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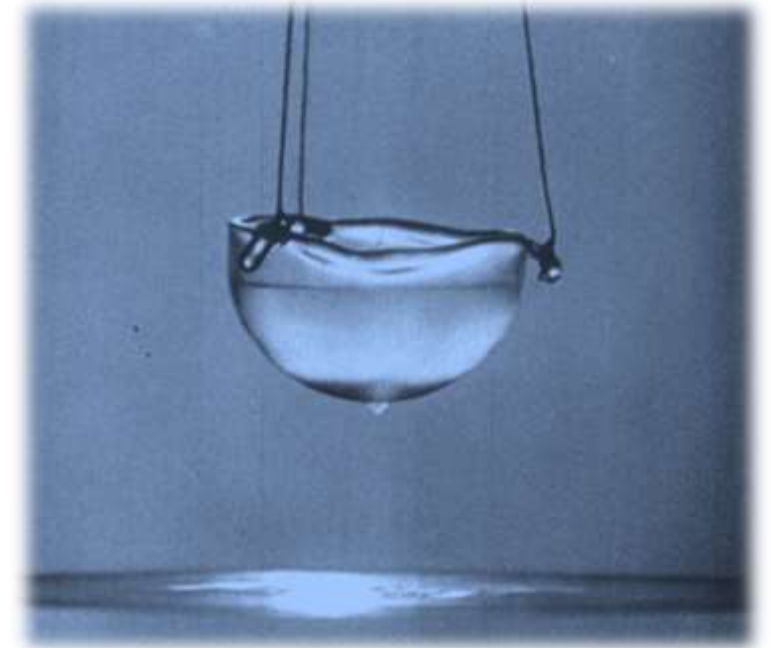
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Superconductivity and Low Temperature Physics II



**Lecture Notes
Summer Semester 2023**

**R. Gross
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Chapter 1

Quantum Liquids



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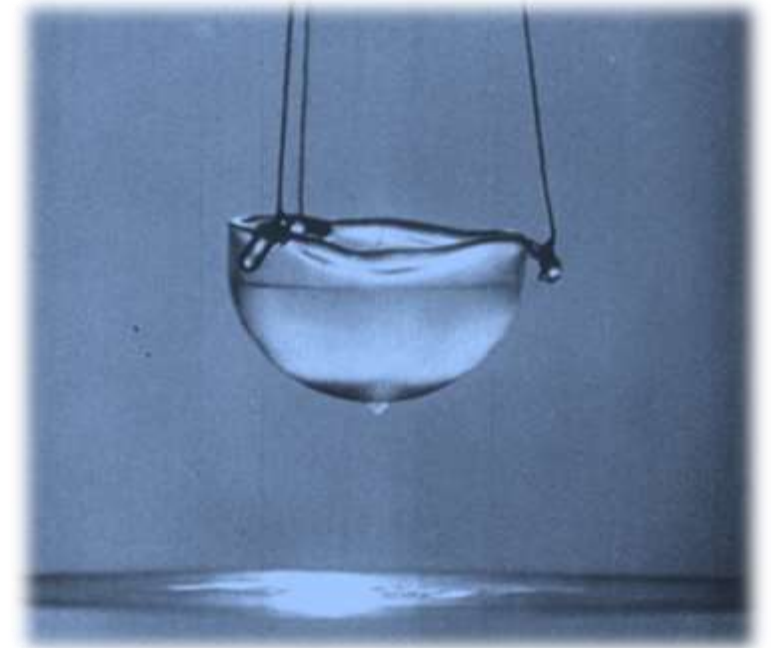
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Superconductivity and Low Temperature Physics II



Lecture No. 1

R. Gross

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Contents Part I: Quantum Liquids

Contents:



I.1 Foundations and General Properties

- I.1.1 Quantum Gases & Liquids
- I.1.2 Helium
- I.1.3 Van der Waals Bonding
- I.1.4 Zero-Point Fluctuations
- I.1.5 Helium under Pressure
- I.1.6 pT-Phase Diagram of ^4He and ^3He
- I.1.7 Characteristic Properties of ^4He and ^3He
- I.1.8 Specific Heat of ^4He and ^3He

I.2 ^4He as an Ideal Bose Gas

- I.2.1 Bose-Einstein Condensation
- I.2.2 Ideal Bose Gas
- I.2.3 Bose Gas with Interactions
- I.2.4 Bose-Einstein Condensation of ^4He

I.3 Superfluid ^4He

- I.3.1 Two-Fluid Model
- I.3.2 Experimental Observations
- I.3.3 Two-Fluid Hydrodynamics
- I.3.4 Excitation Spectrum of ^4He

I.4 Vortices

- I.4.1 Quantization of Circulation
- I.4.2 Experimental Study of Vortices

I.5 ^3He

- I.5.1 normal fluid ^3He
- I.5.2 solid ^3He and Pomeranchuk effect
- I.5.3 superfluid ^3He

I.6 ^3He / ^4He mixtures

I.1 Foundations and General Properties

- **Satyendranath Bose (1924)**

- letter to Einstein: proposes new method to derive Planck's law for spectral density of electromagnetic radiation emitted by a black body
- Bose describes the radiation emitted by a black body in thermal equilibrium by a gas on non-interacting identical particles
→ *photons*
- description is in perfect agreement with Einstein's explanation of the photoelectric effect
- Einstein immediately recognized that this description can be transferred to many other systems:
 - (i) gases of identical particles with finite mass,
 - (ii) phonons
- Einstein strongly supported the publication of Bose's results, he translated the paper to German

Satyendranath Bose: [*Plancks Gesetz und Lichtquantenhypothese*](#).
In: *Zeitschrift für Physik*. 26 (1924), S. 178-181.



Satyendranath Bose (around 1926)

I.1 Foundations and General Properties

What happens if you cool down a gas of **non-interacting** bosons ?

Below a certain temperature the particles feel each other due to their finite de Broglie wavelength, although they are non-interacting !



phase transition driven by statistics of particles
(not by their interactions)

Bose-Einstein Condensation

thermodynamic phase transition into a gas of non-interacting particles:
driven by statistics of particles and not by interactions

I.1 Foundations and General Properties

- 1924: prediction of **Bose-Einstein Condensation (BEC)** by Einstein
- 1995: first observation of BEC on alkali atoms by



Eric A. Cornell



Wolfgang Ketterle



Carl E. Wieman

Nobel Prize in Physics 2001

"for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates"

→ long period between theoretical prediction and experimental realization:

high density, ultra-cold gas of non-interacting atoms required

→ problem: gas liquefies or solidifies due to finite interactions

→ more simple to realize:

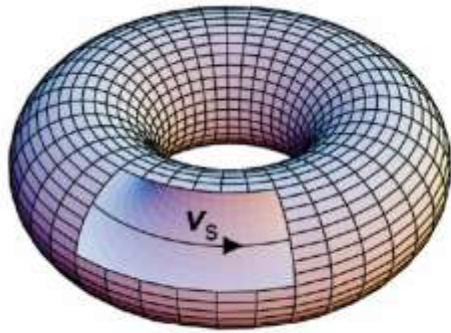
interacting quantum gas, e.g. superfluid ^4He

I.1 Foundations and General Properties

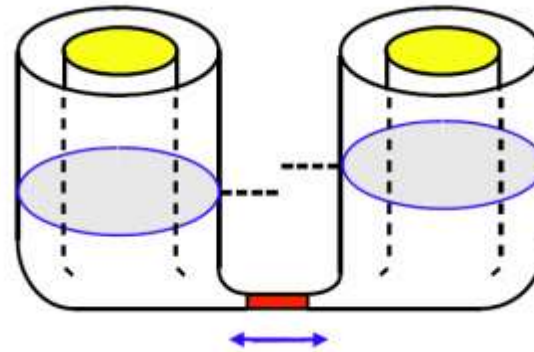
- nomenclature

- system of **non-interacting** particles:
 - classical gas** e.g. classical ideal gas
 - if quantum fluctuations dominate: **quantum gas** e.g. Bose-Einstein Condensate, Fermi gas
- system of **interacting** particles:
 - classical liquid** e.g. water, liquid helium-4
 - if quantum fluctuations dominate: **quantum liquid** e.g. superfluid helium-4, Fermi liquid

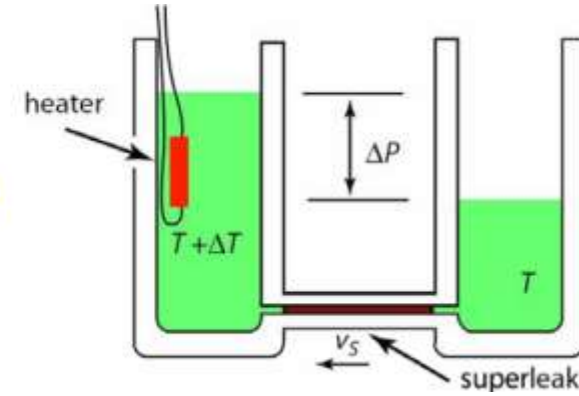
I.1.1 Quantum Liquids



supercurrents

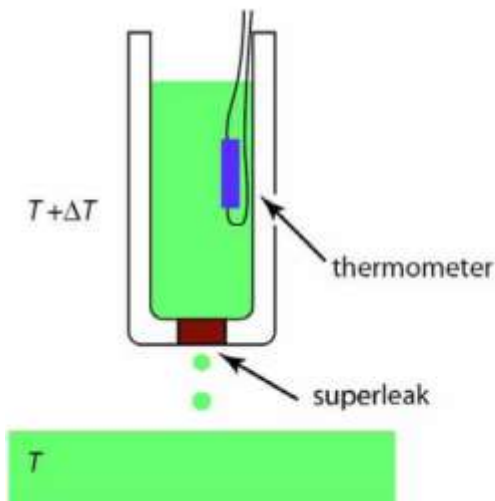


U-tube oscillation

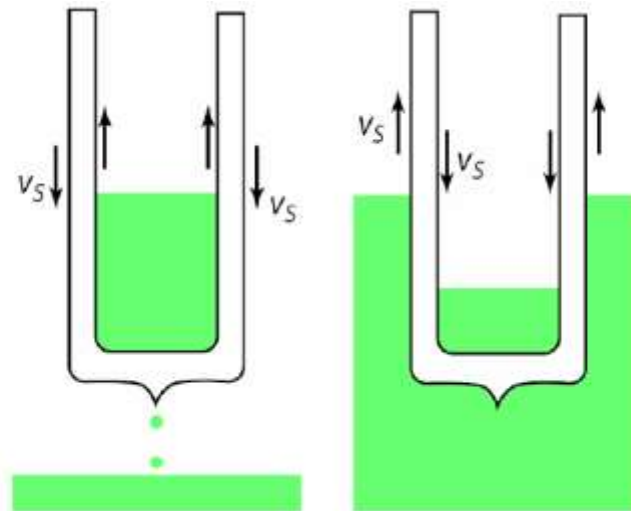


thermomech. effects

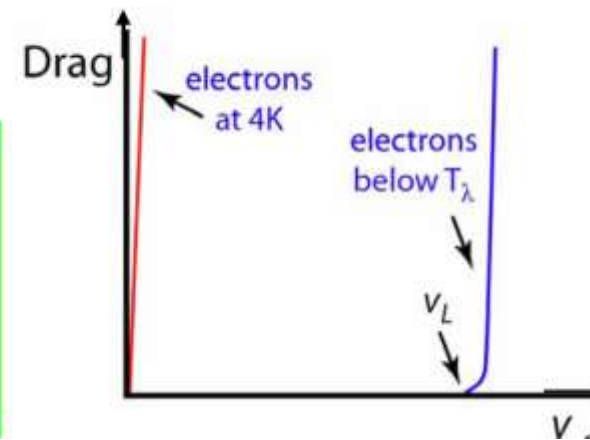
quantum liquids & condensates show fascinating properties → discussed in this lecture



superleaks



beaker film flow



critical velocity

I.1.1 Quantum Liquid

- classical liquids and solids – characteristic energy scales:

characteristic energies: (1) average interaction energy: $\langle V \rangle$ or $\langle E_{\text{pot}} \rangle$
 (2) thermal (kinetic) energy: $k_B T$ or $\langle E_{\text{kin}} \rangle$

criterion for liquid/solid: **liquid:** $k_B T \gg \langle V \rangle$
solid: $k_B T \ll \langle V \rangle$

- characteristic length scales in a gas:

- particle distance: $d = n^{-1/3}$
 - interaction range: $r_0 \simeq a$ (scattering length a)

- de Broglie wavelength:

$$\lambda_T = \sqrt{\frac{h^2}{2\pi m k_B T}} = \sqrt{\frac{2\pi \hbar^2}{m k_B T}}$$

$$\frac{\hbar^2 k_T^2}{2m} \simeq k_B T \quad k_T = \frac{2\pi}{\lambda_T}$$

$$\frac{h^2}{2m \lambda_T^2} \simeq k_B T \rightarrow \lambda_T \simeq \left(\frac{h^2}{2m k_B T} \right)^{1/2}$$

classical non-interacting gas: $d \gg r_0, \lambda_T$, since atoms interact only during collisions

I.1.1 Quantum Liquids

- calculation of the average thermal wavelength of a statistical ensemble (1)

we consider a classical gas of N particles:

→ probability for a specific particular configuration at temperature T is given by Boltzmann distribution

$$P(\mathbf{p}_1, \dots, \mathbf{p}_N, \mathbf{r}_1, \dots, \mathbf{r}_N) = \frac{1}{Z_N} \exp(-\beta \mathcal{H}) \quad \beta = 1/k_B T, \mathcal{H} = \text{Hamiltonian}$$

$$Z_N = \frac{1}{N!} \int d^3 p_1, \dots, d^3 p_N d^3 r_1, \dots, d^3 r_N \exp(-\beta \mathcal{H})$$

classical partition function,
 $\frac{1}{N!}$ results from fact that particles cannot be distinguished

partition function can be factorized into two terms (related to kinetic and potential energy)

$$Z_N = \frac{1}{N!} \underbrace{\prod_i \int d^3 p_i \exp(-p_i^2 / 2mk_B T)}_{\substack{\text{kinetic energy contribution} \\ \text{product of independent factors} \\ [(2\pi mk_B T)^{3/2}]^N}} \cdot \underbrace{\int d^3 r_1, \dots, d^3 r_N \exp\left(-\frac{1}{2}\beta \sum_{i \neq j} V(\mathbf{r}_i - \mathbf{r}_j)\right)}_{\text{potential energy contribution}}$$

(momenta of particles are statistically independent)

I.1.1 Quantum Liquids

- calculation of the average thermal wavelength of a statistical ensemble (2)

probability that particle has momentum in volume d^3p of phase space

$$P(\mathbf{p})d^3p = (2\pi mk_B T)^{-3/2} \exp(-p_i^2/2mk_B T) d^3p$$

probability that particle has momentum between p and $p + dp$ ($\int_{-\infty}^{\infty} d^3p = \int_{-\infty}^{\infty} p^2 dp \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi = 4\pi \int_{-\infty}^{\infty} p^2 dp$)

$$P_{\text{MB}}(p)dp = \frac{4\pi p^2}{\underbrace{(2\pi mk_B T)^{3/2}}_{p_T^3}} \exp(-p^2/2mk_B T) dp$$

Maxwell-Boltzmann distribution

the most likely momentum of the particles is $p = \sqrt{2mk_B T}$, the average momentum is $\bar{p} = \sqrt{8mk_B T/\pi}$

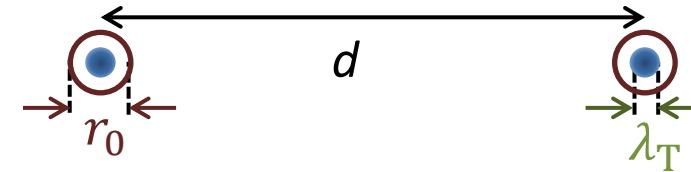
→ thermal de Broglie wavelength

$$\lambda_T = \frac{h}{p_T} = \sqrt{\frac{h^2}{2\pi mk_B T}} = \sqrt{\frac{2\pi \hbar^2}{mk_B T}}$$

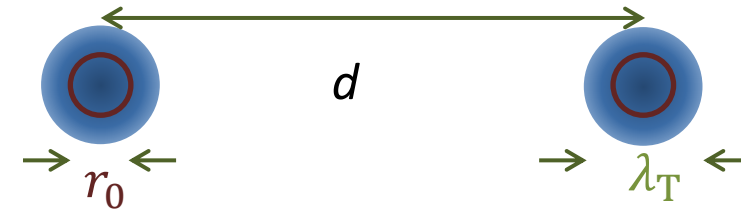
I.1.1 Quantum Liquids

- cooling down a classical gas $\rightarrow \lambda_T = \sqrt{\frac{h^2}{2\pi m k_B T}}$ increases and we can distinguish three regimes

(1) classical regime: $\lambda_T \ll r_0 \ll d$

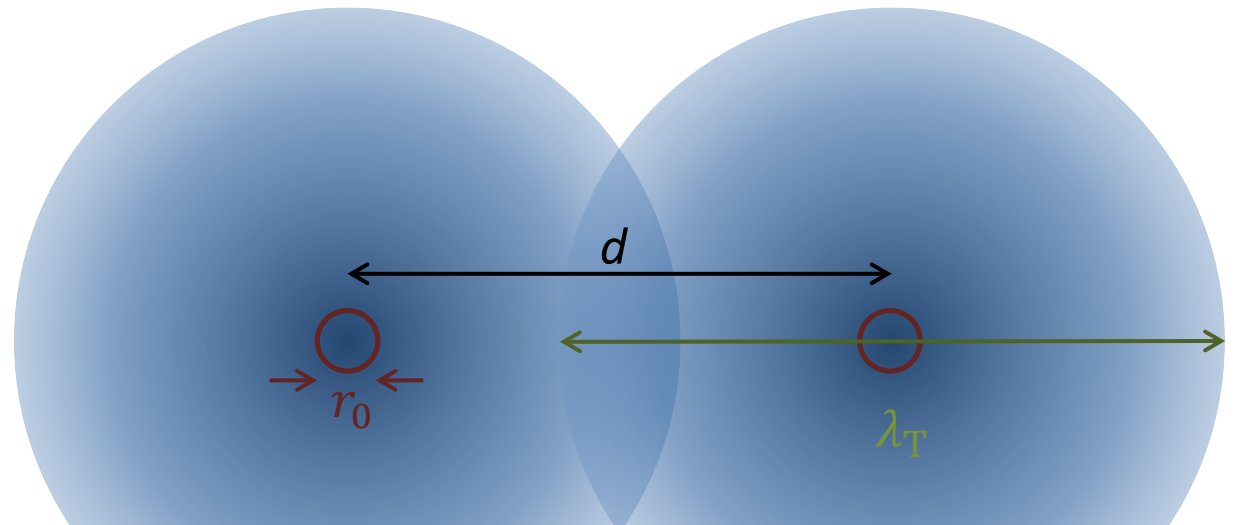


(2) quantum collisions: $r_0 \ll \lambda_T \ll d$
collisions can no longer be treated classically



(3) quantum degeneracy: $r_0 \ll d \leq \lambda_T$

all degrees of freedom of the gas must be treated quantum mechanically



I.1.1 Quantum Liquids

- define new energy scale related to $d = \lambda_T$

(3) zero-point energy E_{zp}

with - particle density $n = N/V$, $d = n^{-1/3}$
 - de Broglie wavelength $\lambda_T = d$

→ momentum uncertainty $\Delta p = \hbar \Delta k = \frac{h}{\lambda_T} = \frac{h}{d} = hn^{1/3}$

→ **zero-point energy**

$$E_{zp} = \frac{\Delta p^2}{2m} = \frac{h^2}{2md^2} = \frac{h^2 n^{2/3}}{2m} = \pi k_B T_{zp}$$

characteristic temperature

for an ideal gas the energy
 $E = \pi k_B T$ is assumed;
 the factor π results from
 the partition function

I.1.1 Quantum Liquids

- **classification of liquids** (interacting liquids with $\langle V \rangle > 0$) :

(a) classical liquid: $T_{zp} \ll T_{\text{melting}} \ll T$ (quantum fluctuation play no role)

(b) quantum liquid: $T_{\text{melting}} < T < T_{zp}$ (quantum fluctuation dominate)


 proportional to $\langle V \rangle$ or $\langle E_{\text{pot}} \rangle$

quantum parameter:

$$\Lambda \equiv \frac{\langle E_{zp} \rangle}{\langle E_{\text{pot}} \rangle}$$

quantum liquid:

$$\Lambda > 1$$

classical liquid:

$$\Lambda < 1$$

I.1.1 Quantum Liquids

- **classification of gases** (non-interacting systems: $\langle V \rangle = 0$) :


(a) classical gas: $T_{zp} < T$

(b) quantum gas: $T < T_{zp}$

$$T_{zp} = \frac{h^2 n^{2/3}}{2\pi m k_B} = \frac{h^2}{2\pi m k_B d^2} \longleftrightarrow T = \frac{h^2}{2\pi m k_B \lambda_T^2}$$

(a) classical gas: $\lambda_T \leq n^{-1/3} \simeq d \rightarrow n \cdot \lambda_T^3 \leq 1$

(b) quantum gas: $\lambda_T \geq n^{-1/3} \simeq d \rightarrow n \cdot \lambda_T^3 \geq 1$

 average particle distance

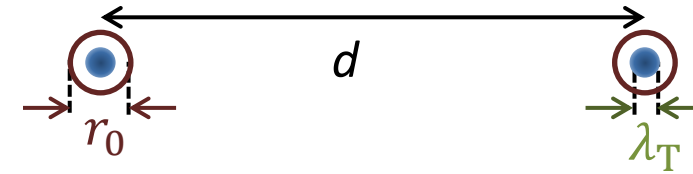
e.g. electrons @ 300 K: $\lambda_T = 4.2$ nm, $d \simeq 0.2$ nm for Cu ($n \simeq 8.5 \times 10^{28} \text{ m}^{-3}$)

→ particles become indistinguishable → degenerate quantum liquid

I.1.1 Quantum Liquids

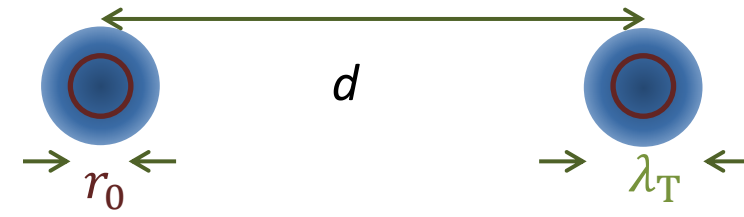
- cooling down a classical gas $\rightarrow \lambda_T = \sqrt{\frac{h^2}{2\pi m k_B T}}$ increases and we can distinguish three regimes

(1) classical regime: $\lambda_T \ll r_0 \ll d$



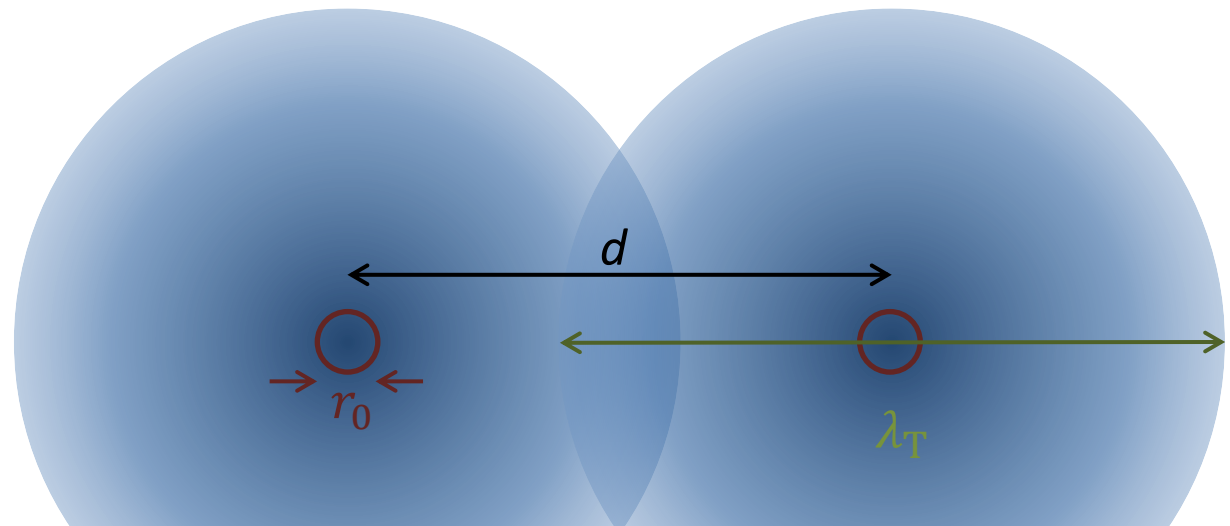
(2) quantum collisions: $r_0 \ll \lambda_T \ll d$

collisions can no longer be treated classically

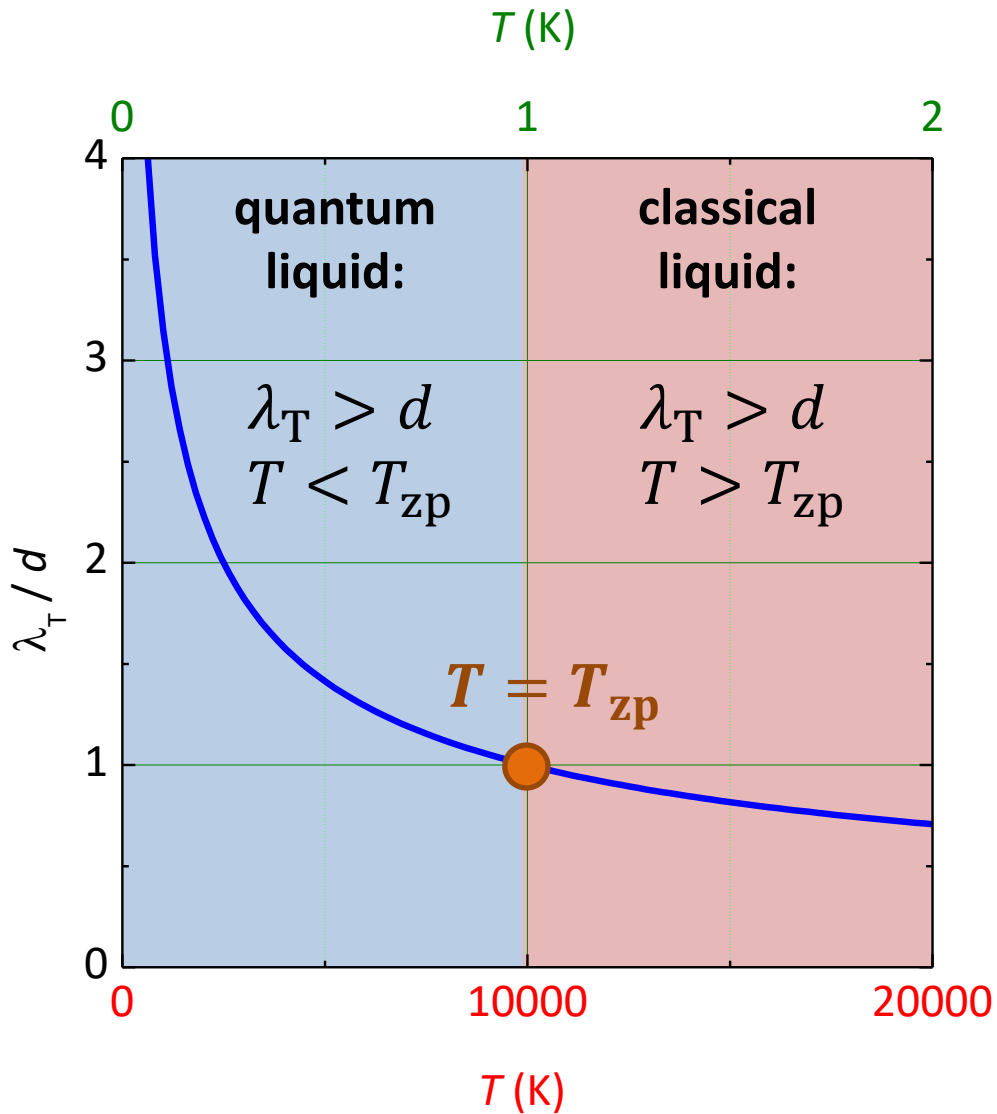


(3) quantum degeneracy: $r_0 \ll d \leq \lambda_T$

all degrees of freedom of the gas must be treated quantum mechanically



I.1.1 Quantum Liquids



← ^4He : $m = 6.65 \times 10^{-27} \text{ kg}$
 $n \simeq 4 \times 10^{27} \text{ m}^{-3}$

$$\lambda_T = \sqrt{\frac{h^2}{2\pi m k_B T}} \propto \frac{1}{\sqrt{T}}$$

← e^- : $m = 9.11 \times 10^{-31} \text{ kg}$
 $n \simeq 3.6 \times 10^{27} \text{ m}^{-3}$

degenerate:
indistinguishable particles

non-degenerate:
distinguishable particles

I.1.1 Quantum Liquids

- quantum statistics:

photons, mesons, ^4He -atoms, ...

integral spin

$$S = 0, \hbar, 2\hbar, 3\hbar, \dots$$

electrons, nucleons, ^3He -atoms, ...

half-integral spin

$$S = \hbar/2, 3\hbar/2, 5\hbar/2, \dots$$

indistinguishability of identical particles:

symmetry of many-particle wavefunction with respect to particle exchange

$$\begin{aligned} \wp \Psi(1,2) &\equiv \Psi(2,1) = e^{i\alpha} \Psi(1,2) \\ \wp \Psi(2,1) &\equiv \Psi(1,2) = (e^{i\alpha})^2 \Psi(1,2) \end{aligned} \quad \Rightarrow \quad e^{i\alpha} = \pm 1$$

- bosons** (integral spin)

$$\Psi_{AB}(1,2) = \frac{1}{\sqrt{2}} [\Psi_A(1)\Psi_B(2) + \Psi_A(2)\Psi_B(1)]$$

- fermions** (semi-integral spin)

$$\Psi_{AB}(1,2) = \frac{1}{\sqrt{2}} [\Psi_A(1)\Psi_B(2) - \Psi_A(2)\Psi_B(1)]$$

- Pauli principle:** *two fermions cannot occupy the same quantum state*

I.1.1 Quantum Liquids

- example: two particles (1,2) in quantum states A and B:

classical:

4 possibilities

double-occupancy ratio: **1/2**

$$A(1)A(2)$$

$$B(1)B(2)$$

$$A(1)B(2)$$

$$B(1)A(2)$$

bosons:

3 possibilities

double-occupancy ratio: **2/3**

$$A(1)A(2)$$

$$\frac{1}{\sqrt{2}} \{A(1)B(2) + A(2)B(1)\}$$

$$B(1)B(2)$$

fermions:

1 possibility

double-occupancy ratio: **0**

$$\frac{1}{\sqrt{2}} \{A(1)B(2) - A(2)B(1)\}$$

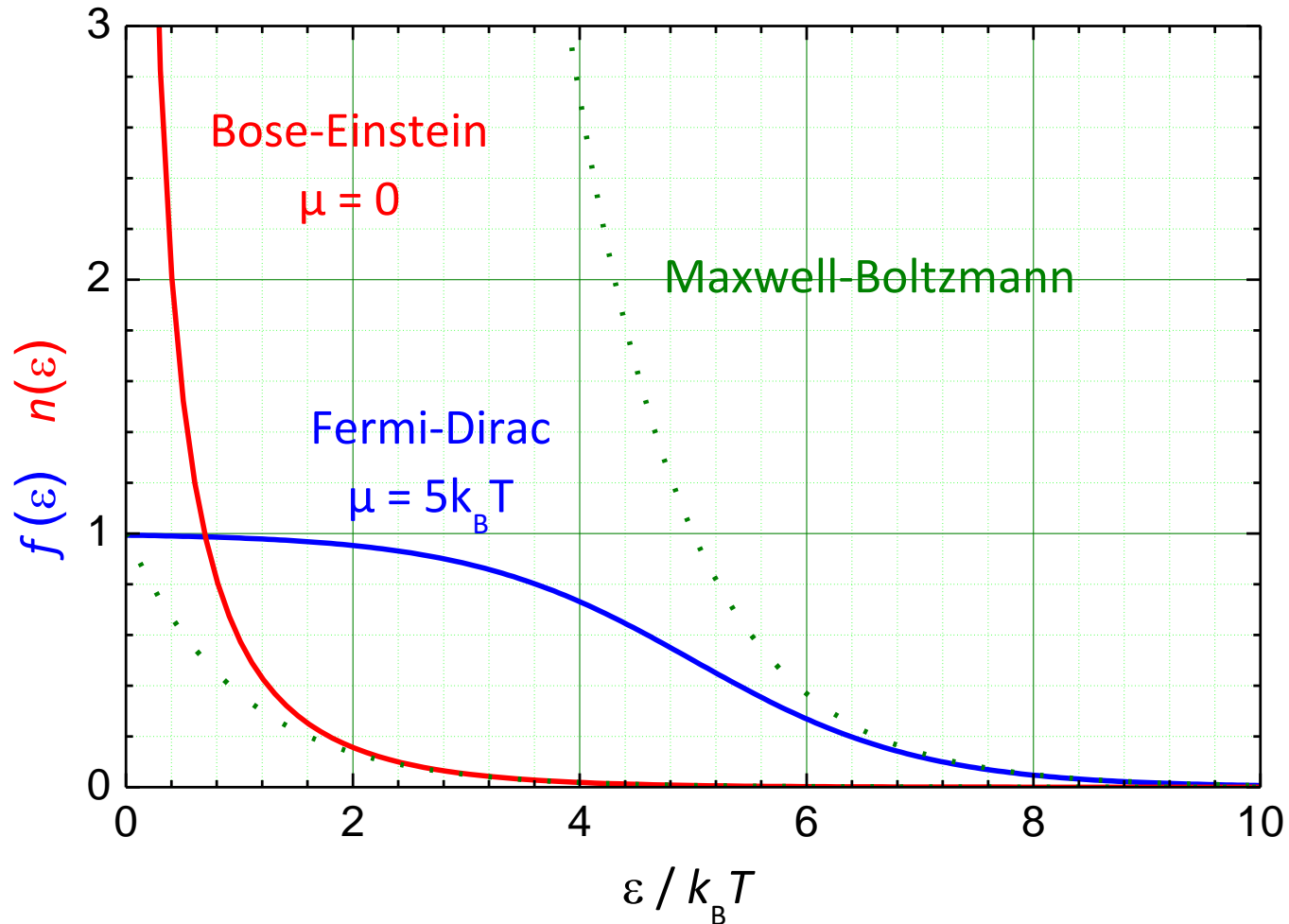
bosons:

tendency to occupy the same quantum state

fermions:

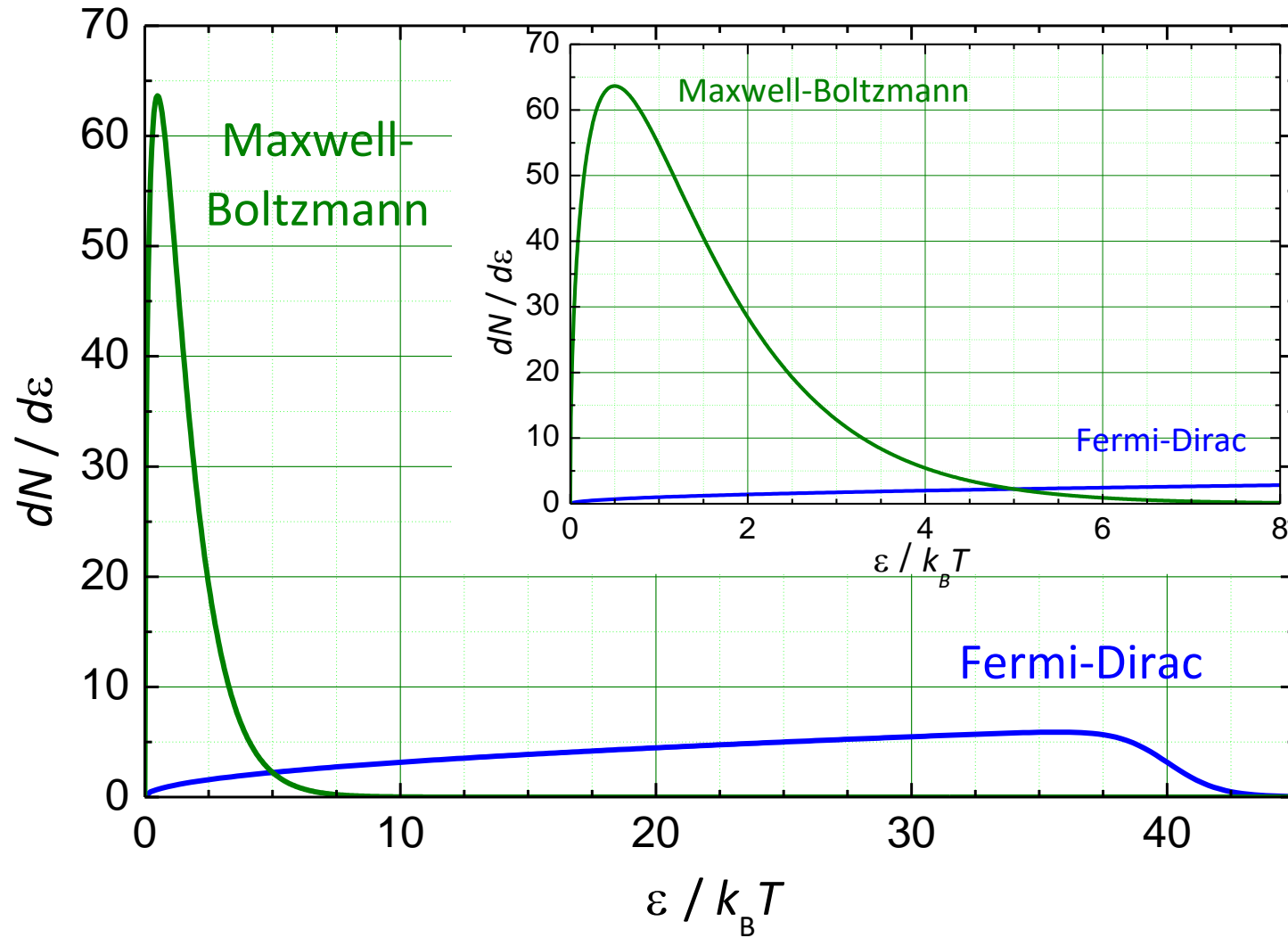
multiple occupancy of the same quantum state is forbidden

I.1.1 Quantum Liquids



Bose-Einstein, Fermi-Dirac and Maxwell-Boltzmann distributions as a function of normalized energy. For the Bose-Einstein distribution we assumed $\mu = 0$, for the Fermi-Dirac distribution $\mu = 5k_B T$ (for a metal μ is several eV corresponding to temperatures of several 10 000 K). For the Maxwell-Boltzmann distributions we also assumed $\mu = 0$ and $5k_B T$ so that the classical distribution coincides with the quantum mechanical distributions for large $\varepsilon \gg k_B T$.

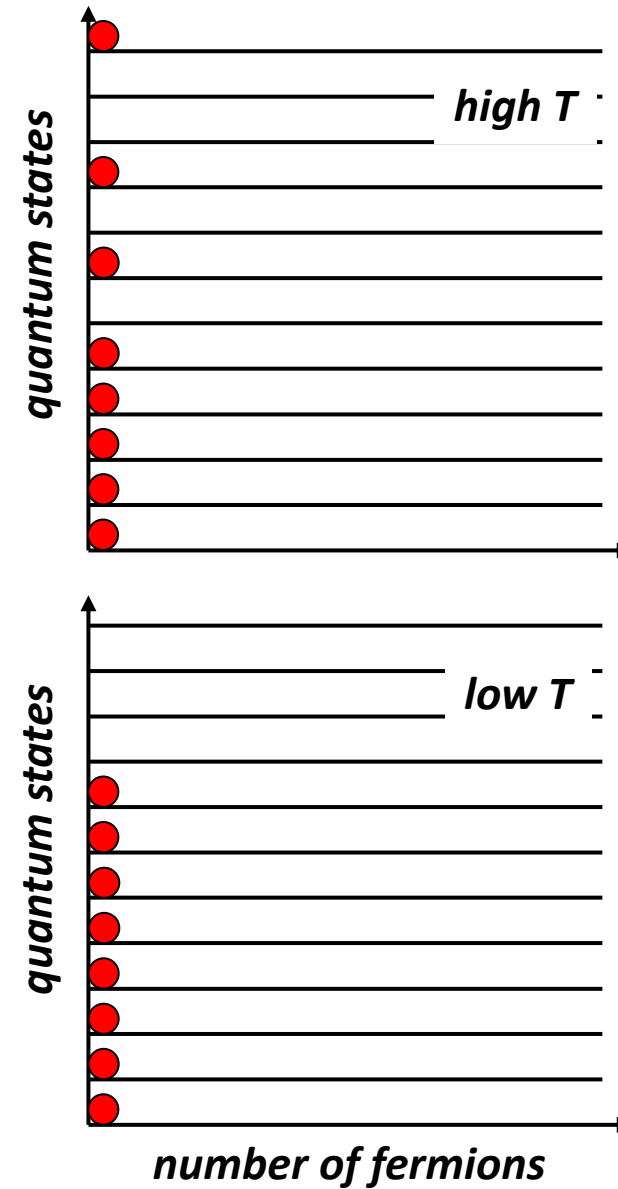
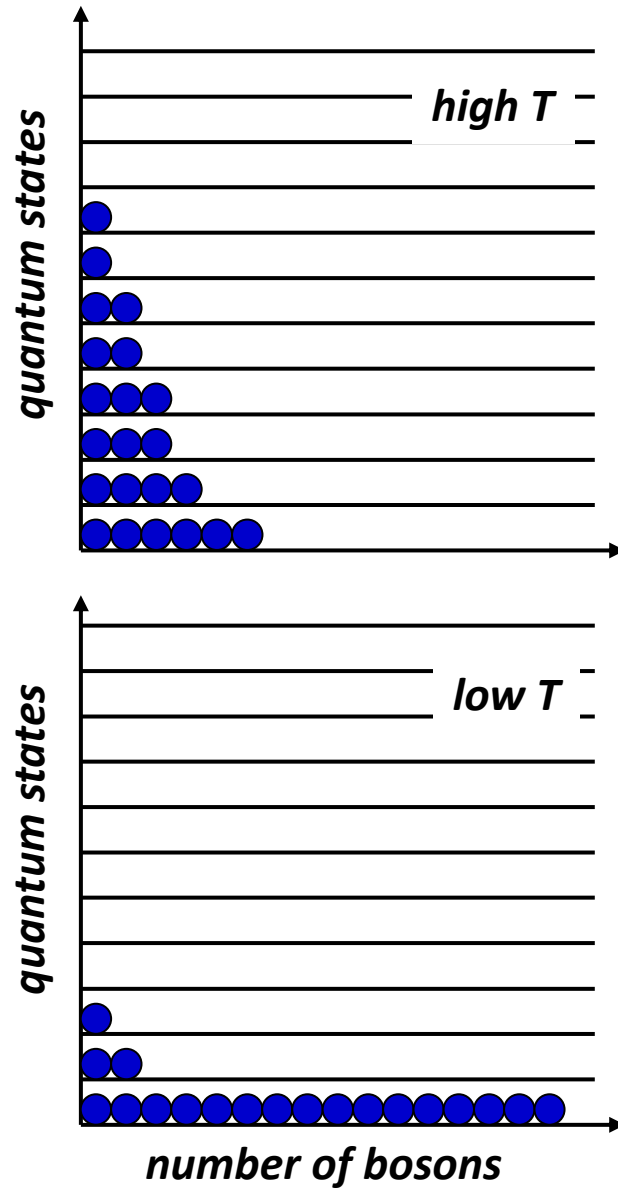
I.1.1 Quantum Liquids



particle number per energy interval: $\frac{dN}{d\varepsilon} = D(\varepsilon) \cdot f(\varepsilon)$

I.1.1 Quantum Liquids

boson



fermions

I.1.1 Quantum Liquids

- well-known quantum liquids:

1. *liquid helium:*

- small mass and d → high T_{zp} (a few K)
- small $\langle V \rangle$ → liquid down to $T = 0$
- ^4He : spin 0 → Bose liquid
- ^3He : spin $\frac{1}{2}$ → Fermi liquid

2. *electrons in metals:*

- “non-interacting” electron gas
- very small electron mass, higher density as for He
 - very high T_{zp} (a few 10^4 K)
 - $T_{zp} \gg T_{\text{melting}}$
- spin $\frac{1}{2}$ → Fermi gas (Fermi liquid due to finite interactions)

3. *nuclear matter:*

- fermions in nuclei, neutron stars, ...
- very high density → very high T_{zp} (a few 10^{12} K)
- spin $\frac{1}{2}$ → Fermi liquid

I.1.1 Quantum Liquids

	mass (kg)	n (m ⁻³)	T_{zp} (K)	statistics
³ He	5.01×10^{-27}	1.63×10^{28}	6.50	Fermi
⁴ He	6.65×10^{-27}	2.18×10^{28}	5.93	Bose
electrons in Na	9.11×10^{-31}	2.65×10^{28}	4.94×10^4	Fermi
nuclear matter	1.67×10^{-27}	1.95×10^{38}	1×10^{12}	Fermi

$$E_{zp} = \frac{\Delta p^2}{2m} = \frac{h^2}{2md^2} = \frac{h^2 n^{2/3}}{2m} = \pi k_B T_{zp}$$

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Contents:

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- I.1.1 Quantum Gases & Liquids
- I.1.2 Helium
- I.1.3 Van der Waals Bonding
- I.1.4 Zero-Point Fluctuations
- I.1.5 Helium under Pressure
- I.1.6 pT-Phase Diagram of ^4He and ^3He
- I.1.7 Characteristic Properties of ^4He and ^3He
- I.1.8 Specific Heat of ^4He and ^3He

I.2 ^4He as an Ideal Bose Gas

- I.2.1 Bose-Einstein Condensation
- I.2.2 Ideal Bose Gas
- I.2.3 Bose Gas with Interactions
- I.2.4 Bose-Einstein Condensation of ^4He

I.3 Superfluid ^4He

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I.6 ^3He / ^4He mixtures



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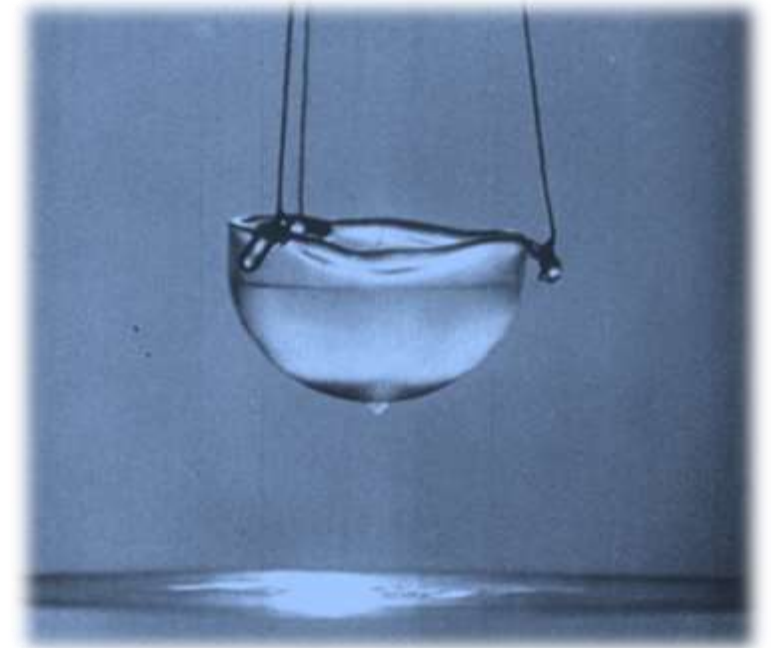


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Lecture No. 2

R. Gross

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I.1.2 Helium

- *history: ^4He*

application of ^4He in low temperature physics

- soon after discovery on earth, race for the liquefaction of He
 - main opponents: **Heike Kamerlingh Onnes** and **James Dewar**
 - Onnes extracted 360 l of He gas out of Monazit sand from North Carolina
- **1908**: first liquefaction: July 10, 1908
- **1925**: first liquefaction of He in Germany (March 7, 1925: 4.2 K, 200 ml) by **Walther Meißner**, 3rd system world-wide besides Leiden and Toronto
- **1932**: Linde supplies first industrial He liquefier to University of Charkov, Ukraine
- **1947**: first commercial liquefier by **S.C. Collins**

Helium was first detected as an unknown, yellow spectral line signature in sunlight during a solar eclipse in 1868 by Georges Rayet *et al.* and was subsequently confirmed by French astronomer Jules Janssen.



Heike Kamerlingh Onnes
 * 21. 09. 1853
 † 21. 02. 1926



James Dewar
 * 20. 09. 1842
 † 27.03.1923



SIR JAMES DEWAR

Sir James Dewar, (1842-1923) was a chemist and physicist, best known for his work with low-temperature phenomena. Dewar was born in Kincardine, Scotland, and educated at the University of Edinburgh. He was professor of experimental natural philosophy at the University of Cambridge, England, in 1875 and professor of chemistry at the Royal Institution of Great Britain in 1877, where he was appointed director of the Davy-Faraday Research Laboratory.

Dewar developed structural formulas for benzene (1867). He studied the specific heat of hydrogen and was the first person to produce hydrogen in liquid form (1898) and to solidify it (1899). He constructed a machine for producing liquid oxygen in quantity (1891). He invented the Dewar flask or thermos (1892) and co-invented cordite (1889), a smokeless gunpowder, with Sir Frederick Abel. His discovery (1905) that cooled charcoal can be used to help create high vacuums later proved useful in atomic physics. Dewar was knighted in 1904.

The Dewar flask or vacuum flask/bottle is a container for storing hot or cold substances, i.e. liquid air. It consists of two flasks, one inside the other, separated by a vacuum. The vacuum greatly reduces the transfer of heat, preventing a temperature change. The walls are usually made of glass because it is a poor conductor of heat; its surfaces are usually lined with a reflective metal to reduce the transfer of heat by radiation. Dewar used silver. The whole fragile flask rests on a shock-absorbing spring within a metal or plastic container, and the air between the flask and the container provides further insulation. The common thermos bottle is an adaptation of the Dewar flask. Dewar invented the Dewar flask in 1892 to aid him in his work with liquid gases.

The vacuum flask was not manufactured for commercial/home use until 1904, when two German glass blowers formed Thermos GmbH. They held a contest to rename the vacuum flask and a resident of Munich submitted "Thermos", which came from the Greek word "Therme" meaning "hot."

I.1.2 Helium

- *history: ^3He*

- **1933**: discovery by *Oliphant, Kinsey* and *Rutherford*
(they believed that ^3He is unstable, disproven by *Alvarez* and *Cornog* in 1939)
- **1947**: first experiments with gaseous ^3He
- **1949**: first condensation of ^3He gas (*Sydoriak and Co.*)
- **1956**: theory of Fermi liquids (*L.D. Landau*, **Nobel Prize in Physics 1962**)
- **1960s**: first experiments with the Fermi liquid ^3He (*various groups*)
- **1963**: first theory of superfluid ^3He (*Anderson and Morel, Balian and Werthamer*)
- **1971**: discovery of superfluid phase transition at 2 mK (*Lee, Osheroff, Richardson*,
Nobel Prize in Physics 1996)
- **1975**: extension of BCS theory to triplet pairing (*Leggett*, **Nobel Prize in Physics 2003**)
- **1977**: first experiments on superfluid ^3He in Germany at WMI (*Eska, Schuberth, Uhlig*)

I.1.2 Helium

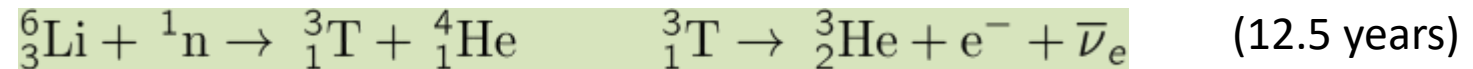
- *He isotops: ^4He and ^3He*

- ^4He : - obtained out of natural gas (e.g. sources in USA, Poland, Algeria contain up to 10%)

- $L = 0, S = 0$, nuclear spin $I = 0 \rightarrow \text{boson}$

- ^3He : - natural amount of ^3He in ^4He is only 0.14 ppm

- artificial generation of ^3He by nuclear reaction



- ^3He is very expensive, several 1000 € per liter


- $L = 0, S = 0$, nuclear spin $I = \frac{1}{2} \rightarrow \text{fermion}$

- despite their simple electronic structure, the liquids of both He isotopes show
 - *a rich variety of exotic phenomena*
 - *pronounced differences due to different quantum statistics*

Contents Part I: Quantum Liquids

Contents:

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- I.1.1 Quantum Gases & Liquids
- I.1.2 Helium
-  I.1.3 Van der Waals Bonding
- I.1.4 Zero-Point Fluctuations
- I.1.5 Helium under Pressure
- I.1.6 pT-Phase Diagram of ^4He and ^3He
- I.1.7 Characteristic Properties of ^4He and ^3He
- I.1.8 Specific Heat of ^4He and ^3He

I.2 ^4He as an Ideal Bose Gas

- I.2.1 Bose-Einstein Condensation
- I.2.2 Ideal Bose Gas
- I.2.3 Bose Gas with Interactions
- I.2.4 Bose-Einstein Condensation of ^4He

I.3 Superfluid ^4He

- I.3.1 Two-Fluid Model
- I.3.2 Experimental Observations
- I.3.3 Two-Fluid Hydrodynamics
- I.3.4 Excitation Spectrum of ^4He

I.4 Vortices

- I.4.1 Quantization of Circulation
- I.4.2 Experimental Study of Vortices

I.5 ^3He

- I.5.1 normal fluid ^3He
- I.5.2 solid ^3He and Pomeranchuk effect
- I.5.3 superfluid ^3He

I.6 ^3He / ^4He mixtures

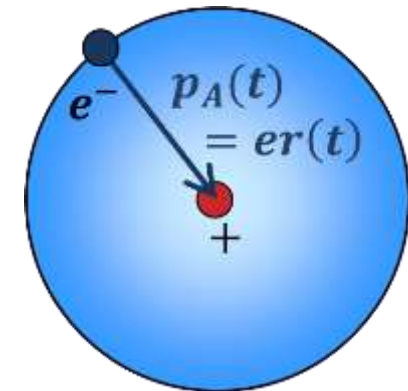
I.1.3 Van der Waals Bonding

- **neutral atom**, average dipole moment vanishes: $\langle \mathbf{p}_A \rangle = 0$

→ dipole moment induced by electric field: $\mathbf{p}_A^{\text{ind}} \propto \mathbf{E}$

proportionality constant is **polarizability** α_A of atom A:

$$\mathbf{p}_A^{\text{ind}} = \alpha_A \mathbf{E}$$



$$\langle \mathbf{p}_A \rangle = 0$$

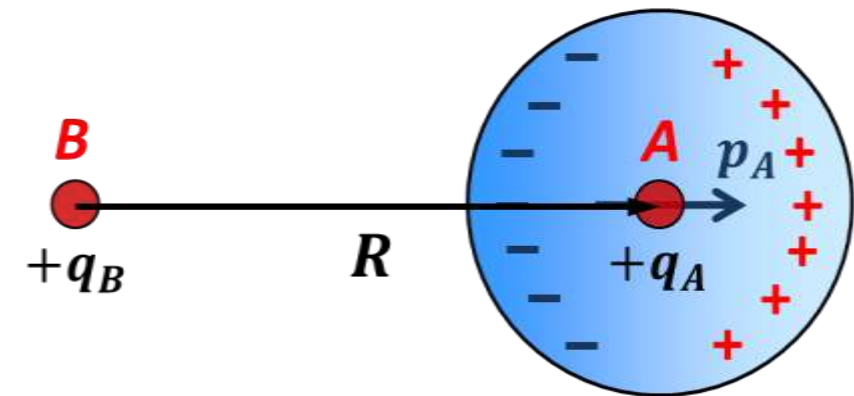
- **assumption:**

\mathbf{E} -field is generated by charge q_B of ion B at distance R

$$\mathbf{p}_A^{\text{ind}} = \alpha_A \mathbf{E}_{B,\text{ion}} = \alpha_A \frac{q_B}{4\pi\epsilon_0 R} \hat{\mathbf{R}}$$

- potential energy of atom A

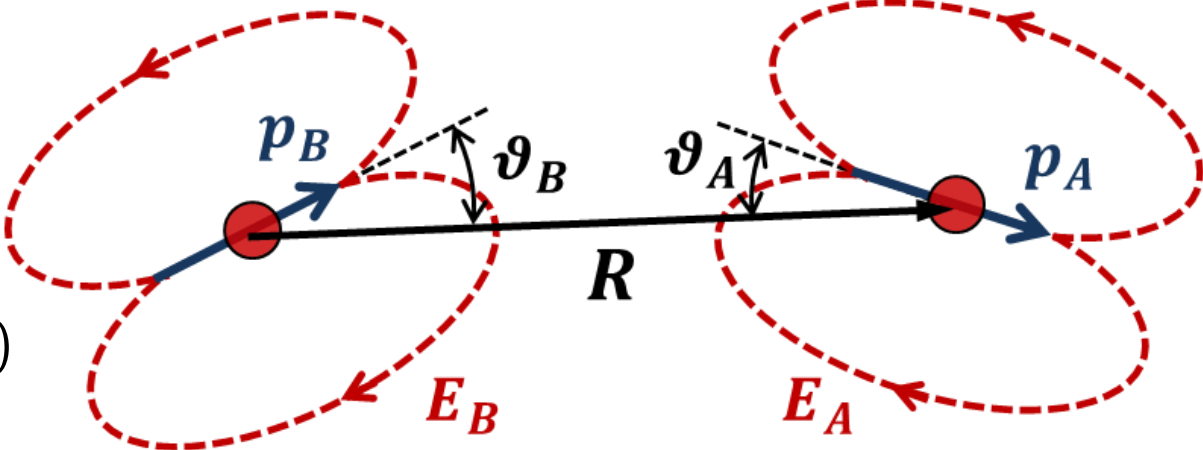
$$E_{\text{pot}} = -\mathbf{p}_A^{\text{ind}} \cdot \mathbf{E}_{B,\text{ion}} = -(\alpha_A \mathbf{E}_{B,\text{ion}}) \cdot \mathbf{E}_{B,\text{ion}} \propto \mathbf{E}_{B,\text{ion}}^2$$



I.1.3 Van der Waals Bonding

- assumption:**

- field $\mathbf{E}(\mathbf{p}_B)$ at position of atom A is generated by atom B with dipole moment \mathbf{p}_B
- $\mathbf{E}(\mathbf{p}_B)$ induces dipole moment \mathbf{p}_A of atom A, which in turn generates field $\mathbf{E}(\mathbf{p}_A)$ at position of atom B



$$\mathbf{E}(\mathbf{p}_B) = \frac{1}{4\pi\epsilon_0 R^3} (3p_B \cos \vartheta_B \hat{\mathbf{R}} - \mathbf{p}_B)$$

$$\mathbf{E}(\mathbf{p}_A) = \frac{1}{4\pi\epsilon_0 R^3} (3p_A \cos \vartheta_A \hat{\mathbf{R}} - \mathbf{p}_A)$$

- Interaction energy** (same atoms: $p_A = p_B = p$, $\alpha_A = \alpha_B = \alpha$)

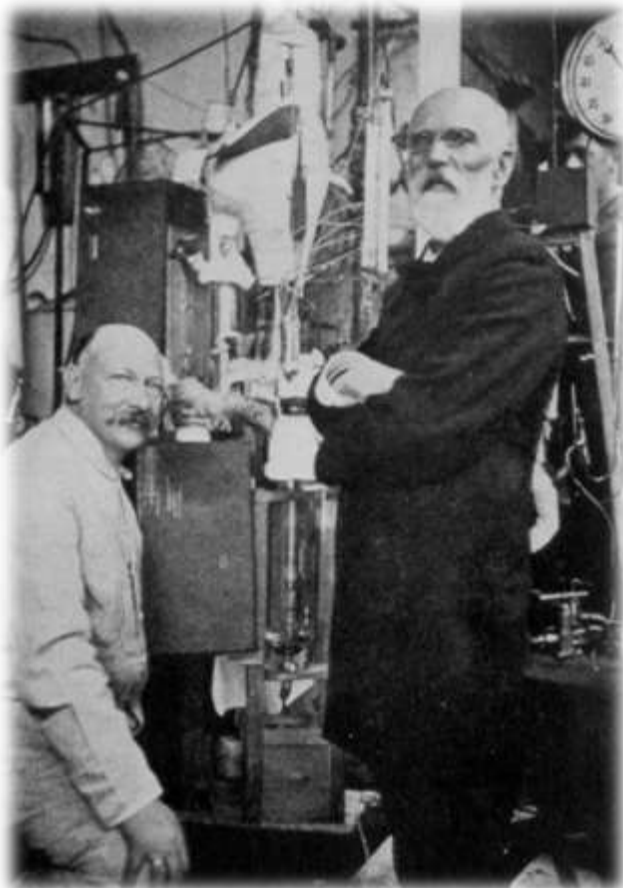
$$E_{\text{pot}}^{\text{dd}}(R) = \frac{1}{4\pi\epsilon_0} \frac{p_A p_B - 3 (\hat{\mathbf{R}} \cdot \mathbf{p}_A)(\hat{\mathbf{R}} \cdot \mathbf{p}_B)}{R^3} \quad \longrightarrow \quad E_{\text{pot}}^{\text{dd}}(R) \propto -\frac{p_A p_B}{R^3} \propto -\frac{p}{R^3} \frac{\alpha p}{R^3} \propto -\frac{\alpha}{R^6}$$

$$E_{\text{pot}}^{\text{dd}}(R) = -C \frac{\alpha}{R^6}$$

induced dipole-dipole interaction

- **attractive interaction**
- weak
- short range

I.1.3 Van der Waals Bonding



Van der Waals (right) and Heike Kamerlingh Onnes in Leiden with the helium 'liquefactor' (1908)



*Johannes Diderik Van der Waals (1837 - 1923)
Nobel Prize in Physics (1910)
"for his work on the equation of state for gases and liquids"*

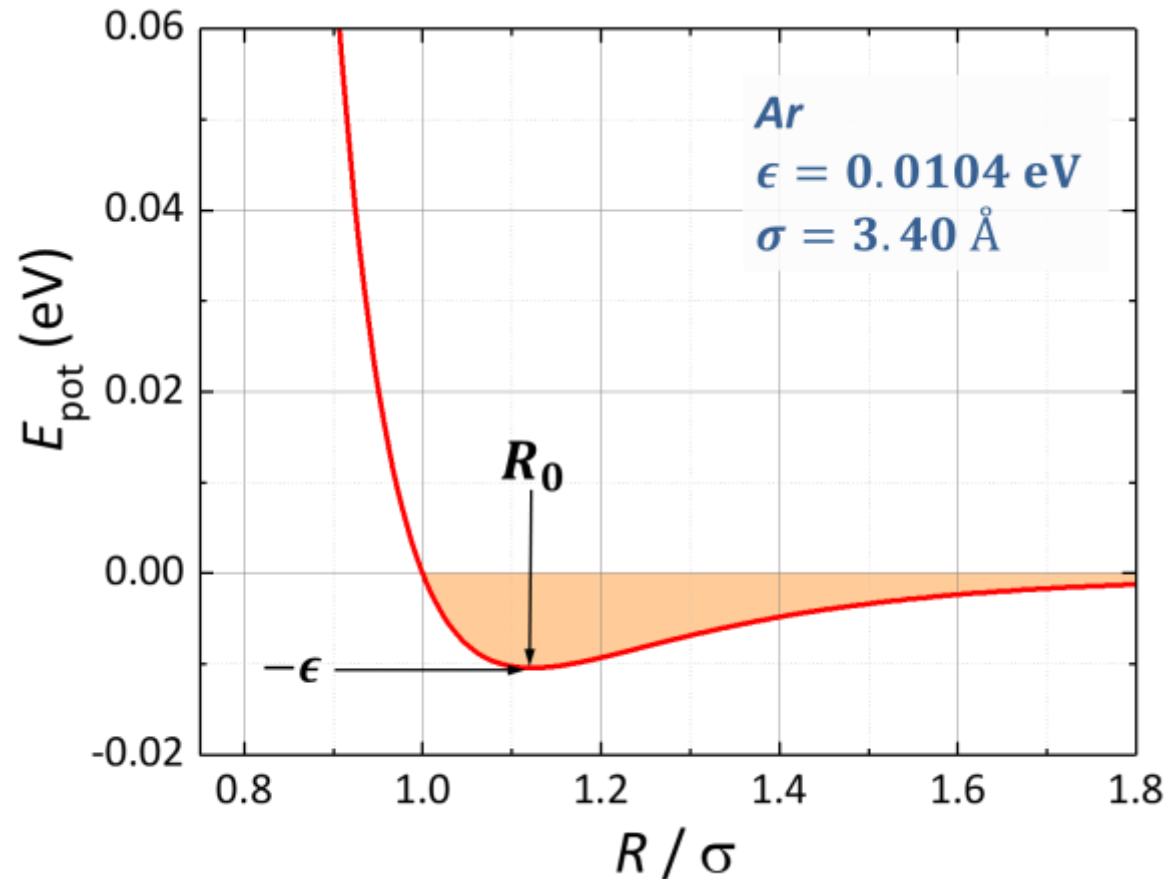


I.1.3 Van der Waals Bonding

- *interaction potential* with short range repulsion due to Pauli principle:

$$E_{\text{pot}}(R) = 4\epsilon \left[\left(\frac{\sigma}{R} \right)^{12} - \left(\frac{\sigma}{R} \right)^6 \right]$$

Lennard-Jones potential



	Ne	Ar	Kr	Xe
ϵ (eV)	0.0031	0.0104	0.0141	0.0200
σ (Å)	2.74	3.40	3.65	3.98
R_0/σ	1.14	1.11	1.10	1.09

11.6 K \Leftrightarrow 1 meV

➔ *Van der Waals bonding is very weak !!*

I.1.3 Van der Waals Bonding

determination of **equilibrium distance** of atoms in lattice: → minimize total potential energy

$$U_{\text{tot}} = \frac{N}{2} \sum_{i \neq j} 4\epsilon \left[\left(\frac{\sigma}{r_{ij}} \right)^{12} - \left(\frac{\sigma}{r_{ij}} \right)^6 \right]$$

Σ = binding energy of a single atom j with all other atoms i .
Total binding energy is obtained by multiplication with $N/2$.

we rewrite

$$U_{\text{tot}}(R) = 2N\epsilon \left[A_{12} \left(\frac{\sigma}{R} \right)^{12} - A_6 \left(\frac{\sigma}{R} \right)^6 \right]$$

R : nearest neighbor distance

with $A_k = \sum_{i \neq j} \frac{1}{\alpha_{ij}^k}, \quad k = 6, 12$ (lattice sum)

$r_{ij} = \alpha_{ij}R$
 k : number of nearest neighbors

equilibrium distance: $\partial U_{\text{tot}}(R)/\partial R = 0$

$$R_0 = \left(2 \frac{A_{12}}{A_6} \right)^{1/6} \cdot \sigma \quad R_0 = 1.09 \cdot \sigma$$



Kristallstruktur	A_6	A_{12}
fcc	14.4539	12.1319
hcp	14.4549	12.1323
bcc	12.253	9.114

$$U_{\text{tot}}(R_0) = -N\epsilon \begin{cases} 8.61016 & \text{fcc - structure} \\ 8.61107 & \text{hcp - structure} \end{cases}$$

hcp has lowest energy but fcc is found experimentally for noble gases

→ effect of zero-point fluctuations

→ more detailed band structure calculation is required

(modification of Van der Waals binding energy due to overlap of excited atomic states with neighboring atoms in crystal structure)

I.1.4 Zero Point Fluctuations

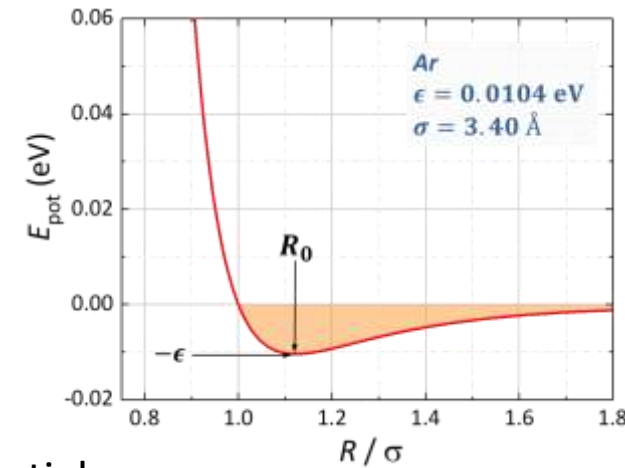
- estimate of **zero-point fluctuation amplitude** for harmonic oscillator:

$$E_{\text{tot}} = E_{\text{kin}} + E_{\text{pot}}$$

- for $x = x_{\text{max}}$:

$$E_{\text{kin}} = 0, \quad E_{\text{pot}} = \frac{1}{2} k x_{\text{max}}^2 = \frac{1}{2} M \omega^2 x_{\text{max}}^2 \quad \omega^2 = k/M$$

spring constant k is given by second derivative of the interaction potential



- by equating $\frac{1}{2} k x_{\text{max}}^2$ with the ground state energy $\frac{1}{2} \hbar \omega$ of the harmonic oscillator, we obtain

$$\Rightarrow x_{\text{max}}^2 = \frac{\hbar}{M\omega} = \frac{\hbar}{\sqrt{kM}}$$

for He: $x_{\text{max}} \approx 0.3 - 0.4$ times the lattice constant a

$$a \approx \left(\frac{V_{\text{mol}}}{N_A} \right)^{1/3} \quad V_{\text{mol}} = \text{mole volume}, \quad N_A = \text{Avogadro number}$$

(**Lindeman** criterion for melting of solid: $x_{\text{max}} \approx 0.3 \cdot a$)

→ Helium stays liquid even at $T = 0$ due to large zero-point fluctuations caused by

- (i) weak Van der Waals forces (small k)
- (ii) small mass M

I.1.4 Zero Point Fluctuations

- we can use uncertainty relation to estimate the **kinetic energy** due to zero-point fluctuations:

$$\Delta p \cdot \Delta x \simeq \hbar \quad \Rightarrow \quad \Delta p \sim \frac{\hbar}{\Delta x} \approx \frac{\hbar}{0.3 a} \approx \frac{h}{a} \quad a = \text{lattice constant}$$

$$E_{\text{zp}} = \frac{\Delta p^2}{2m} \approx \frac{h^2}{2Ma^2} \approx 18 \text{ K}$$

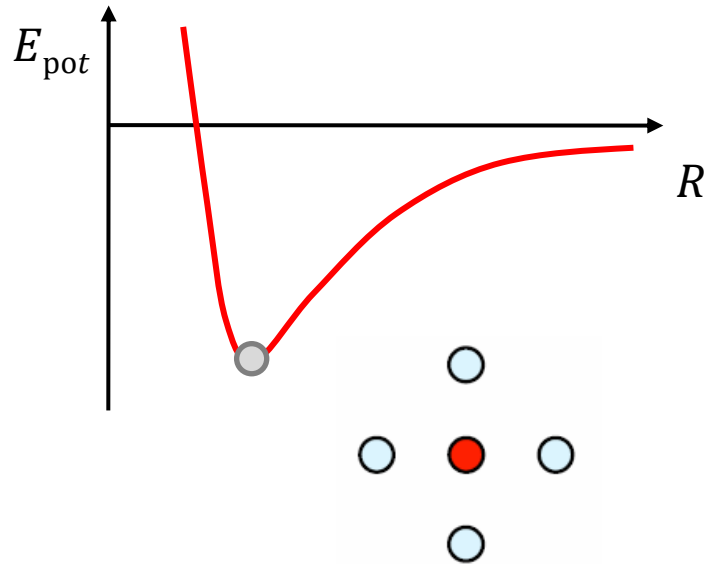
- large quantum mechanical zero-point energy due to small mass M
- kinetic energy due to zero-point fluctuations is comparable to potential energy due to Van der Waals interaction

due to strong influence of quantum fluctuations, helium stays liquid down to temperatures, where $\lambda_T \geq d \simeq a$

→ Helium becomes a **quantum liquid**

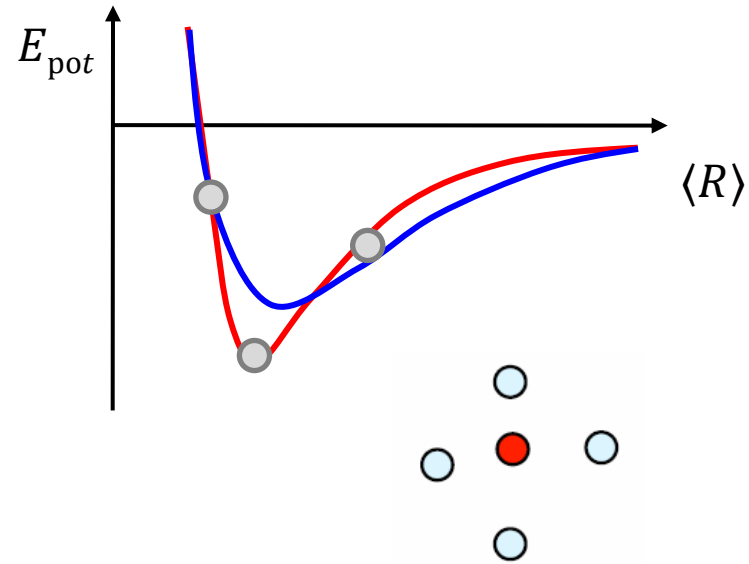
I.1.4 Zero Point Fluctuations

- total potential energy for solid and liquid He



solid: regular lattice

- same distance between atoms
- $E_{\text{pot}}(R) \approx \text{potential } U_{\text{tot}}(R)$



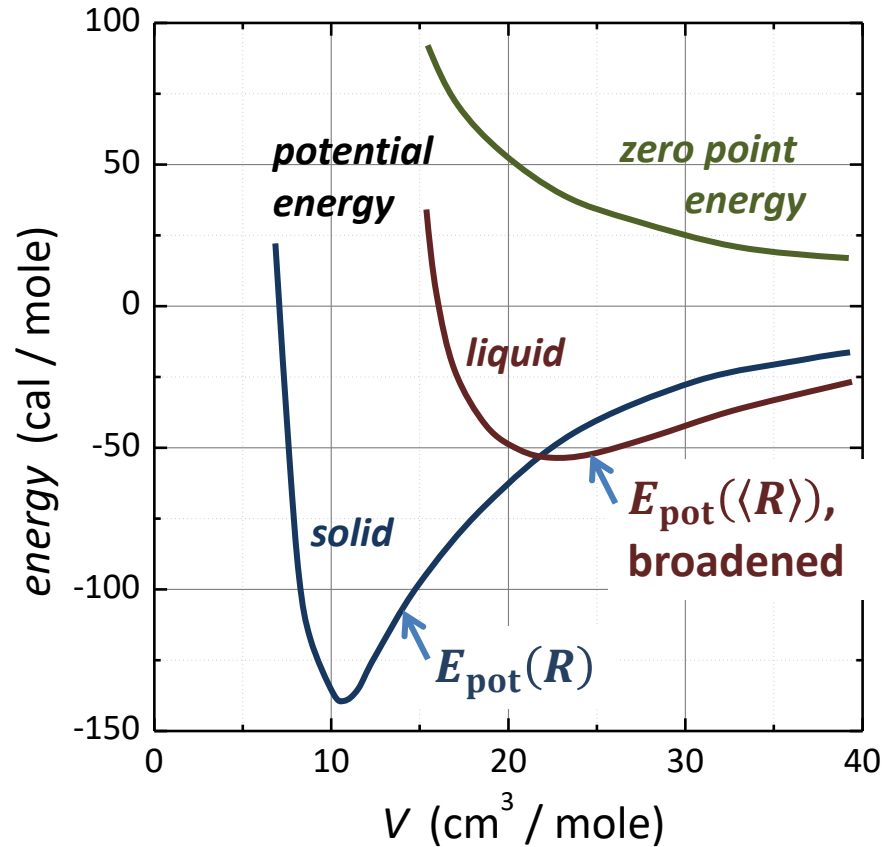
liquid: disordered system

- varying distance between atoms
- E_{pot} is smeared out due to averaging over various distances
- reduced depth and increased width of $E_{\text{pot}}(\langle R \rangle)$

Why does He not become solid at low T ??

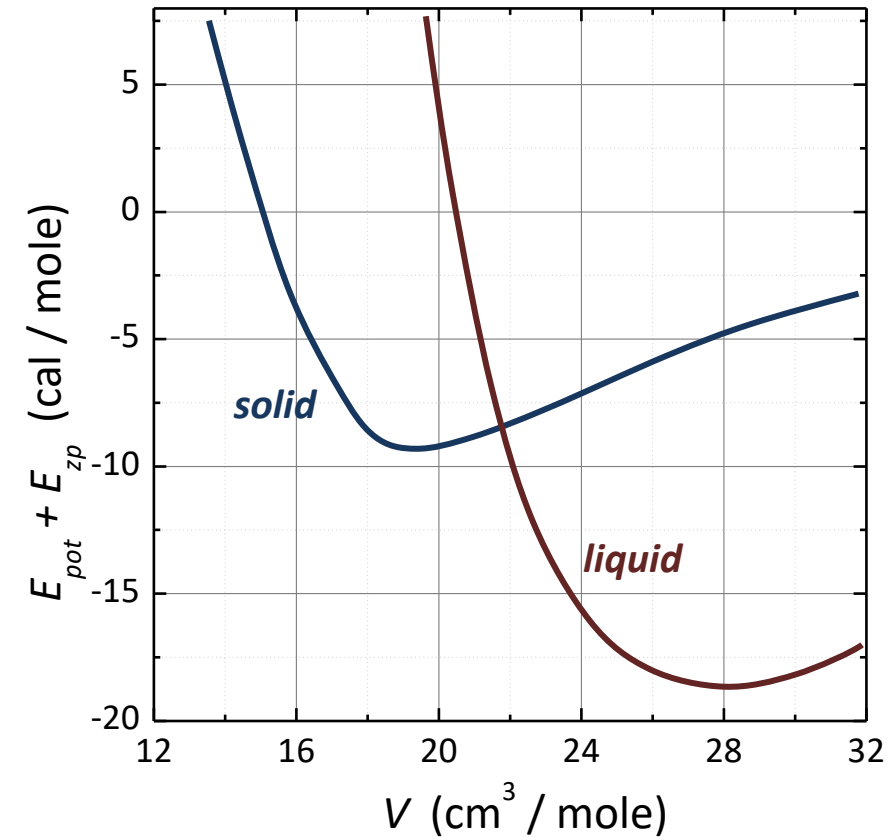
I.1.4 Zero Point Fluctuations

- zero point fluctuations play a crucial role



for liquid helium:

contribution of E_{zp} smaller due to broadened E_{pot}



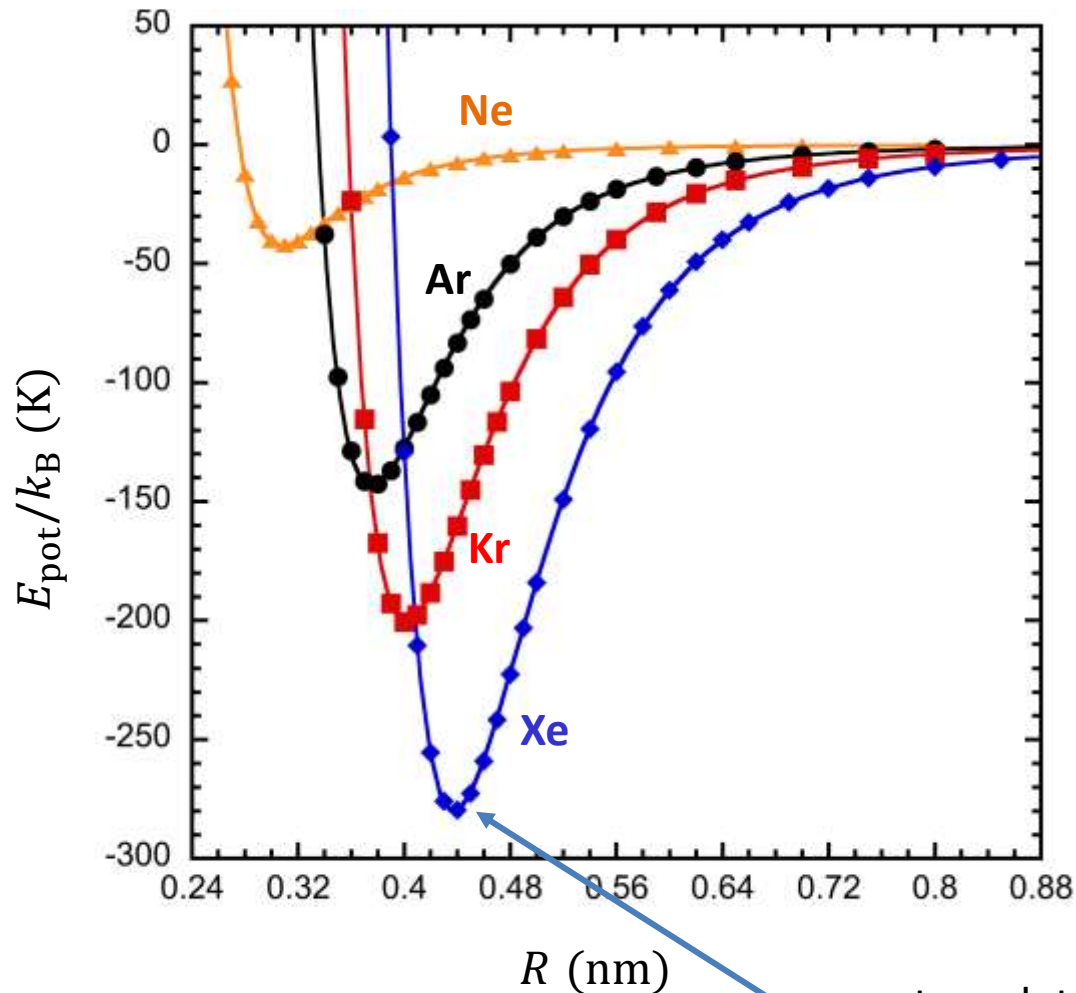
- *without quantum fluctuations:*

⁴He would be solid at a molar volume of **10 cm³**

- *with quantum fluctuations:*

⁴He stays liquid with molar volume of **28 cm³**

I.1.4 Zero Point Fluctuations



Comparison of *ab initio* potential energies for Ne, Ar, Kr, Xe (symbols) with SAAP calculations (solid lines)

U. K. Deiters et al., J. Chem. Phys. **150**, 134504 (2019)
<https://doi.org/10.1063/1.5085420>

comparison of ^4He and Ne:

	He	Ne
potential minimum ε	1.03 meV	3.94 meV
interatomic distance R_0	0.265 nm	0.296 nm

1 meV = 11.6 K

curvature determines spring constant: $k = \frac{1}{2} \frac{\partial^2 E_{\text{pot}}(R)}{\partial R^2} \bigg|_{R=R_0} = \frac{36\varepsilon}{R_0^2}$ (for fcc)

I.1.4 Zero Point Fluctuations

quantum parameter:

$$\Lambda \equiv \frac{\langle E_{zp} \rangle}{\langle E_{pot} \rangle}$$

quantum liquid:

$$\Lambda > 1$$

liquid	Xe	Kr	Ar	Ne	H ₂	⁴ He	³ He
Λ	0.06	0.10	0.19	0.59	1.73	2.64	3.05

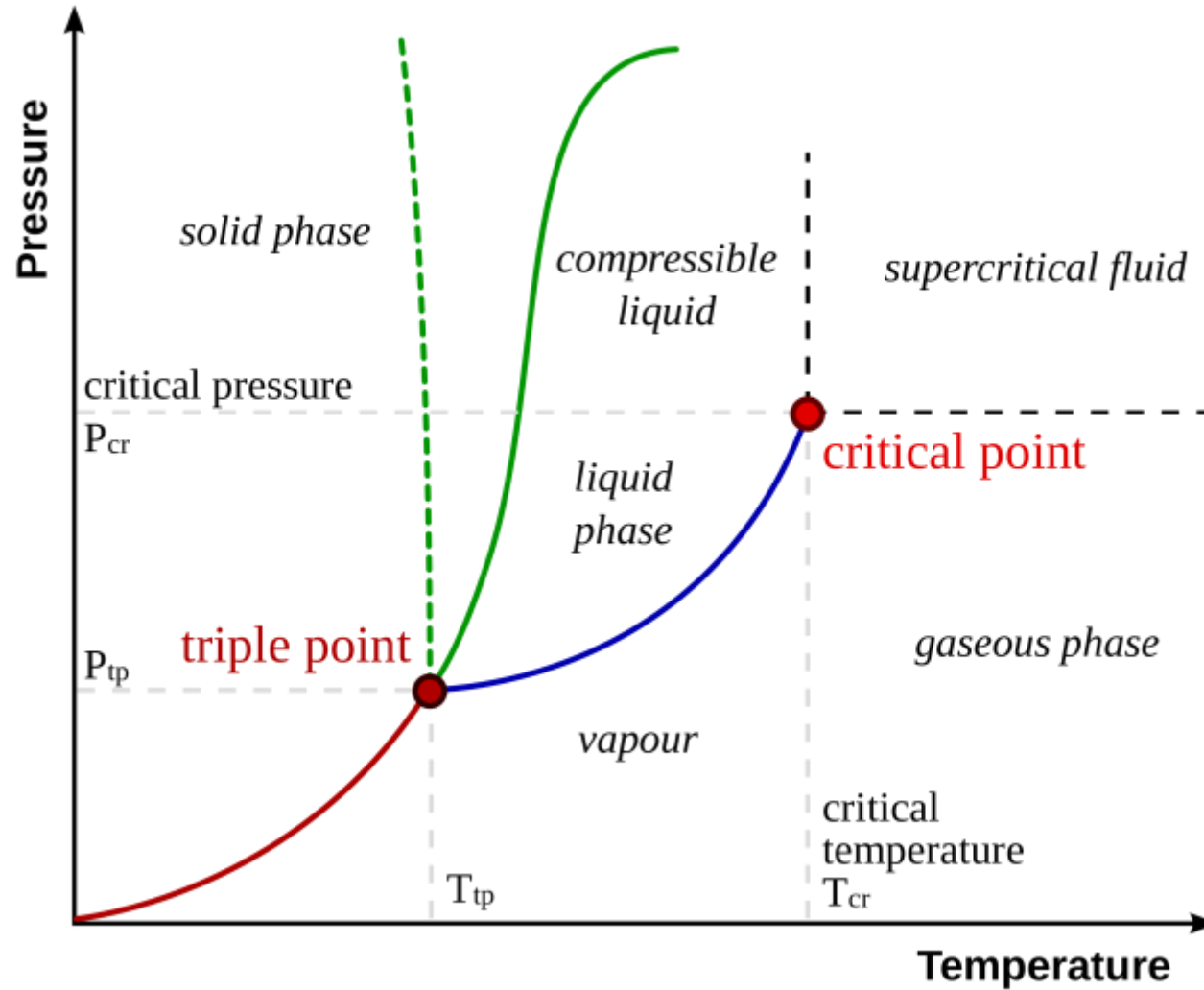
*large E_{zp} ,
but also
large E_{pot}*
*identical E_{pot} ,
but different E_{zp}*

mass of ³He smaller than of ⁴He

- ➔ even stronger effect of quantum fluctuations
 - ➔ larger quantum parameter
 - ➔ larger molar volume
- ³He: **40 cm³/mole**, ⁴He: **28 cm³/mole**

I.1.6 pT phase diagram of ^4He

- general nomenclature, definitions



<https://commons.wikimedia.org/wiki/File:Phase-diag2.svg>

I.1.5 Helium under Pressure

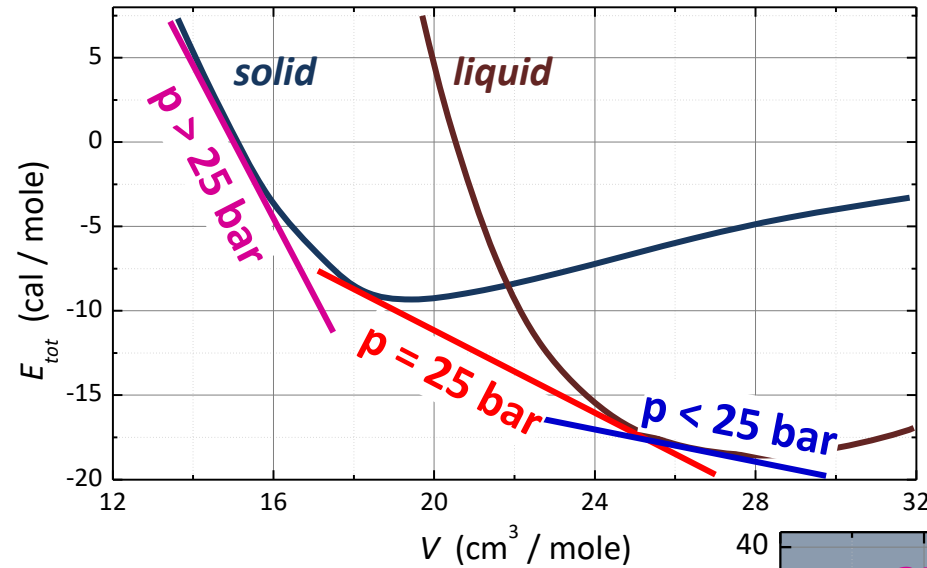
– **enthalpy:** $H = U + pV$
 $H = \text{min} ?$

– use of isenthalpy lines:
 $H = \text{const.}$

– with $U = H - pV$
 → straight lines with slope $-p$

– **lowest isenthalpy:**

→ tangent with slope $-p$



$p > 25 \text{ bar}$: solid

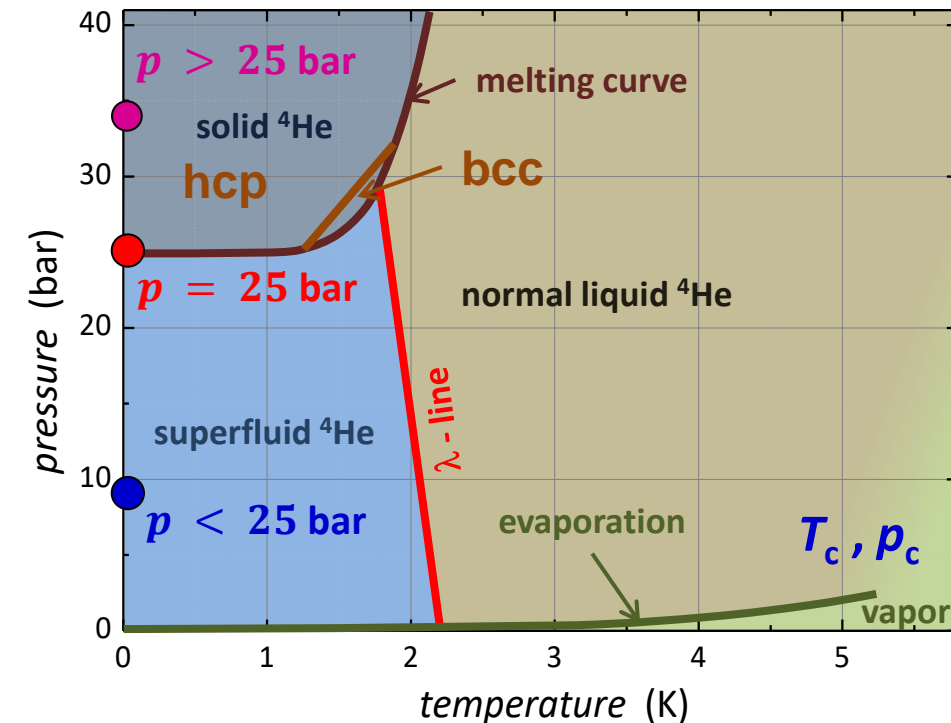
$p = 25 \text{ bar}$: coexistence

$p < 25 \text{ bar}$: liquid

$$H = U + pV$$

$$dH = \underbrace{dU}_{TdS - pdV} + pdV + Vdp$$

$$dH = TdS + Vdp$$



I.1.6 pT phase diagram of ^4He

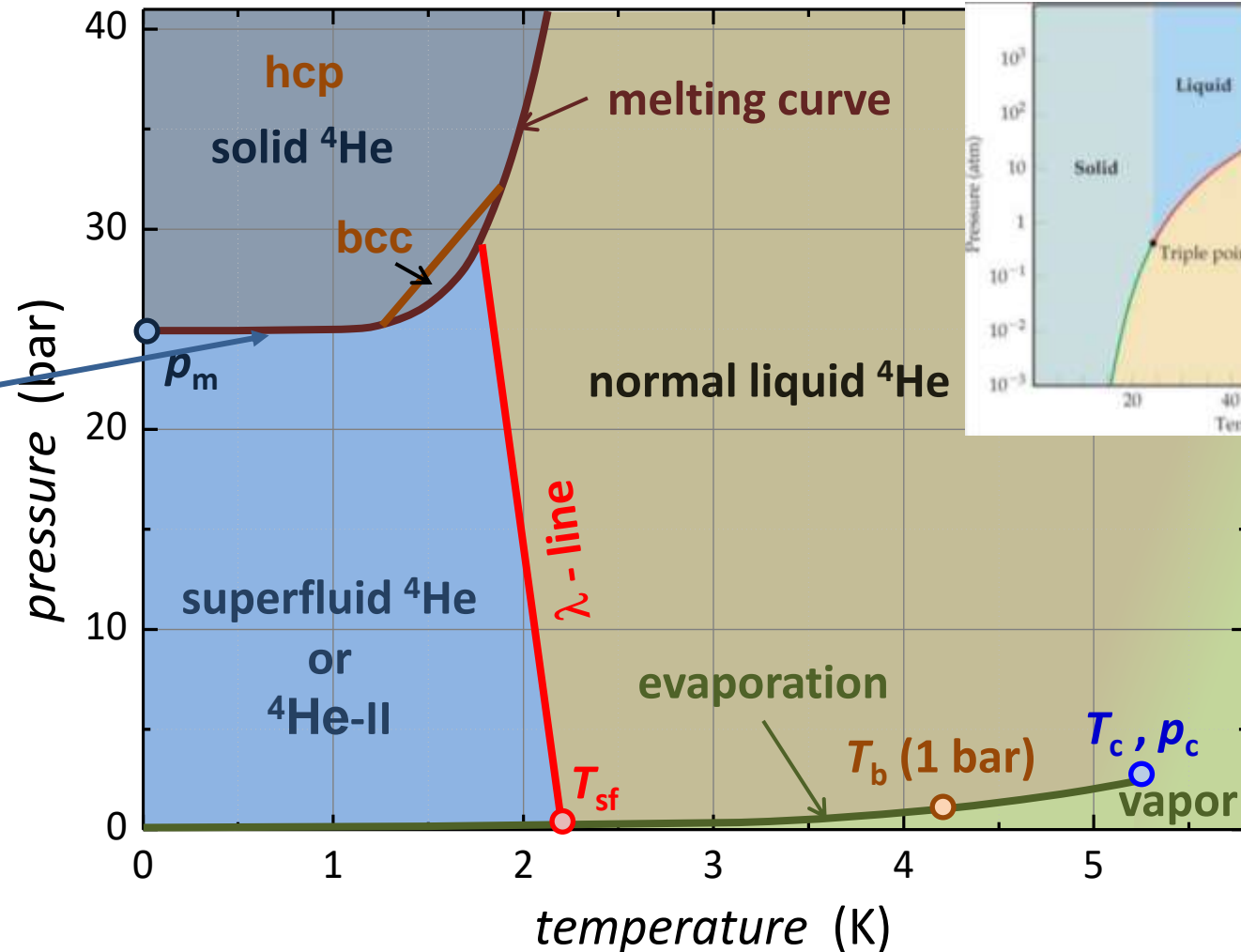
- Clausius-Clapeyron:**

$$\frac{\partial p_m}{\partial T} = \frac{\Delta S_m}{\Delta V_m}$$

$$\frac{\partial p_m}{\partial T} = \frac{\Delta S_m}{\Delta V_m} \approx 0 \quad @ \quad T \rightarrow 0$$

$$\Rightarrow \Delta S_m \rightarrow 0 \quad \text{for} \quad T \rightarrow 0$$

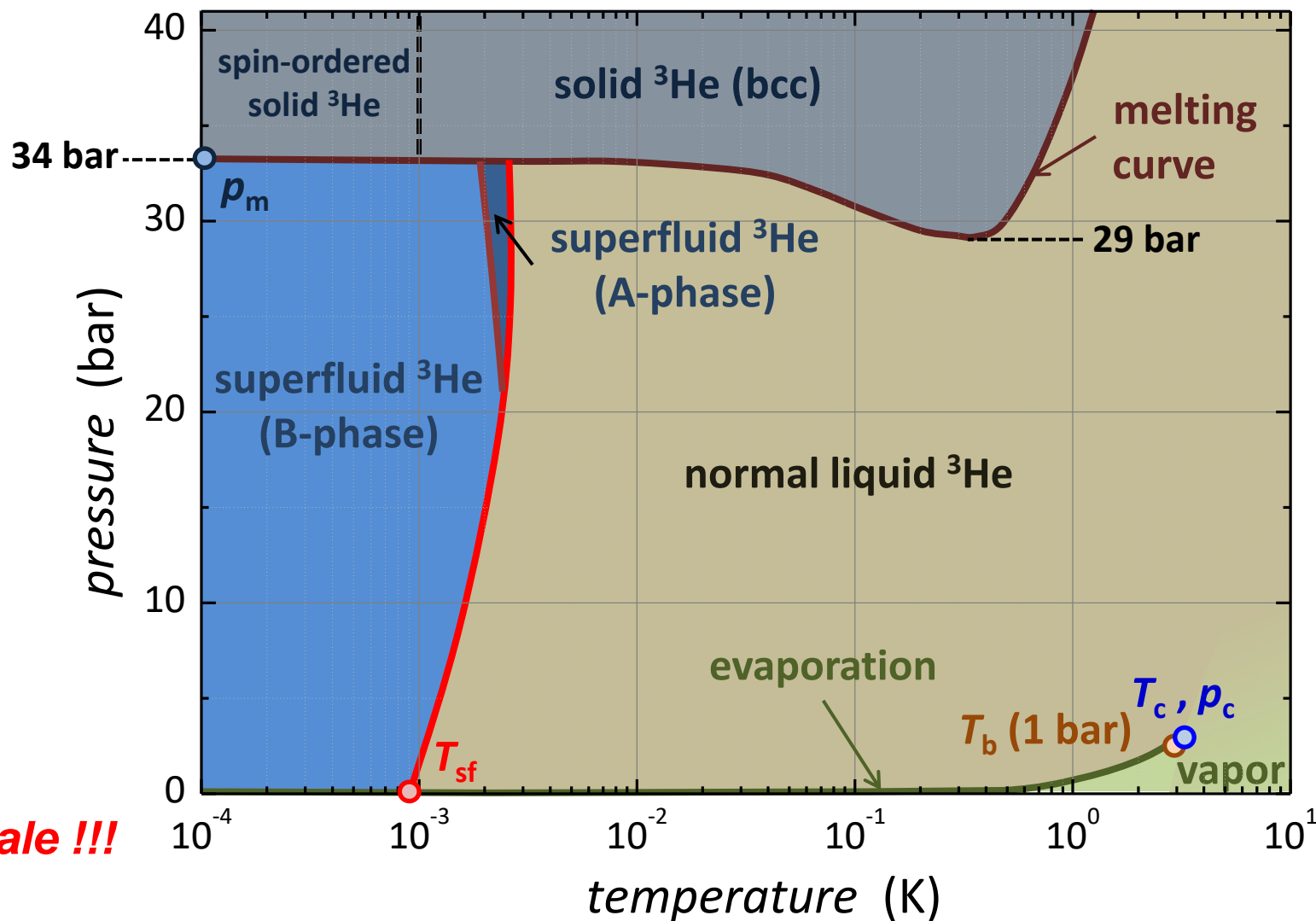
(agrees with 3rd Law of thermodynamics)



Phase diagram of ^4He at low temperatures. ^4He remains liquid at zero temperature if the pressure is below 2.5 MPa (approximately 25 atmospheres) → **no triple point**.

