i2\phi}} \]

phase accumulated during the round trip

\[ \chi(\varepsilon) = 2\varphi(\varepsilon) = 2k(\varepsilon)s \]

\[ k(\varepsilon) = \sqrt{\frac{2m\varepsilon}{\hbar^2}} \]
II.3.2 Double Tunnel Junction

• quantum interference effect in double tunnel junction

- transmission coefficient \( |t_{\text{tot}}|^2 \) depends on energy

\[
T_{\text{tot}}(\epsilon) = |t_{\text{tot}}|^2 = \frac{T_L T_R}{1 + R_L R_R - 2 \sqrt{R_L R_R} \cos[2k(\epsilon)s]}
\]

assume \( T_L = T_R = T \ll 1 \),
\( R_L = R_R = R \approx 1 \)

between peaks: \( T(\epsilon) \approx T^2 \)

maximum value: \( T_{\text{max}} = \frac{T_L T_R}{(1 - \sqrt{R_L R_R})^2} \approx 1 \) @ \( \chi = n \cdot 2\pi \)

maximum value: \( T_{\text{min}} = \frac{T_L T_R}{(1 + \sqrt{R_L R_R})^2} \ll 1 \) @ \( \chi = (n + \frac{1}{2}) \cdot \pi \)

\( \rightarrow \) resonant tunneling
(or Fabry-Perot resonances)

\( \rightarrow \) double barrier structure behaves as an optical interferometer
\( \rightarrow \) resonant tunneling is quantum interference effect
II.3.2 Double Tunnel Junction

- quantum interference effect in double tunnel junction

  - how does the transmission $T(E)$ look like close to the transmission resonances?

    $$\cos \chi = \cos(2ks) \approx 1 - \frac{1}{2} (2ks)^2 \quad \text{for} \quad \chi \ll 1$$

    $$\cos \chi \approx 1 - \frac{\epsilon - \epsilon_{\text{res}}}{2D} (2ks)^2 = \frac{8ms^2(\epsilon - \epsilon_{\text{res}})}{\hbar^2} = \frac{\epsilon - \epsilon_{\text{res}}}{D}$$

  - after some math:

    $$T(\epsilon) = \frac{T_LT_R}{\left(\frac{T_L + T_R}{2}\right)^2 + \left(\frac{\epsilon - \epsilon_{\text{res}}}{D}\right)^2}$$

    transmission assumes Lorentzian shape

    $$T(\epsilon) = \frac{D^2 T_LT_R}{\left(D(T_L + T_R)\right)^2 + (\epsilon - \epsilon_{\text{res}})^2}$$

    energy width of transmission resonance: $d = D (T_L + T_R)$

- interpretation in terms of a particle that moves back and forth between the two potential wells and escapes at a certain tunneling rates $\Gamma_L$ and $\Gamma_R$

- with $d = \hbar(\Gamma_L + \Gamma_R)$ according to uncertainty relation we obtain well-known Breit-Wigner formula
II.3.3 Aharonov-Bohm Effect

- quantum interference effects in multiply connected conductors, e.g. rings
  - phase shift due to magnetic field

\begin{align*}
\Phi_{1,2} &= kL_{1,2} + \frac{e}{\hbar} \int_{1,2} A \cdot dx \\
\varphi_2 - \varphi_1 &= k(L_2 - L_1) + \frac{e}{\hbar} \oint A \cdot dx \\
\varphi_{AB} &= \frac{e}{\hbar} \oint A \cdot dx = \frac{e}{\hbar} \oint B \cdot dF = \frac{2\pi \Phi}{\Phi_0} \\
\Phi_0 &= \frac{\hbar}{e} \quad (\text{"normal" flux quantum})
\end{align*}

(all quantities are periodic in $\Phi/\Phi_0$, even if there is NO magnetic field at the trajectories!)

(in superconductors we have $q_s = -2e$ and therefore $\Phi_0 = \hbar/2e$)
II.3.3 Aharonov-Bohm Effect

- description of Aharonov-Bohm ring by two beam splitters and loop

\[
\begin{pmatrix}
    r \\
    b_1 \\
    d_1
\end{pmatrix} =
\begin{pmatrix}
    0 & 1/\sqrt{2} & 1/\sqrt{2} \\
    1/\sqrt{2} & -1/2 & 1/2 \\
    1/\sqrt{2} & 1/2 & -1/2
\end{pmatrix}
\begin{pmatrix}
    1 \\
    a_1 \\
    c_1
\end{pmatrix}
\]

left beam splitter

\[
\begin{pmatrix}
    t \\
    b_2 \\
    d_2
\end{pmatrix} =
\begin{pmatrix}
    0 & 1/\sqrt{2} & 1/\sqrt{2} \\
    1/\sqrt{2} & -1/2 & 1/2 \\
    1/\sqrt{2} & 1/2 & -1/2
\end{pmatrix}
\begin{pmatrix}
    0 \\
    a_2 \\
    c_2
\end{pmatrix}
\]

right beam splitter

**dynamical phase:**
\[\chi_{1,2} = kL_{1,2}\]

**magnetic phase:**
\[\varphi_{\text{mag,1}} + \varphi_{\text{mag,2}} = \varphi_{\text{AB}} = 2\pi \frac{\Phi}{\Phi_0}\]
II.3.3 Aharonov-Bohm Effect

- description of Aharonov-Bohm ring by two beam splitters and loop

\[
\begin{pmatrix}
  0 & 1/\sqrt{2} & 1/\sqrt{2} \\
  1/\sqrt{2} & -1/2 & 1/2 \\
  1/\sqrt{2} & 1/2 & -1/2
\end{pmatrix}
\begin{pmatrix}
  a_1 \\
  b_1 \\
  c_1
\end{pmatrix}
\]

left beam splitter

\[
\begin{pmatrix}
  0 & 1/\sqrt{2} & 1/\sqrt{2} \\
  1/\sqrt{2} & -1/2 & 1/2 \\
  1/\sqrt{2} & 1/2 & -1/2
\end{pmatrix}
\begin{pmatrix}
  a_2 \\
  b_2 \\
  c_2
\end{pmatrix}
\]

right beam splitter

\[t_1 = \frac{1}{\sqrt{2}} e^{i(\chi_2 + \varphi_{mag,2})} \frac{1}{\sqrt{2}} = \frac{1}{2} e^{i(\chi_2 + \varphi_{mag,2})}\]

\[t_2 = \frac{1}{\sqrt{2}} e^{i(\chi_1 - \varphi_{mag,1})} \frac{1}{\sqrt{2}} = \frac{1}{2} e^{i(\chi_1 - \varphi_{mag,1})}\]

- example 1: electron enters from left, takes lower path and goes out to the right:

\[\chi_1,2 = kL_{1,2}\]

\[\varphi_{mag,1} + \varphi_{mag,2} = \varphi_{AB} = 2\pi \frac{\Phi}{\Phi_0}\]

