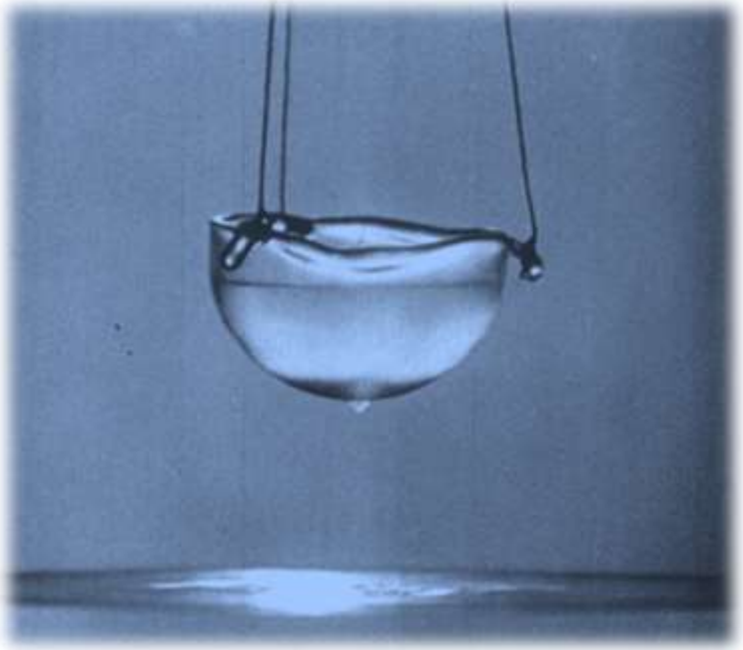
Superconductivity and Low Temperature Physics II



Lecture Notes
Summer Semester 2023

R. Gross

Chapter 2

Quantum Transport
in
Nanostructures

Chapter II: Quantum Transport in Nanostructures

Literature:

1. **Introduction to Mesoscopic Physics**
   Yoseph Imry
   Oxford University Press, Oxford (1997)
2. **Electronic Transport in Mesoscopic Systems**
   Supriyoto Datta
   Cambridge University Press, Cambridge (1995)
3. **Mesoscopic Electronic in Solid State Nanostructures**
   Thomas Heinzel
   Wiley VCH, Weinheim (2003)
4. **Quantum Transport**
   Yuli V. Nazarov, Yaroslav M. Blanter
   Cambridge University Press, Cambridge (2009)
5. **Semiconductor Nanostructures**
   Thomas Ihn
   Oxford University Press (2010)

Walther
Meißner
Institut

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DER
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Technische
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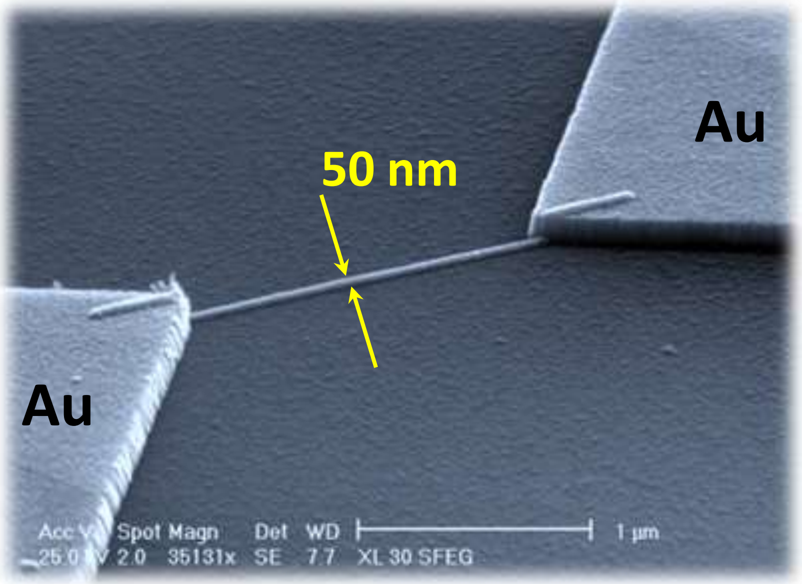
Superconductivity and Low Temperature Physics II



Lecture No. 8

R. Gross

Contents Part II: Quantum Transport in Nanostructures

Contents:

II.1 Introduction

II.1.1 General Remarks

- ***macroscopic solid-state systems***

  - *usually consideration of thermodynamic limit* → $N \rightarrow \infty, \Omega \rightarrow \infty, N/\Omega = const.$

- ***what happens if system size becomes small ?***

  - *discrete spectrum of electronic levels*

  - *coherent motion of electrons*

    → phase memory due to lack of inelastic scattering within system size:
    system size L smaller than *phase coherence length* L_ϕ

    → new *interference phenomena*

  - *validity of Boltzmann theory of electronic transport and concept of resistivity ?*

    → system size L smaller than *mean free path* ℓ*: ballistic transport*

  - *discreteness of electric charge and magnetic flux becomes important*

    → single electron and single flux effects

  - *concept of impurity ensemble breaks down*

    → sample properties show „fingerprint“ of detailed arrangement of impurities

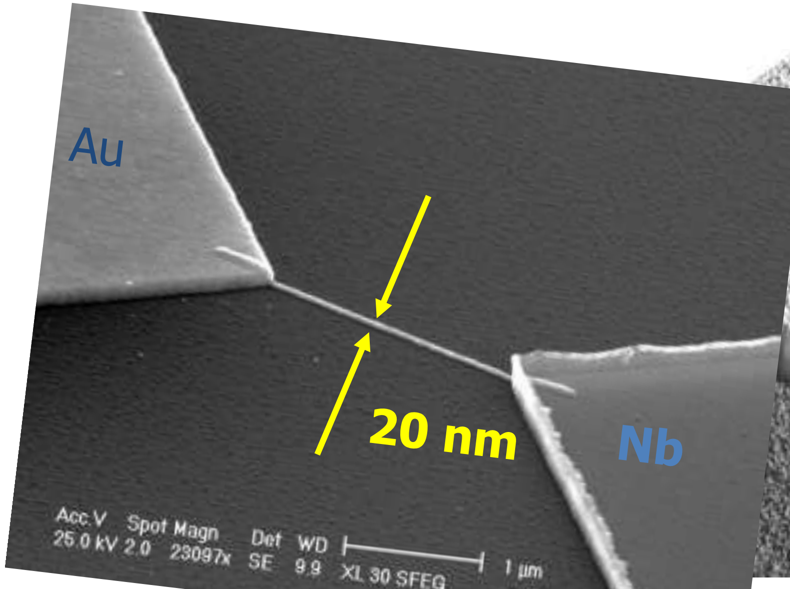
II.1.2 Mesoscopic Systems

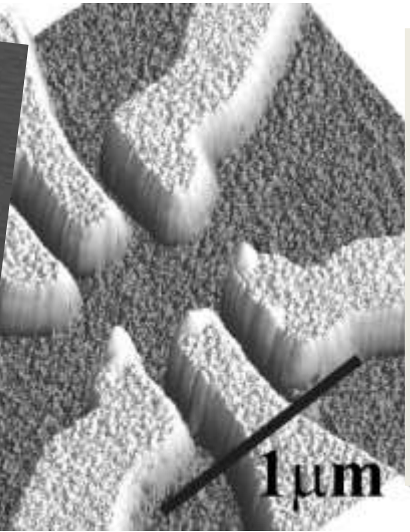
- ***mesoscopic systems*** *(coined by Van Kampen in 1981):* **mesos (*Greek*): between**
  - ***system size is between microscopic (e.g. atom, molecule) and macroscopic system (e.g. bulk solid)***
  - ***system size L is smaller than phase coherence length L_ϕ*** **(typically in nm - µm regime)**
    → phenomena related to phase coherence become important
    → statistical concepts no longer applicable due to smallness of system size
    → still coupling to environment/reservoir present (in contrast to microscopic objects such as atoms)

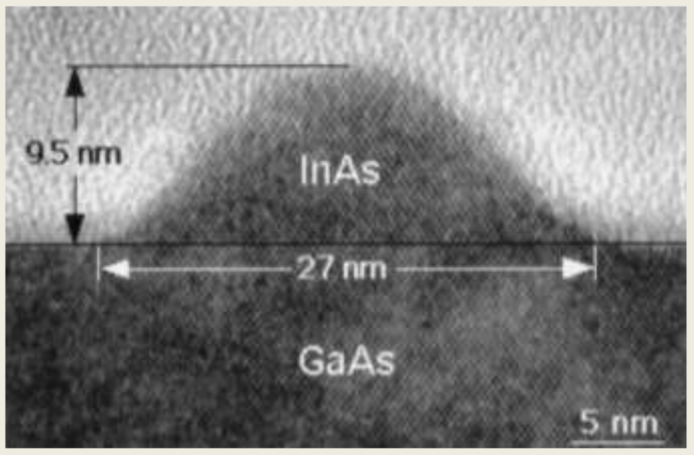
- ***properties of mesoscopic systems are usually studied at low temperatures***
  - *phase coherence length L_ϕ decreases rapidly with increasing T*
    ➔ $L < L_\phi$ can usually be satisfied only at low T
  - *observation of level quantization effects require $k_\mathrm{B}T < \Delta E \simeq 1/L^2$*