The liquid has a phase transition to a **superfluid phase**, also known as He-II, at the temperature of 2.17 K (at vapor pressure). The solid phase has either hexagonal close packed (hcp) or body centered cubic (bcc) symmetry.

I.1.6 pT phase diagram of ^3He



big difference in value of T_{sf}

- normal/superfluid transition dominated by quantum effects
- quantum statistics matters!! (bosons vs. Fermions)
- nature of superfluid phase in ^4He and ^3He is different

log T scale !!!

- phase diagram of ^3He : note the logarithmic temperature scale. There are **two superfluid phases of ^3He , A and B.**
- the line within the solid phase indicates a transition between spin-ordered and spin disordered structures (at low and high temperatures, respectively).

I.1.6 pT phase diagram of ^3He

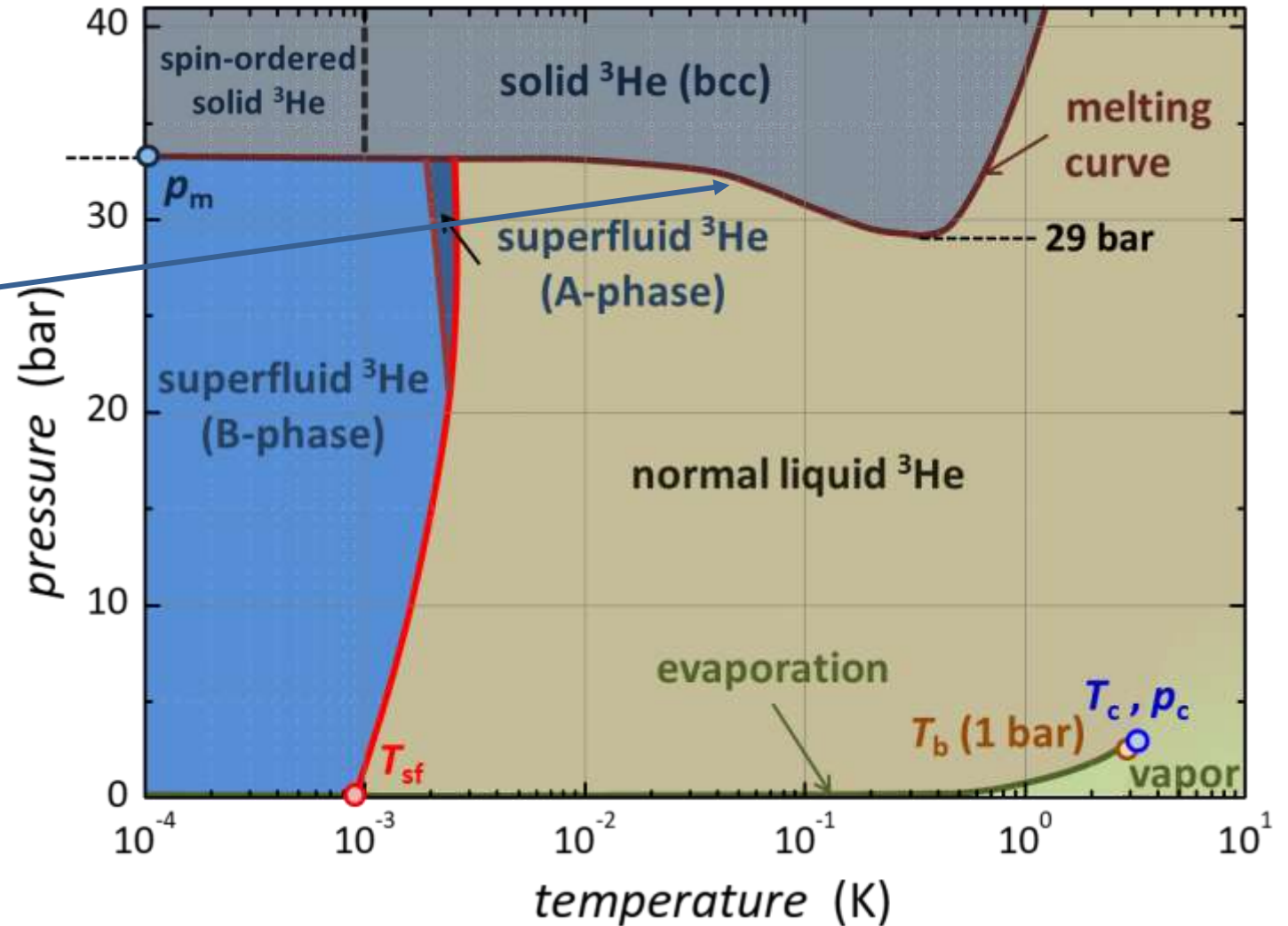
- Clausius-Clapeyron:*

$$\frac{\partial p_m}{\partial T} = \frac{\Delta S_m}{\Delta V_m}$$

$$\frac{\partial p_m}{\partial T} = \frac{\Delta S_m}{\Delta V_m} < 0$$

since volume of liquid state is larger, the entropy of the liquid state must be smaller

- ➔ more ordered liquid state!?
- ➔ explanation follows later:
ordering in momentum space



	³ He	⁴ He
boiling point T_b (K) (@ $p = 1$ bar)	3.19	4.21
critical temperature T_c (K)	3.32	5.20
critical pressure p_c (bar)	1.16	2.29
superfluid transition temperature T_s (K)	0.0025	2.177
density ρ (g/cm³) (for $T \rightarrow 0$ K)	0.076	0.145
density ρ (g/cm³) (@ boiling point)	0.055	0.125
classical molar volume V_m (cm³ /mole) (@ saturated vapor pressure and $T = 0$ K)	12	12
actual molar volume V_m (cm³ /mole) (@ saturated vapor pressure and $T = 0$ K)	36.84	27.58
melting pressure p_m (bar) (@ $T = 0$ K)	34.39	25.36

strong difference between superfluid properties of ⁴He and ³He

→ different nuclear spin → different quantum statistics

^4He	$p^+ \uparrow \quad p^+ \downarrow$	$n \uparrow \quad n \downarrow$	$e^- \uparrow \quad e^- \downarrow$	boson
^3He	$p^+ \uparrow \quad p^+ \downarrow$	$n \uparrow$	$e^- \uparrow \quad e^- \downarrow$	fermion

*different
nuclear spin:
 $^4\text{He}: I = 0$
 $^3\text{He}: I = 1/2$*

*same chemical
properties*

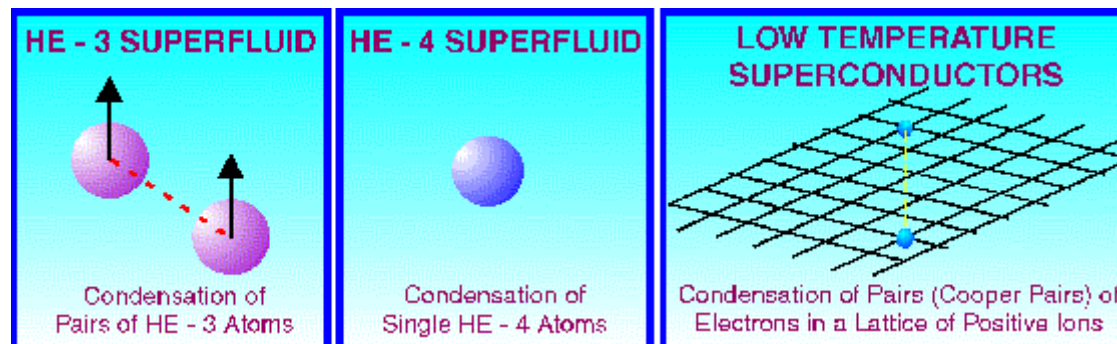
different statistics

^3He : *pairing similar to superconductivity in metals*

weak attractive interaction between Fermions \rightarrow „Cooper pairs“

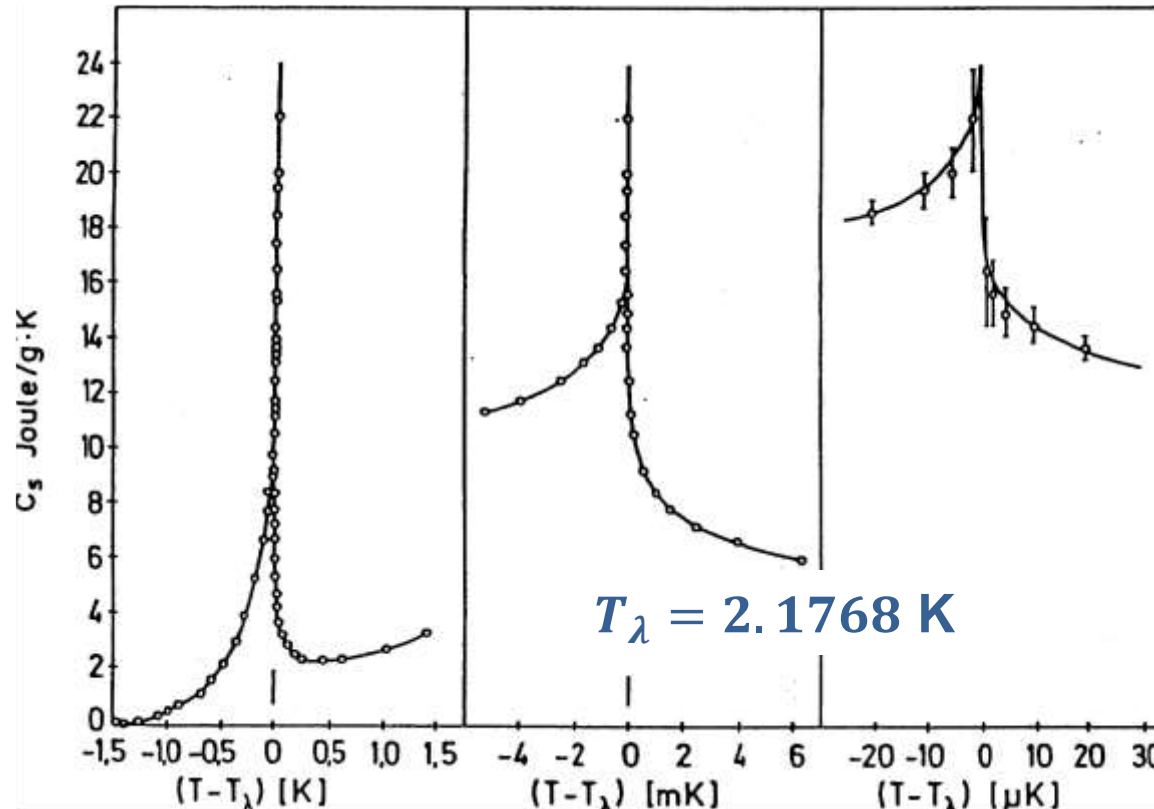
^4He : *Bose-Einstein condensation of Bosons without „pairing interaction“*

\rightarrow no attractive interaction required



I.1.8 Specific Heat of ^4He – Lambda Point

- first measurements by *Dana* and *Kamerlingh Onnes* in 1923
 - anomalous increase of C_p at about $T_\lambda = 2.177\text{ K}$
 - in publication (1926) some data points were omitted, since they were believed to result from exp. artefacts
- confirmation by *Clusius* and *Keesom* in 1932 (introduced denotation **He I** and **He II** → **phase transition**)
 - **He I: normal fluid, He II: superfluid**



specific heat:

$$C_V \equiv \left(\frac{\partial U}{\partial T} \right)_V$$

observation: specific heat diverges at $T = T_\lambda$

→ **1st order phase transition**

→ shape: **Lambda point**

free energy F : **kink**

($F = U - TS \Rightarrow dF = -SdT - pdV + \mu dN$ with $dU = TdS - pdV + \mu dN$)

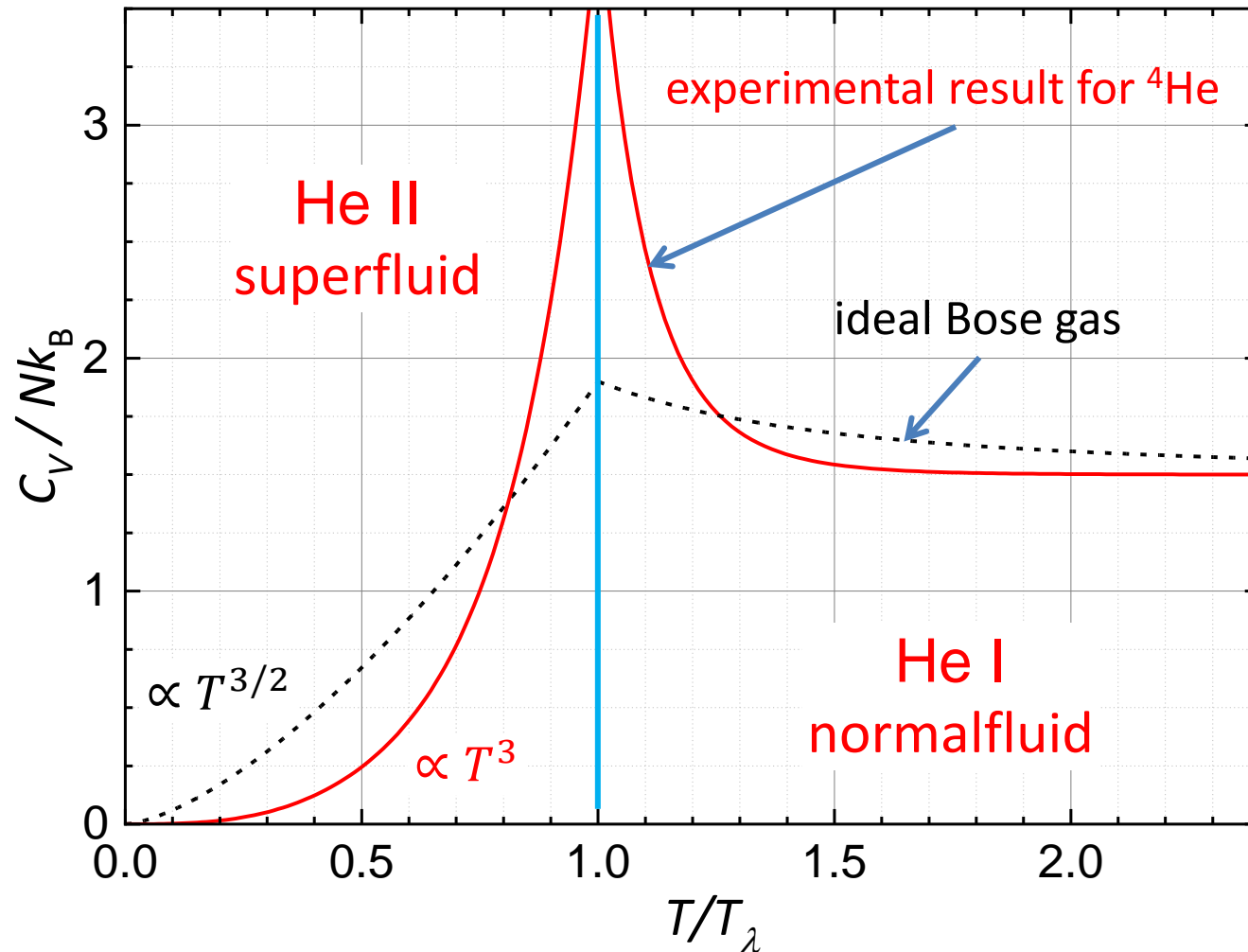
entropy $S \propto (-\partial F / \partial T)_{V,N}$: **jump**

spec. heat $C_V \propto (-\partial^2 F / \partial T^2)_{V,N}$: **divergence**

Buckingham, M. J. and Fairbank, W. M., "The Nature of the Lambda Transition", in Progress in Low Temperature Physics III, 1961.

I.1.8 Specific Heat of ^4He – Lambda Point

- comparison of specific heat of ^4He and that of an ideal Bose-Einstein condensate



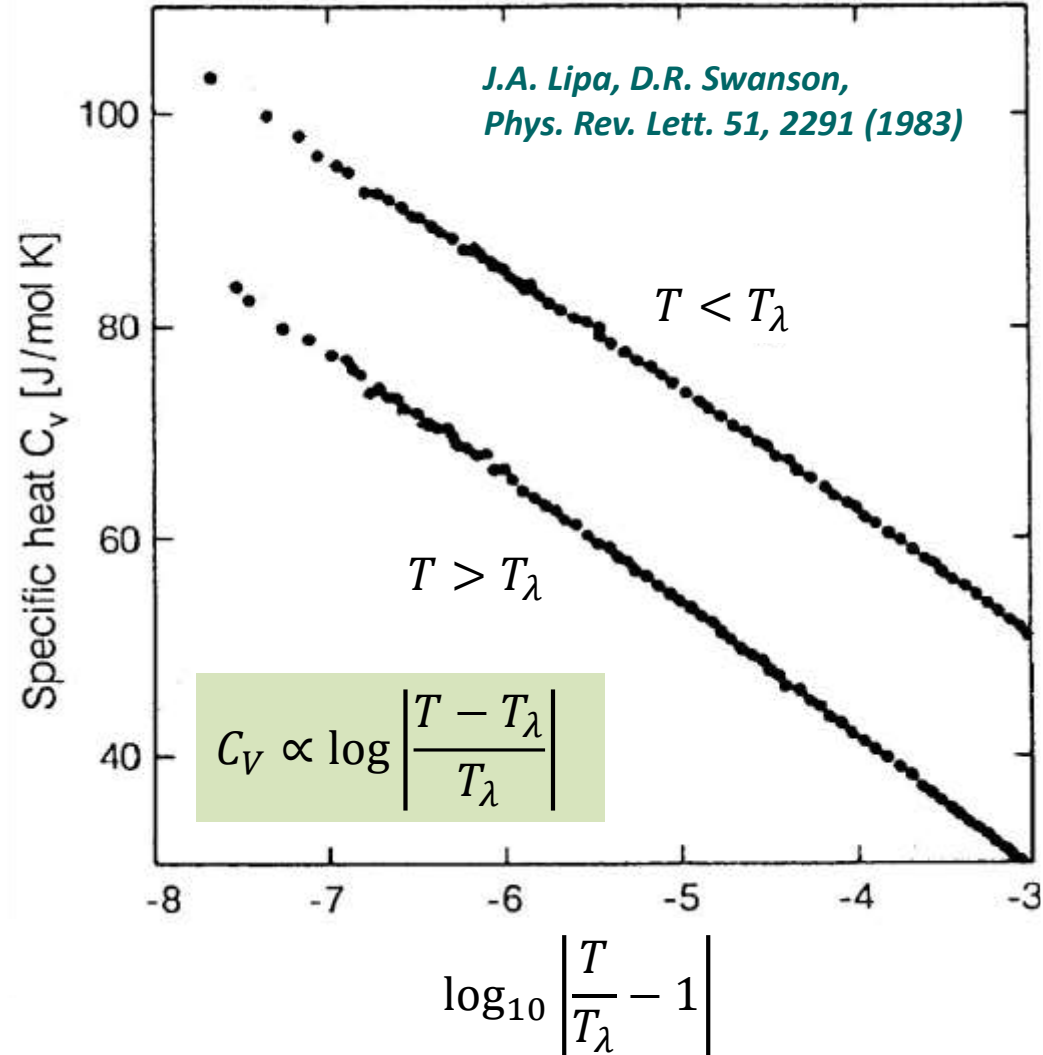
2nd order phase transition is expected for ideal Bose-Einstein condensate

→ no divergence of C_V at $T = T_\lambda$

measured behavior of ^4He near T_λ is more complicated

(detailed discussion follows later)

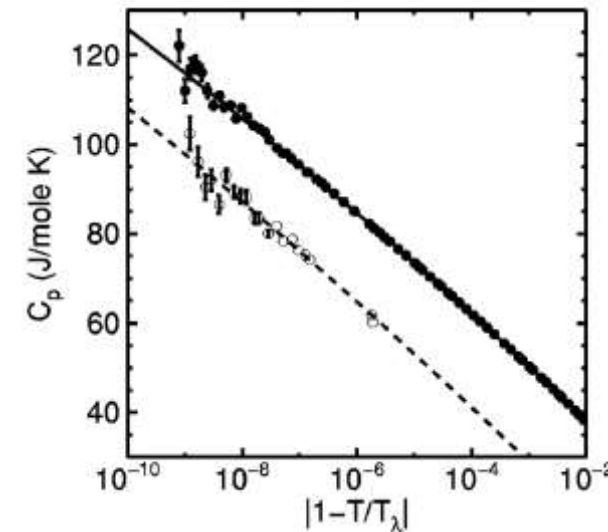
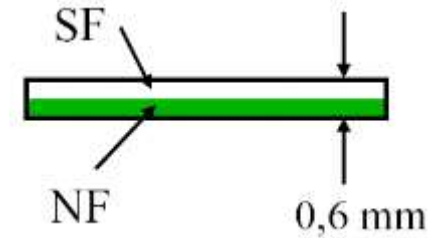
I.1.8 Specific Heat of ^4He – Precision Measurement



- specific heat is measured with very high temperature resolution (\approx nK)
- extreme purity of liquid He is important

→ *model system for testing physics of phase transitions*

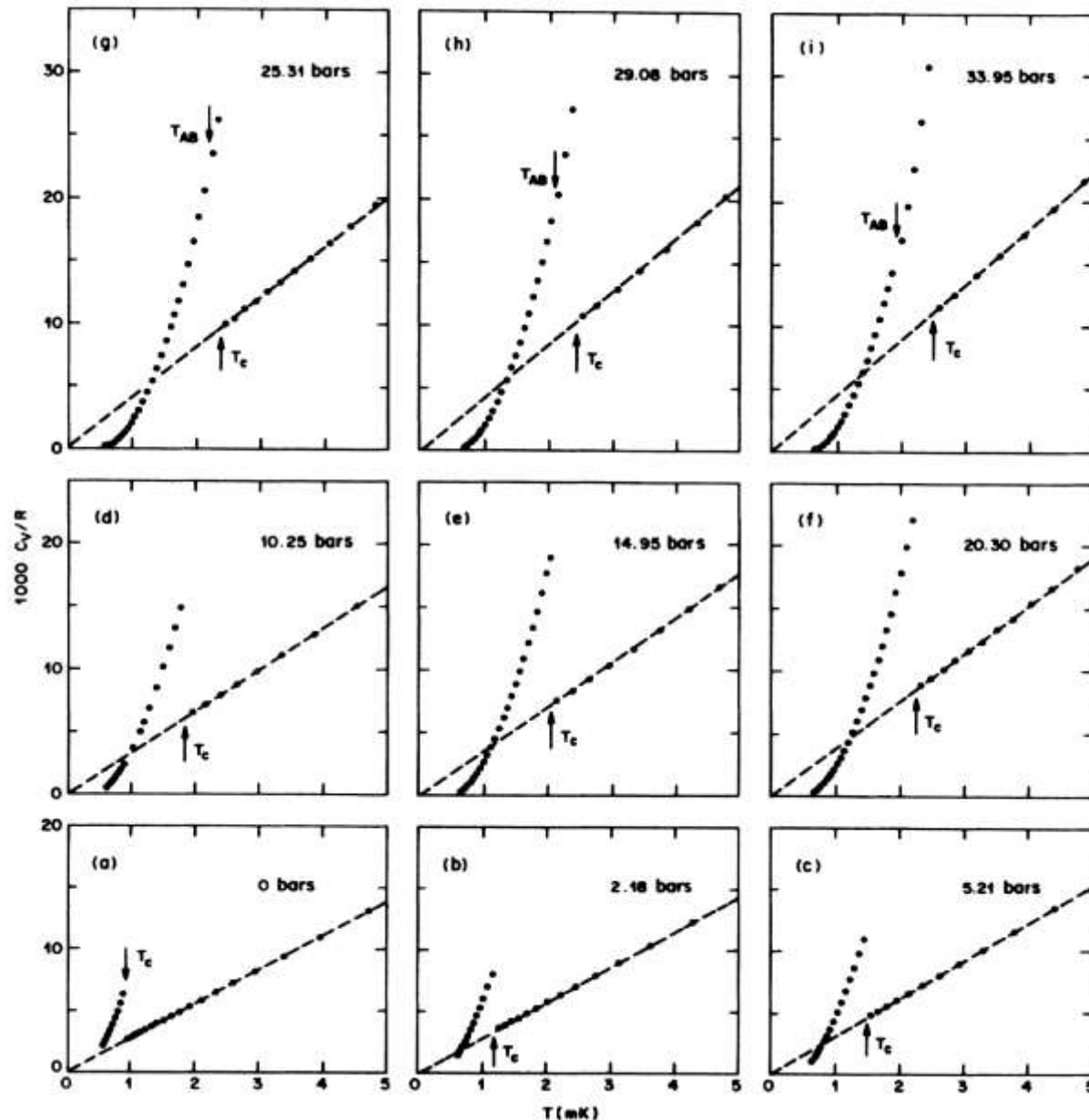
- experimental problem:
 - hydrostatic pressure
 - space experiments



Specific heat of liquid helium in zero gravity very near the lambda point
J. A. Lipa et al., Phys. Rev. B 68, 174518 (2003)

→ *key result: second order phase transition with logarithmic singularity*

I.1.8 Specific Heat of ^3He



$C_V(T)$ similar to a metallic superconductor

- $T > T_c$:
linear $C_V(T)$ of Fermi liquid

$$C_V(T) = \gamma T$$

$$\gamma = \frac{3}{2} \frac{n}{E_F} \frac{\pi^2}{3} k_B^2 \quad (\text{Sommerfeld, 1927})$$

- $T = T_c$:
jump due to superfluid transition
(as for SCs and predicted by BCS theory)
- $T < T_c$:
reduction of C_V due to condensation of
Fermions (similar to superconductors)

Source:

Greywall, Phys. Rev. B 33, 7520 (1986)

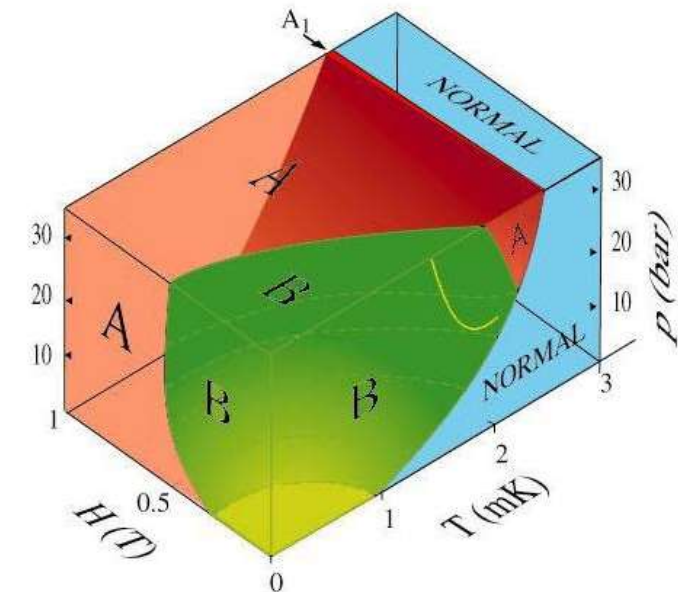
^3He specific heat and thermometry at millikelvin temperatures

I.1 Summary of Experimental Observations

- specific heat of ^4He is large: $C_V \approx 1 \text{ J/g K}$ (compared to about 10^{-5} J/g K of Cu, $C_V \propto n \cdot T/T_F$)
- latent heat of evaporation of ^4He is small (20.59 J/g), but large compared to specific heat of other materials
→ *application as cooling liquid*
- strong anomaly of specific heat at T_λ , origin unknown for more than 30 years
→ originates from *Bose-Einstein condensation* (bosonic character of ^4He , interacting bosonic system)
- shape of anomaly resembles greek letter λ → *Lambda point*
- below T_λ entropy and specific heat decrease rapidly with decreasing T
→ *condensation into superfluid*
- for ^4He temperature variation of C_V below T_λ determined by collective excitations in superfluid
(*phonons, rotons* – discussion later)

I.1 Summary of Experimental Observations

- for ^4He phase transition at $T_\lambda \rightarrow$ **Lambda point** is due to **bosonic character of ^4He**
 \rightarrow no such transition was expected for ^3He due to nuclear spin $I = 1/2$ (**fermionic character of ^3He**)
- only after the development of BCS theory in 1957 (condensation of Cooper pairs into macroscopic quantum state), the **search for the superfluid phase of ^3He** has been intensified
- 1963: **Theory of superfluid ^3He** (Anderson and Morel, Balian and Werthamer)
- 1971: after the temperature regime below 5 mK has been accessed, the **superfluid phase of ^3He has been found** by Osheroff, Richardsen and Lee in NMR experiment
D.D. Osheroff, R.C. Richardsen, D.M. Lee,
Phys. Rev. Lett. 28, 885 (1972)
- there is not only a single superfluid phase, but three, depending on temperature, pressure and applied magnetic field
- 1977: first experiment with ^3He in Germany at WMI
 (Eska, Schuberth, Uhlig,)



Contents Part I: Quantum Liquids

Contents:

I.1 Foundations and General Properties

- I.1.1 Quantum Gases & Liquids
- I.1.2 Helium
- I.1.3 Van der Waals Bonding
- I.1.4 Zero-Point Fluctuations
- I.1.5 Helium under Pressure
- I.1.6 pT-Phase Diagram of ^4He and ^3He
- I.1.7 Characteristic Properties of ^4He and ^3He
- I.1.8 Specific Heat of ^4He and ^3He

I.4 Vortices

- I.4.1 Quantization of Circulation
- I.4.2 Experimental Study of Vortices

I.5 ^3He

- I.5.1 normal fluid ^3He
- I.5.2 solid ^3He and Pomeranchuk effect
- I.5.3 superfluid ^3He

I.6 ^3He / ^4He mixtures

I.2 ^4He as an Ideal Bose Gas

- I.2.1 Bose-Einstein Condensation
- I.2.2 Ideal Bose Gas
- I.2.3 Bose Gas with Interactions
- I.2.4 Bose-Einstein Condensation of ^4He

I.3 Superfluid ^4He

- I.3.1 Two-Fluid Model
- I.3.2 Experimental Observations
- I.3.3 Two-Fluid Hydrodynamics
- I.3.4 Excitation Spectrum of ^4He

I.2 ^4He as Ideal Bose Gas

- basic assumption

^4He is treated as a non-interacting gas of bosons

we already have learnt that there are Van der Waals interactions

- ➔ treatment of ^4He as an ideal bosonic gas is only a *first order approximation*
- ➔ interactions lead to deviations from behavior of ideal gas of non-interacting bosons

I.2 ^4He as Ideal Bose Gas

- high temperatures:

average distance \bar{R} of atoms in gas is large compared to **de Broglie wavelength** $\lambda_T = h/p \propto 1/\sqrt{T}$

→ **classical gas with Maxwell-Boltzmann statistics**

thermal de Broglie wave length:

$$\lambda_T = \sqrt{\frac{h^2}{2\pi m k_B T}} = \sqrt{\frac{2\pi \hbar^2}{m k_B T}}$$

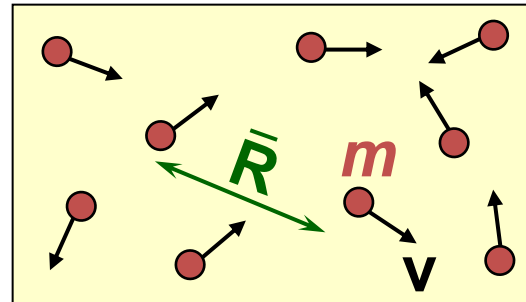
compare I.1.1

- low temperatures:

- quantum mechanical description, as soon as average particle distance $\bar{R} = \lambda_T$
- classical gas of distinguishable particles turns into quantum soup of indistinguishable identical particles

→ **quantum gas with Bose statistics for ^4He (or Fermi statistics for ^3He)**

→ **Bose-Einstein condensation for bosons (^4He)**

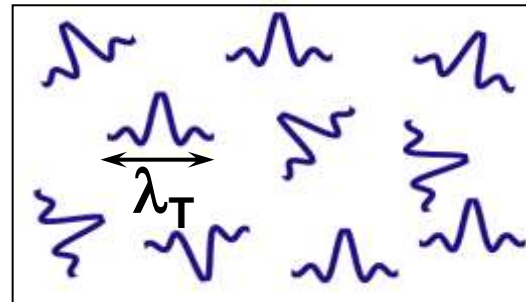


high temperature:

thermal velocity v_T

$$\text{density } \frac{N}{V} = n \approx \frac{1}{\bar{R}^3}$$

billiard balls

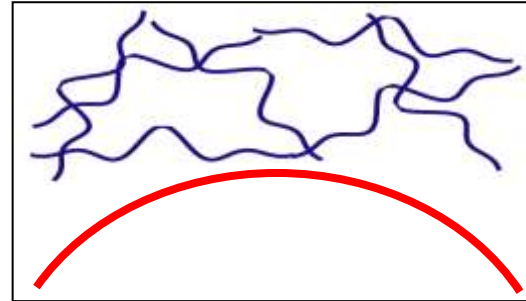


low temperature:

de Broglie wavelength λ_T

$$\lambda_T = h/mv_T \propto 1/\sqrt{T}$$

wave packets

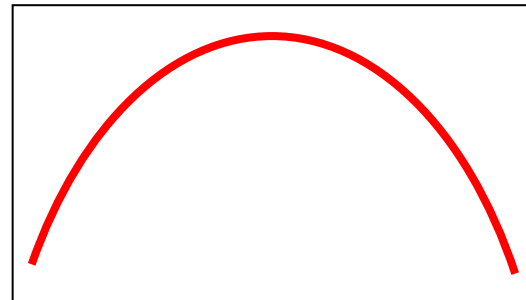


$T = T_{\text{BEC}}$:

Bose-Einstein condensation

$$\lambda_T \approx \bar{R}$$

wave packets overlap



$T \rightarrow 0$:

pure Bose-Einstein condensate

„Super“-mater wave

I.2.1 Bose-Einstein Condensation

**Bose -
Einstein
Condensation**

I.2.1 Bose-Einstein Condensation

- phase transition occurs for *interactionless bosonic particles*
- originates from the *indistinguishability of identical particles*



Albert Einstein und Satyendranath Bose

proposed in 1924 that below a characteristic temperature T_B a macroscopic number of bosons can condense into the state of lowest energy

experimental confirmation in 1995



Eric A. Cornell



Wolfgang Ketterle



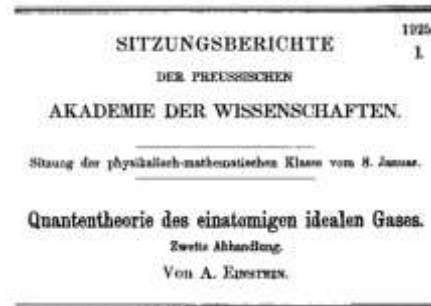
Carl E. Wieman



Nobel Prize in Physics 2001

"for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates"

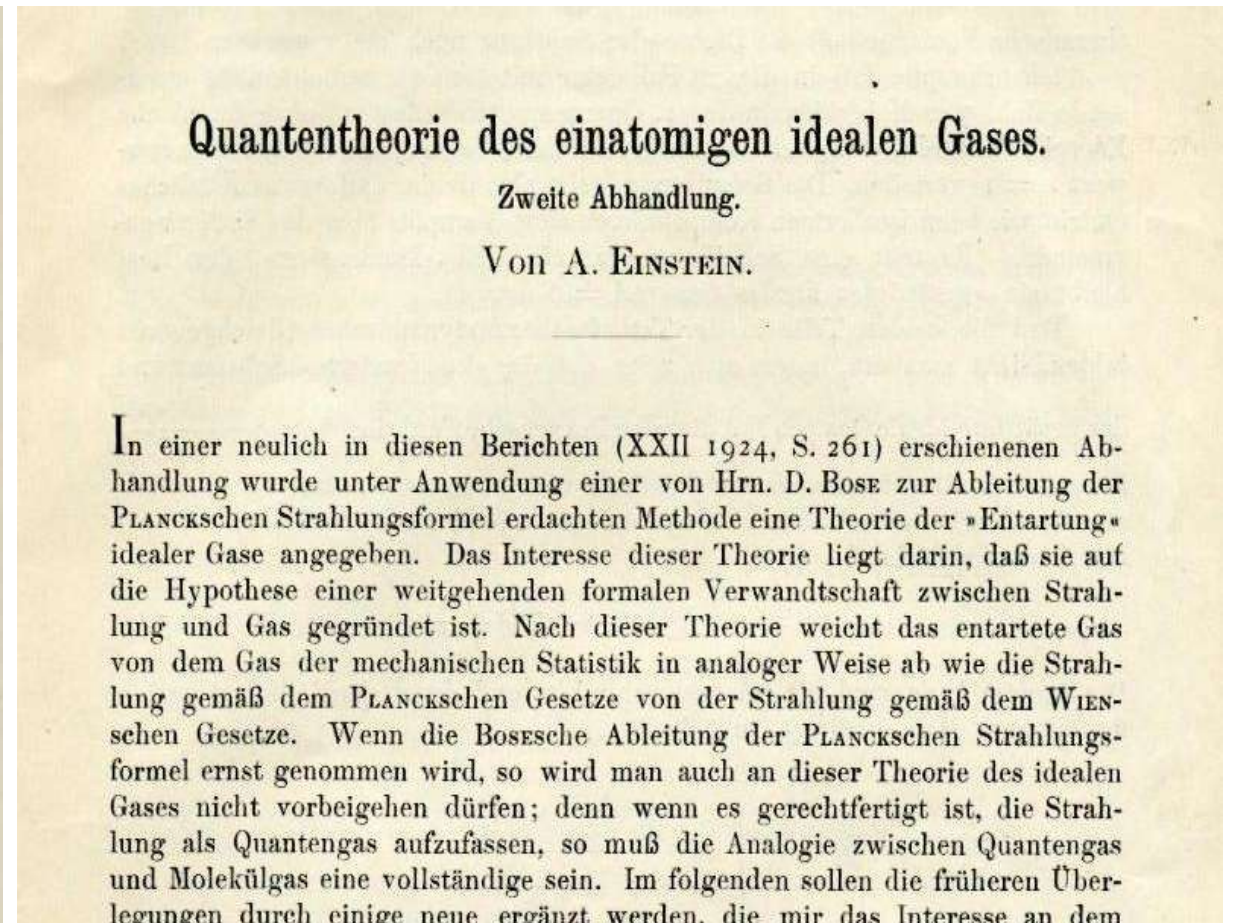
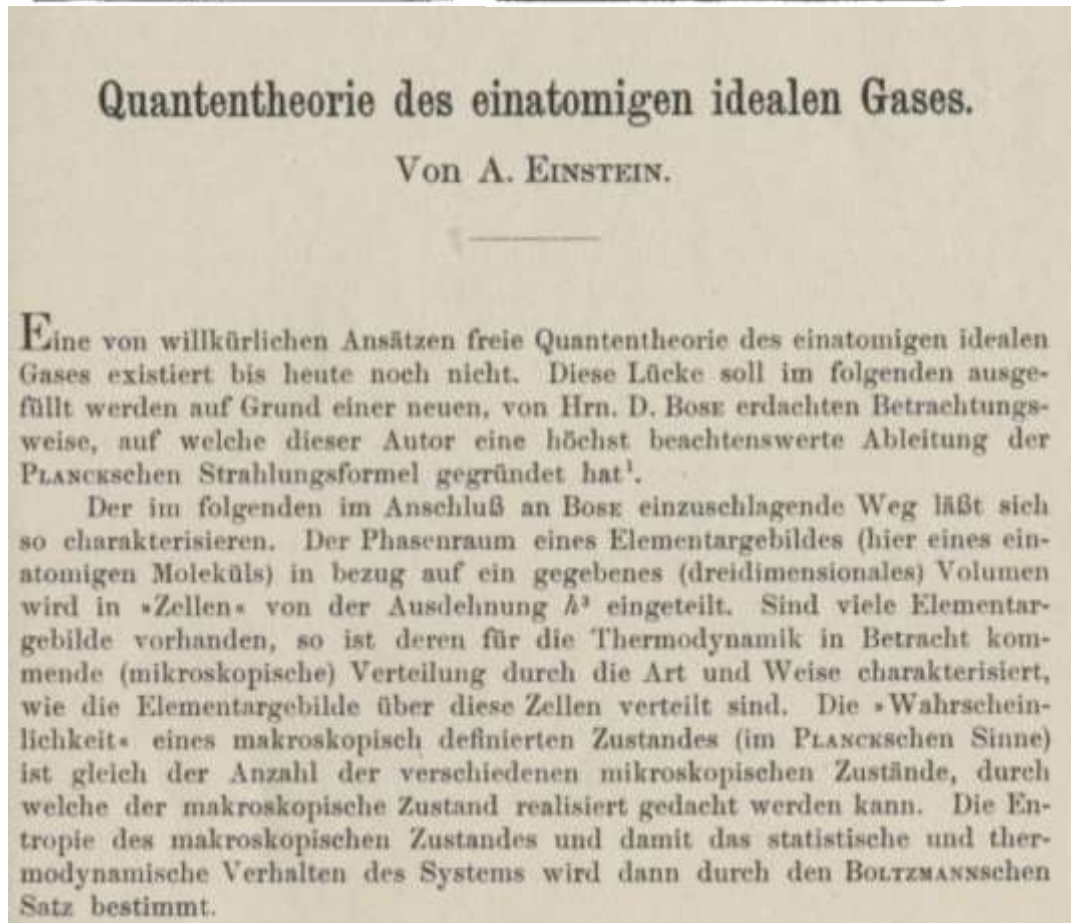
I.2.1 Bose-Einstein Condensation



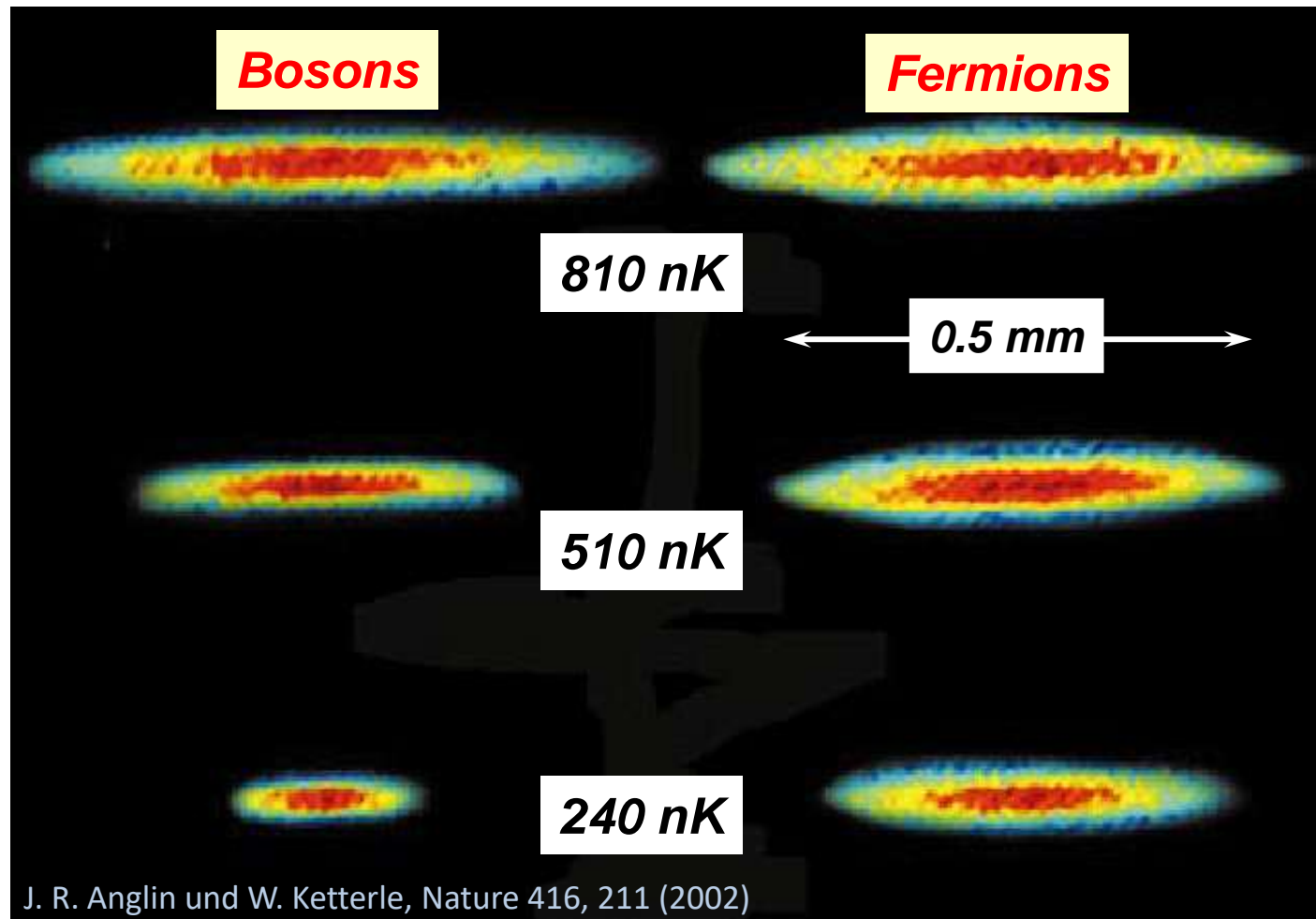
A. Einstein.

Quantentheorie des einatomigen Gases.

SITZUNGSBERICHTE DER PREUSSISCHEN AKADEMIE DER WISSENSCHAFTEN
PHYSIKALISCH-MATHEMATISCHE KLASSE 3, 1925.



I.2.1 Bose-Einstein Condensation



Pauli exclusion principle
does not allow further
compression of fermions in
momentum space
at low T

I.2.1 Bose-Einstein Condensation

- degenerate multiparticle system: Bosons and Fermions (cf. I.1.1)

- at given density n :
particle occupies cube with $d = 1/n^{1/3}$

- classical liquid**: $\lambda_T \leq d$

particles are distinguishable

- quantum liquid**: $\lambda_T \geq d$

particles are indistinguishable

→ *degenerate quantum liquid*

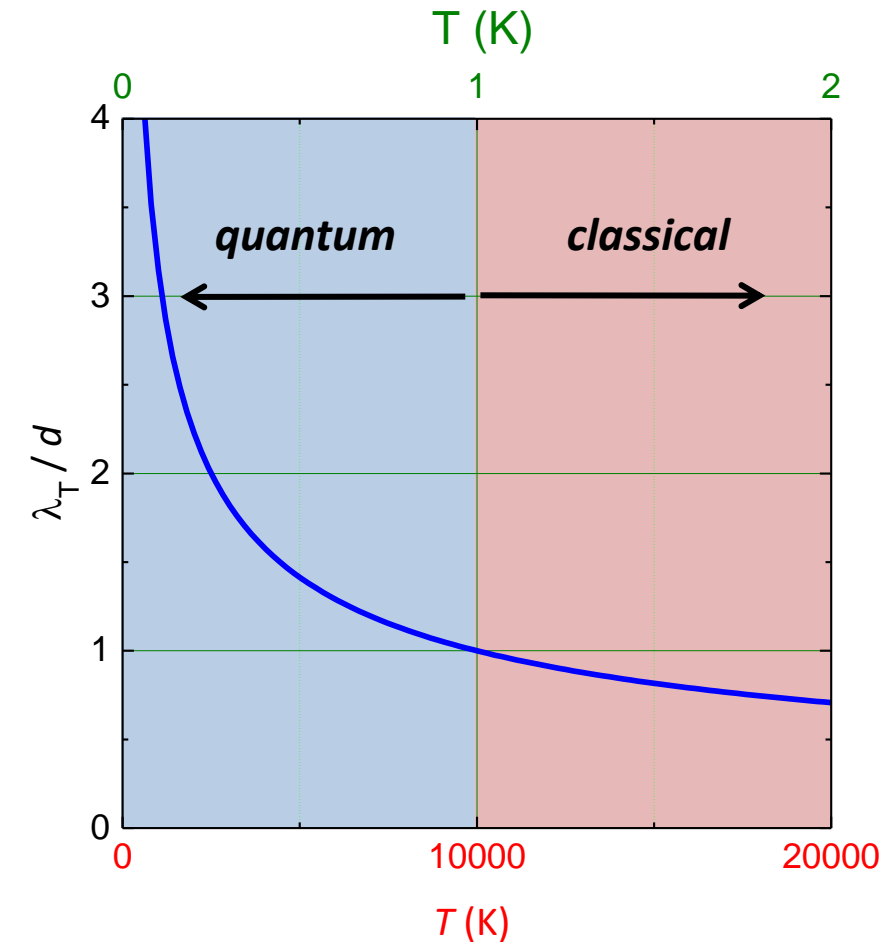
- Bosons**: (^4He ,)

→ *Bose-Einstein condensation*

- Fermions**: (^3He , e, p, n,)

→ *Fermi sea*

^4He : $n = 4 \cdot 10^{21} \text{ cm}^{-3}$,



e^- : $n = 3.6 \cdot 10^{21} \text{ cm}^{-3}$

I.2.1 Bose-Einstein Condensation

- cooling down atoms

$$\lambda_T^3/d^3$$

	temperature	density (1/cm ³)	phase space density
atomic oven	500 K	10 ⁻¹	10 ⁻¹³
Laser cooling	50 μK	10 ¹¹	10 ⁻⁶
evaporation cooling	500 nK	10 ¹⁴	2.612
Bose-Einstein condensation		10 ¹⁵	10 ⁷

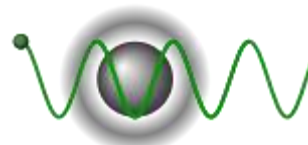
$$\lambda_T = \sqrt{\frac{h^2}{2\pi m k_B T}} = \sqrt{\frac{2\pi \hbar^2}{m k_B T}} \propto \frac{1}{\sqrt{mT}}$$

I.2.1.1 Laser Cooling

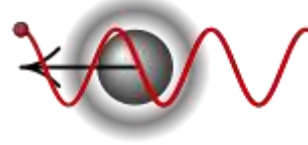
- **Laser cooling:**
techniques in which atomic and molecular samples are cooled down to near absolute zero **through the interaction with one or more laser fields**
- **Laser cooling techniques**
 - Doppler cooling (Schawlow, Hänsch)
T.W. Hänsch, and A.L. Schawlow, *Cooling of gases by laser radiation*, Optics Communications, 13, 68 (1975)
 - other techniques
 - Sisyphus cooling
 - Resolved sideband cooling
 - Velocity selective coherent population trapping (VSCPT)
 - Anti-Stokes inelastic light scattering (typically in the form of fluorescence or Raman scattering)
 - Cavity mediated cooling
 - Sympathetic cooling
- **Nobel Price in Physics 1997:**
„for the development of methods to cool and trap atoms with laser light”
for **Steven Chu**, **Claude Cohen-Tannoudji** und **William D. Phillips**

I.2.1.1 Laser Cooling – Doppler Cooling

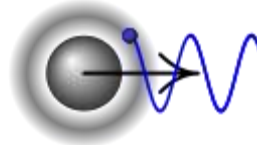
- in **Doppler cooling**, the frequency of *light is tuned slightly below an electronic transition in the atom*
- because the light is detuned to the "red" (i.e., at lower frequency) of the transition, the atoms will absorb more photons if they move towards the light source, due to the Doppler effect



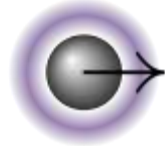
- *a stationary atom sees the laser neither red- nor blue-shifted and does not absorb the photon*



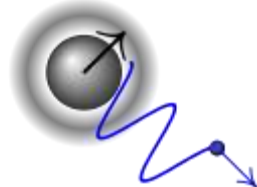
- *an atom moving away from the laser sees it red-shifted and does not absorb the photon*



- *an atom moving towards the laser sees it blue-shifted and absorbs the photon, slowing the atom*



- *the photon excites the atom, moving an electron to a higher quantum state.*



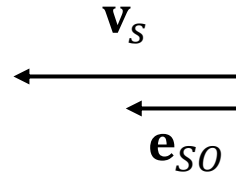
- *the atom re-emits a photon, as its direction is random, there is no net change in momentum over many absorption-emission cycles*

I.2.1.1 Laser Cooling – Doppler Cooling

- **Doppler shift** (how does the moving atom see the laser frequency)



atom = observer (O)
fixed position



laser = signal source (S) with frequency f_S
moving at velocity v_S towards atom

$$f_O = \frac{f_S}{1 - \frac{\mathbf{v}_S \cdot \mathbf{e}_{SO}}{c}}$$

laser frequency seen by the atom is blue-shifted,
if laser and atom move towards each other

➔ red detuned laser light becomes resonant if atom moves towards the laser

- **momentum/velocity change of atom due to photon absorption**

$$\frac{\Delta p}{p} = \frac{\hbar k_{ph}}{mv} = \frac{\Delta v}{v} \quad \Rightarrow \quad \Delta v = \frac{\hbar k_{ph}}{m}$$

I.2.1.1 Laser Cooling – Doppler Cooling

- *atom-laser detuning with atom at rest:*

$$\delta_0 = \omega_{atom} - \omega_{laser}$$

- *atom-laser detuning with atom moving at velocity v :*

$$\delta_{Doppler} = 2\pi(f_0 - f_s) = 2\pi f_0 \frac{\mathbf{v} \cdot \mathbf{e}_{s0}}{c} = 2\pi f_0 \frac{\hbar \mathbf{k} \cdot \mathbf{v}}{\hbar \omega_0} = \mathbf{k} \cdot \mathbf{v}$$

$$f_0 = \frac{f_s}{1 - \frac{\mathbf{v}_s \cdot \mathbf{e}_{s0}}{c}}$$

effective detuning between atom and laser

$$\delta_{eff} = \delta_0 - \delta_{Doppler} = \delta_0 - \mathbf{k} \cdot \mathbf{v}$$

we use:

$$p = \frac{E}{c}, \quad c = \frac{E}{p} = \frac{\hbar \omega}{mv} = \frac{\hbar \omega}{\hbar k}$$

- *force acting on the atom:*

$$\mathbf{F} = \hbar \mathbf{k} \gamma = \hbar \mathbf{k} \frac{\Gamma}{2} \frac{(I/I_0)}{1 + \frac{I}{I_0} + \left(\frac{2\delta_{eff}}{\Gamma}\right)^2}$$

large intensity $I \gg I_0$:

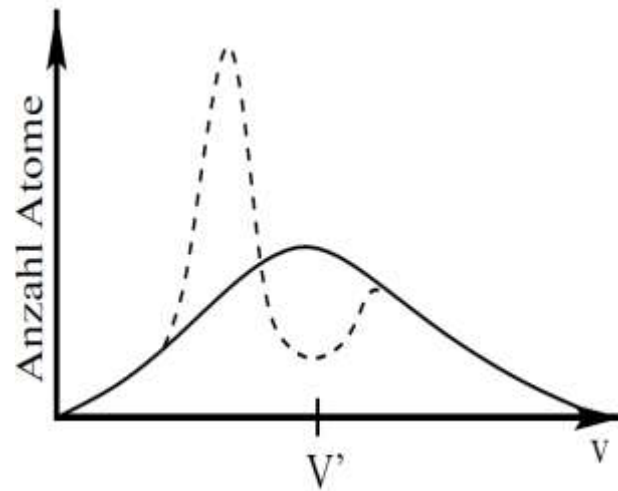
$$\mathbf{F} \rightarrow \hbar \mathbf{k} \frac{\Gamma}{2}$$

γ = absorption rate

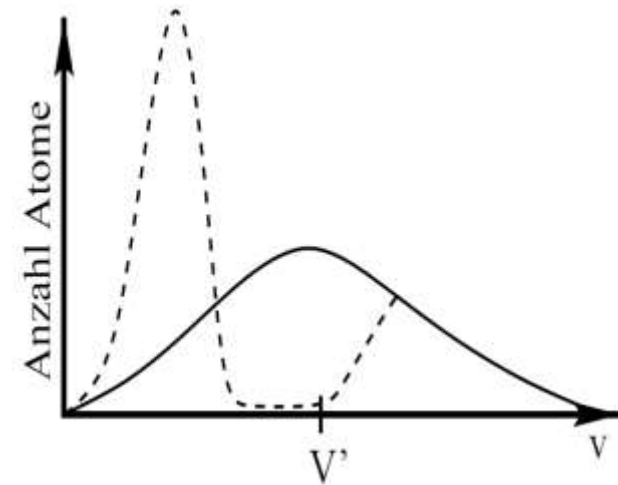
Γ = line width of atom transition,
spontaneous emission rate

I_0 = saturation intensity

I.2.1.1 Laser Cooling – Doppler Cooling



- velocity distribution of an atom beam before and after (dashed line) the interaction with a laser beam of fixed frequency



- velocity distribution of an atom beam before and after (dashed line) the interaction with a laser beam of continuously decreasing frequency

I.2.1.1 Laser Cooling – Doppler Cooling

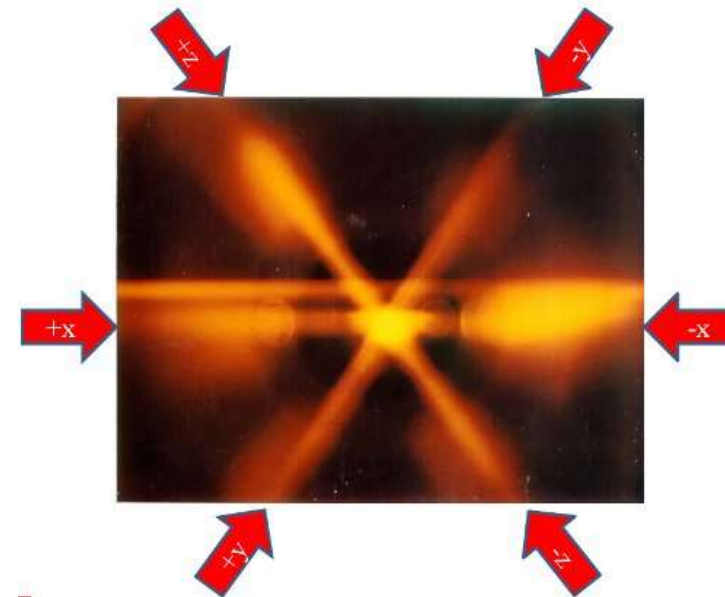
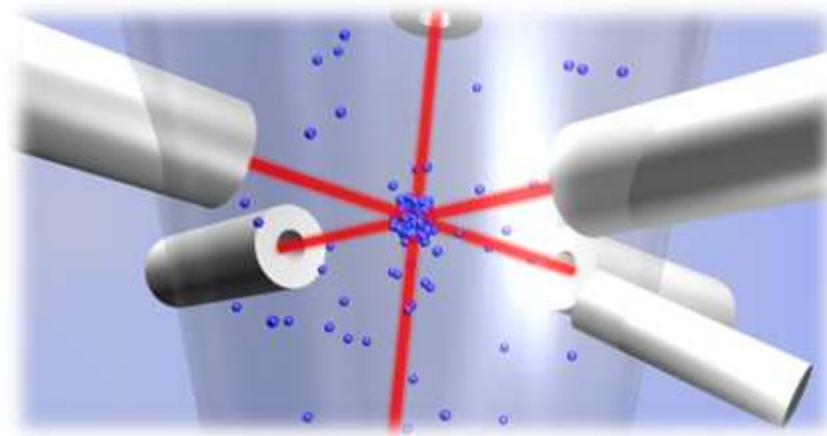
- *achievable minimum temperature in Doppler cooling:*
 - spontaneous absorption and emission are random processes
 - the average force acting on the atoms is statistically fluctuating
 - Brownian motion in phase space
 - minimum temperature is given by the compensation of heating effect due to the random absorption and emission processes and maximum cooling rate

$$k_B T_{\text{Doppler}} = \frac{\hbar \Gamma}{2}$$

- e.g. $T_{\text{Doppler}} = 240 \mu\text{K}$ for Na corresponding to average velocity of 1 km/h
 $T_{\text{Doppler}} = 140 \mu\text{K}$ for Rb

I.2.1.1 Laser Cooling – Optical Molasses

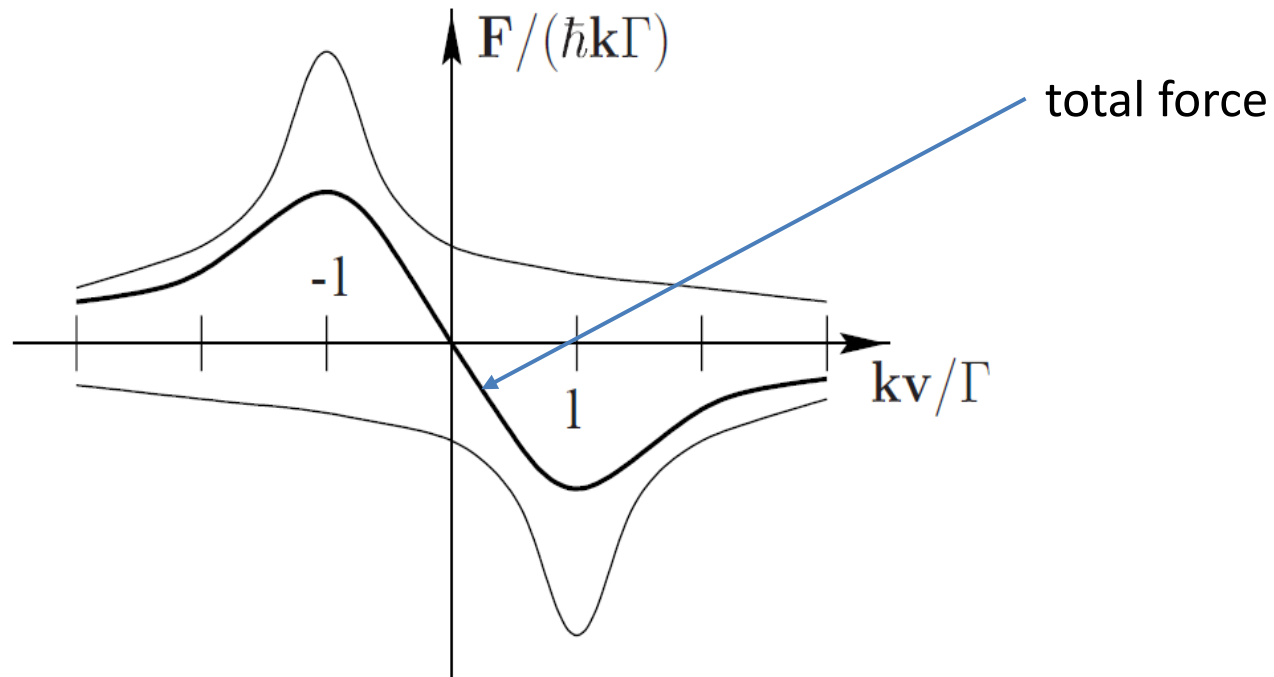
- **optical molasses** (Steven Chu, 1985)
- an **optical molasses** consists of 3 pairs of counter-propagating circularly polarized laser beams intersecting in the region where the atoms are present
- while for simple Doppler cooling the minimum temperature for Na atoms is about $250\ \mu\text{K}$, optical molasses can cool the atoms down to $40\ \mu\text{K}$, i.e. an order of magnitude colder.



I.2.1.1 Laser Cooling – Optical Molasses

- force acting by two laser with opposite directions

$$\mathbf{F} = +\hbar\mathbf{k}\frac{\Gamma}{2}\frac{(I/I_0)}{1 + \frac{I}{I_0} + \left(\frac{2(\delta_0 - \mathbf{k} \cdot \mathbf{v})}{\Gamma}\right)^2} - \hbar\mathbf{k}\frac{\Gamma}{2}\frac{(I/I_0)}{1 + \frac{I}{I_0} + \left(\frac{2(\delta_0 + \mathbf{k} \cdot \mathbf{v})}{\Gamma}\right)^2}$$





Walther
Meißner
Institut

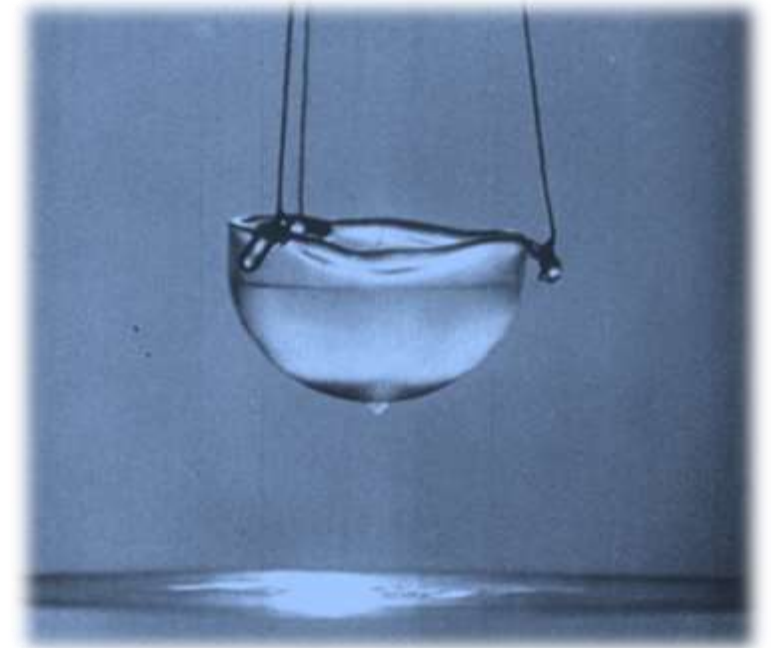
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Superconductivity and Low Temperature Physics II



Lecture No. 3

R. Gross

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Contents Part I: Quantum Liquids

Contents:

I.1 Foundations and General Properties

- I.1.1 Quantum Gases & Liquids
- I.1.2 Helium
- I.1.3 Van der Waals Bonding
- I.1.4 Zero-Point Fluctuations
- I.1.5 Helium under Pressure
- I.1.6 pT-Phase Diagram of ^4He and ^3He
- I.1.7 Characteristic Properties of ^4He and ^3He
- I.1.8 Specific Heat of ^4He and ^3He

I.2 ^4He as an Ideal Bose Gas

- I.2.1 Bose-Einstein Condensation
- I.2.2 Ideal Bose Gas
- I.2.3 Bose Gas with Interactions
- I.2.4 Bose-Einstein Condensation of ^4He

I.3 Superfluid ^4He

- I.3.1 Two-Fluid Model
- I.3.2 Experimental Observations
- I.3.3 Two-Fluid Hydrodynamics
- I.3.4 Excitation Spectrum of ^4He

I.4 Vortices

- I.4.1 Quantization of Circulation
- I.4.2 Experimental Study of Vortices

I.5 ^3He

- I.5.1 normal fluid ^3He
- I.5.2 solid ^3He and Pomeranchuk effect
- I.5.3 superfluid ^3He

I.6 ^3He / ^4He mixtures



I.2.2 Ideal Bose Gas

- consider a gas of N ($\approx 10^{22}$) bosons of mass m in a cube of volume $V = L^3$ ($\approx 1 \text{ cm}^3$), canonical ensemble (given T, V, N)

- eigen energies:

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m} \left(\frac{\pi}{L} \right)^2 n^2 \quad \text{with } n^2 = n_x^2 + n_y^2 + n_z^2$$

- $\Delta\varepsilon/k_B T \leq 10^{-14}$

→ level spacing small compared to thermal energy (e.g. He @ 10 K)
→ no influence on experimental result

- occupation probability:

$$n_k = \frac{1}{\exp\left(\frac{\varepsilon_k - \mu}{k_B T}\right) - 1}$$

Bose-Einstein distribution

μ = chemical potential

important note:

- $\mu \leq \varepsilon_k$ for all k to avoid negative occupation probabilities
- usually one sets $\varepsilon_0 = 0$ for the lowest energy state

- energy and particle number:

$$U(T, V, \mu) = \sum_k \varepsilon_k n_k$$

$$N(T, V, \mu) = \sum_k n_k$$

I.2.2 Ideal Bose Gas

- derivation of Bose-Einstein distribution (1)

- we use the single spin density of states of free particles in 3D: $D(\varepsilon) = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \sqrt{\varepsilon}, \quad N(\varepsilon) = \frac{D(\varepsilon)}{V}$

- total entropy of the particle gas: $S = k_B \ln \Omega$ Ω = number of the available states at given total energy E

- Ω is determined by the number of possibilities to distribute the N particles on the available states
 - we divide the k -space into spherical shells with the shell s containing M_s states
 - the number of possibilities to put N_s particles on M_s states is given by

$$\Omega_s = \frac{(N_s + M_s - 1)!}{N_s! (M_s - 1)!} \quad \text{(we use } M_s = D(\varepsilon) \delta \varepsilon_s \text{ for the number of available states in the shell } s)$$

- the total number Ω of possibilities is obtained by multiplying the numbers Ω_s

$$\Omega = \prod_s \Omega_s = \prod_s \frac{(N_s + M_s - 1)!}{N_s! (M_s - 1)!}$$

I.2.2 Ideal Bose Gas

- derivation of Bose-Einstein distribution (2)

- we use the Stirling approximation $\ln N! = N \ln N - N$ and for $N_s, \Omega_s \gg 1$ we obtain

$$S = k_B \ln \Omega = k_B \sum_s [(N_s + M_s) \ln(N_s + M_s) - N_s \ln N_s - M_s \ln M_s]$$

- in thermal equilibrium the particles are distributed on the shells to maximize S under the boundary conditions $N = \sum_s N_s = \text{const.}$ and $U = \sum_s \varepsilon_s N_s = \text{const.}$

- we maximize S for $N = \text{const.}$ and $U = \text{const.}$ we obtain

$$(dU = TdS + \mu dN)$$

$$\frac{\partial S}{\partial N_s} - \frac{1}{T} \frac{\partial U}{\partial N_s} + \frac{\mu}{T} \frac{\partial N}{\partial N_s} = 0 = \frac{\partial S}{\partial N_s} - k_B \beta \frac{\partial U}{\partial N_s} + k_B \beta \mu \frac{\partial N}{\partial N_s} \quad \beta = 1/k_B T$$

(Lagrangian multipliers)

- performing the differentiation we obtain

$$\ln(N_s + M_s) - \ln N_s - \beta \varepsilon_s + \beta \mu = 0 \quad \Rightarrow \quad N_s = \frac{1}{\exp(\beta(\varepsilon_s - \mu)) - 1} M_s$$

- average particle number occupying state in shell s

$$n_{\text{BE}} = \left[\exp\left(\frac{\varepsilon_s - \mu}{k_B T}\right) - 1 \right]^{-1}$$

I.2.2 Ideal Bose Gas

- how to get rid of the summation over k ?

$$\sum_k \dots \Rightarrow \int Z(k) d^3k = \frac{V}{(2\pi)^3} \int d^3k$$

$$\sum_k \dots \Rightarrow \int D(\varepsilon) d\varepsilon = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int \sqrt{\varepsilon} d\varepsilon$$

$$U(T, V, \mu) = \sum_k \varepsilon_k n_k \Rightarrow \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int \frac{\varepsilon^{3/2}}{\exp\left(\frac{\varepsilon - \mu}{k_B T}\right) - 1} d\varepsilon$$

$$N(T, V, \mu) = \sum_k n_k \Rightarrow \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int \frac{\varepsilon^{1/2}}{\exp\left(\frac{\varepsilon - \mu}{k_B T}\right) - 1} d\varepsilon$$

for free bosons with dispersion $\varepsilon_k = \frac{\hbar^2 k^2}{2m}$:

$$Z(k) = \frac{V}{(2\pi)^3}$$

$$D(\varepsilon) = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \sqrt{\varepsilon}$$

Note:

we assume that the states are not degenerate, otherwise we have to multiply with degeneracy g_k

→ we assume $g_k = 1$

crucial point: the integral is bounded!! If the particle number N is too large and the temperature too low, the equation can no longer be satisfied

I.2.2 Ideal Bose Gas

- evaluation of the equation for the particle number

$$N(T, V, \mu) = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^\infty \frac{\varepsilon^{1/2}}{\exp\left(\frac{\varepsilon - \mu}{k_B T}\right) - 1} d\varepsilon = \frac{V}{4\pi^2} \left(\frac{2mk_B T}{\hbar^2} \right)^{3/2} \int_0^\infty \frac{\gamma e^{-x}}{1 - \gamma e^{-x}} x^{1/2} dx$$

with $x = \varepsilon/k_B T = \beta\varepsilon$ and **fugacity** $\gamma = \exp\left(\frac{\mu}{k_B T}\right)$

Gamma function: $\Gamma(t) = \int_0^\infty e^{-y} y^{t-1} dy$

- to solve the integral we use the series expansion $\frac{\gamma e^{-x}}{1 - \gamma e^{-x}} = \gamma e^{-x} (1 + \gamma e^{-x} + \gamma^2 e^{-2x} + \dots) = \sum_{p=1}^\infty \gamma^p e^{-px}$
- plugging in this expression we can perform the integration by using

$$\int_0^\infty e^{-px} x^{1/2} dx \stackrel{y=px}{=} \frac{1}{p^{3/2}} \underbrace{\int_0^\infty e^{-y} y^{1/2} dy}_{\Gamma(3/2)} = \frac{1}{p^{3/2}} \frac{\sqrt{\pi}}{2} \Rightarrow \int_0^\infty \frac{\gamma e^{-x}}{1 - \gamma e^{-x}} x^{1/2} dx = \frac{\sqrt{\pi}}{2} \underbrace{\sum_{p=1}^\infty \frac{\gamma^p}{p^{3/2}}}_{g_{3/2}(\gamma)}$$

Gamma function: $\Gamma(t) = \int_0^\infty e^{-y} y^{t-1} dy$

$$\Rightarrow N(T, V, \mu) = V \underbrace{\left(\frac{mk_B T}{2\pi \hbar^2} \right)^{3/2}}_{1/\lambda_T^3} g_{3/2}(\gamma) = \frac{V}{\lambda_T^3} g_{3/2}(\gamma)$$

$$\Rightarrow n(T, \mu) \lambda_T^3 = g_{3/2}(\gamma)$$

I.2.2 Ideal Bose Gas

- analysis of the function $g_{3/2}(\gamma)$

$$g_{3/2}(\gamma) = \sum_{p=1}^{\infty} \frac{\gamma^p}{p^{3/2}}$$

- $|\gamma| < 1$: the series converges
- $|\gamma| > 1$: the series diverges
- $|\gamma| = 1$: the series converges and has the value $g_{3/2}(1)$

$$g_{3/2}(1) = \sum_{p=1}^{\infty} \frac{1}{p^{3/2}} = \zeta\left(\frac{3}{2}\right) = 2.612$$

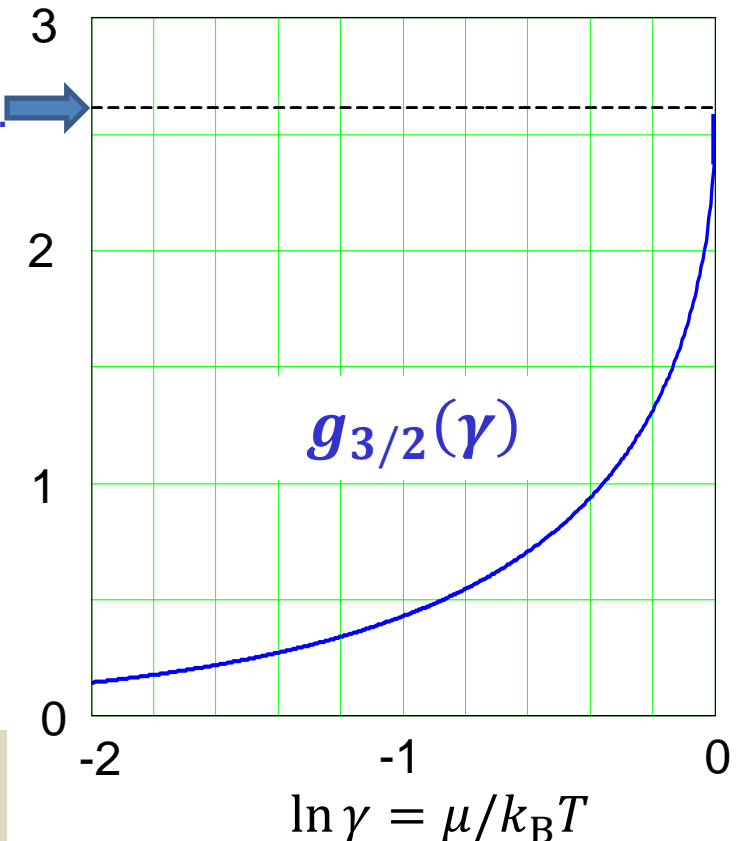
$$\zeta(3/2) = 2.61237 \dots$$

$$\zeta(s) = \sum_{p=1}^{\infty} \frac{1}{p^s}$$

(Rieman Zeta function)

$$\frac{dg_{3/2}(z)}{dz} = \frac{1}{z} \sum_{p=1}^{\infty} \frac{\gamma^p}{p^{1/2}}$$

slope diverges for $\gamma = 1$



important fact: $g_{3/2}(\gamma)$ has maximum value 2.61 for $\gamma \rightarrow 1$, resp. $\ln \gamma \rightarrow 0$
particle number N is bounded to maximum value !!

I.2.2 Ideal Bose Gas

- what happens with decreasing T ?