- example 2: electron enters from left, takes upper path and goes out to the right:

\[\chi_2 - \chi_1 + \varphi_{mag,2} + \varphi_{mag,1} = \chi_2 - \chi_1 + \varphi_{AB}\]

(phase difference in two paths: \(\chi_2 - \chi_1 + \varphi_{mag,2} + \varphi_{mag,1} = \chi_2 - \chi_1 + \varphi_{AB}\) (depends on dynamical phases \(\chi_2, \chi_1\))
**II.3.3 Aharonov-Bohm Effect**

- description of Aharonov-Bohm ring by two beam splitters and loop

**Example 3:** electron takes upper path + full clockwise turn:

\[
t_3 = \frac{1}{\sqrt{2}} e^{i(x_1 - \varphi_{\text{mag},1})} \frac{1}{2} e^{i(x_2 - \varphi_{\text{mag},2})} \frac{1}{2} e^{i(x_1 - \varphi_{\text{mag},1})} \frac{1}{\sqrt{2}} = \frac{1}{8} e^{i(2x_1 + x_2 - 2\varphi_{\text{mag},1} - \varphi_{\text{mag},2})}
\]

**Example 4:** electron takes upper path + full counter-clockwise turn (time-reversed path):

\[
t_4 = \frac{1}{\sqrt{2}} e^{i(x_1 - \varphi_{\text{mag},1})} \left(-\frac{1}{2}\right) e^{i(x_1 + \varphi_{\text{mag},1})} \frac{1}{2} e^{i(x_2 + \varphi_{\text{mag},2})} \frac{1}{\sqrt{2}} = -\frac{1}{8} e^{i(2x_1 + x_2 + \varphi_{\text{mag},2})}
\]

\[\Rightarrow\] phase difference in two paths: \(2\varphi_{\text{mag},2} + 2\varphi_{\text{mag},1} = 2\varphi_{\text{AB}}\) (independent of dynamical phases \(x_2, x_1\))
II.3.3 Aharonov-Bohm Effect

- description of Aharonov-Bohm ring by two beam splitters and loop

\[ P_{AB} = P_{\text{classical}} + 2\sqrt{P_1P_2} \cos \Delta \varphi \]

\[ \Delta \varphi = \chi_2 - \chi_1 + \varphi_{AB} \]

universal conductance fluctuations

\[ P_{AB} \propto \cos(\chi_2 - \chi_1 + \varphi_{AB}) \]

Altshuler-Aronov-Spivak oscillations

\[ P_{AB} \propto \cos(2\varphi_{AB}) \]

\[ \chi_{1,2} = kL_{1,2} \]

\[ \varphi_{\text{mag,1}} + \varphi_{\text{mag,2}} = \varphi_{AB} = 2\pi \frac{\Phi}{\Phi_0} \]

\[ \Phi_0 = \frac{h}{e} \]

+ many other trajectories
II.3.3 Aharonov-Bohm Effect

- description of Aharonov-Bohm ring by two beam splitters and loop
  - summing up (without closed loops):

\[
\chi_1 = \chi_2 = \chi = kL_{1,2} = kL
\]
\[
\varphi_{\text{mag},1} = \varphi_{\text{mag},2} = \varphi_{\text{mag}}
\]
\[
\varphi_{\text{mag},1} + \varphi_{\text{mag},2} = \varphi_{\text{AB}} = 2\pi \frac{\Phi}{\Phi_0}
\]

\[
T = \frac{(1 - \cos 2\chi)(1 + \cos^2 \varphi_{\text{AB}})}{\sin^2 2\chi + \left[\cos 2\chi - \frac{1}{2}(1 + \cos \varphi_{\text{AB}})\right]^2}
\]

\[2\chi = \frac{\pi}{4} \Rightarrow T = \frac{(1 + \cos^2 \varphi_{\text{AB}})}{1 + \frac{1}{4}[1 - \cos \varphi_{\text{AB}}]^2}
\]

**Aharonov-Bohm effect: flux dependent transmission**
II.3.4 Weak Localization

• description of Aharonov-Bohm ring by two beam splitters and loop

\[
\begin{pmatrix}
r \\
b_1 \\
d_1
\end{pmatrix} = \begin{pmatrix}
0 & 1/\sqrt{2} & 1/\sqrt{2} \\
1/\sqrt{2} & -1/2 & 1/2 \\
1/\sqrt{2} & 1/2 & -1/2
\end{pmatrix} \begin{pmatrix}
a_1 \\
c_1
\end{pmatrix}
\]

\[
\begin{pmatrix}
t \\
b_2 \\
d_2
\end{pmatrix} = \begin{pmatrix}
0 & 1/\sqrt{2} & 1/\sqrt{2} \\
1/\sqrt{2} & -1/2 & 1/2 \\
1/\sqrt{2} & 1/2 & -1/2
\end{pmatrix} \begin{pmatrix}
a_2 \\
c_2
\end{pmatrix}
\]

\[r_3 = \frac{1}{\sqrt{2}} e^{i(\chi_1 - \phi_{mag,1})} \frac{1}{2} e^{i(\chi_2 - \phi_{mag,2})} \frac{1}{\sqrt{2}} = \frac{1}{4} e^{i(\chi_1 + \chi_2 - \phi_{mag,1} - \phi_{mag,2})}\]

\[r_4 = \frac{1}{\sqrt{2}} e^{i(\chi_2 + \phi_{mag,2})} \frac{1}{2} e^{i(\chi_1 + \phi_{mag,1})} \frac{1}{\sqrt{2}} = \frac{1}{4} e^{i(\chi_1 + \chi_2 + \phi_{mag,1} + \phi_{mag,2})}\]

\[\phi_{mag,1} + \phi_{mag,2} = \phi_{AB} = 2\pi \frac{\Phi}{\Phi_0}\]

\[P_{AA} \propto \cos \Delta \varphi \propto \cos \left(4\pi \frac{\Phi}{\Phi_0}\right)\]

@ \(B = 0\): enhanced back-scattering due to time-reversed paths

\(\Rightarrow\) weak localization

\[\Rightarrow\) phase difference in two paths: \(2\phi_{mag,2} + 2\phi_{mag,1} = 2\phi_{AB}\) (independent of dynamical phases \(\chi_2, \chi_1\))
II.3.3 Aharonov-Bohm Effect

- Aharonov-Bohm effect: experiments

**Aharonov-Bohm (AB) oscillations:**

- period: $\Phi = \Phi_0 = h/e$
- amplitude: $G_Q = 2e^2/h$
- one channel in Landauer model

Fourier analysis shows that there are also weak oscillations with half period

$\rightarrow$ higher order interferences:

**Altshuler-Aronov-Spivak (AAS) oscillations**

- period: $\Phi = \Phi_0/2 = h/2e$
- Interference of time-reversed traces
- constructive interference for $B = 0$
- coherent backscattering

---

R. Webb et al, PRL 54, 2696 (1985)
II.3.3 Aharonov-Bohm Effect

- Aharonov-Bohm effect: experiments

Aharonov–Bohm like magneto-conductance oscillations (Altshuler-Aronov-Spivak (AAS) oscillations) observed in normally conducting Mg cylinders of diameter 1.5 µm. Left and right resistance scales correspond to samples 1 and 2, respectively. The periodicity of the oscillations corresponds to $\Delta \Phi = h/2e$.