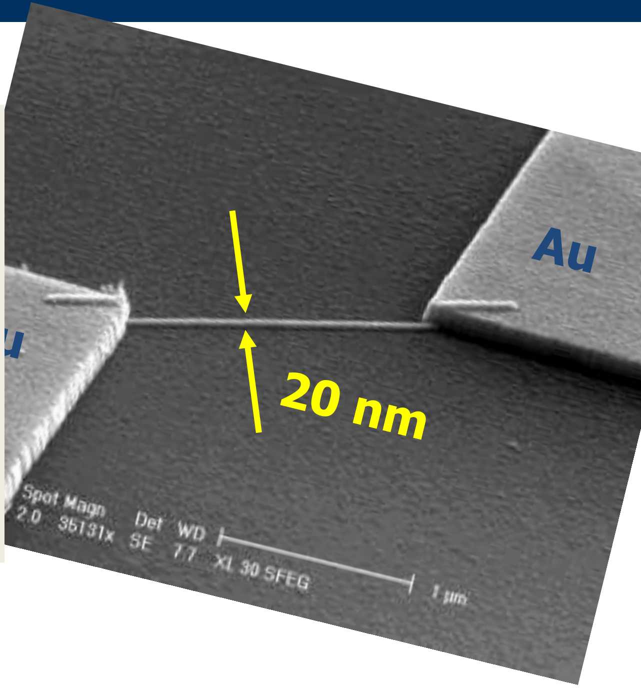
⇨ ***study of nanostructures at low temperature***

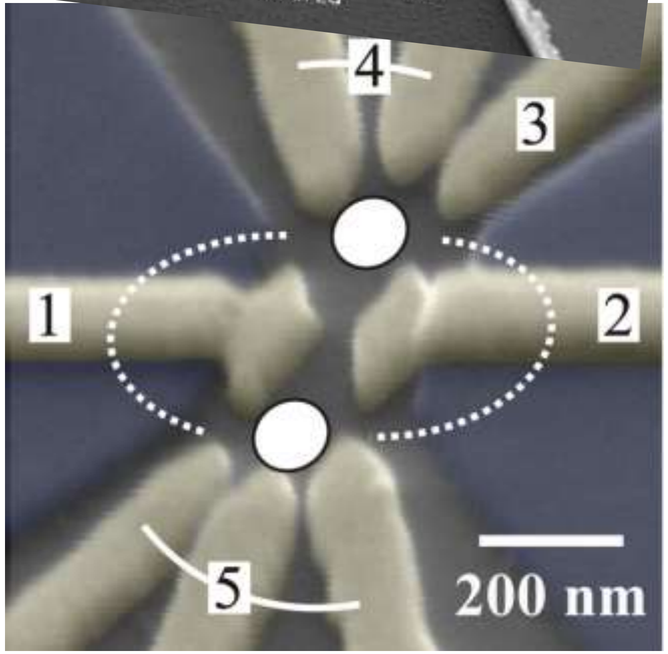
II.1.2 Mesoscopic Systems

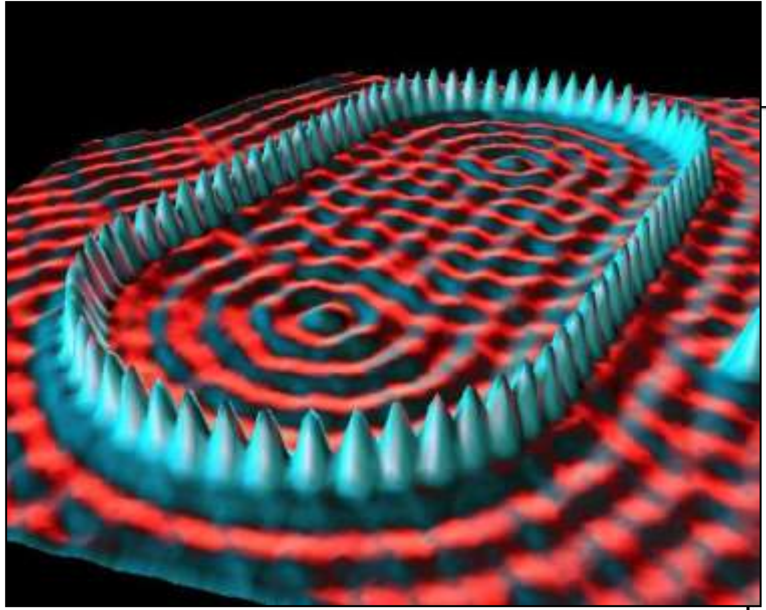


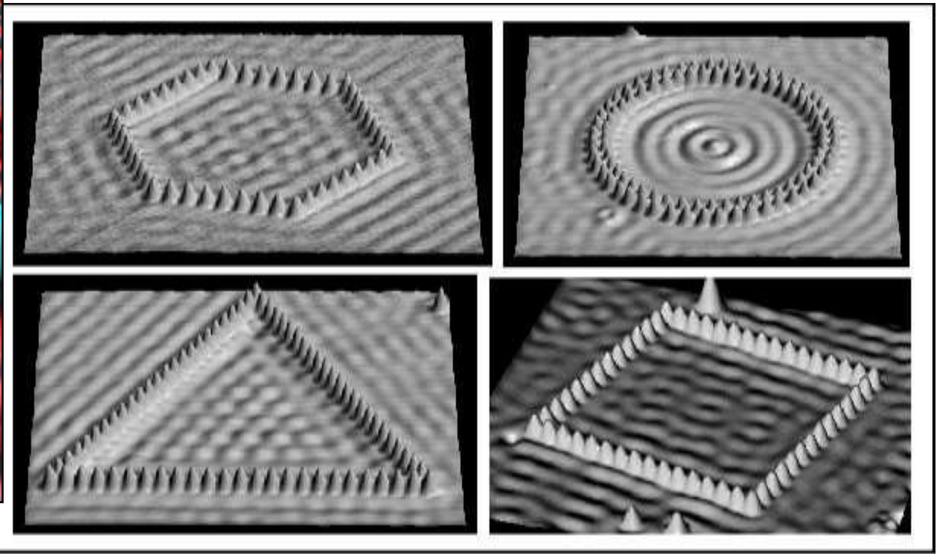












III.8

II.1.2 Mesoscopic Systems

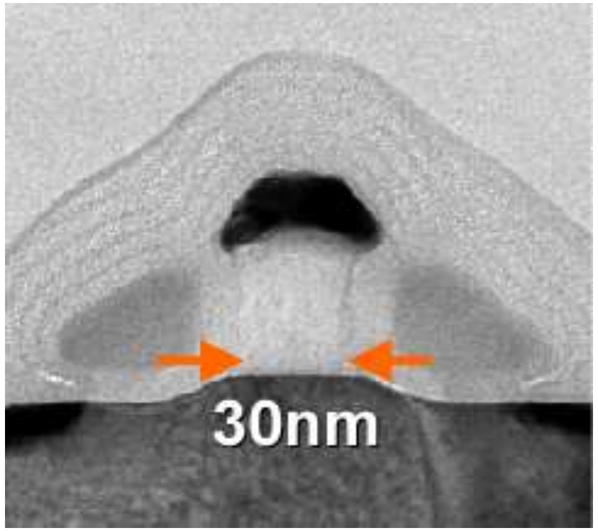
Die folgende graphische Animation zeigt den Anflug auf eine Einzelelektronen-Schaltung.

Sie beginnt mit der Ansicht des gesamten Wafers und endet mit der elektronenmikroskopischen Aufnahme einer realen Struktur.

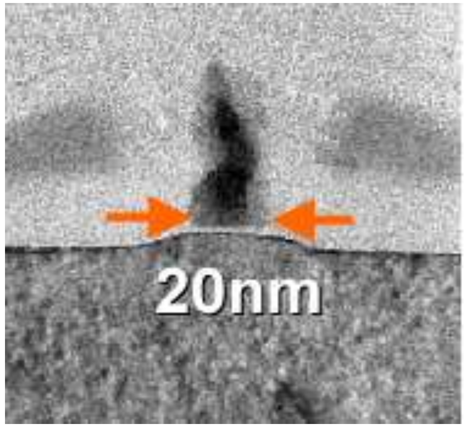
II.1.2 Mesoscopic Systems

- **miniaturization of electronic devices**

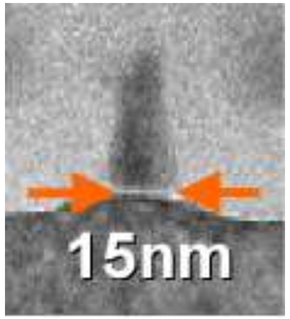
gate length of transistors



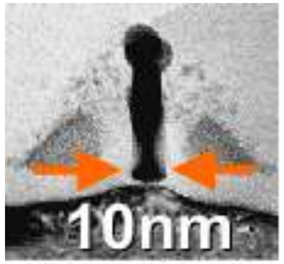
65 nm process
2005



45 nm process
2007



32 nm
2009

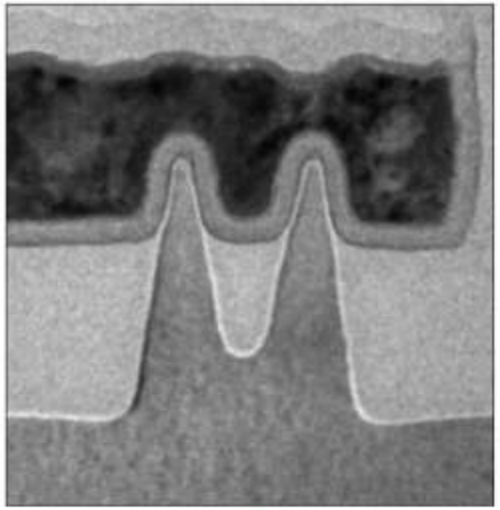


22 nm
2011

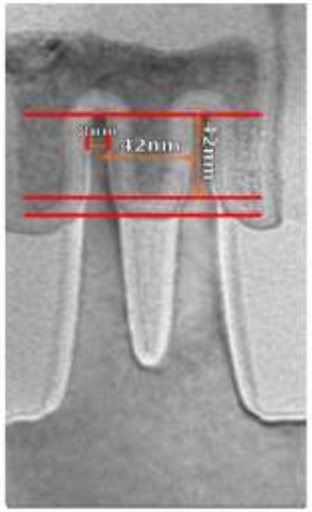
(Source: Intel Inc.)