– λ_T increases $\propto 1/\sqrt{T} \rightarrow g_{3/2}(\gamma)$ has to increase to keep $n(T, \mu) = \frac{1}{\lambda_T^3} g_{3/2}(\gamma)$ constant

- problem:**

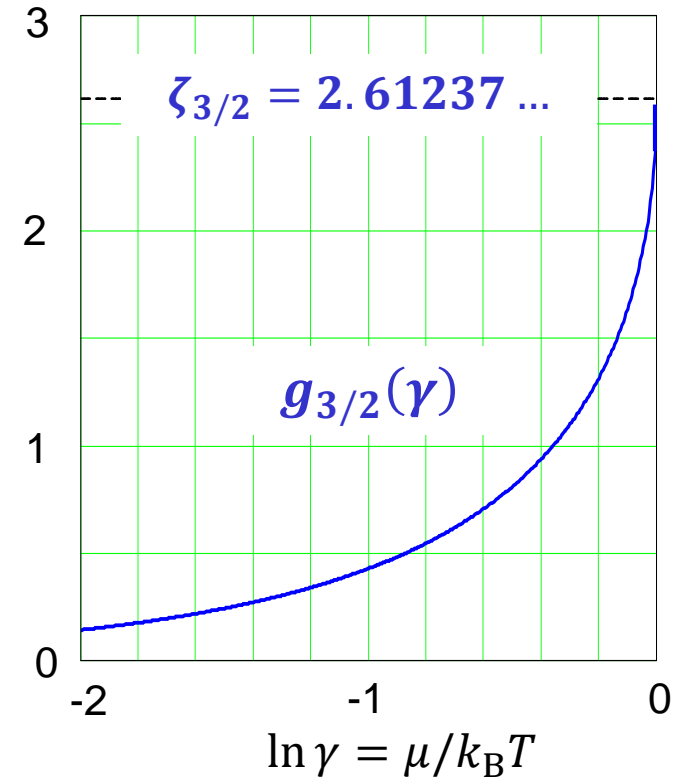
$g_{3/2}(\gamma)$ has maximum value $g_{3/2}(1) = \zeta(3/2) = 2.61$ for $\gamma = 1$, resp. $\mu = 0$,

$\rightarrow g_{3/2}(\gamma = 1)$ is related to **maximum value** $\lambda_T = \lambda_{T,\text{BEC}}$,
resp. to **minimum temperature** T_{BEC}

$$\lambda_{T,\text{BEC}}^3 \cdot n = g_{3/2}(\gamma = 1) = \zeta(3/2) = 2.61237 \dots$$

defines the characteristic temperature T_{BEC} :

for $T < T_{\text{BEC}}$ it is not possible to satisfy the equation $n(T, \mu) = \frac{1}{\lambda_T^3} g_{3/2}(\gamma)$



- with $\lambda_{T,\text{BEC}} = h/\sqrt{2\pi m k_B T_{\text{BEC}}}$, we obtain:

$$n = 2.612 \left(\frac{m k_B T_{\text{BEC}}}{2\pi \hbar^2} \right)^{3/2} \Rightarrow T_{\text{BEC}} = \frac{2\pi \hbar^2}{m k_B} \left(\frac{n}{2.612} \right)^{2/3}$$

Bose-Einstein condensation temperature

$\rightarrow \lambda_{T,\text{BEC}}^3 \cdot n \simeq O(1) \rightarrow$ *critierion for the importance of quantum effects or degeneracy*

I.2.2 Ideal Bose Gas

- how to solve the problem ?

- physical solution of the problem that we cannot satisfy the equation $n(T, \mu) = \frac{1}{\lambda_T^3} g_{3/2}(\gamma)$ below T_{BEC}

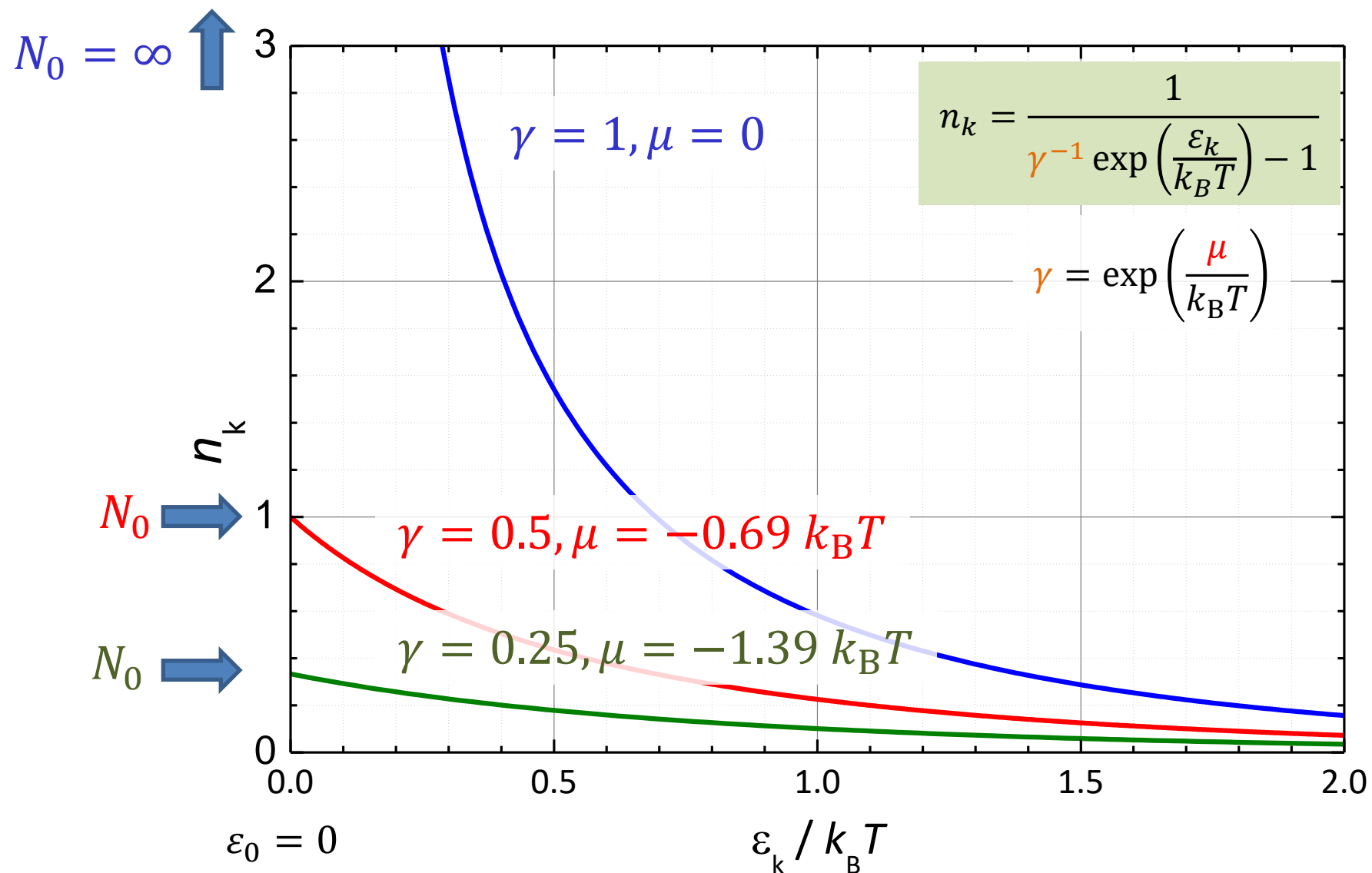
- we have to put a significant number N_0 of the particles in the ground state $k = 0$

- this is in agreement with the Bose-Einstein distribution

- setting $\varepsilon(k = 0) = \varepsilon_0 = 0$, the number of bosons in the ground state is

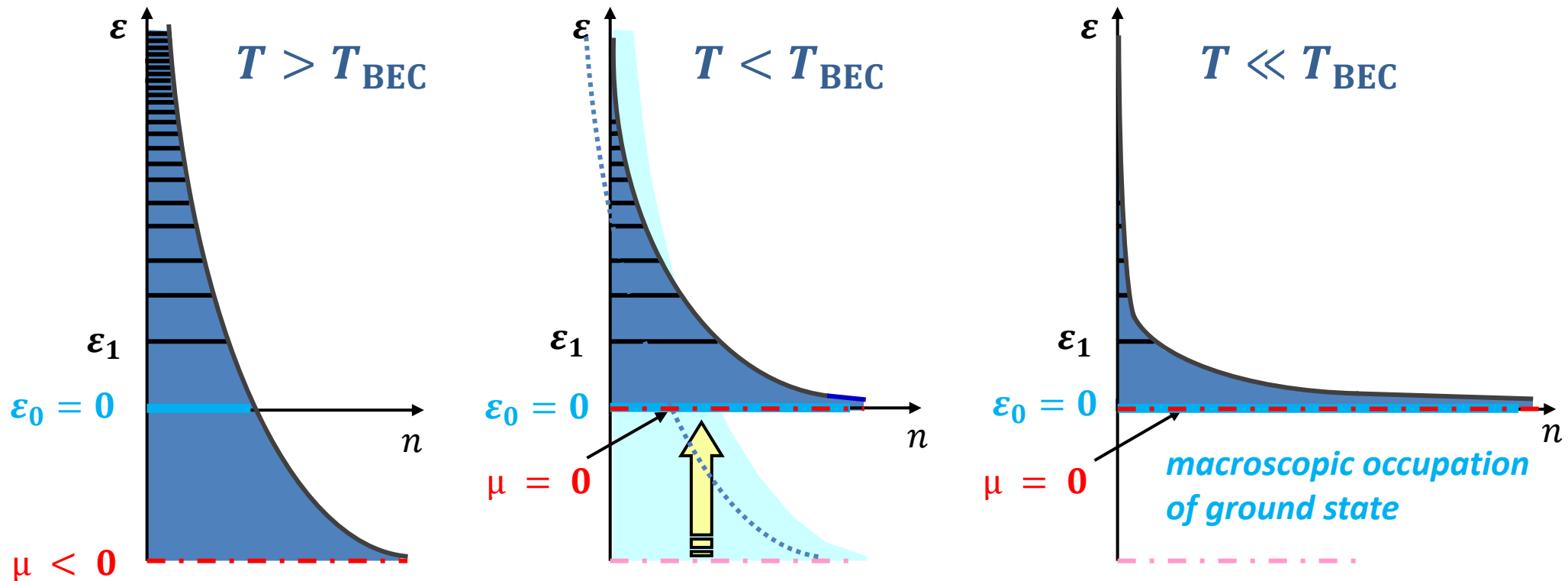
$$N_0(T, \mu) = \frac{1}{\exp\left(\frac{0 - \mu}{k_B T}\right) - 1} \simeq \frac{1}{1 - \frac{\mu}{k_B T} - 1} = \frac{k_B T}{|\mu|} \rightarrow \infty \text{ for } |\mu| = |\varepsilon_0 - \mu| \rightarrow 0$$

I.2.2 Ideal Bose Gas



if $|\varepsilon_0 - \mu| = |0 - \mu| = |\mu| \rightarrow 0$, the occupation number $N_0 = n(k=0)$ of the ground state $\varepsilon_0 = 0$ strongly increases

I.2.2 Ideal Bose Gas



- calculation of $N_0(T) = N(\epsilon_0, T)$ for $(\epsilon_0 - \mu) = (0 - \mu) = -\mu \rightarrow 0$:

$$N_0(T, \mu) = \frac{1}{\exp\left(\frac{\epsilon_0 - \mu}{k_B T}\right) - 1} \simeq \frac{1}{1 + \frac{\epsilon_0 - \mu}{k_B T} - 1} = \frac{k_B T}{\epsilon_0 - \mu} \rightarrow \infty \text{ for } \epsilon_0 - \mu \rightarrow 0$$

→ $N_0(T, \mu)$ grows dramatically for $(\epsilon_0 - \mu) \rightarrow 0$, that is, for $\mu \rightarrow 0$

Note:

$$\epsilon_0 - \mu = k_B T \ln\left(1 + \frac{1}{N_0}\right) \simeq \frac{k_B T}{N_0}$$

→ difference between ground state and chemical potential is much smaller than $k_B T$

I.2.2 Ideal Bose Gas

- detailed analysis of the chemical potential

$$n(T, \mu) = \frac{N(T, V, \mu)}{V} = \left(\frac{mk_B T}{2\pi\hbar^2} \right)^{3/2} g_{3/2}(\gamma) = \frac{1}{\lambda_T^3} g_{3/2}(\gamma) \Rightarrow g_{3/2}(\gamma) = \left(\frac{2\pi\hbar^2}{mk_B T} \right)^{3/2} n(T, \mu)$$

- for high temperatures T and/or small particle density n we can use the approximation $g_{3/2}(\gamma) \simeq \gamma + \dots \simeq e^{\beta\mu} + \dots$

$$\Rightarrow e^{\beta\mu} = \left(\frac{2\pi\hbar^2}{mk_B T} \right)^{3/2} n(T, \mu) \Rightarrow \mu \simeq -\frac{3}{2} k_B T \ln \left(\frac{mk_B T}{2\pi\hbar^2 n^{2/3}} \right)$$

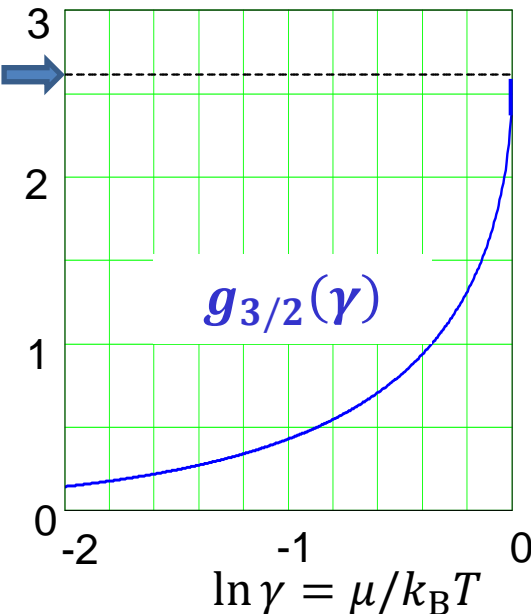
$$\zeta_{3/2} = 2.61237 \dots$$

- for $T \rightarrow \infty$:

$\lambda_T \rightarrow 0, \mu \rightarrow -\infty, \gamma \rightarrow 0, \ln \gamma \rightarrow -\infty$ and therefore $g_{3/2}(\gamma) \rightarrow 0$

- for decreasing T :

λ_T increases, $\mu/k_B T$ increases $\rightarrow 0, \gamma$ increases $\rightarrow 1$ and therefore $g_{3/2}(\gamma) \rightarrow 2.61$



I.2.2 Ideal Bose Gas

- what happens if we decrease T to values below T_{BEC} ?

- for $T < T_{\text{BEC}}$, there is no solution of equation $n(T, \mu) = \frac{1}{\lambda_T^3} g_{3/2}(\gamma)$ as $g_{3/2}(\gamma)$ has maximum value $g_{3/2}(1)$ for $\gamma \rightarrow 1$

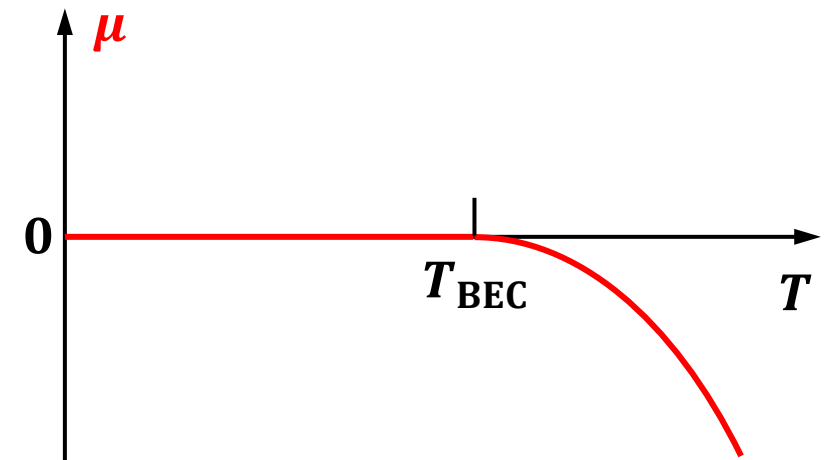
→ for $T < T_{\text{BEC}}$, the particle number N_0 in the ground quantum state ($\varepsilon_0 = 0$) must become large (first recognized by Einstein)

occupation of the ground state $\varepsilon_0 = 0$:
$$N_0 = \frac{1}{\exp(-\mu/k_B T) - 1} \Rightarrow \mu = -k_B T \ln \left(1 + \frac{1}{N_0} \right) \simeq -k_B T \frac{1}{N_0}$$

- from the total particle number N a macroscopically large number N_0 has to be placed in ground state ε_0
- macroscopically large means that N_0 is proportional to system volume V
- if the fraction of particles in the ground state becomes finite at $T = T_{\text{BEC}}$, then in the thermodynamic limit $V \rightarrow \infty$ the particle number $N_0 \rightarrow \infty$ and hence the chemical potential $\mu \rightarrow 0$

→ $\mu = 0$ for $T \leq T_{\text{BEC}}$

as μ corresponds to the average energy required to add a particle to the systems, the result is evident: we are adding particles only to the ground state $\varepsilon_0 = 0$



I.2.2 Ideal Bose Gas

- for $T \leq T_{\text{BEC}}$: how are the N particles distributed among the available states ?

$$- N = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^\infty \frac{\sqrt{\varepsilon}}{\exp\left(\frac{\varepsilon - \mu}{k_B T}\right) - 1} d\varepsilon \text{ yields } \mu \rightarrow 0 \text{ for } T \rightarrow T_{\text{BEC}}$$

$$- \text{for } \mu \rightarrow 0 \text{ (} T \rightarrow T_{\text{BEC}} \text{), we obtain } N_0 = \frac{1}{\exp\left(\frac{\varepsilon_0 - \mu}{k_B T}\right) - 1} \simeq \frac{k_B T}{|\mu|} \rightarrow \infty$$

$$\varepsilon_0 = \frac{\hbar^2}{2m} \left(\frac{\pi}{L} \right)^2 \propto \frac{1}{V^{2/3}} \propto \frac{1}{N^{2/3}} \rightarrow 0 \text{ for } N \rightarrow \infty \text{ (thermodynamic limit)}$$

below T_{BEC} : *macroscopic occupation $N_0 \gg 1$ of the ground state is required*

- key result:
 - at $T \leq T_{\text{BEC}}$ a finite fraction of all particles occupies the ground state
 - this phenomenon is called *Bose-Einstein condensation*
 - the name „*condensation*“ is used since in analogy to the gaseous-liquid transition *the condensed particles do no longer contribute to the pressure* (see discussion below)

I.2.2 Ideal Bose Gas

- for $T \leq T_{\text{BEC}}$: occupation of the first excited state

for $\mu \rightarrow 0$ and $T \sim 1$ K: level spacing for 10^{22} atoms in a box with $L = 1$ cm: $\Delta\varepsilon/k_B T = \varepsilon_1/k_B T \leq 10^{-14}$

$$\Rightarrow n_1(T) = \frac{1}{1 + \exp\left(\frac{\varepsilon_1 - \mu}{k_B T}\right) - 1} \simeq \frac{1}{1 + \frac{\varepsilon_1 - \mu}{k_B T} - 1} \simeq \frac{k_B T}{\varepsilon_1 - \mu} \approx 10^{14}$$

n_1 and all $n_{k>1}$ are small compared to N_0 and the total particle number $N = 10^{22}$

(justifies why only the ground state was singled out in the calculation of T_{BEC})

- more detailed calculation: $n_1/N_0 \propto N^{-1/3}$, i.e. $n_1/N_0 \rightarrow 0$ for $N \rightarrow \infty$

$$\frac{n_1}{N_0} = \frac{k_B T}{\varepsilon_1 - \mu} \cdot \frac{\varepsilon_0 - \mu}{k_B T} = -\frac{\mu}{\varepsilon_1 - \mu} \simeq -\frac{\mu}{\varepsilon_1} \simeq \frac{k_B T}{N_0 \varepsilon_1} \quad \text{we use } N_0(T, \mu) \simeq \frac{k_B T}{\varepsilon_0 - \mu} = \frac{k_B T}{-\mu}$$

for $T < T_{\text{BEC}}/2$ we can use the crude approximations $N_0 \simeq N$, $T \simeq T_{\text{BEC}} \propto N^{2/3}$:

$$\frac{n_1}{N_0} \propto \frac{1}{N^{1/3}} \propto \frac{1}{V^{1/3}} \quad \Rightarrow \text{already at } T \simeq T_{\text{BEC}}/2, \text{ we have } N_0 \gg n_1, n_{k>1}$$

- for $T \rightarrow 0$: $\varepsilon_{k>0}/k_B T \rightarrow \infty$ and hence $n_{k>0} \rightarrow 0$, $\mu/k_B T = 1/N_0 \simeq 0$: **all bosons occupy ground state**

I.2.2 Ideal Bose Gas

- for $T < T_{\text{BEC}}$ we have to treat the ground state ε_0 separately

$$N = \sum_{\mathbf{k}} \frac{1}{\exp\left(\frac{\varepsilon_{\mathbf{k}} - \mu}{k_B T}\right) - 1} = \underbrace{N_0}_{\text{particles in ground state}} + \underbrace{N_{\text{ex}}}_{\text{particles in excited states}} = N_0 + \sum_{\mathbf{k} \neq 0} \left[\exp\left(\frac{\varepsilon_{\mathbf{k}}}{k_B T}\right) - 1 \right]^{-1} = N_0 + \int_0^{\infty} D(\varepsilon) n(\varepsilon) d\varepsilon$$

- we obtain for **free bosons**:

$$N_{\text{ex}} = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^{\infty} \frac{\varepsilon^{1/2}}{\exp\left(\frac{\varepsilon - \mu}{k_B T}\right) - 1} d\varepsilon = \frac{V}{4\pi^2} \left(\frac{2mk_B T}{\hbar^2}\right)^{3/2} \underbrace{\int_0^{\infty} \frac{\gamma e^{-x}}{1 - \gamma e^{-x}} x^{1/2} dx}_{=\Gamma(3/2) \zeta(3/2)}$$

with $x = \beta\varepsilon$ and
fugacity $\gamma = \exp\left(\frac{\mu}{k_B T}\right) = 1$

$$\Rightarrow n = n_0 + n_{\text{ex}} = n_0 + 2.612 \left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2} \Rightarrow \frac{n_0}{n} = 1 - \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2}$$

n_0 : condensate density
 n_{ex} : normal fluid density

$$T_{\text{BEC}} = \frac{2\pi\hbar^2}{mk_B} \left(\frac{n}{2.612}\right)^{2/3} \Rightarrow n = 2.612 \left(\frac{2\pi\hbar^2}{mk_B T_{\text{BEC}}}\right)^{3/2}$$

I.2.2 Ideal Bose Gas

- Summary: temperature dependence of N_0 and N_{ex}

we assume constant particle number:
 $N = N_0(T) + N_{ex}(T) = \text{const.}$

- $T > T_{\text{BEC}}$: $N_0 = O(1) \rightarrow$ can be neglected

$$N = N_{ex} = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^\infty \frac{\sqrt{\varepsilon}}{\exp\left(\frac{\varepsilon - \mu}{k_B T}\right) - 1} d\varepsilon \quad \text{is valid}$$

- $T = T_{\text{BEC}}$: $\mu \rightarrow 0$, $\lambda_{T,\text{BEC}}^3 \cdot n = 2.612 \rightarrow N = N_{ex} = 2.612 \cdot \frac{V}{\lambda_{T,\text{BEC}}^3}$

- $T \leq T_{\text{BEC}}$: $\mu = 0$, $N = N_0 + N_{ex} = N_0 + 2.612 \cdot \frac{V}{\lambda_T^3} = N_0 + 2.612 \cdot \frac{V}{\lambda_T^3} \cdot \frac{\lambda_{T,\text{BEC}}^3}{\lambda_{T,\text{BEC}}^3}$

$$N = N_0 + 2.612 \cdot \frac{V}{\lambda_{T,\text{BEC}}^3} \cdot \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} = N_0 + N \cdot \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2}$$

$$\rightarrow \frac{N_0}{N} = 1 - \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} \quad \frac{N_{ex}}{N} = \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2}$$

I.2.2 Ideal Bose Gas

- normal fluid density ($T < T_{\text{BEC}}$)

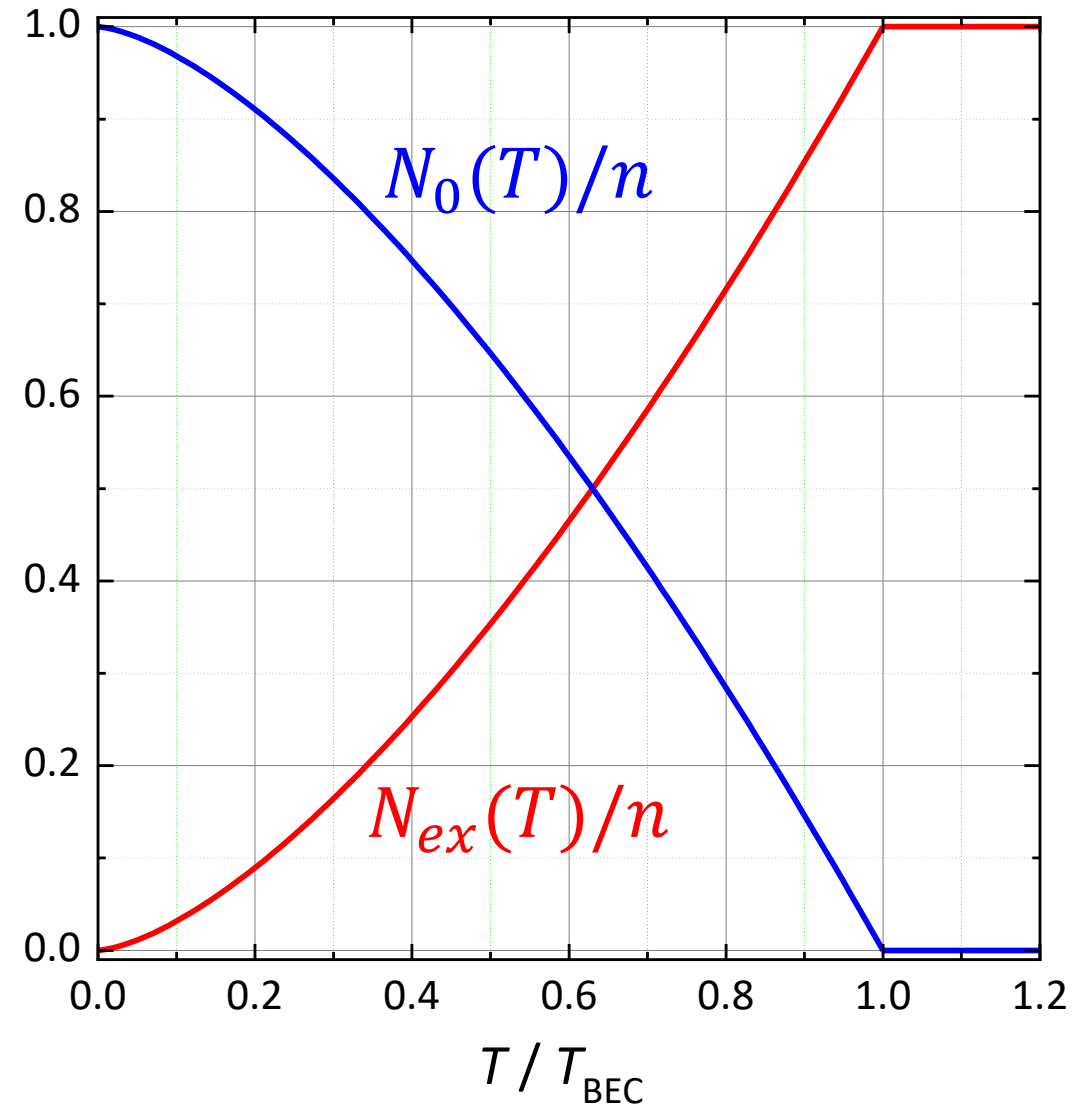
$$\frac{N_{\text{ex}}}{N} = \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2}$$

- superfluid or condensate density

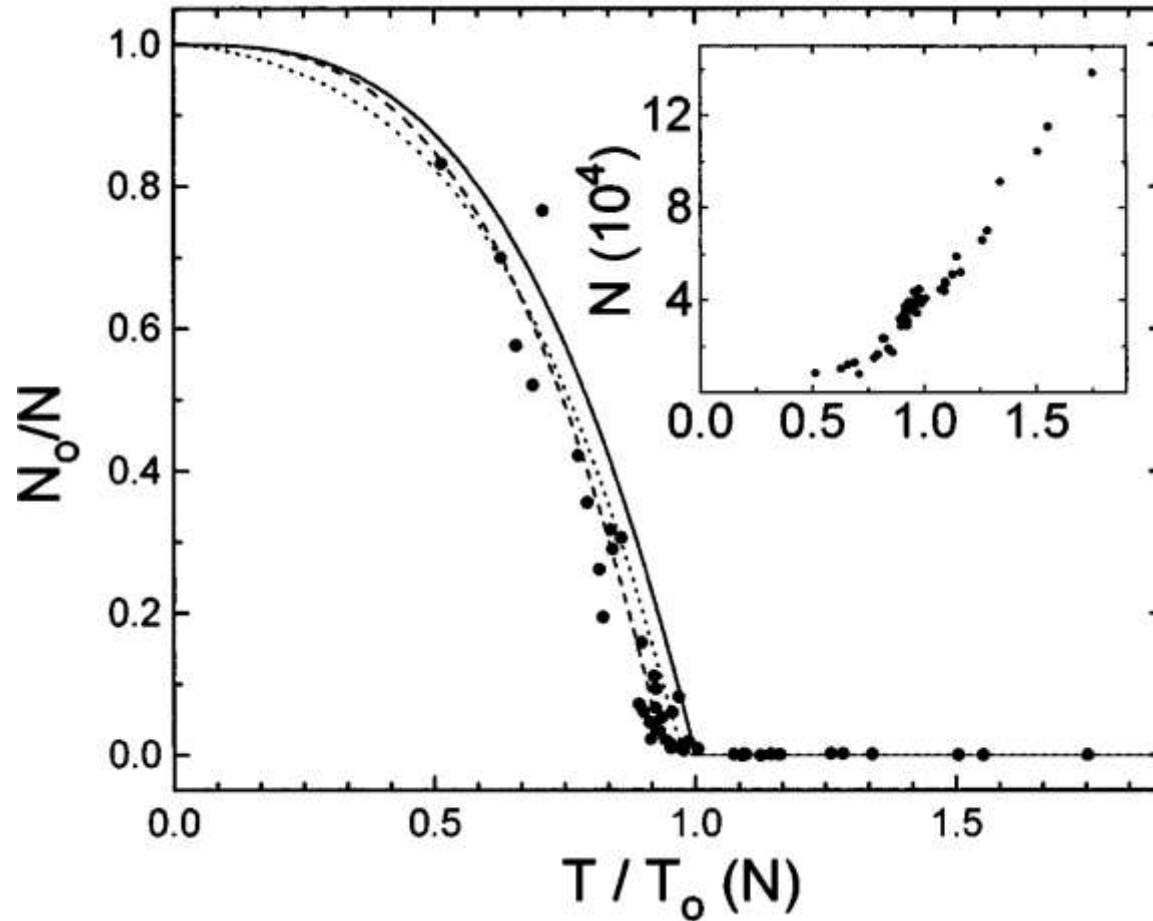
$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2}$$

$$T_{\text{BEC}} = \frac{2\pi\hbar^2}{mk_B} \left(\frac{n}{2.612} \right)^{2/3}$$

$$\lambda_T^3 = \left(\frac{2\pi\hbar^2}{mk_B T} \right)^{3/2}$$



I.2.2 Ideal Bose Gas



Total number N (inset) and ground-state fraction N_0/N as a function of T/T_{BEC} .

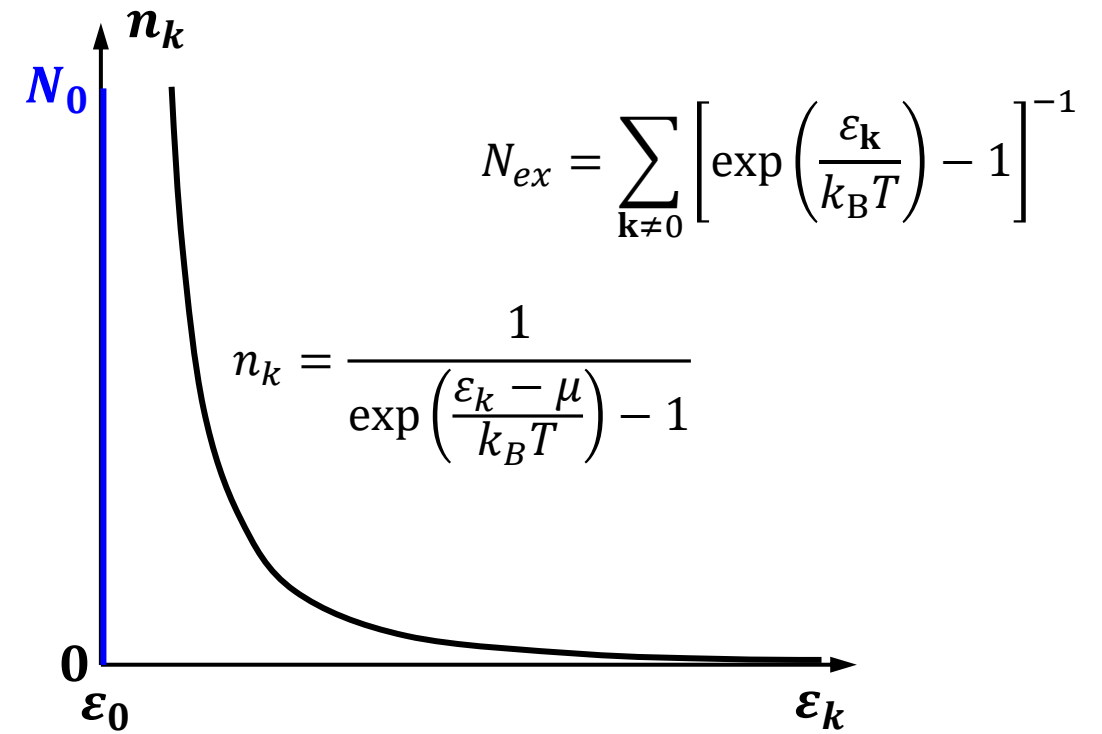
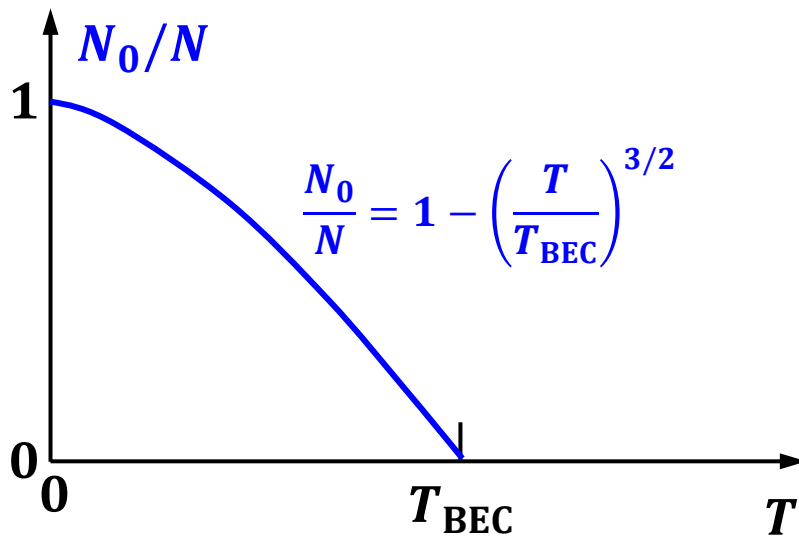
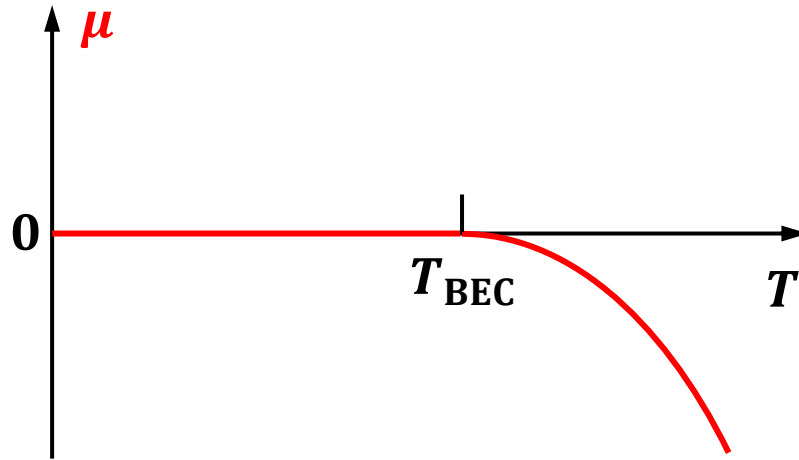
The solid (dotted) line shows the infinite (finite) N theory curves. At the transition, the cloud consists of 40 000 ^{87}Rb atoms at 280 nK.

Bose-Einstein Condensation in a Dilute Gas: Measurement of Energy and Ground-State Occupation

J. R. Ensher, D. S. Jin, M. R. Matthews, C. E. Wieman, and E. A. Cornell

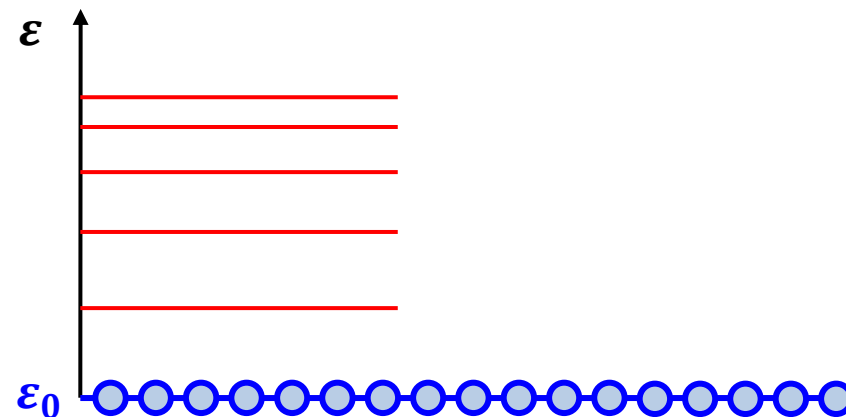
Phys. Rev. Lett. 77, 4984 (1996)

I.2.2 Ideal Bose Gas



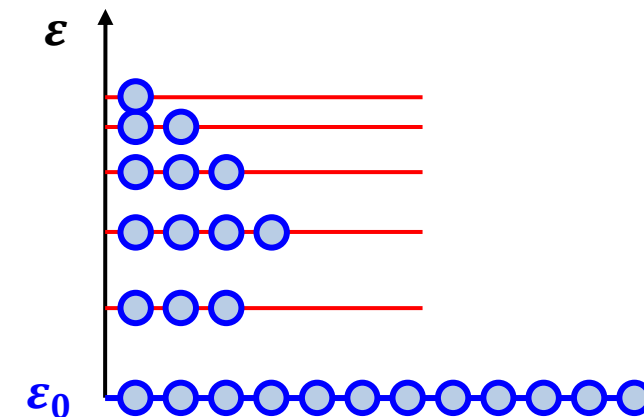
I.2.2 Ideal Bose Gas

- Bose-Einstein condensation can be viewed as condensation of gas in momentum space
 → *all atoms have the same momentum*
- Bose-Einstein condensation corresponds to order-disorder phase transition (first order phase transition)



$$T = 0$$

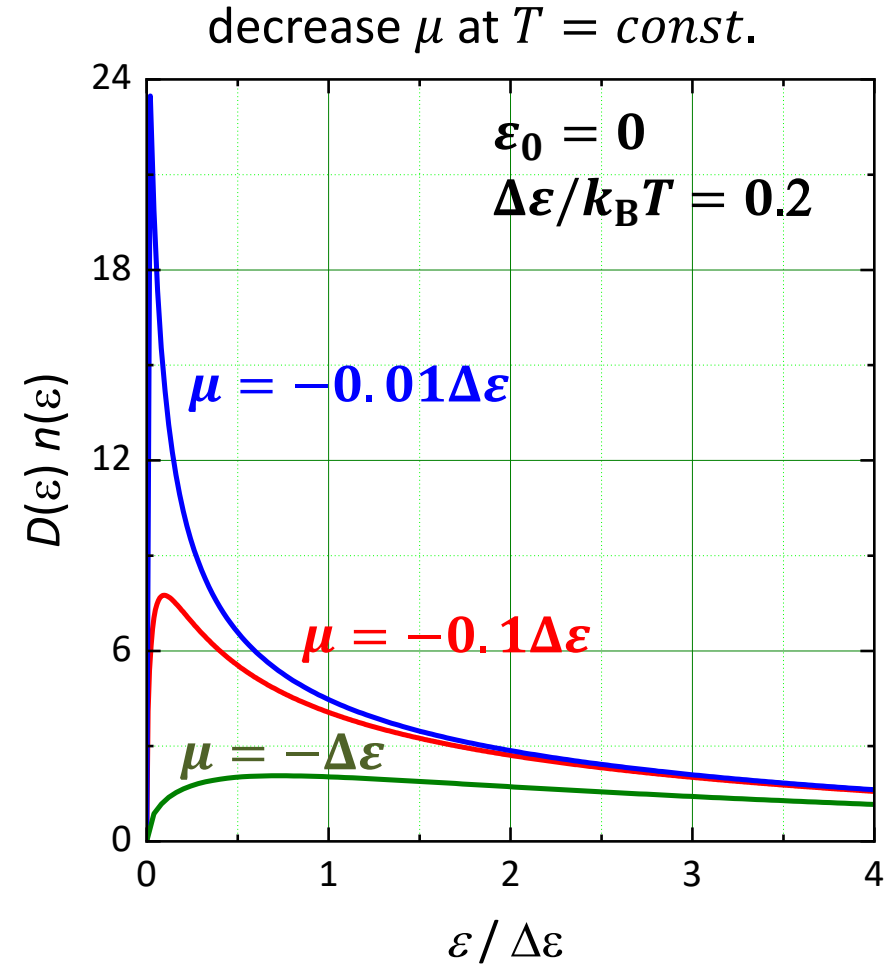
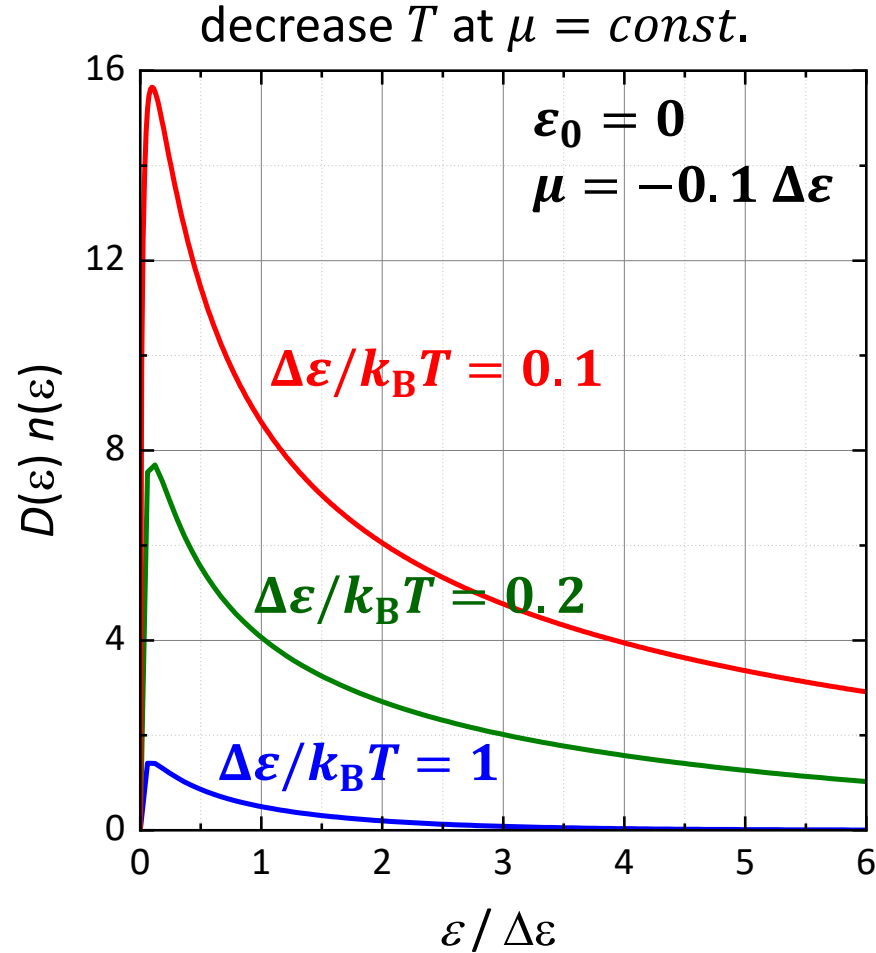
only ground state
is occupied



$$0 < T < T_{\text{BEC}}$$

still macroscopic
occupation of the
ground state

I.2.2 Ideal Bose Gas

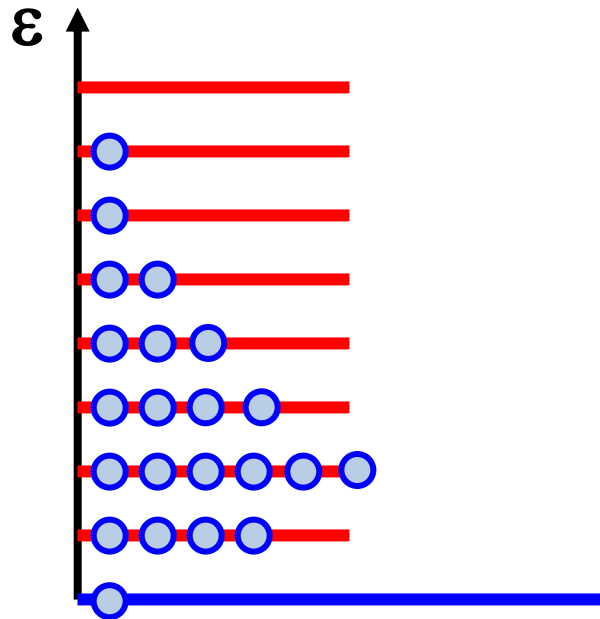


we defined the Bose-Einstein temperature T_{BEC} as the temperature, where $N_{\text{ex}}/N = 1$:

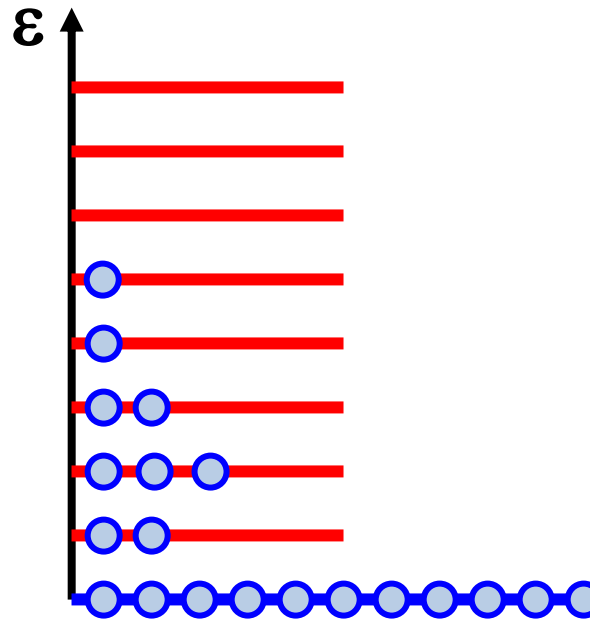
$$T_{\text{BEC}} = \frac{2\pi\hbar^2}{mk_B} \left(\frac{n}{2.612} \right)^{2/3}$$

I.2.2 Ideal Bose Gas

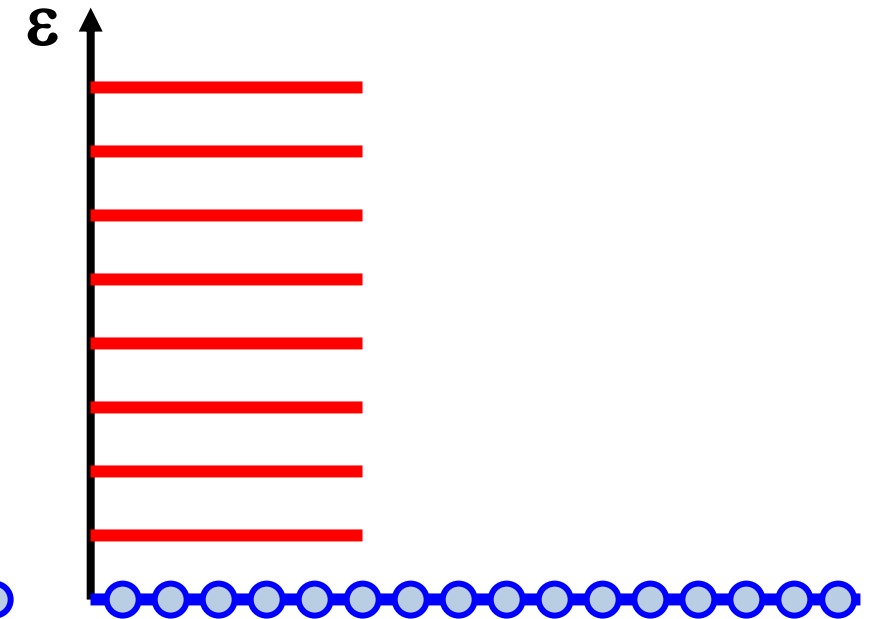
$$T > T_{\text{BEC}}$$



$$0 < T < T_{\text{BEC}}$$



$$T = 0$$



— excitations

— condensate

I.2.2 Ideal Bose Gas

- thermodynamic properties of ideal Bose gas: inner energy U

free bosons:

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m}$$

$$D(\varepsilon) = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \sqrt{\varepsilon}$$

$$U = \sum_k \varepsilon_k n_k \Rightarrow U = \int D(\varepsilon) \varepsilon d\varepsilon = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int \frac{\varepsilon^{3/2}}{\exp\left(\frac{\varepsilon - \mu}{k_B T}\right) - 1} d\varepsilon$$

$$U = \frac{V}{4\pi^2} k_B T \left(\frac{2mk_B T}{\hbar^2} \right)^{3/2} \int_0^\infty \frac{\gamma e^{-x}}{1 - \gamma e^{-x}} x^{3/2} dx$$

with $x = \varepsilon/k_B T = \beta\varepsilon$ and **fugacity** $\gamma = \exp\left(\frac{\mu}{k_B T}\right)$

we use again $\frac{\gamma e^{-x}}{1 - \gamma e^{-x}} = \gamma e^{-x} (1 + \gamma e^{-x} + \gamma^2 e^{-2x} + \dots) = \sum_{p=1}^\infty \gamma^p e^{-px}$ and $\int_0^\infty e^{-px} x^{3/2} dx = \frac{1}{p^{5/2}} \underbrace{\int_0^\infty e^{-y} y^{3/2} dy}_{\Gamma(t)=\int_0^\infty e^{-y} y^{t-1} dy} = \frac{1}{p^{5/2}} \underbrace{\frac{3\sqrt{\pi}}{4}}_{\Gamma(5/2)}$

$$\Rightarrow \int_0^\infty \frac{\gamma e^{-x}}{1 - \gamma e^{-x}} x^{3/2} dx = \underbrace{\frac{3\sqrt{\pi}}{4}}_{\Gamma(5/2)} \underbrace{\sum_{p=1}^\infty \frac{\gamma^p}{p^{5/2}}}_{g_{5/2}(\gamma)} = \Gamma(5/2) g_{5/2}(\gamma) \Rightarrow U = V \underbrace{\frac{3}{2} k_B T \left(\frac{mk_B T}{2\pi\hbar^2} \right)^{3/2}}_{1/\lambda_T^3} g_{5/2}(\gamma)$$

with $V = N\lambda_T^3 / g_{3/2}(\gamma)$, resp. $V = N\lambda_{T,\text{BEC}}^3 / g_{3/2}(1)$ we obtain

$$U = \frac{3}{2} N k_B T \frac{g_{5/2}(\gamma)}{g_{3/2}(\gamma)}$$

$$U = \frac{3}{2} N k_B T \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} \frac{g_{5/2}(\gamma)}{g_{3/2}(1)}$$

I.2.2 Ideal Bose Gas

- thermodynamic properties of ideal Bose gas: inner energy U

i. $T > T_{\text{BEC}}$:
$$U(T) = \frac{3}{2} N k_B T \frac{g_{5/2}(\gamma)}{g_{3/2}(\gamma)}$$

- ii. $T \gg T_{\text{BEC}}$: it is sufficient to use only the first term in the series evolution of g :

$$g_{5/2}(\gamma) = \sum_{p=1}^{\infty} \frac{\gamma^p}{p^{5/2}} \simeq \gamma; \quad g_{3/2}(\gamma) = \sum_{p=1}^{\infty} \frac{\gamma^p}{p^{3/2}} \simeq \gamma \quad \text{since } \gamma \rightarrow 1$$

$$U(T) = \frac{3}{2} N k_B T$$

result for gas of classical non-interacting particles
(quantum statistics does not play any role at high T)

iii. $T < T_{\text{BEC}}$:
$$U(T) = \frac{3}{2} N k_B T \frac{g_{5/2}(1)}{g_{3/2}(1)}$$

$$g_{5/2}(1) = \zeta\left(\frac{5}{2}\right) = 1.341, \quad g_{3/2}(1) = \zeta\left(\frac{3}{2}\right) = 2.612$$

$$U(T) = \frac{3}{2} N_{\text{ex}} k_B T \frac{g_{5/2}(1)}{g_{3/2}(1)} = \frac{3}{2} N \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} k_B T \frac{g_{5/2}(1)}{g_{3/2}(1)} = 0.770 N \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} k_B T$$

we use $\frac{N_0}{N} = 1 - \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} \Rightarrow N_{\text{ex}} = N - N_0 = N \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2}$

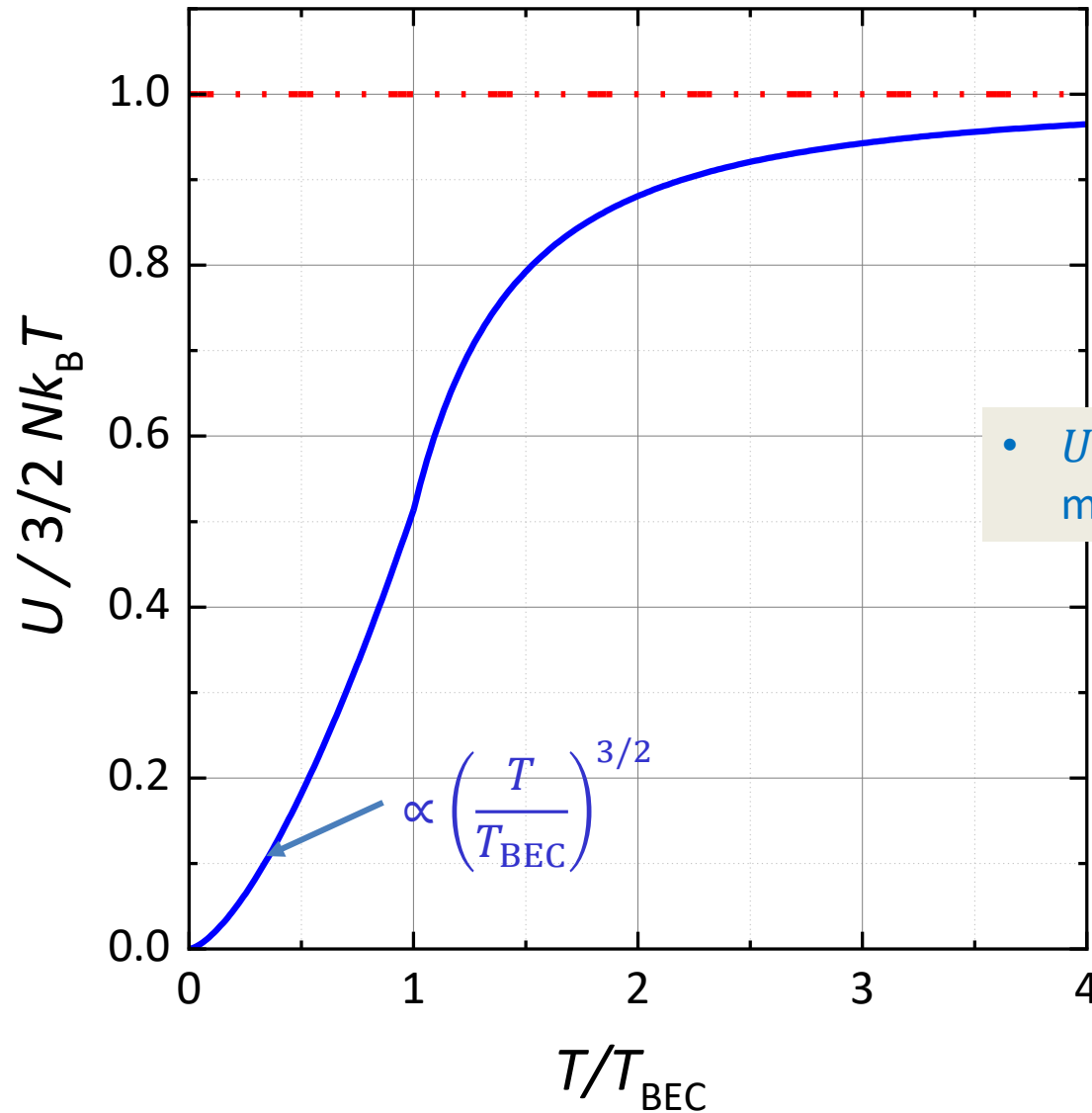
particles in ground state $\varepsilon_0 = 0$ do not contribute!

α	$\Gamma(\alpha)$	$\zeta_\alpha(\gamma = 1)$
1	1	∞
1.5	$\sqrt{\pi}/2 = 0.886$	2.612
2	1	$\pi^2/6 = 1.645$
2.5	$3\sqrt{\pi}/4 = 1.329$	1.341
3	2	1.202
3.5	$15\sqrt{\pi}/8 = 3.323$	1.127
4	6	$\pi^4/90 = 1.082$

Γ = Gamma function

ζ = Riemann Zeta function

I.2.2 Ideal Bose Gas



$$U(T) = \frac{3}{2} N k_B T \quad \text{classical result}$$

- $U(T)$ decreases towards lower T since more and more particles condense into the ground state ε_0

I.2.2 Ideal Bose Gas

- thermodynamic properties of ideal Bose gas: specific heat: $C_V(T) = \left(\frac{\partial U}{\partial T} \right)_{V,N}$

i. $T > T_{\text{BEC}}$: $\mu < 0, \gamma < 1$, $g_{5/2}(\gamma)$ and $g_{3/2}(\gamma)$ are functions of T and $n = N/V$

with $V = N\lambda_T^3 / g_{3/2}(\gamma)$, resp. $V = N\lambda_{T,\text{BEC}}^3 / g_{3/2}(1)$ we obtain

$$U(T) = \frac{3}{2} N k_B T \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} \frac{g_{5/2}(\gamma)}{g_{3/2}(1)} = \frac{3}{2} N k_B T \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} \frac{g_{5/2}(\gamma)}{\zeta(3/2)}$$

$$C_V(T) = \left(\frac{\partial U}{\partial T} \right)_{V,N} = \frac{15}{4} N k_B \frac{g_{5/2}(\gamma)}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} + \frac{3}{2} N k_B \frac{T}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} \underbrace{\frac{\partial g_{5/2}(\gamma)}{\partial \gamma}}_{\gamma^{-1} g_{3/2}(\gamma)} \left(\frac{\partial \gamma}{\partial T} \right)_N$$

to derive $\left(\frac{\partial \gamma}{\partial T} \right)_N$ we use $\lambda_T^3 \cdot \frac{N}{V} = \lambda_T^3 \cdot n = g_{3/2}(\gamma) \Rightarrow 1 = \frac{g_{3/2}(\gamma)}{\lambda_T^3 \cdot n} = \frac{g_{3/2}(\gamma)}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2}$

temperature derivative yields: $0 = \frac{3}{2T} \frac{g_{3/2}(\gamma)}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} + \frac{1}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} \underbrace{\frac{\partial g_{3/2}(\gamma)}{\partial \gamma}}_{\gamma^{-1} g_{1/2}(\gamma)} \left(\frac{\partial \gamma}{\partial T} \right)_N = \frac{3}{2T} g_{3/2}(\gamma) + \gamma^{-1} g_{1/2}(\gamma) \left(\frac{\partial \gamma}{\partial T} \right)_N$

$$\Rightarrow \left(\frac{\partial \gamma}{\partial T} \right)_N = - \frac{3\gamma}{2T} \frac{g_{3/2}(\gamma)}{g_{1/2}(\gamma)}$$

$$U(T) = \frac{3}{2} N k_B T \frac{g_{5/2}(\gamma)}{g_{3/2}(\gamma)}$$

$$g_{3/2}(1) = \zeta(3/2)$$

I.2.2 Ideal Bose Gas

- thermodynamic properties of ideal Bose gas: specific heat: $C_V(T) = \left(\frac{\partial U}{\partial T} \right)_{V,N}$

i. $T > T_{\text{BEC}}$:

$$C_V(T) = \left(\frac{\partial U}{\partial T} \right)_{V,N} = \frac{15}{4} N k_B \frac{g_{5/2}(\gamma)}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} + \frac{3}{2} N k_B \frac{T}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} \underbrace{\frac{\partial g_{5/2}(\gamma)}{\partial \gamma}}_{\gamma^{-1} g_{3/2}(\gamma)} \left(\frac{\partial \gamma}{\partial T} \right)_N$$

$$\left(\frac{\partial \gamma}{\partial T} \right)_N = - \frac{3\gamma}{2T} \frac{g_{3/2}(\gamma)}{g_{1/2}(\gamma)}$$

$$C_V(T) = \frac{15}{4} N k_B \frac{g_{5/2}(\gamma)}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} + \frac{3}{2} N k_B \frac{T}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} \frac{g_{3/2}(\gamma)}{\gamma} \left(- \frac{3\gamma}{2T} \frac{g_{3/2}(\gamma)}{g_{1/2}(\gamma)} \right)$$

$$C_V(T) = N k_B \frac{g_{5/2}(\gamma)}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} \left[\frac{15}{4} - \frac{9}{4} \frac{g_{3/2}(\gamma)}{g_{5/2}(\gamma)} \frac{g_{3/2}(\gamma)}{g_{1/2}(\gamma)} \right]$$

$$C_V(T) = N k_B \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} \left[\frac{15}{4} \frac{g_{5/2}(\gamma)}{\zeta(3/2)} - \frac{9}{4} \frac{\underbrace{g_{3/2}(\gamma)}_{\left(\frac{T_{\text{BEC}}}{T} \right)^{3/2}}}{\zeta(3/2)} \frac{g_{3/2}(\gamma)}{g_{1/2}(\gamma)} \right]$$

we use $\lambda_T^3 \cdot \frac{N}{V} = \lambda_T^3 \cdot n = g_{3/2}(\gamma) \Rightarrow 1 = \frac{g_{3/2}(\gamma)}{\lambda_T^3 \cdot n} = \frac{g_{3/2}(\gamma)}{\zeta(3/2)} \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2}$

$$C_V(T) = \frac{15}{4} N k_B \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} \frac{g_{5/2}(\gamma)}{\zeta(3/2)} - \frac{9}{4} N k_B \frac{g_{3/2}(\gamma)}{g_{1/2}(\gamma)}$$

I.2.2 Ideal Bose Gas

- thermodynamic properties of ideal Bose gas: specific heat: $c_V(T) = \left(\frac{\partial U}{\partial T} \right)_{V,N}$

ii. $T \gg T_{\text{BEC}}$:

$$c_V(T) = \frac{3}{2} N k_B, \quad c_V^{\text{mol}}(T) = \frac{3}{2} N_A k_B = \frac{3}{2} R$$

result for classical gas of non-interacting particles

(quantum statistics does not play any role at high T)

$$U(T) = \frac{3}{2} N k_B T$$

iii. $T < T_{\text{BEC}}$:

$$c_V(T) = \frac{15}{4} N k_B \frac{\zeta_{5/2}(1)}{\zeta_{3/2}(1)} \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} = 1.925 \cdot N k_B \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2}$$

$$U(T) = \frac{3}{2} N \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2} k_B T \frac{g_{5/2}(1)}{g_{3/2}(1)}$$

$$g_{5/2}(1) = \zeta(5/2) \quad g_{3/2}(1) = \zeta(3/2)$$

interpretation: all $N_{\text{ex}} = N \left(\frac{T}{T_{\text{BEC}}} \right)^{3/2}$ non-condensed particles contribute $O(k_B)$ to the specific heat

$$c_V(T) = \frac{5}{2} \frac{U(T)}{T}$$

$$\longleftrightarrow$$

$$c_V = T \partial S / \partial T$$

$$S(T) = \frac{5}{3} \frac{U(T)}{T}$$

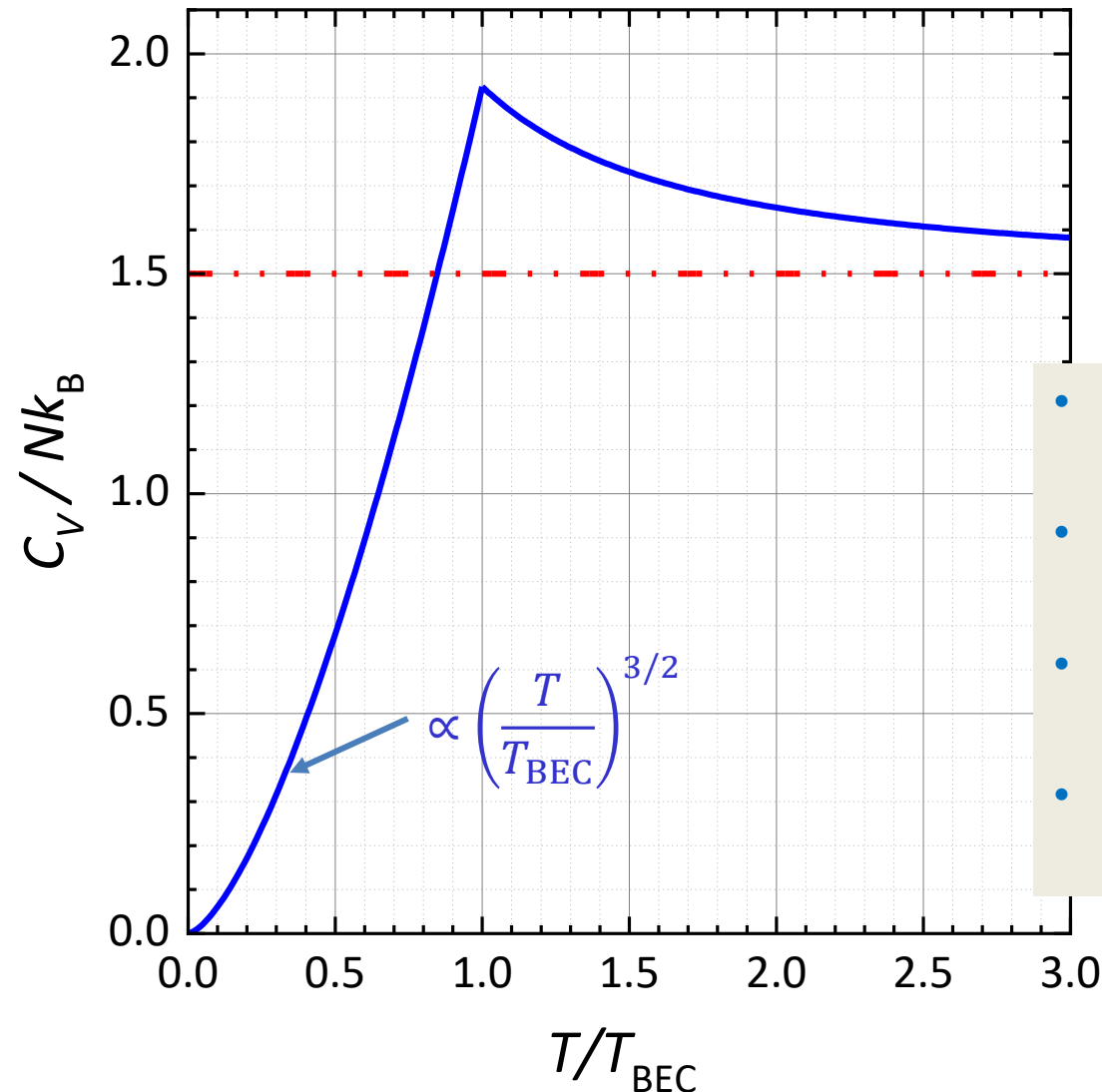
$$\longleftrightarrow$$

$$F = U - TS$$

$$F(T) = -\frac{2}{3} U(T)$$

I.2.2 Ideal Bose Gas

- thermodynamic properties of ideal Bose gas: specific heat: $C_V(T) = \left(\frac{\partial U}{\partial T} \right)_{V,N}$



$$C_V(T) = \frac{3}{2} Nk_B \quad \text{classical result}$$

- $C_V(T)$ decreases towards lower T since more and more particles condense into the ground state ε_0
- temperature T_{BEC} represents phase transition temperature
- $C_V(T)$ shows cusp at T_{BEC} , indicating the free energy is not analytical at T_{BEC}
- phase transition is usually interpreted as 2nd order phase transition

I.2.2 Ideal Bose Gas

- thermodynamic properties of ideal Bose gas: pressure $p = \left(\frac{\partial F}{\partial V} \right)_{N,T}$
- we discuss only the case $T < T_{\text{BEC}}$:

$$p = - \left(\frac{\partial F}{\partial V} \right)_{N,T} = \frac{2}{3} \left(\frac{\partial U}{\partial V} \right)_{N,T}$$

with $V = N \lambda_{T,\text{BEC}}^3 / g_{3/2}(1)$ we obtain

$$U = \frac{3}{2} N k_B T \frac{g_{5/2}(1)}{g_{3/2}(1)} = \frac{3}{2} \frac{V}{\lambda_{T,\text{BEC}}^3} g_{3/2}(1) k_B T \frac{g_{5/2}(1)}{g_{3/2}(1)} = \frac{3}{2} \frac{V}{\lambda_{T,\text{BEC}}^3} k_B T g_{5/2}(1)$$

$$p = \frac{2}{3} \left(\frac{\partial U}{\partial V} \right)_{N,T} = \frac{k_B T}{\lambda_T^3} g_{5/2}(1)$$

pressure of the gas does not depend on V in the BEC regime
 → compressibility of the BEC phase is infinite

this pathological behavior is removed by the inclusion of finite interactions

$$\begin{aligned} F &= U - TS \\ dF &= TdS - pdV + \mu dN - TdS - SdT \\ dF &= -pdV + \mu dN - SdT \\ \left(\frac{\partial F}{\partial V} \right)_{N,T} &= -p, \left(\frac{\partial F}{\partial N} \right)_{V,T} = \mu, \left(\frac{\partial F}{\partial T} \right)_{N,V} = -S \end{aligned}$$

$$U(T) = \frac{3}{2} N k_B T \frac{g_{5/2}(1)}{g_{3/2}(1)}$$

$$F(T) = -\frac{2}{3} U(T)$$

I.2.2 Ideal Bose Gas

- experimental realization with ultra-cold atomic gases

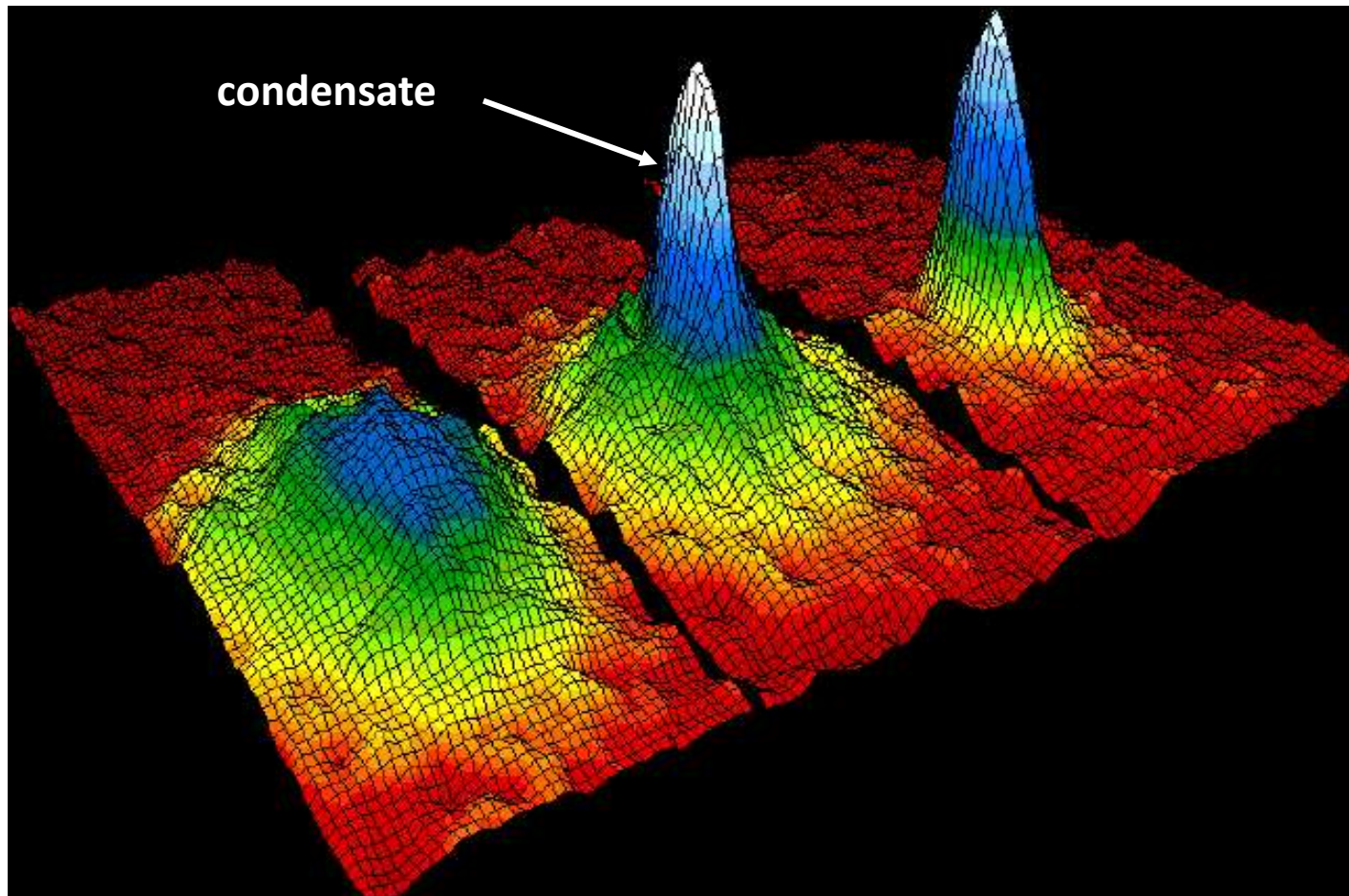
- first realization in 1995 with trapped alkali metal atoms, e.g. ^{87}Rb (Eric A. Cornell, Carl E. Wieman, Wolfgang Ketterle)
- typical densities: $n \simeq 10^{11} - 10^{15} \text{ cm}^{-3} \rightarrow T_{\text{BEC}} \simeq 10 \text{ nK} - 1 \text{ }\mu\text{K}$ due to large atomics mass (compared to ^4He)
- ^{87}Rb : electron configuration: $(\text{Kr})5s^1 \Rightarrow J = \frac{1}{2}$, nuclear spin $I = \frac{3}{2} \Rightarrow$ total spin is even \rightarrow **boson**
similar behavior: ^7Li , ^{23}Na
- **important**: alkali metal atoms interact (short length repulsion, attractive Van der Waals interaction)
 - atoms do not cluster since most interactions are elastic two-particle interactions:
 \rightarrow particles cannot bind since they cannot get rid of their kinetic energy
 - rate for three-particle interactions is very small
 - description of pairwise interactions: $V(\mathbf{r}_1 - \mathbf{r}_2) \simeq g \delta(\mathbf{r}_1 - \mathbf{r}_2)$

g = strenght of the scatterer

$$a_s = \frac{\hbar^2}{mg} = \text{s-wave scattering length}$$

(interactions and effect of trapping potential will be discussed later)

I.2.2 Ideal Bose Gas



Anderson, M. H.; Ensher, J. R.; Matthews, M. R.; Wieman, C. E.; Cornell, E. A.
Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor
Science **269**, 198-201 (1995).

velocity distribution for a gas of rubidium atoms, $n = 2.5 \times 10^{12} \text{ cm}^{-3}$

The measured distribution confirmed the existence of a Bose-Einstein condensate.

Left: $T > T_{\text{BEC}} \simeq 170 \text{ nK}$

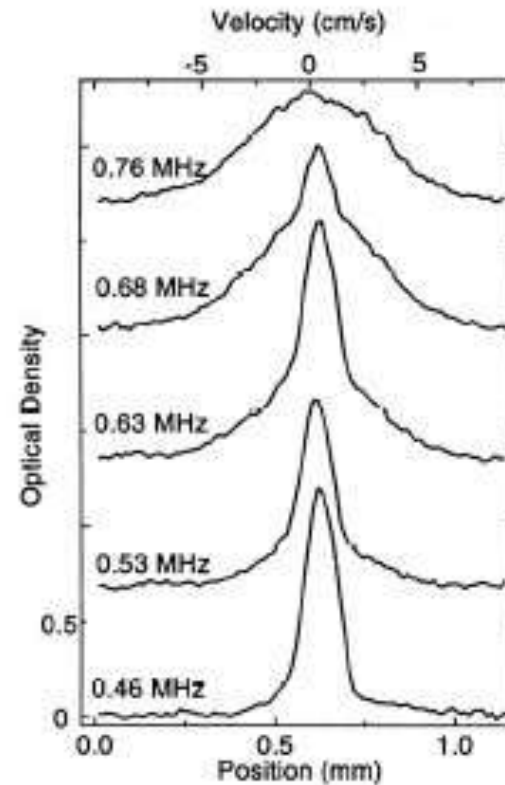
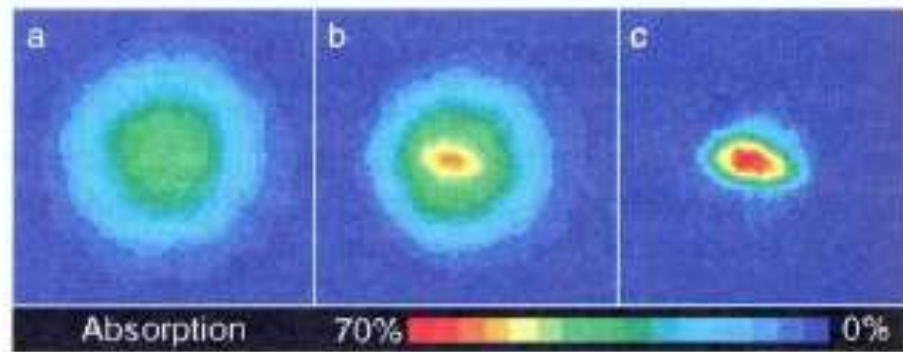
Center: $T < T_{\text{BEC}}$

Right: nearly pure condensate

experimental technique:

- switch-off the trap potential and let the condensate freely expand
- image the position of the atoms after a specific time by absorption of laser light by atom cloud
- measured spatial distribution is proportional to velocity distribution in condensate

I.2.2 Ideal Bose Gas



velocity distribution for a gas of sodium atoms, $n \sim 10^{14} \text{ cm}^{-3}$

the measured distribution confirmed the existence of a Bose-Einstein condensate.

Left: $T > T_{\text{BEC}} \approx 2 \mu\text{K}$

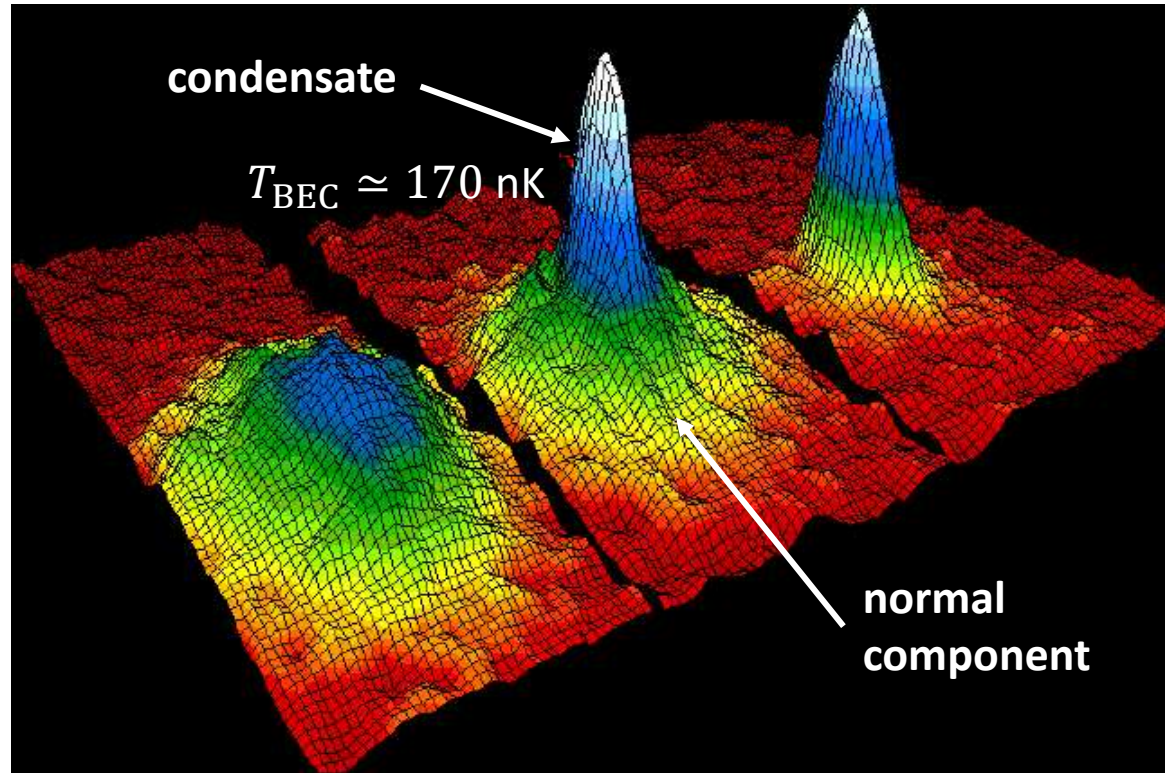
Center: $T < T_{\text{BEC}}$

Right: after further evaporation cooling

K. B. Davis, M. -O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle
Bose-Einstein Condensation in a Gas of Sodium Atoms
 Phys. Rev. Lett. **75**, 3969 (1995)

I.2.2 Ideal Bose Gas

- what determines the width of the central peak associated with the condensate



Anderson, M. H.; Ensher, J. R.; Matthews, M. R.; Wieman, C. E.; Cornell, E. A.
Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor
 Science **269**, 198-201 (1995).

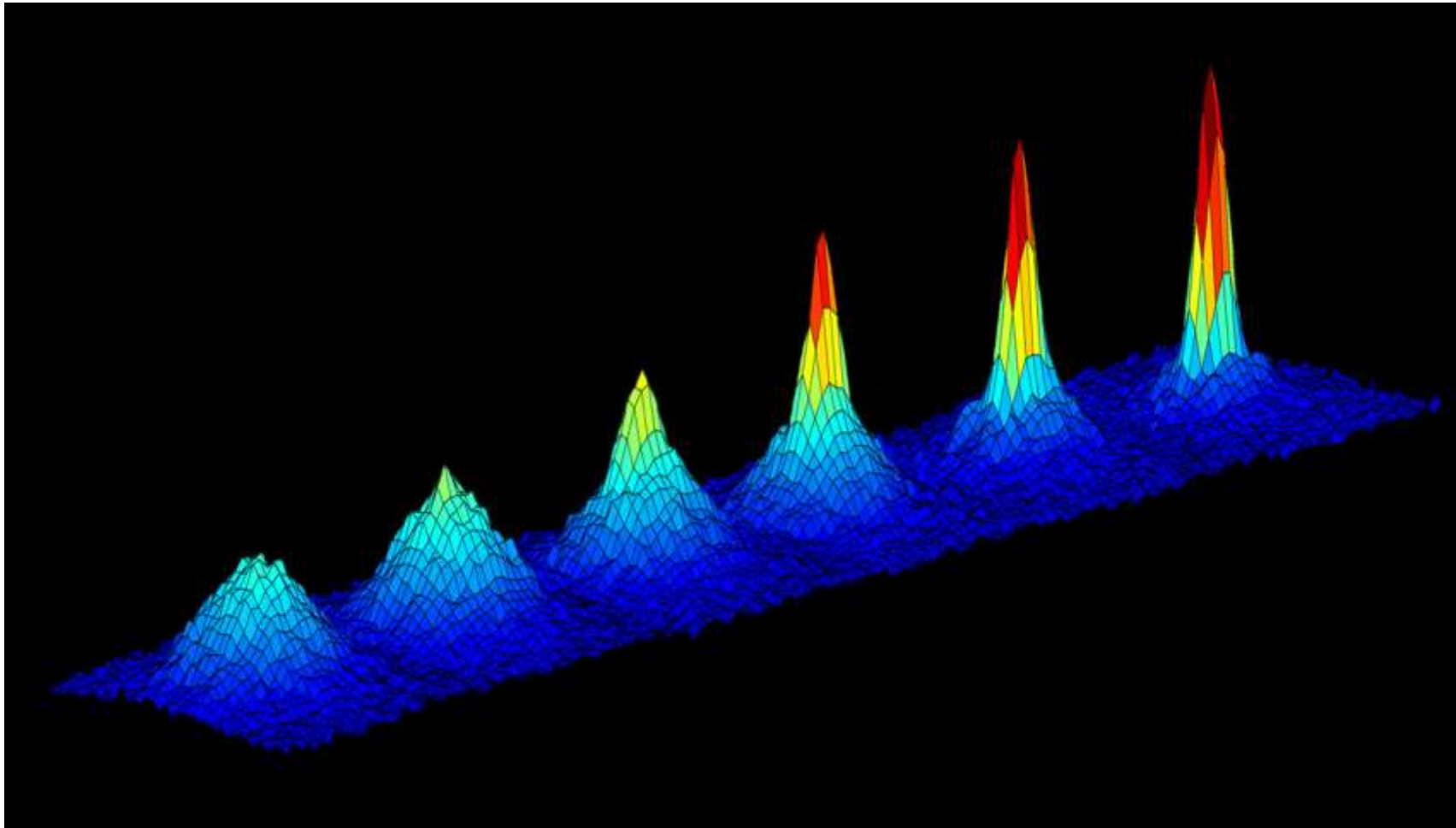
expected:

- delta function distribution since condensate atoms occupy the same state

observed:

- finite width Δx of the condensate due to trapping of condensate in trapping potential of finite width Δx
- resulting finite momentum distribution:
 $\Delta p \simeq \hbar/\Delta x \quad \Rightarrow \quad \Delta v \simeq \hbar/m\Delta x$
- width of the normal component:
 $\frac{1}{2}m \langle v_x^2 \rangle = \frac{1}{2}k_B T \quad \Rightarrow \quad \Delta v \simeq \sqrt{k_B T/m}$

I.2.2 Ideal Bose Gas



Space Station Experiment (2018)

Changing density of a cloud of rubidium atoms as it is cooled to lower and lower temperatures (going from left to right) approaching absolute zero. The emergence of a sharp peak in the later graphs confirms the formation of a Bose-Einstein condensate occurring here at about $T_{\text{BEC}} = 130 \text{ nK}$

I.2.2 Ideal Bose Gas

- concluding remarks
 - Bose-Einstein condensates have been realized with many different atoms
 - ideal model system, as most parameters can be well controlled and changed
 - it became evident that the finite interactions between the particles have an important impact
 - ideal **non-interacting Bose gas is not a superfluid** that can show frictionless flow
 - ➔ no persistent current of the condensate
 - ➔ no vortices
 - we will see later that this is due to the fact that the **Landau critical velocity** of the superfluid is zero



Walther
Meißner
Institut

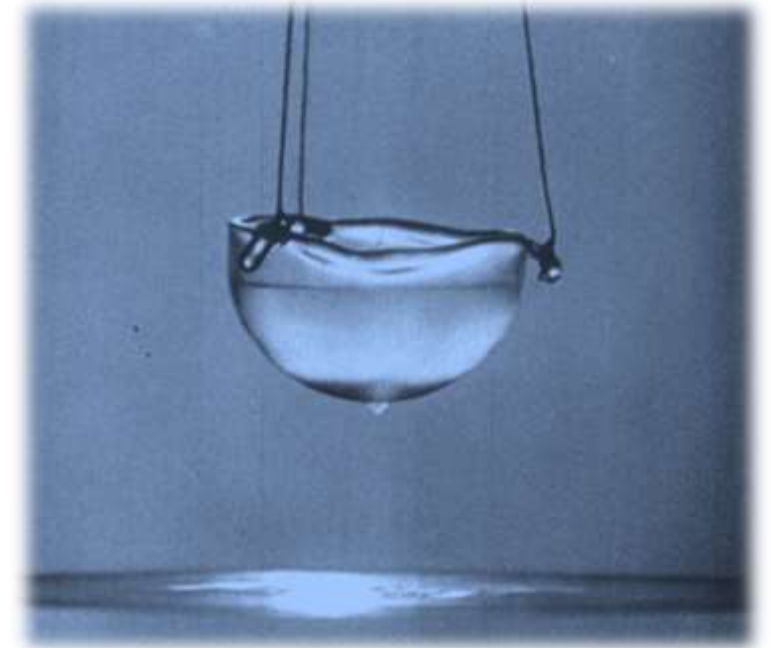
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München



Superconductivity and Low Temperature Physics II



Lecture No. 4

R. Gross

© Walther-Meißner-Institut

Contents Part I: Quantum Liquids

Contents:

I.1 Foundations and General Properties

- I.1.1 Quantum Gases & Liquids
- I.1.2 Helium
- I.1.3 Van der Waals Bonding
- I.1.4 Zero-Point Fluctuations
- I.1.5 Helium under Pressure
- I.1.6 pT-Phase Diagram of ^4He and ^3He
- I.1.7 Characteristic Properties of ^4He and ^3He
- I.1.8 Specific Heat of ^4He and ^3He

I.2 ^4He as an Ideal Bose Gas

- I.2.1 Bose-Einstein Condensation
- I.2.2 Ideal Bose Gas
- I.2.3 Bose Gas with Interactions
- I.2.4 Bose-Einstein Condensation of ^4He

I.3 Superfluid ^4He

- I.3.1 Two-Fluid Model
- I.3.2 Experimental Observations
- I.3.3 Two-Fluid Hydrodynamics
- I.3.4 Excitation Spectrum of ^4He

I.4 Vortices

- I.4.1 Quantization of Circulation
- I.4.2 Experimental Study of Vortices

I.5 ^3He

- I.5.1 normal fluid ^3He
- I.5.2 solid ^3He and Pomeranchuk effect
- I.5.3 superfluid ^3He

I.6 ^3He / ^4He mixtures



I.2.3 Bose Gas with Interactions

- How to treat the finite interaction between particles ?