II.3.3 Aharonov-Bohm Effect

- Aharonov-Bohm effect: experiments

- Conductance of a Cu ring in units of $G_Q = \frac{e^2}{h}$, as a function of magnetic field at $T = 100 \text{ mK}$.
- Narrow AB oscillations $\Delta B \approx 2.5 \text{ mT}$ are superimposed on larger and broader universal conductance fluctuations.

F. Pierre et. al., PRL 89, 206804 (2002)
II.3.3 Aharonov-Bohm Effect

- Aharonov-Bohm effect: experiments

Benzene ring

Large Electron Positron Collider at CERN (Geneva)

AB effect: one flux quantum \((h/e)\) through ring area:

\[
\frac{h}{e} \frac{1}{\pi r^2} = 5000 \, T
\]

\[
\frac{h}{e} \frac{1}{\pi r^2} = 7 \times 10^{-23} \, T
\]
**II.3.4 Weak Localization**

**Weak localization:**

interference of time reversed electron paths
II.3.4 Weak Localization

- quantum interference of time-reversed trajectories

\[ P_{AB} = |A_1 + A_2|^2 = \frac{|A_1|^2}{P_1} + \frac{|A_2|^2}{P_2} + \frac{A_1 A_2^* + A_1^* A_2}{2 \text{Re} [A_1 A_2^*]} \]

\[ 2 |A_1 A_2| \cos \Delta \varphi \]

\[ \langle \cos \Delta \varphi \rangle = 0 \quad !? \]

\textit{time-reversed trajectories:}

we consider a closed loop with \( A = B \)

\( \rightarrow \) the amplitude \( A_2 \) is just a time reversal of \( A_1 \)

\[ |A_1 + A_2|^2 = |A_1 + A_1^*|^2 = 4 |A_1|^2 \]

- the backscattering probability is enhanced by factor \( 2 \) for all time-reversed paths!!
- this is a predecessor of localization

\[ A_1 \]
\[ A_2 \]
\[ A \]
\[ B \]
II.3.4 Weak Localization

- quantum interference of time-reversed trajectories

  - increased backscattering probability to original position makes self-intersecting scattering paths important
  - interference effects make it more likely that a charge carrier is doing closed paths than without any interference ➔ increased net resistivity
  - applied magnetic field reduces backscattering probability ➔ decrease of resistivity with increasing field
II.3.4 Weak Localization

- Magnetic field dependence of weak localization

  - Calculate phase difference of time reversed paths:

    \[ \varphi_{\text{mag},A_2} - \varphi_{\text{mag},A_1} = \frac{2e}{\hbar} \oint A \cdot ds = 2\varphi_{AB} \]

  - Loss of constructive interference due to additional \( \varphi_{AB} \)

    \[ \varphi_{\text{mag},A_2} - \varphi_{\text{mag},A_1} = \frac{2e}{\hbar} \oint A \cdot ds = 2\varphi_{AB} = 4\pi \frac{\Phi}{\Phi_0} \]

    \[ \Phi = B F = \text{flux enclosed in the loop} \]

    \[ F = \text{area of the enclosed loop} \]

  - Characteristic field defined by \( \varphi_{\text{mag},A_2} - \varphi_{\text{mag},A_1} = 2\pi \) (complete dephasing):

    \[ B^* = \frac{\Phi_0}{2F} = \frac{\Phi_0}{2\pi L_{\varphi}^2} = \frac{\hbar}{2e L_{\varphi}^2} \]
II.3.4 Weak Localization

- weak localization: important facts
  - coherent backscattering: called the weak localization (the relative number of contributing closed loops is small)
  - effect is important, since it is sensitive to weak magnetic fields:
    - small fields: contributions of large rings oscillate rapidly, phase difference in small rings almost unchanged
    - the larger the field, the fewer loops/rings contribute to constructive backscattering
    - resistance drops to classical value for large fields, if phase shift in smallest rings is about $2\pi$
  - weak localization has to be distinguished from strong localization (due to strong disorder)
**II.3.4 Weak Localization**

- weak localization: experiments

- **requirement:**
  sample larger than elastic scattering length: \( L > \ell \)

- **observations:**
  - conductivity is reduced by \( \approx 2e^2/h \) for \( B = 0 \)
  - large \( B \): Shubnikov de-Haas oscillations

weak localization in SiGe 2-dimensional quantum well with hole gas

II.3.4 Weak Localization

• weak localization: measurement of phase coherence time
  
  - as dependence of magnitude of WL on the coherence time is known to be \( \tau_\varphi \approx L_\varphi^2 / D \)

  \[ \Rightarrow \text{weak localization experiments can be used to determine } \tau_\varphi \]

Senz et al., PRB 61, 5082 (2000)
II.3.3 Aharonov-Bohm Effect

• weak localization: in combination with Aharonov-Bohm effect

S. Pedersen, A.E. Hansen, A. Kristensen, C.B. Sørensen, P.E. Lindelof, Aharonov–Bohm effect in GaAs/GaAlAs ring interferometers, Materials Science and Engineering: B 74, 234-238 (2000)
II.3.5 Universal Conductance Fluctuations

Universal Conductance Fluctuations:

fluctuation of conductance due to different configuration of scatters

\[ P_{AB} = \left| \sum_p A_p e^{i\chi_p} \right|^2 = \sum_p A_p^2 + \sum_{p \neq p'} A_p A_{p'} e^{i(\chi_p - \chi_{p'})} \]

phases \( \chi_p \) depend on specific configuration of scatters in each sample

\( P_{AB} \) and hence conductance is fingerprint of this configuration
II.3.5 Universal Conductance Fluctuations