II.1.2 Mesoscopic Systems

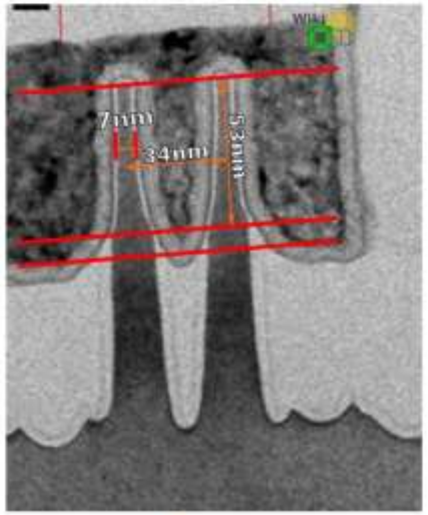
Transistor Fin Improvement



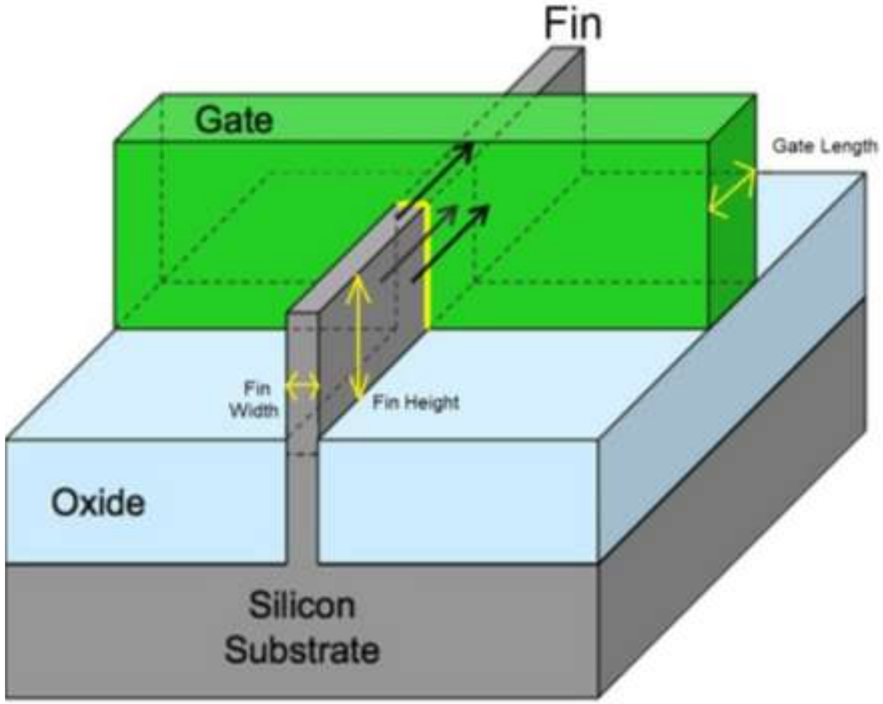
22 nm 1st Generation
Tri-gate Transistor



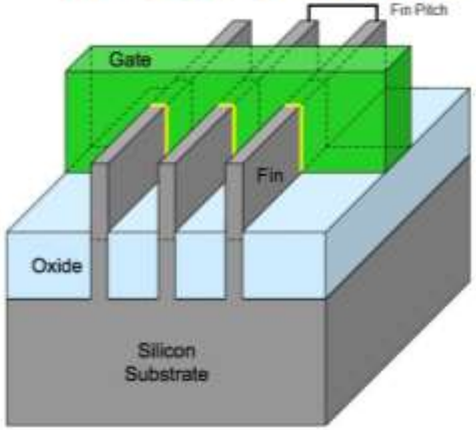
14 nm 2nd Generation
Tri-gate Transistor



10 nm 3rd Generation
Tri-gate Transistor



Tri-Gate Transistor



II.1.3 Characteristic Length Scales

- **from microscopic to macroscopic systems**

microscopic ↔ **mesoscopic** ↔ macroscopic

Fermi wave length: $\lambda_{\mathrm{F}} < 1$ nm (for metals)

➔ *"size" of charge carrier*

electron mean free path: $\ell \approx$ 10 - 100 nm

➔ *distance between (elastic) scattering events*

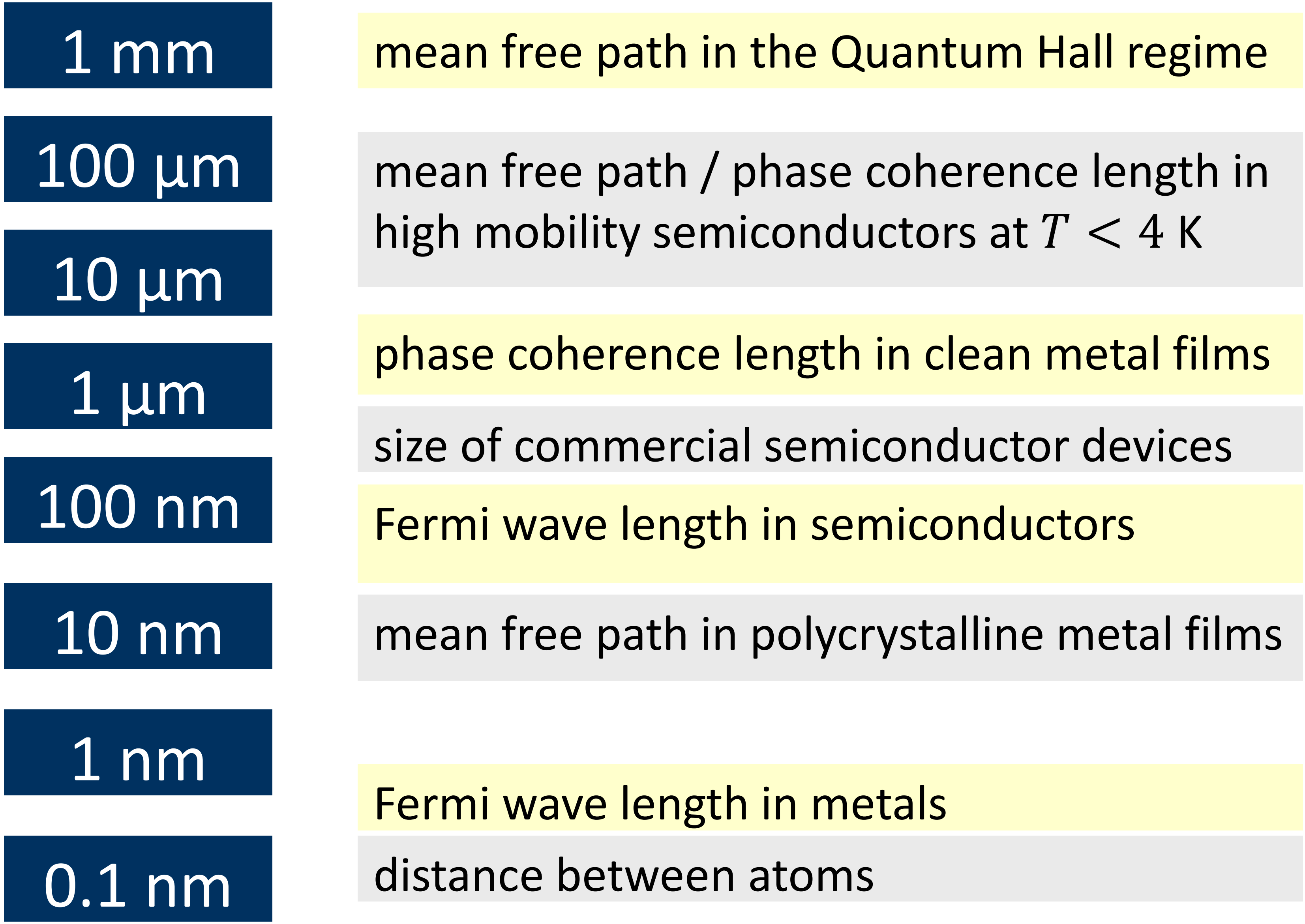
phase coherence length: $L_\varphi \approx 1\ \mu$m

➔ *loss of phase memory*

sample size: $L, W \approx$ 0.01 - 1 μm

mesoscopic regime: $L < L_\varphi(T)$

II.1.3 Characteristic Length Scales



1 mm

100 µm

10 µm

1 µm

100 nm

10 nm

1 nm

0.1 nm

mean free path in the Quantum Hall regime

mean free path / phase coherence length in high mobility semiconductors at $T < 4$ K

phase coherence length in clean metal films

size of commercial semiconductor devices

Fermi wave length in semiconductors

mean free path in polycrystalline metal films

Fermi wave length in metals

distance between atoms

II.1.3 Characteristic Length Scales

- **electron wavelength**: $\boldsymbol{\lambda_{\mathrm{F}} = \frac{h}{\sqrt{2m^\star \varepsilon_{\mathrm{F}}}} = \frac{2\pi}{(3\pi^2 n)^{1/3}}}$ *(Fermi wavelength)*

- **mean free path:** $\boldsymbol{\ell = v_{\mathrm{F}} \cdot \tau_m}$ $\boldsymbol{\tau_m^{-1} = \tau_c^{-1} \cdot \alpha_m}$ ← *effectiveness of collision:* $0 < \alpha_m < 1$

  ↑ *collision time*

ballistic

- **phase relaxation length:** $\boldsymbol{L_\varphi = v_F \tau_\varphi}$ $\boldsymbol{\tau_\varphi^{-1} = \tau_c^{-1} \cdot \alpha_\varphi}$ ← *effectiveness of collision in destroying phase coherence:* $0 < \alpha_\varphi < 1$

diffusive ⟶ $\boldsymbol{L_\varphi = \sqrt{D\tau_\varphi} = \sqrt{\frac{1}{3} v_{\mathrm{F}}^2 \tau_m \tau_\varphi}}$

→ *elastic impurity scattering*: $\tau_\varphi \to \infty$ or $\alpha_\varphi \to 0$

→ *electron-phonon scattering*: $\tau_\varphi \approx \tau_{e-ph}$ *??*

→ *electron-electron scattering:* $\tau_\varphi \approx \tau_{e-e}$ *??*

→ *electron-impuritiy scattering* *(with internal degree of freedom, e.g. spin)*

II.1.3 Characteristic Length Scales

- ***question***: what is the effectiveness of an ***inelastic scattering process*** regarding destruction of phase coherence ?