➤ description by **pairwise interaction**: $V(\mathbf{r}_1 - \mathbf{r}_2) \simeq g \delta(\mathbf{r}_1 - \mathbf{r}_2)$

g = strength of the interaction

$a_s = \frac{\hbar^2}{mg}$ = s-wave scattering length

➤ modeling of pairwise interactions by **average „mean field“ potential** seen by every particle due to interaction with all others

$$V_{\text{eff}}(\mathbf{r}) \simeq g n(\mathbf{r})$$

➤ in addition, particles see finite **trapping potential** $V_{\text{trap}}(\mathbf{r})$ of e.g. a magnetic trap

➤ Schrödinger equation for particles in trap:

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}}(\mathbf{r}) + V_{\text{eff}}(\mathbf{r}) \right) \psi_k(\mathbf{r}) = \varepsilon_k \psi_k(\mathbf{r})$$

important:

Schrödinger equation is nonlinear since $V_{\text{eff}}(\mathbf{r})$ depends on $n(\mathbf{r})$ and hence on wave function $\psi_k(\mathbf{r})$ itself

➔ self-consistent solution: **Gross-Pitaevskii equations**

particle density and number:

$$n(\mathbf{r}) = \sum_k \left[\exp \left(\frac{(\varepsilon_k - \mu)}{k_B T} \right) - 1 \right]^{-1} |\psi_k(\mathbf{r})|^2$$

$$N = \int n(\mathbf{r}) d^3r = \sum_k \left[\exp \left(\frac{(\varepsilon_k - \mu)}{k_B T} \right) - 1 \right]^{-1}$$

I.2.3 Bose Gas with Interactions

- Now: more detailed discussion of finite interactions between particles

assumptions: (i) **weak interaction** (allows perturbative treatment)
(ii) **repulsive interaction** (attractive interaction may lead to instability)

we will show: **repulsive interaction stabilizes the Bose-Einstein condensate**
(counter-intuitive, as a repulsive interaction is expected to push particles out of a single-particle ground state)

- **Gedanken experiment:**
we consider two bosons and show that in the presence of a repulsive interaction it is more favorable to put them into the same eigenstate

I.2.3 Bose Gas with Interactions

- We first consider the case without any interaction:

– bosons = particles in a box described by normalized wave functions $\phi(\mathbf{r}) = \exp(i\mathbf{k} \cdot \mathbf{r})$, $\tilde{\phi}(\mathbf{r}) = \exp(i\tilde{\mathbf{k}} \cdot \mathbf{r})$,

- **two bosons in the same eigenstate**
($\mathbf{k} = \tilde{\mathbf{k}}, \phi = \tilde{\phi}, |\phi(\mathbf{r}_1)|^2 = |\phi(\mathbf{r}_2)|^2 = 1$)

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \phi(\mathbf{r}_1)\phi(\mathbf{r}_2) = \exp(i\mathbf{k} \cdot (\mathbf{r}_1 + \mathbf{r}_2))$$

symmetric wave function: $\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi(\mathbf{r}_2, \mathbf{r}_1)$

$$\int |\psi(\mathbf{r}_1, \mathbf{r}_2)|^2 d^3r_1 d^3r_2 = 1$$

- **two bosons in different eigenstates:**

$$(\mathbf{k} \neq \tilde{\mathbf{k}}, \phi \neq \tilde{\phi}, |\phi(\mathbf{r}_1)|^2 = |\tilde{\phi}(\mathbf{r}_1)|^2 = 1)$$

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}} [\phi(\mathbf{r}_1)\tilde{\phi}(\mathbf{r}_2) + \phi(\mathbf{r}_2)\tilde{\phi}(\mathbf{r}_1)]$$

symmetric superposition of the two possibilities

$\phi(\mathbf{r}_1)\tilde{\phi}(\mathbf{r}_2)$ and $\phi(\mathbf{r}_2)\tilde{\phi}(\mathbf{r}_1)$

$$\int |\psi(\mathbf{r}_1, \mathbf{r}_2)|^2 d^3r_1 d^3r_2 = 1$$

I.2.3 Bose Gas with Interactions

- what happens if we add repulsive potential $V(\mathbf{r}_1 - \mathbf{r}_2)$ between the two bosons?

➤ energy change for the two bosons in the same eigenstate: $\psi(\mathbf{r}_1, \mathbf{r}_2) = \phi(\mathbf{r}_1)\phi(\mathbf{r}_2) = \exp(i\mathbf{k} \cdot (\mathbf{r}_1 + \mathbf{r}_2))$

$$(\mathbf{k} = \tilde{\mathbf{k}}, \phi = \tilde{\phi}, |\phi(\mathbf{r}_1)|^2 = |\phi(\mathbf{r}_2)|^2 = 1)$$

$$\delta E = \int d^3r_1 d^3r_2 |\psi(\mathbf{r}_1, \mathbf{r}_2)|^2 V(\mathbf{r}_1 - \mathbf{r}_2) = \int d^3r_1 d^3r_2 V(\mathbf{r}_1 - \mathbf{r}_2) = \int d^3r V(\mathbf{r}) = E_0 > 0$$

we use $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ and $d^3r_1 d^3r_2 = d^3R d^3r$

note that $\delta E = \int d^3r V(\mathbf{r}) = E_0 > 0$ for a purely repulsive potential

I.2.3 Bose Gas with Interactions

➤ energy change for the two bosons in different eigenstates: $\psi(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}} [\phi(\mathbf{r}_1)\tilde{\phi}(\mathbf{r}_2) + \phi(\mathbf{r}_2)\tilde{\phi}(\mathbf{r}_1)]$

$$\delta E = \int d^3r_1 d^3r_2 |\psi(\mathbf{r}_1, \mathbf{r}_2)|^2 V(\mathbf{r}_1 - \mathbf{r}_2) \quad (\mathbf{k} \neq \tilde{\mathbf{k}}, \phi \neq \tilde{\phi}, |\phi(\mathbf{r}_1)|^2 = |\tilde{\phi}(\mathbf{r}_1)|^2 = 1)$$

$$= \frac{1}{2} \int d^3r_1 d^3r_2 \left[|\phi(\mathbf{r}_1)|^2 |\tilde{\phi}(\mathbf{r}_2)|^2 + |\phi(\mathbf{r}_2)|^2 |\tilde{\phi}(\mathbf{r}_1)|^2 + \phi(\mathbf{r}_1)\tilde{\phi}(\mathbf{r}_2)\phi^*(\mathbf{r}_2)\tilde{\phi}^*(\mathbf{r}_1) + c. c. \right] V(\mathbf{r}_1 - \mathbf{r}_2)$$

- since $|\phi(\mathbf{r}_1)|^2 = 1$ and $|\phi(\mathbf{r}_2)|^2 = 1$, the first two terms in [] yields the contribution $\int d^3r V(\mathbf{r}) = E_0$
- the last term in [] comes from the fact that the wave function must be symmetric with respect to the exchange of particles (bosons) since they are indistinguishable → results in additional **exchange energy**
- since $\phi(\mathbf{r}_1)\tilde{\phi}(\mathbf{r}_2)\phi^*(\mathbf{r}_2)\tilde{\phi}^*(\mathbf{r}_1) = \exp[i(\mathbf{k} - \tilde{\mathbf{k}}) \cdot (\mathbf{r}_1 - \mathbf{r}_2)]$, last term in [] results in an energy change $V_{\mathbf{k}-\tilde{\mathbf{k}}}/2$ with

$$V_{\mathbf{k}-\tilde{\mathbf{k}}} = \int d^3r e^{i(\mathbf{k}-\tilde{\mathbf{k}}) \cdot \mathbf{r}} V(\mathbf{r})$$

Fourier transform of $V(\mathbf{r})$

(we assume $V(\mathbf{r}) = V(-\mathbf{r})$ so that the FT is real: $V_{\mathbf{k}}^* = V_{\mathbf{k}} \rightarrow$ the complex conjugate yields the same contribution $V_{\mathbf{k}-\tilde{\mathbf{k}}}/2$)

➔ $\delta E = E_0 + V_{\mathbf{k}-\tilde{\mathbf{k}}}$ (if we assume two bosons with different wave numbers)

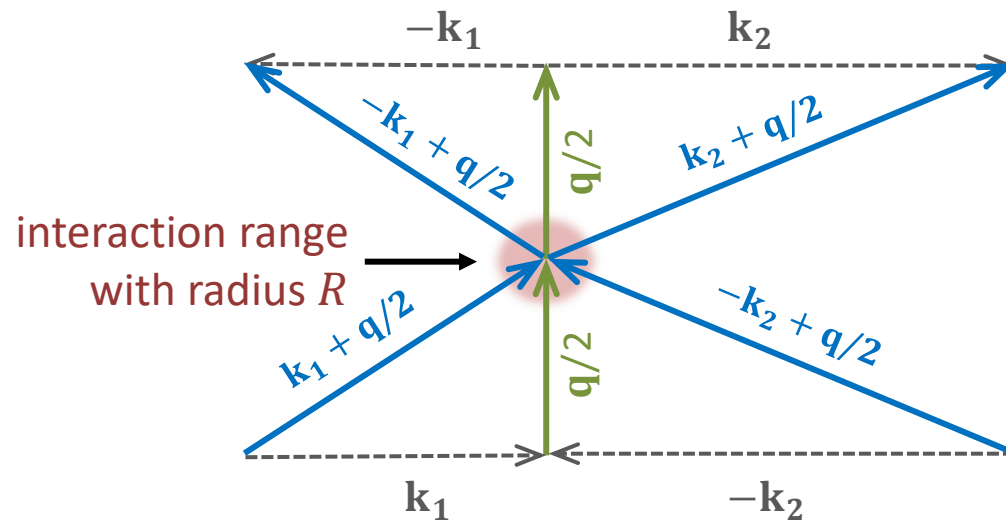
➔ $\delta E = E_0$ (if we assume two bosons with the same wave number)

due to the bosonic nature of the particles, it is energetically favorable to put them into the same quantum state in the presence of a repulsive interaction potential

I.2.3 Bose Gas with Interactions

- description of finite interaction between particles within second quantization formalism

$$\mathcal{H}_{\text{int}} = \frac{V_0}{2} \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{q}} \hat{b}_{\mathbf{k}_2 + \mathbf{q}/2}^\dagger \hat{b}_{-\mathbf{k}_1 + \mathbf{q}/2}^\dagger \hat{b}_{-\mathbf{k}_2 + \mathbf{q}/2} \hat{b}_{\mathbf{k}_1 + \mathbf{q}/2}$$



describes scattering process:

particle $|\mathbf{k}_1 + \mathbf{q}/2\rangle$ is destroyed and $|\mathbf{-k}_1 + \mathbf{q}/2\rangle$ created and
particle $|\mathbf{-k}_2 + \mathbf{q}/2\rangle$ is destroyed and $|\mathbf{k}_2 + \mathbf{q}/2\rangle$ created

I.2.3 Bose Gas with Interactions

- description of finite interaction between particles within second quantization formalism

$$\mathcal{H}_{\text{int}} = \frac{V_0}{2} \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{q}} \hat{b}_{\mathbf{k}_2 + \mathbf{q}/2}^\dagger \hat{b}_{-\mathbf{k}_1 + \mathbf{q}/2}^\dagger \hat{b}_{-\mathbf{k}_2 + \mathbf{q}/2} \hat{b}_{\mathbf{k}_1 + \mathbf{q}/2}$$

we have made the following simplifying assumptions:

- we assume a **local interaction** $V(\mathbf{r}_1 - \mathbf{r}_2) \rightarrow$ Fourier transform $V_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{q}}$ does not depend on \mathbf{k}_1 and \mathbf{k}_2 :
 $V_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{q}} \simeq V_{\mathbf{q}}$
- since we are at very low T , the involved velocities and hence wavevectors \mathbf{q} are very small
- if the interaction range R of the potential is small, $qR \ll 1$, then the FT $V_{\mathbf{q}}$ is essentially independent of \mathbf{q} and we can use the approximation $V_{\mathbf{q}} = V_0$
- as above, we assume two bosons with the same wave number $\mathbf{k}_1 = \mathbf{k}_2$ and set it equal to zero
- we describe the two states of the interacting bosons with the bosonic operators \hat{b} and $\hat{\tilde{b}}$

$$\mathcal{H}_{\text{int}} = \frac{V_0}{2} \left[\hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + \hat{b}^\dagger \hat{\tilde{b}}^\dagger \hat{b} \hat{\tilde{b}} + \hat{\tilde{b}}^\dagger \hat{b}^\dagger \hat{b} \hat{\tilde{b}} + \hat{b}^\dagger \hat{\tilde{b}}^\dagger \hat{\tilde{b}} \hat{b} + \hat{\tilde{b}}^\dagger \hat{b}^\dagger \hat{\tilde{b}} \hat{b} \right]$$

we do not include terms like $\hat{\tilde{b}}^\dagger \hat{\tilde{b}}^\dagger \hat{\tilde{b}} \hat{\tilde{b}}$, as we do not consider states occupied with two $\hat{\tilde{b}}$ bosons

I.2.3 Bose Gas with Interactions

- relevant relations

- both states occupied with single boson: $|1,1\rangle = \hat{b}^\dagger \hat{b}^\dagger |0\rangle$ with $|0\rangle =$ normalized vacuum state
- $\langle 1,1|1,1\rangle = \langle 0|\hat{b}\hat{b} \hat{b}^\dagger \hat{b}^\dagger |0\rangle = 1,$ as \hat{b} and \hat{b} commute and $\hat{b}\hat{b}^\dagger |0\rangle = 0$
- one state occupied with two bosons and the other is empty: $|2,0\rangle = \hat{b}^\dagger \hat{b}^\dagger |0\rangle = (\hat{b}^\dagger)^2 |0\rangle$
- $\langle 2,0|2,0\rangle = \langle 0|b\hat{b}\hat{b}^\dagger \hat{b}^\dagger |0\rangle = \langle 0|b(1 + \hat{b}^\dagger b)\hat{b}^\dagger |0\rangle = 2,$ as $[b, \hat{b}^\dagger] = 1$ and $b\hat{b}^\dagger |0\rangle = 0$
 $\Rightarrow |2,0\rangle = \frac{1}{\sqrt{2}} (\hat{b}^\dagger)^2 |0\rangle$ correctly normalized state

- **interaction energy** for the states $|1,1\rangle$ and $|2,0\rangle$:

$|2,0\rangle$: only the first term in $\frac{V_0}{2} [\hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b}]$ contributes

$$\langle 0|\hat{b}\hat{b}(\hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b})\hat{b}^\dagger \hat{b}^\dagger |0\rangle = 2\langle 0|\hat{b}\hat{b}(\hat{b}^\dagger \hat{b}^\dagger)|0\rangle = 2 \quad \rightarrow \langle 2,0|\mathcal{H}_{int}|2,0\rangle = V_0$$

$|1,1\rangle$: all four last terms in $\frac{V_0}{2} [\hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b}]$ contribute equally, since \hat{b} and \hat{b} commute

$$4\langle 0|\hat{b}\hat{b}(\hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b})\hat{b}^\dagger \hat{b}^\dagger |0\rangle = 4\langle 0|\hat{b}\hat{b}^\dagger \hat{b} \hat{b}^\dagger |0\rangle \langle 0|\hat{b}\hat{b}^\dagger \hat{b} \hat{b}^\dagger |0\rangle = 4 \quad \rightarrow \langle 1,1|\mathcal{H}_{int}|1,1\rangle = 2V_0$$

I.2.3 Bose Gas with Interactions

- generalization to states with N bosons (without calculation)

$$\langle N, 1 | \mathcal{H}_{int} | N, 1 \rangle - \langle N + 1, 0 | \mathcal{H}_{int} | N + 1, 0 \rangle = N V_0$$

- it is energetically more favorable to put an additional boson into the same state as occupied by all the others rather than putting it into another state
- energy difference scales with the particle number N and is therefore an extensive variable

I.2.3 Bose Gas with Interactions

- **Bogoliubov theory of weakly interacting Bose gas** (N.N Bogoliubov, J. Phys. USSR **11**, 23 (1947))
 - **starting hypothesis:** condensation of a large number of bosons into ground state also exists at finite interactions
 - since ground state $k = 0$ is occupied by large particle number, we have to deal with large quantum numbers
 - quantum fluctuations become negligible
 - the value of the commutator $[\hat{b}_0, \hat{b}_0^\dagger] = 1$ is small compared to these large values
 - we can set $[\hat{b}_0, \hat{b}_0^\dagger] = 0$, i.e. the operators commute like classical quantities
 - number of bosons in ground state, $N_0 = \langle \hat{b}_0^\dagger \hat{b}_0 \rangle \simeq \hat{b}_0^\dagger \hat{b}_0$, is large and also the numbers \hat{b}_0^\dagger and \hat{b}_0 themselves
 - as \hat{b}_0^\dagger and \hat{b}_0 are complex conjugate, we can take them real by eliminating an irrelevant phase factor:

$$N_0 = \hat{b}_0^\dagger \hat{b}_0 = \hat{b}_0^2 \text{ and } \hat{b}_0^\dagger = \hat{b}_0 = \sqrt{N_0}$$
 - **interaction energy:**

$$\mathcal{H}_{\text{int}}^{(0)} = \frac{V_0}{2} \left[\underbrace{\hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b}}_{\hat{b}_0^2 \hat{b}_0^2 = N_0^2} + \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} \right]$$

dominating term is the one with all wave numbers equal and zero

➔

$$E_{\text{int}}^{(0)} = \frac{V_0}{2} N_0^2$$

I.2.3 Bose Gas with Interactions

– interaction energy, next order term in powers of \hat{b}_0 :

→ we cannot have three \hat{b}_0 factors due to momentum conservation

→ we have to consider only terms with two \hat{b}_0 factors in the next order

→ momentum conservation requires that we have factors $\hat{b}_k^\dagger \hat{b}_k$ (4 possibilities), $\hat{b}_k^\dagger \hat{b}_{-k}^\dagger$ (1 possibility) and $\hat{b}_k \hat{b}_{-k}$ (1 possibility)

$$\mathcal{H}_{\text{int}}^{(1)} = \frac{V_0}{2} \hat{b}_0^2 \sum_{\mathbf{k}} (4\hat{b}_k^\dagger \hat{b}_k + \hat{b}_k^\dagger \hat{b}_{-k}^\dagger + \hat{b}_k \hat{b}_{-k})$$

$\mathcal{H}_{\text{int}}^{(1)}$ contains a contribution $\hat{b}_0^2 \sum_{\mathbf{k}} \hat{b}_k^\dagger \hat{b}_k = N_0 \sum_{\mathbf{k}} \hat{n}_k$, where \hat{n}_k is the particle number operator for bosons in state \mathbf{k}

→ we have to take into account the bosons that are not in the ground state $\mathbf{k} = \mathbf{0}$ (as already done for the non-interacting Bose gas)

$$N = N_0 + \sum_{\mathbf{k}} \hat{b}_k^\dagger \hat{b}_k \Rightarrow \frac{V_0}{2} (\hat{b}_0^2)^2 = \frac{V_0}{2} N_0^2 = \frac{V_0}{2} \left(N - \sum_{\mathbf{k}} \hat{b}_k^\dagger \hat{b}_k \right)^2 \simeq \frac{V_0}{2} \left(N^2 - 2N \sum_{\mathbf{k}} \hat{b}_k^\dagger \hat{b}_k \right) \simeq \underbrace{\frac{V_0}{2} N^2}_{\simeq E_{\text{int}}^{(0)} = \frac{V_0}{2} N_0^2} - \frac{V_0}{2} 2N \sum_{\mathbf{k}} \hat{b}_k^\dagger \hat{b}_k$$



$$\mathcal{H} = \underbrace{\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \hat{b}_k^\dagger \hat{b}_k}_{\text{kinetic energy}} + \underbrace{N \frac{V_0}{2} \sum_{\mathbf{k}} (2\hat{b}_k^\dagger \hat{b}_k + \hat{b}_k^\dagger \hat{b}_{-k}^\dagger + \hat{b}_k \hat{b}_{-k})}_{\text{additional ground state energy}}$$

the last two terms do not conserve particle number:
additional states are excited out of the condensate or absorbed into the condensate

I.2.3 Bose Gas with Interactions

- diagonalization of the Hamiltonian by Bogoliubov transformation

discussed in detail
in lecture SC+LTP 1

- introduction of new operators being linear combinations of the original bosonic operators

spinors of Bogoliubov operators:

$$\mathcal{B}_{\mathbf{k}}^{\dagger} = (\alpha_{\mathbf{k}}^{\dagger}, \beta_{-\mathbf{k}}) = \underbrace{\mathcal{C}_{\mathbf{k}}^{\dagger}}_{(\hat{b}_{\mathbf{k}}^{\dagger}, \hat{b}_{-\mathbf{k}})} \mathcal{U}_{\mathbf{k}} \quad \mathcal{B}_{\mathbf{k}} = (\alpha_{\mathbf{k}}, \beta_{-\mathbf{k}}^{\dagger}) = \mathcal{U}_{\mathbf{k}}^{\dagger} \underbrace{\mathcal{C}_{\mathbf{k}}}_{(\hat{b}_{\mathbf{k}}, \hat{b}_{-\mathbf{k}}^{\dagger})}^T$$

$$\alpha_{\mathbf{k}}^{\dagger} = u_{\mathbf{k}} \hat{b}_{\mathbf{k}}^{\dagger} - v_{\mathbf{k}} \hat{b}_{-\mathbf{k}}$$

$$\alpha_{\mathbf{k}} = u_{\mathbf{k}}^* \hat{b}_{\mathbf{k}} - v_{\mathbf{k}}^* \hat{b}_{-\mathbf{k}}^{\dagger}$$

$$\beta_{-\mathbf{k}} = v_{\mathbf{k}}^* \hat{b}_{\mathbf{k}}^{\dagger} + u_{\mathbf{k}} \hat{b}_{-\mathbf{k}}$$

$$\beta_{-\mathbf{k}}^{\dagger} = v_{\mathbf{k}} \hat{b}_{\mathbf{k}} + u_{\mathbf{k}} \hat{b}_{-\mathbf{k}}^{\dagger}$$

- appropriate unitary matrix to make transformed energy matrix $\tilde{\mathcal{E}}_{\mathbf{k}} = \mathcal{U}_{\mathbf{k}}^{\dagger} \mathcal{E}_{\mathbf{k}} \mathcal{U}_{\mathbf{k}}$ diagonal:

$$\mathcal{U}_{\mathbf{k}} = \begin{pmatrix} u_{\mathbf{k}} & v_{\mathbf{k}}^* \\ -v_{\mathbf{k}} & u_{\mathbf{k}}^* \end{pmatrix} \quad \mathcal{U}_{\mathbf{k}}^{\dagger} = \begin{pmatrix} u_{\mathbf{k}}^* & -v_{\mathbf{k}}^* \\ v_{\mathbf{k}} & u_{\mathbf{k}} \end{pmatrix}$$

$$\Rightarrow \mathcal{U}_{\mathbf{k}}^{\dagger} \mathcal{E}_{\mathbf{k}} \mathcal{U}_{\mathbf{k}} = \tilde{\mathcal{E}}_{\mathbf{k}} = \begin{pmatrix} E_{\mathbf{k}} & D_{\mathbf{k}} \\ -D_{\mathbf{k}} & -E_{\mathbf{k}} \end{pmatrix}$$

choose $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ such that off-diagonal terms vanish

I.2.3 Bose Gas with Interactions

- diagonalization of the Hamiltonian by Bogoliubov transformation (without detailed calculation)

$$\Rightarrow u_{\mathbf{k}} = \sqrt{\frac{1}{2} \frac{(\varepsilon_{\mathbf{k}} + NV_0)}{E_{\mathbf{k}}} + 1} \quad v_{\mathbf{k}} = \sqrt{\frac{1}{2} \frac{(\varepsilon_{\mathbf{k}} + NV_0)}{E_{\mathbf{k}}} - 1} \quad u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 = 1$$

with
$$E_{\mathbf{k}} = \sqrt{(\varepsilon_{\mathbf{k}} + NV_0)^2 - (NV_0)^2} = \sqrt{\varepsilon_{\mathbf{k}} (\varepsilon_{\mathbf{k}} + 2NV_0)}$$

resulting Hamiltonian

$$\mathcal{H} = \underbrace{\mathcal{H}_0}_{\text{ground state energy}} + \underbrace{\sum_{\mathbf{k}} \{E_{\mathbf{k}} \alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + E_{\mathbf{k}} \beta_{-\mathbf{k}}^\dagger \beta_{-\mathbf{k}}\}}_{\text{elementary excitations}}$$

elementary excitations of the interacting Bose gas are non-interacting bosonic quasiparticles described by the creation and annihilation operators $\alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}}$ and $\beta_{-\mathbf{k}}^\dagger \beta_{-\mathbf{k}}$

I.2.3 Bose Gas with Interactions

- nature of excitations

- large \mathbf{k} : kinetic energy dominates: $\varepsilon_{\mathbf{k}} \gg NV_0 \Rightarrow E_{\mathbf{k}} \simeq \varepsilon_{\mathbf{k}}, u_{\mathbf{k}} \simeq 1, v_{\mathbf{k}} \simeq 0,$
 $\alpha_{\mathbf{k}}^{\dagger} = u_{\mathbf{k}} \hat{b}_{\mathbf{k}}^{\dagger} - v_{\mathbf{k}} \hat{b}_{-\mathbf{k}} \simeq \hat{b}_{\mathbf{k}}^{\dagger}$

excitations are identical to bare bosons

- $\mathbf{k} \rightarrow \mathbf{0}$: kinetic energy vanishes: $\varepsilon_{\mathbf{k}} \ll NV_0 \Rightarrow E_{\mathbf{k}} \simeq \sqrt{2NV_0\varepsilon_{\mathbf{k}}} = \sqrt{NV_0/m} \hbar k = \hbar k c_s$
 $c_s = \text{sound velocity}$

propagating sound waves are expected for interacting system, correctly described by Bogoliubov theory \rightarrow quantized sound waves: phonons

$$u_{\mathbf{k}} \simeq v_{\mathbf{k}} \text{ with } u_{\mathbf{k}}^2 \simeq v_{\mathbf{k}}^2 = 1/2$$

excitations are equal superpositions of $\hat{b}_{\mathbf{k}}^{\dagger}$ and $\hat{b}_{-\mathbf{k}}$ ($\hat{b}_{-\mathbf{k}}^{\dagger}$ and $\hat{b}_{\mathbf{k}}$) both adding (removing) a momentum $\hbar\mathbf{k}$ to the system
 \rightarrow consistent with excitation of quantized sound waves

Contents Part I: Quantum Liquids

Contents:

I.1 Foundations and General Properties

- I.1.1 Quantum Gases & Liquids
- I.1.2 Helium
- I.1.3 Van der Waals Bonding
- I.1.4 Zero-Point Fluctuations
- I.1.5 Helium under Pressure
- I.1.6 pT-Phase Diagram of ^4He and ^3He
- I.1.7 Characteristic Properties of ^4He and ^3He
- I.1.8 Specific Heat of ^4He and ^3He

I.2 ^4He as an Ideal Bose Gas

- I.2.1 Bose-Einstein Condensation
- I.2.2 Ideal Bose Gas
- I.2.3 Bose Gas with Interactions
- I.2.4 Bose-Einstein Condensation of ^4He



I.3 Superfluid ^4He

- I.3.1 Two-Fluid Model
- I.3.2 Experimental Observations
- I.3.3 Two-Fluid Hydrodynamics
- I.3.4 Excitation Spectrum of ^4He

I.4 Vortices

- I.4.1 Quantization of Circulation
- I.4.2 Experimental Study of Vortices

I.5 ^3He

- I.5.1 normal fluid ^3He
- I.5.2 solid ^3He and Pomeranchuk effect
- I.5.3 superfluid ^3He

I.6 ^3He / ^4He mixtures

I.2.4 Bose-Einstein Condensation of ^4He

- What do we already know about ^4He ?

- ^4He does not become solid at ambient pressure due to the weak Van der Waals interaction and the large zero-point fluctuations
- ^4He is a boson and shows superfluidity
discovered in 1937 by **Pyotr Kapitsa** ("*Viscosity of Liquid Helium Below the λ -Point*", Nature **141**, 74 (1938)) and **John F. Allen**, and **Don Misener** ("*Flow of Liquid Helium II*", Nature **142**, 643 (1938))
- the properties of superfluid ^4He often deviate from those of an ideal Bose-Einstein condensate
 - $C_V \propto T^3$ instead of $C_V \propto T^{3/2}$ at low T
 - behavior of C_V around T_{BEC} is different – sharp λ -shaped peak instead of "cusp": ^4He shows critical behavior consistent with 3D-XY universality class
- description of superfluid ^4He by macroscopic wave function $\psi(\mathbf{r})$

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r})e^{i\theta(\mathbf{r})}, \quad |\psi_0(\mathbf{r})|^2 = n_0(\mathbf{r}), \quad N_0 = \int |\psi_0(\mathbf{r})|^2 d^3r$$

$$n_0 = \frac{N_0}{V} = \text{particle density in condensate}$$

I.2.4 Bose-Einstein Condensation of ^4He

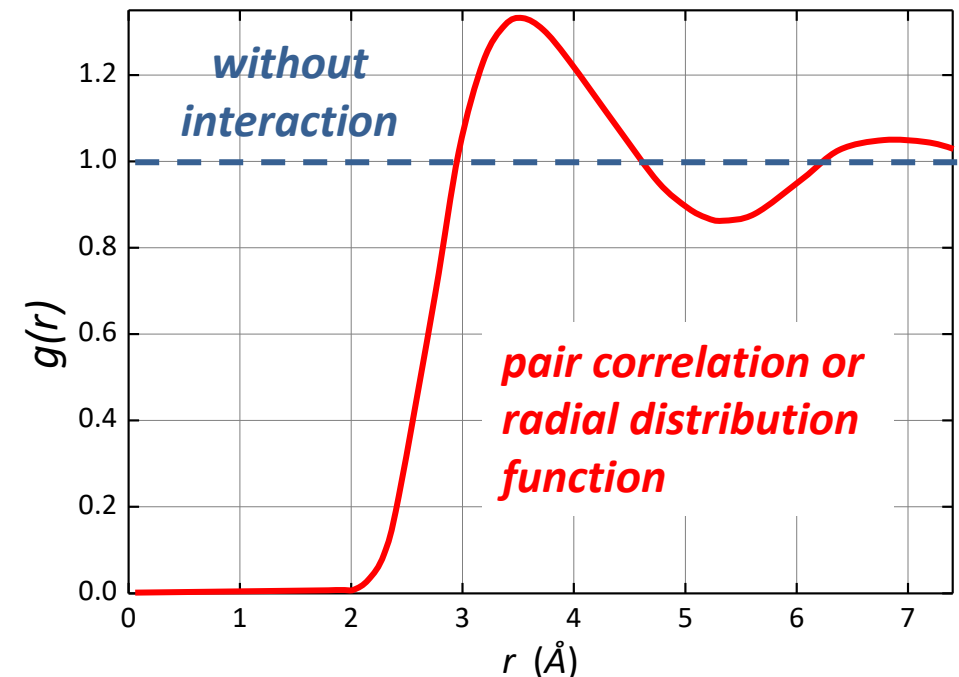
- How well does the ideal Bose gas describe the normal-superfluid transition temperature?

$$T_{\text{BEC}} = \frac{2\pi\hbar^2}{mk_B} \left(\frac{n}{2.612} \right)^{2/3}$$

- $T_{\text{BEC}} \simeq 0.5 \text{ K}$ by using data of gaseous ^4He
(is well below normal-superfluid transition temperature)
- $T_{\text{BEC}} \simeq 3.1 \text{ K}$ by using data of liquid ^4He
(agrees only qualitatively with measured $T_\lambda = 2.17 \text{ K}$)

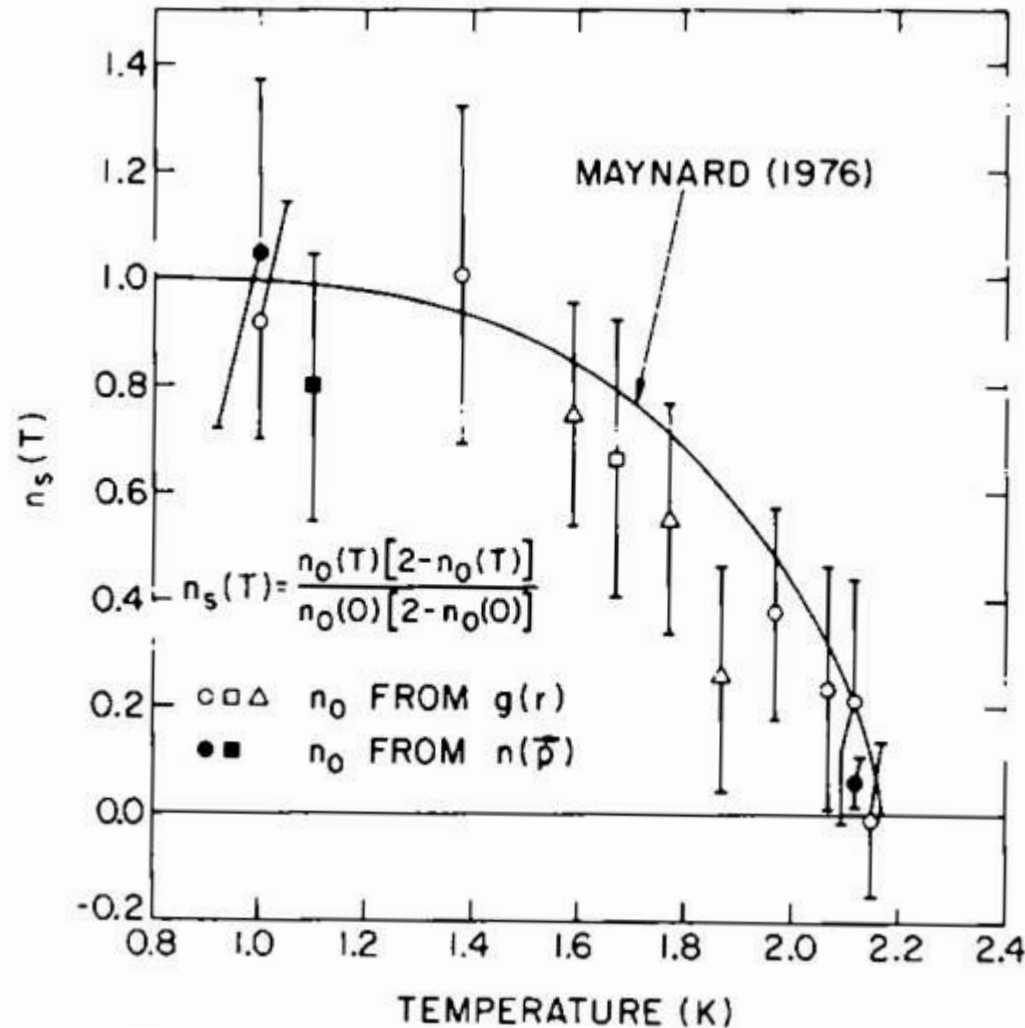
^4He is not an ideal Bose gas:
interaction of He atoms

- hard core repulsion
- admixture of higher \mathbf{k} -states (is equivalent to larger k -uncertainty)
- reduction of particle number in ground state



I.2.4 Bose-Einstein Condensation of ^4He

- superfluid fraction of ^4He is only about 14% for $T \rightarrow 0$: $n_0(T = 0) \simeq 0.14 n$



experimental observations

$$T \ll T_\lambda: n_s(T) \simeq n_s(0) - AT^\alpha \text{ with } \alpha \simeq 4$$

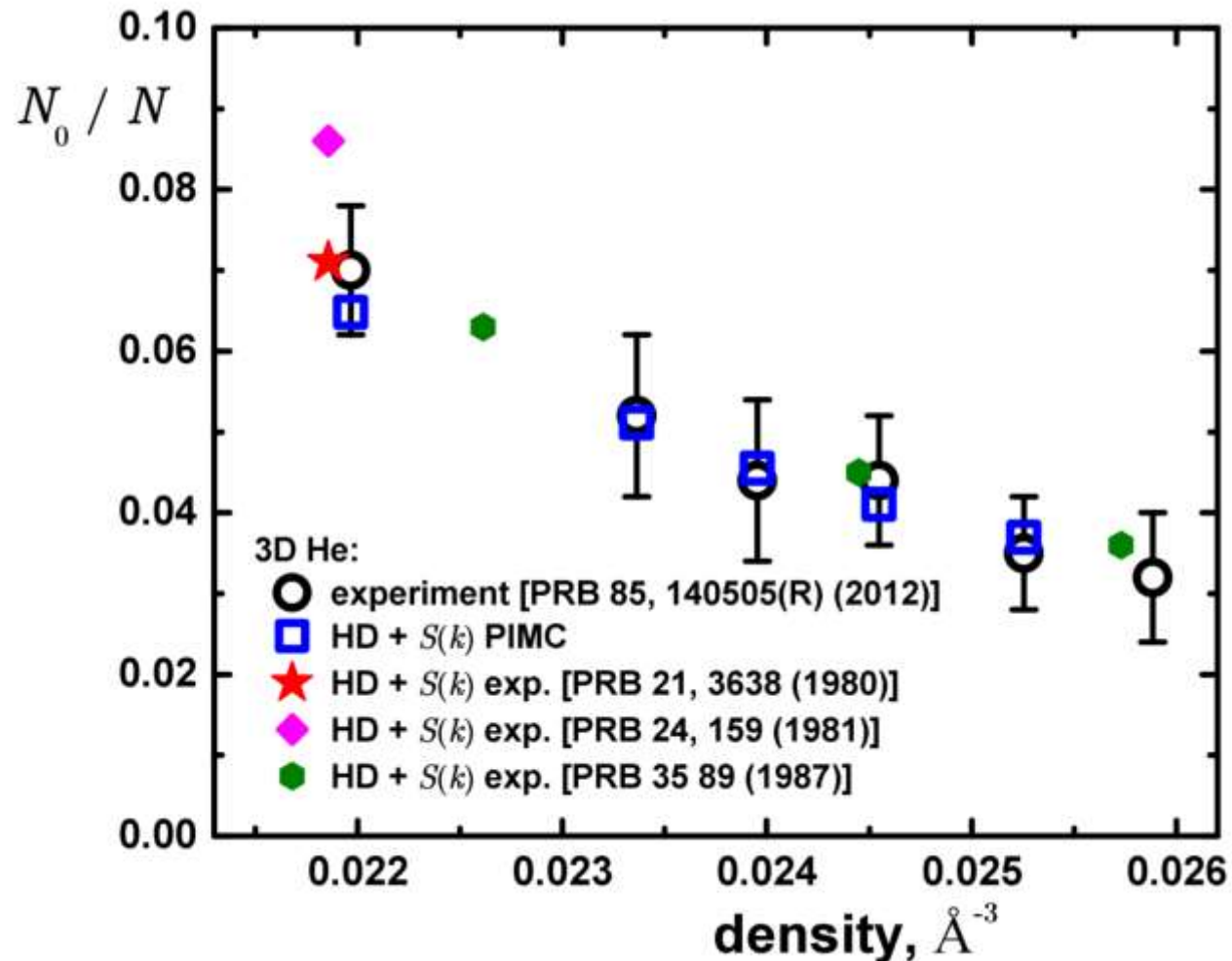
$$T \sim T_{\text{BEC}}: n_s(T) \simeq \begin{cases} B(T_\lambda - T)^\beta & \text{for } T < T_\lambda \\ 0 & \text{for } T > T_\lambda \end{cases}$$

data obtained from x-ray scattering and neutron scattering

J.S. Brooks and R.J. Donnelly, J. Phys. Chem. Ref. Data **6**, 51 (1977) .

I.2.4 Bose-Einstein Condensation of ^4He

- superfluid fraction of ^4He



condensate fraction of superfluid ^4He as a function of the density. Open circles are experimental results obtained from deep inelastic neutron scattering, squares are HD predictions at the same densities. The star diamond, and hexagonal points correspond to quantum-field hydrodynamics (HD) results using the experimental measures of the static structure factor $S(\mathbf{p})$

Lozovik, Yurii & Kurbakov, I. & Astrakharchik, G. & Boronat, J. (2020). *Universal relation between the static structure factor and the condensate fraction.*

I.2.4 Bose-Einstein Condensation of ^4He

- superfluid fraction of ^4He – more detailed treatment (1)

- reduced superfluid fraction is caused by the fact that superfluid ^4He is a strongly interacting Bose gas described by

$$\mathcal{H} = \sum_{i=1}^N \frac{\hbar^2}{2m} \nabla_i^2 + \frac{1}{2} \sum_{i \neq j} V(\mathbf{r}_i - \mathbf{r}_j)$$

- solutions are symmetric N -particle wave functions

$$\psi(\mathbf{r}_1, \dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots, \mathbf{r}_N) = \psi(\mathbf{r}_1, \dots, \mathbf{r}_j, \dots, \mathbf{r}_i, \dots, \mathbf{r}_N)$$

$$\mathcal{H}\psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = E_n \psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$$

- occupation probability of eigenstate with energy E_n

$$P_n = \frac{1}{Z_N} \exp(-E_n/k_B T) \quad \text{with partition function} \quad Z_N = \sum_n \exp(-E_n/k_B T)$$

- if particle number is not fixed (grand canonical ensemble): $Z_N = \sum_{n,N} \exp\left(-\frac{E_n^{(N)}}{k_B T} - \mu N\right)$

I.2.4 Bose-Einstein Condensation of ^4He

- superfluid fraction of ^4He – more detailed treatment (2)

- for grand canonical ensemble:

$$\Omega(T, \mu) = -k_B T \ln Z$$

grand canonical potential

$$\langle N \rangle = k_B T \frac{\partial \ln Z}{\partial \mu}$$

average particle number

$$U = \langle \mathcal{H} \rangle = \mu \langle N \rangle - k_B T \frac{\partial \ln Z}{\partial \beta}$$

inner energy (with $\beta = 1/k_B T$)

- calculations have to be performed numerically, e.g. by perturbation theory (if interactions are weak) or quantum Monte Carlo simulations in the case of strong interactions (as for superfluid ^4He)

key result: superfluid ^4He forms a condensate of He atoms in the ground state, however, not all atoms occupy ground state at $T = 0$

see e.g. D.M. Ceperley, Rev. Mod. Phys. 67, 279-355 (1995)

I.2.4 Bose-Einstein Condensation of ^4He

- superfluid fraction of ^4He – more detailed treatment (3)

- single particle density matrix:

$$\rho(\mathbf{r}_1 - \mathbf{r}'_1) = N \int \psi_0^*(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) \psi_0(\mathbf{r}'_1, \mathbf{r}_2, \dots, \mathbf{r}_N) d^3r_2 \dots d^3r_N$$

- corresponds to correlation function between the many-body wave functions at the coordinates $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N$ and $\mathbf{r}'_1, \mathbf{r}_2, \dots, \mathbf{r}_N$
- by integrating of all coordinates despite \mathbf{r}_1 and \mathbf{r}'_1 , we average over all configurations of the particles despite the first ones

- properties if the single particle density matrix:

- depends only on relative coordinate $\mathbf{r}_1 - \mathbf{r}'_1$ due to translational invariance
- for $\mathbf{r}_1 = \mathbf{r}'_1$ we obtain: $\int \rho_1(0) d^3r_1 = \frac{N}{V} = n$

- for ideal non-interacting Bose gas, we have $\psi_0(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \psi_0(\mathbf{r}_1) \psi_0(\mathbf{r}_2) \dots \psi_0(\mathbf{r}_N)$

$$\rho(\mathbf{r}_1 - \mathbf{r}'_1) = N \psi_0^*(\mathbf{r}_1) \psi_0(\mathbf{r}'_1) \int |\psi_0(\mathbf{r}_1)|^2 |\psi_0(\mathbf{r}_2)|^2 \dots |\psi_0(\mathbf{r}_N)|^2 d^3r_2 \dots d^3r_N = N \psi_0^*(\mathbf{r}_1) \psi_0(\mathbf{r}'_1)$$

$$\rho(\mathbf{r}_1 - \mathbf{r}'_1) = \frac{N}{V} = n \quad \text{since } \psi_0(\mathbf{r}) = \frac{1}{\sqrt{V}} \exp(i\mathbf{k} \cdot \mathbf{r})$$

I.2.4 Bose-Einstein Condensation of ^4He

- superfluid fraction of ^4He – more detailed treatment (4)

- for interacting Bose gas we obtain more complicated result

- $\rho(\mathbf{r}_1 - \mathbf{r}'_1)$ is no longer independent of $\mathbf{r}_1 - \mathbf{r}'_1$

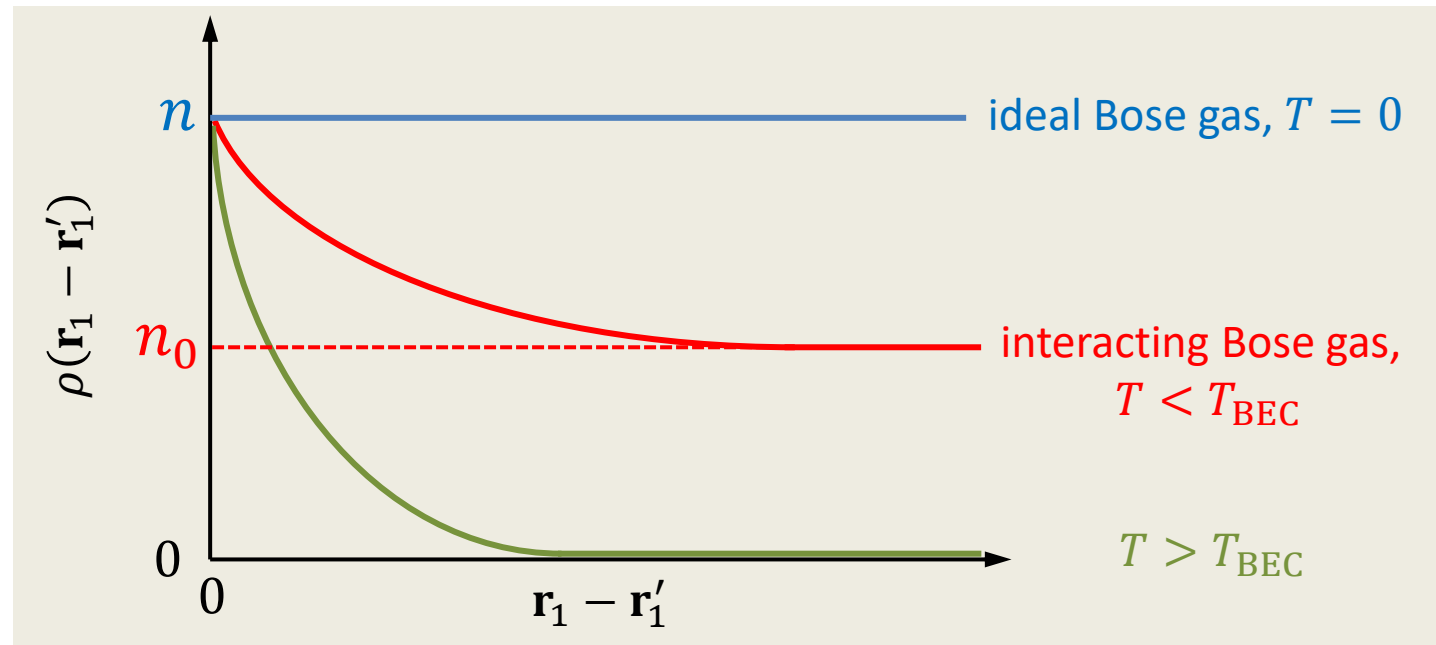
- $\rho(\mathbf{r}_1 - \mathbf{r}'_1)$ decreases with increasing $\mathbf{r}_1 - \mathbf{r}'_1$ and converges towards a constant value at large $\mathbf{r}_1 - \mathbf{r}'_1$

- condensate density:

$$n_0 = \lim_{|\mathbf{r}_1 - \mathbf{r}'_1| \rightarrow \infty} \rho(\mathbf{r}_1 - \mathbf{r}'_1)$$

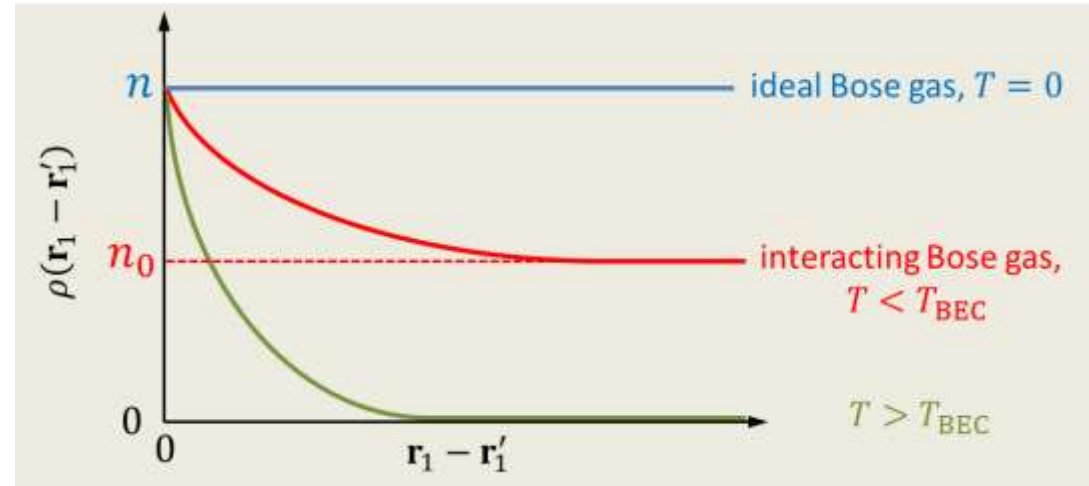
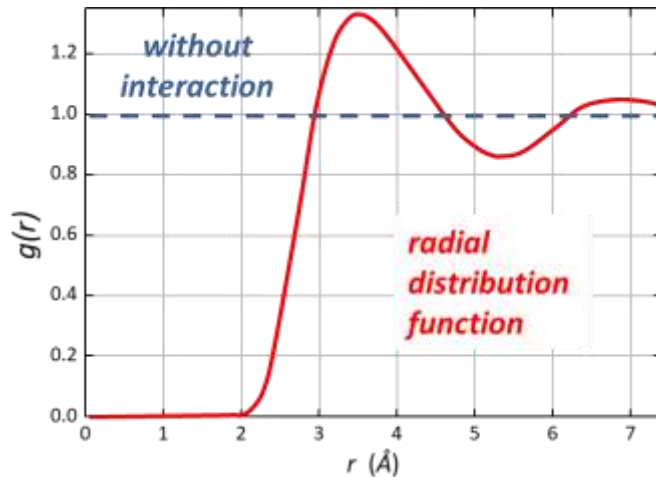
for superfluid ^4He :

$$n_0(T = 0) \simeq 0.14 n$$



I.2.4 Bose-Einstein Condensation of ^4He

- superfluid fraction of ^4He – more detailed treatment (5)
 - plausibility consideration by considering spatial distribution function



- without interactions particles can be described by plane waves with $k = k_0$
 - ➔ $g(\mathbf{r}_1 - \mathbf{r}_2) = 1$, all particles in ground state: $n_0 = n$
- in the presence of interactions, $g(\mathbf{r}_1 - \mathbf{r}_2) \neq 1$ for short distances
 - ➔ we have to admix states with higher k to describe the situation: spatial localization requires finite Δk
 - ➔ some states with $k > k_0$ are occupied: $n_0 < n$
 - ➔ n_0 becomes smaller, if $g(\mathbf{r}_1 - \mathbf{r}_2)$ deviates more strongly from $g(\mathbf{r}_1 - \mathbf{r}_2) = 1$, i.e. with increasing interactions

Contents Part I: Quantum Liquids

Contents:

I.1 Foundations and General Properties

- I.1.1 Quantum Gases & Liquids
- I.1.2 Helium
- I.1.3 Van der Waals Bonding
- I.1.4 Zero-Point Fluctuations
- I.1.5 Helium under Pressure
- I.1.6 pT-Phase Diagram of ^4He and ^3He
- I.1.7 Characteristic Properties of ^4He and ^3He
- I.1.8 Specific Heat of ^4He and ^3He

I.2 ^4He as an Ideal Bose Gas

- I.2.1 Bose-Einstein Condensation
- I.2.2 Ideal Bose Gas
- I.2.3 Bose Gas with Interactions
- I.2.4 Bose-Einstein Condensation of ^4He



I.3 Superfluid ^4He

- I.3.1 Two-Fluid Model
- I.3.2 Experimental Observations
- I.3.3 Two-Fluid Hydrodynamics
- I.3.4 Excitation Spectrum of ^4He

I.4 Vortices

- I.4.1 Quantization of Circulation
- I.4.2 Experimental Study of Vortices

I.5 ^3He

- I.5.1 normal fluid ^3He
- I.5.2 solid ^3He and Pomeranchuk effect
- I.5.3 superfluid ^3He

I.6 ^3He / ^4He mixtures

I.3 Superfluid ^4He

- nomenclature and history

- superfluid ^4He can flow through tiny capillaries without any friction → nomenclature *superfluidity*

- **discovery of superfluidity** in 1937 by two groups:
 - (i) *P. Kapitza, Nature 141, 74 (1938)*
 - (ii) *J.F. Allen, A.D. Misener, Nature 141, 75 (1938)*

- **theory of superfluidity:**

- **F. London** (1938): superfluidity caused by ordered state in momentum space
F. London, Nature 141, 643 (1938); Phys. Rev. 54, 947 (1938).
- **L. Tisza** (1938): two-fluid model: prediction of second sound
L. Tisza, Nature 141, 913 (1938).
- **L.D. Landau** (1941-47): two-fluid hydrodynamics of superfluid ^4He
→ *superfluidity as a consequence of the excitation spectrum*
L.D. Landau, J. Phys. USSR 5, 71 (1941); ibid. 8, 1 (1944); ibid. 11, 91 (1947).
- **R.P. Feynman** (1953): excitation spectrum postulated by Landau follows from quantum mechanics
R.P. Feynman, Phys. Rev. 90, 1116 (1953).

I.3.1 Two-Fluid Model

- *two-fluid model allows phenomenological description of superfluid ^4He*
microscopic description is difficult \rightarrow treatment using hydrodynamics

- already in 1938: anomalous properties of superfluid ^4He can be well described by two-fluid model:
mixture of two non-interacting fluid components *(L. Tisza, J. de Phys. et Radium **1**, 164 (1938))*
- formal description of superfluid ^4He as the sum of a normal and a superfluid component

$$\rho = \rho_s + \rho_n \quad \rho_s, \rho_n = \text{superfluid and normal fluid densities}$$

- $T = 0$: $\rho_s = \rho$ and $\rho_n = 0$
- $T = T_\lambda$: $\rho_s = 0$ and $\rho_n = \rho$

- superfluid component: *no entropy: $S_s = 0$* *zero viscosity: $\eta_s = 0$*
- normal fluid component: *carries total entropy: $S_n = S$, finite viscosity: $\eta_n = \eta$*

I.3.1 Two-Fluid Model

- **superfluid component**

- description of superfluid component by macroscopic wave function

as done in superconductivity for electronic system: charged superfluid

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r})e^{i\theta(\mathbf{r})}, \quad |\psi_0(\mathbf{r})|^2 = n_0(\mathbf{r}), \quad N_0 = \int |\psi_0(\mathbf{r})|^2 d^3r \quad n_0 = \frac{N_0}{V} = \text{particle density in condensate}$$

- use quantum mechanics to derive expression for the particle current density (probability current density in 3D)

$$J_\rho = \frac{\hbar}{2mi} [\psi^*(\mathbf{r})\nabla\psi(\mathbf{r}) - \psi(\mathbf{r})\nabla\psi^*(\mathbf{r})]$$

$$J_\rho = \frac{\hbar}{2mi} \left[\underbrace{\sqrt{n_0} e^{-i\theta(\mathbf{r})}}_{\psi^*(\mathbf{r})} \underbrace{\left(e^{i\theta(\mathbf{r})}\nabla\sqrt{n_0} + i\sqrt{n_0}e^{i\theta(\mathbf{r})}\nabla\theta(\mathbf{r}) \right)}_{\nabla\psi(\mathbf{r})} - \underbrace{\sqrt{n_0} e^{i\theta(\mathbf{r})}}_{\psi(\mathbf{r})} \underbrace{\left(e^{-i\theta(\mathbf{r})}\nabla\sqrt{n_0} - i\sqrt{n_0}e^{-i\theta(\mathbf{r})}\nabla\theta(\mathbf{r}) \right)}_{\nabla\psi^*(\mathbf{r})} \right]$$

$$J_\rho(\mathbf{r}) = n_0 \underbrace{\frac{\hbar}{m} \nabla\theta(\mathbf{r})}_{v_s} = n_0 v_s(\mathbf{r})$$

current density – phase relation (as in superconductivity)

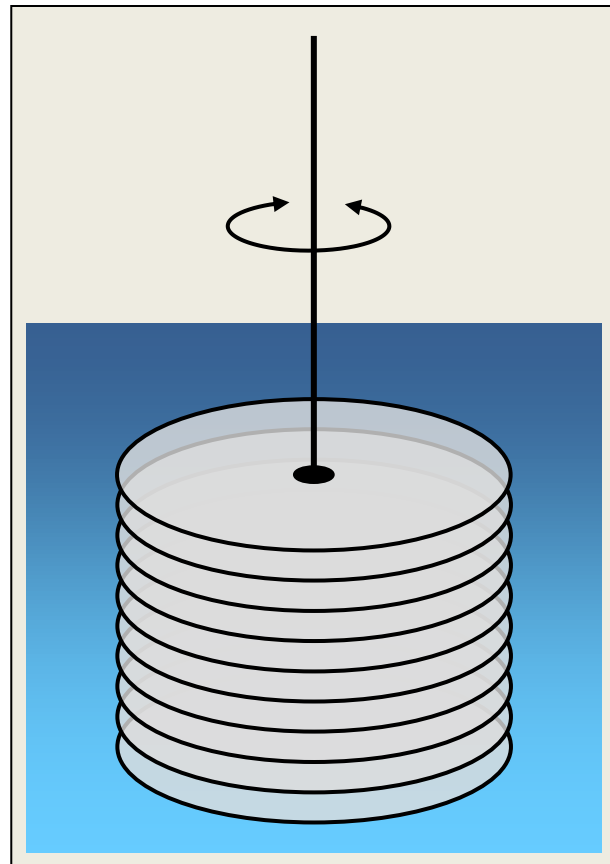
v_s = superfluid velocity

I.3.1 Two-Fluid Model

- what is the ratio of the normal und superfluid component? How does it depend on temperature?

- torsional pendulum experiment: 50 Al disks, thickness: 13 μm , diameter: 3.5 cm, disk separation: 210 μm

(E.L. Andronikashvili, Zh. Eksperim. i. Teor. Fiz. 18, 424 (1948))



- only normal component is moved during rotation

- skin depth of viscous wave:

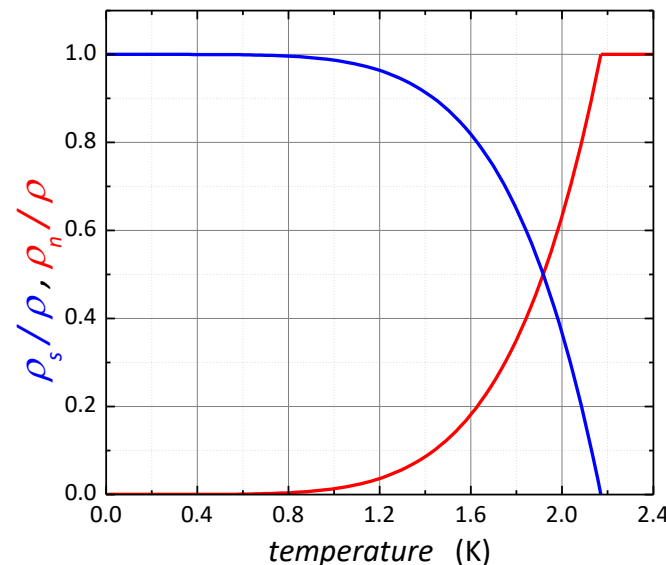
$$\sqrt{2\eta_n/\rho_n\omega}$$

→ large compared to disk separation below T_λ

→ complete normal component is moved

- moved mass of normal component results in change of moment of inertia

→ change of oscillation frequency: $\omega \propto 1/\text{mass}$



empirical relation by fitting data:

$$\frac{\rho_n}{\rho(T_\lambda)} = \left(\frac{T}{T_\lambda}\right)^{5.6}$$

cf. normal fluid density of ideal BEC:

$$\frac{\rho_n}{\rho} = \left(\frac{T}{T_{\text{BEC}}}\right)^{3/2}$$

I.3.1 Two-Fluid Model

- physical consequence of macroscopic wave function description

$$J_{\rho}(\mathbf{r}) = n_0 \frac{\hbar}{m} \nabla \theta(\mathbf{r}) = n_0 v_s(\mathbf{r}) \Rightarrow v_s(\mathbf{r}) = \frac{\hbar}{m} \nabla \theta(\mathbf{r})$$

- since the curl of the gradient of a scalar function vanishes, we obtain

$$\nabla \times v_s(\mathbf{r}) = 0$$

$v_s(\mathbf{r})$ cannot take the form $\Omega \times \mathbf{r}$ corresponding to a rotating liquid

➔ it is impossible to set the superfluid into rotation

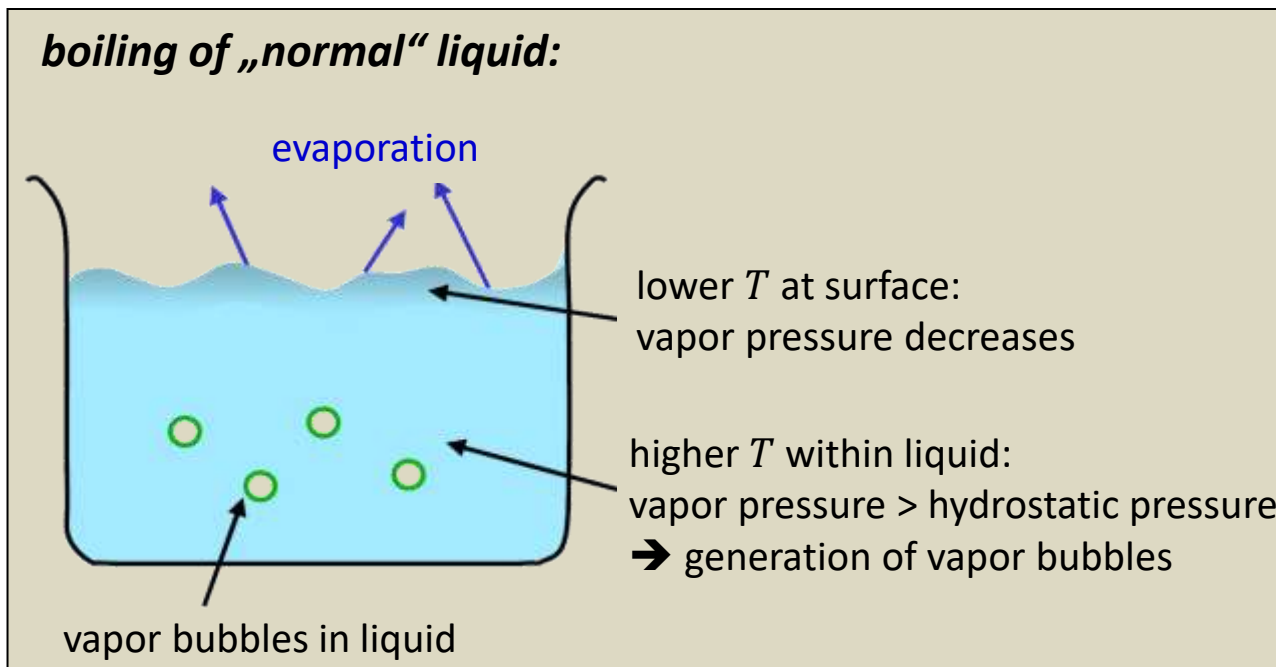
note:

we will see later that this is true only for slow enough rotations of the vessel containing the superfluid. At large rotation speed, we are generating vortices in the superfluid
➔ the critical rotation speed corresponds to the lower critical field in superconductors!!

I.3.2 Experimental Observations

- boiling of liquid helium

- $T > T_\lambda$:
 - liquid He is boiling
 - latent heat of evaporation is removed from bath
 - temperature is decreasing
- $T \simeq T_\lambda$:
 - temperature is decreasing only slowly → large specific heat
 - temperature does not stay constant → no latent heat, 2. order phase transition
 - **boiling suddenly stops !!!**



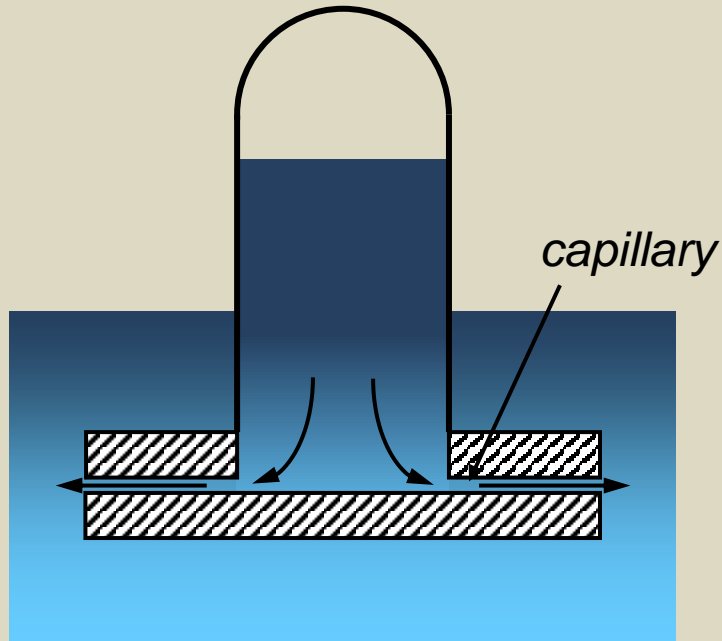
boiling of superfluid helium:

- **very high thermal conductivity**
 - low T not only at surface, but also within liquid
 - no vapor bubbles in liquid
- **evaporation only from surface, no boiling**

I.3.2 Experimental Observations

- viscosity η of superfluid ^4He : *two different measuring methods*

flow through narrow capillary



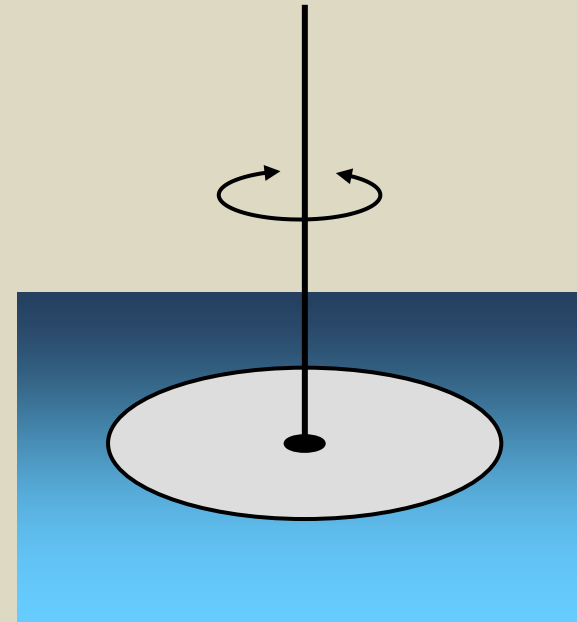
$$\eta < 10^{-11} \text{ poise}$$

(cf. water: 8.90×10^{-5} poise @ 25°C)

- flow without measurable friction
- viscosity $\eta \approx 0$
- *superfluid component flows out*

(1 Poise = $1 \text{ g/cm s} = 0.1 \text{ Pa} \cdot \text{s}$)

torsional pendulum



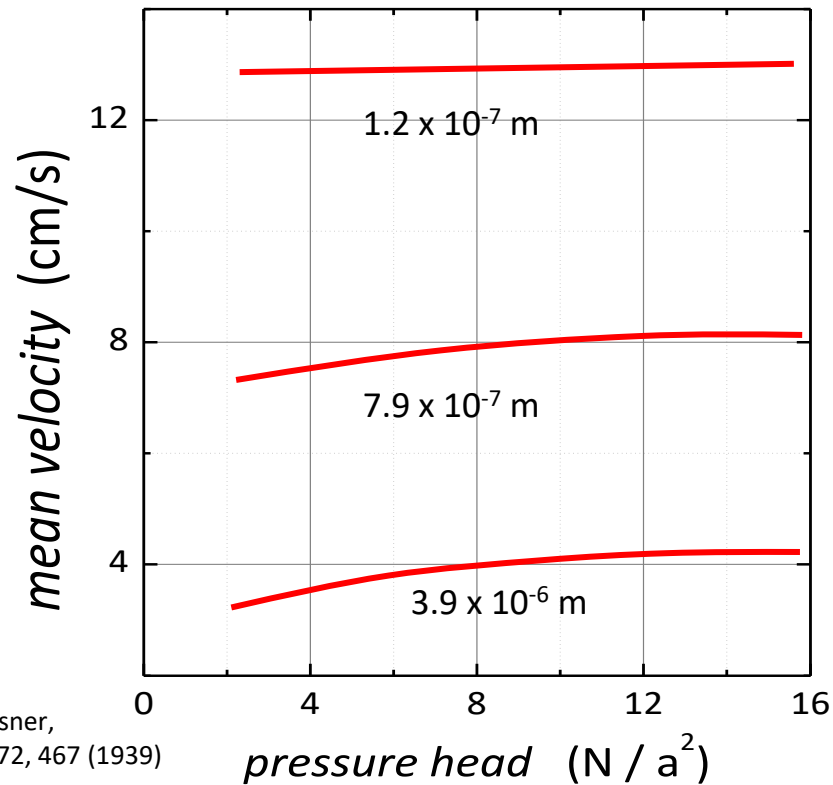
$$\eta \approx 10^{-5} \text{ poise}$$

- damping of oscillation: $\eta > 0$
- scattering of **phonons** and **rotons**
- *normal component picks up momentum*

I.3.2 Experimental Observations

- viscosity η of superfluid ^4He : *experimental results*

➤ flow through narrow capillary

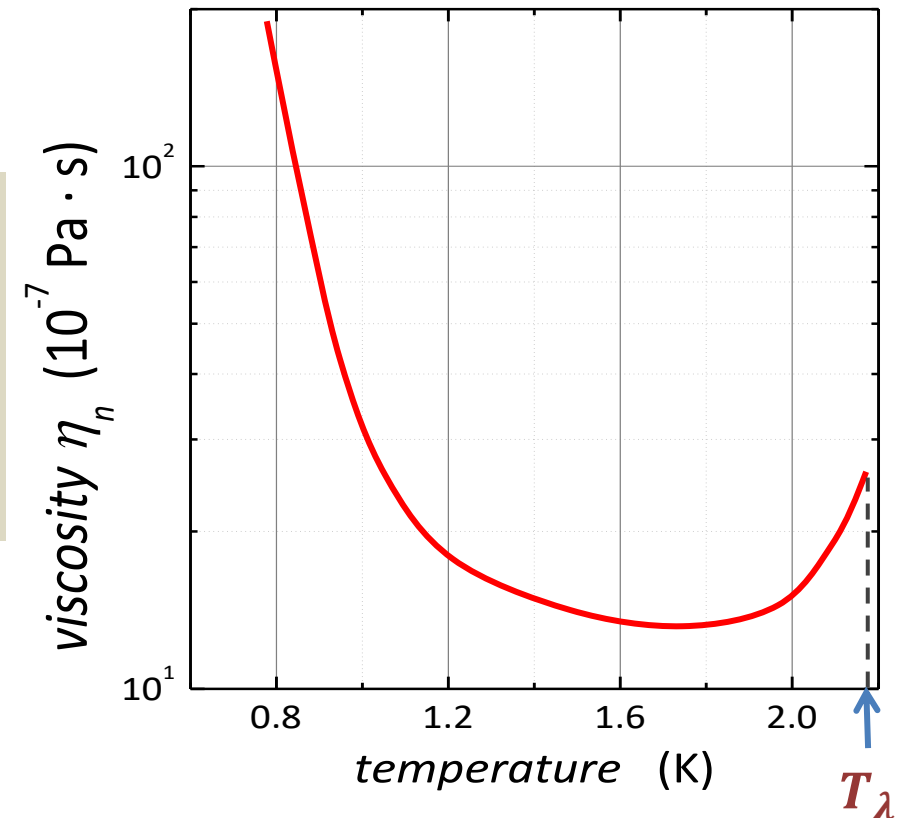


*experimental result
depends on
measuring method !!*



*explanation by two-
fluid model*

➤ torsional pendulum



J.F. Allen, A.D. Miesner,
Proc. Roy. Soc. A 172, 467 (1939)

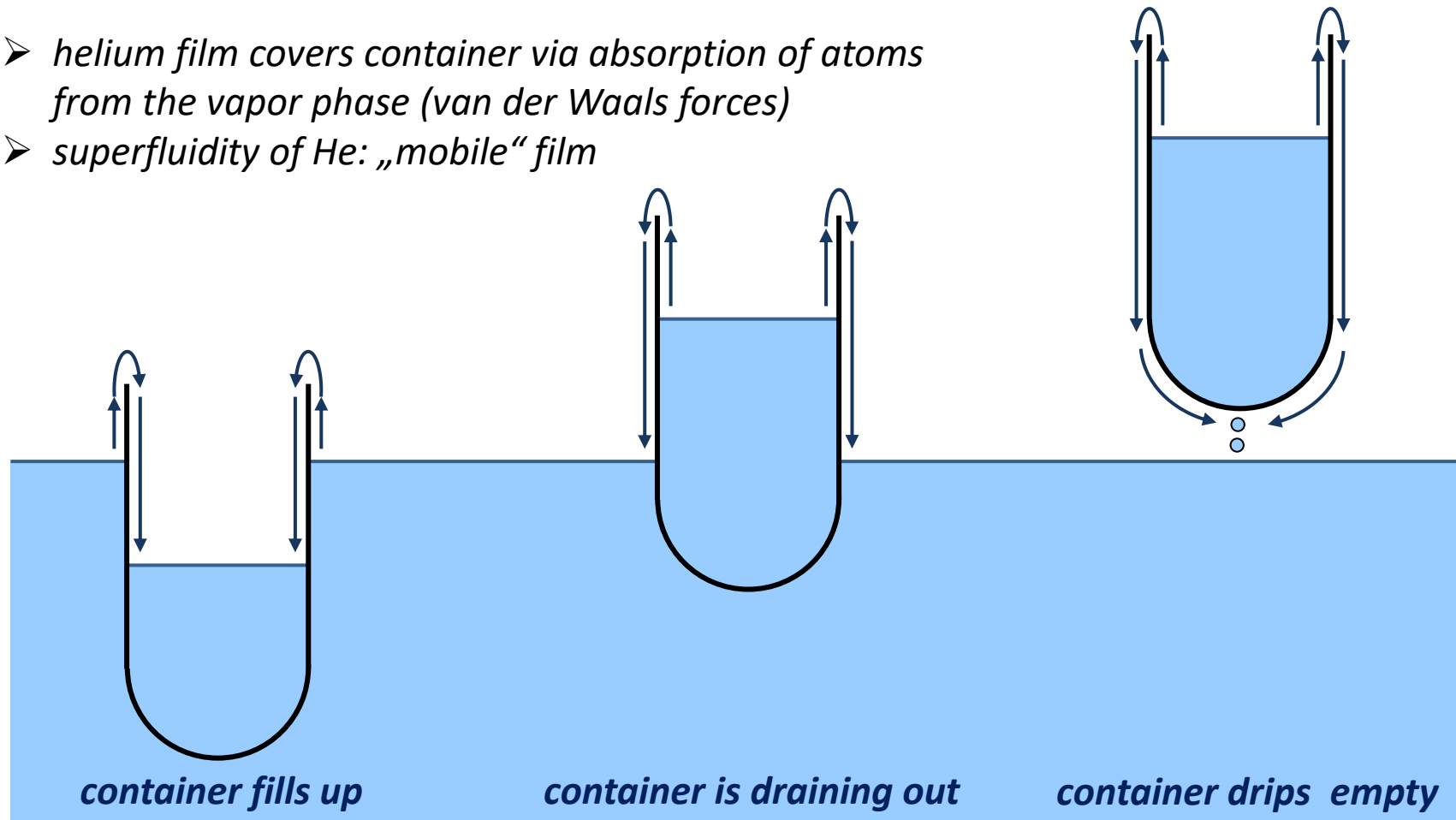
- flow velocity v about independent of pressure
- flow velocity increases v with decreasing diameter a of capillary

expected for viscous flow through tube of length L and diameter a : $\Delta p / L \sim \eta (v / a^4)$ (laminar flow, Hagen-Poiseuille law)
experimental result of Kapitza: $\Delta p \simeq 0$ independent of flow velocity $v \rightarrow \eta \simeq 0$

I.3.2 Experimental Observations

- superfluid film flow

- helium film covers container via absorption of atoms from the vapor phase (van der Waals forces)
- superfluidity of He: „mobile“ film



overshooting during fill up and draining out → oscillation of filling height

I.3.2 Experimental Observations

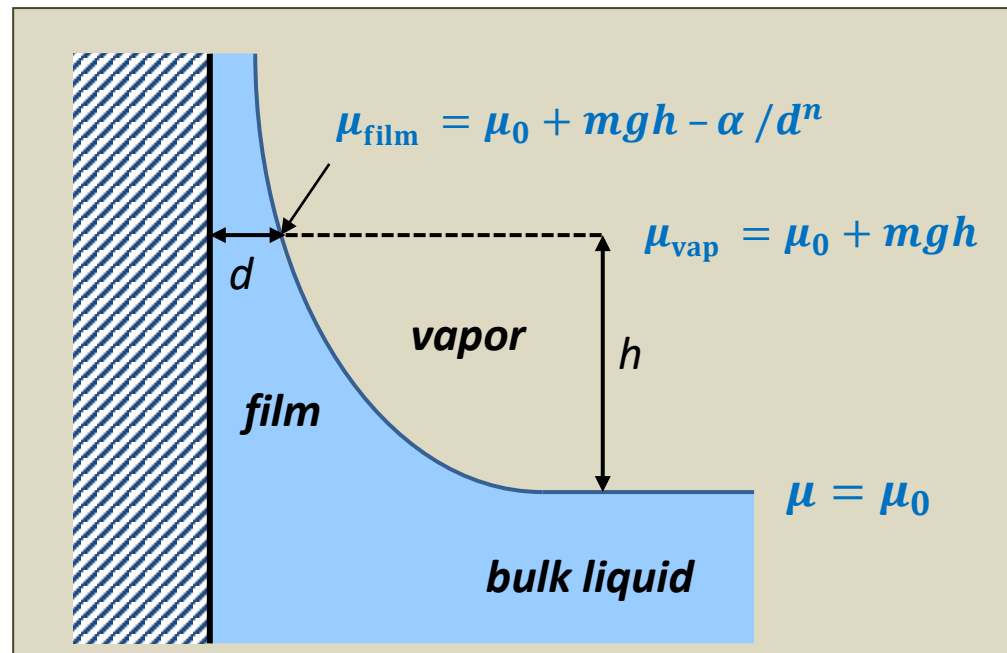
- **superfluid film flow: estimate of superfluid film thickness**

- in equilibrium **chemical potentials** are identical: $\mu_{\text{film}} = \mu_{\text{vap}} = \mu_0$ (for film in saturated vapor)

➤ for film: $\mu_{\text{film}} = \mu_0 + \underbrace{mgh - \alpha/d^n}_{=0}$ (n ≈ 3 for d < 5 nm, α = Hamaker constant, α is determined by dielectric properties of wall and He atoms)

bulk liquid
gravitation
van der Waals potential

➤ for vapor: $\mu_{\text{vap}} = \mu_0 + mgh$



equilibrium condition: $\mu_{\text{film}} = \mu_0$

$$d = \left(\frac{\alpha}{mgh} \right)^{1/3}$$

typical values:

$$d \text{ (nm)} \approx 30/[h \text{ (cm)}]^{1/3}$$

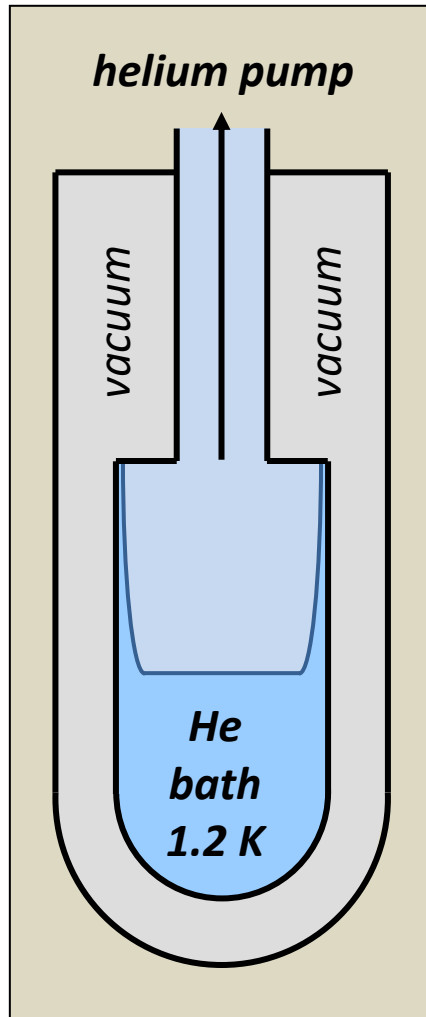
$$h = 5 \text{ cm} \rightarrow d = 20 \text{ nm}$$

normal fluid blocked in thin film

➔ **superleak**

I.3.2 Experimental Observations

- superfluid film flow: *acting as a heat leak*



- film creeps up along the container walls and pumping line
- film creeps up to the position, where $T = 2.18$ K
- evaporation from film dominates, since vapor pressure at height h is much lower
- estimate of film thickness for „*unsaturated film*“

→ d depends on gas pressure p (kinetic gas theory)

$$p(h) = p_{\text{sat}} \exp\left(-\frac{mgh}{k_B T}\right) \Rightarrow h = -\frac{k_B T}{mg} \ln\left(\frac{p(h)}{p_{\text{sat}}}\right)$$

$$d = \left(\frac{\alpha}{mgh}\right)^{1/3} \Rightarrow d = \left(\frac{-\alpha}{k_B T \ln(p/p_{\text{sat}})}\right)^{1/3}$$

- estimate of the flow of liquid for a critical superfluid velocity of $v_s = 30$ cm/s:

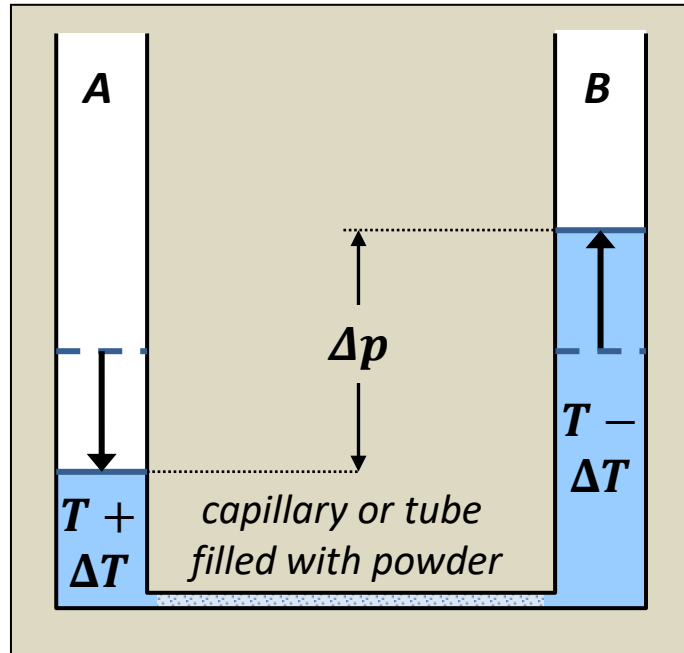
$$\dot{V}_{\text{liq}} = 2\pi R d v_s \approx 1 \text{ cm}^3/\text{h} \text{ for } R = 5 \text{ mm} \Rightarrow \dot{V}_{\text{gas}} \approx 1 \text{ l/h}$$

- min. pressure achieved by vacuum pump with $\dot{V}_{\text{pump}} \approx 10^4$ l/h

$$p_{\text{min}} = p \dot{V}_{\text{gas}}/\dot{V}_{\text{pump}} \approx 10^{-4} \text{ bar} \Rightarrow T_{\text{min}} \approx 1 \text{ K}$$

I.3.2 Experimental Observations

- thermomechanical effect: *generate ΔT by applying Δp*



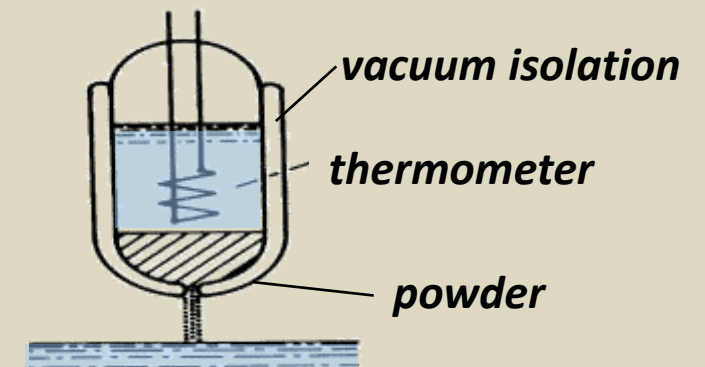
only superfluid can flow from A to B through narrow capillary

increase/decrease of normal fluid concentration in A/B

- starting point: same filling level and temperature T of container A and B
- apply $\Delta p \rightarrow$ only superfluid component flows through capillary from A to B, but cannot transport heat from A to B
- *temperature decreases in B and increases in A, since normal fluid fraction increases in A and decreases in B*
- remove $\Delta p \rightarrow$ original state is recovered
- note: mass flow is not accompanied by heat flow since only superfluid component can flow

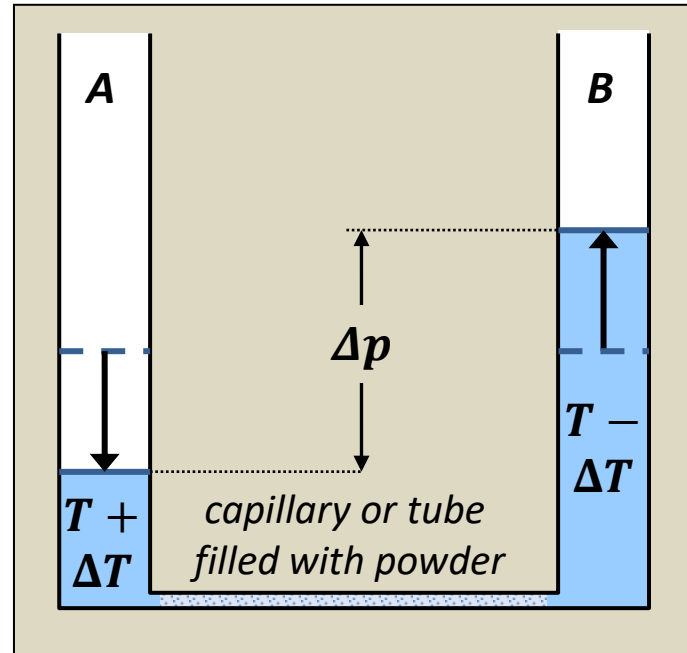
similar experiment:

- only superfluid flows through powder
- increase of normal fluid concentration
- increase of T in container



I.3.2 Experimental Observations

- thermomechanical effect: *generate ΔT by applying Δp*



only superfluid can flow from A to B through narrow capillary

➤ since $S = S_n$

$$\Delta p = \frac{S_n}{V} \Delta T = \frac{S_n}{M_n} \frac{M_n}{V} \Delta T = \sigma_n \rho_n \Delta T$$

$$\Delta p = \sigma_n \rho_n \Delta T \quad \Rightarrow \Delta p \rightarrow 0 \quad \text{for} \quad \rho_n \rightarrow 0$$

$$G = U - TS + pV$$

$$dG = TdS - pdV + \mu dN - TdS - SdT + pdV + Vdp$$

$$dG = -SdT + Vdp + \mu dN$$

- the configuration can be considered as two systems A and B in equilibrium:
 $\Rightarrow \mu_A = \mu_B = \mu$ and particle number is conserved: $N = N_A + N_B$