- experimental study of universal conductance fluctuations
  - would require fabrication of many samples with different (random) configuration of scatters
  - ergodicity theorem: same result is obtained for a single sample measured at different applied magnetic fields

\[ P_{AB} = |\sum_p A_p e^{i\chi_p}|^2 = \sum_p A_p^2 + \sum_{p \neq p'} A_p A_{p'} e^{i(\chi_p - \chi_{p'})} \]

\[
\begin{align*}
\Delta G(e^2/h) &\quad 200\text{mK} \quad 800\text{mK} \\
B(T) \quad -8 &\quad -6 &\quad -4 &\quad -2 &\quad 0 &\quad 2 &\quad 4 &\quad 6 &\quad 8
\end{align*}
\]

random phase shifts

position of scatters becomes important

+\(\varphi_{\text{mag}}\)
II.3.5 Universal Conductance Fluctuations

- experimental observations and facts
  - irregular conductance variations as a function of field ($B$), carrier density ($n$), and voltage ($V$)
  - conductance variations are symmetric with respect to $B$ (2 probe setup)
  - different in each individual sample ("magnetic fingerprint"), fluctuations characterize impurity configuration
  - caused by quantum interference
  - amplitude of conductance variations is of the order $e^2/h$, not noise
  - theory based on ergodicity theorem

**Variance of ensemble conductance:**

Consider an ensemble of macroscopically identical but microscopically different samples (different configurations of scattering centers)

$$\langle (G - \langle G \rangle)^2 \rangle = \frac{e^4}{h^2} \left( \sum_{mn} T_{mn} - \sum_{mn} \langle T_{mn} \rangle \right)^2$$

$$T_{mn} = |t_{mn}|^2$$

→ complicated calculation
II.3.5 Universal Conductance Fluctuations

- UCFs in Au wires

red and blue curve taken at different days without warming up the sample ➔ no noise effect !!
**II.3.5 Universal Conductance Fluctuations**

- **UCFs in Au wires**

  UCF in gold nanowire
  
  \[ L = 600 \text{ nm} \]
  
  \[ W = 60 \text{ nm} \]

  UCF amplitude decreases with increasing \( T \) as phase coherence length becomes smaller than sample length

II.3.5 Universal Conductance Fluctuations

- UCFs in GaAs quantum wire

Data from Heinzel (2003)
## Contents Part II: Quantum Transport in Nanostructures

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II.4 From Quantum Mechanics to Ohm’s Law

• two different points of view:

  ➔ quantum transport
    (electron waves, scattering/transfer matrix)

  ➔ classical transport
    (electric currents, charged particles, friction due to scattering, Ohm’s law)

What is the bridge between these limiting cases ??
II.4 From Quantum Mechanics to Ohm’s Law

• consider two conductors with transmission probabilities $T_1$ and $T_2$ connected in series

\[ T_1 \quad R_1 \quad T_2 \quad R_2 \]

- what is the transmission probability $T_{12}$ ?

- if $T_{12} = T_1 T_2$, then for a chain of scatterers we would expect the transmission probability to drop exponentially with the length of the chain:

\[ T(L) = \exp(-L/L_0) \]

as $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$

→ no Ohm’s law

- problem: if we assume $T_{12} = T_1 T_2$, then we do not take into account multiple reflections

→ to obtain the correct result we have to add the probabilities of multiply reflected paths
II.4  From Quantum Mechanics to Ohm’s Law

- two scatterers in series

\[ T_1 T_2 + T_1 T_2 R_1 R_2 + T_1 T_2 R_1^2 R_2^2 + \cdots \]

\[ T_{12} = \frac{T_1 T_2}{1 - R_1 R_2} \]

\[ \frac{1 - T_{12}}{T_{12}} = \frac{1 - T_1}{T_1} + \frac{1 - T_2}{T_2} \]

with \( T_1 = 1 - R_1 \) and \( T_2 = 1 - R_2 \)

transmission probabilities

incoherent processes

additive property
II.4 From Quantum Mechanics to Ohm’s Law

- \( N \) scatterers in series

\[
\frac{1 - T(N)}{T(N)} = N \frac{1 - T}{T}
\]

\[
T(N) = \frac{T}{N(1 - T) + T}
\]

- number of scatterers in conductor of length \( L \) can be written as \( N = nL \), where \( n \) is the linear density

\[
T(L) = \frac{L_0}{L + L_0}
\]

with \( L_0 = \frac{T}{n(1 - T)} \)

- \( L_0 \) is of the order of the mean free path \( \ell \)

\[
\ell = \frac{1}{n(1 - T)}
\]

\[
\ell = \frac{1}{n(1 - T)} \approx \frac{T}{n(1 - T)} = L_0 \quad \text{(for } T \text{ close to 1)}
\]
II.4 From Quantum Mechanics to Ohm’s Law

- quantum conductance for $N$ channels

- wide conductor with $M \approx k_F W / \pi$ modes:
  \[ G \approx 2 G_Q M T = \frac{2 e^2}{h} M T \approx \frac{e^2 W}{\pi} T \frac{2k_F}{h} \]

- 2D density of transverse modes:
  \[ n_{2D} = \frac{1}{2\pi} \frac{2m}{h^2} \quad \Rightarrow \quad n_{2D} \nu_F = \frac{1}{2\pi} \frac{2m}{h^2} \frac{\hbar k_F}{m} = \frac{2k_F}{h} \]

- using $T(L) = \frac{L_0}{L + L_0}$ yields:
  \[ G \approx \frac{W}{L + L_0} e^{2n_{2D} \nu_F L_0 \pi} \approx \sigma \text{ (Einstein relation)} \]

\[
G \approx \frac{W}{L + L_0} \sigma \quad \text{or} \quad R = \frac{1}{G} \approx \frac{L + L_0}{W} \frac{1}{\sigma} = \frac{L}{\sigma W} + \frac{L_0}{\sigma W}
\]

resistance obeying Ohm’s law

length independent interface resistance
II.4 From Quantum Mechanics to Ohm’s Law

• conclusions

− Ohm’s law is obtained from the expression for the quantum conductance

  → by summing up *probabilities of multiply reflected paths*

  → note that by summing up probabilities *coherence effects are neglected*
    *(of course these are not contained in Ohm’s law, incoherent transport)*

− sample size $L \gg \text{phase coherence length } L_\phi$: large phase shifts (also affected by disorder)

  formally identical samples:
  - very different phase shifts,
  - but same ohmic resistance, since interference effects average out for $L \gg L_\phi$

− $L < L_\phi$: interference effects play important role
  → deviation from Ohm’s law
  → different resistance for formally identical samples due to different impurity configurations
II.4 From Quantum Mechanics to Ohm’s Law

• Where is the resistance??
  – expression for quantum conductance: \( G = 2 \frac{e^2}{h} M T \)
    \[ \Rightarrow \text{scatterers give rise to resistance by reducing } T \]
  – example: waveguide with \( M \) modes and a single scatterer
    \[
    \frac{1}{G} = \frac{h}{2e^2 M} + \frac{h}{2e^2 M} \frac{1}{T} \]
    "interface“ resistance "scatterer“ resistance
    \[ \Rightarrow \text{scatterer resistance determined by properties of scatterer via its transmissivity} \]
  – remaining questions:
    \[ \Rightarrow \text{can we associate a resistance with the scatterer?} \]
    \[ \Rightarrow \text{what about the potential drop? Does it occur across the scatterer?} \]
    \[ \Rightarrow \text{what about Joule heating? Dissipation at the scatterer?} \]
II.4 From Quantum Mechanics to Ohm’s Law