  - ***Altshuler, Aronov, Khmelnitsky (1982):***

    if $\hbar\omega$ is the characteristic energy of an ***inelastic process*** (e.g. phonon energy), then the mean-squared energy spread of electron after collisions is

$$\langle \Delta E \rangle^2 = \underbrace{(\hbar\omega)^2}_{\text{square of energy change}} \overbrace{\frac{\tau_\varphi}{\tau_c}}^{\text{number of scattering events}}$$

number of scattering events

square of energy change

τ_φ is time required to acquire a phase change of $\approx 2\pi$

$$\Delta\varphi \approx \frac{\Delta E}{\hbar}\,\tau_\varphi \approx 2\pi \quad \Rightarrow \quad \tau_\varphi \approx \left(\frac{\tau_c}{\omega^2}\right)^{1/3}$$

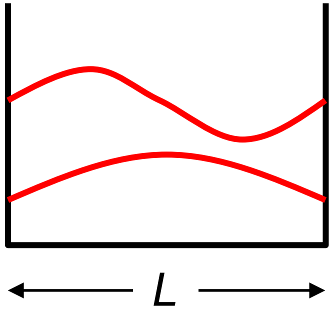
low-frequency excitations are less effective in destroying phase coherence !!

  - at low T: e-e scattering is dominating

II.1.4 Characteristic Energy Scales

- **size quantization**

  – electron in a box:



level spacing: $\Delta E = \frac{h^2}{2m^\star}\left(\frac{1}{L}\right)^2$

1 nm	↔	10.000 K	↔	800 meV
10 nm	↔	100 K	↔	8 meV
100 nm	↔	1 K	↔	0.08 meV

  – Fermi wavelength: $\lambda_\mathrm{F} = \frac{h}{\sqrt{2m^\star\varepsilon_\mathrm{F}}} = \frac{2\pi}{(3\pi^2 n)^{1/3}}$

if $\boldsymbol{\lambda_\mathrm{F} > L_x, L_y, L_z}$ → reduction of dimension by ***size quantization***

3D → 2D → 1D → 0D

for metals: $n \approx 10^{22} - 10^{23}$ cm⁻³ → $\lambda_\mathrm{F} \approx$ 1 nm

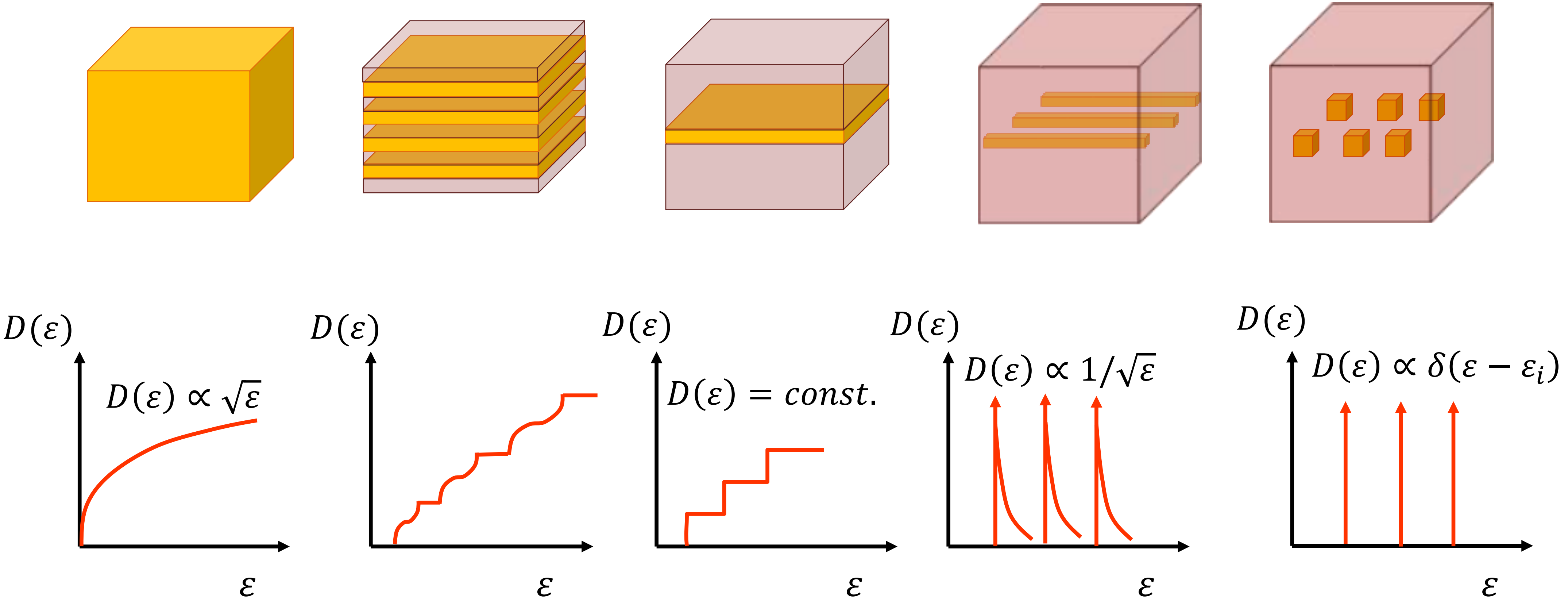
for semiconductors: $n \approx 10^{16} - 10^{19}$ cm⁻³ → $\lambda_\mathrm{F} \approx$ 10 - 100 nm

  – single charge/flux effects: $\boldsymbol{\frac{e^2}{2C} > k_\mathrm{B}T, \quad \frac{\Phi_0^2}{2L} > k_\mathrm{B}T}$

II.1.4 Characteristic Energy Scales

- **size quantization: DOS in 3D, 2D, 1D, and 0D**

bulk
superlattice
quantum well
quantum wire
quantum dot

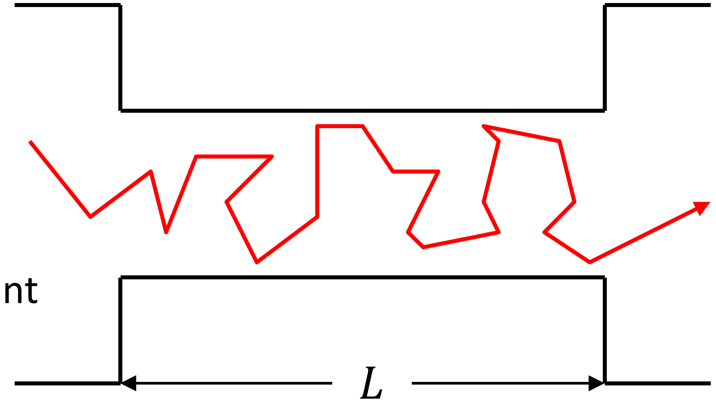


3-dim
2-dim
1-dim
0-dim

II.1.4 Characteristic Energy Scales

- **Thouless energy**

  – how long does it take for an electron to diffuse through a sample of length L



$$L = \sqrt{Dt} \Longrightarrow t = \frac{L^2}{D}$$

$D = v^2\tau$: diffusion constant

  – mean diffusion time is related to the characteristic energy (uncertainty relation)

$$\varepsilon_{\mathrm{Th}} = \frac{\hbar}{t} = \frac{\hbar D}{L^2}$$

(Thouless energy)

➢ macroscopic samples: $\varepsilon_{\mathrm{Th}} \ll k_{\mathrm{B}}T$

➢ mesoscopic samples: $\varepsilon_{\mathrm{Th}} > k_{\mathrm{B}}T$

$$L < \sqrt{\frac{\hbar D}{k_{\mathrm{B}}T}}$$

– low T
– small L
– *clean samples* (large D)

  – ballistic transport regime (see below):

$$t = \frac{L}{v_{\mathrm{F}}} \Longrightarrow \varepsilon_{\mathrm{Th}} = \frac{\hbar}{t} = \frac{\hbar v_{\mathrm{F}}}{L}$$

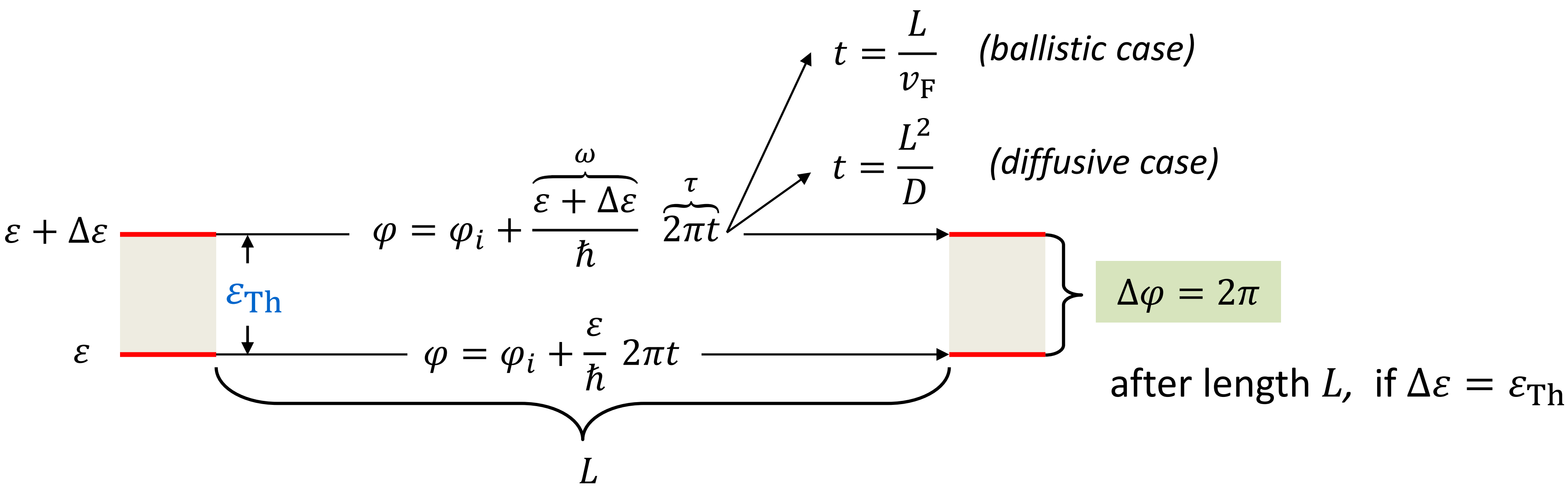
(v_{F}: Fermi velocity)