- Gibbs-Duhem relation:**

$$G(T, p, \mu_A, \mu_B) = N_A \mu_A + N_B \mu_B = N \mu$$

$$\Rightarrow dG = \mu dN + N d\mu$$

- comparison with $dG = -SdT + Vdp + \mu dN$ yields

$$-SdT + Vdp - Nd\mu = 0$$

$$-S(T_B - T_A) + V(p_B - p_A) - N \underbrace{(\mu_B - \mu_A)}_{=0} = 0$$

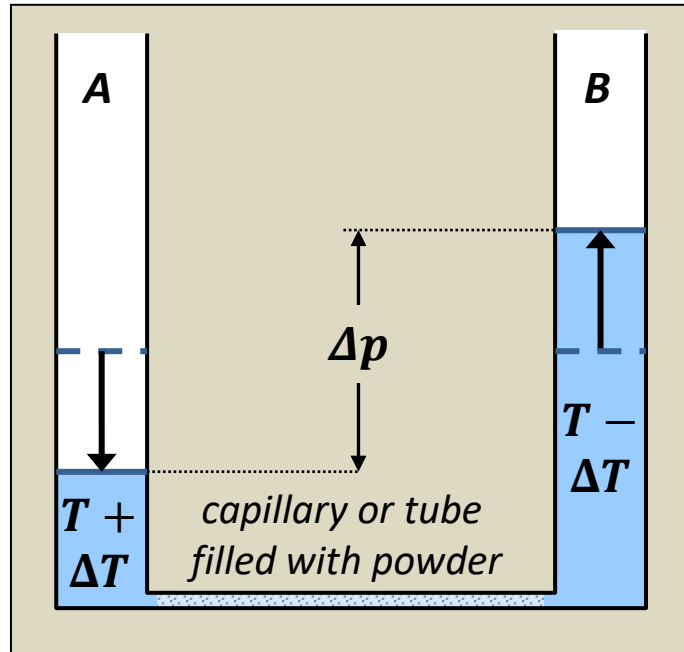


$$\underbrace{V(p_B - p_A)}_{\text{work associated with mass flow}} = \underbrace{S(T_B - T_A)}_{\text{compensating heat change}}$$

- since superfluid flow does not carry heat, there is no mechanism to equilibrate T_A and T_B
- nevertheless, the particle flow results in $\mu_A = \mu_B = \mu$

I.3.2 Experimental Observations

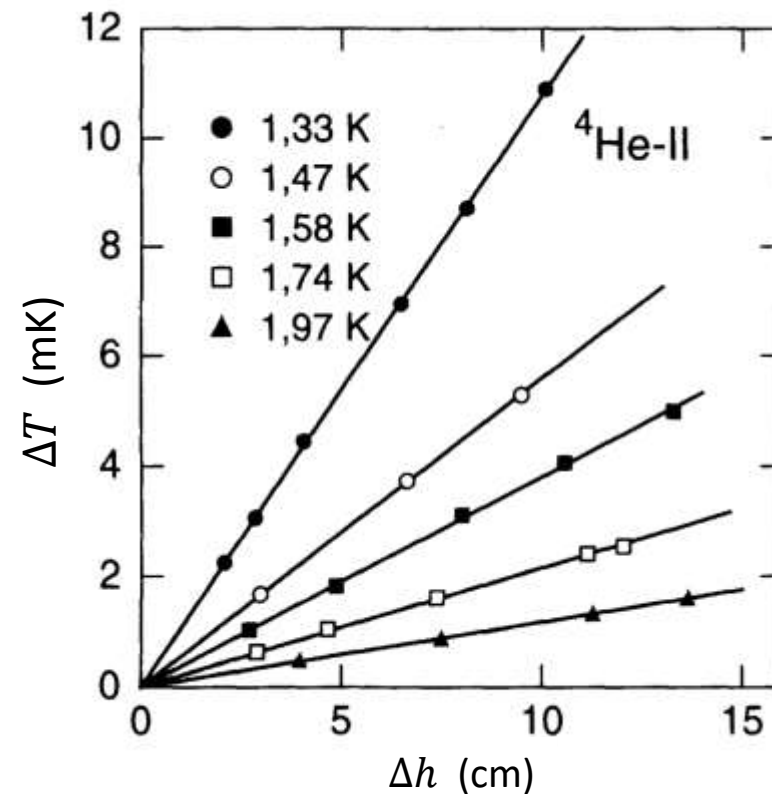
- thermomechanical effect: *generate ΔT by applying Δp*



➤ since $S = S_n$

$$\Delta p = \frac{S_n}{V} \Delta T = \frac{S_n}{M_n} \frac{M_n}{V} \Delta T = \sigma_n \rho_n \Delta T$$

$$\Delta p = \sigma_n \rho_n \Delta T \quad \rightarrow \Delta p \rightarrow 0 \text{ for } \rho_n \rightarrow 0$$

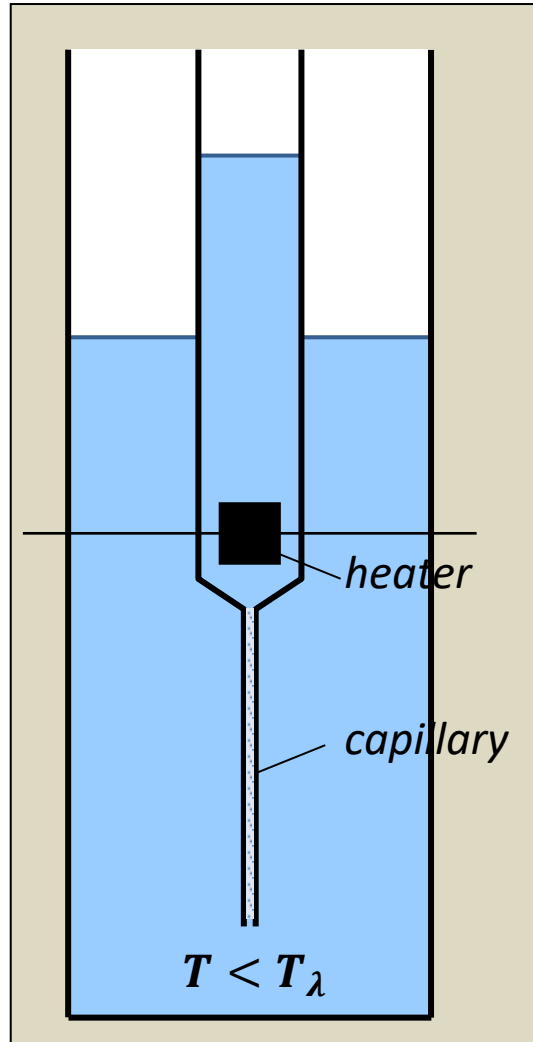


Enns, Hunklinger,
Low Temperature Physics, Springer (2000)

I.3.2 Experimental Observations

- inverse thermomechanical effect: *generate Δp by applying ΔT*

$$\Delta p = \sigma_n \rho_n \Delta T$$

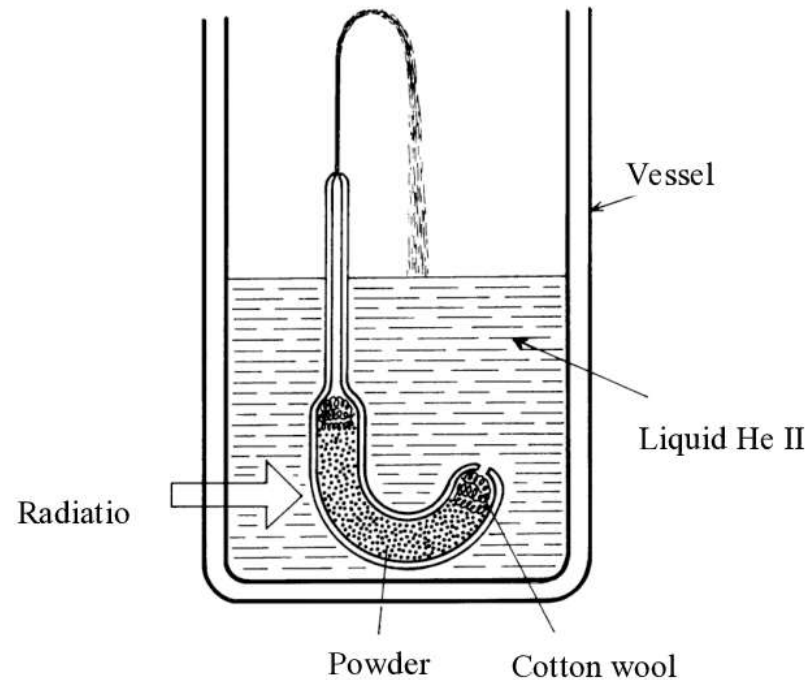


- heating generates increase of normal fluid concentration
- superfluid components flows through capillary to establish balance
- analogous to osmotic pressure
- superfluid flows very rapidly through capillary so that a fountain pressure can be generated

→ ***fountain effect***

I.3.2 Experimental Observations

- inverse thermomechanical effect: *fountain effect*



- light is absorbed by emery powder → heating
- lower opening is closed by cotton wool → only superfluid can enter
- a He fountain is generated by the thermomechanical effect (typical height up to 30 cm)



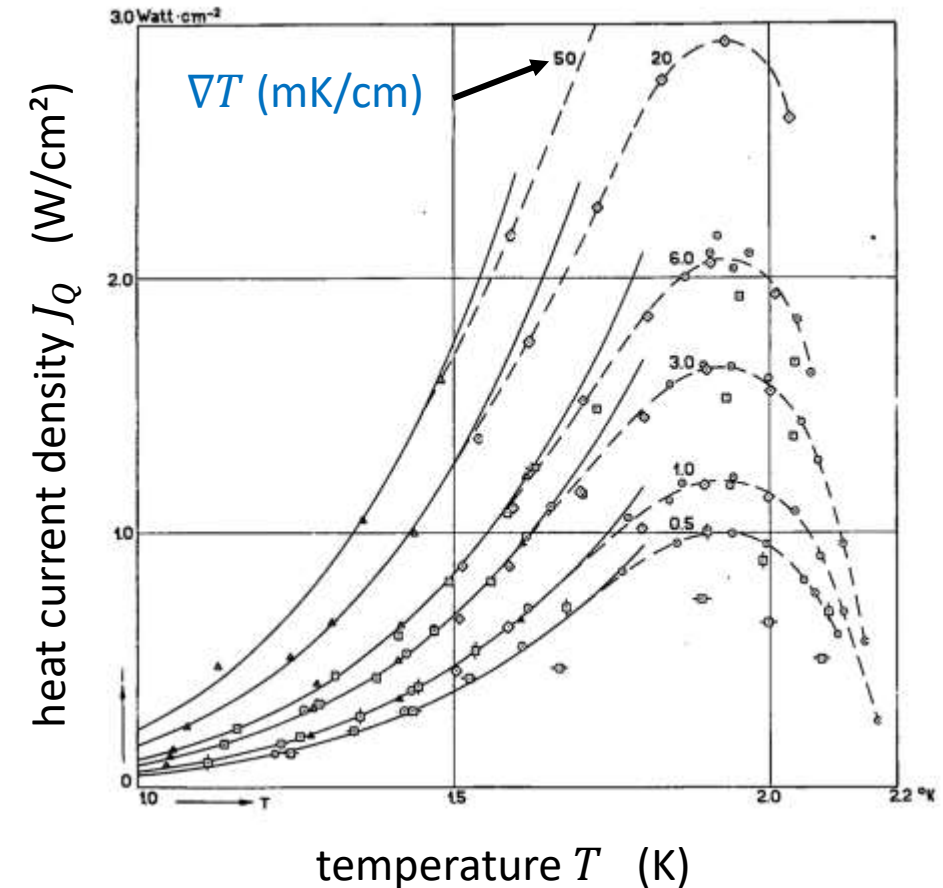
I.3.2 Experimental Observations

- thermal conductivity

- thermal conductivity of superfluid ^4He is by more than five orders of magnitude larger than that of normal liquid ^4He
- explains disappearance of boiling below T_λ
- heat conduction through narrow capillaries has maximum at $\approx 1.8\text{ K}$
- heat current density varies as $J_Q \propto \nabla T^{1/3}$
(if we would use $J_Q \simeq -\lambda \nabla T$, then $\lambda \propto \nabla T^{-2/3}$ and would diverge for small ∇T)
- linear $J_Q \propto \nabla T$ dependence only for small heat current densities

➔ *no Umklapp (U) processes possible in superfluid ^4He*
(no periodic lattice)

➔ *excitations (phonons, rotons) move ballistically, only surface scattering*



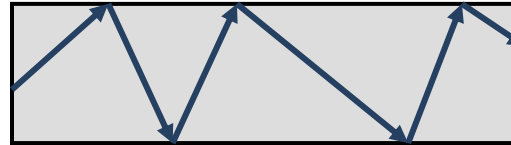
W.H. Keesom, B.F. Saris, L. Meyer, Physica **7**, 817 (1940)

I.3.2 Experimental Observations

- thermal conductivity: comparison to thermal conductivity of solids

solids at low T :

- ballistic phonons
- N processes do not cause momentum relaxation
- only surface scattering



kinetic gas theory:



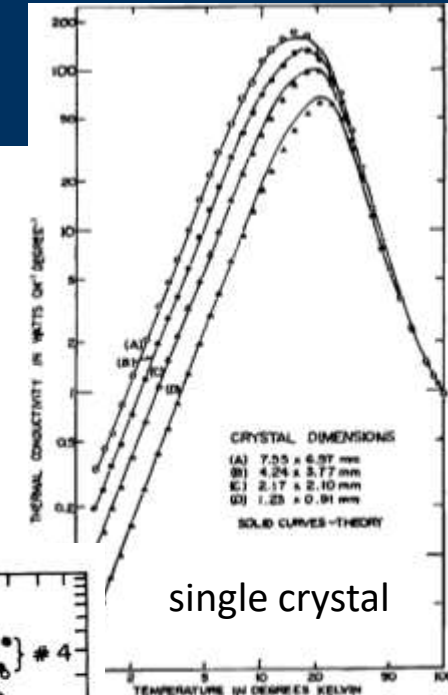
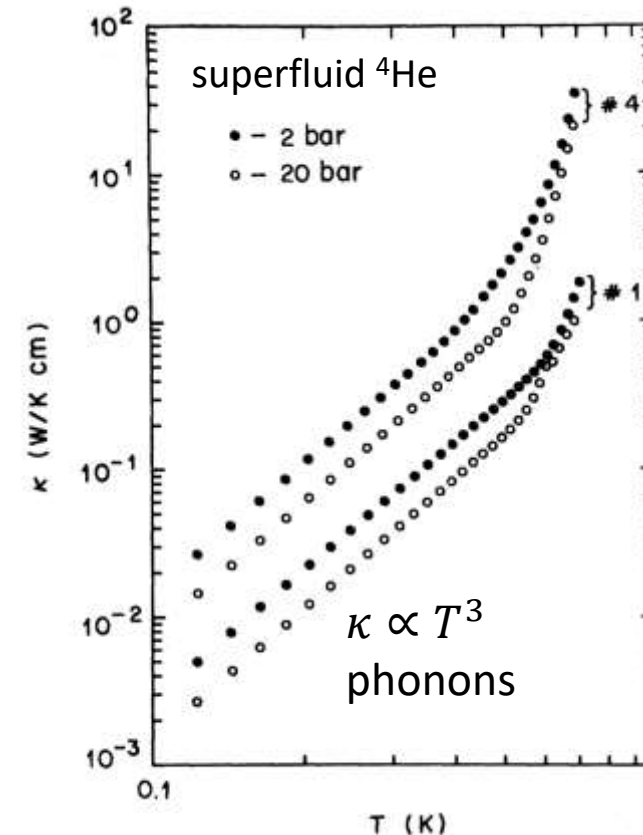
$$\kappa = \frac{1}{3} \langle v \rangle \ell C_V$$

function of d
(e.g. Hagen-Poiseuille)

$n \cdot c_V^*$

thermal conductivity κ is proportional to

- mean velocity of the particles
- mean free path
- density of the particles
- specific heat carried by particles



Dennis S. Greywall
Phys. Rev. B 23, 2152 (1981)



Walther
Meißner
Institut

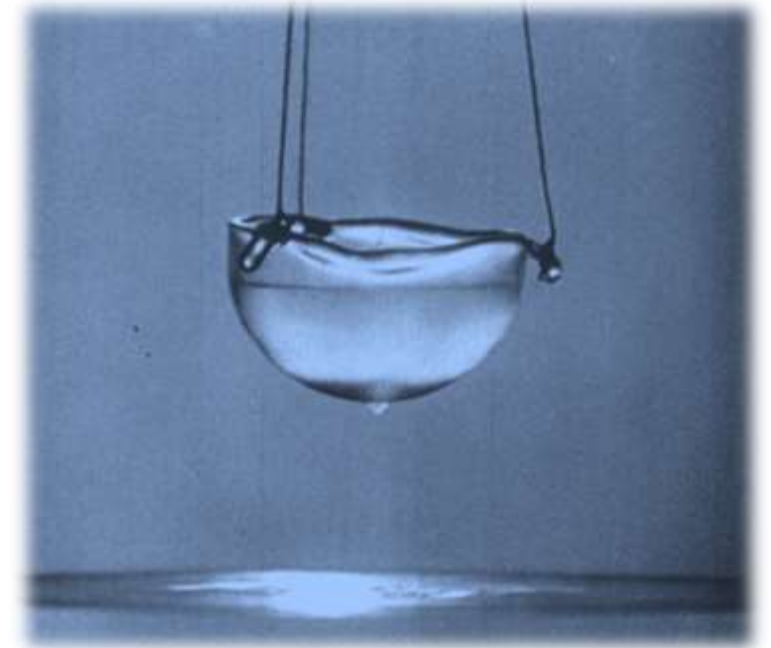
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München



Superconductivity and Low Temperature Physics II



Lecture No. 5

R. Gross

© Walther-Meißner-Institut

Contents Part I: Quantum Liquids

Contents:

I.1 Foundations and General Properties

- I.1.1 Quantum Gases & Liquids
- I.1.2 Helium
- I.1.3 Van der Waals Bonding
- I.1.4 Zero-Point Fluctuations
- I.1.5 Helium under Pressure
- I.1.6 pT-Phase Diagram of ^4He and ^3He
- I.1.7 Characteristic Properties of ^4He and ^3He
- I.1.8 Specific Heat of ^4He and ^3He

I.2 ^4He as an Ideal Bose Gas

- I.2.1 Bose-Einstein Condensation
- I.2.2 Ideal Bose Gas
- I.2.3 Bose Gas with Interactions
- I.2.4 Bose-Einstein Condensation of ^4He

I.3 Superfluid ^4He

- I.3.1 Two-Fluid Model
- I.3.2 Experimental Observations
- I.3.3 Two-Fluid Hydrodynamics
- I.3.4 Excitation Spectrum of ^4He

I.4 Vortices

- I.4.1 Quantization of Circulation
- I.4.2 Experimental Study of Vortices

I.5 ^3He

- I.5.1 normal fluid ^3He
- I.5.2 solid ^3He and Pomeranchuk effect
- I.5.3 superfluid ^3He

I.6 ^3He / ^4He mixtures



I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics

- starting point for hydrodynamic description are **Navier-Stokes equations**:

➤ **normal fluid:**

$$\underbrace{\rho_n \frac{D\mathbf{v}_n}{Dt}}_{\text{inertia}} = - \underbrace{\frac{\rho_n}{\rho} \nabla p}_{\text{pressure gradient}} - \underbrace{\frac{\rho_n \sigma_n}{\rho} \nabla T}_{\text{temperature gradient}} - \underbrace{\frac{\rho_s \rho_n}{2\rho} \nabla (\mathbf{v}_n - \mathbf{v}_s)^2}_{\text{add. term due to compressibility}} + \underbrace{\eta_n \nabla^2 \mathbf{v}_n}_{\text{viscosity}}$$

entropy per mass

ρ_s, ρ_n = superfluid and normal fluid densities
 $\mathbf{v}_s, \mathbf{v}_n$ = superfluid and normal fluid velocities
 η_n = normal fluid viscosity

terms have unit force/volume

with the definition of the material derivative $D\mathbf{v}/Dt$:

$$\frac{D\mathbf{v}_n}{Dt} = \frac{\partial \mathbf{v}_n}{\partial t} + \left(\frac{\partial \mathbf{v}_n}{\partial \mathbf{x}} \right) \cdot \mathbf{v}_n \quad \text{and} \quad \left(\frac{\partial \mathbf{v}_n}{\partial \mathbf{x}} \right) = \nabla \left(\frac{\|\mathbf{v}_n\|^2}{2} \right) + \underbrace{(\nabla \times \mathbf{v}_n) \times \mathbf{v}_n}_{\text{curl of the velocity (called vorticity)}}$$

the tensor derivative of the velocity vector

curl of the velocity (called vorticity)
disappears for irrotational flow
(no vortices)

I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics

➤ **superfluid (no viscous friction):**

$$\underbrace{\rho_s \frac{D\mathbf{v}_s}{Dt}}_{\text{inertia}} = - \underbrace{\frac{\rho_s}{\rho} \nabla p}_{\text{pressure gradient}} - \underbrace{\rho_s \sigma_s \nabla T}_{=0 \text{ as superfluid carries no entropy}} - \underbrace{\frac{\rho_s \rho_n}{2\rho} \nabla (\mathbf{v}_n - \mathbf{v}_s)^2}_{\text{add. term due to compressibility}} + \underbrace{\eta_s \nabla^2 \mathbf{v}_s}_{=0 \text{ as viscosity } \eta_s=0}$$

ρ_s, ρ_n = superfluid and normal fluid densities
 $\mathbf{v}_s, \mathbf{v}_n$ = superfluid and normal fluid velocities
 η_n = normal fluid viscosity

terms have unit force/volume

with the definition of the material derivative $D\mathbf{v}/Dt$:

$$\frac{D\mathbf{v}_s}{Dt} = \frac{\partial \mathbf{v}_s}{\partial t} + \left(\frac{\partial \mathbf{v}_s}{\partial \mathbf{x}} \right) \cdot \mathbf{v}_s \quad \text{and} \quad \left(\frac{\partial \mathbf{v}_s}{\partial \mathbf{x}} \right) = \nabla \left(\frac{\|\mathbf{v}_s\|^2}{2} \right) + \underbrace{(\nabla \times \mathbf{v}_s) \times \mathbf{v}_s}_{\text{curl of the velocity (called vorticity) disappears for irrotational flow (no vortices)}}$$

the tensor derivative of the velocity vector

➔ **Euler type equation for superfluid** (if there are no vortices)

➔ detailed treatment including vortices: **Gross-Pitaevskii equation**

I.3.3 Two-Fluid Hydrodynamics

- **two-fluid hydrodynamics (1)** - a simplified description

- mass flow density:
$$\mathbf{J} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s$$

\nearrow
normal

\uparrow
superfluid component

- conservation of mass:
$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$$

- viscosity of normal component is very small and can be neglected in most situations
 - in first order approximation we treat superfluid He as ideal liquid
 - description by **Euler equation** (*motion of a compressible, inviscid fluid*)

$$\frac{\partial \mathbf{J}}{\partial t} + \underbrace{\rho \mathbf{v} \cdot \nabla \mathbf{v}}_{\approx 0} = -\nabla p \quad (p = \text{pressure})$$

→ if \mathbf{v} is small, quadratic terms in \mathbf{v} can be neglected

- since we are neglecting dissipative effects (no viscous friction), entropy is conserved:

$$\frac{\partial S}{\partial t} = \frac{\partial(\rho\sigma)}{\partial t} = -\nabla \cdot (\rho\sigma) \mathbf{v}_n \quad \sigma = \text{entropy per mass}$$

I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics (2)

- exact derivation of equation of motion of normal and superfluid phase is difficult
→ we use approach following „Gedanken“ experiment of Landau
- we are adding superfluid component at constant volume and consider change of inner energy
→ the resulting operation only changes the mass of the superfluid component, no entropy and volume change)

$$dU = TdS - pdV + \underbrace{\mu_m}_{\text{potential energy per mass}} dm \Rightarrow -\nabla(\mu_m dm) = \mathbf{F} = dm \frac{d\mathbf{v}_s}{dt}$$

- since adding particles to the superfluid does not change the energy, we have

$$0 = TdS - pdV + \mu_m dm \Rightarrow \frac{d\mathbf{v}_s}{dt} = \frac{\Delta S}{\Delta m} \nabla T - \frac{\Delta V}{\Delta m} \nabla p = \sigma \nabla T - \frac{1}{\rho} \nabla p$$

- with $\frac{d\mathbf{v}_s}{dt} = \frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{v}_s \simeq \frac{\partial \mathbf{v}_s}{\partial t}$ (neglecting quadratic terms in \mathbf{v}_s) we obtain

$$\frac{\partial \mathbf{v}_s}{\partial t} = \sigma \nabla T - \frac{1}{\rho} \nabla p$$

I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics (3)

- for the normal fluid component we obtain

$$\frac{\partial \mathbf{J}}{\partial t} = \frac{\partial(\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s)}{\partial t} = -\nabla \mathbf{p} \quad \Rightarrow \quad \frac{\partial \mathbf{v}_n}{\partial t} = -\frac{1}{\rho_n} \nabla \mathbf{p} - \frac{\rho_s}{\rho_n} \frac{\partial \mathbf{v}_s}{\partial t}$$

$$\frac{\partial \mathbf{v}_s}{\partial t} = \sigma \nabla T - \frac{1}{\rho} \nabla p$$

with $\frac{\partial \mathbf{v}_s}{\partial t} = \sigma \nabla T - \frac{1}{\rho} \nabla p$ we then obtain

$$\frac{\partial \mathbf{v}_n}{\partial t} = -\frac{1}{\rho_n} \nabla \mathbf{p} - \frac{\rho_s}{\rho_n} \sigma \nabla T + \frac{\rho_s}{\rho_n} \frac{1}{\rho} \nabla p = -\frac{\rho_s}{\rho_n} \sigma \nabla T + \frac{\rho_s - \rho}{\rho_n \rho} \nabla p = -\frac{\rho_s}{\rho_n} \sigma \nabla T - \frac{1}{\rho} \nabla p$$

$$\frac{\partial \mathbf{v}_n}{\partial t} = -\frac{\rho_s}{\rho_n} \sigma \nabla T - \frac{1}{\rho} \nabla p$$

- equations of motion

$$\frac{\partial \mathbf{v}_s}{\partial t} = \sigma \nabla T - \frac{1}{\rho} \nabla p$$

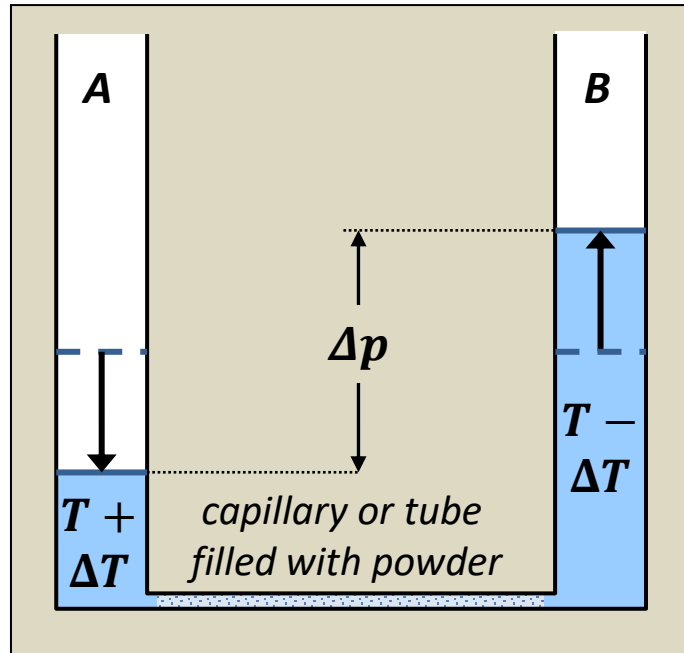
$$\frac{\partial \mathbf{v}_n}{\partial t} = -\frac{\rho_s}{\rho_n} \sigma \nabla T - \frac{1}{\rho} \nabla p$$

note:

the equations of motion are valid only for the linear regime !

I.3.3 Two-Fluid Hydrodynamics

- example: explanation of thermomechanical effect:



$$\frac{\partial \mathbf{v}_s}{\partial t} = \sigma \nabla T - \frac{1}{\rho} \nabla p$$

– in steady state: $\frac{\partial \mathbf{v}_s}{\partial t} = 0$

$$0 = \sigma \nabla T - \frac{1}{\rho} \nabla p$$

$$\Delta p = \sigma \rho \Delta T$$

I.3.3 Two-Fluid Model

- two-fluid hydrodynamics can be used to understand sound propagation in superfluid helium

- *first sound:*

normal and superfluid component move *in-phase*:

$$v_n = v_s$$

$$v_1 = 238 \text{ m/s for } T \rightarrow 0$$

- *second sound:*

normal and superfluid component move *out-of-phase*:
(Peshkov 1944)

$$\rho_n v_n + \rho_s v_s = 0$$

$$v_2 = \frac{v_1}{\sqrt{3}} \simeq 137 \text{ m/s for } T \rightarrow 0$$

- *third sound: waves propagating in thin He films*

- *fourth sound: compression wave propagating in a super leak*

use two-fluid hydrodynamics to understand this behavior

(super leak: very small opening or capillary, in which normal fluid cannot move because of its finite viscosity)

I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics – sound propagation (1)

- take time derivative of $\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$ and insert into $\frac{\partial \mathbf{J}}{\partial t} + \underbrace{\rho \mathbf{v} \cdot \nabla \mathbf{v}}_{\approx 0} = -\nabla p$

$$\frac{\partial^2 \rho}{\partial t^2} = -\nabla \cdot \frac{\partial \mathbf{J}}{\partial t} = -\nabla \cdot (-\nabla p) \Rightarrow \frac{\partial^2 \rho}{\partial t^2} = \nabla^2 p$$

- elimination of terms with \mathbf{v}_n and \mathbf{v}_s in $\frac{\partial S}{\partial t} = \frac{\partial(\rho\sigma)}{\partial t} = -\nabla \cdot (\rho\sigma) \mathbf{v}_n$ and $\frac{\partial \mathbf{v}_s}{\partial t} = \sigma \nabla T - \frac{1}{\rho} \nabla p$ (cannot be observed experimentally)

→ after some steps and neglecting higher order terms one obtains

$$\frac{\partial^2 \sigma}{\partial t^2} = \frac{\rho_s}{\rho_n} \sigma^2 \nabla^2 T$$

$$\frac{\partial \mathbf{v}_s}{\partial t} = \sigma \nabla T - \frac{1}{\rho} \nabla p$$

$$\frac{\partial \mathbf{v}_n}{\partial t} = -\frac{\rho_s}{\rho_n} \sigma \nabla T - \frac{1}{\rho} \nabla p$$

- in the two equations we have four variables: ρ, σ, p and $T \rightarrow$ only two of them are independent, we choose ρ and σ
- dependence of p and T on ρ and σ can be expressed as

$$\delta p = \left(\frac{\partial p}{\partial \rho} \right)_\sigma \delta \rho + \left(\frac{\partial p}{\partial \sigma} \right)_\rho \delta \sigma \quad \delta T = \left(\frac{\partial T}{\partial \rho} \right)_\sigma \delta \rho + \left(\frac{\partial T}{\partial \sigma} \right)_\rho \delta \sigma$$

we insert relations into expression for $\frac{\partial^2 \rho}{\partial t^2}$ and $\frac{\partial^2 \sigma}{\partial t^2}$

I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics – sound propagation (2)

- insertion yields

$$\frac{\partial^2 \rho}{\partial t^2} = \left(\frac{\partial p}{\partial \rho} \right)_\sigma \nabla^2 \rho + \left(\frac{\partial p}{\partial \sigma} \right)_\rho \nabla^2 \sigma$$

$$\frac{\partial^2 \sigma}{\partial t^2} = \frac{\rho_s}{\rho_n} \sigma^2 \left[\left(\frac{\partial T}{\partial \rho} \right)_\sigma \nabla^2 \rho + \left(\frac{\partial T}{\partial \sigma} \right)_\rho \nabla^2 \sigma \right]$$

2nd order partial differential equations

- to solve these equations we use the ansatz (1D for simplicity)

$$\rho = \rho_0 + \rho_1 \exp \left[i\omega \left(t - \frac{x}{v} \right) \right] \quad \sigma = \sigma_0 + \sigma_1 \exp \left[i\omega \left(t - \frac{x}{v} \right) \right]$$

- inserting the ansatz yields

*linear system
of equations*

$$\left[\left(\frac{v}{v_1} \right)^2 - 1 \right] \rho_1 + \left(\frac{\partial p}{\partial \sigma} \right)_\rho \left(\frac{\partial \rho}{\partial p} \right)_\sigma \sigma_1 = 0$$

$$\left(\frac{\partial T}{\partial \rho} \right)_\sigma \left(\frac{\partial \sigma}{\partial T} \right)_\rho \rho_1 + \left[\left(\frac{v}{v_2} \right)^2 - 1 \right] \sigma_1 = 0$$

with abbreviations

$$v_1^2 = \left(\frac{\partial p}{\partial \rho} \right)_\sigma$$

$$v_2^2 = \frac{\rho_s}{\rho_n} \sigma^2 \left(\frac{\partial T}{\partial \sigma} \right)_\rho$$

I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics – sound propagation (3)

- constraint equation for coefficients

$$\left[\left(\frac{v}{v_1} \right)^2 - 1 \right] \cdot \left[\left(\frac{v}{v_2} \right)^2 - 1 \right] = \left(\frac{\partial p}{\partial \sigma} \right)_\rho \left(\frac{\partial \rho}{\partial p} \right)_\sigma \left(\frac{\partial T}{\partial \rho} \right)_\sigma \left(\frac{\partial \sigma}{\partial T} \right)_\rho$$

- the r.h.s. can be rewritten by using known thermodynamic relations

$$\left[\left(\frac{v}{v_1} \right)^2 - 1 \right] \cdot \left[\left(\frac{v}{v_2} \right)^2 - 1 \right] = \frac{C_p - C_V}{C_p} \quad C_p, C_V: \text{specific heat at constant pressure/volume}$$

- for superfluid helium the difference $C_p - C_V$ is very small \rightarrow we can use the approximation

$$\left[\left(\frac{v}{v_1} \right)^2 - 1 \right] \cdot \left[\left(\frac{v}{v_2} \right)^2 - 1 \right] \simeq 0$$

$$v_1^2 = \left(\frac{\partial p}{\partial \rho} \right)_\sigma$$

$$v_2^2 = \frac{\rho_s}{\rho_n} \sigma^2 \left(\frac{\partial T}{\partial \sigma} \right)_\rho$$

I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics: **first sound**

- propagation of usual sound waves with velocity $v = v_1$ is given by

$$\left[\left(\frac{v}{v_1} \right)^2 - 1 \right] \rho_1 + \left(\frac{\partial p}{\partial \sigma} \right)_\rho \left(\frac{\partial \rho}{\partial p} \right)_\sigma \sigma_1 = 0 \quad \text{if we set } \rho_1 \neq 0 \text{ and } \sigma_1 = 0$$

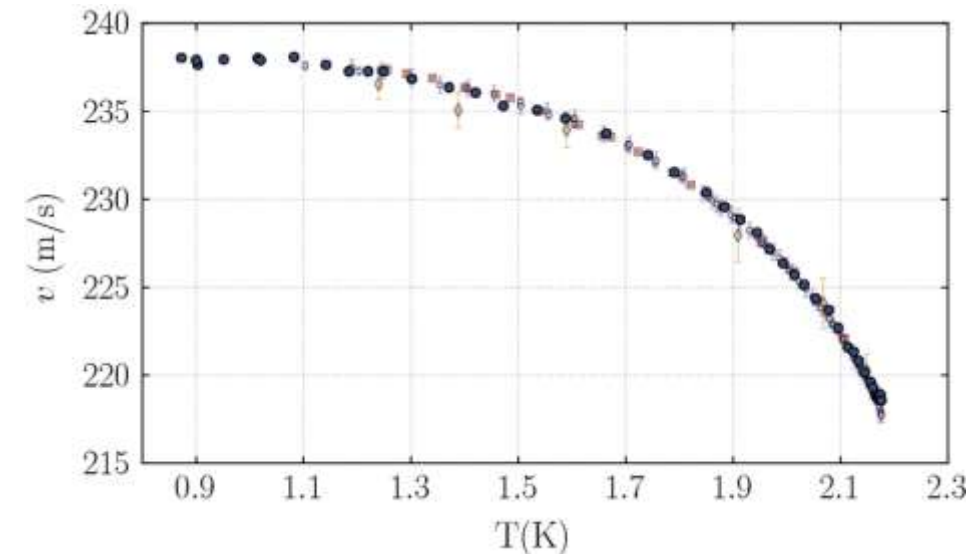
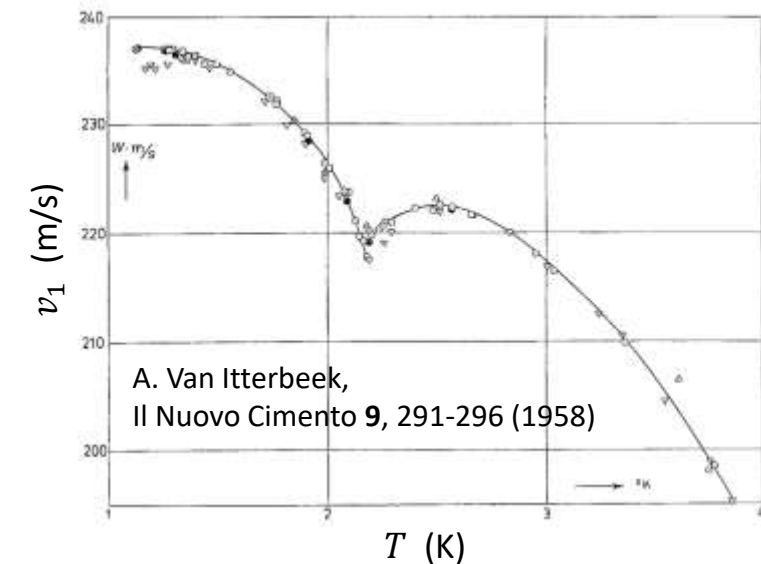
under these conditions we have $\nabla T = 0$ as for a usual sound waves (**first sound**)

- from $\frac{\partial \mathbf{J}}{\partial t} = -\nabla \mathbf{p}$ and $\frac{\partial \mathbf{v}_s}{\partial t} = \sigma \nabla T - \frac{1}{\rho} \nabla p$ we obtain

$$v_n = v_s \quad (\text{acoustic mode})$$

→ **normal and superfluid component move in phase** (**first sound**)

- density changes are adiabatic
- superfluid He behaves as usual liquid
- sound velocity $v = v_1 = 238 \text{ m/s}$ for $T \rightarrow 0$
- $v_1(T)$ shows anomaly at $T = T_\lambda$



L. Djadaojee, A. Douillet, J. Grucker, Jules, J. Low Temp. Phys. 203 (2021)

I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics: **second sound**

- if we set $\rho_1 = 0$ and $\sigma_1 \neq 0$ we obtain temperature wave propagating with velocity $v = v_2$

$$\left(\frac{\partial T}{\partial \rho}\right)_\sigma \left(\frac{\partial \sigma}{\partial T}\right)_\rho \rho_1 + \left[\left(\frac{v}{v_2}\right)^2 - 1\right] \sigma_1 = 0$$

- from $\frac{\partial^2 \rho}{\partial t^2} = \nabla^2 p$ we obtain $\nabla p = 0$

- together with Euler's equation $\frac{\partial \mathbf{J}}{\partial t} + \underbrace{\rho \mathbf{v} \cdot \nabla \mathbf{v}}_{\approx 0} = -\nabla p$ we obtain: $\frac{\partial \mathbf{J}}{\partial t} = \frac{\partial(\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s)}{\partial t} = 0$

- since \mathbf{J} cannot be constant in a closed container $\rightarrow \mathbf{J} = 0$

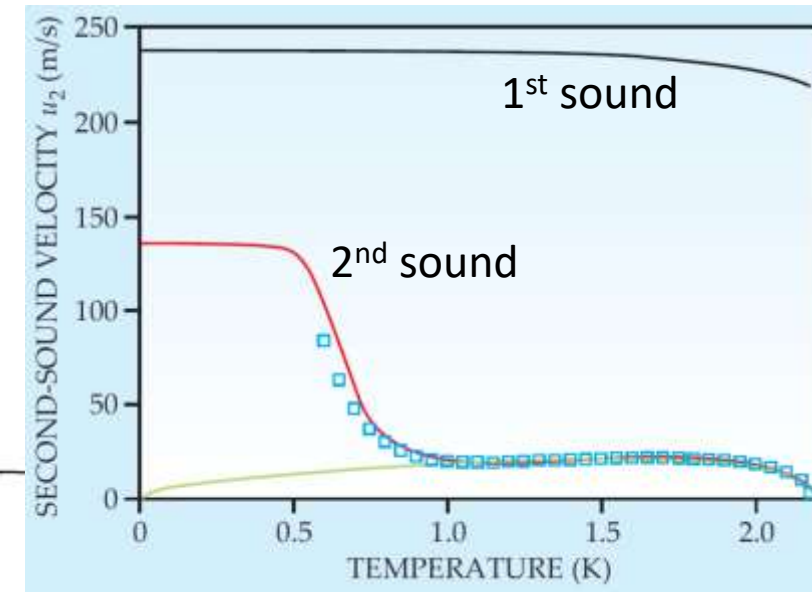
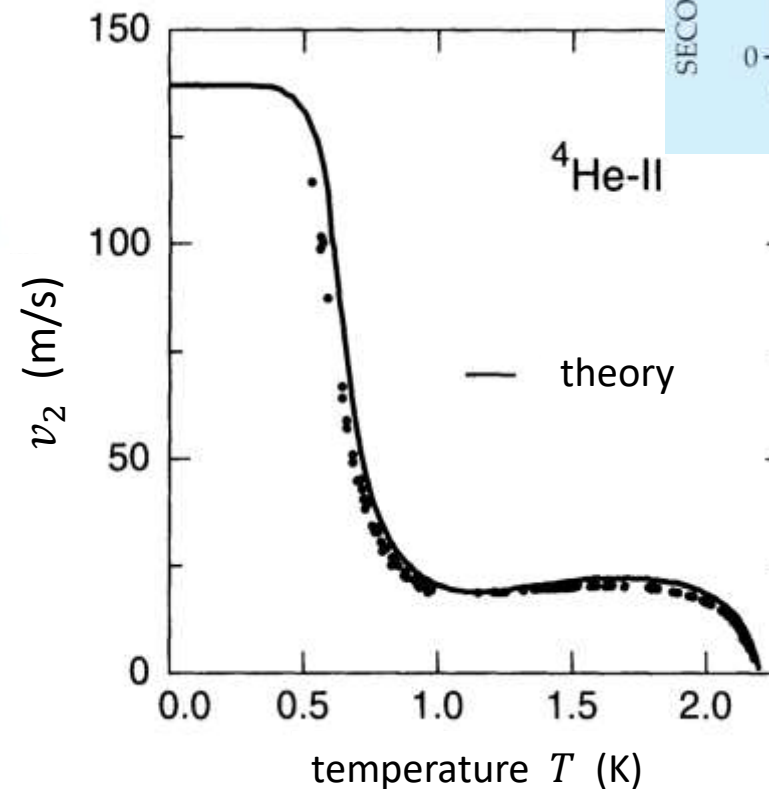
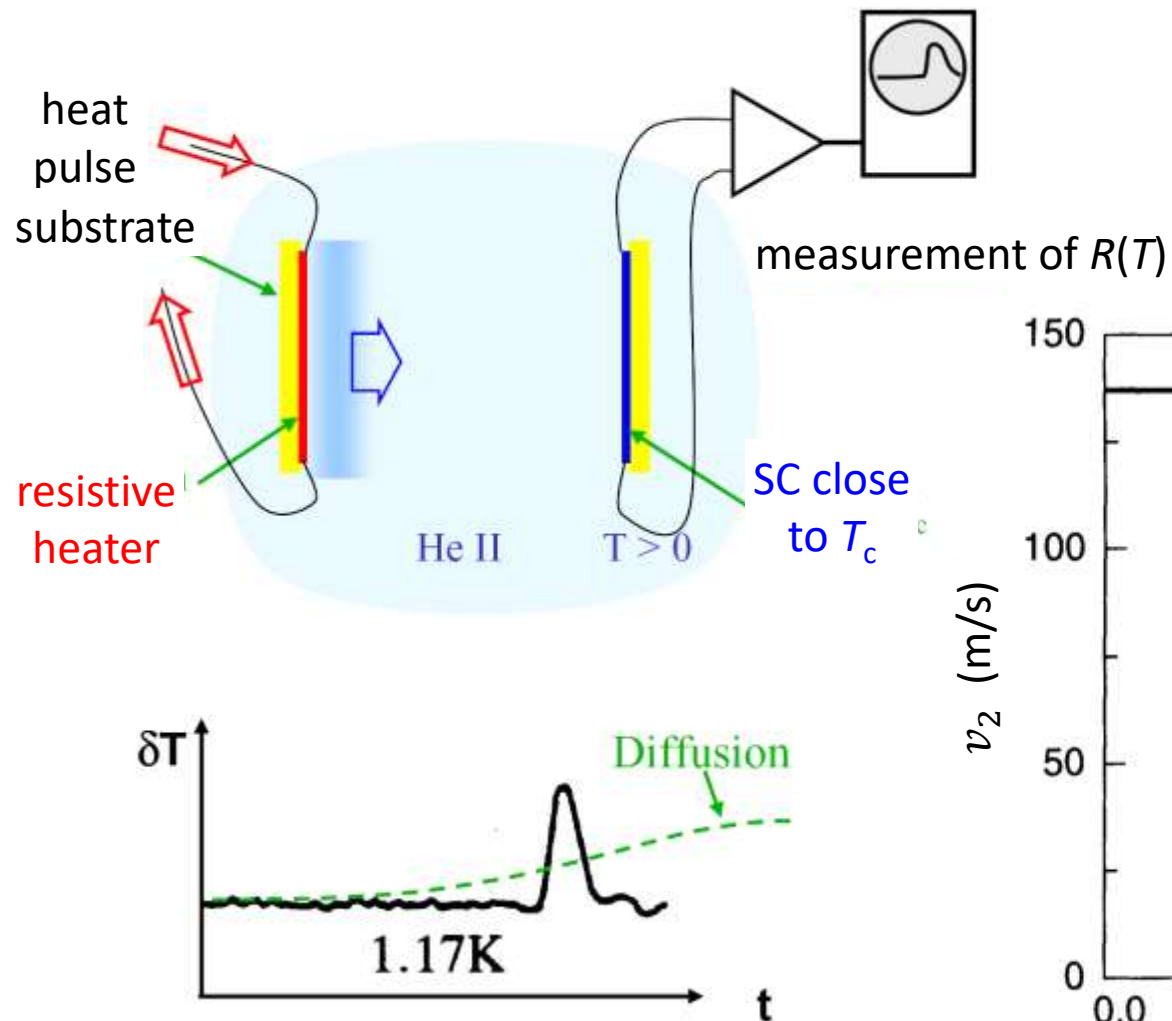
$$\rho_n v_n + \rho_s v_s = 0 \quad (\text{optical mode})$$

- velocity of the temperature wave

$$v_2^2 = \frac{\rho_s}{\rho_n} \sigma^2 \left(\frac{\partial T}{\partial \sigma}\right)_\rho = \frac{\rho_s}{\rho_n} \sigma^2 \frac{T}{C_p}, \quad v_2 = \frac{v_1}{\sqrt{3}} \simeq 137 \text{ m/s for } T \rightarrow 0$$

I.3.3 Two-Fluid Hydrodynamics

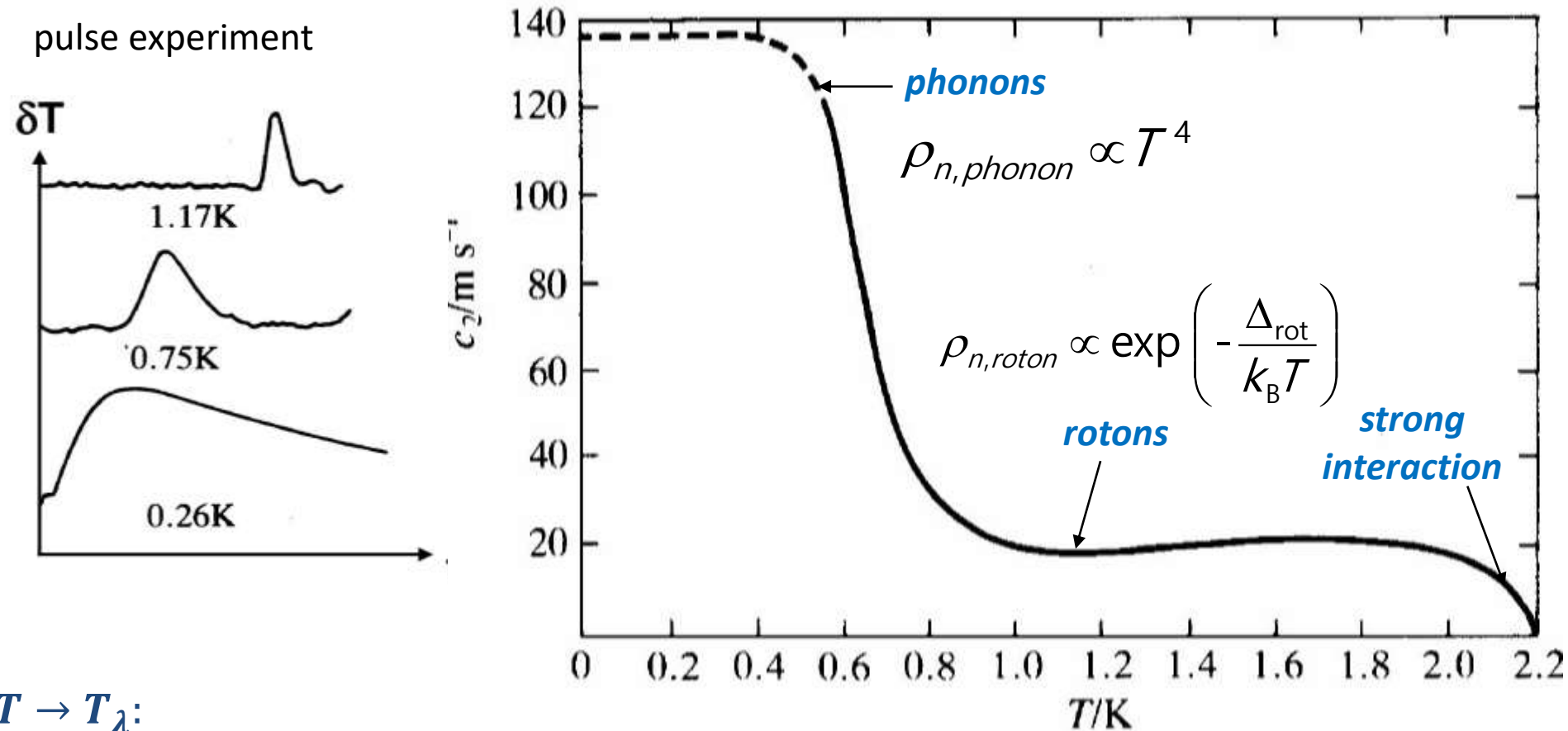
- two-fluid hydrodynamics: **second sound** – experiment



Enns, Hunklinger,
Low Temperature Physics, Springer (2000)

I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics: **second sound** – experiment



$T \rightarrow T_\lambda$:

- collision probability increases
- picture of elementary excitations no longer valid
- vortex rings

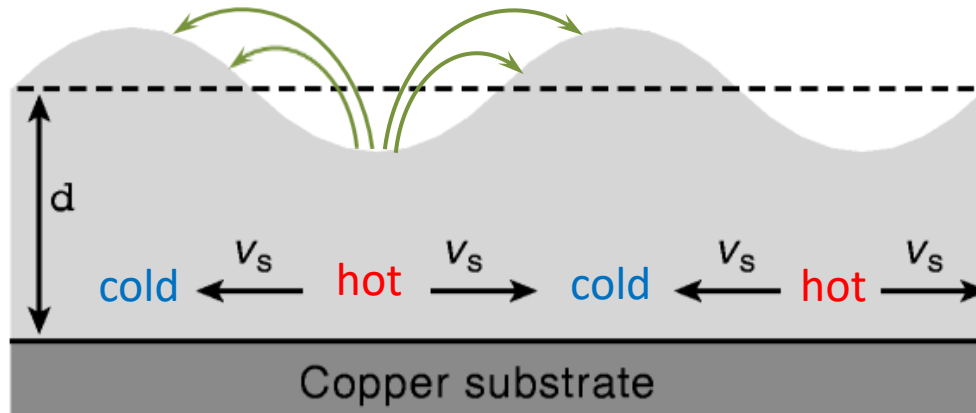
I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics: **third sound – thickness waves in thin helium films**

- we already discussed the thickness of helium films on surfaces

$$d = \left(\frac{-\alpha}{k_B T \ln(p/p_{\text{sat}})} \right)^{1/3} = \left(\frac{\alpha}{k_B T (\ln p_{\text{sat}} - \ln p)} \right)^{1/3}$$

- film thickness depends on pressure and temperature
- local periodic heating causes thickness variation propagating along surface

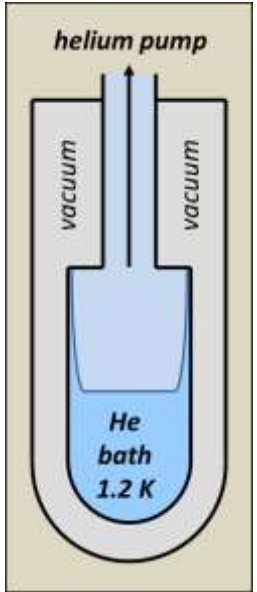


- viscous normal component is fixed at surface
- superfluid component oscillates freely parallel to surface

wave crest: surplus of superfluid → **cold**

wave valley: deficit of superfluid → **hot**

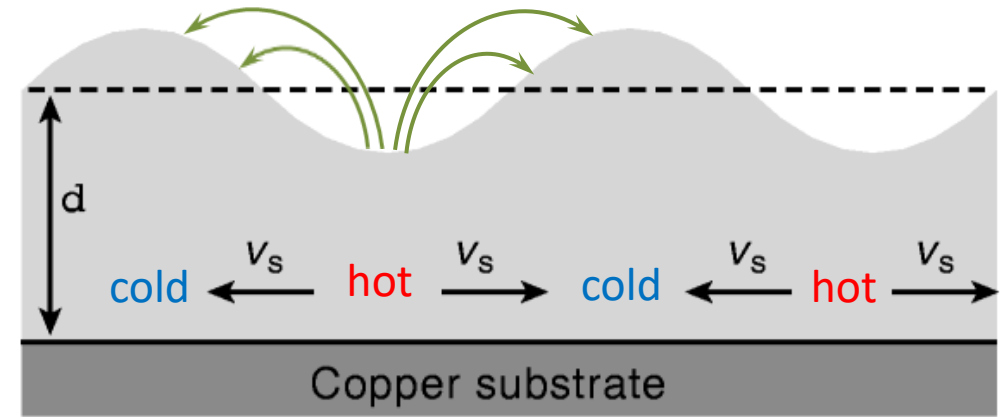
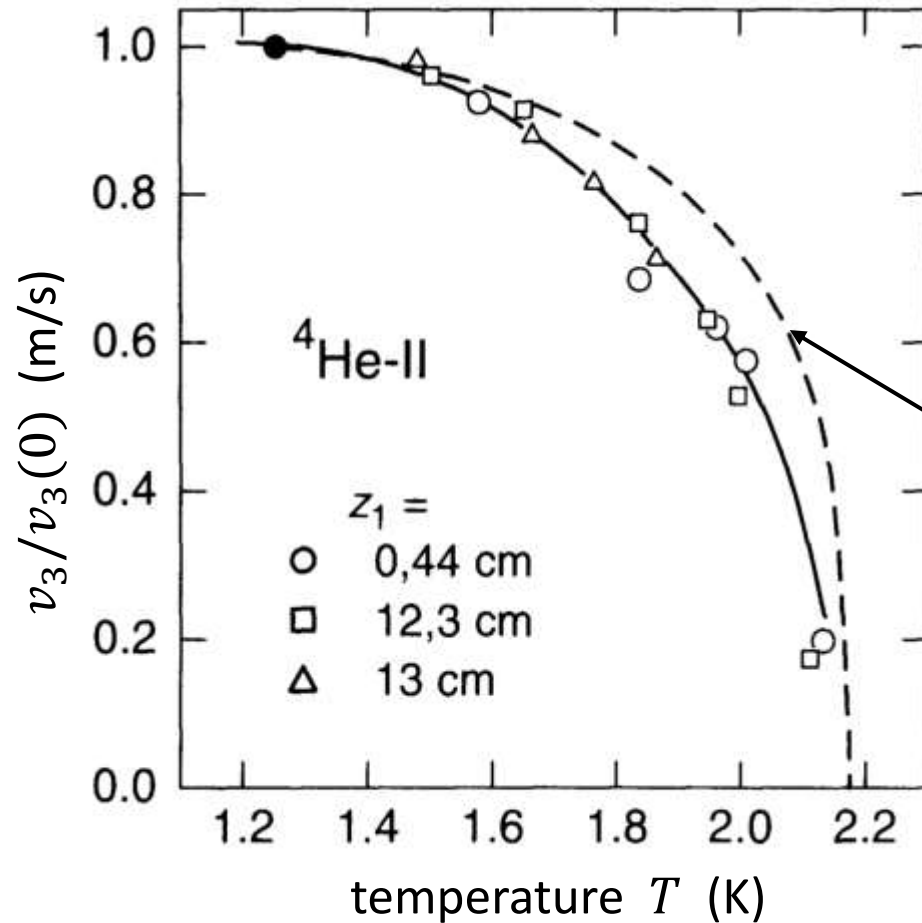
sound velocity: $v_3 \sim 10 \dots 100 \text{ m/s}$



detailed discussion complicated due to additional evaporation and condensation at surface

I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics: **third sound – thickness waves in thin helium films**



$$v_3^2 = \frac{3\alpha}{d^3} \frac{\rho_s}{\rho} \left(1 + \frac{\sigma T}{L} \right)$$

L = latent heat

Enns, Hunklinger,
Low Temperature Physics, Springer (2000)

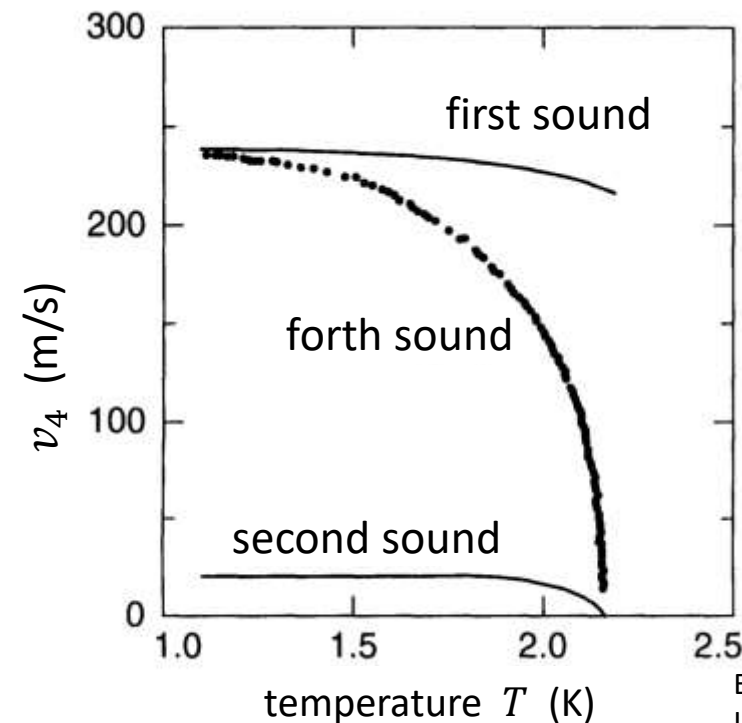
I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics: **forth sound – compression waves in superleaks**

(super leak = very small opening or capillary, in which normal fluid cannot move because of its finite viscosity)

- only superfluid can move, whereas normal fluid is blocked due to its finite viscosity
- compression waves are not only density oscillations, but also oscillations of the superfluid density ρ_s/ρ_n , the pressure p , the temperature T , and the entropy σ
- sound velocity (without derivation)

$$v_4^2 \simeq \frac{\rho_s}{\rho} v_1^2 + \frac{\rho_n}{\rho} v_2^2$$



Enns, Hunklinger,
Low Temperature Physics, Springer (2000)

I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics: **heat transport**

- example: mass flow \dot{V}_n through capillary of diameter d :
 - the normal fluid component flows from hot to cold side
 - this mass flow is compensated by opposite flow of superfluid component
 - due to different entropies of both components this results in a finite heat flow (limited by η_n)
- mass flow \dot{V}_n of normal component according to classical hydrodynamics

$$\dot{V}_n = \frac{\beta}{\eta_n} \frac{\Delta p}{L}$$

η_n : viscosity of normal component

L : length of capillary

Δp : pressure difference

- the parameter β depends on the geometry of the capillary: e.g. $\beta \propto r^4$ according to Hagen-Poiseuille law
- heat flow = mass flow times heat/mass: $\dot{Q} = \rho_n \dot{V}_n \cdot \sigma_n T = \dot{V}_n S_n T$

$$\dot{Q}_n = \rho_n \underbrace{\dot{V}_n}_{\frac{\beta}{\eta_n} \frac{\Delta p}{L}} \cdot \sigma_n T = \rho_n \frac{\beta}{\eta_n} \frac{\Delta p}{L} \sigma_n T \quad \text{with } \Delta p = \rho_n \sigma_n \Delta T \Rightarrow$$

$$\dot{Q}_n = \frac{\beta T}{\eta_n L} (\rho_n \sigma_n)^2 \Delta T$$

I.3.3 Two-Fluid Hydrodynamics

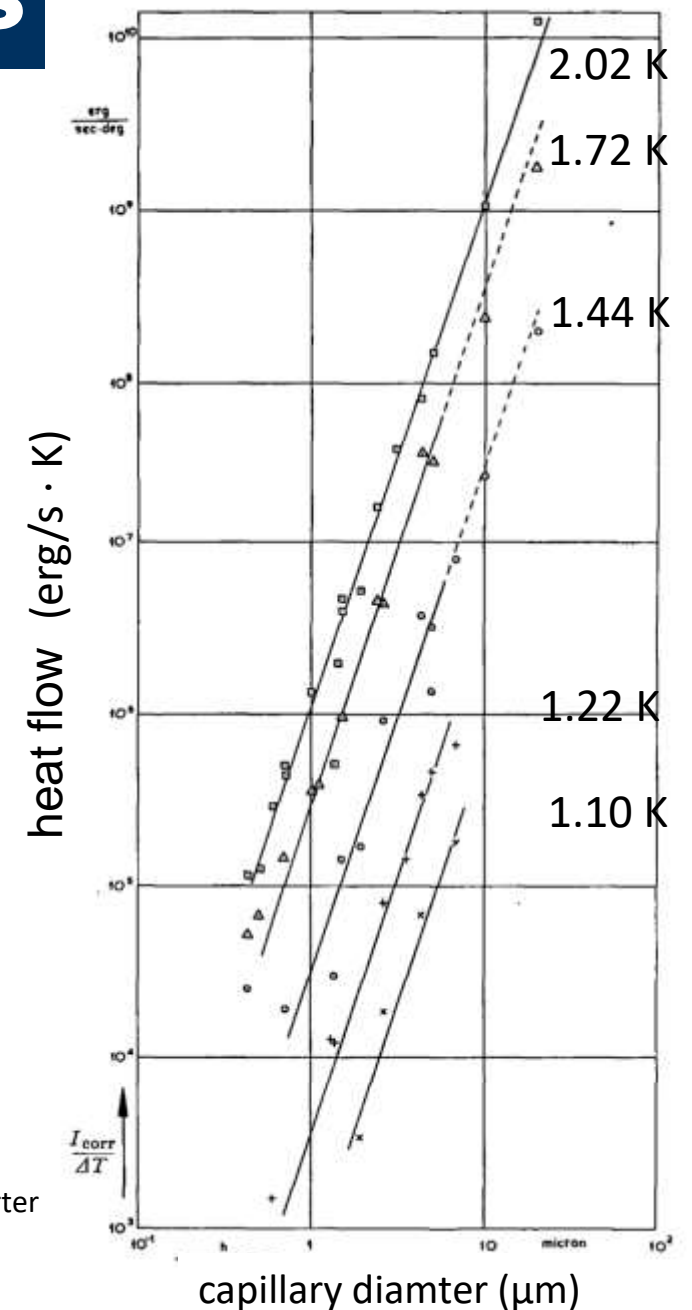
- two-fluid hydrodynamics: **heat transport**

$$\dot{Q}_n = \frac{\beta T}{\eta_n L} (\rho_n \sigma_n)^2 \Delta T$$

- experiment:

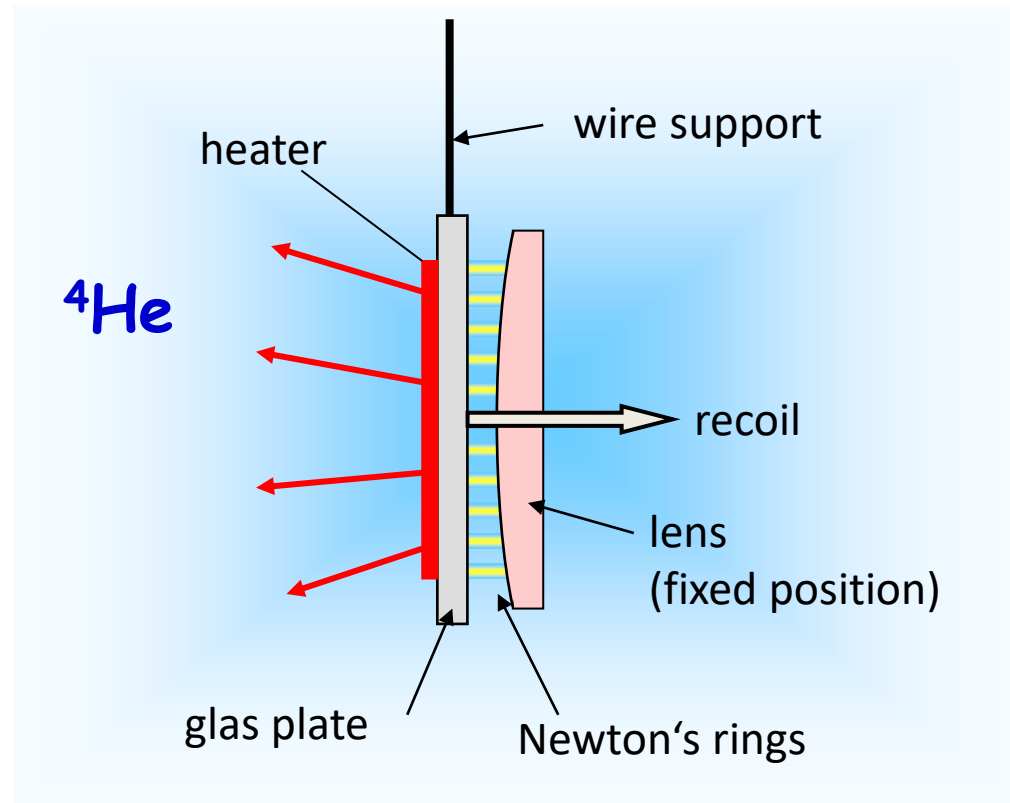
heat flow is found to be proportional to $\beta \propto d^3$ (for slit of width d)

A. Broese Van Groenou, J.D. Poll, A.M.G. Delsing, C.J. Gorter
Physica 22, 905-910 (1956)



I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics: **momentum flow associated with heat flow**



- heater generates normal fluid: phonons and rotons
- radiation to the left
- recoil to the right
- measurement of displacement of the glass plate via Newton's rings between glass plate and lens

– momentum flow (momentum / area and time): $\rho \mathbf{v} \cdot \mathbf{v}$

– resulting pressure on heat source:

$$p = \rho_n v_n^2 + \rho_s v_s^2$$

– with heat flow per area:

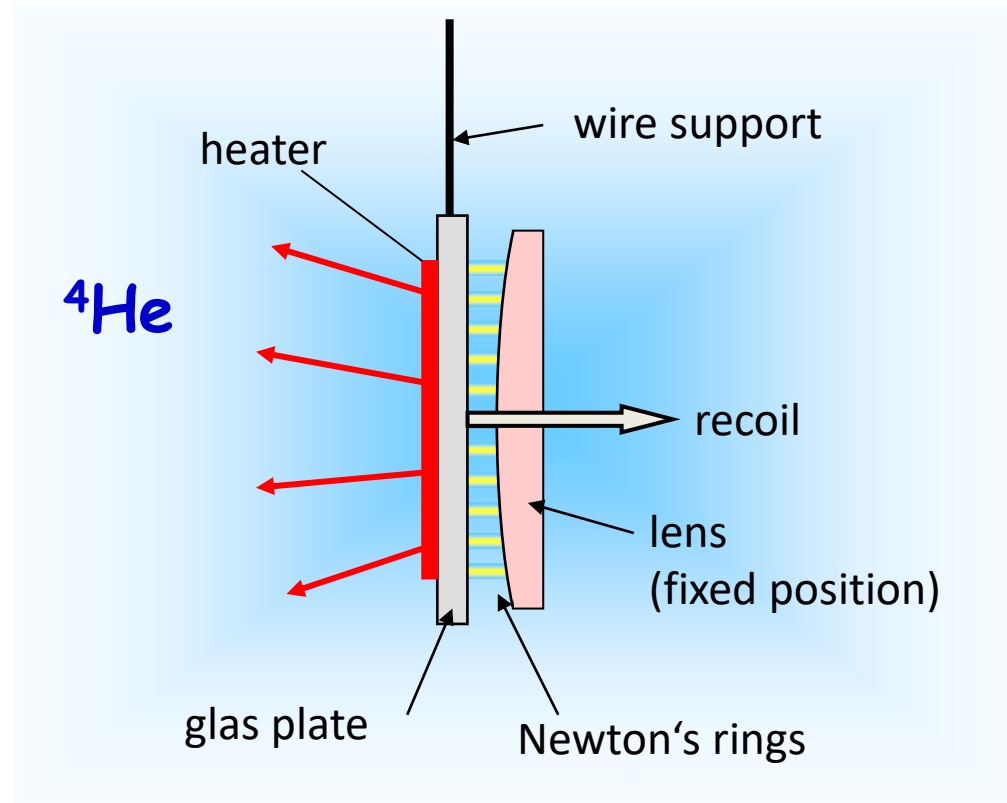
$$J_h = \rho \sigma T v_n$$

– and the fact that there is no mass transport, $\rho_n v_n + \rho_s v_s = 0$, we obtain pressure

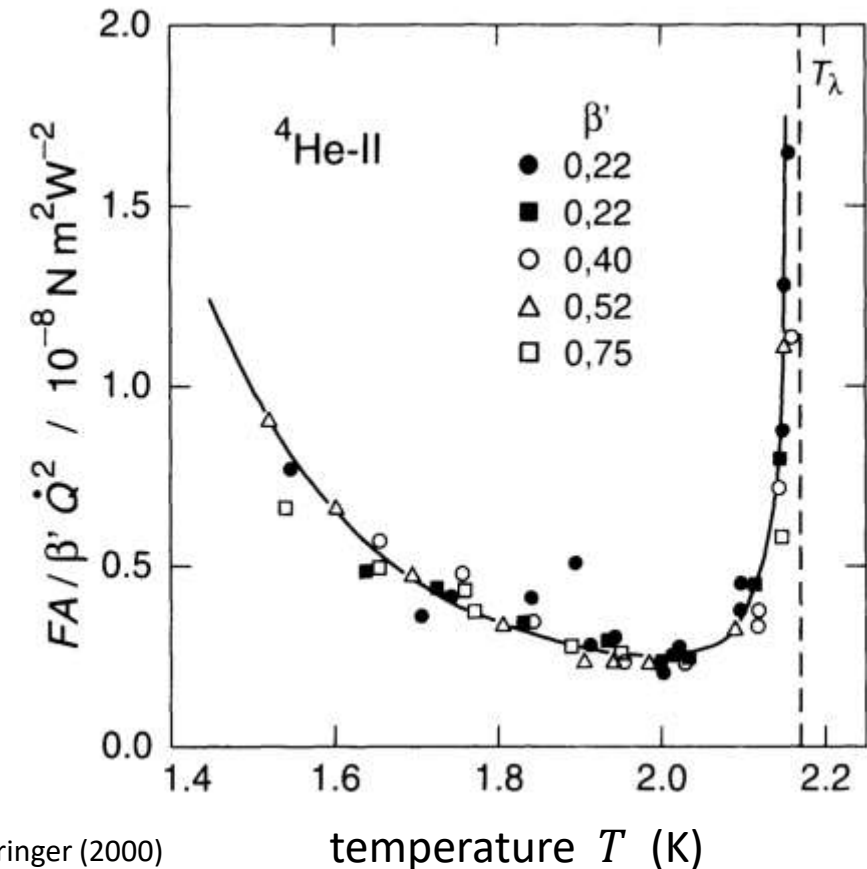
$$p = \frac{\rho_n J_h^2}{\rho_s \rho T^2 \sigma^2} = \frac{J_h^2}{v_2^2 \rho C_p T}$$

I.3.3 Two-Fluid Hydrodynamics

- two-fluid hydrodynamics: **momentum flow associated with heat flow**



$$p = \frac{\rho_n J_H^2}{\rho_s \rho T^2 \sigma^2} = \frac{J_h^2}{v_2^2 \rho C_p T}$$



Enns, Hunklinger,
Low Temperature Physics, Springer (2000)

Contents Part I: Quantum Liquids

Contents:

I.1 Foundations and General Properties

- I.1.1 Quantum Fluids
- I.1.2 Helium
- I.1.3 Van der Waals Bonding
- I.1.4 Zero-Point Fluctuations
- I.1.5 Helium under Pressure
- I.1.6 pT-Phase Diagram of ^4He and ^3He
- I.1.7 Characteristic Properties of ^4He and ^3He
- I.1.8 Specific Heat of ^4He and ^3He

I.2 ^4He as an Ideal Bose Gas

- I.2.1 Bose-Einstein Condensation
- I.2.2 Ideal Bose Gas
- I.2.3 Bose Gas with Interactions
- I.2.4 Bose-Einstein Condensation of ^4He

I.3 Superfluid ^4He

- I.3.1 Two-Fluid Model
- I.3.2 Experimental Observations
- I.3.3 Two-Fluid Hydrodynamics
- I.3.4 Excitation Spectrum of ^4He

I.4 Vortices

- I.4.1 Quantization of Circulation
- I.4.2 Experimental Study of Vortices

I.5 ^3He

- I.5.1 normal fluid ^3He
- I.5.2 solid ^3He and Pomeranchuk effect
- I.5.3 superfluid ^3He

I.6 ^3He / ^4He mixtures



I.3.4 Excitation Spectrum of Superfluid ^4He

- **important question:** **does the formation of a condensate automatically result in superfluidity?**
Does an ideal Bose-Einstein condensate show superfluidity as $^4\text{He-II}$?

Answer: *Structure of the excitation spectrum is crucial for observability of superfluidity*
(Lev Landau, 1947)

– **note:**

- we have used a phenomenological two-fluid model to describe superfluid current density in $^4\text{He-II}$

$$\mathbf{J}_\rho(\mathbf{r}) = n_0 \frac{\hbar}{m} \underbrace{\nabla\theta(\mathbf{r})}_{v_s} = n_0 \mathbf{v}_s(\mathbf{r})$$

– **still unknown:**

- is there a maximum value for the superfluid velocity \mathbf{v}_s ?
- if yes, what determines the value of \mathbf{v}_s ?
- is there a qualitative difference between a condensate of a non-interacting Bose gas and an interacting Bose gas ?

I.3.4 Excitation Spectrum of Superfluid ^4He

- Gedanken experiment by Lev Landau

- what happens, if we move an object (e.g. sphere) of mass M at velocity \mathbf{v}_i through the condensate/superfluid ?
 - at $T = 0$, the deceleration of the object is possible only by the generation of excitations
 - at which velocity can the moving object generate an **excitation** of energy E_p and momentum $\mathbf{p} = \hbar \mathbf{k}$?

- with velocity difference $v_i - v_f$ of object we obtain (*energy and momentum conservation*):

$$E_p = \frac{1}{2} M (v_i^2 - v_f^2) \quad \mathbf{p} = M (\mathbf{v}_i - \mathbf{v}_f)$$

- elimination of v_f yields

$$E_p = \mathbf{p} \cdot \mathbf{v}_i - \frac{1}{2M} p^2 = \mathbf{p} \cdot \mathbf{v}_i - \mathcal{O}\left(\frac{1}{M}\right) \quad \Rightarrow$$

- there may be a minimum value of v_i for which the equation can be satisfied
- mass of object can be large → we can neglect 2nd term on rhs

$$v_L = \min \left\{ \frac{E_p}{|\mathbf{p}|} \right\} \quad \text{Landau critical velocity}$$

v_L = smallest v_i for momentum \mathbf{p} of the excitation parallel to \mathbf{v}_i

I.3.4 Excitation Spectrum of Superfluid ^4He

- Gedanken experiment by Lev Landau

- for free bosons: $E_p = p^2/2m$ (m = mass of the boson)

$$\frac{p^2}{2m} = \mathbf{p} \cdot \mathbf{v}_i$$

equation **always satisfied** for cone of momentum vectors $|p| = 2mv_i \cos \vartheta$, where $\vartheta = \angle(\mathbf{p}, \mathbf{v}_i)$

$$E_p = \mathbf{p} \cdot \mathbf{v}_i - \frac{1}{2M} p^2 = \mathbf{p} \cdot \mathbf{v}_i - \mathcal{O}\left(\frac{1}{M}\right)$$

$$v_L = \min \left\{ \frac{E_P}{|\mathbf{p}|} \right\}$$

➔ object (e.g. a sphere) moving in superfluid can always transfer momentum to superfluid and is thereby decelerated ➔ $v_L = 0$

ideal Bose-Einstein condensate of free (non-interacting) bosons with dispersion $E_p = p^2/2m$ does not show superfluidity !!

I.3.4 Excitation Spectrum of Superfluid ^4He

- motivation for Landau critical velocity

- deceleration of moving object by generation of excitations with energy E_p only possible if $\frac{p^2}{2m} = \mathbf{p} \cdot \mathbf{v}_i$ can be satisfied

$$\Rightarrow E_p = \hbar\omega = \frac{\partial E_p}{\partial p} \cdot \mathbf{p} = \hbar \underbrace{\mathbf{v}_i}_{\mathbf{v}_i} \cdot \mathbf{k} = \hbar v_i k \cos \vartheta \leq \hbar v_i k$$

$$v_i \geq \frac{\omega}{k} = v_{\text{ph}}$$

v_{ph} = Phasengeschwindigkeit

$$v_L = \min(v_{\text{ph}})$$

v_L = **Landau critical velocity**
 $\min(v_{\text{ph}}) = 0$ for free bosons

no deceleration of object, if $v_i < v_L$

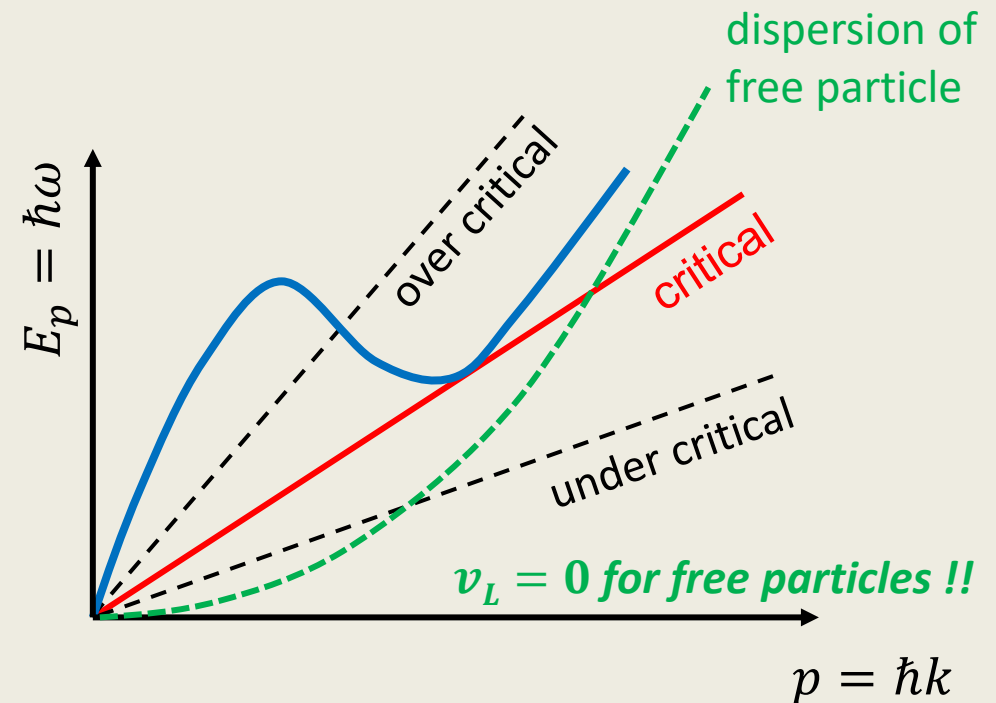
Laudau:

in order to have a superfluid, we need a dispersion relation differing significantly from that of free bosons !

$$E_p = \mathbf{p} \cdot \mathbf{v}_i - \frac{1}{2M} p^2 \simeq \mathbf{p} \cdot \mathbf{v}_i$$

$$v_L = \min \left\{ \frac{E_P}{|\mathbf{p}|} \right\}$$

how to obtain the blue dispersion curve ??



I.3.4 Excitation Spectrum of Superfluid ^4He

- excitation spectrum of superfluid ^4He : historical aspects (1)

dispersion relation must deviate from that of free bosons, $E_p = p^2/2m$, as in this case $v_L = 0$!!



Nikolai Bogoliubov (1947):

- interaction effects drastically change the nature of elementary excitations
collective excitations (many particles move at the same time)
- even for weak repulsive interaction: $E_p = c |\mathbf{p}|$ (phonon-like dispersion)



Lev Landau (1947) and **Richard Feynman** (1953):

- discussion of situation for superfluid ^4He (stronger interaction)
- postulation of the following excitation spectrum (phonons and rotons):

(i) small p :

$$\lim_{p \rightarrow 0} E_p = c_s |\mathbf{p}|$$

phonon-like collective excitations



(ii) higher p :

$$\lim_{|\mathbf{p}| \rightarrow p_0} E_p = \Delta_{\text{rot}} + \frac{(|\mathbf{p}| - p_0)^2}{2m_{\text{rot}}}$$

roton-like collective excitations
(finite energy gap Δ_{rot} for excitation)

expression „**roton**“ used by Feynman due to analogy with „**smoke ring**“, since it is connected with a forward motion of a particle accompanied by a ring of back-flowing particles

→ **however: detailed nature of rotons still not clarified**

I.3.4 Excitation Spectrum of Superfluid ^4He

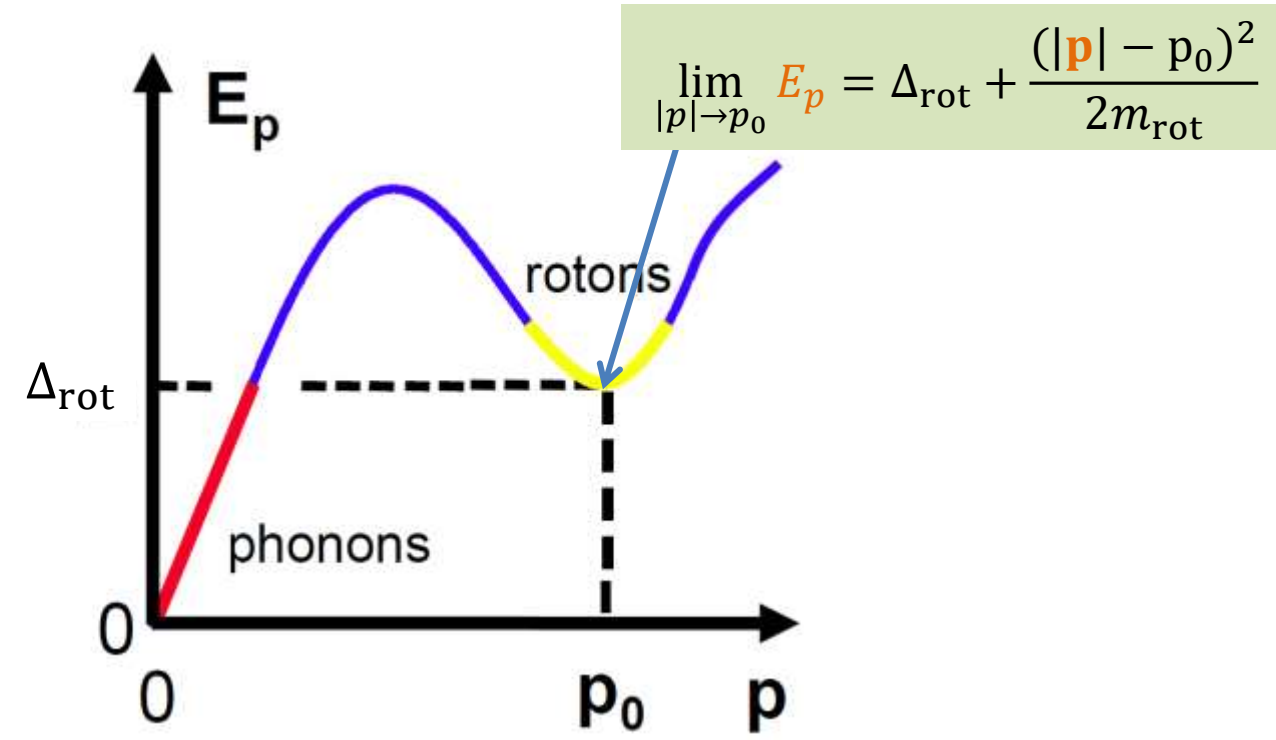
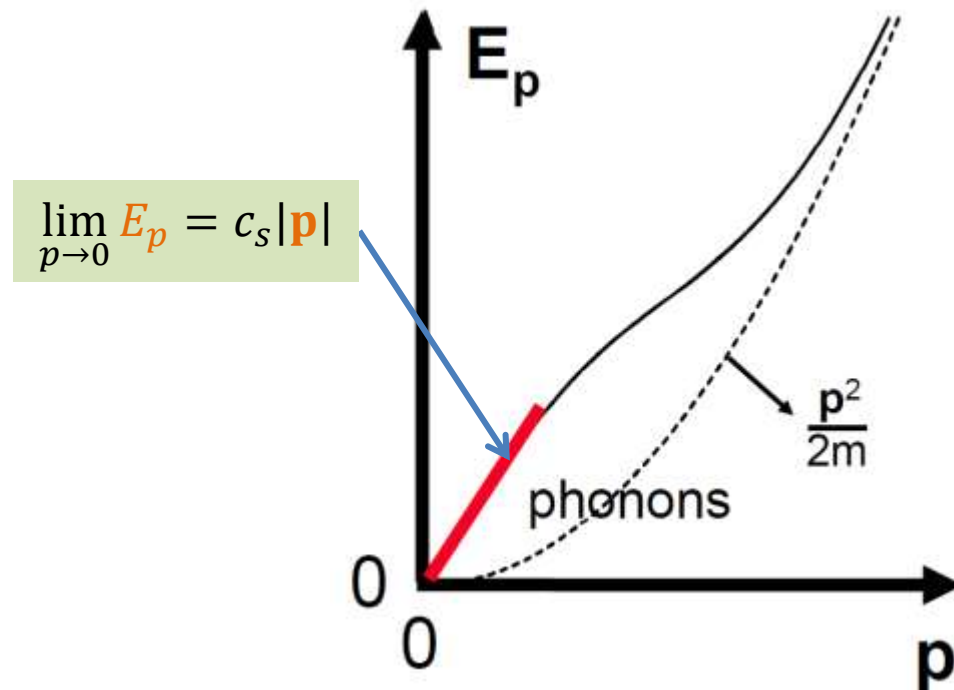
- excitation spectrum of superfluid ^4He : historical aspects (2)

interacting Bose gas
 \rightarrow finite stiffness
 \rightarrow solid like excitations

weak interactions:
phonons
 [Bogoliubov, 1947]

strong interactions:
phonons and rotons

[Landau, 1947, Feynman & Cohen, 1957]



I.3.4 Excitation Spectrum of Superfluid ^4He

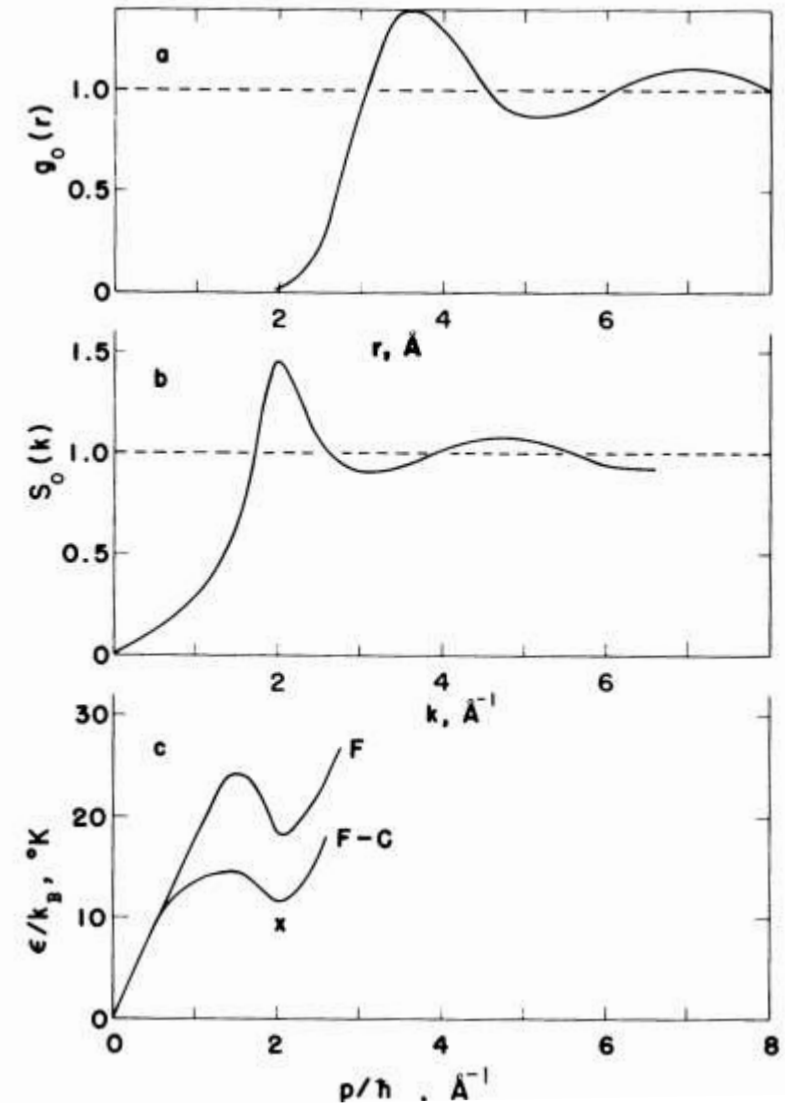
- excitation spectrum of superfluid ^4He : historical aspects (3)

Richard P. Feynman:

(from: „The beat of a different drum“
Jagdish Mehra, 1994)

„I cannot remember, how it happened.
I was walking along the street... and
zing! I understood it!“ ...