- energy distribution of the electrons

**reservoirs:**

\[ f^+ = \mathcal{G}(\mu_1 - E) \]
\[ f^- = \mathcal{G}(\mu_2 - E) \]

**near left and right:**

\[ f^+ = \mathcal{G}(\mu_1 - E) + T\{\mathcal{G}(\mu_1 - E) - \mathcal{G}(\mu_2 - E)\} \]
\[ f^- = \mathcal{G}(\mu_2 - E) + (1-T)\{\mathcal{G}(\mu_1 - E) - \mathcal{G}(\mu_2 - E)\} \]

**far left and right:**

\[ F' = \mu_2 + (1-T)\{\mu_1 - \mu_2\} \]
\[ F'' = \mu_2 + T\{\mu_1 - \mu_2\} \]

(step functions)

**partial filling of states for**

\[ \mu_2 < E < \mu_1 \]

(follows from the conservation of the number of electrons)
II.4 From Quantum Mechanics to Ohm’s Law

• spatial variation of the electrochemical potential

  • left and right to the scatterer (after energy relaxation):

    \[ F^+ = \mu_1 \quad \text{(left)} \]
    \[ F^+ = \mu_2 + T(\mu_1 - \mu_2) \quad \text{(right)} \]
    \[ F^- = \mu_2 + (1-T)(\mu_1 - \mu_2) \quad \text{(left)} \]
    \[ F^- = \mu_2 \quad \text{(right)} \]

  • close to scatterer (nonequilibrium distribution, \( F \) can be defined via the number of electrons)

- drop of electrochemical potential across scatterer \( \rightarrow \) localized „scatterer“ resistance
- drop close to contact \( \rightarrow \) contact resistance
Superconductivity
and
Low Temperature
Physics II

Lecture No. 11
21 July 2022

R. Gross
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II.5 Coulomb Blockade

- charge quantization and charging energy
  - electric charge is quantized for an isolated island
  - charging energy:
    \[
    \varepsilon = \frac{Q^2}{2C} = \frac{N^2e^2}{2C} = n^2E_c \quad \text{with} \quad \varepsilon_c = \frac{e^2}{2C}
    \]
  - how large is \( \varepsilon_c \) for island of size \( L \) (bring charge \( e \) from \( \infty \) to island)
    \[
    \varepsilon_c \approx \frac{e^2}{\varepsilon_0L} \approx \frac{10 \text{ eV}}{L \text{ [nm]}}
    \]
    typically in meV regime for 100 nm-sized samples
  - level splitting in nm-sized island:
    \[
    \delta\varepsilon \approx \frac{\varepsilon_F}{N_{\text{atom}}} \approx \frac{1 \text{ eV}}{L^3 \text{ [nm}^3]}\]
    typically in \( \mu \text{eV regime} \) for 100 nm-sized samples
II.5 Coulomb Blockade

- capacitance model for metallic island

\[ Q_0 = \sum_{i=1}^{k} C_i V_i + \bar{Q}_0 \]

charge for all \( V_i = 0 \) „background charge“

- potential \( V_0 \) of the island is not known, but its charge \( Q_0 \) is known to be \( Ne \) ➔ electrostatic potential of the island:

\[ V_0(Q_0) = \frac{Q_0 - \bar{Q}_0}{C_{\Sigma}} - \sum_{i=1}^{k} \frac{C_i}{C_{\Sigma}} V_i \quad \text{with} \quad C_{\Sigma} = \sum_{i=1}^{k} C_i \]

- electrostatic energy needed to put additional charge \( \Delta Q = Ne \) on island

\[ \varepsilon_{el}(\Delta Q) = \int_{Q_0}^{\bar{Q}_0+Ne} V_0(Q_0) \, dQ_0 = \frac{(Ne)^2}{2C_{\Sigma}} - eN \sum_{i=1}^{k} \frac{C_i}{C_{\Sigma}} V_i \]

- energy needed to charge the island with one additional charge \( \Delta Q = e \)

\[ \varepsilon_{el}(N + 1) - \varepsilon_{el}(N) = \frac{e^2}{C_{\Sigma}} \left( N + \frac{1}{2} \right) - e \sum_{i=1}^{k} \frac{C_i}{C_{\Sigma}} V_i \]
II.5 Coulomb Blockade

- capacitance model for metallic island – only a single capacitance
  - electrostatic energy \( \epsilon_{el}(\Delta Q) = \frac{(Ne)^2}{2C} - eNCV = \frac{e^2}{2C} \left[ \left( N - \frac{CV}{e} \right)^2 - \left( \frac{CV}{2} \right)^2 \right] \)

\[ \epsilon_{el}(\Delta Q) = \frac{e^2}{2C} \left( N - \frac{CV}{e} \right)^2 = E_C \left( N - \frac{CV}{e} \right)^2 \]

\[ \epsilon_{el}(\Delta Q) = \frac{(Ne)^2}{2C^2} - eN \sum_{i=1}^{k} \frac{C_i}{C} V_i \]
II.5 Coulomb Blockade

• capacitance model for 2-terminal device

\[
\varepsilon_{el}(\Delta Q) = \frac{(Ne)^2}{2C_\Sigma} - eN \sum_{i=1}^{k} \frac{C_i}{C_\Sigma} V_i
\]

\[C_\Sigma = C_S + C_D\]

- electrostatic energy barrier for removing one electron to drain: \( \Delta Q = +e \)

\[
\varepsilon_{el}(\Delta Q) = \frac{e^2}{2C_\Sigma} - e \left( \frac{C_S V_{SD}}{2C_\Sigma} - e \frac{C_D V_{SD}}{2} \right) = \frac{e^2}{2C_\Sigma} - e \frac{V_{SD}}{2}
\]

- electrostatic energy barrier for adding an electron from source: \( \Delta Q = -e \)

\[
\varepsilon_{el}(\Delta Q) = \frac{e^2}{2C_\Sigma} - e \left( \frac{C_S V_{SD}}{2C_\Sigma} + e \frac{C_D V_{SD}}{2} \right) = \frac{e^2}{2C_\Sigma} - e \frac{|V_{SD}|}{2}
\]

- at \( T = 0 \), current transport sets in if energy barrier is reduced to zero

**threshold SD-voltage:**

\[
|V_{SD}^{thres}| = \frac{e}{2C_\Sigma}
\]