II.1.4 Characteristic Energy Scales

- **physical meaning of the Thouless energy**

$$\boxed{\varepsilon_{\mathrm{Th}} = \frac{\hbar}{t} = \frac{\hbar D}{L^2}}$$

→ electrons in energy interval $\Delta\varepsilon = \varepsilon_{\mathrm{Th}}$ stay phase coherent in sample of length L



$t = \dfrac{L}{v_{\mathrm{F}}}$ *(ballistic case)*

$t = \dfrac{L^2}{D}$ *(diffusive case)*

$\varepsilon + \Delta\varepsilon$

$\varepsilon_{\mathrm{Th}}$

ε

$$\varphi = \varphi_i + \frac{\overbrace{\varepsilon + \Delta\varepsilon}^{\omega}}{\hbar}\,\overbrace{2\pi t}^{\tau}$$

$$\varphi = \varphi_i + \frac{\varepsilon}{\hbar}\,2\pi t$$

L

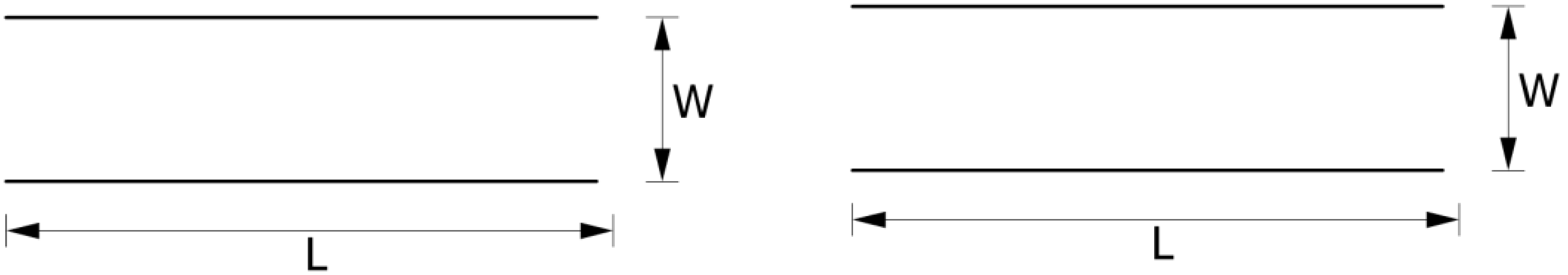
$\Delta\varphi = 2\pi$

after length L, if $\Delta\varepsilon = \varepsilon_{\mathrm{Th}}$

⇨ ***if $\Delta\varepsilon \leq \varepsilon_{\mathrm{Th}}$, the acquired phase shift is less than 2π***

example: $D = 10^3$ cm²/s, $L = 1$ µm → $\varepsilon_{\mathrm{Th}}\,/k_{\mathrm{B}} \approx 1$ K

II.1.5 Transport Regimes



macroscopic sample	*mesoscopic sample*
diffusive: $L, W \gg \ell$	ballistic: $L, W < \ell$
	quasi-ballistic: $W < \ell$
incoherent: $L \gg L_\varphi$	coherent: $L < L_\varphi$

- @ 300 K: $\ell \sim 10$ nm due to e-ph scattering
- @ at low T: ℓ is limited by impurity and e-e scattering → sample quality matters
- L_φ is limited by inelastic processes: e-ph and e-e scattering:

  strong T dependence: L_φ increases with decreasing T

  $L_\varphi \approx 1\ \mu m$ @ 1K

Contents Part II: Quantum Transport in Nanostructures

Contents:

II.2 Description of Electron Transport by Scattering of Waves

II.2.1 Electron Waves and Waveguides

(true only in vacuum)

$$\Psi(\mathbf{r},t) = \frac{1}{\sqrt{V}} \exp\left(\imath\mathbf{k}\cdot\mathbf{r} - \frac{\imath}{\hbar}\,\varepsilon(\mathbf{k})t\right)$$

$\Psi(\mathbf{r},t)$	wave function
$\lvert\Psi(\mathbf{r},t)\rvert^2$	probability to find electron at position $\mathbf{r}$ at time t
V	normalization volume
$\mathbf{k}$	wave vector
$\mathbf{p} = \hbar\mathbf{k}$	momentum
$\varepsilon(\mathbf{k}) = \frac{\hbar^2k^2}{2m}$	energy

II.2.1 Electron Waves and Waveguides

- **electrons as fermions:**

→ *Pauli principle (state either occupied by single electron or empty)*

→ *density of states in k-space:* $2\frac{V}{(2\pi)^3}$ *(factor 2 due to spin)*

→ *fraction of filled states:* $f(\mathbf{k},T)$

– important quantities:

$$\begin{matrix}\text{density}\\ \text{energy density}\\ \text{current density}\end{matrix} = \begin{bmatrix}\rho\\ \varepsilon\\ \mathbf{J}\end{bmatrix} = 2\int\frac{\mathrm{d}^3k}{(2\pi)^3}\begin{bmatrix}1\\ \varepsilon(\mathbf{k})\\ e\mathbf{v}(\mathbf{k})\end{bmatrix} f(\mathbf{k})$$

– f determined by statistics:

$$f(\mathbf{k},T) = \left[\exp\left(\frac{\varepsilon(\mathbf{k})-\mu}{k_\mathrm{B}T}\right)+1\right]$$

Fermi statistics for electrons

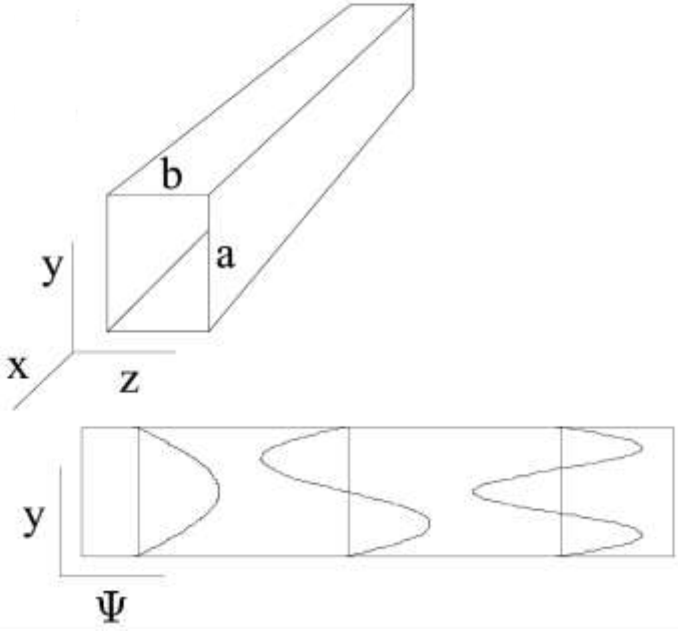
II.2.1 Electron Waves and Waveguides

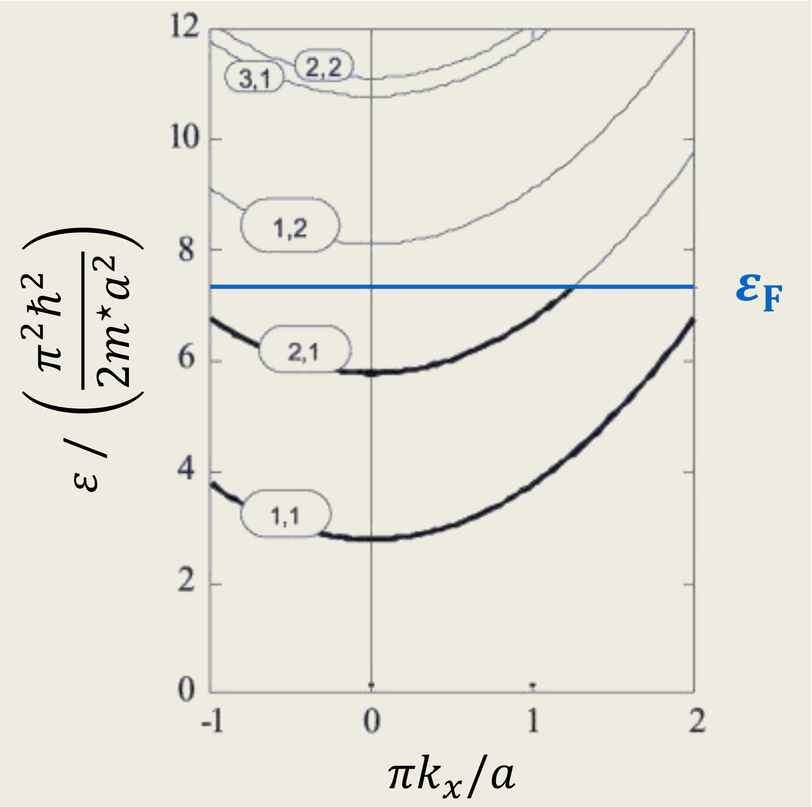
- **ballistic conductor as waveguide**
  - **example**: *1D free motion of charge carriers, e.g. in x-direction with confinement in y, z-direction*

$$\Psi_{k_x,n,m}(\boldsymbol{r},t) = \phi_{n,m}(y,z)\exp[i(k_x x - \omega t)]$$

mode index n, m　*standing wave*　*plane wave*

$$\varepsilon_{n,m}(k_x) = \frac{\hbar^2 k_x^2}{2m^\star} + \varepsilon_{n,m} \qquad \varepsilon_{n,m} = \frac{\pi^2\hbar^2}{2m^\star}\left(\frac{n_y^2}{a^2} + \frac{m_z^2}{b^2}\right)$$