... „But of course, that is just what liquids
must do. If you measure X-ray diffraction,
then because of the spatial structure of
the liquid, *which is almost like a solid*,
there will be a maximum corresponding
to the first diffraction ring of the X-ray
pattern! That was a terrific moment!“

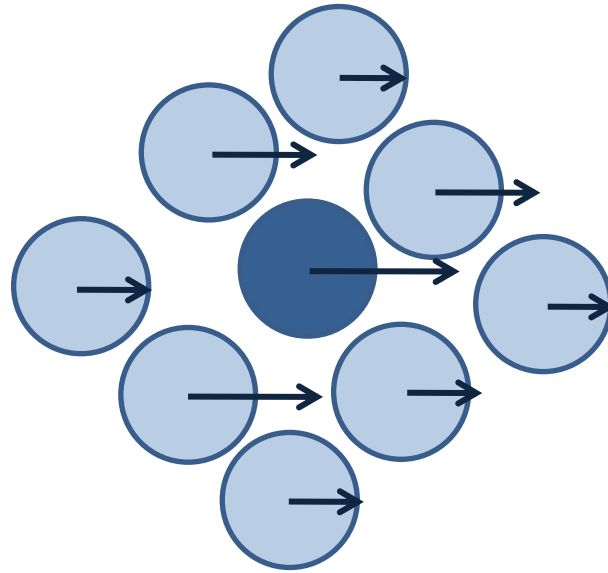


I.3.4 Excitation Spectrum of Superfluid ^4He

- excitation spectrum of superfluid ^4He : plausibility considerations

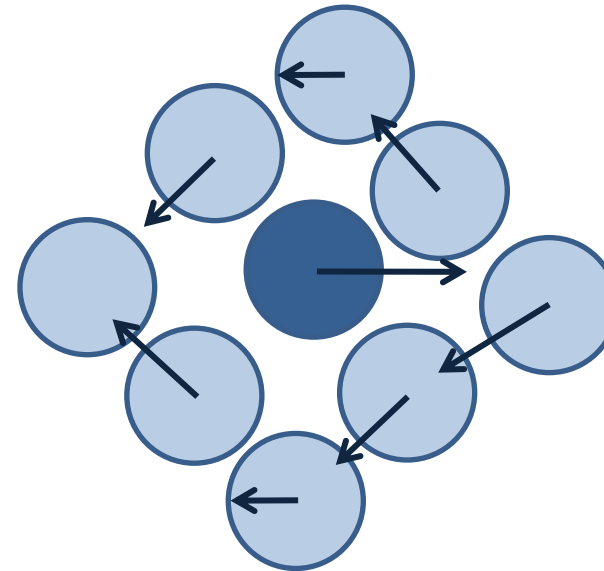
physical interpretation of the phonon and roton part of the excitation spectrum

phonon



- wavelength $\lambda = 2\pi/k \gg$ atomic distance
- coupled motion of group of atoms (like for acoustic phonon in solid)

roton



- wavelength $\lambda = 2\pi/k \simeq$ interatomic distance
- central atom moves forward while closely packed neighbors move (backward) out of the way in circular motion (like smoke ring)

in both cases we have collective excitations with dispersion $E_p(k) \rightarrow$ quasiparticles

I.3.4 Excitation Spectrum of Superfluid ^4He

- excitation spectrum of superfluid ^4He : measurement by inelastic scattering experiments

- **experimental study:**
inelastic neutron scattering
(coupling to density)

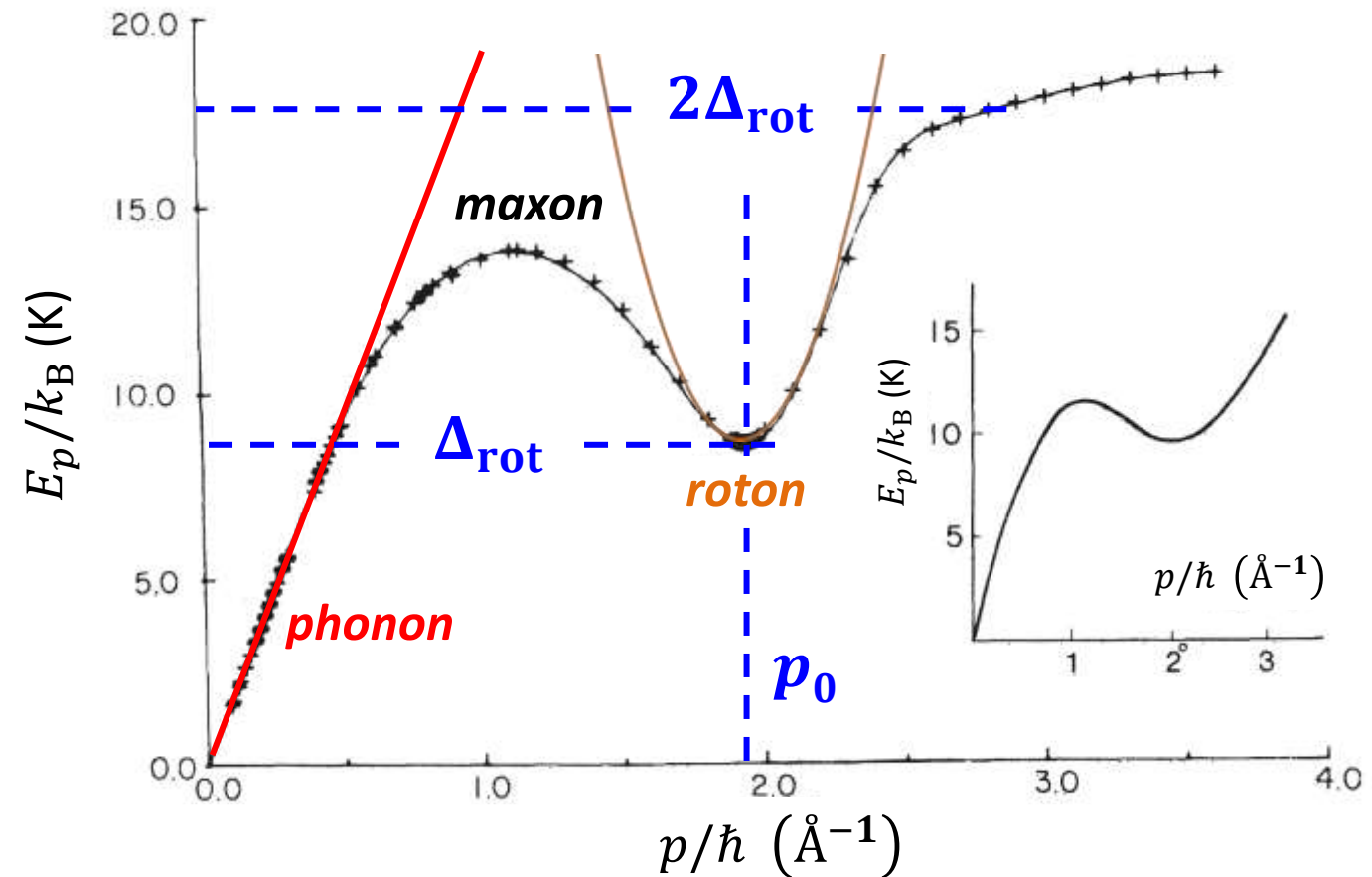
- **experimental parameter:**

$$\frac{\Delta_{\text{rot}}}{k_B} = 8.61 \text{ K}, \frac{p_0}{\hbar} = 1.92 \text{ \AA}^{-1}, m_{\text{rot}} = 0.14 m_{\text{He}}$$

TABLE IV. Zero pressure roton parameters (this work) compared to previous results: Woods *et al.* [58], Stirling [14,62,64], Andersen *et al.* [68–72], Gibbs *et al.* [73,74], and Pearce *et al.* [76]. Δ_R at $P = 0$ (shown below in italics) is taken from Refs. [62,64] in our instrument calibration procedure.

	Δ_R (meV)	k_R (\AA^{-1})	μ_R
This work	<i>0.7418(10)</i>	1.918(2)	0.141(2)
Woods 1977	0.7426(10)	1.926(5)	0.126(30)
Stirling 1991	0.7418(10)	1.920(2)	0.136(5)
Andersen 1992-1994	0.743(1)	1.931(3)	0.144(3)
Gibbs 1999	0.7426(21)	1.929(2)	0.161(4)
Pearce 2001	0.7440(20)	1.926(-)	0.166(10)

H. Godfrin et al., Phys. Rev. B **103**, 104516 (2021)

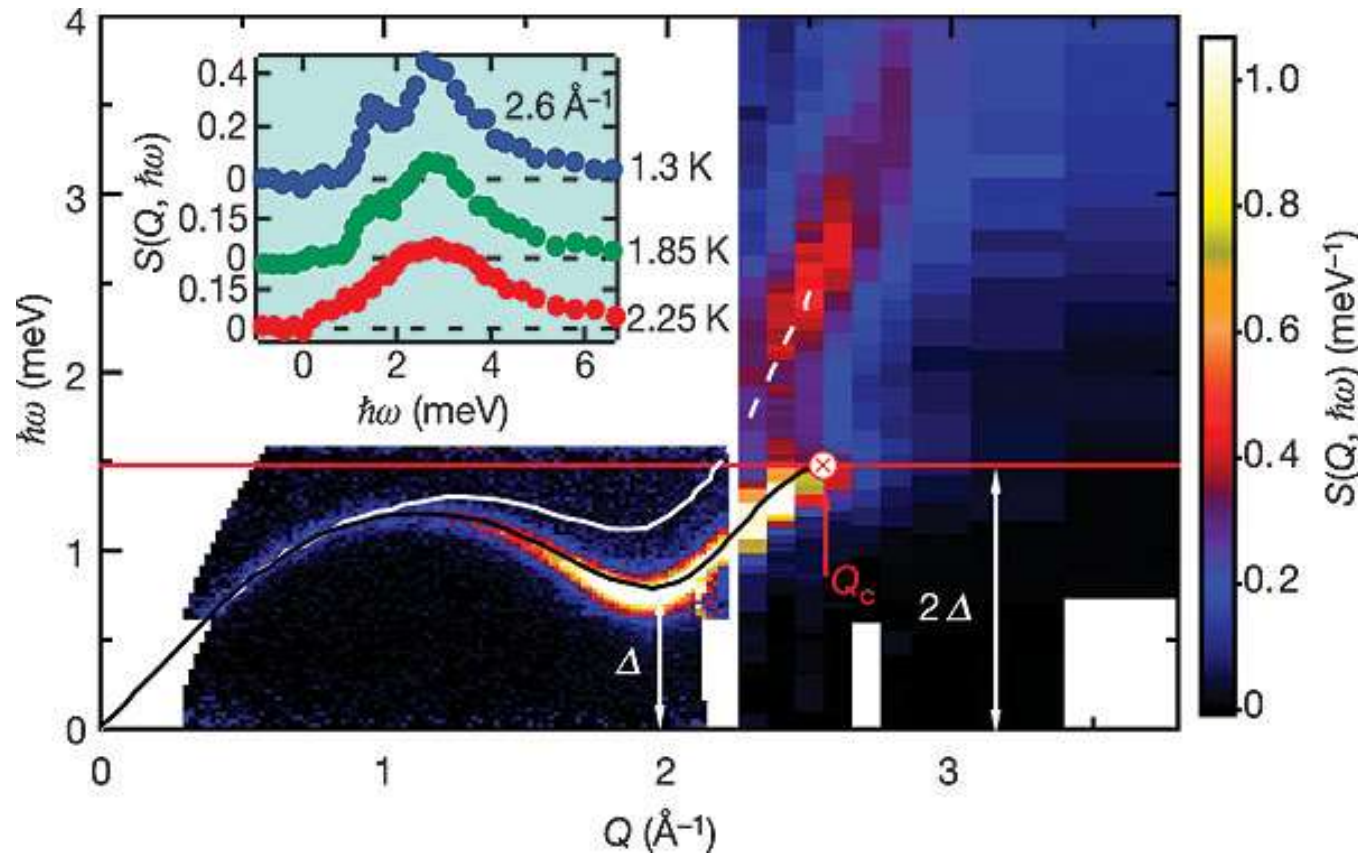


R. J. Donnelly, J. A. Donnelly, R. N. Hills,
J. Low Temp. Phys. **44**, 471 (1981).

Inset:
L. D. Landau, J. Phys. U.S.S.R. **11**, 91 (1947).

I.3.4 Excitation Spectrum of Superfluid ^4He

- excitation spectrum of superfluid ^4He : measurement by inelastic scattering experiments



Liquid helium excitation spectrum $S(Q, \hbar\omega)$ from inelastic neutron scattering measurements.

Main panel, excitation spectrum in ^4He for $1.5 < T < 1.8$ K. Data for wave vector $Q \geq 2.3 \text{ \AA}^{-1}$ are reproduced from ref. 13, data at smaller Q are from C.L.B. and S.-H. Lee, unpublished results. Solid black line, dispersion from ref. 13; red circle with cross, spectrum termination point at $Q = Q_c$ and $\hbar\omega = 2\Delta$. White line, Feynman–Cohen bare dispersion in absence of decays; horizontal red line at $\hbar\omega = 2\Delta$, onset of two-roton states for $\hbar\omega \geq 2\Delta$. Inset, excitations near termination point, at $Q = 2.6 \text{ \AA}^{-1} \approx Q_c$, for several temperatures.

[Quasiparticle breakdown in a quantum spin liquid](#)

Matthew B. Stone, Igor A. Zaliznyak, Tao Hong, Collin L. Broholm and Daniel H. Reich
Nature **440**, 187-190 (9 March 2006)

I.3.4 Excitation Spectrum of Superfluid ^4He

- excitation spectrum of superfluid ^4He : qualitative understanding

periodic crystal:

\mathbf{k} and $\mathbf{k} + \mathbf{G}$ are equivalent

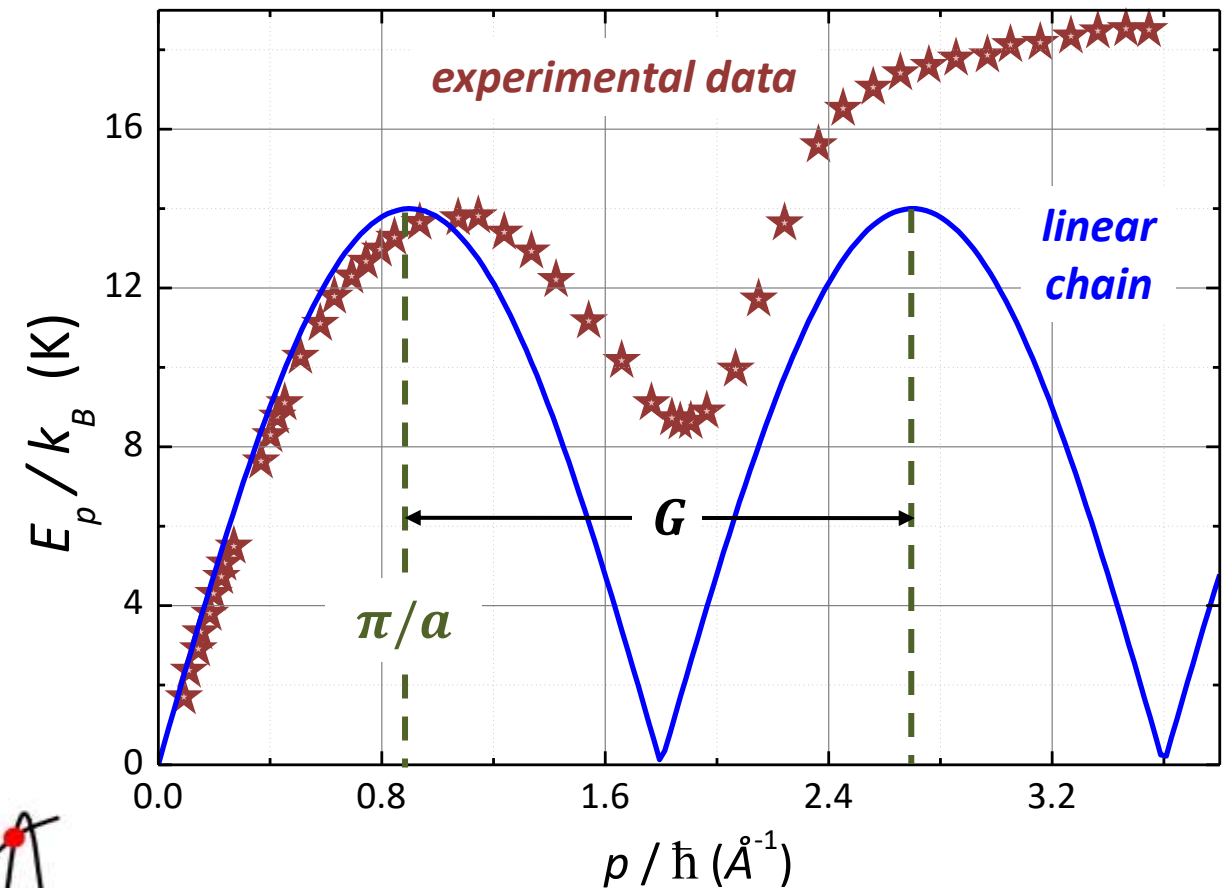
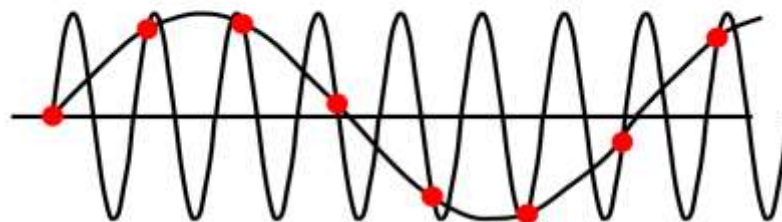
liquid:

only short distance ordering

→ a becomes smeared out

→ also $G = 2\pi/a$ is smeared out

$a = 3.6 \text{ \AA} \rightarrow 2\pi/a \approx 1.8 \text{ \AA}^{-1}$
(reasonable agreement)



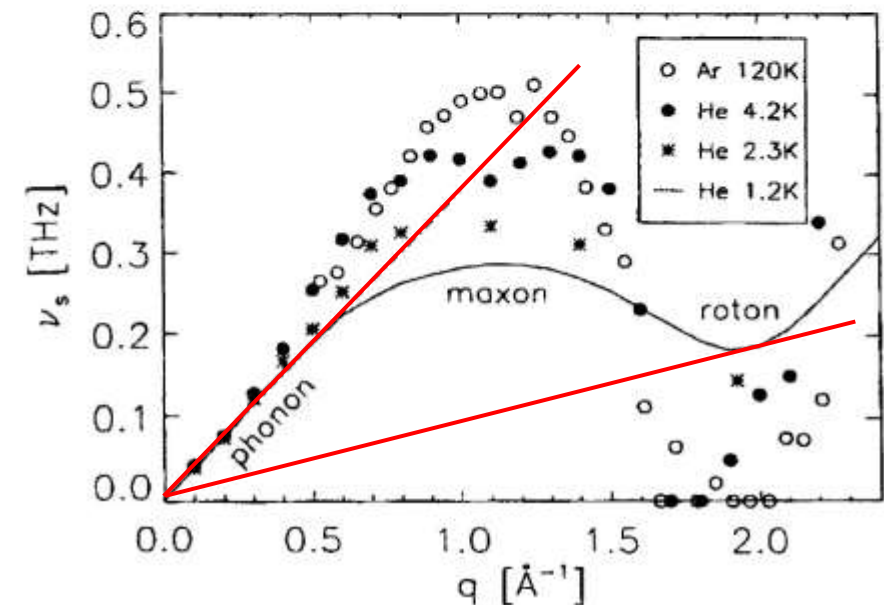
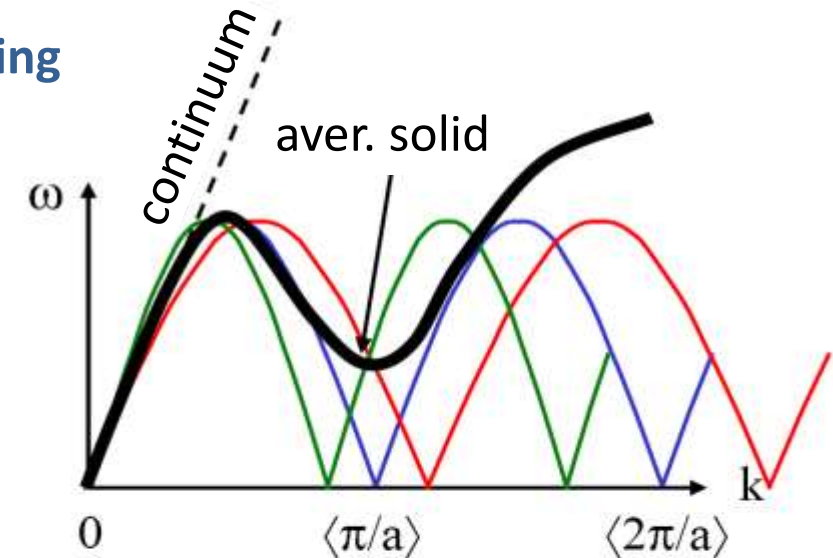
I.3.4 Excitation Spectrum of Superfluid ^4He

- excitation spectrum of superfluid ^4He : qualitative understanding

- smearing of $G = 2\pi/a$**
→ modelling by average solid + continuum
- other liquids:**
→ rotons strongly damped
- superfluid He:**
→ no damping, since there are no single particle excitations
- critical velocities:**
→ $v_L = 238 \text{ m/s}$ for phonons
→ $v_L \approx 60 \text{ m/s}$ for rotons

measured v_L usually smaller

→ **excitation of vortices**



I.3.4 Excitation Spectrum of Superfluid ^4He

- measurement of Landau critical velocity

*measurement of drag
on ions in $^4\text{He-II}$*

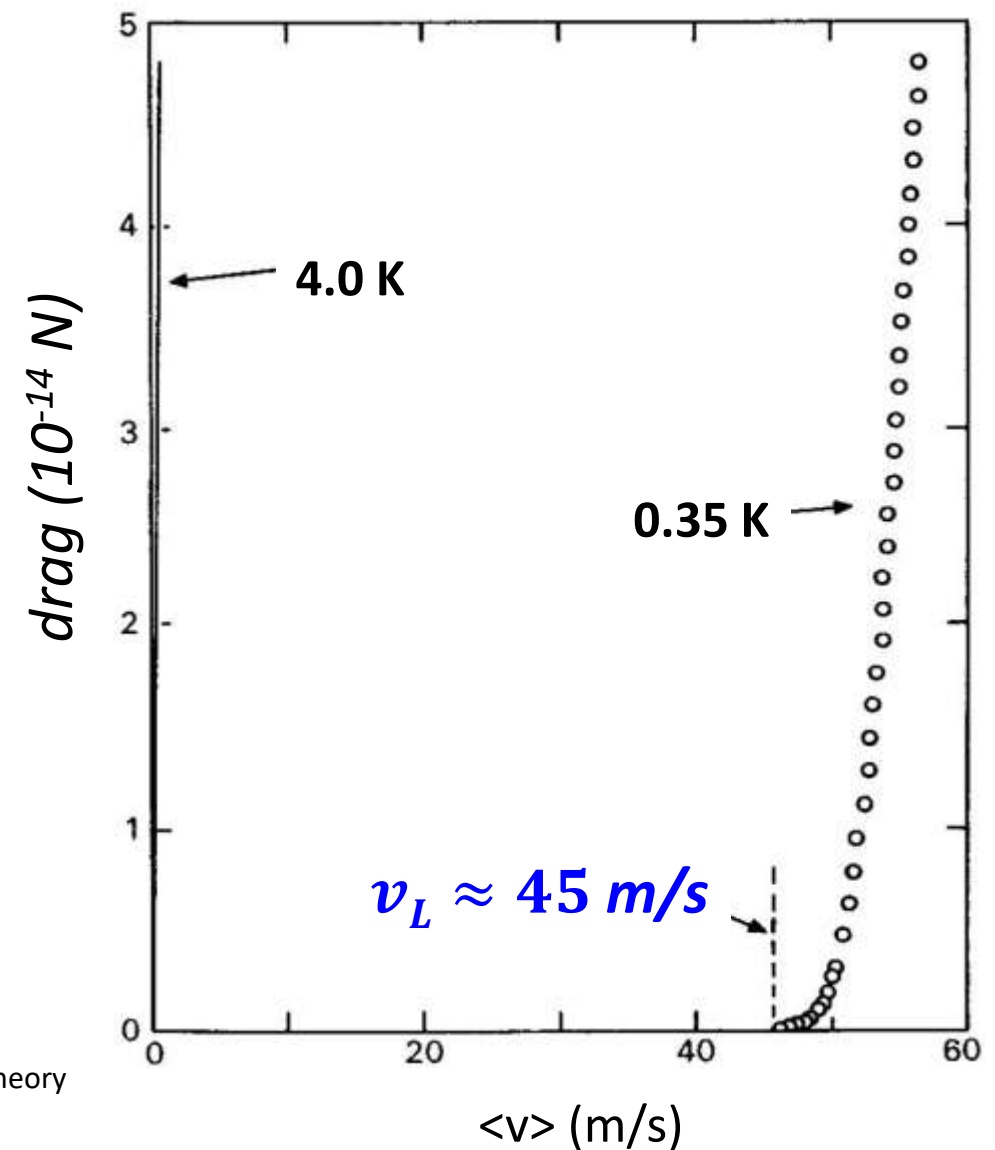
→ *no generation of excitations
for $v < v_L$*

→ *no friction*

key result:

*there is a finite Landau critical
velocity in $^4\text{He-II}$!*

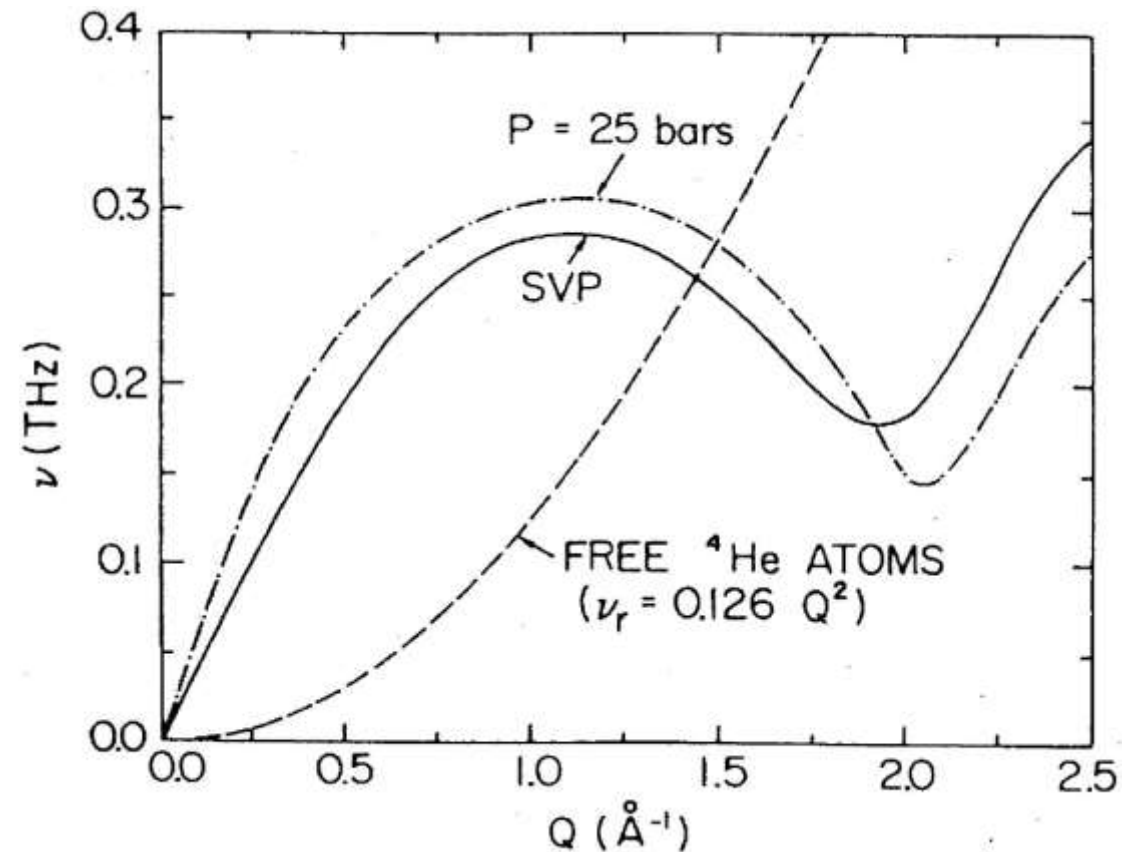
The breakdown of superfluidity in liquid ^4He : an experimental test of Landau's theory
D. R. Allum, Peter V. E. McClintock, A. Phillips, R. M. Bowley
Philosophical Transactions of the Royal Society of London A **284**, 179-224 (1977).



I.3.4 Excitation Spectrum of Superfluid ^4He

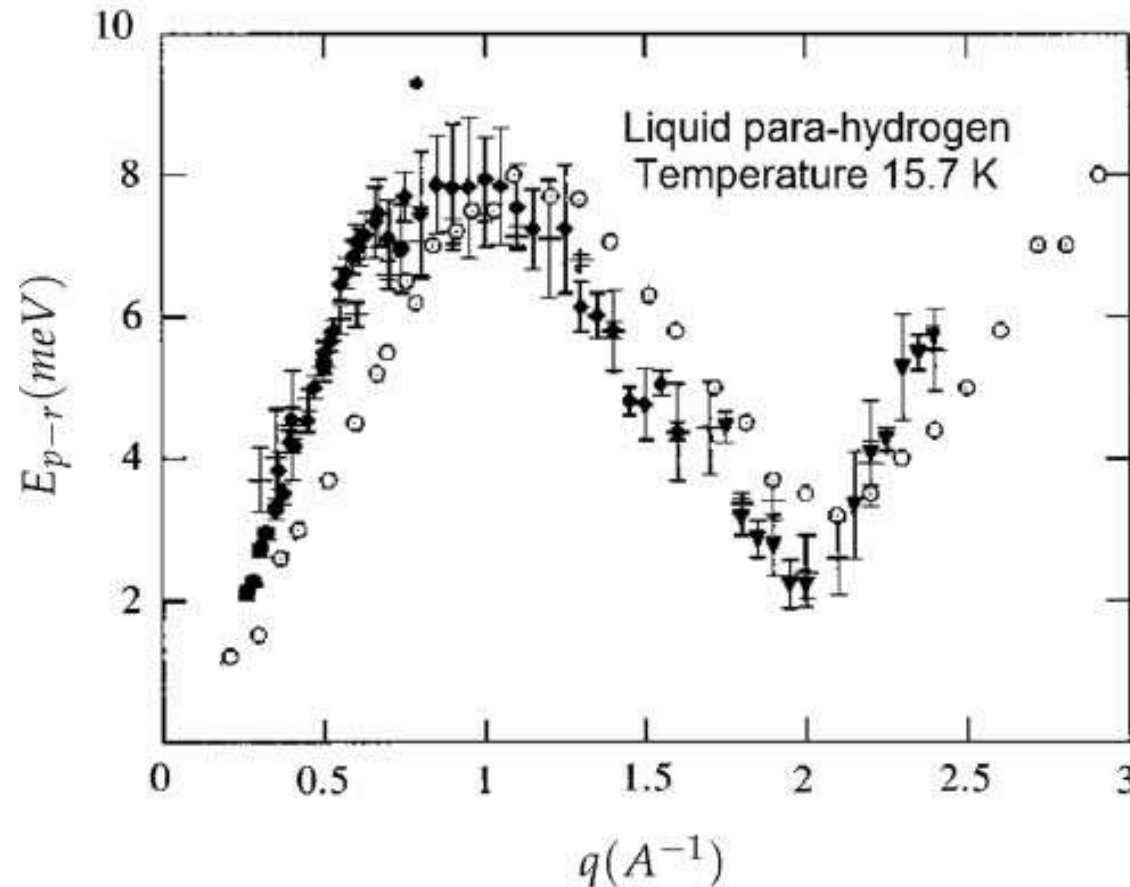
- excitation spectrum of superfluid ^4He : pressure dependence

- average distance a becomes smaller
- He becomes harder
→ *sound velocity increases*
- ordering increases
→ *roton gap decreases*



I.3.4 Excitation Spectrum of Superfluid ^4He

- excitation spectrum of superfluid ^4He : pressure dependence



The experimental INS phonon–roton curve for molecular para-hydrogen at $T = 15.7 \text{ K}$. The filled symbols (inverted triangles, diamonds and crosses) depict data measured at different spectrometers and the open circles with a dot show the simulation results.

J. Dawidowski, F. J. Bermejo, M. L. Ristig, B. Fak, C. Cabrillo, et al., Phys. Rev. **B 60**, 15154 (1999).

I.3.4 Excitation Spectrum of Superfluid ^4He

- excitation spectrum of superfluid ^4He : pressure dependence

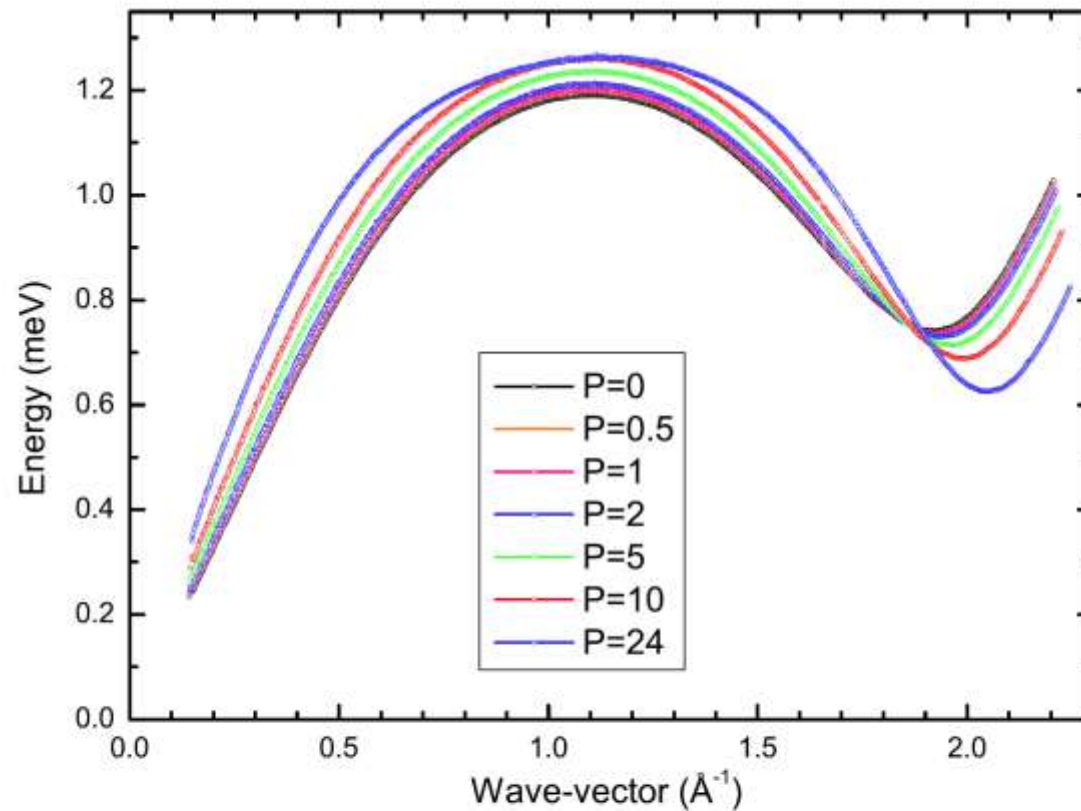
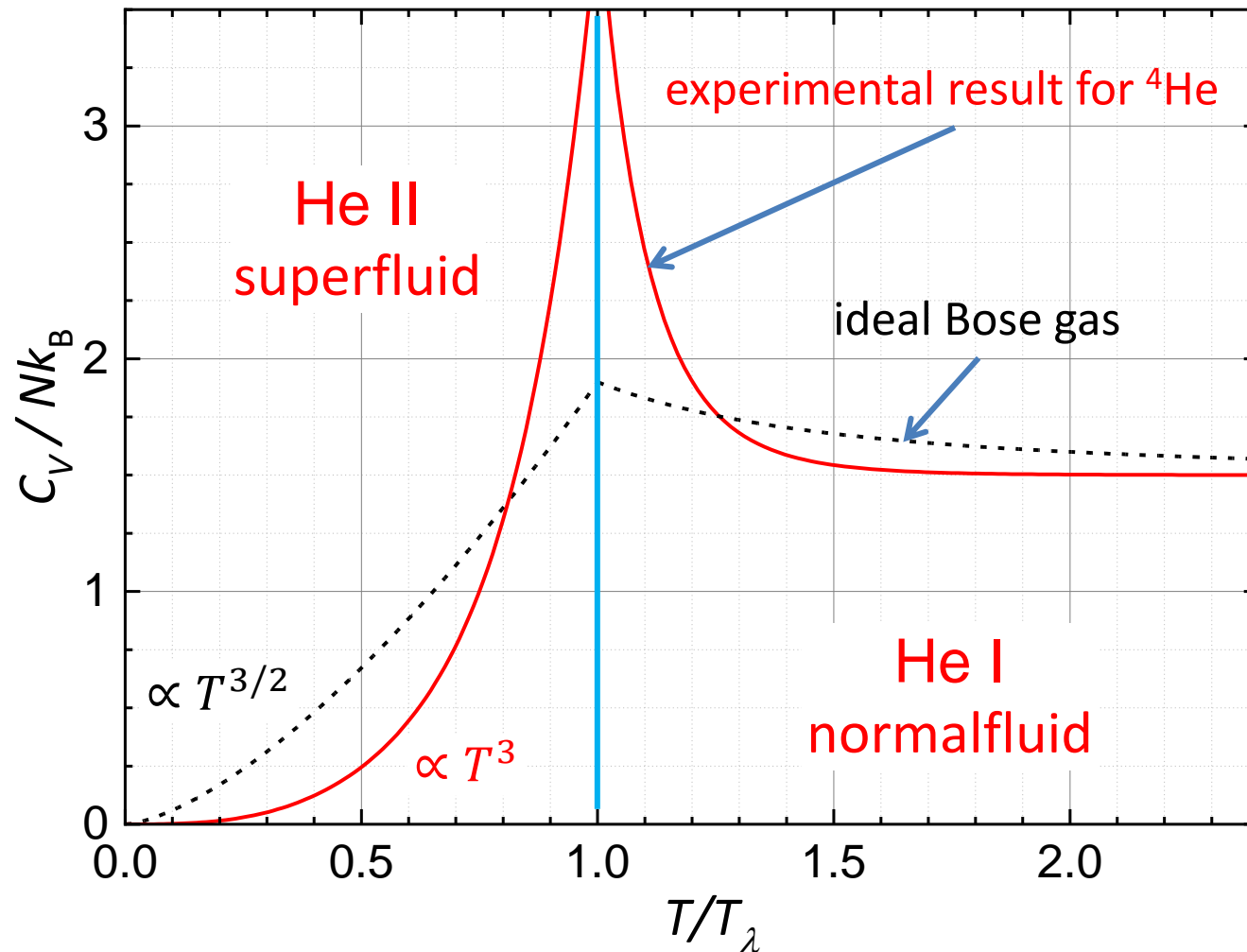


FIG. 17. Dispersion curves $\epsilon(k)$ measured for several pressures in the 0 to 24 bar range. The individual data points are represented by small circles (best seen on-line). Accurate values for the pressures are given in Table I.

H. Godfrin, K. Beauvois, A. Sultan, E. Krotscheck, J. Dawidowski, B. Fåk, and J. Ollivier
 Phys. Rev. B **103**, 104516 (2021)

I.3.4 Excitation Spectrum of Superfluid ^4He

- specific heat of ^4He revisited



deviations from ideal Bose gas are expected due to finite interactions
 → *collective excitations: phonons, rotons*

$C_V(T)$ determined by freeze out of phonons and rotons at $T < T_\lambda$

2nd order phase transition is expected for interacting Bose gas

→ however, no jump of C_V at $T = T_\lambda$ is observed, as expected for 2nd order phase transition

I.3.4 Excitation Spectrum of Superfluid ^4He

- specific heat of ^4He revisited: temperature dependence below T_λ

I. $T < 0.6 \text{ K}$:

in this temperature regime the dominant excitations are long wave length phonons

$$C_{V,ph} \propto T^3 \quad (\text{corresponds to Debye model in solid state physics})$$

$$\frac{C_V}{M} = \frac{2}{15} \frac{\pi^2 k_B^4}{\hbar^3 v_s^3} \frac{V}{M} T^3 \quad \text{single crystal, for single acoustic phonon branch}$$

$$\frac{C_{V,ph}}{M} = \frac{2}{15} \frac{\pi^2 k_B^4}{\hbar^3 v_s^3} \frac{V}{M} T^3 \quad \text{phononic specific heat of } ^4\text{He-II}$$

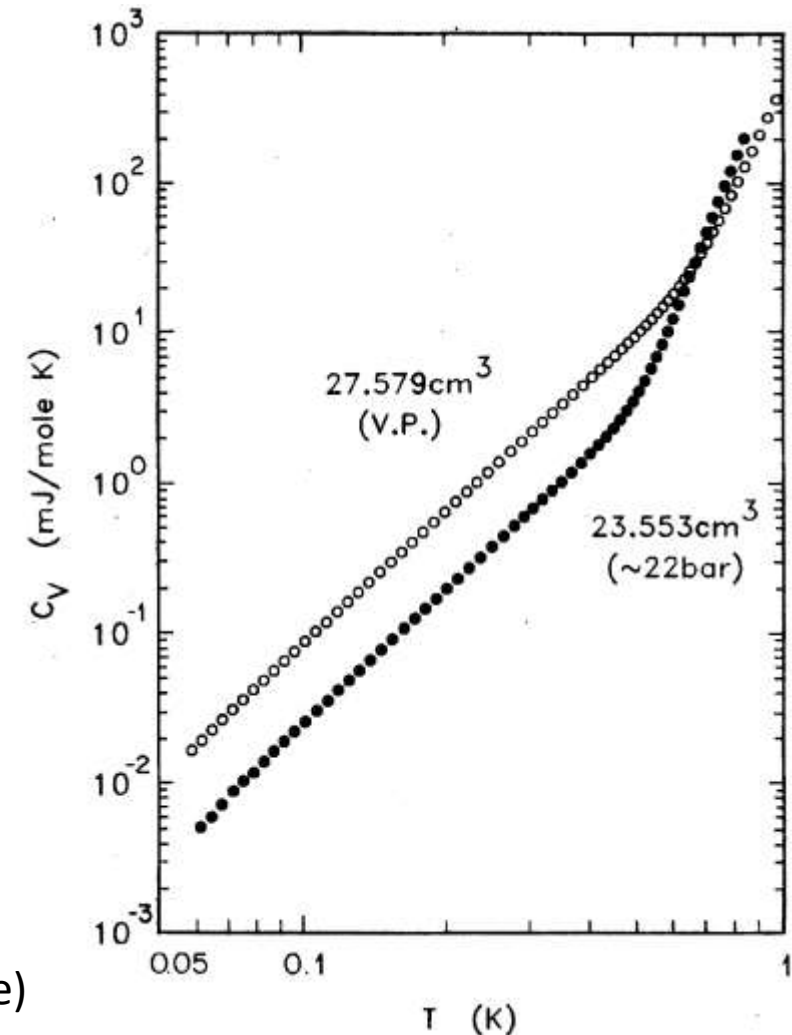
thermal conductivity:

$$\kappa = \frac{1}{3} C_V v \ell \simeq \frac{1}{3} C_V v f(d) \propto T^3$$

if $\ell \simeq d$ (sample dimension, Casimir regime)

absolute value of κ depends on details:

e.g., $\kappa \propto d^3$ for flow through narrow slit of width d



Specific Heat and Phonon Dispersion of liquid ^4He
D. S. Greywall, Phys. Rev. B **18**, 2127 (1978);
see also Phys. Rev. B **21**, 1329 (1979)

I.3.4 Excitation Spectrum of Superfluid ^4He

- specific heat of ^4He revisited: temperature dependence below T_λ

II. $0.6 \text{ K} < T < 1.2 \text{ K}$:

in this temperature regime the dominant excitations are rotons,
number N_R of rotons increases with $T \rightarrow$ thermal activation

$$C_{V,R} \propto N_R \propto \frac{1}{\sqrt{T}} \exp\left(-\frac{\Delta_{\text{rot}}}{k_B T}\right)$$

III. $1.2 \text{ K} < T < T_\lambda$:

- lifetime broadening of roton states comparable to Δ_{rot}
- additional excitations
- more complicated behavior

I.3.4 Excitation Spectrum of Superfluid ^4He

- normal fluid density

- within two-fluid model, the normal component can be considered as formed by the excitations out of the ground state (quasiparticles)

→ normal fluid density is determined by phonon and roton density

- temperature dependence of phonon and roton density is given by

$$\rho_{n,\text{ph}} = \frac{1}{V} \int D_{\text{ph}}(E) n_{\text{ph}}(E, T) dE$$

$n_{\text{ph}}, n_{\text{rot}}$ = occupation probability

$$\rho_{n,\text{rot}} = \frac{1}{V} \int D_{\text{rot}}(E) n_{\text{rot}}(E, T) dE$$

$D_{\text{ph}}, D_{\text{rot}}$ = density of states

- after some math

$$\rho_{n,\text{ph}} = \frac{2\pi^2 k_B^4}{45 \hbar^3 v_{\text{ph}}^5} T^4$$

$$\rho_{n,\text{rot}} = \frac{2p_0^4}{3\hbar^3} \sqrt{\frac{m_{\text{rot}}}{(2\pi)^3 k_B T}} \exp\left(-\frac{\Delta_{\text{rot}}}{k_B T}\right)$$

experimental parameter:

$$\frac{\Delta_{\text{rot}}}{k_B} = 8.61 \text{ K}$$

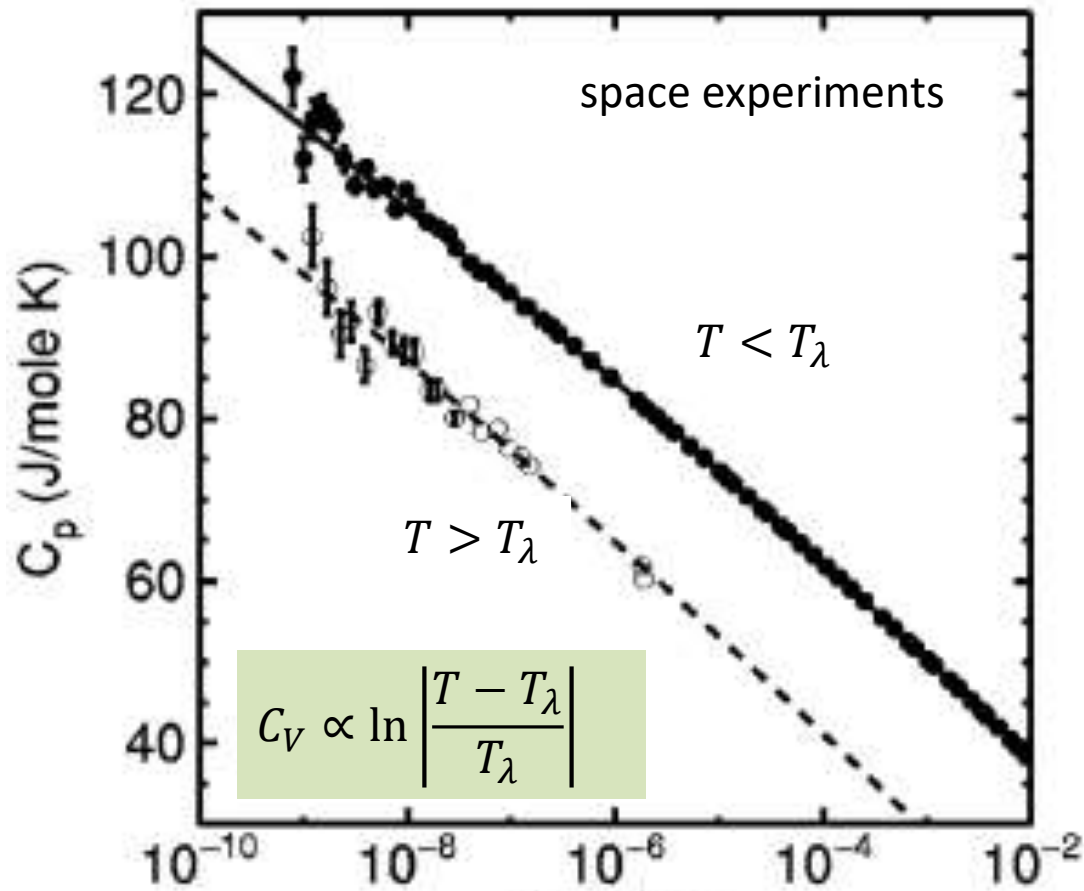
$$\frac{p_0}{\hbar} = 1.92 \text{ \AA}^{-1}$$

$$m_{\text{rot}} = 0.15 m_{\text{He}}$$

$$v_{\text{ph}} = 238 \text{ m/s}$$

I.3.4 Excitation Spectrum of Superfluid ^4He

- specific heat of ^4He revisited: temperature dependence close to T_λ



experiment:

second order phase transition with logarithmic singularity

theory of phase transitions:

$$C_p = a^\pm + b^\pm \frac{t^{-\alpha}}{\alpha} (1 - c^\pm t^\delta + \dots) \quad \text{with } t = \frac{T}{T_\lambda} - 1$$

$\alpha \simeq -0.0127 \pm 0.0003$ (critical exponent), $\delta = 0.529$

$a^\pm, b^\pm, c^\pm = \text{constants}$ (agree well with 3D XY universality class)

note that $t^{-\alpha} = \exp(-\alpha \ln t) \simeq 1 - \alpha \ln t$, since $\alpha \ln t \ll 1$

$$\Rightarrow C_p \propto b^\pm \frac{t^{-\alpha}}{\alpha} = \frac{b^\pm}{\alpha} - b^\pm \ln t$$

- T_λ depends on pressure, i.e. on the filling height of the He container
→ results in an experimental broadening of the cusp

→ *precision experiment in space shuttle*

- deviations also appear, if the size of the container becomes smaller than the coherence length, which diverges at T_λ

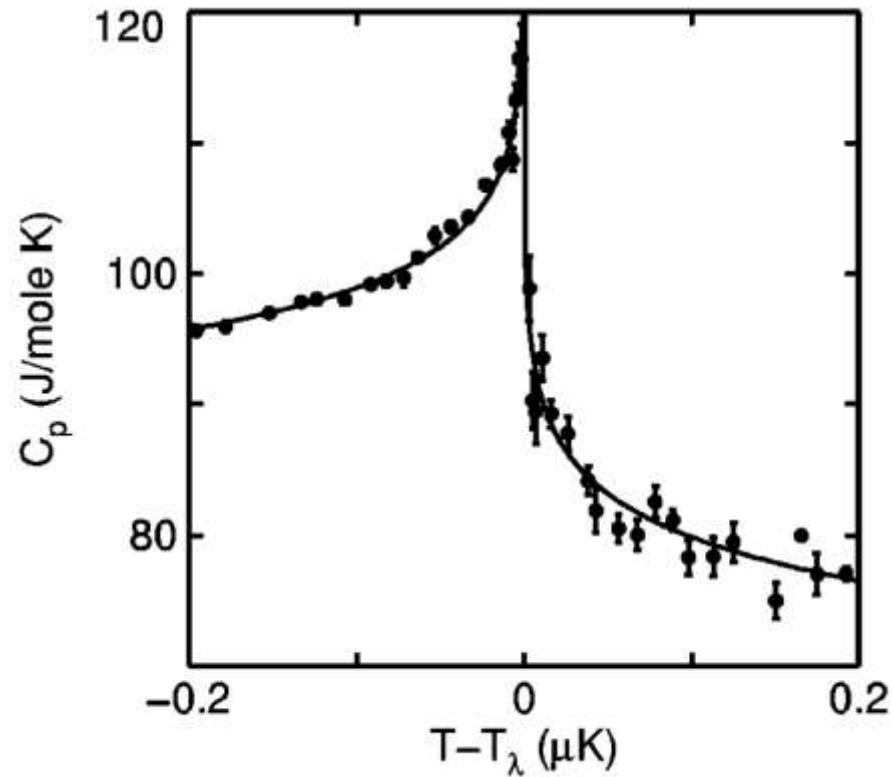
Specific heat of liquid helium in zero gravity very near the lambda point

J. A. Lipa et al., Phys. Rev. B 68, 174518 (2003)

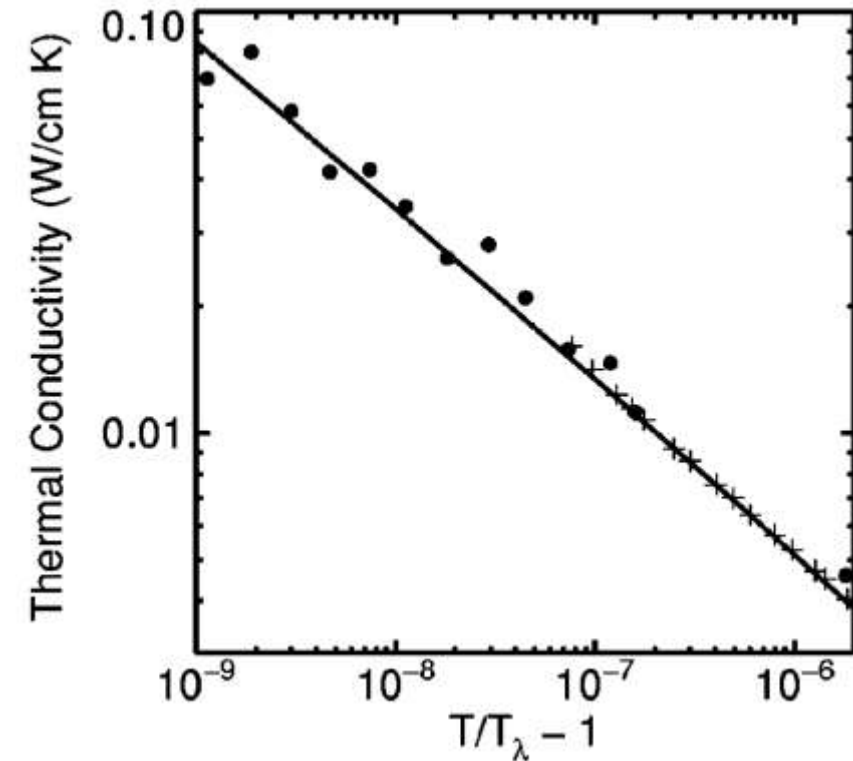
$$\left| \frac{T}{T_\lambda} - 1 \right|$$

I.3.4 Excitation Spectrum of Superfluid ^4He

- specific heat and thermal conductivity of ^4He revisited: temperature dependence close to T_λ



Averaged data close to the transition. Line shows the best-fit function.



Log-log plot of thermal conductivity vs reduced temperature above the lambda point. Filled circles: data, curve: theory model.

Specific heat of liquid helium in zero gravity very near the lambda point
J. A. Lipa et al., Phys. Rev. B 68, 174518 (2003)



Walther
Meißner
Institut

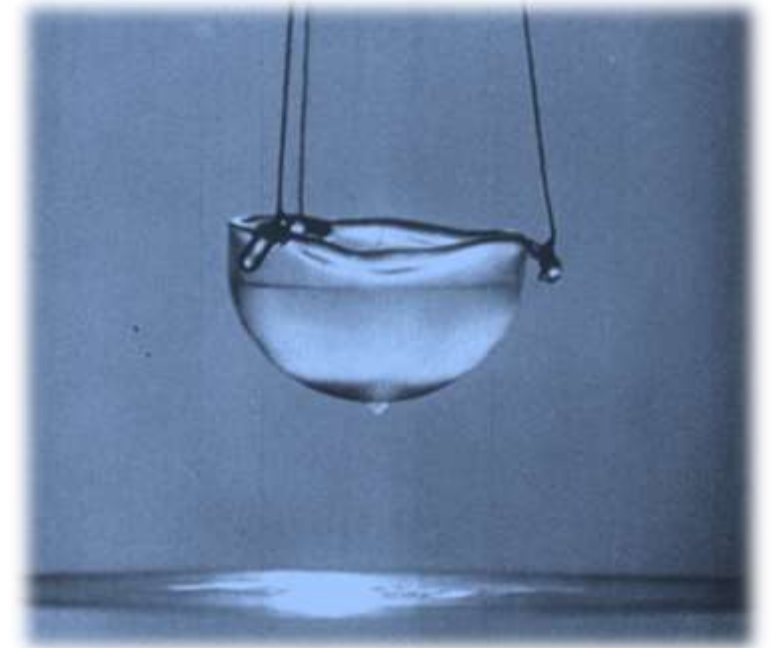
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Superconductivity and Low Temperature Physics II



Lecture No. 6

R. Gross

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Contents Part I: Quantum Liquids

Contents:

I.1 Foundations and General Properties

- I.1.1 Quantum Gases & Liquids
- I.1.2 Helium
- I.1.3 Van der Waals Bonding
- I.1.4 Zero-Point Fluctuations
- I.1.5 Helium under Pressure
- I.1.6 pT-Phase Diagram of ^4He and ^3He
- I.1.7 Characteristic Properties of ^4He and ^3He
- I.1.8 Specific Heat of ^4He and ^3He



I.4 Vortices

- I.4.1 Quantization of Circulation
- I.4.2 Experimental Study of Vortices

I.5 ^3He

- I.5.1 normal fluid ^3He
- I.5.2 solid ^3He and Pomeranchuk effect
- I.5.3 superfluid ^3He

I.2 ^4He as an Ideal Bose Gas

- I.2.1 Bose-Einstein Condensation
- I.2.2 Ideal Bose Gas
- I.2.3 Bose Gas with Interactions
- I.2.4 Bose-Einstein Condensation of ^4He

I.6 ^3He / ^4He mixtures

I.3 Superfluid ^4He

- I.3.1 Two-Fluid Model
- I.3.2 Experimental Observations
- I.3.3 Two-Fluid Hydrodynamics
- I.3.4 Excitation Spectrum of ^4He



- vortices in superfluid ^4He

- in the discussion of the two-fluid model, we assumed the absence of turbulence in \mathbf{J}_s : $\Rightarrow \nabla \times \mathbf{v}_s = 0$
- 1941: **Landau** proposed to check this assumption in experiments with superfluid He in rotating containers
- 1949: **Onsager** predicted the appearance of vortices in rotating superfluid He
- 1953: **Feynman** noted that circulation in superfluid ^4He should be quantized
- 1961: first experimental proof of quantization by **Vinen**
 - ➔ *quantized vortices in superfluid ^4He as in superconductors*
 - ➔ *difference: superfluid ^4He is not charged, neutral circulation*

I.4.1 Quantization of Circulation

- vortices – nomenclature and some definitions

- ideal vortex in z-direction at position $\mathbf{r} = 0$: $\nabla \times \mathbf{v}(\mathbf{r}) = \Gamma \cdot \delta(x) \cdot \delta(y)$

- circulation Γ :

$$\int_A \nabla \times \mathbf{v}(\mathbf{r}) dA = \oint_S \mathbf{v}(\mathbf{r}) \cdot d\mathbf{r} = \Gamma = \text{const.}$$

Stokes

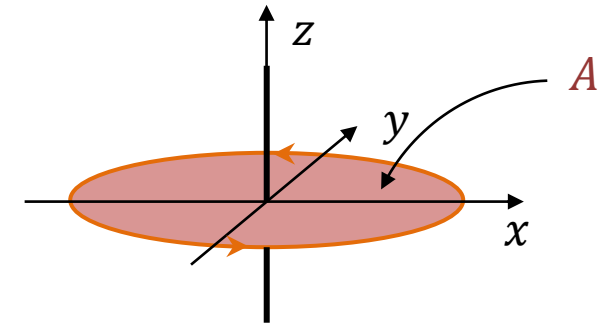
Thomson circulation theorem

- if A is a circle, then $\Gamma = 2\pi r \cdot v_\phi$

and $v_\phi = \frac{\Gamma}{2\pi r} \hat{e}_\phi$

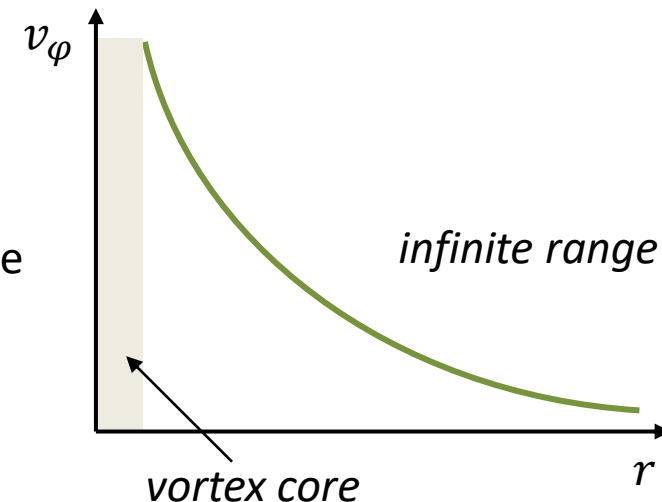
note that $\nabla \times \mathbf{v}_\phi = \mathbf{0}$ apart from the vortex core

→ for superfluid He: circulation Γ is quantized



$$\nabla \times \mathbf{v} = \begin{pmatrix} \left[\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right] \hat{x} \\ \left[\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right] \hat{y} \\ \left[\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right] \hat{z} \end{pmatrix}$$

$$\nabla \times \mathbf{v} = \begin{pmatrix} \left[\frac{1}{r} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{e}_r \\ \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right] \hat{e}_\phi \\ \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_\phi) - \frac{\partial v_r}{\partial \phi} \right] \hat{e}_z \end{pmatrix}$$



I.4.1 Quantization of Circulation

- description of superfluid component by macroscopic wave function

as done in superconductivity for electronic system: charged superfluid

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r})e^{i\theta(\mathbf{r})}, \quad |\psi_0(\mathbf{r})|^2 = n_0(\mathbf{r}), \quad N_0 = \int |\psi_0(\mathbf{r})|^2 d^3r \quad n_0 = \frac{N_0}{V} = \text{particle density in condensate}$$

- phase is related to velocity $\mathbf{v}_s(\mathbf{r})$ of superfluid:

$$\mathbf{J}_\rho(\mathbf{r}) = n_0 \underbrace{\frac{\hbar}{m_{\text{He}}} \nabla \theta(\mathbf{r})}_{\mathbf{v}_s(\mathbf{r})} = n_0 \mathbf{v}_s(\mathbf{r}) \quad \text{current density – phase relation (as in superconductivity)}$$

\mathbf{v}_s = superfluid velocity, n_0 = superfluid density

- analogy to superconductors

$$\mathbf{J}_s(\mathbf{r}) = n_s q_s \underbrace{\left[\frac{\hbar}{m_s} \nabla \theta(\mathbf{r}) - \frac{q_s}{m_s} \mathbf{A}(\mathbf{r}) \right]}_{\mathbf{v}_s} \quad \mathbf{v}_s(\mathbf{r}) = \frac{\hbar}{m_s} \nabla \theta(\mathbf{r}) - \frac{q_s}{m_s} \mathbf{A}(\mathbf{r})$$

superfluid He is uncharged: $q_s = 0$ $\mathbf{v}_s(\mathbf{r}) = \frac{\hbar}{m_{\text{He}}} \nabla \theta(\mathbf{r})$

I.4.1 Quantization of Circulation

- circulation of superfluid component

$$\Gamma = \oint_S \mathbf{v}_s(\mathbf{r}) \cdot d\mathbf{r} = \frac{\hbar}{m_{\text{He}}} \oint_S \underbrace{\nabla\theta(\mathbf{r}) \cdot d\mathbf{r}}_{=2\pi n} = \frac{\hbar}{m_{\text{He}}} \cdot 2\pi n = \frac{h}{\underbrace{m_{\text{He}}}_{\kappa}} \cdot n$$

→ circulation is quantized in units of $\hbar/m_{\text{He}} = \kappa$ (**vorticity**)

→ quantum number n corresponds to **topological winding number**

(counts the 2π windings along a closed contour)

- generation of vortices by rotation:

$$\text{angular frequency } \Omega = \Gamma/A \Leftrightarrow \text{magnetic field } B = \Phi/A$$

$$\text{Coriolis force } \mathbf{F}_{\text{cor}} = 2m_{\text{He}}\mathbf{v}_s \times \boldsymbol{\Omega} \Leftrightarrow \text{Lorentz force } \mathbf{F}_L = q_s\mathbf{v}_s \times \mathbf{B}$$

$$2m_{\text{He}}\Omega \Leftrightarrow q_s B$$

I.4.1 Quantization of Circulation

- consequences of macroscopic quantum description of superfluid

- superfluid velocity: $\mathbf{v}_s(\mathbf{r}) = \frac{\hbar}{m_{\text{He}}} \nabla \theta(\mathbf{r}) \Rightarrow \nabla \times \mathbf{v}_s(\mathbf{r}) = \frac{\hbar}{m_{\text{He}}} \underbrace{\nabla \times \nabla \theta(\mathbf{r})}_{=0} = 0$

$$\Rightarrow \oint \mathbf{v}(\mathbf{r}) \cdot d\mathbf{r} = \Gamma = 0 \quad \text{an uncharged superfluid cannot rotate homogeneously}$$

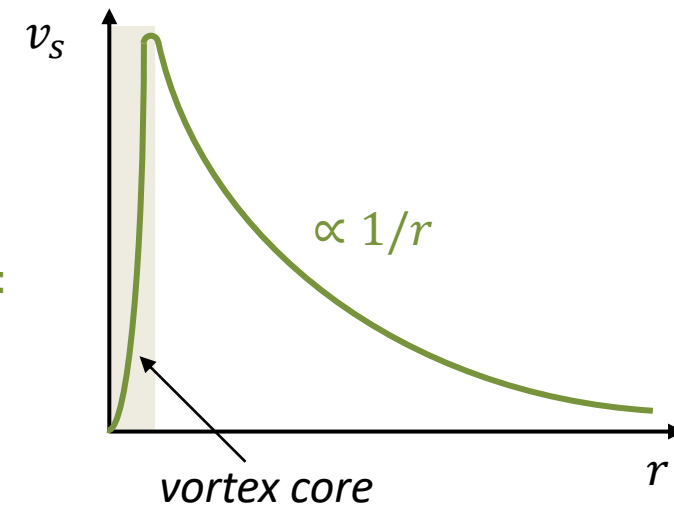
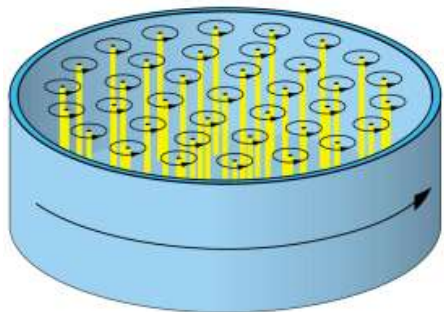
- experimental observation:

- if we rotate a cylinder filled with superfluid ^4He at angular frequency ω , the surface of the superfluid does not stay flat but shows curvature due to centrifugal force

parabolic radial height profile: $z = \frac{\omega^2}{2g} r^2$

- apparent contradiction is caused by the presence of vortices

- $\nabla \times \mathbf{v}_s(\mathbf{r}) = 0$ is satisfied apart inside the vortex core at $r = 0$: macroscopic wave function $|\psi_0(0)|^2 = 0$, phase θ is not defined, superfluid density $J_s = n_0 v_s = 0$



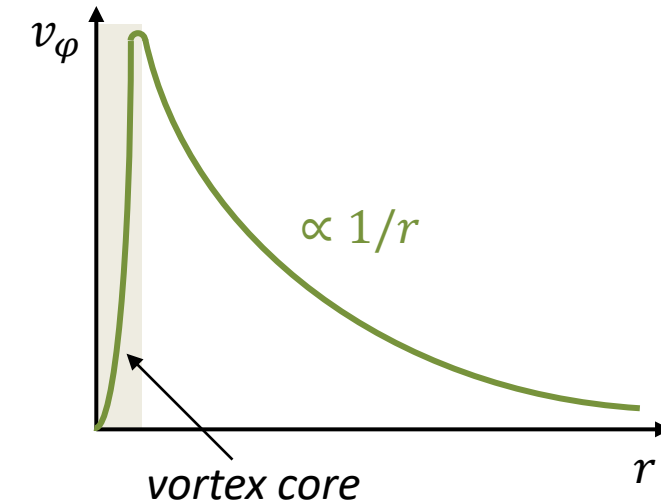
I.4.1 Quantization of Circulation

- some estimates

- magnitude of the superfluid velocity (in circular vortex)

$$\Gamma = \oint \mathbf{v}(\mathbf{r}) \cdot d\mathbf{r} = \frac{h}{m_{\text{He}}} n = 2\pi r v_{\phi}$$

$$\Rightarrow v_{\phi} = \frac{h}{m_{\text{He}}} n \frac{1}{2\pi r} = \frac{\kappa}{2\pi r} n \simeq \frac{1.5 \times 10^{-8}}{r} n \left[\frac{\text{m}}{\text{s}} \right]$$



- diameter d_{vc} of vortex core

$$\Rightarrow d_{\text{vc}} = \frac{h}{m_{\text{He}}} n \frac{1}{2\pi v_L} = \frac{\kappa}{2\pi v_L} n \sim 3 \text{ \AA}$$

for $n = 1$ and $v_L \simeq 45 \frac{\text{m}}{\text{s}}$ = critical (Landau) velocity for roton formation

- energy of vortex is proportional to v_{ϕ}^2

$$E_v \approx E_{\text{kin}} \propto \kappa^2 \propto n^2$$

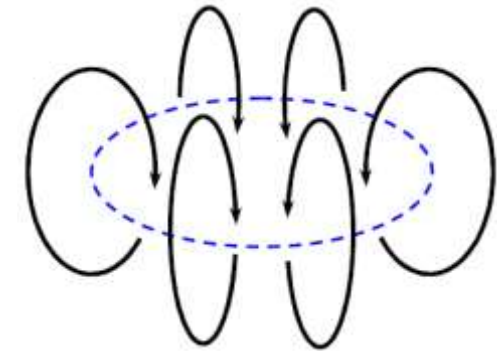
→ it is more favorable to generate two vortices with $n = 1$ instead of a single vortex with $n = 2$

I.4.1 Quantization of Circulation

- are there vortices in thermal equilibrium ?

- excitation energy $\approx E_{\text{kin}}$ should be as small as possible:

- minimum v_s $\rightarrow n = 1$
- minimum length \rightarrow vortex rings
- smallest ring \rightarrow atomic dimension



- vortex still macroscopic object

- $\rightarrow E_V \gg k_B T$ except for $T \sim T_\lambda$

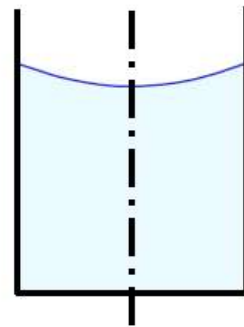
- \rightarrow momentum is large $\rightarrow v_L = E / p$ is small for vortex rings

- critical velocity v_L is determined by excitation of vortex rings

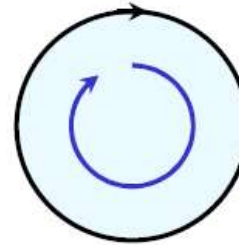
I.4.2 Experimental Study of Vortices

- cooling down a rotating ^4He container below T_λ

- above T_λ :
rigid rotation

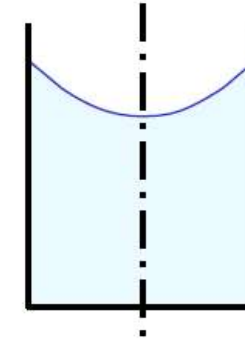


weak circulation



first experiments by Vinen in 1961

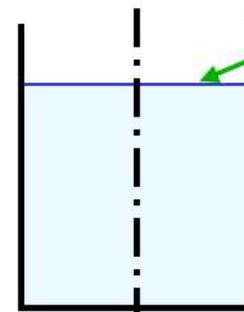
The detection of a single quantum of circulation in liquid helium II, W.F. Vinen, *Proc. Roy. Soc. (Lond.) A* **260**,218 (1961).



strong circulation

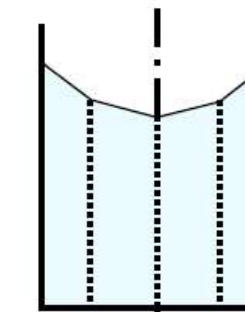
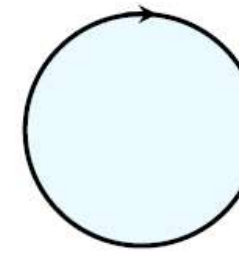


- below T_λ :

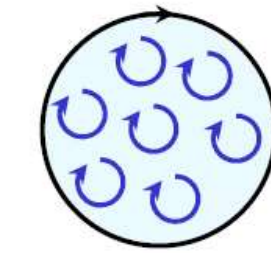


*screening of circulation
„Meißner State“*

flat surface



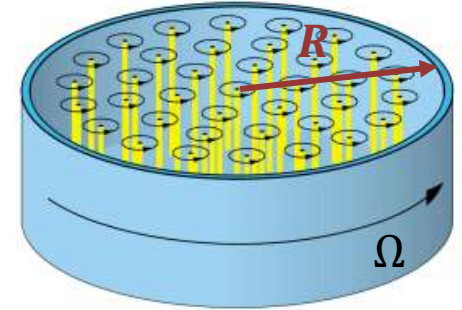
*penetration of circulation
„Mixed State“*



I.4.2 Experimental Study of Vortices

- what is the density of vortices inside the container ?
 - circulation at the edge of the container with radius R :

$$\Gamma = \oint_S \mathbf{v}(\mathbf{r}) \cdot d\mathbf{r} = 2\pi R \ v(R) = 2\pi R \ \Omega R = 2\pi R^2 \Omega$$



- circulation must be equal to number of vortices N_V \times vorticity κ \times area πR^2

$$\Gamma = N_V \kappa \pi R^2 = 2\pi R^2 \Omega$$

$$N_V = \frac{2\Omega}{\kappa} = \frac{2m_{\text{He}}\Omega}{h} = \frac{m_{\text{He}}\Omega}{\pi\hbar}$$

$$N_V \simeq \frac{20}{\text{mm}^2} \times \Omega \text{ [Hz]}$$

I.4.2 Experimental Study of Vortices

- rotating cryostats

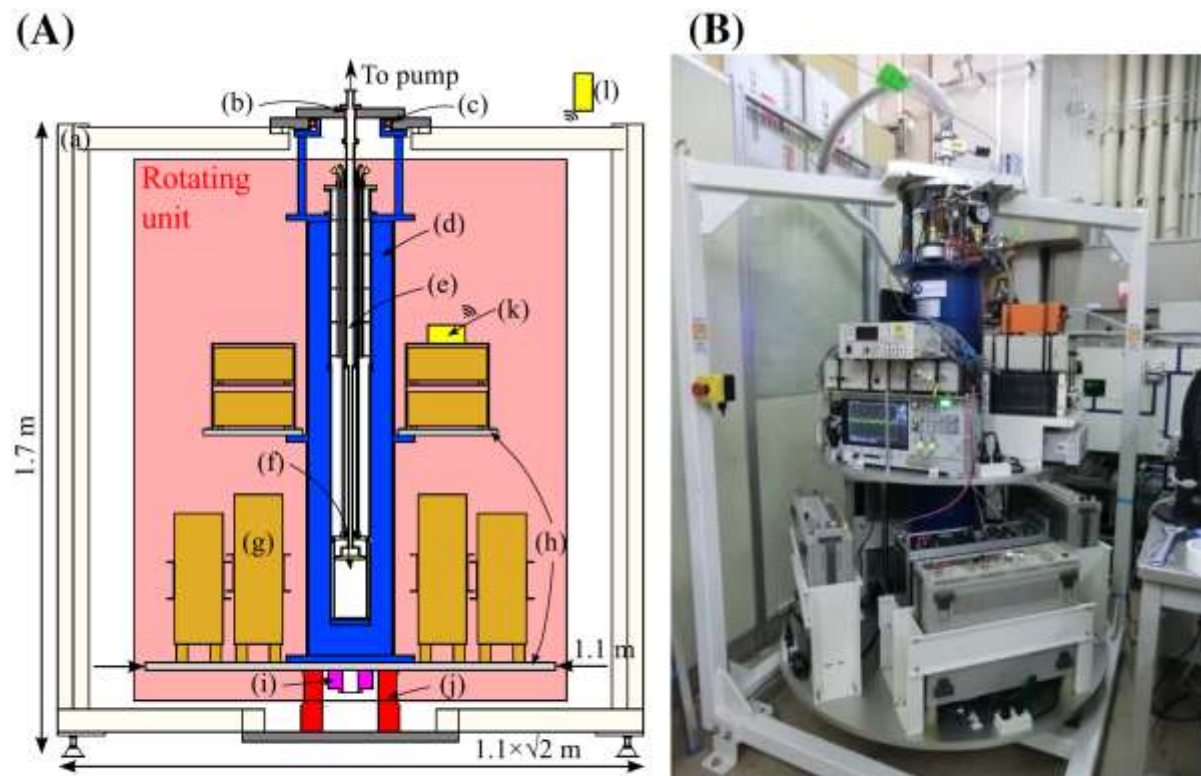


Fig. 1 **a** Schematic vertical section of the rotating cryostat: (a) Steel frame, (b) rotating o-ring seal, (c) ball bearing, (d) aluminum/FRP dewar, (e) 1K cryostat insert, (f) sample space, (g) instruments, (h) duralumin round tables, (i) slip ring for power supply, (j) AC servomotor, (k) GPIB-USB interface and wireless USB device server and (l) wireless LAN access point. **b** Photograph of the rotating cryostat (Color figure online)

A Compact Rotating 1K Cryostat for ^4He Studies
 Takahiko Makiuchi, Satoshi Murakawa & Keiya Shirahama
 J. Low Temp. Phys. **187**, 633-638 (2017)



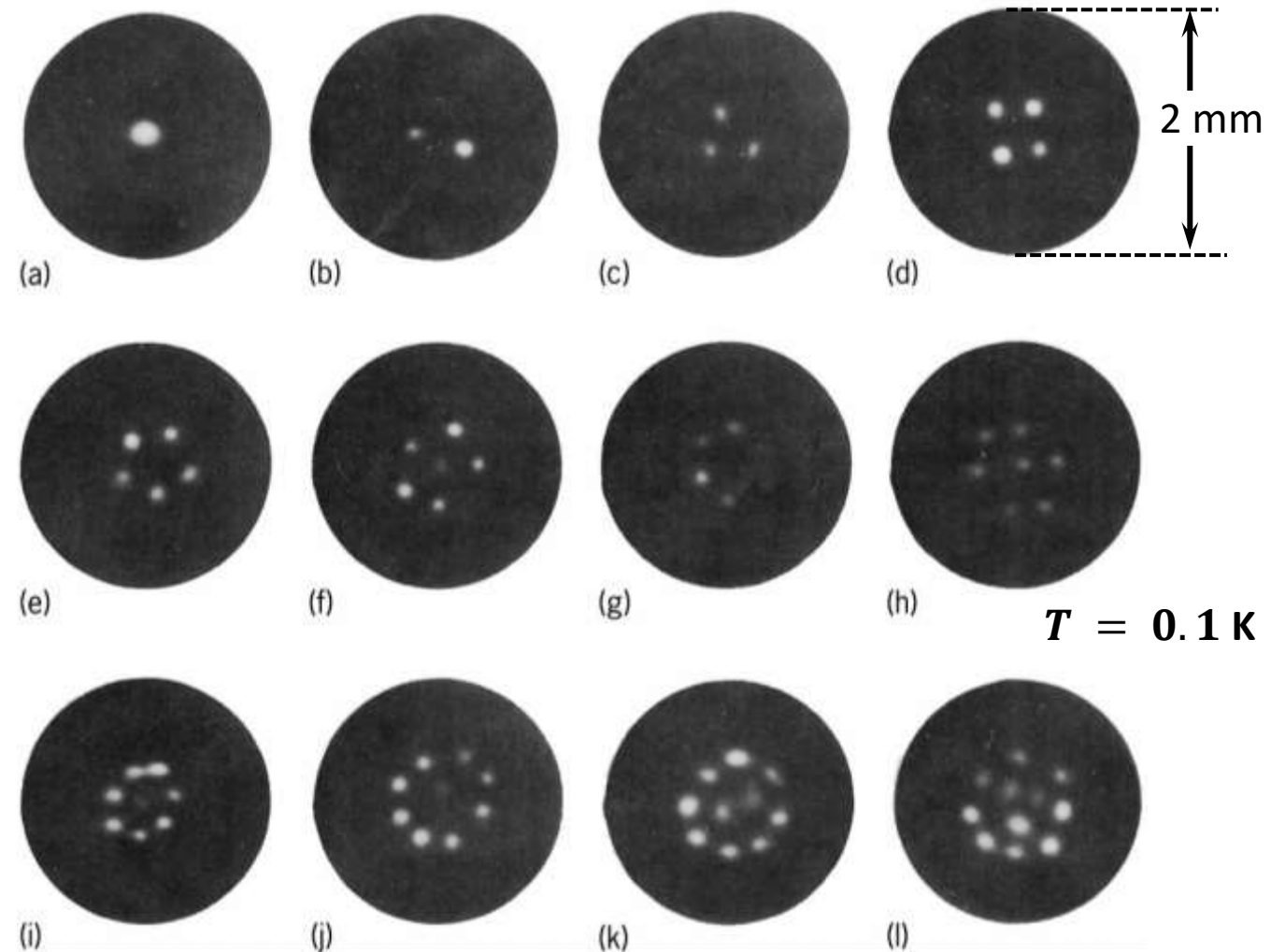
I.4.2 Experimental Study of Vortices

- imaging of vortices in superfluid ^4He

- *He is covered by electrons*
- *electrons are trapped in vortex cores*
- *on applying an electric field, electrons are extracted to fluorescent screen*

*pattern and
camera are
rotating*

*pinning of vortices
at bottom of
container*



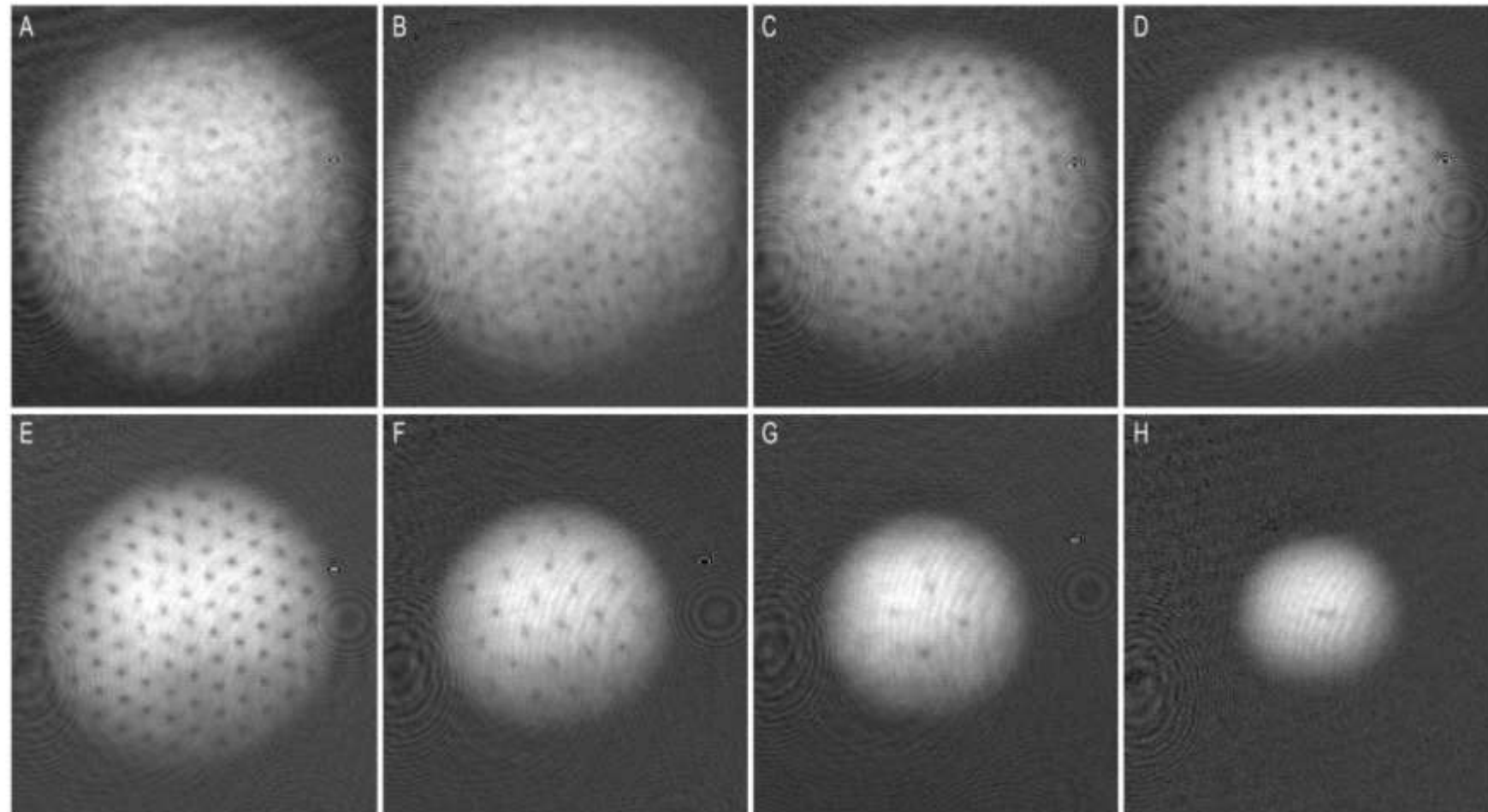
E.J. Yarmchuk, M.J.V. Gordon, R.E. Packard, Phys. Rev. Lett. **43**, 214 (1979)

E.J. Yarmchuk, R.E. Packard, J. Low Temp. Phys. **46**, 479 (1982)

I.4.2 Experimental Study of Vortices

- imaging of vortices in a Bose-Einstein condensate of Na atoms

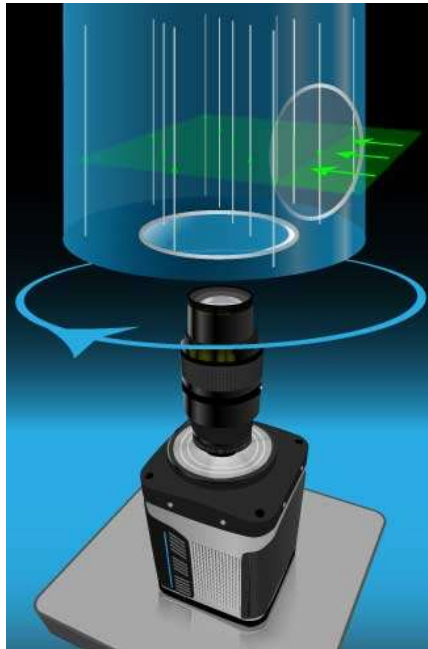
Formation and decay of a vortex lattice. The condensate was rotated for 400 ms and then equilibrated in the stationary magnetic trap for various hold times. **(A)** 25 ms, **(B)** 100 ms, **(C)** 200 ms, **(D)** 500 ms, **(E)** 1 s, **(F)** 5 s, **(G)** 10 s, and **(H)** 40 s. The decreasing size of the cloud in (E) to (H) reflects a decrease in atom number due to inelastic collisions. The field of view is about 1 mm by 1.15 mm.