**Coulomb blockade effect**

for \(|V_{SD}| \leq |V_{SD}^{thres}|\)
II.5 Coulomb Blockade

- capacitance model for SET: electrostatic energy

\[
\varepsilon_{\text{el}}(\Delta N) = \frac{(\Delta N e)^2}{2C_\Sigma} - \Delta N e \frac{C_S}{C_\Sigma} \left| \frac{V_{SD}}{2} \right| - \Delta N e \frac{C_D}{C_\Sigma} \left| \frac{V_{SD}}{2} \right| - \Delta N e \frac{C_G}{C_\Sigma} V_G = \left( \frac{\Delta N e^2}{2C_\Sigma} - \Delta N e \left( \frac{C_S}{C_\Sigma} |V_{SD}| + \frac{C_G}{C_\Sigma} V_G \right) \right)_{C_S=C_D=C}
\]

- charging the neutral island by \( \Delta Q = -\Delta Ne \) from source at constant \( V_G \)

\[
\varepsilon_{\text{el}}(\Delta Q) = \frac{(Ne)^2}{2C_\Sigma} - eN \sum_{i=1}^{k} \frac{C_i}{C_\Sigma} V_i
\]

- electrostatic energy difference between adding \( \Delta N = N + 1 \) and \( \Delta N = N \) electrons

\[
\varepsilon_{\text{el}}(N+1) - \varepsilon_{\text{el}}(N) = \left( N + \frac{1}{2} \right) \frac{e^2}{C_\Sigma} - e \left( \frac{C_S}{C_\Sigma} |V_{SD}| + \frac{C_G}{C_\Sigma} V_G \right)
\]
## II.5 Coulomb Blockade

- Capacitance model for SET: electrostatic energy

\[
\varepsilon_{el}(\Delta N) = \frac{(\Delta Ne)^2}{2C_\Sigma} - \Delta Ne \frac{C_S|V_{SD}|}{2C_\Sigma} - \Delta Ne C_D \frac{|V_{SD}|}{2C_\Sigma} - \Delta Ne \frac{C_G}{C_\Sigma} V_G = \frac{(\Delta Ne)^2}{2C_\Sigma} - \Delta Ne \left(\frac{C}{C_\Sigma} |V_{SD}| + \frac{C_G}{C_\Sigma} V_G \right)
\]

- Charging the island by removing \(\Delta Q = -\Delta Ne\) to source at constant \(V_G\) (corresponds to adding \(\Delta Q = +\Delta Ne\) to island)

\[
\varepsilon_{el}(\Delta N) = \left(N - \frac{1}{2}\right) e^2 C_\Sigma - e \left(\frac{C}{C_\Sigma} |V_{SD}| + \frac{C_G}{C_\Sigma} V_G\right)
\]

Electrostatic energy difference between adding \(\Delta N = N - 1\) and \(\Delta N = N\) electrons

\[
\varepsilon_{el}(N+1) - \varepsilon_{el}(N) - [\varepsilon_{el}(N) - \varepsilon_{el}(N-1)] = \frac{e^2}{C_\Sigma}
\]

Energy ladder with fixed spacing \(e^2/C_\Sigma\)
II.5 Coulomb Blockade

- capacitance model for 3-terminal device: single electron transistor (SET)

\[ \varepsilon_{el}(\Delta Q) = \frac{(Ne)^2}{2C_\Sigma} - eN \sum_{i=1}^{k} \frac{C_i}{C_\Sigma} V_i \]

- electrostatic energy barrier for removing one electron to drain (\(\Delta Q = +e\)) or adding one electron from source (\(\Delta Q = -e\)) at finite \(V_G\)

\[ \varepsilon_{el}(\Delta Q) = \frac{e^2}{2C_\Sigma} - e \frac{C_S}{C_\Sigma} V_{SD} - e \frac{C_D}{C_\Sigma} V_{SD} - e \frac{C_G}{C_\Sigma} V_G \]

\(\text{with } C_\Sigma = C_S + C_D + C_G\)

- at \(V_{SD} \approx 0\): \(\varepsilon_{el}(\Delta Q) = \frac{e^2}{2C_\Sigma} - e \frac{V_{SD}}{2} \overset{\approx 0}{=} - e \frac{C_G}{C_\Sigma} V_G\)

\(\Rightarrow\) transport allowed for

\[ V_{G}^{\text{trans}} = \frac{e}{2C_G} \]

- analog result for adding one electron from source (\(\Delta Q = -e\)) at finite \(V_G\)

\(\Rightarrow\) periodic peaks in \(I_{SD}\) at \(V_G = N \cdot \frac{e}{2C_G}\)

additional \(V_G\) shifts potential energy of island
II.5 Coulomb Blockade

- capacitance model for SET: current flow at $V_{SD} \approx 0$ as a function of $V_G$

For $V_{SD} \approx 0$:

- periodic peaks in SD-current $I_{SD}$ at $V_G = N \cdot \frac{e}{2C_G}$
II.5 Coulomb Blockade

- Capacitance model for SET: current flow at finite $V_{SD}, V_G$

1. From the left: $N \rightarrow N + 1$  \( \Delta \varepsilon_{FL}(N) = \varepsilon_{el}(N + 1) - \varepsilon_{el}(N) \)
2. To the left: $N \rightarrow N - 1$  \( \Delta \varepsilon_{TL}(N) = \varepsilon_{el}(N - 1) - \varepsilon_{el}(N) \)
3. From the right: $N \rightarrow N + 1$  \( \Delta \varepsilon_{FR}(N) = \varepsilon_{el}(N + 1) - \varepsilon_{el}(N) \)
4. To the right: $N \rightarrow N - 1$  \( \Delta \varepsilon_{TR}(N) = \varepsilon_{el}(N - 1) - \varepsilon_{el}(N) \)

At a given $N$ on island, four different electron transfer processes are possible.
II.5 Coulomb Blockade

- capacitance model for SET: current flow at finite $V_{SD}, V_G$

allowed and forbidden electron transfer processes:

- $T > 0$: all transfer processes are allowed (by thermal activation)
- $T = 0$: only transfer processes with $\Delta \varepsilon < 0$ are allowed

\[
\Delta \varepsilon_{FL,TL,FR,TR}(N) > 0
\]

\[
\begin{align*}
\Delta \varepsilon_{FL}(N) &< 0 \\
\Delta \varepsilon_{TR}(N) &< 0 \\
\Delta \varepsilon_{FL}(N + 1) &> 0 \\
\Delta \varepsilon_{TR}(N - 1) &> 0 \\
\end{align*}
\]

no second additional or missing electron on island !!
II.5 Coulomb Blockade

- capacitance model for SET: current flow at finite $V_{SD}, V_G$
  - in which range of $V_{SD}$ and $V_G$ is the electron transport blocked?
  - assumptions: $C_S = C_D = C$, symmetric SD voltage bias

\[
\begin{align*}
\epsilon_{el}(N + 1) - \epsilon_{el}(N) &= \left(N + \frac{1}{2}\right) \frac{e^2}{C_{\Sigma}} - e \left(\frac{C}{C_{\Sigma}} V_{SD} + \frac{C_G}{C_{\Sigma}} V_G\right) \\
\epsilon_{el}(N) - \epsilon_{el}(N - 1) &= \left(N - \frac{1}{2}\right) \frac{e^2}{C_{\Sigma}} - e \left(\frac{C}{C_{\Sigma}} V_{SD} + \frac{C_G}{C_{\Sigma}} V_G\right)
\end{align*}
\]