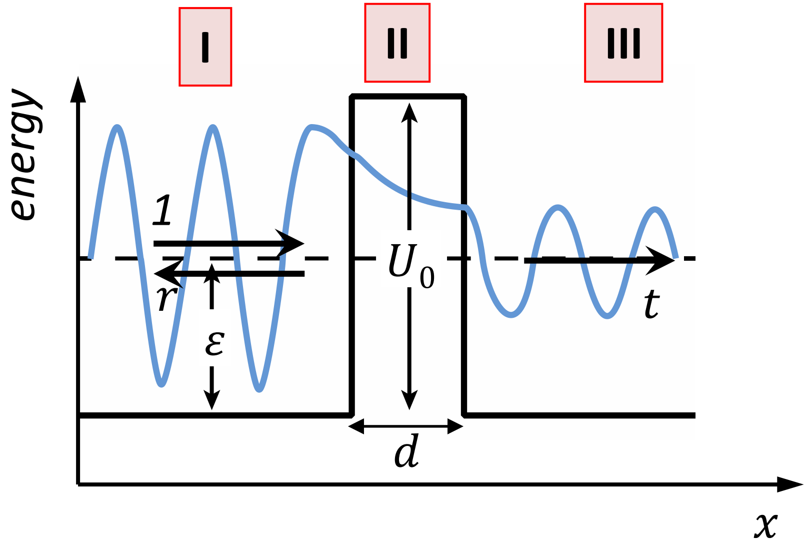




Source: Handouts Nazarov, TU Delft

II.2.1 Electron Waves and Waveguides

- **wave guide with potential barrier**



$$\varepsilon_{n,m}(k_x) = \frac{\hbar^2 k_x^2}{2m^\star} + \varepsilon_{n,m}(0)$$

$$\varepsilon_{n,m}(k_x) - U_0 = \frac{\hbar^2 \kappa^2}{2m^\star}$$

I $\Psi(x) = 1 \cdot \exp(ik_x x) + r \cdot \exp(-ik_x x)$

II $\Psi(x) = A \cdot \exp(i\kappa x) + B \cdot \exp(-i\kappa x)$

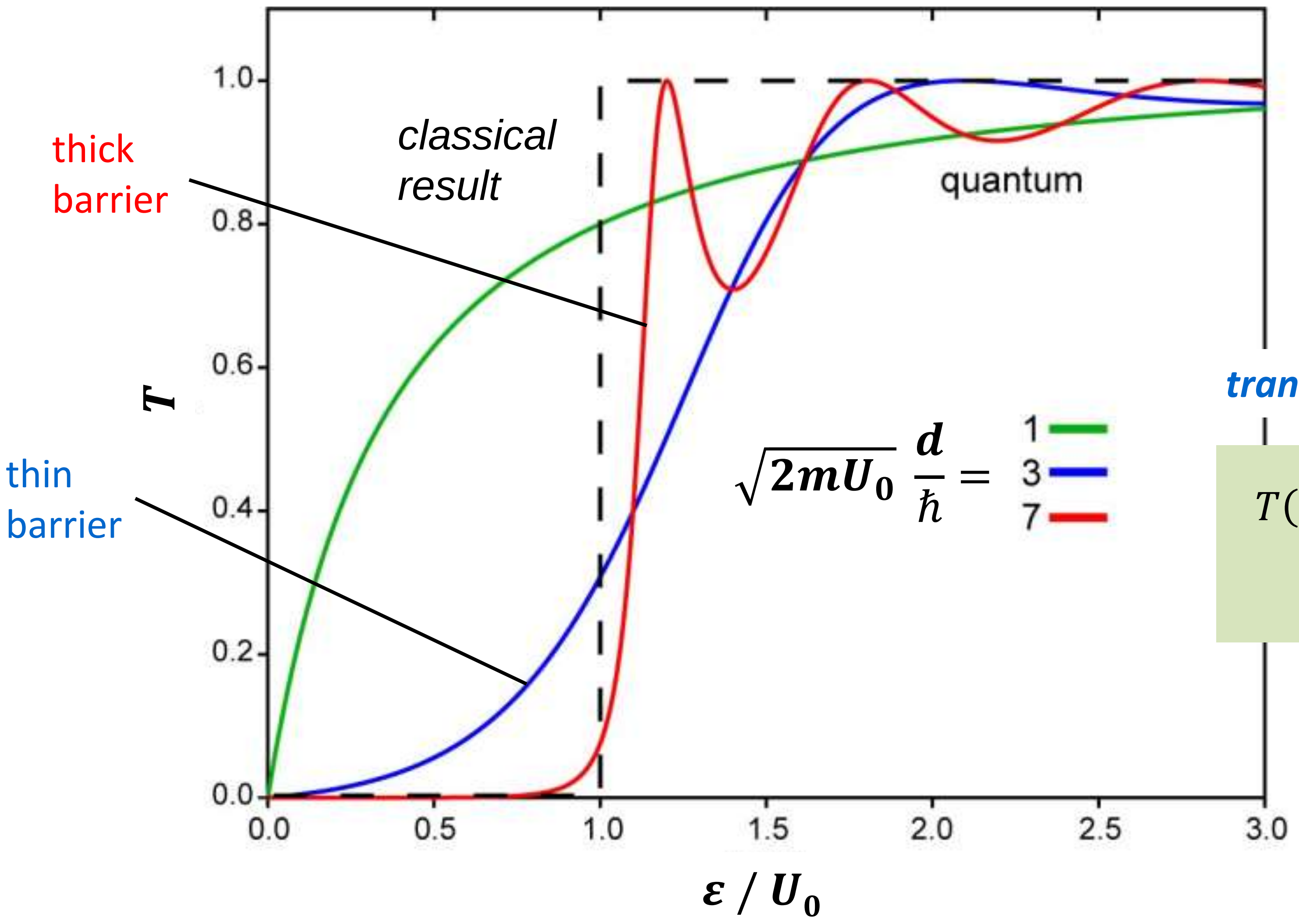
III $\Psi(x) = t \cdot \exp(ik_x x)$

4 unknown variables:
$\boldsymbol{A, B, r, t}$
t: transmission amplitude
r: reflection amplitude

4 equations
(wave function matching
at interfaces)

II.2.1 Electron Waves and Waveguides

- **wave guide with potential barrier** → example: rectangular barrier



transmission probability/coefficient:

$$T(\varepsilon) \equiv |t^2| = \frac{1}{1 + \left(\frac{k_x^2 - \kappa^2}{2k_x\kappa}\right)^2 \sinh^2 \kappa d}$$

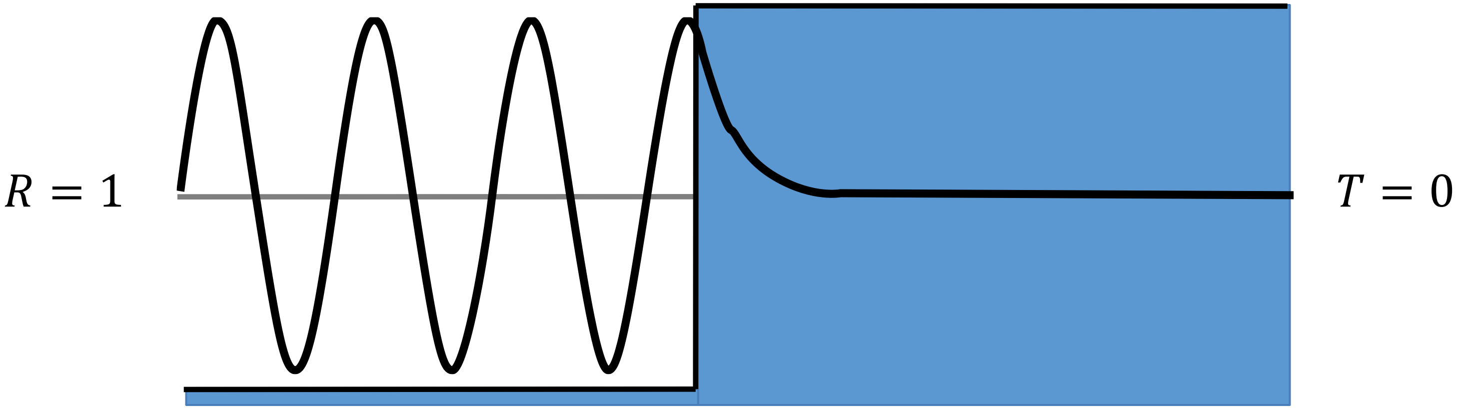
for $\kappa d \gg 1$:

$$\sinh^2(\kappa d) = [\exp(\kappa d) - \exp(-\kappa d)]^2$$
$$\simeq \exp(2\kappa d)$$