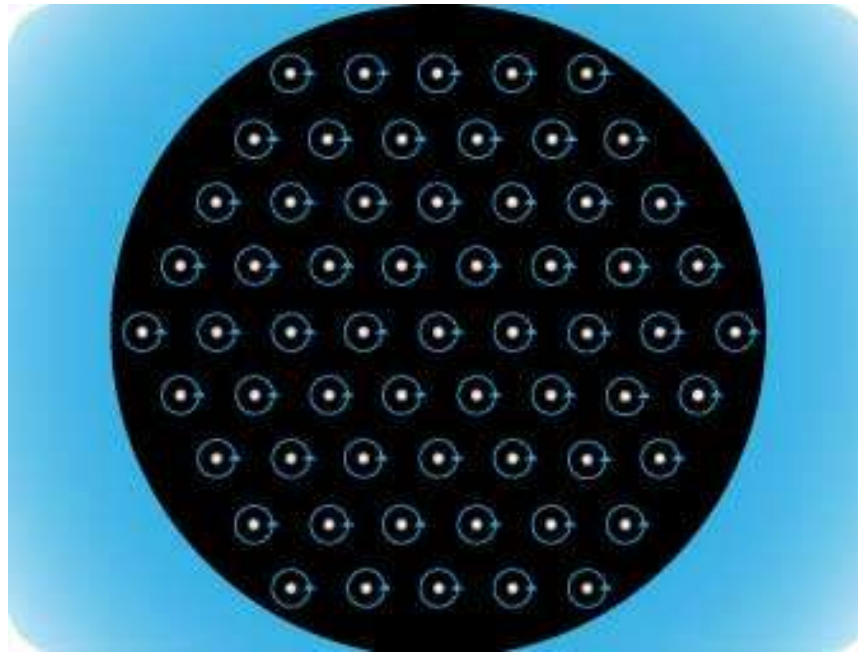
Observation of Vortex Lattices in Bose-Einstein Condensates
W. Ketterle et al., Science **292**, 476 (2001)

I.4.2 Experimental Study of Vortices

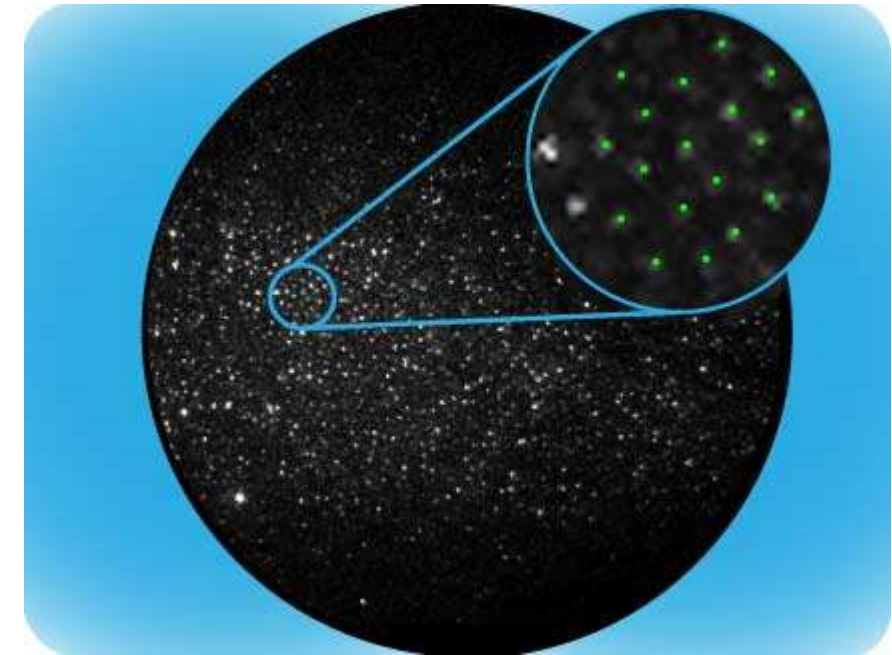
- imaging of vortices



A sheet of laser light illuminates frozen hydrogen particles trapped on vortex cores, which are then captured on video.



Quantized vortices arrange themselves in a triangular lattice pattern.



A still image captures the vortex lattice along the axis of rotation in a vat of superfluid helium. A perfect lattice is not observed due to waves and other boundary effects.

<https://engage.aps.org/dfd/resources/virtual-pressroom/2009-pressroom/language-papers-2009/tornadoes>

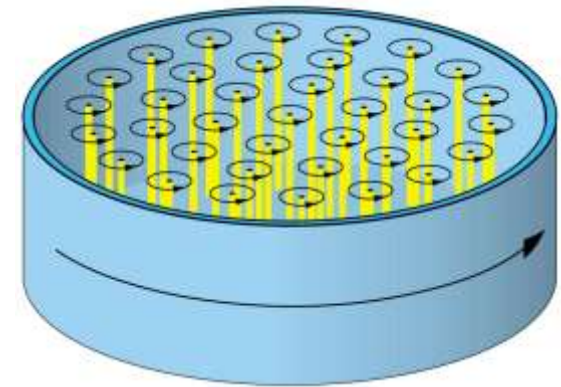
I.4.2 Experimental Study of Vortices

- generation of persistent current

- rotate a normal fluid at $T > T_\lambda$
- cool down the rotating normal fluid to $T < T_\lambda \rightarrow$ the rotation is frozen in and does not decay

- generation of circulation jumps or phase slip events

- start with a non-rotating fluid at $T < T_\lambda$: $\kappa = 0, n = 0$
- rotate the cylindrical vessel containing the fluid:
 - \rightarrow normal fluid rotates at the speed of the vessel
 - \rightarrow superfluid stays in rest
- reduce the temperature:
 - \rightarrow more and more particles of the normal fluid enter the superfluid
 - \rightarrow the lost angular momentum is transferred to the superfluid, which starts to rotate
 - \rightarrow the circulations and the vorticity κ do not change continuously but in jumps of h/m_{He}
 - \rightarrow jumps are associated with **additional vortex** corresponding to phase change of 2π called **phase slip**
 - \rightarrow phase slips correspond to **transition between two quantum states** with different quantum number n



Contents Part I: Quantum Liquids

Contents:

I.1 Foundations and General Properties

- I.1.1 Quantum Gases & Liquids
- I.1.2 Helium
- I.1.3 Van der Waals Bonding
- I.1.4 Zero-Point Fluctuations
- I.1.5 Helium under Pressure
- I.1.6 pT-Phase Diagram of ^4He and ^3He
- I.1.7 Characteristic Properties of ^4He and ^3He
- I.1.8 Specific Heat of ^4He and ^3He



I.4 Vortices

- I.4.1 Quantization of Circulation
- I.4.2 Experimental Study of Vortices

I.5 ^3He

- I.5.1 normal fluid ^3He
- I.5.2 solid ^3He and Pomeranchuk effect
- I.5.3 superfluid ^3He

I.2 ^4He as an Ideal Bose Gas

- I.2.1 Bose-Einstein Condensation
- I.2.2 Ideal Bose Gas
- I.2.3 Bose Gas with Interactions
- I.2.4 Bose-Einstein Condensation of ^4He

I.6 ^3He / ^4He mixtures

I.3 Superfluid ^4He

- I.3.1 Two-Fluid Model
- I.3.2 Experimental Observations
- I.3.3 Two-Fluid Hydrodynamics
- I.3.4 Excitation Spectrum of ^4He

I.5 ^3He

- basic properties of ^3He

^4He	$p^+\uparrow p^+\downarrow$	$n\uparrow n\downarrow$	$e^-\uparrow e^-\downarrow$	boson
^3He	$p^+\uparrow p^+\downarrow$	$n\uparrow$	$e^-\uparrow e^-\downarrow$	fermion

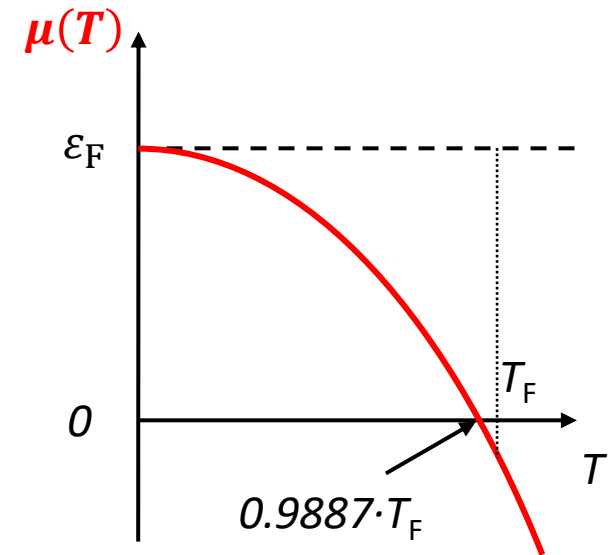
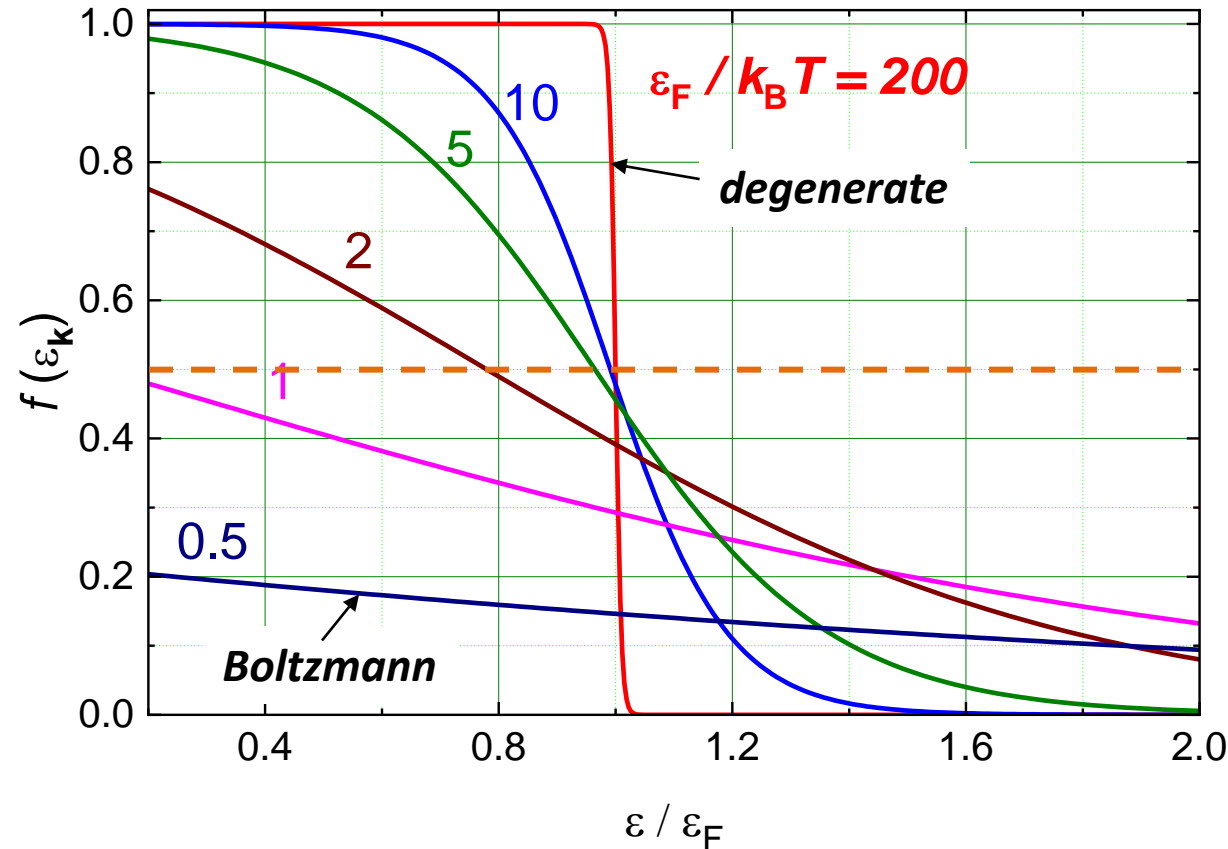
- ^3He has spin $\frac{1}{2}$ → **Fermion**
 - fluid ^3He : **interacting Fermi gas**
 - Fermi liquid (*ideal non-interacting Fermi gas with finite interaction*)
 - theoretical description by Landau (1956-1958)
 - L.D. Landau, Soviet. Phys. JETP 3, 920 (1957)
 - L.D. Landau, Soviet. Phys. JETP 5, 101 (1957)
 - solid ^3He shows nuclear magnetism
- due to different quantum statistics:
liquid ^3He has completely different properties than liquid ^4He

I.5.1 Normal Fluid ³Helium

- **first approximation:** description by an *ideal Fermi gas* (*non-interacting Fermions*)

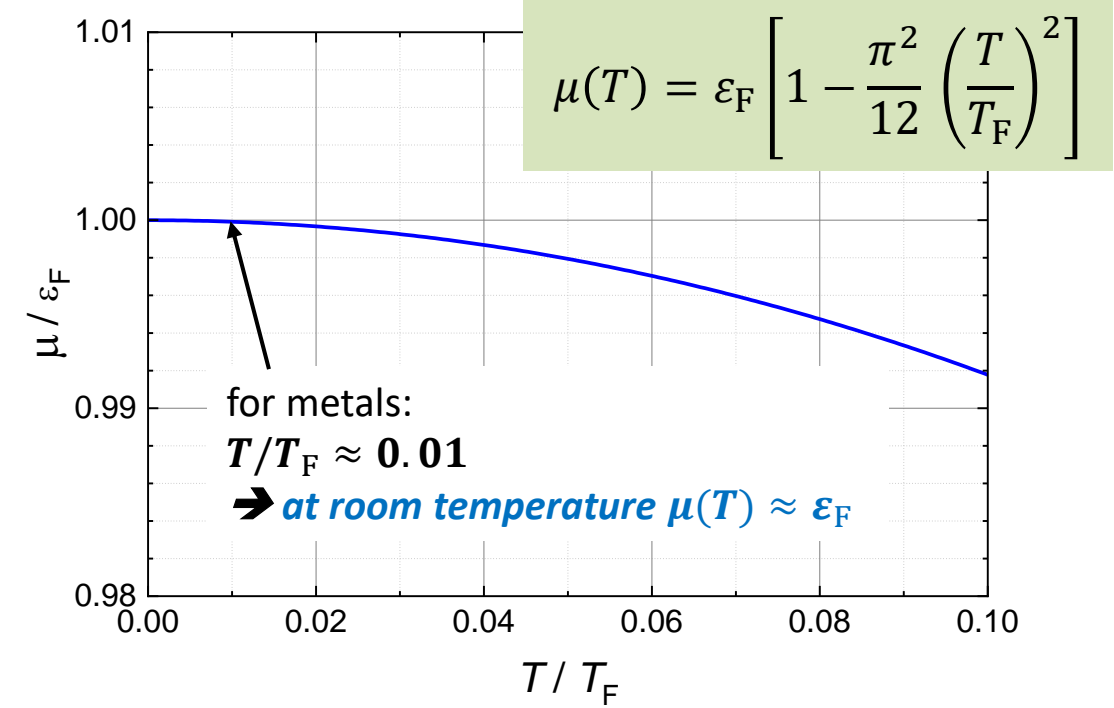
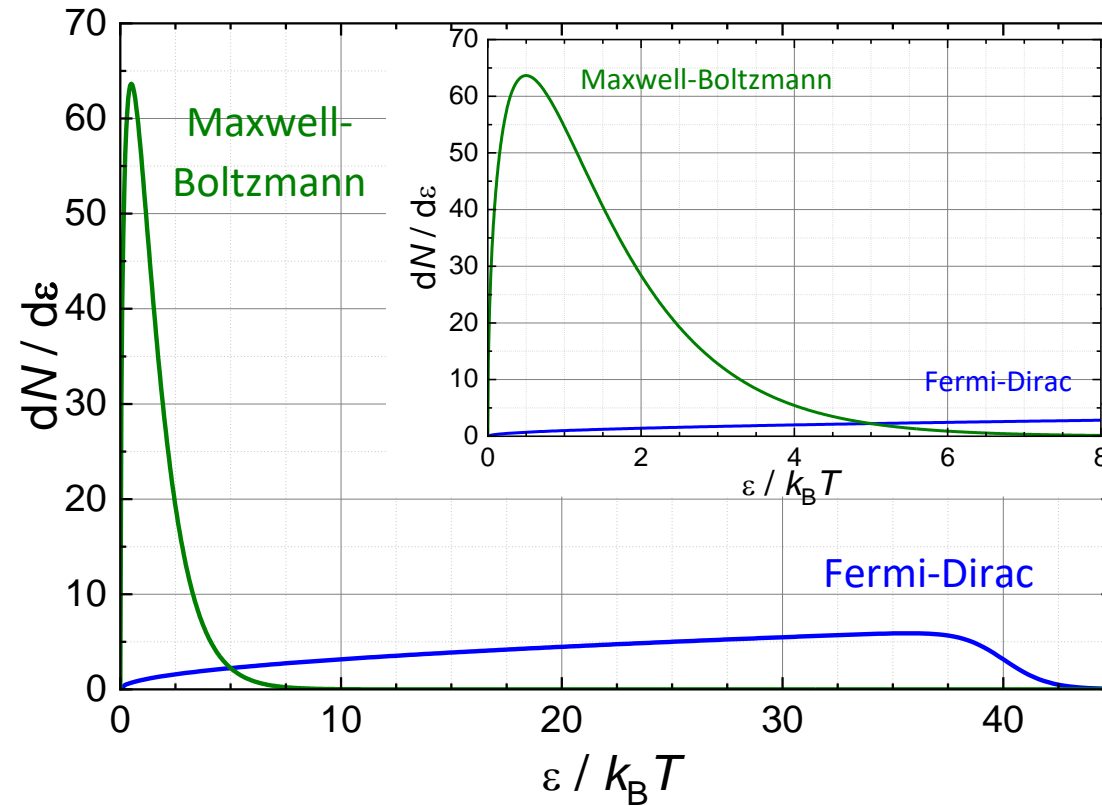
– Fermi-Dirac distribution:

$$f(\varepsilon_{\mathbf{k}}) = \frac{1}{\exp\left(\frac{\varepsilon_{\mathbf{k}} - \mu}{k_B T}\right) + 1}$$



I.5.1 Normal Fluid ^3He

- ideal (non-interacting) Fermi gas



$$N = \int D(\varepsilon) f(\varepsilon) d\varepsilon$$

$$\frac{dN}{d\varepsilon} = D(\varepsilon) f(\varepsilon)$$

I.5.1 Normal Fluid ^3He

- ideal (non-interacting) Fermi gas

$$\varepsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m} (3\pi^3 n)^{2/3} \quad \text{Fermi energy}$$

$$T_F = \varepsilon_F / k_B \quad \text{Fermi temperature}$$

$$\lambda_F = 2\pi / k_F \quad \text{Fermi wavelength}$$

$$v_F = \frac{p_F}{m} = \frac{\hbar k_F}{m} \quad \text{Fermi velocity}$$

N = particle number
 n = particle density [$1/\text{m}^3$]

$m = m_{\text{He}}$ is large \rightarrow expected Fermi temperature (energy) is small: $T_F \simeq 4.2 \text{ K}$

comparison to electron gas in metal:

electrons in metals:

$n \approx 10^{23} \text{ cm}^{-3}$ is large, $m = m_e$ is small
 $\rightarrow T_F \approx 10^5 \text{ K}$ is large

^3He :

$n \approx 10^{22} \text{ cm}^{-3}$ is slightly smaller, $m = m_{\text{He}}$ is large
 $\rightarrow T_F \approx \text{few K}$ is small

$$T_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2mk_B} (3\pi^3 n)^{2/3}$$

I.5.1 Normal Fluid ³Helium

- ideal (non-interacting) Fermi gas: specific heat

– classical: $C_V^{\text{class}} = \left. \frac{\partial \langle U \rangle}{\partial T} \right|_V = N \cdot 3 \cdot \frac{1}{2} k_B = \frac{3}{2} N k_B$ (Dulong-Petit)

kinetic degrees of freedom
contribution per degree of freedom (equipartition theorem)

– quantum mechanical: $U = \int_0^\infty \varepsilon D(\varepsilon) f(\varepsilon) d\varepsilon$

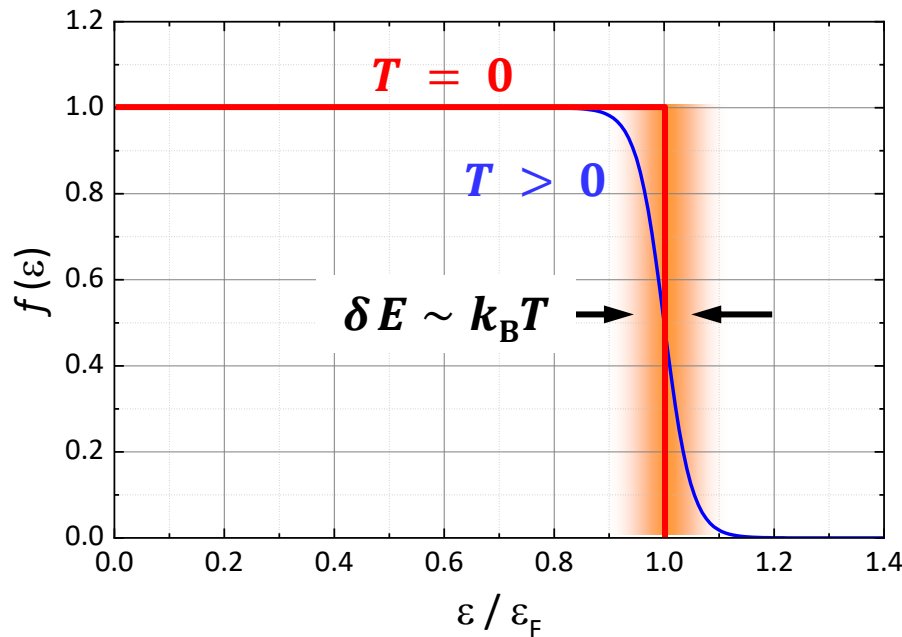
Sommerfeld expansion $U = U(T=0) + (k_B T)^2 \frac{\pi^2}{6} D(\varepsilon_F)$

$\frac{D(\varepsilon_F)}{V} = \frac{3}{2} \frac{n}{\varepsilon_F}$

$\frac{C_V^{\text{qm}}}{V} = \frac{\pi^2}{2} n k_B^2 \frac{T}{\varepsilon_F} = \gamma \cdot T$

Sommerfeld coefficient

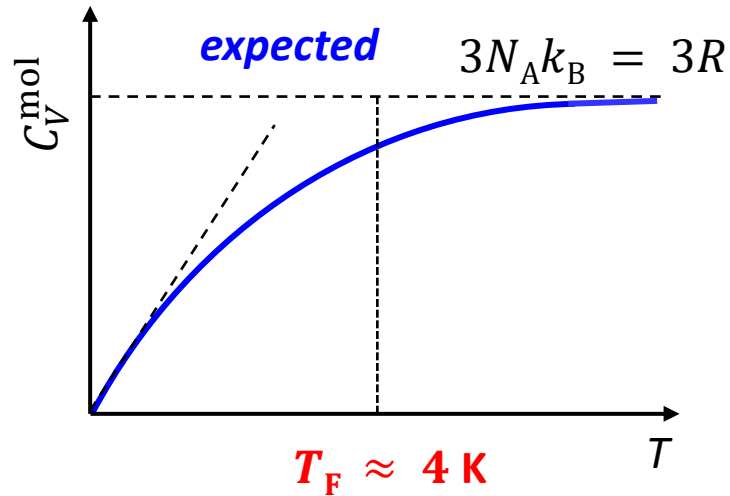
$\gamma = \frac{\pi^2}{3} k_B^2 \frac{D(\varepsilon_F)}{V} = \frac{\pi^2}{2} \frac{n k_B^2}{\varepsilon_F}$



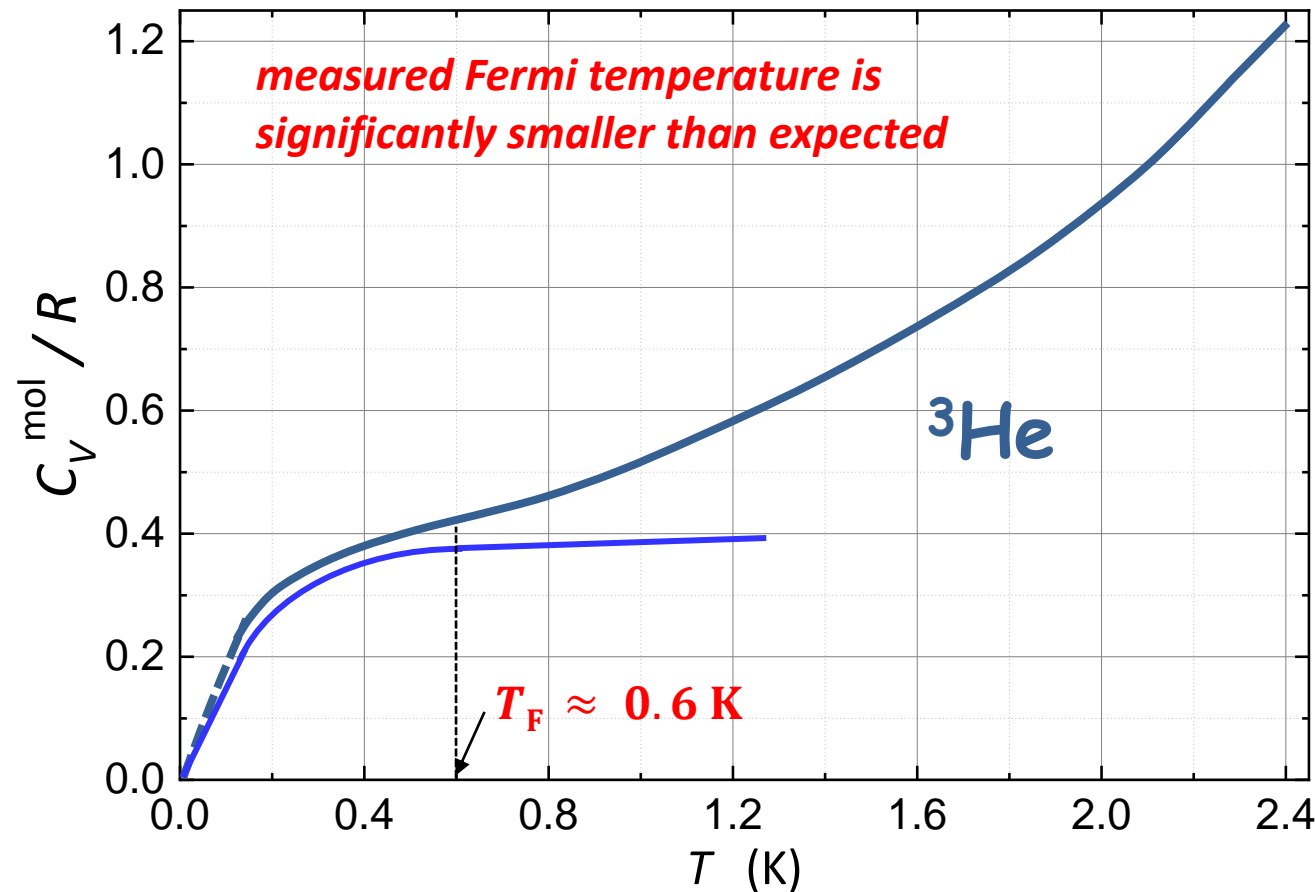
- only small fraction $\simeq T/T_F$ of particles in energy interval $k_B T$ around E_F contributes to specific heat
- energy change per particle $\simeq k_B T$
- total energy change $\simeq k_B T \cdot N k_B \left(\frac{T}{T_F} \right) \propto D(E_F) T^2$
- $C_V \propto D(E_F) T$

I.5.1 Normal Fluid ^3He

- ideal (non-interacting) Fermi gas: specific heat

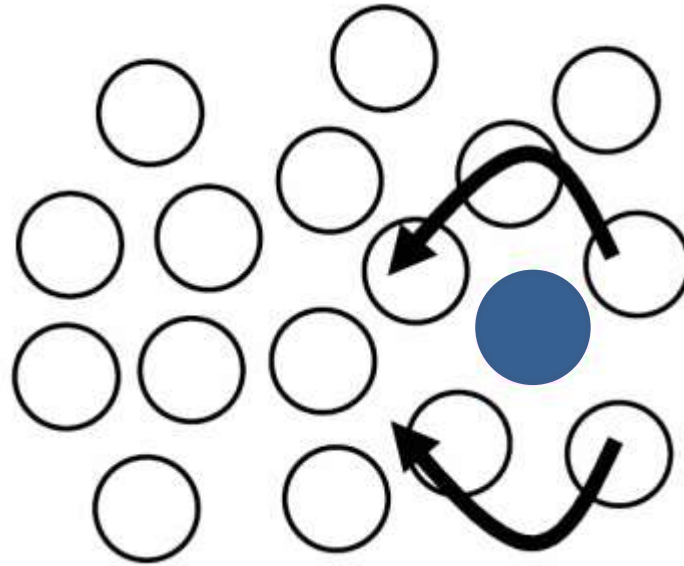


- interacting Fermi gas \rightarrow **Fermi liquid** (Landau)
- increased effective mass m^*
 - \rightarrow smaller Fermi temperature $T_F \propto 1/m^*$
 - $m^*/m_{\text{He}} \simeq 2.8$ (@ $p = 1 \text{ bar}$)
 - $\simeq 5.5$ (@ $p = 30 \text{ bar}$)
- additional contribution due to **spin fluctuations (paramagnons)**
 - \rightarrow strong repulsive interaction at short distance \rightarrow antisymmetric orbital wavefunction
 - \rightarrow symmetric spin wavefunction \rightarrow parallel orientation of neighboring spins



I.5.1 Normal Fluid ^3He

- intuitive argument for increased effective mass:



interaction causes „backflow“ of particles

→ increased effective mass m^*

→ reduced Fermi temperature $T_F \propto 1/m^*$

I.5.1 Normal Fluid ³Helium

- magnetic susceptibility: nuclear spin $I = 1/2$ determines magnetic properties

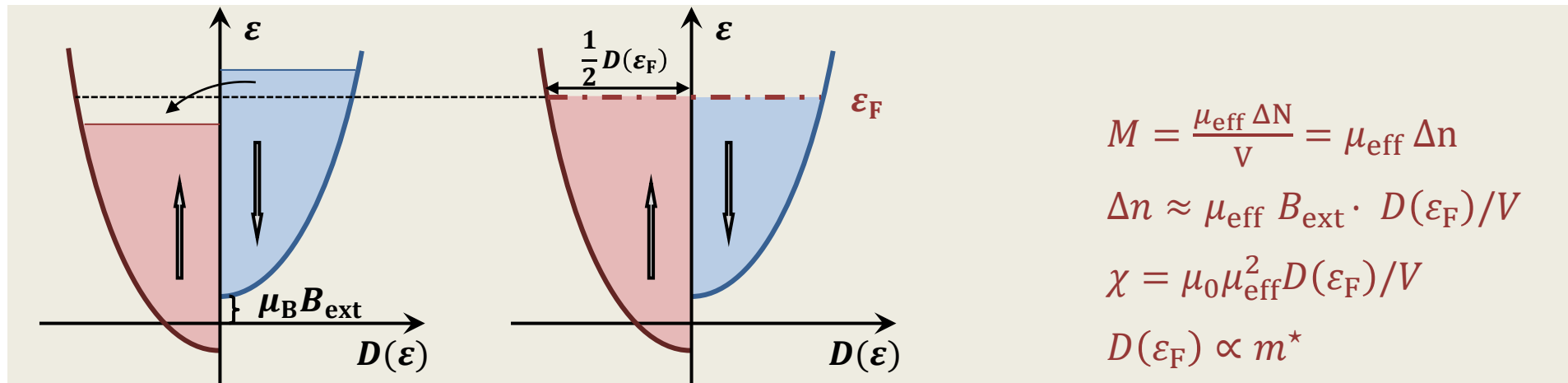
- $T > T_F$: **non-degenerate Fermi gas** → magnetic susceptibility follows **Curie law**: $\chi \propto C/T$

$$\chi = \mu_0 \left(\frac{\partial M}{\partial B_{\text{ext}}} \right)_{T,V} = \frac{\mu_0 n \mu_{\text{eff}}^2}{k_B T} = \frac{C}{T} \quad \mu_{\text{eff}} = g_n \mu_n \quad \begin{array}{l} g_n = \text{Landé factor} \\ \mu_n = \text{magnetic moment of } ^3\text{He} \end{array}$$

- $T < T_F$: **degenerate Fermi gas** → temperature independent **Pauli spin susceptibility** (only fraction T/T_F contributes)

for electron liquid: $\chi_P = \mu_0 \left(\frac{\partial M}{\partial B_{\text{ext}}} \right)_{T,V} = \mu_0 \mu_B^2 \frac{D(\epsilon_F)}{V} = \frac{3}{2} \frac{\mu_0 n \mu_B^2}{k_B T_F} \simeq \text{const.}$

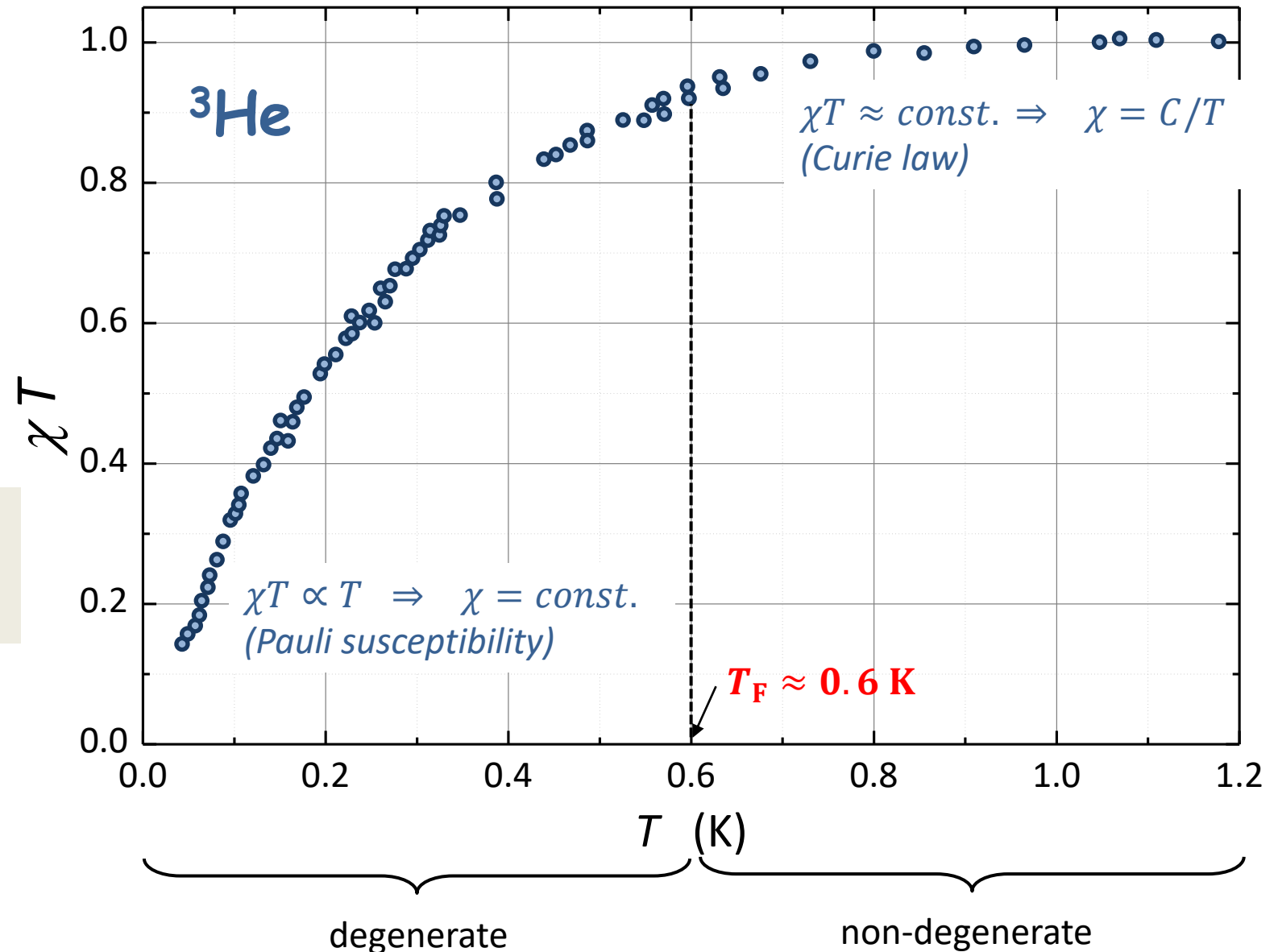
for ³He liquid: $\chi_P = \mu_0 \mu_{\text{eff}}^2 \frac{D(\epsilon_F)}{V}$ with $\mu_{\text{eff}} = g_n \mu_n$



I.5.1 Normal Fluid ^3He

- magnetic susceptibility

cannot be measured for an electron gas system in a metal since $T_F \gg$ melting temperature



I.5.1 Normal Fluid ^3He

- transport properties

- description by *kinetic gas theory*:

→ viscosity: $\eta = \frac{1}{3} \rho \, v \, \ell$ (ℓ = mean free path)

→ diffusion coefficient: $D = \frac{1}{3} v \, \ell$ (self-diffusion)

→ thermal conductivity: $\kappa = \frac{1}{3} C_V \, v \, \ell$

- classical gas: v_{th} = thermal velocity $v_{\text{th}} = \sqrt{2k_B T/m}$ (Maxwell-Boltzmann statistics)

- Fermi gas: v_F = Fermi velocity $v_F = \frac{\hbar}{m} (3\pi^3 n)^{1/3}$ (Fermi statistics)

- Fermi gas, with $\ell = v_F \tau$

$$\eta = \frac{1}{3} \rho \tau v_F^2 \propto \frac{1}{T^2}$$

$$D = \frac{1}{3} \tau v_F^2 \propto \frac{1}{T^2}$$

$$\kappa = \frac{1}{3} C_V \tau v_F^2 \propto \frac{1}{T}$$

$$\tau^{-1} \propto \left(\frac{T}{T_F} \right)^2$$

particle-particle scattering rate

Contents Part I: Quantum Liquids

Contents:

I.1 Foundations and General Properties

- I.1.1 Quantum Gases & Liquids
- I.1.2 Helium
- I.1.3 Van der Waals Bonding
- I.1.4 Zero-Point Fluctuations
- I.1.5 Helium under Pressure
- I.1.6 pT-Phase Diagram of ^4He and ^3He
- I.1.7 Characteristic Properties of ^4He and ^3He
- I.1.8 Specific Heat of ^4He and ^3He

I.2 ^4He as an Ideal Bose Gas

- I.2.1 Bose-Einstein Condensation
- I.2.2 Ideal Bose Gas
- I.2.3 Bose Gas with Interactions
- I.2.4 Bose-Einstein Condensation of ^4He

I.3 Superfluid ^4He

- I.3.1 Two-Fluid Model
- I.3.2 Experimental Observations
- I.3.3 Two-Fluid Hydrodynamics
- I.3.4 Excitation Spectrum of ^4He

I.4 Vortices

- I.4.1 Quantization of Circulation
- I.4.2 Experimental Study of Vortices

I.5 ^3He

- I.5.1 normal fluid ^3He
- I.5.2 solid ^3He and Pomeranchuk effect
- I.5.3 superfluid ^3He



I.6 ^3He / ^4He mixtures

I.5.1 Normal Fluid ^3He

- comparison of properties of liquid ^3He with those of ideal (non-interacting) Fermi gas

	^3He	Fermi Gas	ratio
$C_V^{\text{mol}}/N_A k_B T$	2.78	1.0	2.78
v_F (m/s)	56	157	2.80
$\chi/\mu_0 \mu_{\text{eff}}^2$ (Jm ³) ⁻¹	$3.3 \cdot 10^{51}$	$3.6 \cdot 10^{50}$	9.2

significant deviations due to finite interactions → Landau Fermi liquid
→ effective mass

mole volume: 36.84 cm^3 @ $p = 0 \text{ bar}$, $T = 0 \text{ K}$ → $n = 1.63 \times 10^{28} \text{ m}^{-3}$
mass: $m_{^3\text{He}} = 3.016 \text{ amu} = 5.006 \times 10^{-27} \text{ kg}$

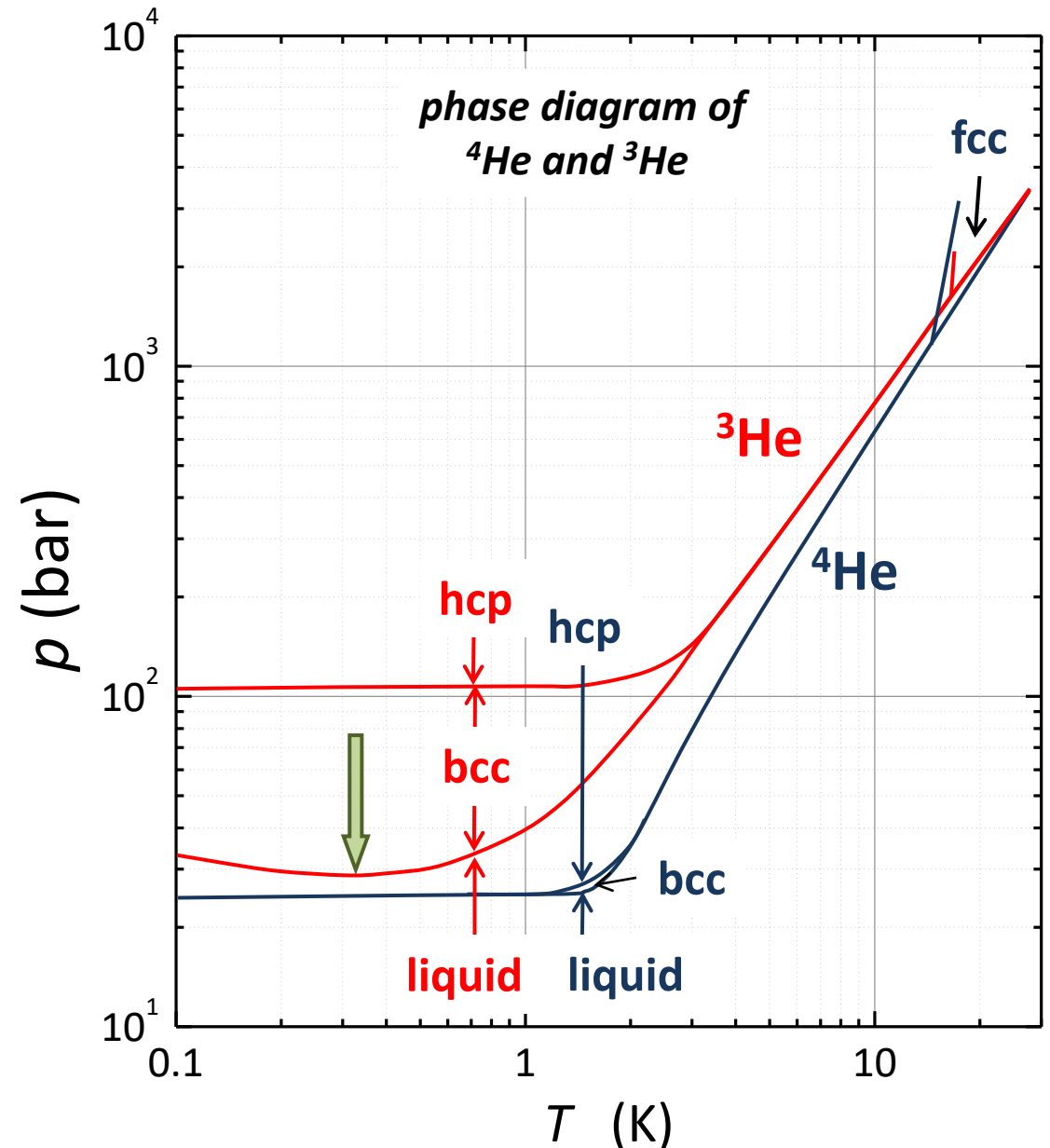
I.5.2 Solid ^3He and Pomerantchuk Effect

- comparison of phase diagrams of ^3He and ^4He

- ^4He mostly hcp, small region with bcc, at high T : fcc
- ^3He shows bcc-hcp transition at 100 bar, at high T : fcc
- ^3He shows **minimum in melting curve** at **$T = 0.32 \text{ K}$**

→ can be used for cooling of ^3He
Pomerantchuk effect

Isaak Jakowlewitsch Pomerantschuk
 (Исаак Яковлевич Померанчук;
 geb: 20. Mai 1913 in Warschau;
 gest: 14. Juli 1966 in Moskau)
 russischer Physiker.



I.5.2 Solid ^3He and Pomerantchuk Effect

– Clausius-Clapeyron equation: $\left. \frac{\partial p_m}{\partial T} \right|_{\text{melting}} = \frac{S_{\text{liq}} - S_{\text{sol}}}{V_{\text{liq}} - V_{\text{sol}}} = \frac{\Delta S_m}{\Delta V_m} = \frac{\Delta Q_m}{\Delta V_m T}$ and $V_{\text{liq}} > V_{\text{sol}}$

for $T < 0.32 \text{ K}$: $\left. \frac{\partial p_m}{\partial T} \right|_{\text{melting}} < 0 \Rightarrow S_{\text{liq}} < S_{\text{sol}} \rightarrow \text{disorder larger in solid than in liquid phase ??}$

– explanation:

→ **solid phase**: atoms are ordered,

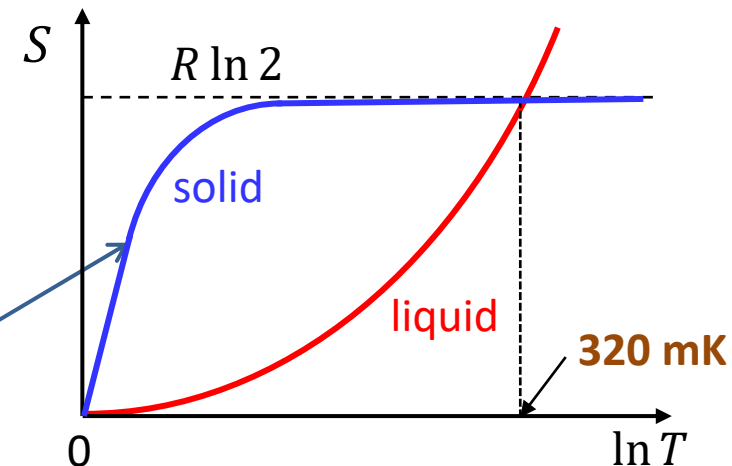
→ spins are disordered and determine entropy: $S_{\text{solid}} = N_A k_B \ln 2 = R \ln 2$
at low T : antiparallel ordering of spins, S decreases towards zero

→ **liquid phase**: atoms are spatially disordered, but ordering in k -space (Fermi liquid)

→ entropy of Fermi liquid:

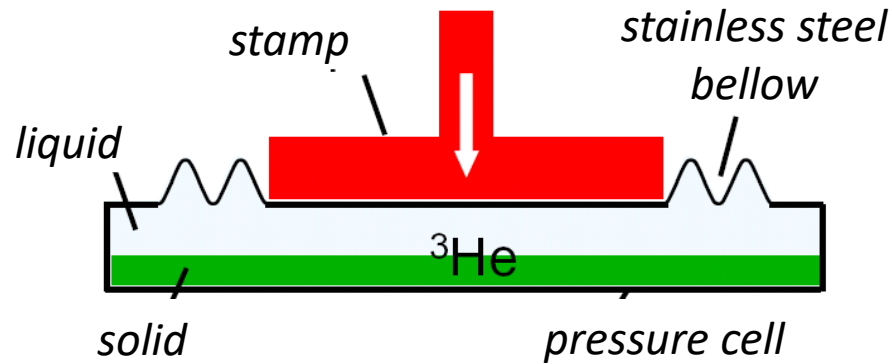
$$S_{\text{liquid}} = \int_0^T \frac{C_V}{T'} dT' \propto \frac{T}{T_F}$$

drop due to interactions

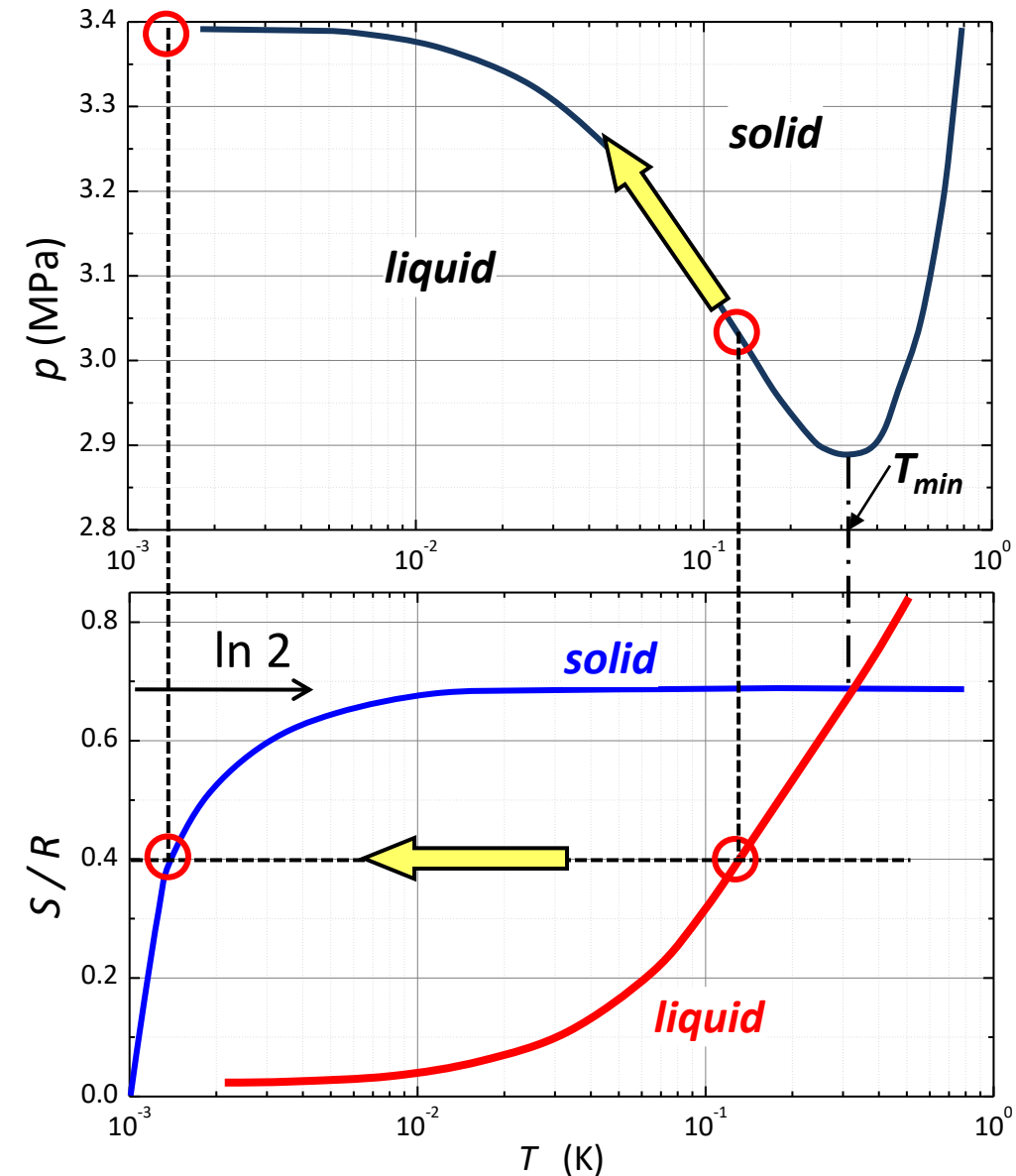


I.5.2 Solid ^3He and Pomerantchuk Effect

- Pomerantchuk cooling

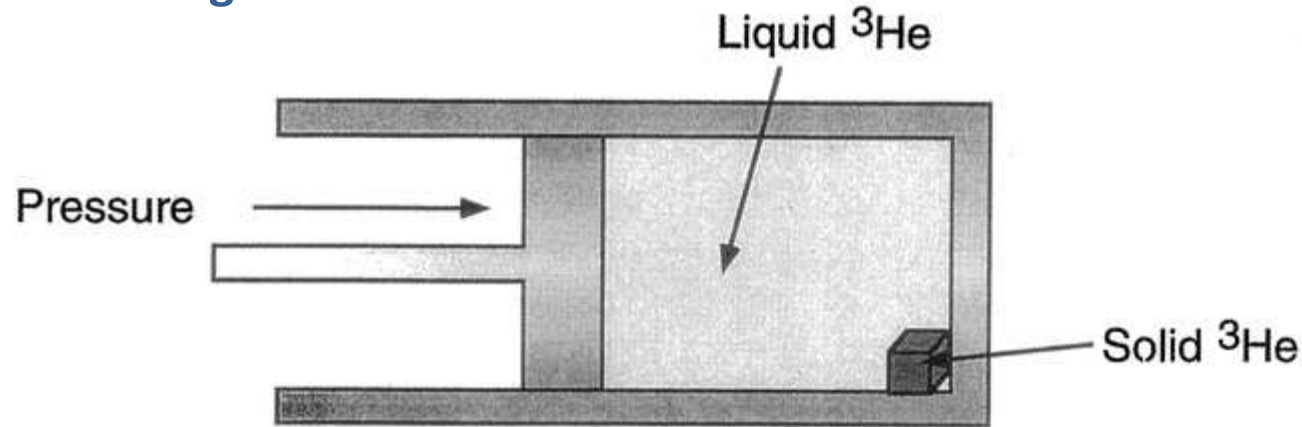


- precooling to $T < T_{\min}$
- adiabatic compression
→ *solidification and cooling*
- lowest T : ≈ 1.5 mK
- *limitation due to antiparallel spin ordering in solid ^3He*



I.5.2 Solid ^3He and Pomerantchuk Effect

- Pomerantchuk cooling



Richardson,
Nobel Prize Lecture (1996)

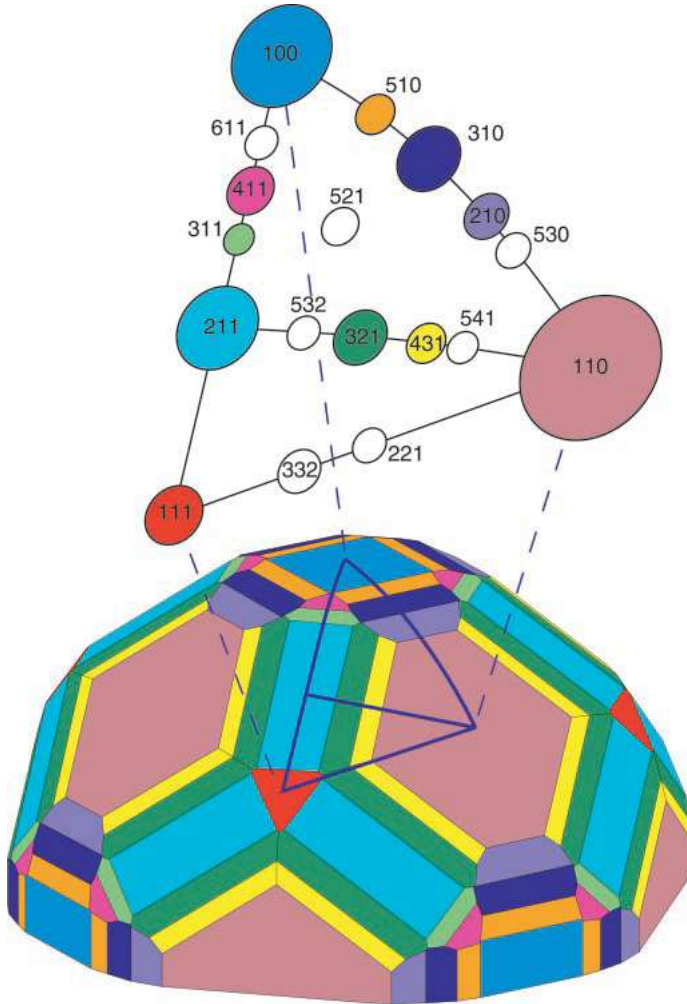
$$\frac{\Delta V}{V} \simeq 5 \% \quad \text{volume change for complete conversion of liquid to solid}$$

Pomeranchuk's suggestion for cooling a melting mixture of ^3He :

The solid phase has a higher entropy than the liquid at low temperatures. As the liquid-solid mixture is compressed, heat is removed from the liquid phase as solid crystallites form. The fractional change of volume required to completely convert liquid into solid is approximately 5%. Unlike melting water, [the solid phase forms at the hottest part of the container.](#)

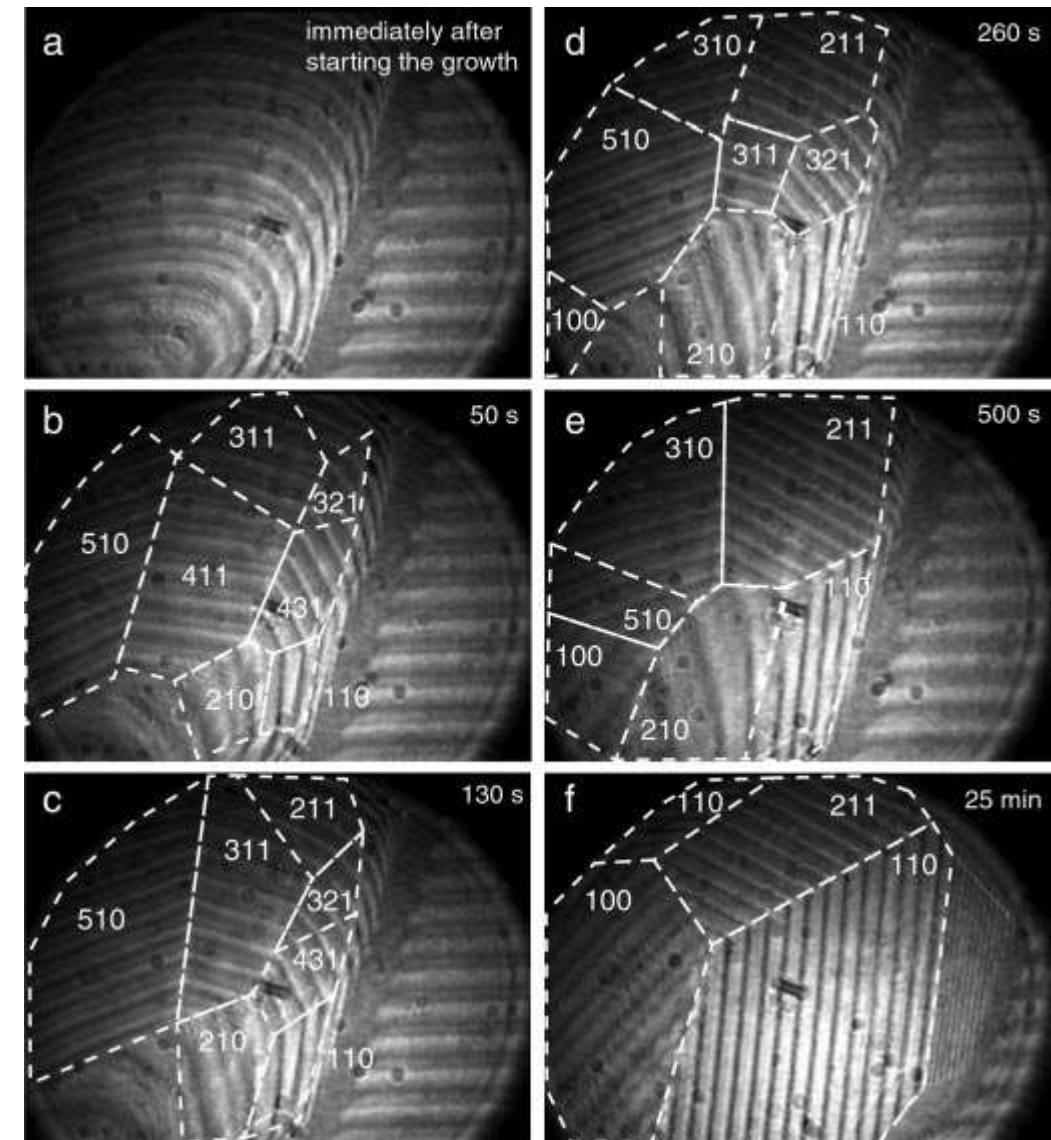
I.5.2 Solid ^3He and Pomerantchuk Effect

- solid ^3He : single crystals



computer-generated shape of a bcc crystal presented together with an elementary patch of a crystal habit where the facets are labeled with Miller indices.

H. Alles et al., PNAS **99**, 1796-1800 (2002)



Sequence of interferograms of a growing ^3He crystal at $T = 0.55$ mK. The dashed white lines outline the identified facets marked with Miller indices.



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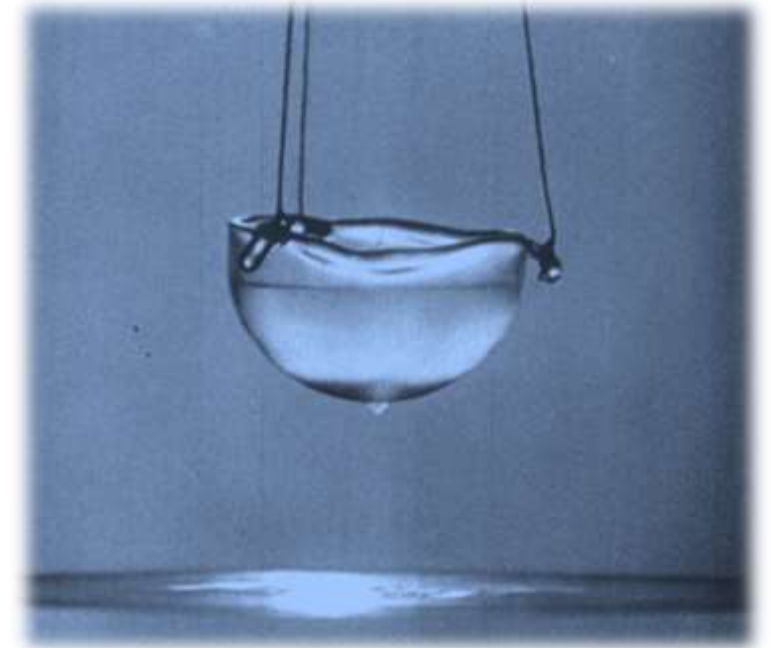
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München



Superconductivity and Low Temperature Physics II



Lecture No. 7

R. Gross

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Contents Part I: Quantum Liquids

Contents:

I.1 Foundations and General Properties

- I.1.1 Quantum Gases & Liquids
- I.1.2 Helium
- I.1.3 Van der Waals Bonding
- I.1.4 Zero-Point Fluctuations
- I.1.5 Helium under Pressure
- I.1.6 pT-Phase Diagram of ^4He and ^3He
- I.1.7 Characteristic Properties of ^4He and ^3He
- I.1.8 Specific Heat of ^4He and ^3He

I.2 ^4He as an Ideal Bose Gas

- I.2.1 Bose-Einstein Condensation
- I.2.2 Ideal Bose Gas
- I.2.3 Bose Gas with Interactions
- I.2.4 Bose-Einstein Condensation of ^4He

I.3 Superfluid ^4He

- I.3.1 Two-Fluid Model
- I.3.2 Experimental Observations
- I.3.3 Two-Fluid Hydrodynamics
- I.3.4 Excitation Spectrum of ^4He

I.4 Vortices

- I.4.1 Quantization of Circulation
- I.4.2 Experimental Study of Vortices

I.5 ^3He

- I.5.1 normal fluid ^3He
- I.5.2 solid ^3He and Pomeranchuk effect
- I.5.3 superfluid ^3He



I.6 ^3He / ^4He mixtures

I.5.3 Superfluid ^3He

- **history of superfluid ^3He**
 - **initial believe:** no superfluid ^3He due to Pauli's exclusion principle for fermions
→ no Bose-Einstein condensation of ^3He , no superfluid phase
 - **after development of BCS theory:** **pairing of ^3He atoms to Cooper pairs ??**
→ problem: liquid ^3He is system of strongly interacting particles
→ formation of Cooper pairs out of „naked“ ^3He atoms not possible (nature of exchange boson ??)
→ Landau: weakly interacting quasiparticles in Fermi liquid
 - **1963:** critical pairing temperature of about 100 mK was expected
 - **1971:** discovery of superfluid ^3He at $T < 2.7$ mK by **Osheroff, Richardson** and **Lee** (used Pomerantschuk cooling)
(*Nobel Prize in Physics 1996 „for their discovery of superfluid ^3He “*)
D.D. Osheroff, R.C. Richardson, D.M. Lee, Phys. Rev. Lett. **28**, 855 (1972),
D.D. Osheroff, W.J. Gully, R.C. Richardson, D.M. Lee, Phys. Rev. Lett. **29**, 920 (1972)
 - **initially:** two superfluid phases **A** and **B**
later: three phases (in magnetic field additional **A₁** phase)
 - experimental result: pairs have spin **$S = 1$** , **spin triplet pairs**
→ pairs with **$S = 1$** must have **$L = 1$** (see below)
→ occupation probability for same position is negligible (*reduction of short length repulsion*)

I.5.3 Superfluid ^3He

- discovery of superfluid ^3He in 1971



Douglas D. Osheroff,
Stanford University,
Stanford, California, USA



David M. Lee,
Cornell University, Ithaca,
New York, USA



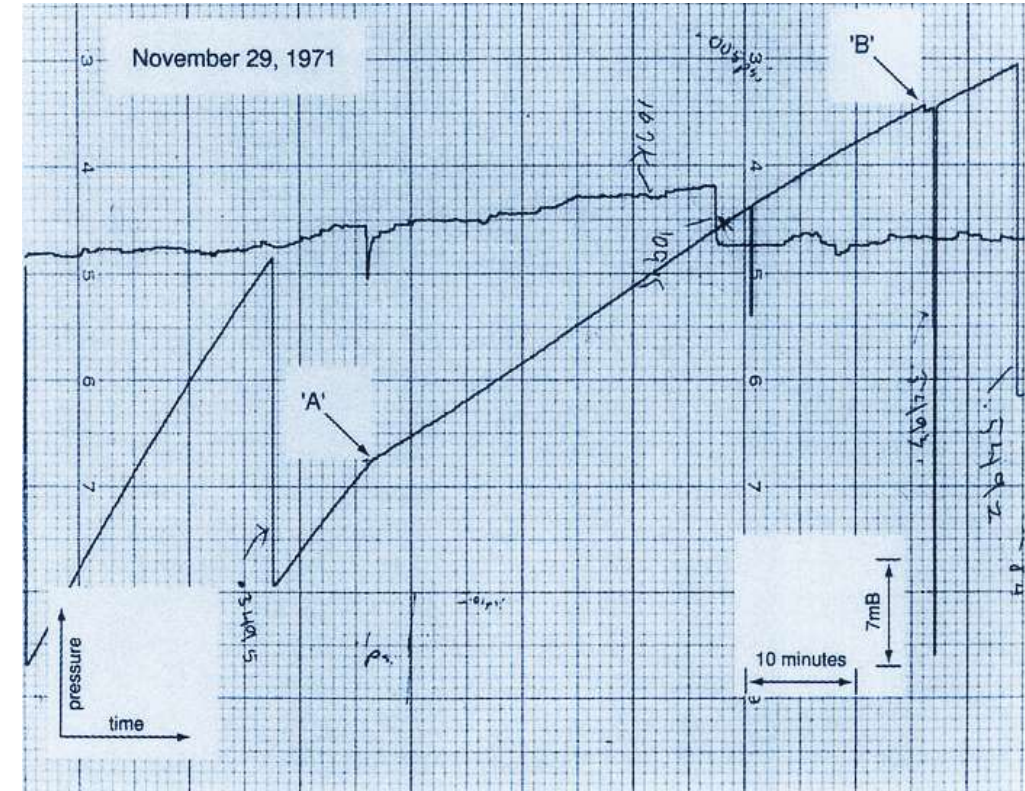
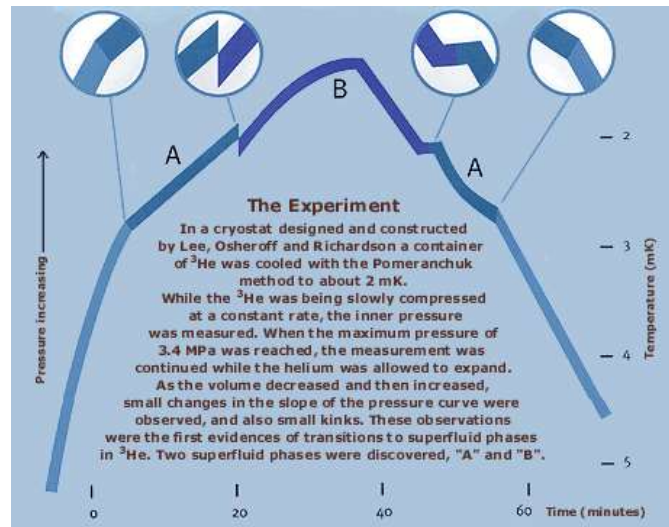
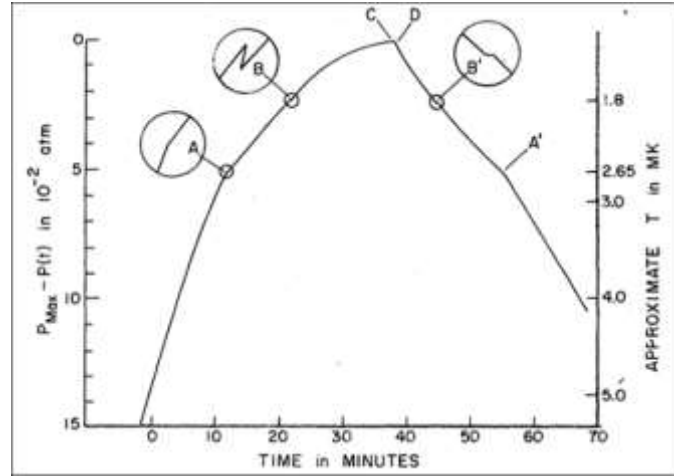
Robert C. Richardson,
Cornell University, Ithaca,
New York, USA

The Nobel Prize in Physics 1996

"for their discovery of superfluidity in helium-3"

I.5.3 Superfluid ^3He

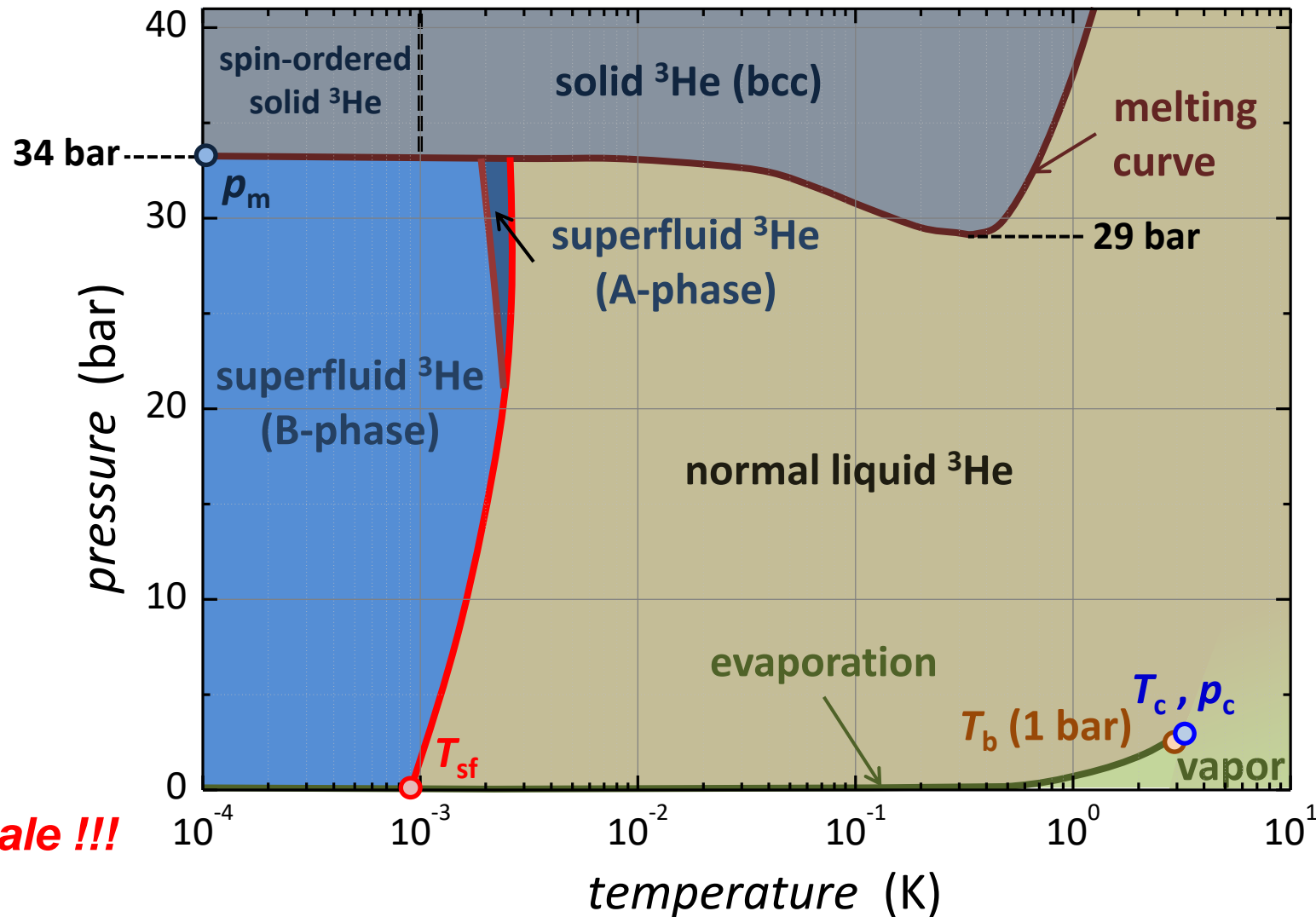
- discovery of superfluid ^3He in 1971



- cool down of ^3He via Pomeranchuk effect
- observation of melting curve
→ *anomalies give hint to phase transitions*

source: Nobel Prize Lecture Lee/Osheroff (1996)

I.5.3 Superfluid ^3He : phase diagram for $B = 0$



big difference in value of T_{sf} compared to ^4He

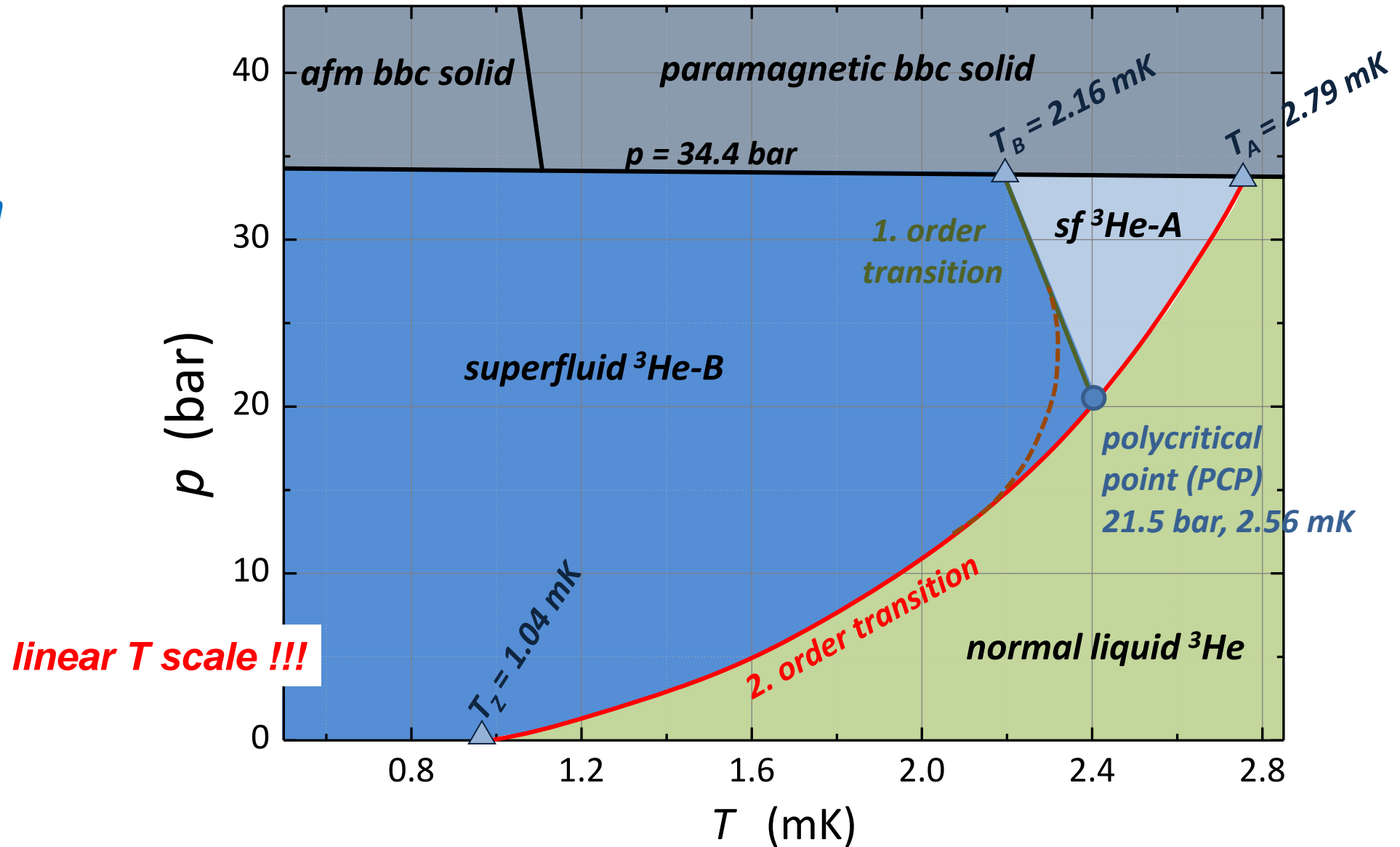
- normal/superfluid transition is dominated by quantum effects
- quantum statistics matters!! (bosons vs. fermions)
- nature of superfluid phase in ^4He and ^3He is different

log T scale !!!

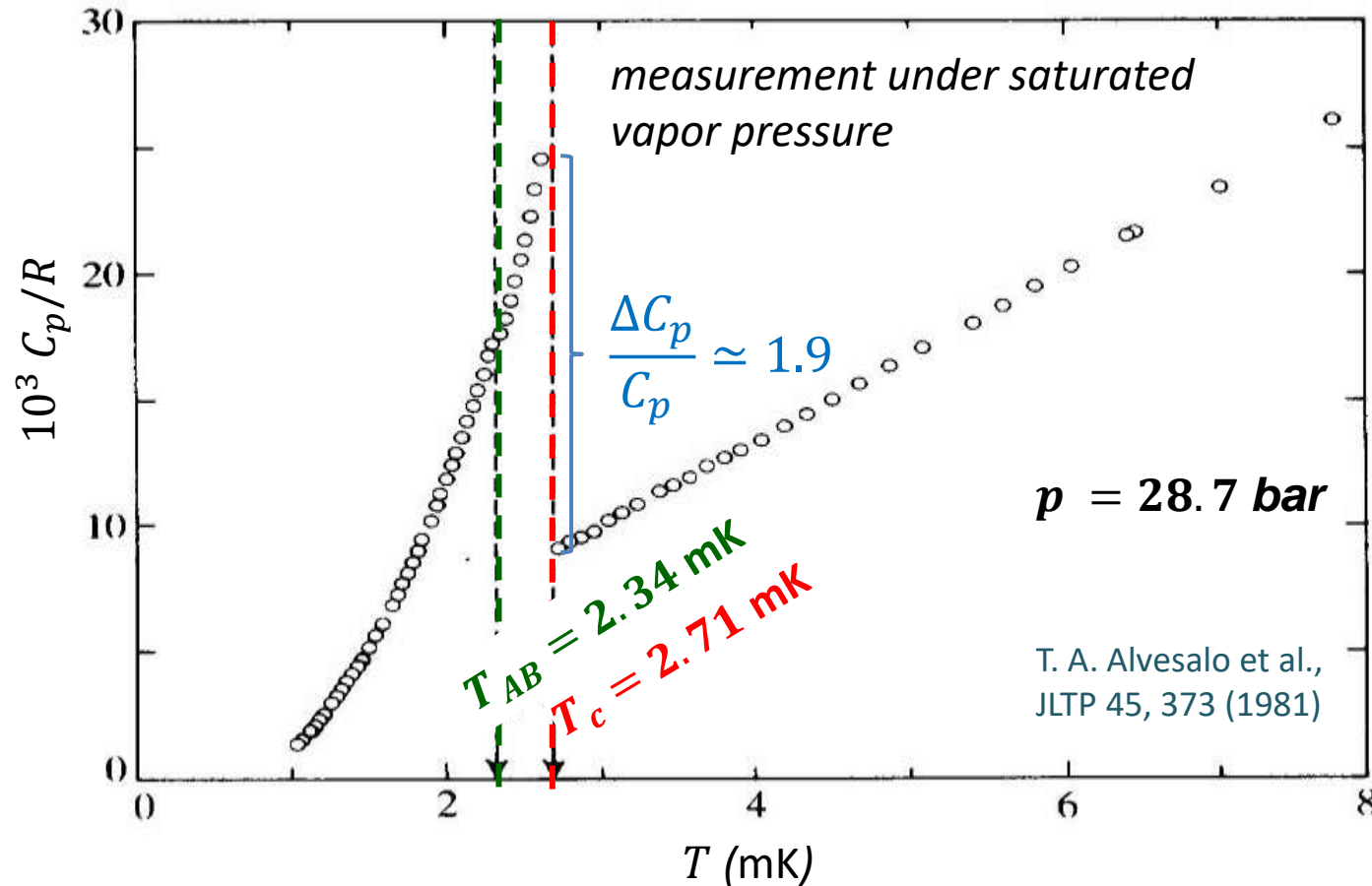
- phase diagram of ^3He : note the logarithmic temperature scale. There are **two superfluid phases of ^3He , A and B.**
- the line within the solid phase indicates a transition between spin-ordered and spin-disordered structures (at low and high temperatures, respectively).