1. from the left: $N \rightarrow N + 1$
   \[\Delta \epsilon_{FL}(0) = \epsilon(+1) - \epsilon(0) = \frac{e^2}{C_{\Sigma}} \left(\frac{1}{2} - \frac{C_G}{C_{\Sigma}} eV_G\right) - \frac{C}{C_{\Sigma}} eV_{SD}\]

2. to the left: $N \rightarrow N - 1$
   \[\Delta \epsilon_{TL}(0) = \epsilon(-1) - \epsilon(0) = \frac{e^2}{C_{\Sigma}} \left(\frac{1}{2} + \frac{C_G}{C_{\Sigma}} eV_G\right) + \frac{C}{C_{\Sigma}} eV_{SD}\]

3. from the right: $N - 1 \rightarrow N$
   \[\Delta \epsilon_{FR}(0) = \epsilon(0) - \epsilon(-1) = \frac{e^2}{C_{\Sigma}} \left(-\frac{1}{2} - \frac{C_G}{C_{\Sigma}} eV_G\right) - \frac{C}{C_{\Sigma}} eV_{SD}\]

4. to the right: $N + 1 \rightarrow N$
   \[\Delta \epsilon_{TR}(0) = \epsilon(0) - \epsilon(1) = \frac{e^2}{C_{\Sigma}} \left(-\frac{1}{2} + \frac{C_G}{C_{\Sigma}} eV_G\right) + \frac{C}{C_{\Sigma}} eV_{SD}\]
II.5 Coulomb Blockade

- capacitance model for SET: current flow at finite $V_{SD}, V_G$

1. from the left: $N \to N + 1$  \[ \Delta \varepsilon_{FL}(0) = \varepsilon(+1) - \varepsilon(0) = \frac{e^2}{C_\Sigma} \left( \frac{1}{2} - \frac{C_G}{C_\Sigma} eV_G \right) - \frac{C}{C_\Sigma} eV_{SD} \]

2. to the left: $N \to N - 1$  \[ \Delta \varepsilon_{TL}(0) = \varepsilon(-1) - \varepsilon(0) = \frac{e^2}{C_\Sigma} \left( \frac{1}{2} + \frac{C_G}{C_\Sigma} eV_G \right) + \frac{C}{C_\Sigma} eV_{SD} \]

3. from the right: $N - 1 \to N$  \[ \Delta \varepsilon_{FR}(0) = \varepsilon(0) - \varepsilon(-1) = \frac{e^2}{C_\Sigma} \left( - \frac{1}{2} - \frac{C_G}{C_\Sigma} eV_G \right) - \frac{C}{C_\Sigma} eV_{SD} \]

4. to the right: $N + 1 \to N$  \[ \Delta \varepsilon_{TR}(0) = \varepsilon(0) - \varepsilon(1) = \frac{e^2}{C_\Sigma} \left( - \frac{1}{2} + \frac{C_G}{C_\Sigma} eV_G \right) + \frac{C}{C_\Sigma} eV_{SD} \]

blue areas mark blockade regimes: "Coulomb diamonds"
II.5 Coulomb Blockade

- capacitance model for SET: current flow at finite $V_{SD}, V_G$

Conductance $G$ ($\mu S$)

Data: ETH Zurich

Single Electron Transistor – Coulomb Diamonds:

blue regions of vanishing conductance correspond to the Coulomb blockade regime (no current flow)
II.5 Coulomb Blockade

- capacitance model for SET: current flow at finite $V_{SD}, V_G$

![Diagram of SET model](image)

**tunneling barriers**
(characterized by tunneling resistance $R$)

- weak coupling of island to metallic leads (reservoirs)
  - too weak: no electron transfer
  - too strong: no conservation of charge number, no single electron effects

- too little
- just right
- too much
II.5 Coulomb Blockade

- requirements for the experimental observation of the Coulomb blockade:

  - thermal fluctuations must be small enough:
    \[ E_c = \frac{e^2}{2C} > k_B T \quad \Rightarrow \quad C < \frac{e^2}{2k_B T} \approx 1 \text{ fF} @ 1 \text{ K} \]

  - quantum fluctuations must be small enough:
    \[ E_c = \frac{\hbar}{\tau} \approx \frac{\hbar}{RC} \quad \Rightarrow \quad R > \frac{\hbar}{e^2} = R_Q \approx 25 \text{ k}\Omega \]

    \( R_Q = \text{quantum resistance} \)

  - requirement for voltage:
    \[ E_c > eV \quad \Rightarrow \quad V < \frac{e}{2C} \approx 80 \mu \text{V} @ 1 \text{ fF} \]
II.5 Coulomb Blockade

- **SET: current-voltage characteristics**
  - **facts:**
    1. Charging state is determined by $N$
    2. No quantum coherence between different states
  - **probability $p_N(t)$** to find system in state $N$ at time $t$:
    \[ p_N(t) \Rightarrow \text{given by Master equation} \]

\[
\frac{d}{dt} p_N(t) = -\left[ \Gamma_F(N) + \Gamma_T(N) \right] p_N(t) + \Gamma_T(N-1)p_{N-1}(t) + \Gamma_F(N+1)p_{N+1}(t)
\]

with tunneling rates $\Gamma_F = \Gamma_{FL} + \Gamma_{FR}$ and $\Gamma_T = \Gamma_{TL} + \Gamma_{TR}$

- If we know $p_N$ for stationary state, we get currents as

\[
\begin{align*}
I_L &= e \sum_N \left[ \Gamma_{FL}(N) - \Gamma_{TL}(N) \right] p_N \\
I_R &= e \sum_N \left[ \Gamma_{TR}(N) - \Gamma_{FR}(N) \right] p_N \\
I &= I_L + I_R
\end{align*}
\]
II.5 Coulomb Blockade

- SET: current-voltage characteristics

**tunneling rates for single tunnel junction:**

**tunneling without CB**

\[ I = G_{\text{tun}} V \]

\[ \Gamma_{\text{tun}} = \frac{I}{e} = \frac{G_{\text{tun}}}{e^2} eV \]