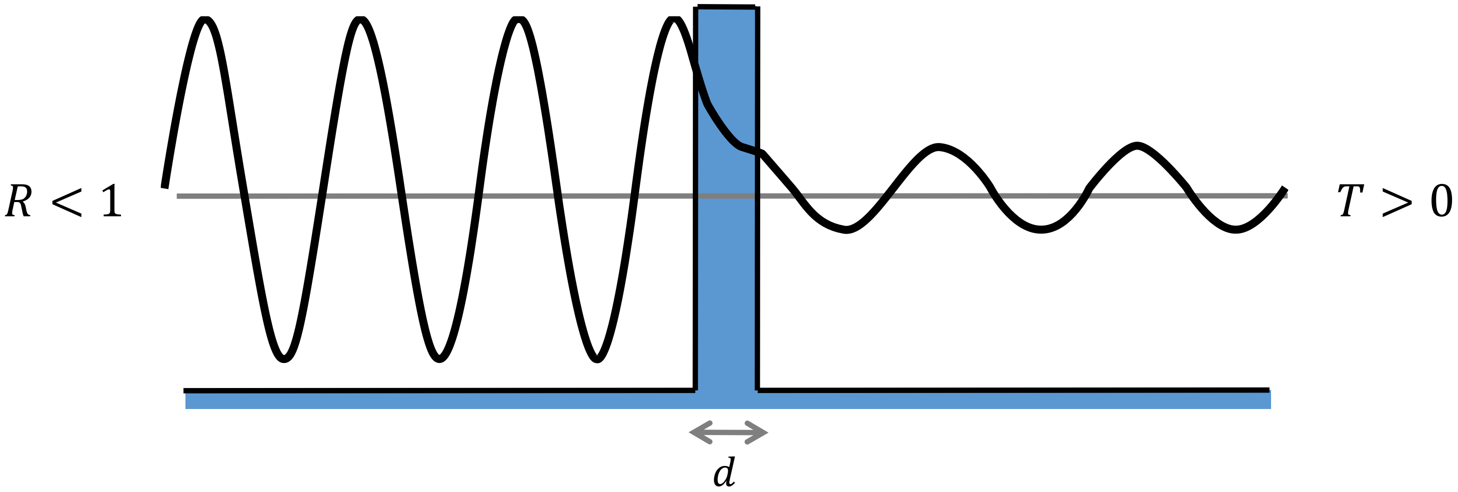
II.2.1 Electron Waves and Waveguides

- **quantum tunneling through a thin potential barrier**

  - total reflection at boundary (barrier with infinite thickness)

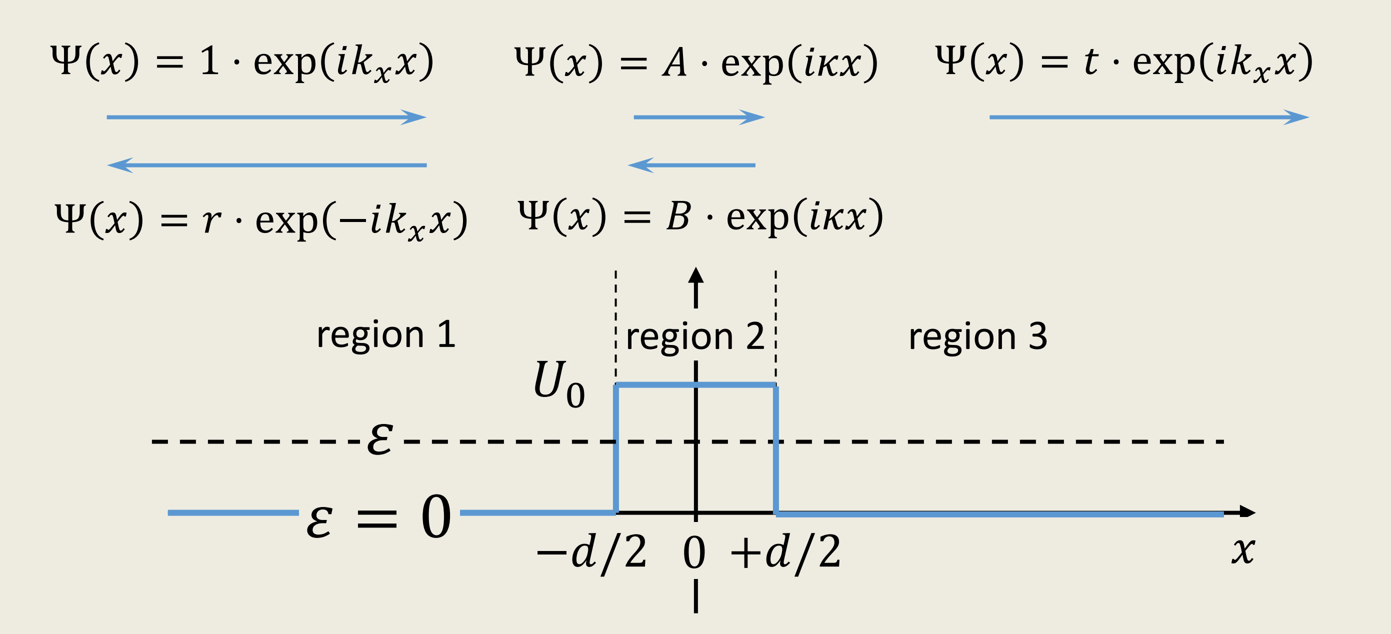


  - partial reflection/tunneling at barrier of finite thickness



II.2.1 Electron Waves and Waveguides

- **quantum tunneling through a thin potential barrier: a rectangular barrier**



$\Psi(x) = 1 \cdot \exp(i k_x x)$ $\Psi(x) = A \cdot \exp(i\kappa x)$ $\Psi(x) = t \cdot \exp(i k_x x)$

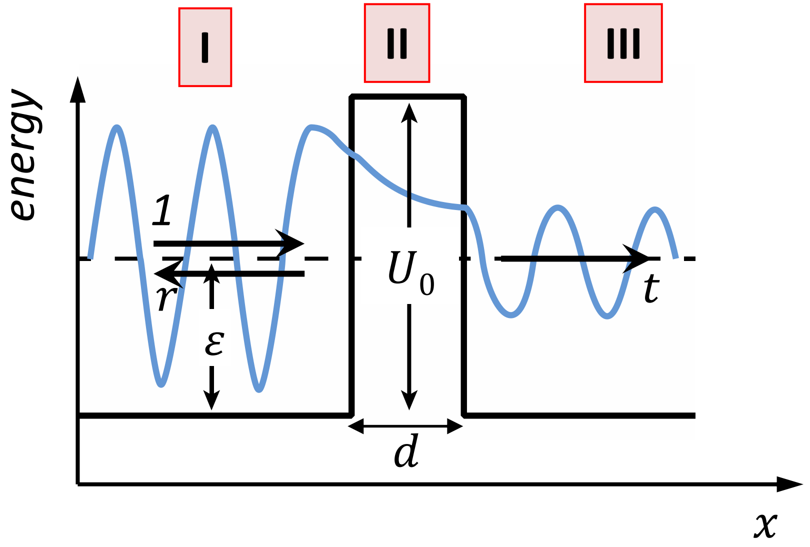
$\Psi(x) = r \cdot \exp(-i k_x x)$ $\Psi(x) = B \cdot \exp(i\kappa x)$

– in regions 1 and 3: $$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = \varepsilon\,\Psi(x) \quad \Rightarrow \quad k_x^2 = \frac{2m\varepsilon}{\hbar^2}$$

– in region 2: $$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = (\varepsilon - U_0)\,\Psi(x) \quad \Rightarrow \quad \kappa^2 = \frac{2m(\varepsilon - U_0)}{\hbar^2}$$

II.2.1 Electron Waves and Waveguides

- **quantum tunneling through a thin potential barrier: a rectangular barrier**



$$T(\varepsilon) \equiv |t^2| = \frac{1}{1 + \left(\frac{k_x^2 - \kappa^2}{2k_x\kappa}\right)^2 \sinh^2 \kappa d}$$

$$T(\varepsilon) \equiv |t^2| = \frac{1}{1 + \frac{U_0^2}{4\varepsilon(U_0 - \varepsilon)} \sinh^2 \kappa d}$$

for $\kappa d \gg 1$:

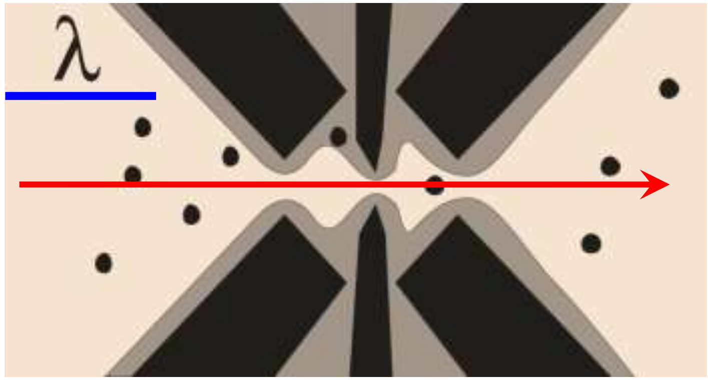
$$\sinh^2(\kappa d) = [\exp(\kappa d) - \exp(-\kappa d)]^2 \simeq \exp(2\kappa d)$$

$$\boxed{T(\varepsilon) \equiv |t^2| = \frac{1}{1 + \frac{U_0^2}{4\varepsilon(U_0 - \varepsilon)} \exp(2\kappa d)}}$$