I.5.3 Superfluid ^3He : phase diagram for $B = 0$

pressure increases
pairing interaction



I.5.3 Superfluid ^3He : specific heat



$$\Delta C_p/C_p \approx 2$$

@ $p = 30$ bar

$$\Delta C_p/C_p \approx 1.4$$

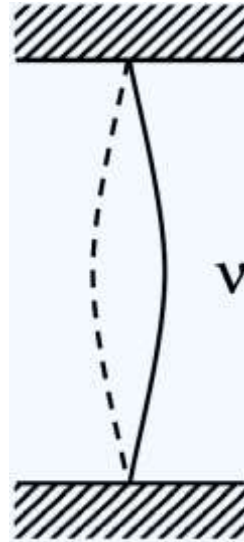
@ $p = 1$ bar

weak coupling
BCS theory:
 $\Delta C_p/C_p = 1.43$

- T_c : 2. order phase transition \rightarrow jump of C_p as for superconductor
- T_{AB} : 1. order phase transition \rightarrow small change of temperature dependence of C_p
 \rightarrow hysteresis, latent heat
- PCP: first and second order phase transition merge

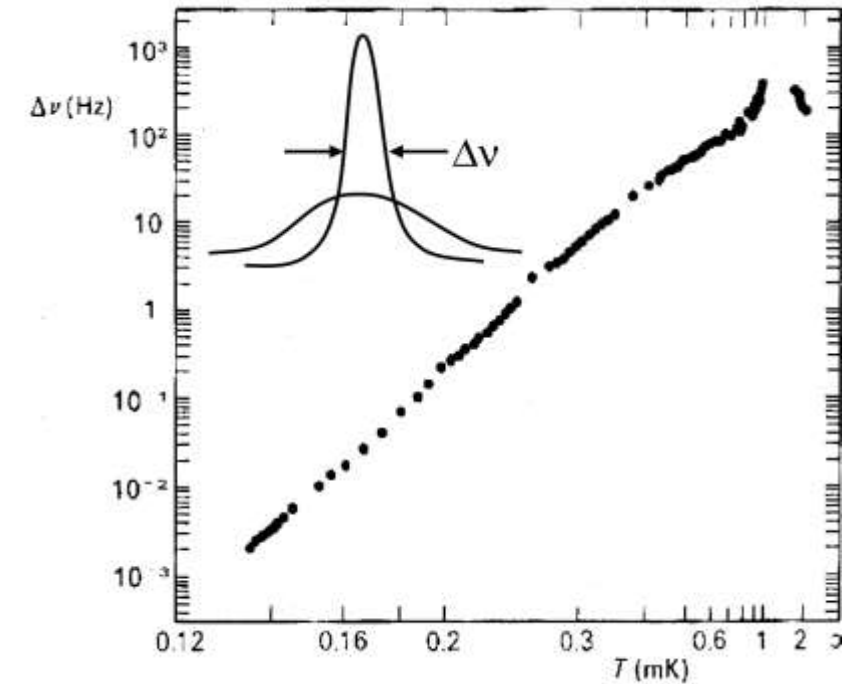
I.5.3 Superfluid ^3He : proof of superfluidity

- use of **vibrating wire**, measurement of quality factor $\nu / \Delta\nu$



*viscosity changes
by 5 orders of
magnitude*

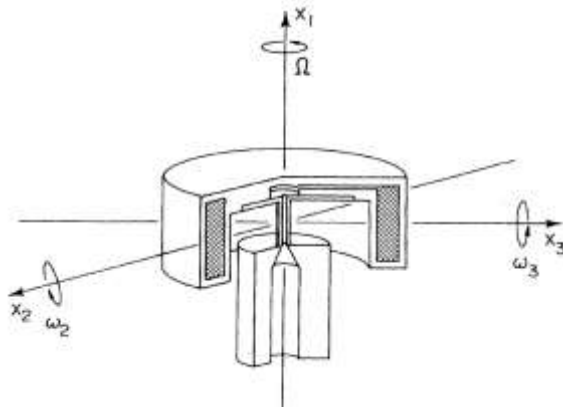
*normal fluid
freezes out*



- **other experiments:**

→ *persistent flow experiments:*

- viscosity of B phase changes at least by 12 orders of magnitude
- persistent flow in A phase decays slowly (appearance of textures)
- small critical velocity: 1 to 100 mm/s



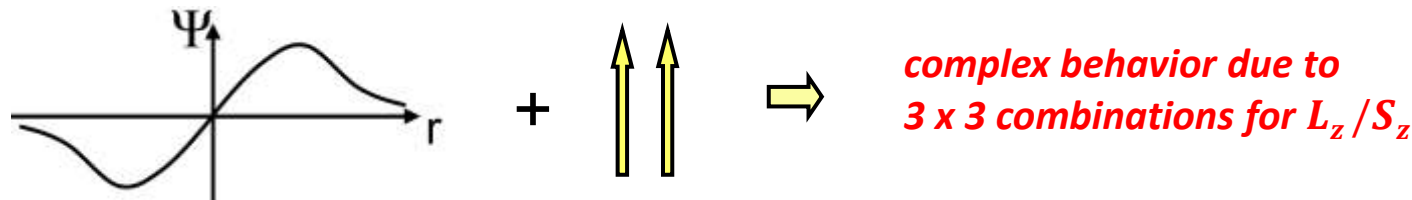
Persistent-Current Experiments on Superfluid He 3 -B and He 3 -A
J. P. Pekola, J. T. Simola, K. K. Nummila, O. V. Lounasmaa, R. E. Packard,
Phys. Rev. Lett. **53**, 70 (1984)

I.5.3 Superfluid ^3He : pairing

- ^3He is **strongly interacting Fermi liquid** (different to electrons in metals)
 - exchange interaction favors parallel spin alignment
 - Fermi statistics favors anti-parallel spin orientation
- ^3He forms Cooper pairs with spin $S = 1$ (**spin triplet pairs**)
in contrast to electrons in metals: $S = 0$ (**spin singlet pairs pairs**)
- note the general requirement: **antisymmetric pair wave function** for fermions

symmetric spin wave function	→ antisymmetric orbital wave function ($L = 1, 3, \dots$)
antisymmetric spin wave function	→ symmetric orbital wave function ($L = 0, 2, \dots$)

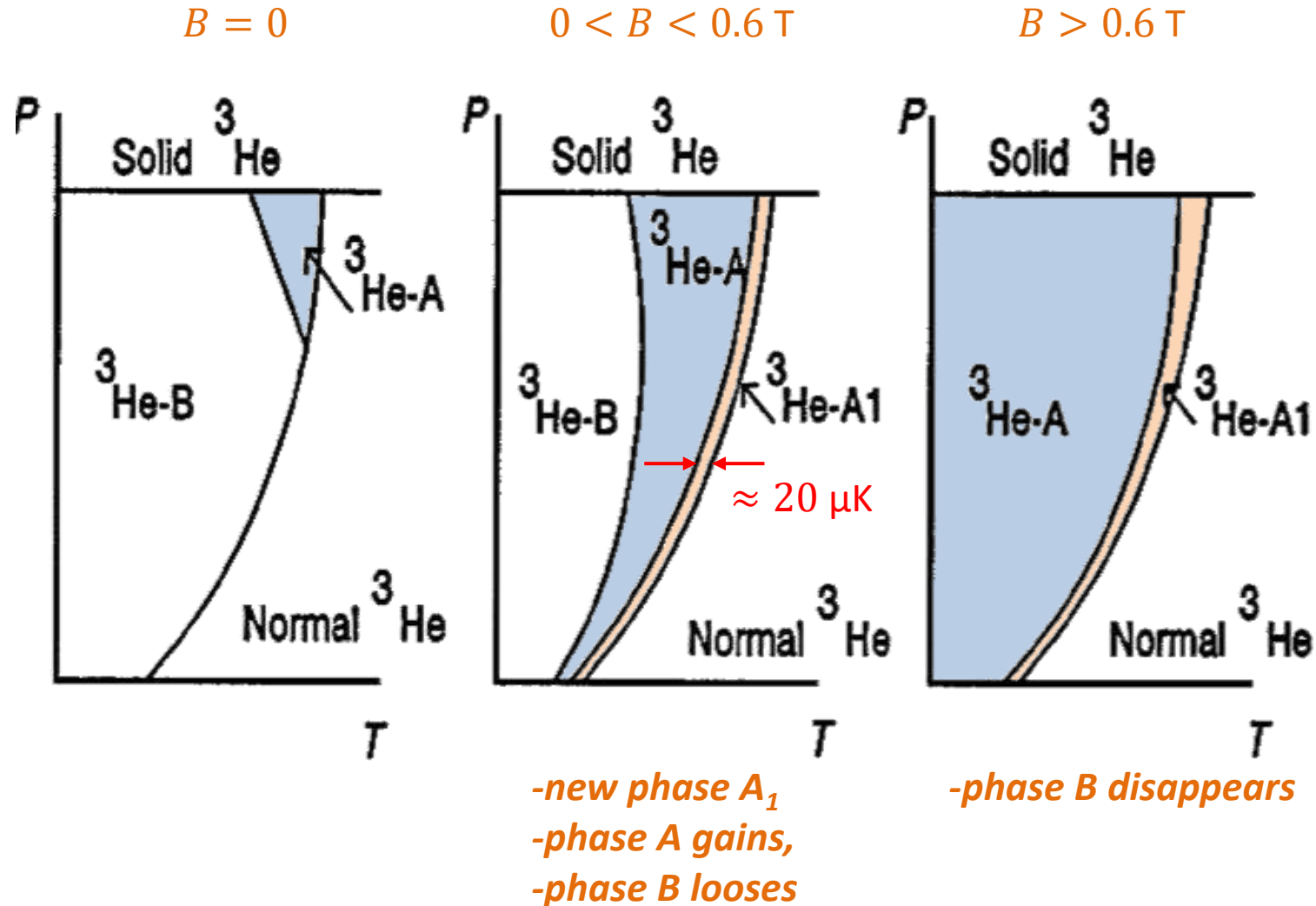
- examples:
- (i) Cooper pairs in metallic superconductors ($L = 0, S = 0$) → **s-wave, spin-singlet pairs**
 - (ii) Cooper pairs in high T_c superconductors ($L = 2, S = 0$) → **d-wave, spin-singlet pairs**
 - (iii) Cooper pairs in superfluid ^3He ($L = 1, S = 1$) → **p-wave, spin-triplet pairs**



p-type orbital wave function minimizes hard core repulsion

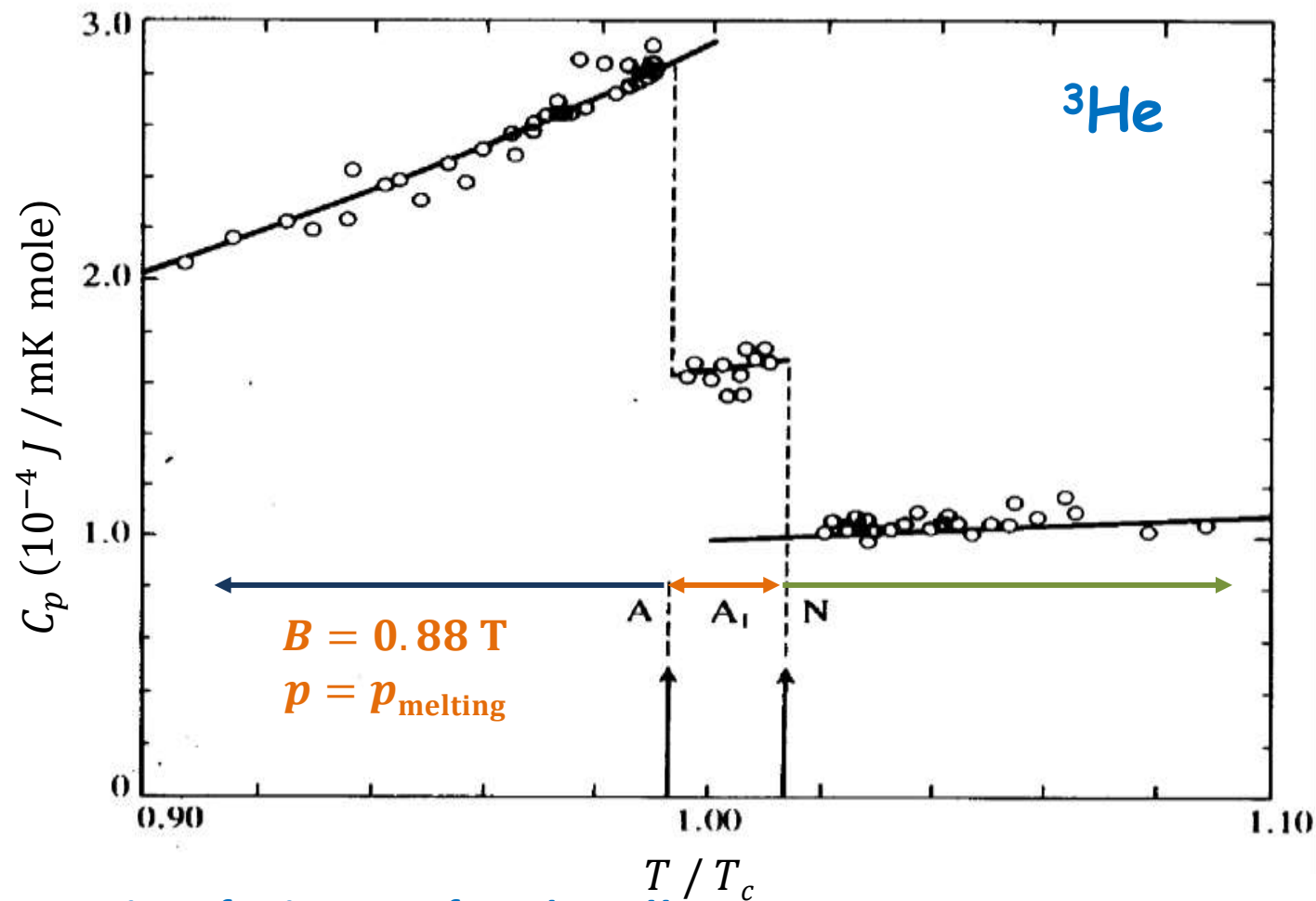
I.5.3 Superfluid ^3He : in $B \neq 0$

spin triplet pairs have magnetic moment \rightarrow *phase diagram depends on magnetic field*

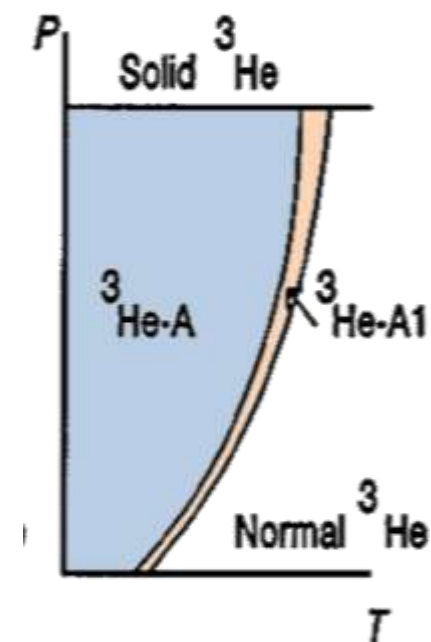


I.5.3 Superfluid ^3He : in $B \neq 0$

- specific heat



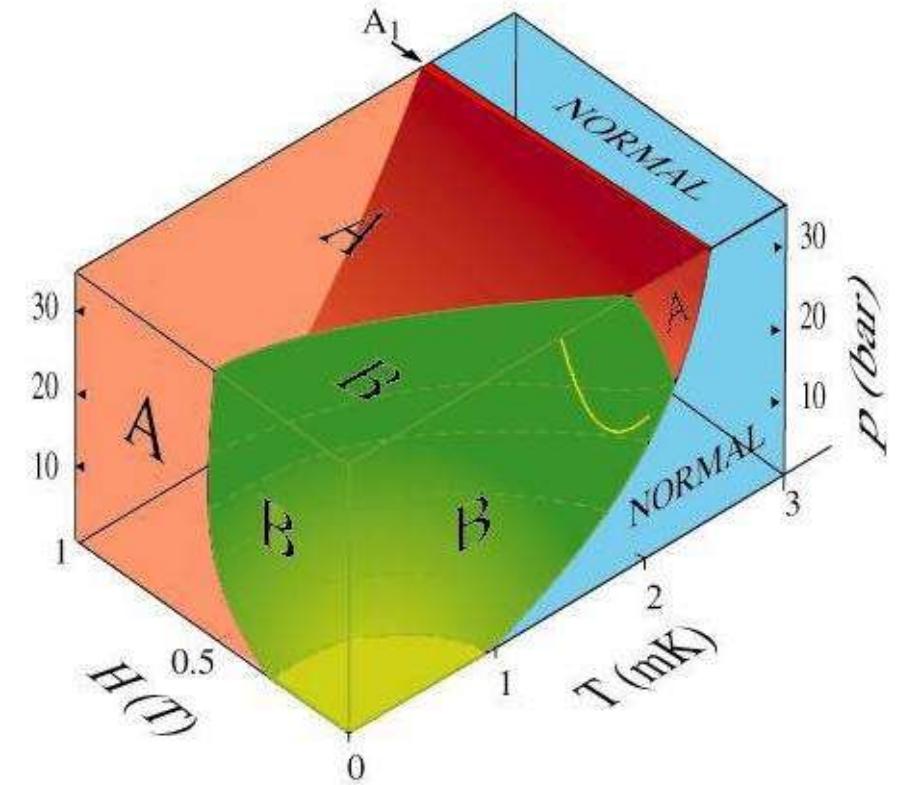
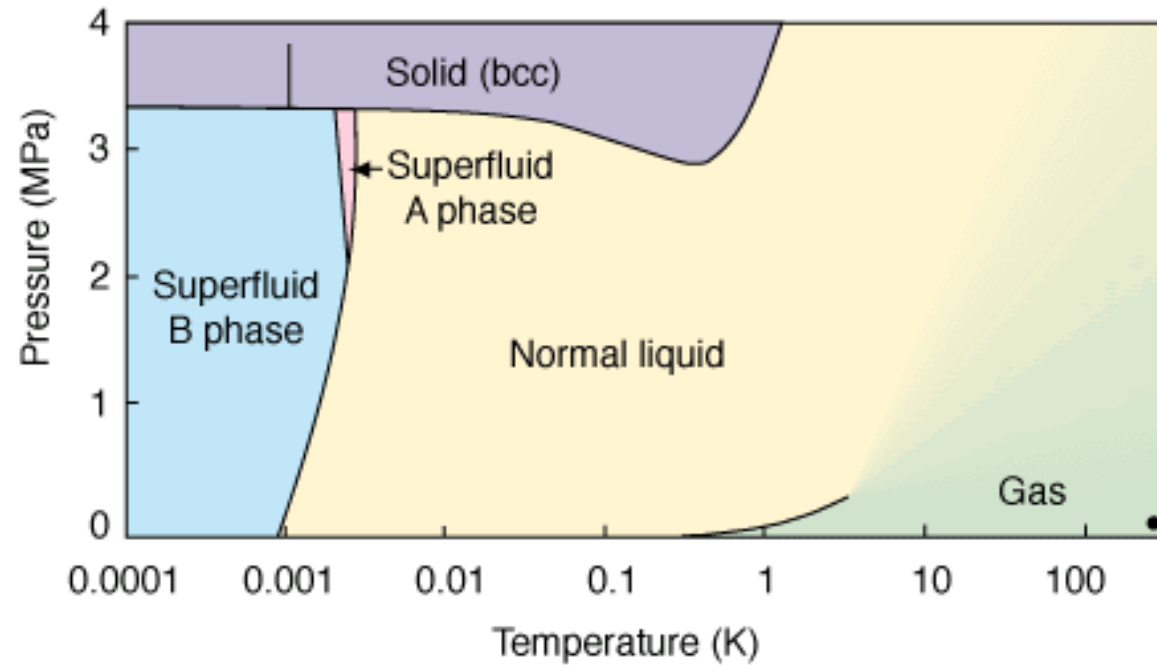
clear demonstration of existence of A_1 phase !!



W.P. Halperin et al.,
Phys. Rev. B **13**, 2124 (1974)

I.5.3 Superfluid ^3He : in $B \neq 0$

- phase diagram of ^3He



source: <http://lth.tkk.fi/research/theory/helium.html>

I.5.3 Superfluid ^3He : Fermi liquid description

- description of liquid ^3He by a Fermi liquid (interacting Fermi gas)

Landau: what happens, if we increase the interaction strength in a non-interacting systems continuously?

- **non-interacting system:** addition/removal of one particle with momentum \mathbf{k} and spin σ to ground state $|\tilde{\Psi}_0\rangle$

$$|\tilde{\Psi}_{\mathbf{k}\sigma}\rangle = c_{\mathbf{k}\sigma}^\dagger |\tilde{\Psi}_0\rangle \quad \tilde{\varepsilon}_{\mathbf{k}} = \frac{\hbar^2(k^2 - k_F^2)}{2m} \quad f_{\mathbf{k}\sigma}(T=0) = \begin{cases} 0 & \text{for } k < k_F \\ 1 & \text{for } k > k_F \end{cases}$$

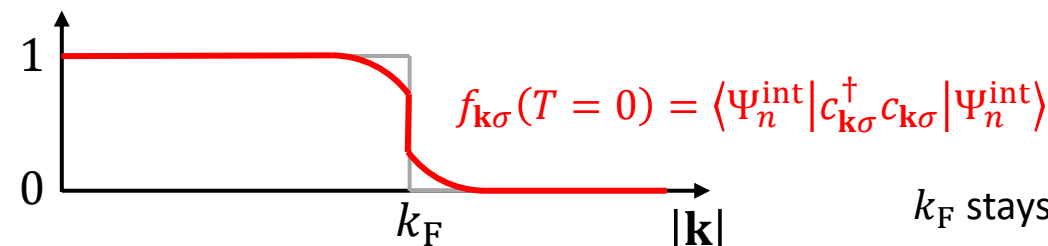
- **interacting system:** addition/removal of one particle changes energies of all other particles due to interaction

Landau's argument of *adiabatic continuity* (wave function changes continuously, when switching on interaction):

quasiparticle energy has similar k -dependence: $\varepsilon_{\mathbf{k}} = \frac{\hbar^2(k^2 - k_F^2)}{2m^*} \quad m^* = \text{effective mass}$

for k close to k_F : $\varepsilon_{\mathbf{k}} = \left. \frac{\partial \varepsilon_{\mathbf{k}}}{\partial k} \right|_{k=k_F} (|\mathbf{k}| - k_F) = \frac{\hbar k_F}{m^*} (|\mathbf{k}| - k_F)$

$f_{\mathbf{k}\sigma}(T=0)$ shows discontinuity, but height of the discontinuity is reduced to smaller value $Z < 1$



k_F stays unchanged: Luttinger theorem

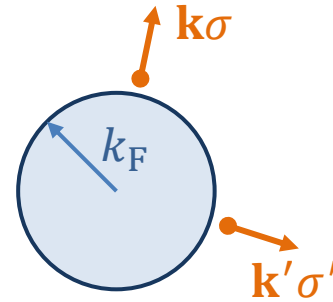
I.5.3 Superfluid ^3He : Fermi liquid description

- description of liquid ^3He by a Fermi liquid (interacting Fermi gas)

➤ total energy:
$$E = \underbrace{E_0}_{\text{ground state}} + \underbrace{\sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}} n_{\mathbf{k}\sigma}}_{\text{quasiparticle excitations}} + \underbrace{\frac{1}{2} \sum_{\mathbf{k}\sigma, \mathbf{k}'\sigma'} \eta(\mathbf{k}\sigma, \mathbf{k}'\sigma') n_{\mathbf{k}\sigma} n_{\mathbf{k}'\sigma'}}_{\text{interaction energy}}$$

➤ two particle energy: $\varepsilon_{\mathbf{k}} + \varepsilon_{\mathbf{k}'} + \underbrace{\eta(\mathbf{k}\sigma, \mathbf{k}'\sigma')}_{\text{effective interaction}} \neq \text{sum of single particle energies } \varepsilon_{\mathbf{k}}, \varepsilon_{\mathbf{k}'}$

describes effective interaction between two quasiparticles:
is usually split in parts $\eta_1(\mathbf{k}, \mathbf{k}')$ and $\eta_2(\mathbf{k}, \mathbf{k}')$ for singlet and triplet contribution



- at low T : $|\mathbf{k}|, |\mathbf{k}'| \simeq k_F$ and due to spherical symmetry $\eta(\mathbf{k}, \mathbf{k}') \propto \mathbf{k} \cdot \mathbf{k}' = k_F^2 \cos \theta$
➔ we can express $\eta_1(\mathbf{k}, \mathbf{k}')$ and $\eta_2(\mathbf{k}, \mathbf{k}')$ by Legendre polynoms $P_\ell(\cos \theta)$

$$g(\varepsilon_F) \eta_1(\mathbf{k}, \mathbf{k}') = \sum_{\ell} F_{\ell} P_{\ell}(\cos \theta) \qquad g(\varepsilon_F) \eta_2(\mathbf{k}, \mathbf{k}') = \sum_{\ell} Z_{\ell} P_{\ell}(\cos \theta)$$

➔ $g(\varepsilon_F)$ = density of states (states per energy interval) ➔ F_{ℓ}, Z_{ℓ} are dimensionless

Theory by Leggett: ➤ F_0, F_1 are large and positive ➔ $m^* = m(1 + F_1/3)$ with $F_1 \simeq 5.4$ (increases with pressure)

- $Z_0 \simeq -3$ and Z_1 is small ➔ magnetic susceptibility $\chi = \frac{m^*}{m} \frac{\chi_0}{1 + (Z_0/4)}$ with $Z_0 \simeq -2.78$
➔ $\chi \simeq 9.2 \chi_0$: He is close to ferromagnetic instability

I.5.3 Superfluid ^3He : pairing interaction

- interaction of ^3He atoms due to van der Waals potential $V(\mathbf{r}_i - \mathbf{r}_j) = V(\mathbf{r})$

$$V(\mathbf{k}, \mathbf{k}') = \int e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} V(\mathbf{r}) d^3r \quad \text{pairing potential in } k\text{-space}$$

- at low T : $|\mathbf{k}|, |\mathbf{k}'| \simeq k_F$ and due to spherical symmetry $V(\mathbf{k}, \mathbf{k}') \propto \mathbf{k} \cdot \mathbf{k}' = k_F^2 \cos \theta$
 ➔ we can express $V(\mathbf{k}, \mathbf{k}')$ by Legendre polynoms $P_\ell(\cos \theta)$

$$V(\mathbf{k}, \mathbf{k}') = \sum_{\ell} \frac{2\ell+1}{2} V_{\ell} P_{\ell}(\cos \theta)$$

$$\text{with } V_{\ell} = \frac{1}{4\pi} \int V(\hat{\mathbf{n}}(\theta, \varphi), \hat{e}_z) P_{\ell}(\cos \theta) \sin \theta d\theta d\varphi$$

$$\frac{2\ell+1}{2} = \int P_{\ell}^2(\cos \theta) \sin \theta d\theta \quad (\text{normalization})$$

- calculation of V_{ℓ} for **van der Waals potential** (e.g. by Leggett 1975):
 $V_0 = \text{strongly repulsive}, V_1, V_2 = \text{weakly attractive} \rightarrow \text{cannot explain pairing in superfluid } ^3\text{He}$
- attractive interaction results from exchange of **virtual spin fluctuations**

one ^3He atom polarizes neighboring nuclear spins which then result in an effective attractive interaction for 2nd ^3He atom

$$V(\mathbf{k}, \mathbf{k}') \propto \frac{1}{g(\varepsilon_F)} \frac{\chi_0}{1 + (Z_0/4)} \hat{\mathbf{s}}_{\mathbf{k}} \cdot \hat{\mathbf{s}}_{\mathbf{k}'}$$

(Leggett 1975)

$$\hat{\mathbf{s}}_{\mathbf{k}} \cdot \hat{\mathbf{s}}_{\mathbf{k}'} = +1/4 \text{ for triplet}$$

$$\hat{\mathbf{s}}_{\mathbf{k}} \cdot \hat{\mathbf{s}}_{\mathbf{k}'} = -3/4 \text{ for singlet}$$

I.5.3 Superfluid ^3He : superfluid phases

- starting point: BCS Hamiltonian

formalism of second quantization is used (>1927, Dirac, Fock, Jordan et al.)

BCS Hamiltonian for N interacting electrons

$$\mathcal{H}_{\text{BCS}} = \sum_{\sigma} \sum_{i=1}^N \left(\underbrace{-\frac{\hbar^2}{2m} \nabla_i^2}_{\text{kinetic energy}} + \underbrace{V_{\text{ext}}(\mathbf{r})}_{\text{potential energy}} \right) + \frac{1}{2} \sum_{\sigma} \sum_{i,j=1}^N \underbrace{V_{\text{int}}(\mathbf{r}_i - \mathbf{r}_j)}_{\text{interaction energy}}$$



insertion of field operators and integration over volume \rightarrow FT of \mathcal{H}_{BCS} into k -space
(see R. Gross, A. Marx, „Festkörperphysik“, 4. Auflage, appendix H.2)

$$\mathcal{H}_{\text{BCS}} = \underbrace{\sum_{\sigma} \sum_{\mathbf{k}} (\varepsilon_{\mathbf{k}} - \mu) \hat{c}_{\mathbf{k}\sigma}^{\dagger} \hat{c}_{\mathbf{k}\sigma}}_{\text{energy of non-interacting free electron gas}} + \underbrace{\frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} \sum_{\mathbf{k}, \mathbf{k}'} V_{\alpha\beta\gamma\delta}(\mathbf{k}, \mathbf{k}') \hat{c}_{\mathbf{k}'\alpha}^{\dagger} \hat{c}_{-\mathbf{k}'\beta}^{\dagger} \hat{c}_{-\mathbf{k}\gamma} \hat{c}_{\mathbf{k}\delta}}_{\text{interaction energy}}$$

operator describes scattering from state $(-\mathbf{k}\gamma, \mathbf{k}\delta)$ into $(\mathbf{k}'\alpha, -\mathbf{k}'\beta)$ by exchange of exchange boson with wave vector $(\mathbf{k} - \mathbf{k}')$

$$\varepsilon_{\mathbf{k}} - \mu = \frac{\hbar^2(k - k_{\text{F}})^2}{2m}$$

$$V(\mathbf{k}, \mathbf{k}') = \frac{1}{\Omega} \int_{\Omega} V(\mathbf{r}) e^{i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{r}} dV$$

factor $\frac{1}{2}$ avoids double counting

I.5.3 Superfluid ^3He : superfluid phases

- symmetry of the pair wavefunction

- **important:** pair consistst of two fermions \rightarrow **total wavefunction must be antisymmetric: minus sign for particle exchange**

$$\Psi(\mathbf{r}_1, \boldsymbol{\sigma}_1, \mathbf{r}_2, \boldsymbol{\sigma}_2) = \underbrace{\frac{1}{\sqrt{V}} e^{i \mathbf{K}_s \cdot \mathbf{R}_s}}_{\substack{\text{center of mass motion} \\ \text{we assume } \mathbf{K}_s = 0}} \underbrace{f(\mathbf{q}, \mathbf{r})}_{\substack{\text{orbital} \\ \text{part}}} \underbrace{\chi(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)}_{\substack{\text{spin} \\ \text{part}}} = -\Psi(\mathbf{r}_2, \boldsymbol{\sigma}_2, \mathbf{r}_1, \boldsymbol{\sigma}_1)$$

$$\mathbf{R}_s = (\mathbf{r}_1 + \mathbf{r}_2)/2$$

$$\mathbf{r} = (\mathbf{r}_1 - \mathbf{r}_2)$$

$$\mathbf{K}_s = (\mathbf{k} + \mathbf{k}')/2$$

$$\mathbf{q} = (\mathbf{k} - \mathbf{k}')$$

- possible **spin wavefunctions** $\chi(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$ for electron pairs

$$S = \begin{cases} 0 & m_s = 0 & \chi^a = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) & \rightarrow \text{singlet pairing,} & \text{antisymmetric spin wavefunction} \\ & & & & \text{symmetric orbital function:} \\ & & & & L = 0, 2, \dots (s, d, \dots) \\ 1 & m_s = \begin{cases} -1 & \chi^s = \downarrow\downarrow \\ 0 & \chi^s = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ +1 & \chi^s = \uparrow\uparrow \end{cases} & \rightarrow \text{triplet pairing,} & \text{symmetric spin wavefunction} \\ & & & & \text{antisymmetric orbital function:} \\ & & & & L = 1, 3, \dots (p, f, \dots) \end{cases}$$

I.5.3 Superfluid ^3He : superfluid phases

- BCS mean-field approximation

pairing amplitude:

$$\hat{c}_{-\mathbf{k}\alpha}\hat{c}_{\mathbf{k}\beta} = \underbrace{\langle \hat{c}_{-\mathbf{k}\alpha}\hat{c}_{\mathbf{k}\beta} \rangle}_{g_{\alpha\beta}(\mathbf{k})} + \underbrace{\hat{c}_{-\mathbf{k}\alpha}\hat{c}_{\mathbf{k}\beta} - \langle \hat{c}_{-\mathbf{k}\alpha}\hat{c}_{\mathbf{k}\beta} \rangle}_{\delta g_{\alpha\beta}(\mathbf{k})} \quad \hat{c}_{\mathbf{k}\alpha}^\dagger \hat{c}_{-\mathbf{k}\beta}^\dagger = \underbrace{\langle \hat{c}_{\mathbf{k}\alpha}^\dagger \hat{c}_{-\mathbf{k}\beta}^\dagger \rangle}_{g_{\alpha\beta}^*(\mathbf{k})} + \underbrace{\hat{c}_{\mathbf{k}\alpha}^\dagger \hat{c}_{-\mathbf{k}\beta}^\dagger - \langle \hat{c}_{\mathbf{k}\alpha}^\dagger \hat{c}_{-\mathbf{k}\beta}^\dagger \rangle}_{\delta g_{\alpha\beta}^*(\mathbf{k})}$$

$$g_{\alpha\beta}(\mathbf{k}) \equiv \langle \hat{c}_{-\mathbf{k}\alpha}\hat{c}_{\mathbf{k}\beta} \rangle \neq 0$$

$$g_{\alpha\beta}^*(\mathbf{k}) \equiv \langle \hat{c}_{\mathbf{k}\alpha}^\dagger \hat{c}_{-\mathbf{k}\beta}^\dagger \rangle \neq 0$$

– make use of **pairing potential** $\Delta_{\alpha\beta}(\mathbf{k}) \equiv - \sum_{\mathbf{k}', \gamma, \delta} V_{\alpha\beta\gamma\delta}(\mathbf{k}, \mathbf{k}') g_{\alpha\beta}(\mathbf{k}')$

four component function due to two values α, β of spin

$\Delta_{\alpha\beta}^*(\mathbf{k}) \equiv - \sum_{\mathbf{k}', \gamma, \delta} V_{\alpha\beta\gamma\delta}(\mathbf{k}, \mathbf{k}') g_{\alpha\beta}^*(\mathbf{k}')$

spin structure
orbital structure



$$\mathcal{H}_{\text{BCS}} - \mathcal{N}\mu = \sum_{\mathbf{k}, \sigma} (\varepsilon_{\mathbf{k}} - \mu) c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}, \alpha, \beta} \left[\Delta_{\alpha\beta}(\mathbf{k}) \hat{c}_{\mathbf{k}\alpha}^\dagger \hat{c}_{-\mathbf{k}\beta}^\dagger + \Delta_{\alpha\beta}^*(\mathbf{k}) \hat{c}_{\mathbf{k}\alpha} \hat{c}_{-\mathbf{k}\beta} \right]$$

– with $\alpha, \beta = \uparrow, \downarrow$: two fermions with $\mathbf{s} = 1/2$ and $\alpha, \beta = m_s = \uparrow, \downarrow$

$$g(\mathbf{k}) \equiv \begin{pmatrix} \langle \hat{c}_{-\mathbf{k}\uparrow}\hat{c}_{\mathbf{k}\uparrow} \rangle & \langle \hat{c}_{-\mathbf{k}\uparrow}\hat{c}_{\mathbf{k}\downarrow} \rangle \\ \langle \hat{c}_{-\mathbf{k}\downarrow}\hat{c}_{\mathbf{k}\uparrow} \rangle & \langle \hat{c}_{-\mathbf{k}\downarrow}\hat{c}_{\mathbf{k}\downarrow} \rangle \end{pmatrix} \quad g^*(\mathbf{k}) \equiv \begin{pmatrix} \langle \hat{c}_{-\mathbf{k}\uparrow}^\dagger \hat{c}_{\mathbf{k}\uparrow}^\dagger \rangle & \langle \hat{c}_{-\mathbf{k}\uparrow}^\dagger \hat{c}_{\mathbf{k}\downarrow}^\dagger \rangle \\ \langle \hat{c}_{-\mathbf{k}\downarrow}^\dagger \hat{c}_{\mathbf{k}\uparrow}^\dagger \rangle & \langle \hat{c}_{-\mathbf{k}\downarrow}^\dagger \hat{c}_{\mathbf{k}\downarrow}^\dagger \rangle \end{pmatrix} \quad \Delta(\mathbf{k}) = \begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} \quad \Delta^*(\mathbf{k}) = \begin{pmatrix} \Delta_{\uparrow\uparrow}^*(\mathbf{k}) & \Delta_{\uparrow\downarrow}^*(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}^*(\mathbf{k}) & \Delta_{\downarrow\downarrow}^*(\mathbf{k}) \end{pmatrix}$$

4 components covering all spin orientations before and after interaction

I.5.3 Superfluid ^3He : superfluid phases

- diagonalization of mean-field Hamiltonian by Bogoliubov transformation

- BCS energy gap equation:
(Nambu, 1962)

$$\begin{array}{c}
 \begin{array}{cc} \text{particles} & \text{pairs} \end{array} \\
 \begin{pmatrix}
 \varepsilon_{\mathbf{k}} - \mu & 0 & \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\
 0 & \varepsilon_{\mathbf{k}} - \mu & \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \\
 \Delta_{\uparrow\uparrow}^*(\mathbf{k}) & \Delta_{\uparrow\downarrow}^*(\mathbf{k}) & -(\varepsilon_{\mathbf{k}} - \mu) & 0 \\
 \Delta_{\downarrow\uparrow}^*(\mathbf{k}) & \Delta_{\downarrow\downarrow}^*(\mathbf{k}) & 0 & -(\varepsilon_{\mathbf{k}} - \mu)
 \end{pmatrix}
 \begin{pmatrix} u_{\mathbf{k}\uparrow n} \\ u_{\mathbf{k}\downarrow n} \\ v_{\mathbf{k}\uparrow n} \\ v_{\mathbf{k}\downarrow n} \end{pmatrix} = E_{\mathbf{k}n} \begin{pmatrix} u_{\mathbf{k}\uparrow n} \\ u_{\mathbf{k}\downarrow n} \\ v_{\mathbf{k}\uparrow n} \\ v_{\mathbf{k}\downarrow n} \end{pmatrix} \\
 \begin{array}{cc} \text{pairs} & \text{holes} \end{array}
 \end{array}$$

- 4 x 4 matrix equation instead of 2 x 2 matrix equation for spin singlet superconductor
 - diagonalization by suitable Bogoliubov transformation
 - two positive eigenvalues $E_{\mathbf{k}n}$ ($n = 1, 2$) and two negative eigenvalues $-E_{\mathbf{k}n}$
 - general gap equation allows for both spin singlet and spin triplet pairing
- symmetry properties of pairing amplitude and pairing potential (follows from anti-commutation relation of fermionic operators)

$$g_{\alpha\beta}(\mathbf{k}) = \langle \hat{c}_{-\mathbf{k}\alpha} \hat{c}_{\mathbf{k}\beta} \rangle = -\langle \hat{c}_{\mathbf{k}\beta} \hat{c}_{-\mathbf{k}\alpha} \rangle = -g_{\beta\alpha}(-\mathbf{k})$$

$$\Delta_{\alpha\beta}(\mathbf{k}) = -\Delta_{\beta\alpha}(-\mathbf{k})$$

I.5.3 Superfluid ^3He : superfluid phases

- representation of the energy gap matrix $\Delta_{\alpha\beta}(\mathbf{k})$ by a scalar gap function $\Delta(\mathbf{k})$ and a vector $\mathbf{d}(\mathbf{k})$

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = i [\Delta(\mathbf{k}) \mathbf{I} + \mathbf{d}(\mathbf{k}) \cdot \boldsymbol{\sigma}] \sigma_y$$

scalar gap function $\Delta(\mathbf{k})$ $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **d-vector** $\mathbf{d}(\mathbf{k})$ Pauli spin matrix $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
 $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ = vector of Pauli matrices
 $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -d_x(\mathbf{k}) + i d_y(\mathbf{k}) & \Delta(\mathbf{k}) + d_z(\mathbf{k}) \\ -\Delta(\mathbf{k}) + d_z(\mathbf{k}) & d_x(\mathbf{k}) + i d_y(\mathbf{k}) \end{pmatrix}$$

$$\mathbf{d} = \hat{\mathbf{d}} |\mathbf{d}| e^{i\phi}$$

direction of spin wave function $\hat{\mathbf{d}}$
 superfluid density $|\mathbf{d}|$
 phase factor $e^{i\phi}$

- amplitude $|\mathbf{d}|$ (\mathbf{d} is vector with complex components):
pair density $|d|^2 = d_x^2 + d_y^2 + d_z^2 = n_s$
- direction $\hat{\mathbf{d}} = \mathbf{d}/|\mathbf{d}|$:
spatial orientation of spin wave function: $\hat{\mathbf{d}} \cdot \mathbf{S} = 0$

I.5.3 Superfluid ^3He : superfluid phases

- parity consideration

– with $g_{\alpha\beta}(\mathbf{k}) = -g_{\beta\alpha}(-\mathbf{k})$ and $\Delta_{\alpha\beta}(\mathbf{k}) = -\Delta_{\beta\alpha}(-\mathbf{k})$ we obtain

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -d_x(\mathbf{k}) + id_y(\mathbf{k}) & \Delta(\mathbf{k}) + d_z(\mathbf{k}) \\ -\Delta(\mathbf{k}) + d_z(\mathbf{k}) & d_x(\mathbf{k}) + id_y(\mathbf{k}) \end{pmatrix}$$

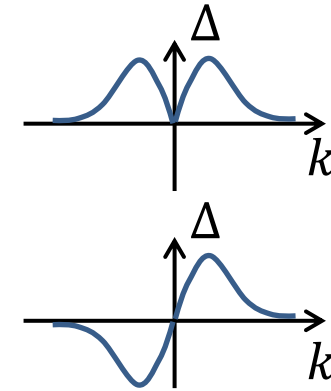
$$\Delta(\mathbf{k}) = \Delta(-\mathbf{k})$$

scalar component is **even** with respect to inversion $\mathbf{k} \rightarrow -\mathbf{k}$

$$\mathbf{d}(\mathbf{k}) = -\mathbf{d}(-\mathbf{k})$$

vector component is **odd** with respect to inversion $\mathbf{k} \rightarrow -\mathbf{k}$

orbital structure of the pairing potential



$L = 0, 2, 4, \dots$

$L = 1, 3, 5, \dots$

– anti-symmetry of pair wave function requires

- spin singlet ($S = 0$):

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} 0 & \Delta(\mathbf{k}) \\ -\Delta(\mathbf{k}) & 0 \end{pmatrix}$$

- spin triplet ($S = 1$):

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -d_x(\mathbf{k}) + id_y(\mathbf{k}) & +d_z(\mathbf{k}) \\ +d_z(\mathbf{k}) & d_x(\mathbf{k}) + id_y(\mathbf{k}) \end{pmatrix}$$

I.5.3 Superfluid ^3He : superfluid phases

- representation of the scalar function $\Delta(\mathbf{k})$ and vector $\mathbf{d}(\mathbf{k})$
 - we can restrict ourselves to wave vectors near the Fermi level
 - ➔ due to spherical symmetry, we can use spherical harmonics Y_{lm} to represent

$$d_i(\mathbf{k}) = \sum h_{ilm} Y_{lm}(\theta_{\mathbf{k}}, \varphi_{\mathbf{k}}) \quad i = x, y, z$$

- spin singlet: $l = 0, 2, 4, \dots$
- spin triplet: $l = 1, 3, 5, \dots$

- in general, one has to sum up over all angular momentum quantum numbers l
- practically, one can restrict oneself to a particular quantum number (dominating pairing channel, all other channels have much smaller T_c)
- for superfluid ^3He , the **dominating pairing channel is the $l = 1$ channel** mediated by spin fluctuations



$$d_i(\mathbf{k}) = \sum h_{im} p_j(\theta_{\mathbf{k}}, \varphi_{\mathbf{k}})$$

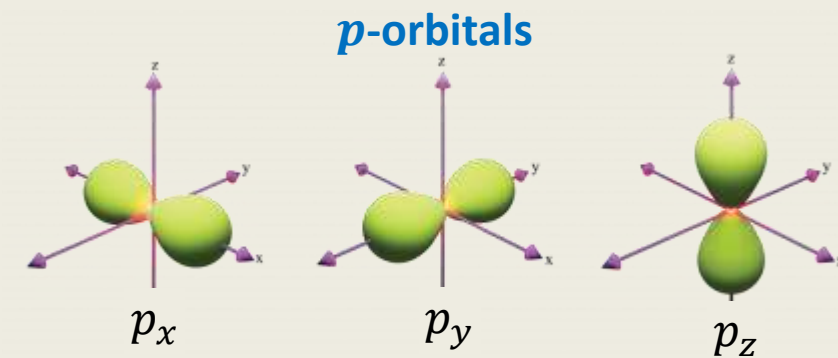
$$p_x(\theta_{\mathbf{k}}, \varphi_{\mathbf{k}}) = \sqrt{3/4\pi} \cos \varphi_{\mathbf{k}} \sin \theta_{\mathbf{k}}$$

$$p_y(\theta_{\mathbf{k}}, \varphi_{\mathbf{k}}) = \sqrt{3/4\pi} \sin \varphi_{\mathbf{k}} \cos \theta_{\mathbf{k}}$$

$$p_z(\theta_{\mathbf{k}}, \varphi_{\mathbf{k}}) = \sqrt{3/4\pi} \cos \theta_{\mathbf{k}}$$

$h_{im} = 3 \times 3$ matrix as $i = x, y, z$ and $m = -1, 0, +1$

complex matrix elements have to be determined by solving BCS gap equation



I.5.3 Superfluid ^3He

- superfluid phases of ^3He

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -d_x(\mathbf{k}) + \imath d_y(\mathbf{k}) & +d_z(\mathbf{k}) \\ +d_z(\mathbf{k}) & d_x(\mathbf{k}) + \imath d_y(\mathbf{k}) \end{pmatrix}$$

$$d_i(\mathbf{k}) = \sum h_{im} p_m(\theta_{\mathbf{k}}, \varphi_{\mathbf{k}})$$

$$h_{im} = \begin{pmatrix} h_{xx} & h_{xy} & h_{xz} \\ h_{yx} & h_{yy} & h_{yz} \\ h_{zx} & h_{zy} & h_{zz} \end{pmatrix}$$

- Anderson-Brinkman-Morrel (ABM or A) phase

$$h_{im} = h \begin{pmatrix} 1 & \imath & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow d_x(\mathbf{k}) = p_x + \imath p_y$$

$$|\psi\rangle_{\text{ABM}} = (p_x + \imath p_y) |\downarrow\downarrow\rangle - (p_x - \imath p_y) |\uparrow\uparrow\rangle$$

linear combination of $S_z = \pm 1$

(i) $B = 0: |h_{xx}| = |h_{xy}|$

$$\Rightarrow \langle S_z \rangle = 0$$

(ii) $B \neq 0: |h_{xx}| \neq |h_{xy}|$

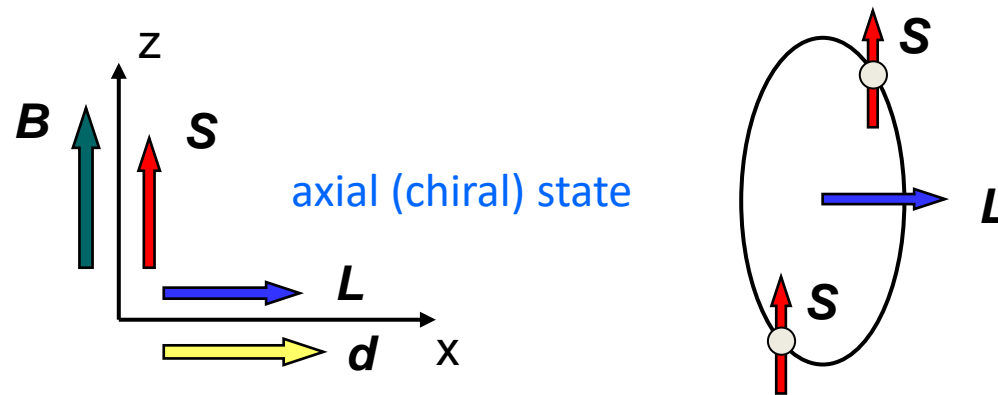
$$\Rightarrow \langle S_z \rangle \neq 0 \quad (\text{anisotropic, paramagnetic})$$

I.5.3 Superfluid ^3He

– Anderson-Brinkman-Morrel (ABM or A) phase

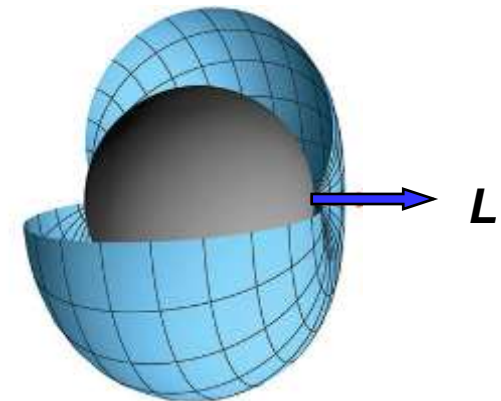
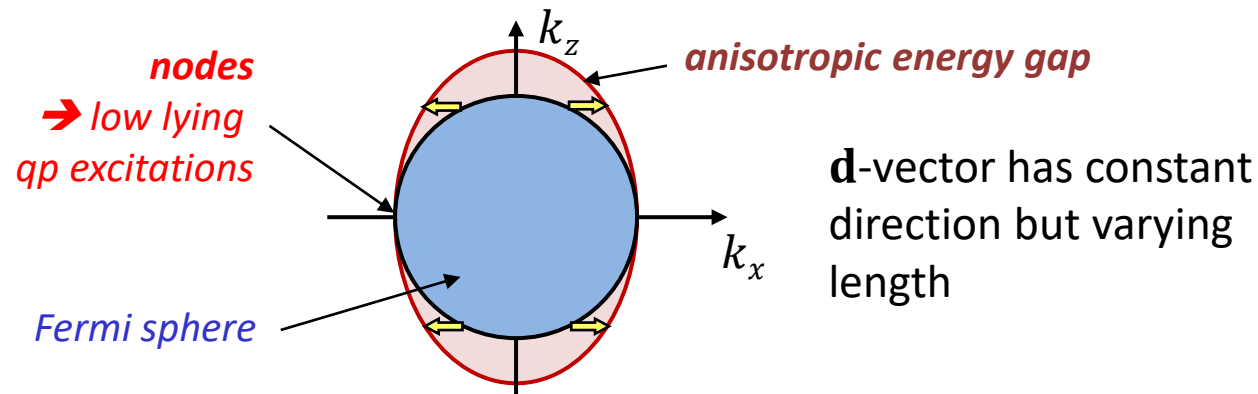
$$|\psi\rangle_{\text{ABM}} = (p_x + ip_y) |\downarrow\downarrow\rangle - (p_x - ip_y) |\uparrow\uparrow\rangle$$

- both spins are always parallel
- all paired atoms are moving in plane perpendicular to \mathbf{L}



note that $\mathbf{L} \parallel \mathbf{S}$ would result in repulsive spin-orbit interaction

- unpaired atoms moving in \mathbf{L} direction do not perturb → no energy gap in \mathbf{L} -direction



I.5.3 Superfluid ^3He

- superfluid phases of ^3He

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -d_x(\mathbf{k}) + id_y(\mathbf{k}) & +d_z(\mathbf{k}) \\ +d_z(\mathbf{k}) & d_x(\mathbf{k}) + id_y(\mathbf{k}) \end{pmatrix}$$

$$d_i(\mathbf{k}) = \sum h_{im} p_m(\theta_{\mathbf{k}}, \varphi_{\mathbf{k}})$$

$$h_{im} = \begin{pmatrix} h_{xx} & h_{xy} & h_{xz} \\ h_{yx} & h_{yy} & h_{yz} \\ h_{zx} & h_{zy} & h_{zz} \end{pmatrix}$$

- Balian-Werthamer (BW or B) phase

$$h_{im} = h \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow d_x(\mathbf{k}) = p_x, \quad d_y(\mathbf{k}) = p_y, \quad d_z(\mathbf{k}) = p_z,$$

$$|\psi\rangle_{\text{BW}} = (p_x + ip_y) |\downarrow\downarrow\rangle - (p_x - ip_y) |\uparrow\uparrow\rangle + p_z (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

linear combination of $S_z = -1, 0, +1$ (all spin states are represented equally)

$$|h_{xx}| = |h_{yy}| = |h_{zz}| \neq 0 \quad \Rightarrow \quad \langle S_z \rangle = 0 \quad (\text{isotropic, non-magnetic})$$

$$B \neq 0: \quad S_z = 0 \text{ component is non-magnetic} \Rightarrow \chi_{\text{BW}} = \frac{2}{3} \chi_{\text{ABM}}$$

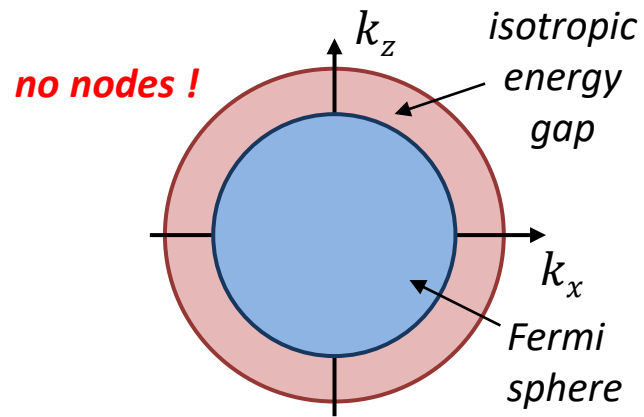
BW phase energetically less favorable than ABM phase in applied magnetic field !!

I.5.3 Superfluid ^3He

– Balian-Werthamer (BW or B) phase

$$|\psi\rangle_{\text{BW}} = (p_x + ip_y) |\downarrow\downarrow\rangle - (p_x - ip_y) |\uparrow\uparrow\rangle + p_z (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

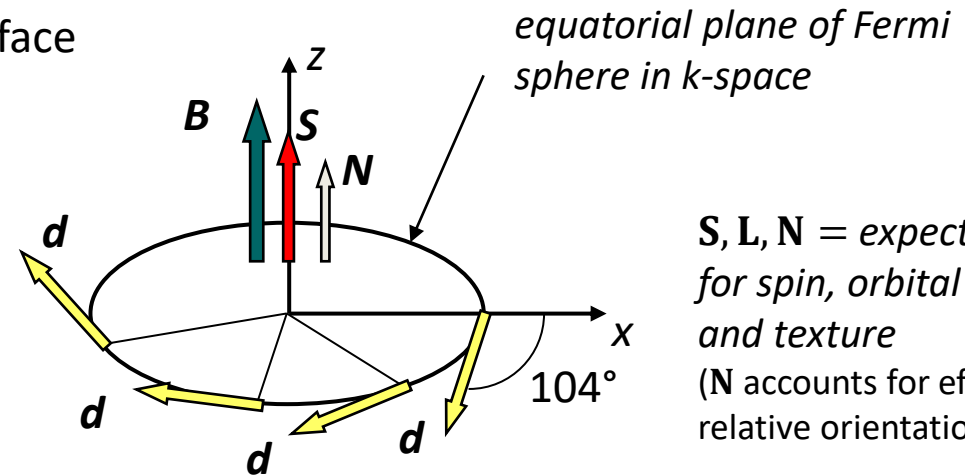
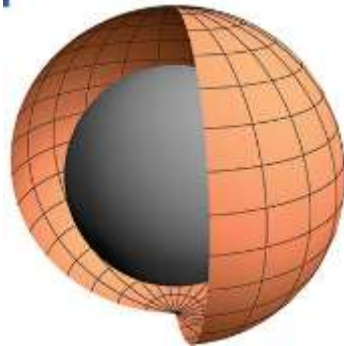
- **d**-vector has constant length all over the Fermi surface
➔ constant magnitude of the energy gap



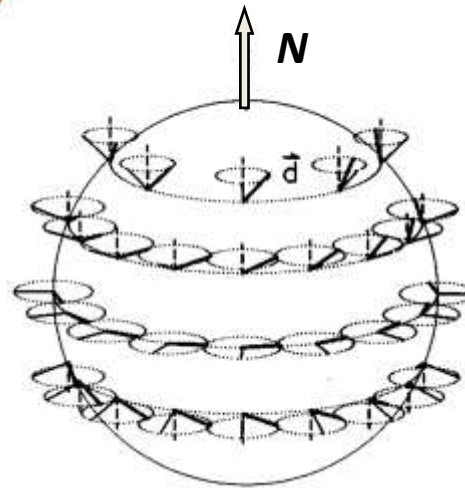
pseudo-isotropic state

(constant gap as s-wave superconductor)

$$C_V(T) \propto \exp(-\Delta(T)/k_B T)$$



S, L, N = expectation values for spin, orbital momentum and texture
(N accounts for effects related to relative orientation of \mathbf{d} and \mathbf{L})



\mathbf{d} -vectors (thick lines) are rotated by 104° about a vector defining the radial directions (thin lines) for all points on the Fermi sphere.
D.M. Lee, Reviews of Modern Physics **69**, 645 (1997)

I.5.3 Superfluid ^3He

- superfluid phases of ^3He

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -d_x(\mathbf{k}) + \imath d_y(\mathbf{k}) & +d_z(\mathbf{k}) \\ +d_z(\mathbf{k}) & d_x(\mathbf{k}) + \imath d_y(\mathbf{k}) \end{pmatrix}$$

$$d_i(\mathbf{k}) = \sum h_{im} p_m(\theta_{\mathbf{k}}, \varphi_{\mathbf{k}})$$

$$h_{im} = \begin{pmatrix} h_{xx} & h_{xy} & h_{xz} \\ h_{yx} & h_{yy} & h_{yz} \\ h_{zx} & h_{zy} & h_{zz} \end{pmatrix}$$

- A_1 phase (only for $B > 0$)

$$h_{im} = h \begin{pmatrix} 1 & \imath & 0 \\ \imath & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow d_x(\mathbf{k}) = p_x + \imath p_y$$

$$\rightarrow d_y(\mathbf{k}) = \imath p_x + p_y$$

$$|\psi\rangle_{A_1} = 2(p_x + \imath p_y) |\uparrow\uparrow\rangle$$

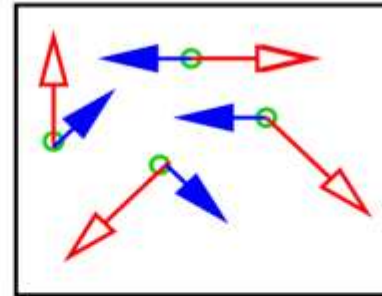
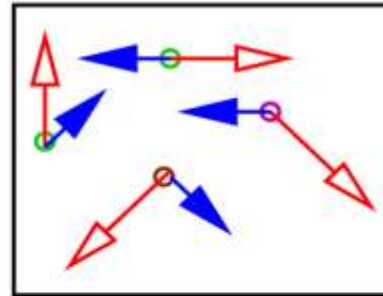
only state $|\uparrow\uparrow\rangle$ contributes: $\langle S_z \rangle = 1$ (“ferro”magnetic)

→ long-range ordered magnetic liquid

I.5.3 Superfluid ^3He : broken symmetries

disordered state

(isotropic with respect to orientation of both degrees of freedom)

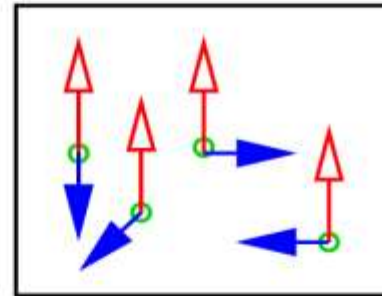
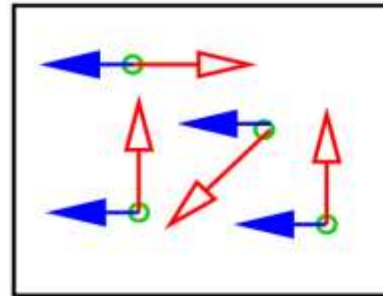


broken gauge symmetry
"superfluid"

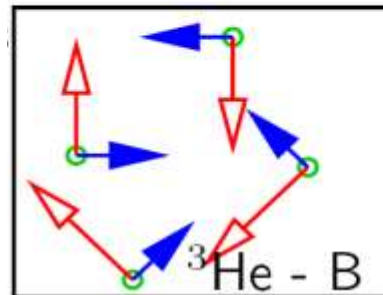
↑ spin
↑ orbital moment

broken gauge and rotational symmetry in spin space

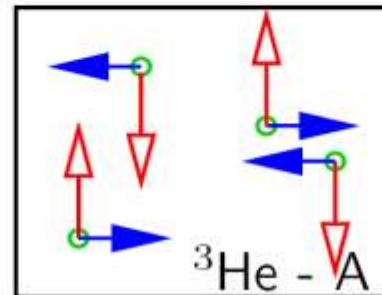
"ferromagnetic superfluid"



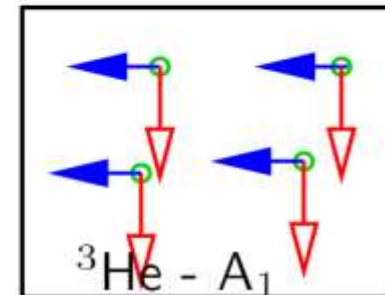
broken gauge and rotational symmetry in orbital space
"ferroorbital superfluid"



only symmetry related to relative orientation is broken



rotational symmetry is broken separately in spin and orbital space



rotational symmetry is broken separately in spin and orbital space

Contents Part I: Quantum Liquids

Contents:

I.1 Foundations and General Properties

- I.1.1 Quantum Gases & Liquids
- I.1.2 Helium
- I.1.3 Van der Waals Bonding
- I.1.4 Zero-Point Fluctuations
- I.1.5 Helium under Pressure
- I.1.6 pT-Phase Diagram of ^4He and ^3He
- I.1.7 Characteristic Properties of ^4He and ^3He
- I.1.8 Specific Heat of ^4He and ^3He

I.2 ^4He as an Ideal Bose Gas

- I.2.1 Bose-Einstein Condensation
- I.2.2 Ideal Bose Gas
- I.2.3 Bose Gas with Interactions
- I.2.4 Bose-Einstein Condensation of ^4He

I.3 Superfluid ^4He

- I.3.1 Two-Fluid Model
- I.3.2 Experimental Observations
- I.3.3 Two-Fluid Hydrodynamics
- I.3.4 Excitation Spectrum of ^4He

I.4 Vortices

- I.4.1 Quantization of Circulation
- I.4.2 Experimental Study of Vortices

I.5 ^3He

- I.5.1 normal fluid ^3He
- I.5.2 solid ^3He and Pomeranchuk effect
- I.5.3 superfluid ^3He



I.6 ^3He / ^4He mixtures

I.6 $^3\text{He}/^4\text{He}$ mixtures

- **why interesting:**

- model system for testing Fermi liquid theories

- (^3He is fermion, can be diluted in passive ^4He background → change of interaction strength)

- technical application in cooling machines

- H.E. Hall, P.J. Ford and K. Thomson, *Cryogenic* **6** (1966) 80

- B.S. Neganov, N.S. Borisov and M.Yu. Liburg, *Zh. Exp. Teor. Fiz.* **50** (1966) 1445.

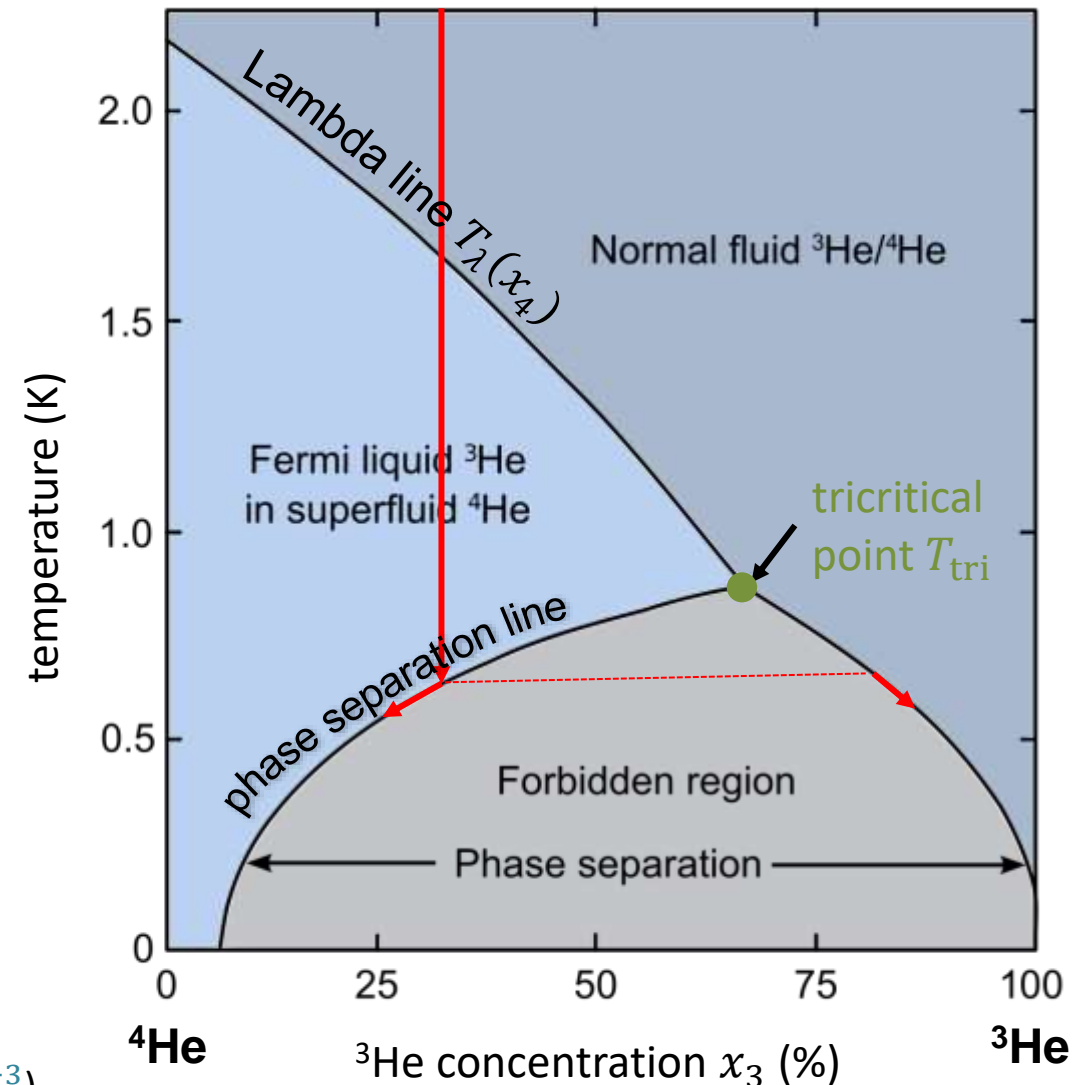
- **experimental observations:**

- specific heat vs. T curve of ^4He is shifted to lower T on adding ^3He , but does not change its shape

- phase separation of ^4He and ^3He at low T

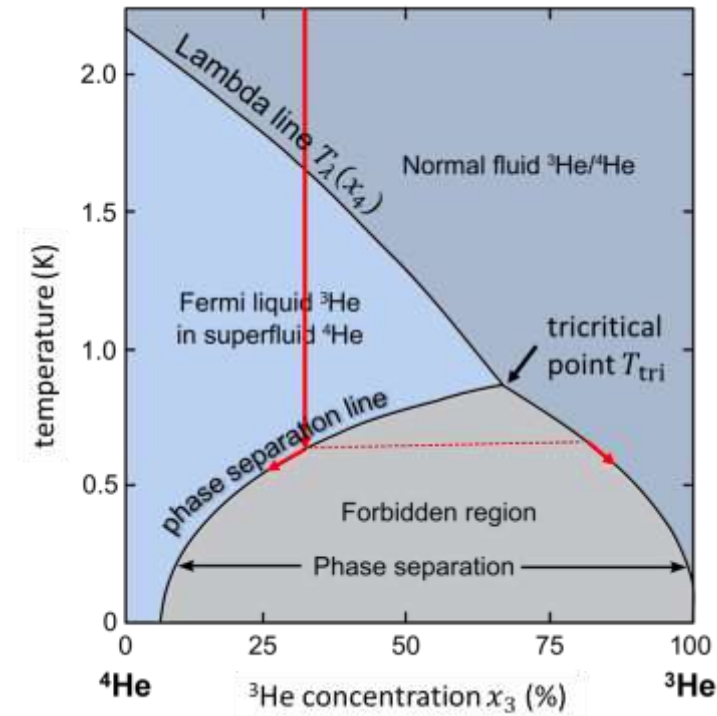
I.6 $^3\text{He}/^4\text{He}$ mixtures: phase diagram

- high T :
 ^3He reduces $T_\lambda \propto n_4$ (^4He concentration)
- tricritical point $T_{\text{tri}} = 0.87 \text{ K}$
- $T < T_{\text{tri}} = 0.87 \text{ K}$:
 → **phase separation**
 ^3He rich phase: lighter
 ^3He poor phase: heavier
- $T \rightarrow 0$:
 → **incomplete phase separation**
 6.5% of ^3He still dissolved in ^4He
- concentrations: $x_3 = \frac{n_3}{n_3+n_4}$; $x_4 = 1 - x_3 = \frac{n_4}{n_3+n_4}$
- limiting concentrations:
 $x_4 = aT^{3/2}e^{-b/T}$ ($a = 0.85 \text{ K}^{-3/2}$, $b = 0.56 \text{ K}^{-1}$)
 $x_3 = 0.0648 (1 + cT^2 + dT^3)$ ($c = 8.4 \text{ K}^{-2}$, $d = 9.4 \text{ K}^{-3}$)



I.6 $^3\text{He}/^4\text{He}$ mixture as diluted Fermi liquid

- at $T_{\text{phase sep}} < T < T_\lambda(x_4)$:
 - ^4He forms superfluid with negligible amounts of excitations (phonons, rotons)
 - viscosity, entropy and specific heat go to zero:
 - $^4\text{He} = \text{inert superfluid background}$
- ^3He obeys Fermi statistics
 - Fermi temperature is about 1 K for pure ^3He
 - dissolving ^3He in ^4He reduces the Fermi temperature

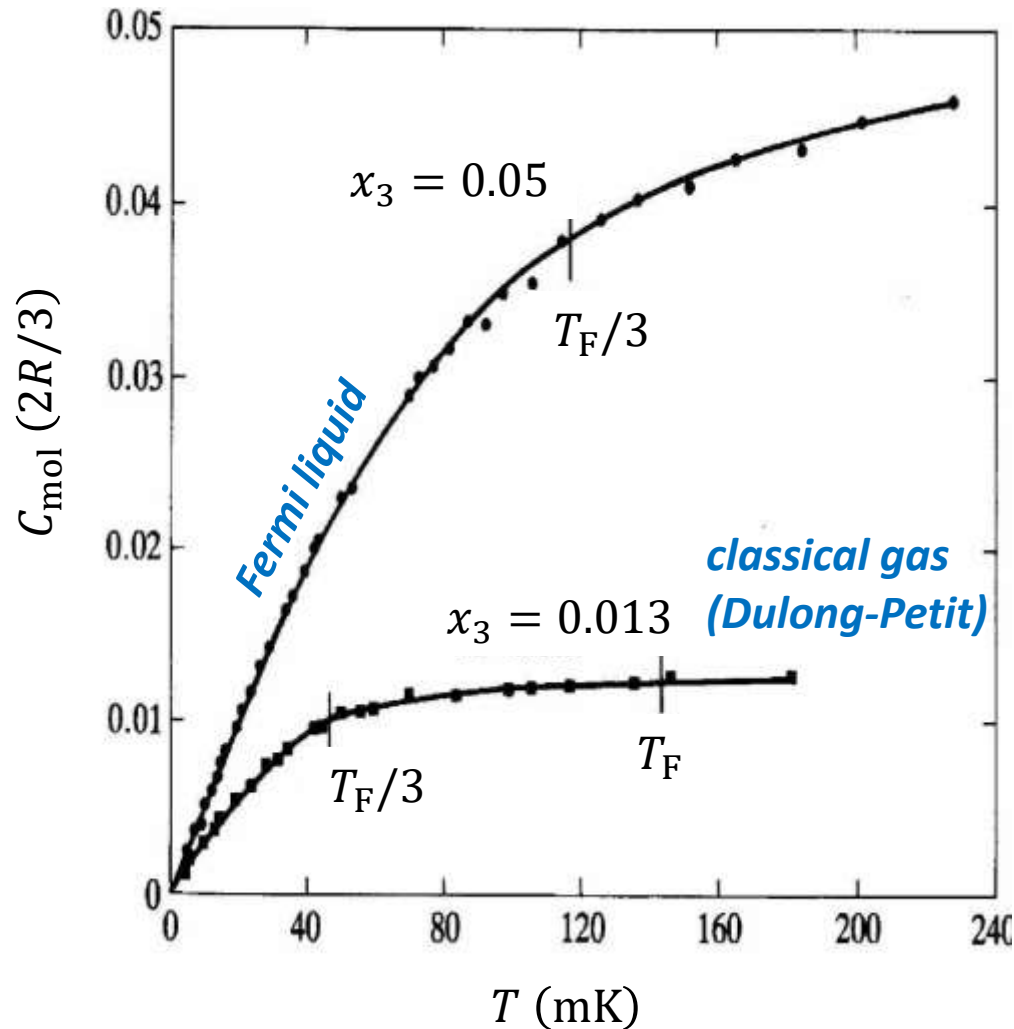


$$T_F = \frac{\hbar^2}{2m_{^3\text{He}}^*} (3\pi^3 n_{^3\text{He}})^{2/3} \propto x_3^{2/3}$$

$m_{^3\text{He}}^*$: effective mass of ^3He dissolved in ^4He
 $n_{^3\text{He}} = \frac{N_{^3\text{He}}}{V} = \frac{x_3 N}{V}$: density of ^3He atoms

I.6 $^3\text{He}/^4\text{He}$ mixture as diluted Fermi liquid

- specific heat



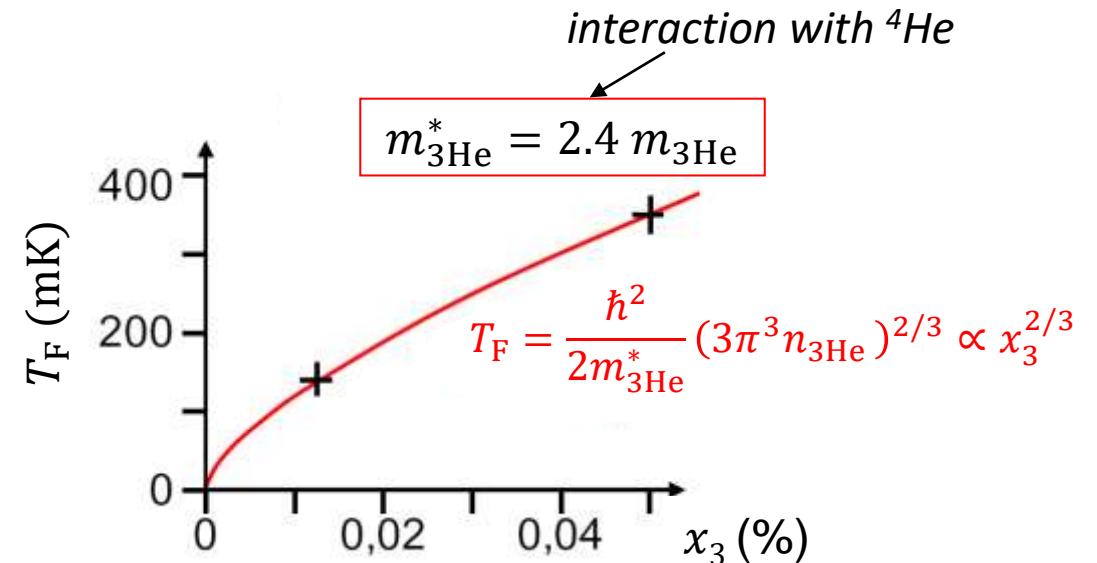
- Fermi gas ($T \ll T_F$)

$$C_{\text{mol}} = \frac{\pi^2}{2} R \frac{T}{T_F}$$

- classical gas ($T > T_F$)

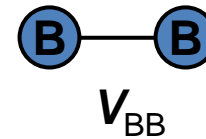
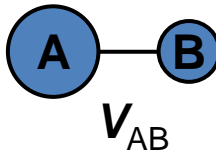
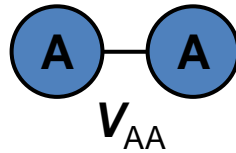
$$C_{\text{mol}} = \frac{3}{2} N_A k_B = \frac{3}{2} R$$

- T_F increases with x_3



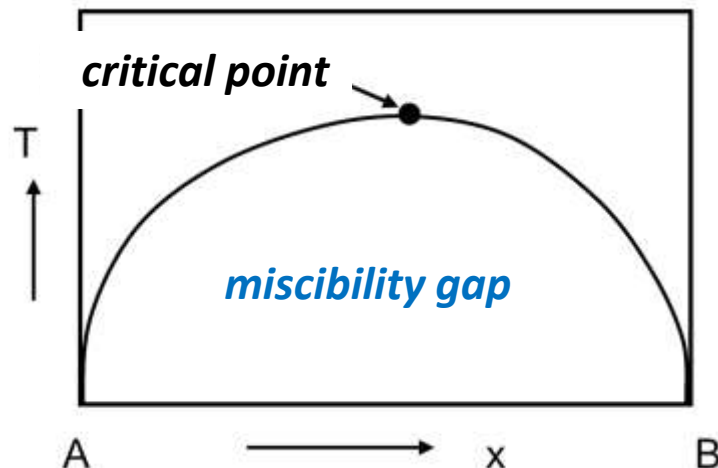
I.6 $^3\text{He}/^4\text{He}$ mixtures: miscibility gap in binary systems

- bonding:



- $V_{AB} > \frac{1}{2}(V_{AA} + V_{BB})$ **complete miscibility** (e.g. water and alcohol)

- $\frac{1}{2}(V_{AA} + V_{BB}) > V_{AB}$ **phase separation** (e.g. water and petrol) \rightarrow **optimization of binding total energy U**
complete phase separation @ $T = 0$



increasing mixing with increasing T :

$$F = U - TS \rightarrow \min$$

binding
energy

thermal
motion

\rightarrow **minimization of free energy**

I.6 $^3\text{He}/^4\text{He}$ mixtures: finite solubility of ^3He in ^4He

- binding energy of ^3He in ^3He (V_{33}) and ^4He (V_{34}):

$$L_3 = U_{\text{vap}} - U_{\text{liq}} + p\Delta V$$

$p\Delta V$ = work done against ambient pressure

- single ^3He atom in liquid ^3He :** average binding energy is given by the latent heat of evaporation L_3 :

binding energy for single ^3He atom: $\epsilon_{3,c} = -\frac{L_3}{N_A} = \mu_{3,c}$ $\mu_{3,c}$ = chemical potential of pure (concentrated) liquid

- single ^3He atom in liquid ^4He :**

binding energy for single ^3He atom: $\epsilon_{3,d}(x_3 \rightarrow 0) = \mu_{3,d}(x_3 \rightarrow 0)$ $\mu_{3,d}$ = chemical potential of dilute phase

– $|\epsilon_{3,d}(0)| > |\epsilon_{3,c}|$ or vice versa ?

^3He has smaller mass \rightarrow larger zero point fluctuations \rightarrow occupies larger volume

\rightarrow binding of ^3He is larger in ^4He than in ^3He due to larger density of ^4He

$\rightarrow |\epsilon_{3,d}(0)| > |\epsilon_{3,c}|$ (it is energetically more favorable to add a ^3He atom to the ^4He superfluid than to pure ^3He)

\rightarrow corresponds to case $V_{AB} > \frac{1}{2}(V_{AA} + V_{BB}) \rightarrow$ **complete miscibility expected,**
why miscibility only up to 6.5% ??

I.6 $^3\text{He}/^4\text{He}$ mixtures: finite solubility of ^3He in ^4He

- questions

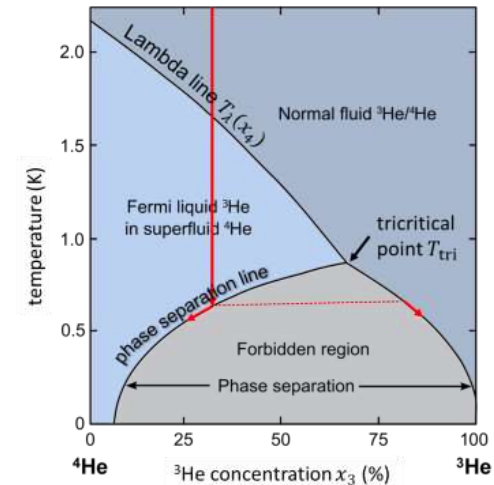
- why can't we dissolve more than 6.5% of ^3He in ^4He at $T = 0$?

we have to take into account two effects:

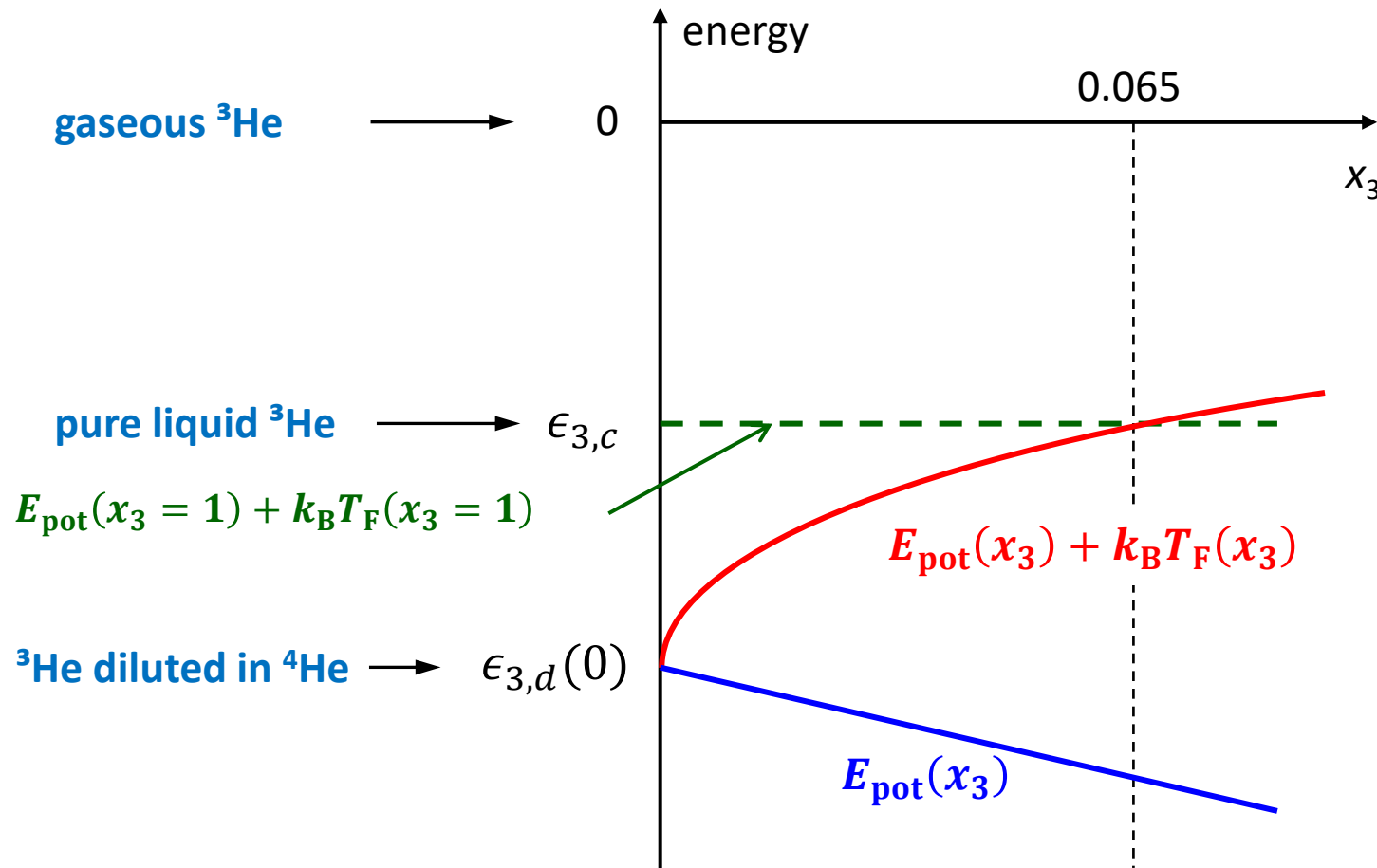
- ^3He forms degenerate Fermi liquid
 - T_F increases with x_3
 - for $x_3 > 6.5\%$, the increase of Fermi energy exceeds the gain in binding energy
- effective attraction between two ^3He atoms (*magnetic and volume effect*)
 - $|\epsilon_{3,d}(x_3)| > |\epsilon_{3,d}(x_3 = 0)|$

- why don't we have a complete phase separation into pure ^3He and pure ^4He at $T = 0$?

- incomplete phase separation results in finite disorder, violation of 3rd law of thermodynamic ?
- no: we have **degenerate Fermi gas, ordering in k -space**



I.6 $^3\text{He}/^4\text{He}$ mixtures: finite solubility of ^3He in ^4He



for $x_3 > 6.5\%$:

$$|E_{\text{pot}}(x_3) + k_B T_F(x_3)| < |\epsilon_{3,c}| = |E_{\text{pot}}(x_3 = 1) + k_B T_F(x_3 = 1)|$$

→ separation of pure ^3He

I.6 $^3\text{He}/^4\text{He}$ mixtures: cooling effect

- $^3\text{He}/^4\text{He}$ dilution refrigeration

- operation principle:

- remove ^3He atoms from the dilute phase below $T_{\text{tri}} = 0.87 \text{ K}$
- transport of ^3He atoms across phase boundary to maintain equilibrium concentration
- corresponds to evaporation of ^3He from concentrated phase
- **cooling effect as the latent heat of evaporation is removed**

- for Fermi liquid:

$$\rightarrow \mathbf{C_{V,concentrated}} < \mathbf{C_{V,diluted}} \quad (x_3 = 0.065) \quad (C_V \propto T/T_F, \quad T_F \propto n_{^3\text{He}}^{2/3})$$

- with $U = Q = \int_0^T C_V(T') dT' = \frac{\gamma}{2} T^2$ we therefore obtain ($C_V = \gamma T$, γ = Sommerfeld coefficient):

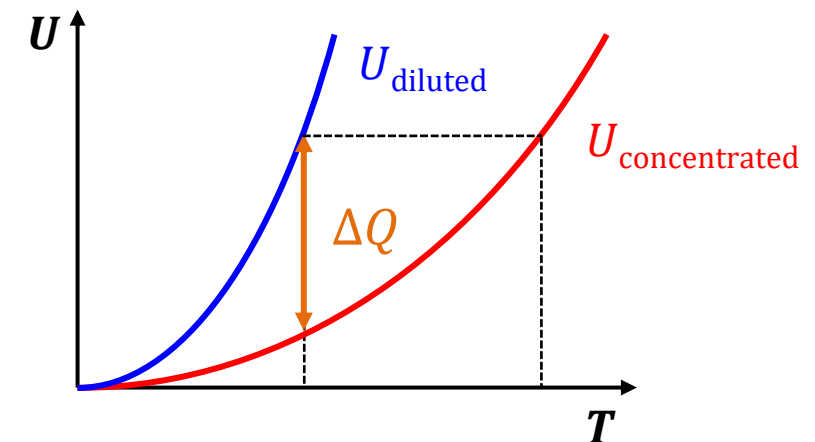
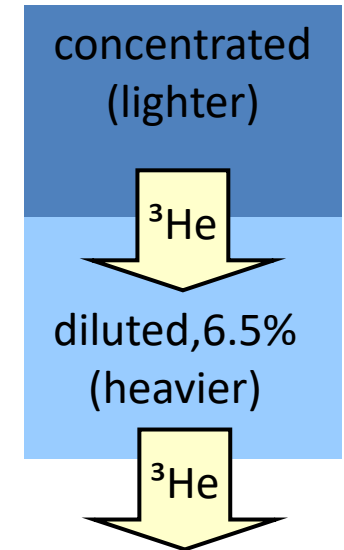
$$\mathbf{U_{concentrated}(T)} < \mathbf{U_{diluted}(T)}$$

- on transition across phase boundary:

$$dG = 0 = dU - TdS \Rightarrow dU = TdS = dQ$$

- removal of heat $\Delta Q = T\Delta S = T [S_{\text{dil}}(T) - S_{\text{con}}(T)]$
- **cooling effect**

- $^3\text{He}/^4\text{He}$ dilution refrigerator



I.6 $^3\text{He}/^4\text{He}$ mixtures: cooling effect

- $^3\text{He}/^4\text{He}$ dilution refrigeration: cooling power

- assumption: *one mole of ^3He crosses boundary between diluted and concentrated phase*

- removed heat:

$$\Delta Q = T \Delta S = T [S_{\text{dil}}(T) - S_{\text{con}}(T)]$$

- cooling power: $\Delta \dot{Q} = \left(\frac{\Delta Q}{n_{^3\text{He}}} \right) \cdot \dot{n}_{^3\text{He}} \Rightarrow$

$$\Delta \dot{Q} = \dot{n}_{^3\text{He}} T [s_{\text{dil}}(T) - s_{\text{con}}(T)]$$

$$S_{\text{dil}} = \frac{S_{\text{dil}}}{n_{^3\text{He}}}, S_{\text{con}} = \frac{S_{\text{con}}}{n_{^3\text{He}}}$$

- since there is no volume change:

$$dQ = dU = T dS = C_V dT, \quad U(T) = \int_0^T C_V dT', \quad S(T) = \int_0^T \frac{C_V}{T'} dT'$$

➤ with $C_{\text{mol}} = \frac{\pi^2 N_A k_B^2 T}{2 \varepsilon_F}$ and $\varepsilon_F = \frac{\hbar^2}{2m_{^3\text{He}}^*} \left(\frac{3\pi^2 N_{^3\text{He}}}{V} \right)^{2/3} = \frac{\hbar^2}{2m_{^3\text{He}}^*} \left(\frac{3\pi^2 N_A}{V_{\text{mol}}} \right)^{2/3}$ (standard expressions for Fermi liquid)

we obtain the entropy

$$S_{\text{mol}} = \frac{\pi^2 N_A k_B^2 m_{^3\text{He}}^*}{\hbar^2} \left(\frac{V}{3\pi^2 N_{^3\text{He}}} \right)^{2/3} T \propto \frac{T}{n_{^3\text{He}}}$$

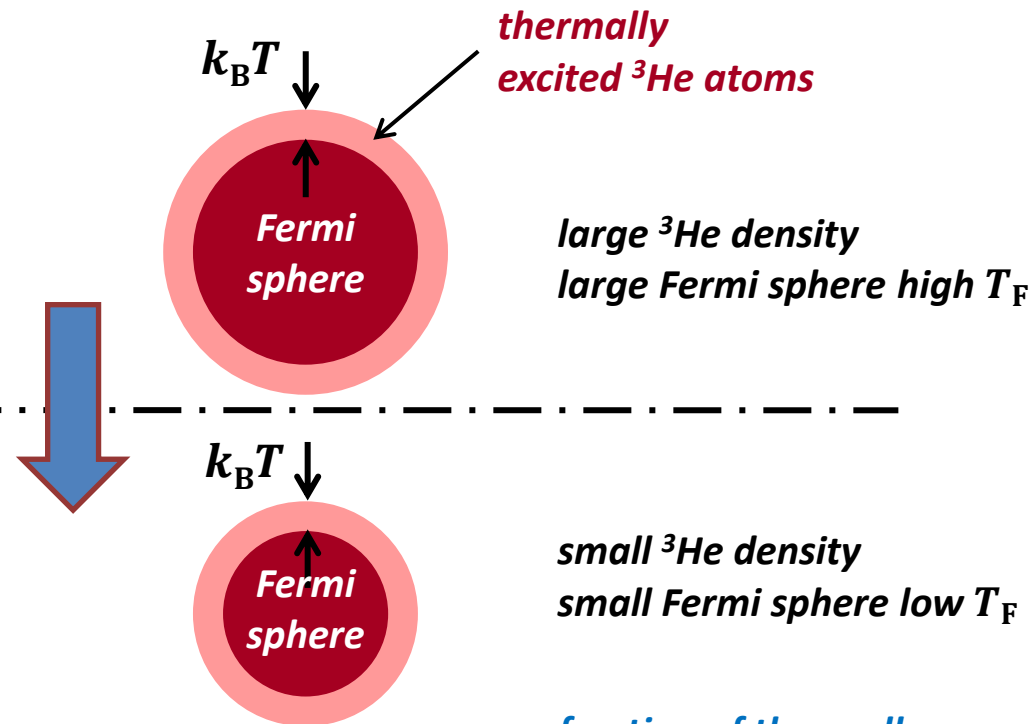
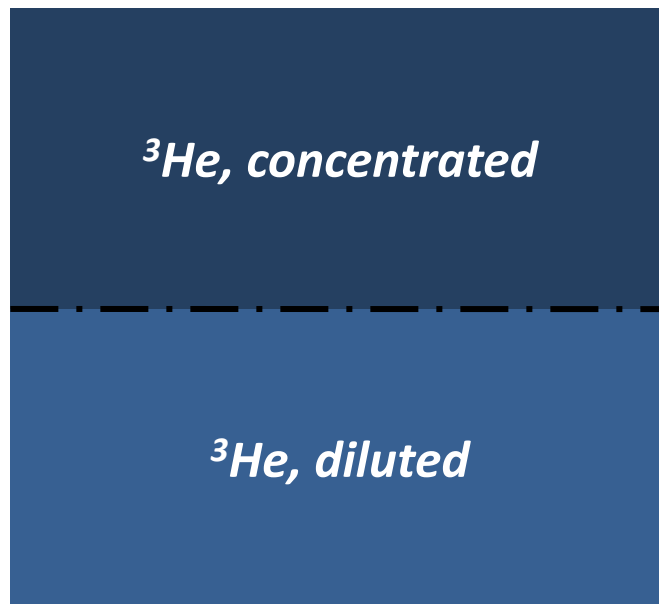
$$\Rightarrow S_{\text{con}} < S_{\text{dil}} \quad (\text{as } n_{^3\text{He,con}} > n_{^3\text{He,dil}})$$

$$\Rightarrow \dot{Q}(T) = 84 \dot{n}_{^3\text{He}} T^2 [\text{W}] \quad (\text{cooling power} \propto ^3\text{He throughput } \dot{n}_3 [\text{mol/s}])$$

I.6 $^3\text{He}/^4\text{He}$ mixtures: cooling effect

- $^3\text{He}/^4\text{He}$ dilution refrigeration: plausibility consideration

plausibility consideration



fraction of thermally excited ^3He atoms increases ($\propto T/T_F$)

entropy increases going from concentrated to diluted phase

removed heat: $dQ = TdS$