**tunneling with CB**

\[ eV - E_C \] for available final states

\[ \Gamma_{\text{tun}} = 0 \] for \( eV < E_C \)

\[ \Gamma_{\text{tun}} = \frac{G_{\text{tun}}}{e^2} (eV - E_C) \] for \( eV < E_C \)
II.5 Coulomb Blockade

- SET: current-voltage characteristics

\[ 2R_{\text{tun}} C_{\text{tun}} / e \]

IVC for tunneling with Coulomb blockade

- \( R = 0 \)
- \( R \) sufficiently large
WMI II.5 Coulomb Blockade

- SET: tunneling rates and IVC
  - electrostatic energy changes as electron tunnels
    \( \rightarrow \) determine tunneling rate at electron energy change of \( \Delta \varepsilon \):

\[
\Gamma_{i \rightarrow f} = \frac{2\pi}{\hbar} |\langle i | H_{\text{tun}} | f \rangle|^2 \delta(\varepsilon_f - \varepsilon_i - \Delta \varepsilon)
\]

Fermi’s Golden Rule

- total transition rate from conductor 1 (source) to 2 (island):
  - tunneling rate proportional to density of states \( D(\varepsilon) \)
  - occupation probability given by Fermi functions \( f(\varepsilon) \)
  - integration over all energies

\[
\Gamma_{i \rightarrow f}(\Delta \varepsilon) = \frac{2\pi}{\hbar} \int_{-\infty}^{\infty} d\varepsilon \ |\langle i | H_{\text{tun}} | f \rangle|^2 \left( D_i(\varepsilon) f(\varepsilon) \frac{D_f(\varepsilon + \Delta \varepsilon)}{\text{occupied initial states}} - D_f(\varepsilon + \Delta \varepsilon) [1 - f(\varepsilon + \Delta \varepsilon)] \frac{D_f(\varepsilon + \Delta \varepsilon)}{\text{empty final states}} \right)
\]
II.5 Coulomb Blockade

- **SET:** tunneling rates and IVC
  - simplifying assumptions:
    - $H_{\text{tun}}$ is energy independent
    - $D(\varepsilon)$ is energy independent

  \[
  \Gamma_{i\rightarrow f}(\Delta \varepsilon) = \frac{1}{e^2 R_{\text{tun}}} \frac{\Delta \varepsilon}{\exp(\Delta \varepsilon / k_B T) - 1}
  \]

  \[
  I = e \left[ \Gamma_{1\rightarrow2} (\Delta \varepsilon_{1\rightarrow2}) - \Gamma_{2\rightarrow1} (\Delta \varepsilon_{2\rightarrow1}) \right]
  \]

  current from current source (1) to island (2) in steady state

  \[
  I = e \sum_N p(N) \left[ \Gamma_{1\rightarrow2} [\Delta \varepsilon_{1\rightarrow2}(N)] - \Gamma_{2\rightarrow1} [\Delta \varepsilon_{2\rightarrow1}(N)] \right]
  \]

  (equivalent expression for current from island to drain)

- net current
  \[
  I = e \left[ \Gamma_{1\rightarrow2} (\Delta \varepsilon_{1\rightarrow2}) - \Gamma_{2\rightarrow1} (\Delta \varepsilon_{2\rightarrow1}) \right]
  \]

  \[
  R_{\text{tun}} = \frac{\hbar}{2\pi e^2} D^2 |\langle i| H_{\text{tun}} |f\rangle|^2
  \]

  at low $T$: Fermi functions $\approx$ step functions
II.5 Coulomb Blockade

- SET: current-voltage characteristics

![Diagram showing SET: current-voltage characteristics and Coulomb diamonds]
II.5 Coulomb Blockade

- SET: current-voltage characteristics - Coulomb staircase

movie shows variation of IVC with varying gate voltage

1st step in IVC

\[ |V_{SD}|/2 \]

2nd step in IVC

\[ |V_{SD}|/2 \]

\[ V_G \]

\[ \text{ground} \]

source: lt.px.tsukuba.ac.jp

R. Gross © Walther-Meißner-Institut (2004 - 2022) - additional topic
II.5 Coulomb Blockade

- SET: variation of the gate voltage – Coulomb oscillations:

- gate voltage shifts up and down the energy levels of the island

- at small SD-voltages: conductance can be varied considerably by gate voltage

⇒ Coulomb Oscillations
II.5 Coulomb Blockade

- SET: variation of the gate voltage – Coulomb oscillations:

\[ \frac{eV_{SD}}{2EC} = 1.1, 1, 0.9, 0.75, 0.5, 0.25 \]

**note:**
- large \( dI_{SD} / dV_G \)

\[ \Rightarrow \text{use as ultra-sensitive electrometer} \]
II.5 Coulomb Blockade

- SET: variation of the gate voltage – Coulomb oscillations:

Experimental data on Al/AlO\textsubscript{x}/Al/AlO\textsubscript{x}/Al - SET

$V_{SD}$ is varied for different curves

J. Schuler, Ph.D. Thesis (WMI 2005)
II.5 Coulomb Blockade

- SET Coulomb oscillations - effect of single fluctuating background charges

![Graph showing SET Coulomb oscillations](image)

Shift of $I_{SD}(V_G)$ curve due to fluctuating background charge

J. Schuler, Ph.D. Thesis (WMI 2005)
II.5 Coulomb Blockade

- SET fabrication by two-angle shadow evaporation

fabrication of sub-μm Josephson Junctions by shadow evaporation technique
II.5 Coulomb Blockade

- SET fabrication by two-angle shadow evaporation

**Optical Lithography**

![Optical Lithography Image](image)

**Electron Beam Lithography**

![Electron Beam Lithography Image](image)
II.5 Coulomb Blockade

- SET fabrication by two-angle shadow evaporation
II.5 Coulomb Blockade

- SET application: single electron detector

Prototype of a self-referenced quantum current source developed at PTB with four semiconductor single-electron current sources ("single-electron pumps") connected in series and three metallic single electron detectors.
II.5 Coulomb Blockade

- SET applications
  - sensitive electrometers: $\frac{\Delta Q}{Q} \approx 10^{-5} e$
  - electron pumps
    $\rightarrow$ transporting electrons one by one: counting of electrons
    $\rightarrow$ current standard: $I = e \cdot f$
    $\rightarrow$ application of oscillating gate voltage
  - charge Qubits
    $\rightarrow$ basic element for quantum information systems

*Quantum oscillations in two coupled charge qubits*
Nature 421, 823-826 (20 February 2003)
II.5 Coulomb Blockade

• SET applications – the quantum metrology triangle

\[ K_J = \frac{2e}{h} = \frac{1}{\Phi_0} \]
\[ = 483\,597.848\,4 \ldots \text{GHz/V} \]

\[ K_1 = \frac{1}{e} \text{ not yet available} \]

\[ R_K = \frac{h}{e^2} = 25\,812.807\,45 \ldots \Omega \]