II.2.1 Electron Waves and Waveguides

- **modelling of nanostructures as complex waveguides**

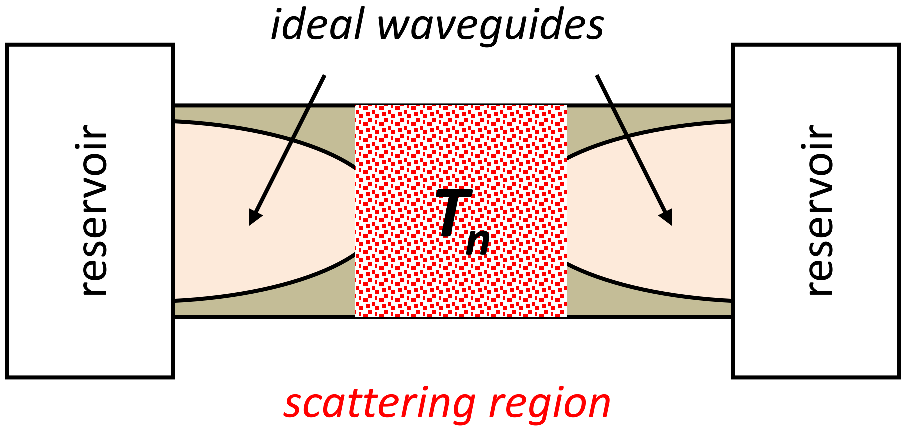
→ ***transport channels + potential barrier***



– description of transport by a ***set of transmission coefficients*** T_n

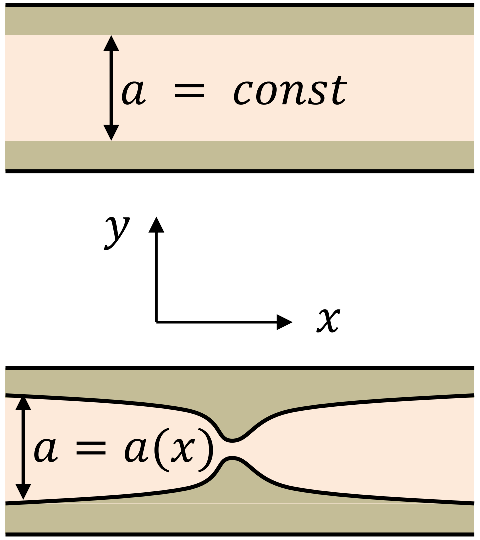
sufficient to describe transport !!

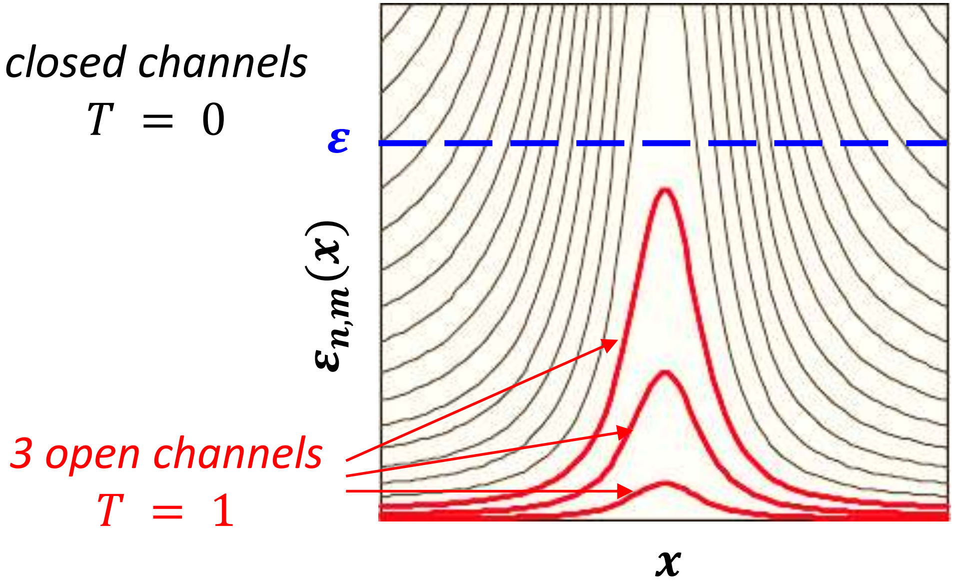
examples:
(i) adiabatic quantum transport
(ii) quantum point contact



II.2.1 Electron Waves and Waveguides

- **modelling of nanostructures as complex waveguides**
  - example: ***adiabatic quantum transport*** → ***constriction as a potential barrier***





closed channels
$T = 0$

3 open channels
$T = 1$

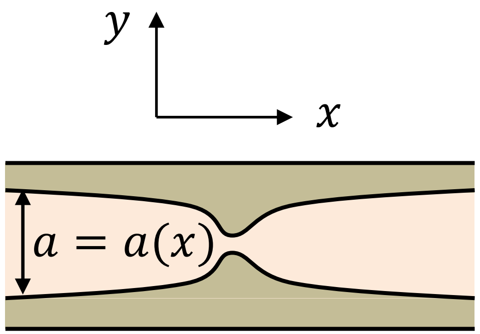
$$\varepsilon_{n,m}(k_x, x) = \frac{\hbar^2 k_x^2}{2m^\star} + \frac{\pi^2\hbar^2}{2m^\star}\left(\frac{n_y^2}{a^2(x)} + \frac{m_z^2}{b^2(x)}\right)$$

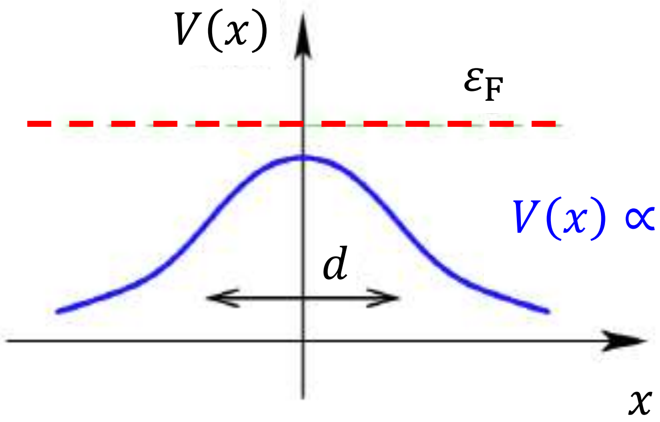
adiabatic waveguide:
variation of dimensions occurs on length scale large compared to width
→ waveguide walls can be assumed parallel locally

II.2.1 Electron Waves and Waveguides

- **modelling of nanostructures as complex waveguides**

  – example: ***adiabatic quantum transport*** → ***constriction as a potential barrier***





$$V(x) \propto \frac{\pi^2\hbar^2}{2m^\star a^2(x)}$$

$$\varepsilon_{n,m}(k_x, x) = \frac{\hbar^2 k_x^2}{2m^\star} + \underbrace{\frac{\pi^2\hbar^2}{2m^\star}\left(\frac{n_y^2}{a^2(x)} + \frac{m_z^2}{b^2(x)}\right)}_{=V(x)}$$

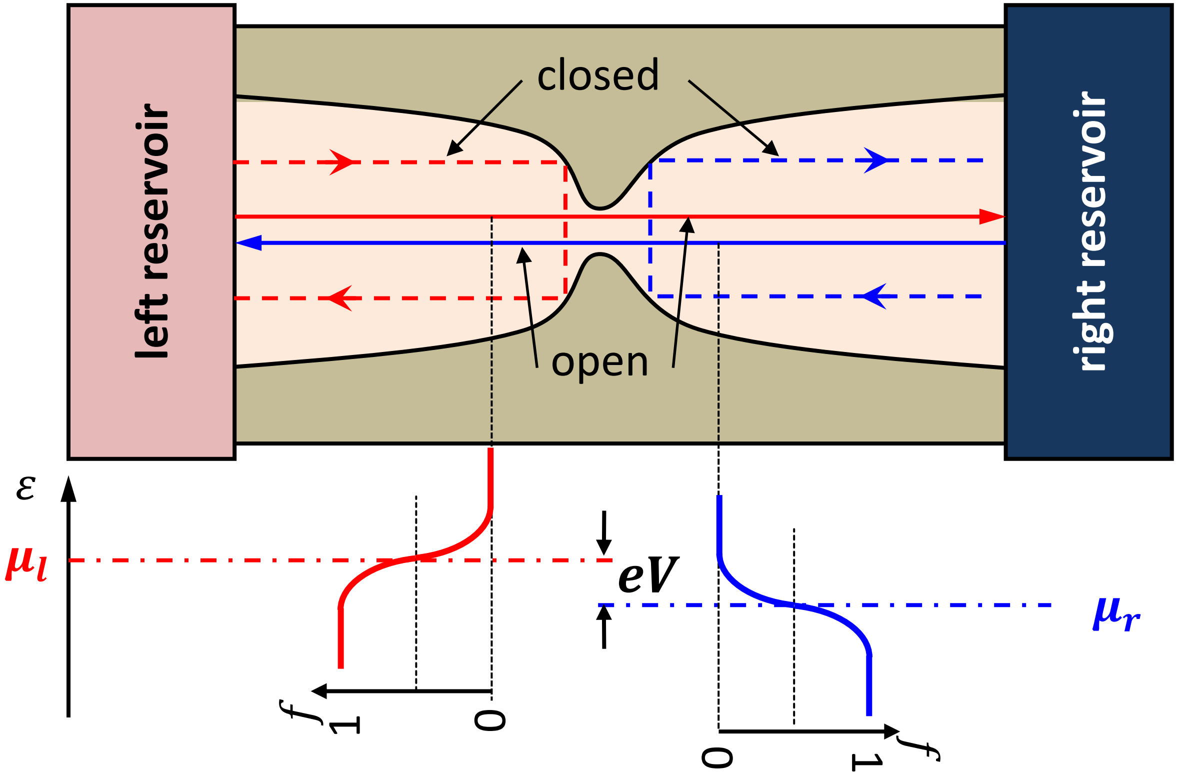
– parabolic approximation of potential step $V(x) \simeq -\frac{1}{2} m\Omega^2\, x^2$

transmission probability: $$T(\varepsilon_F) = \frac{1}{\exp\left(-\frac{2\pi\varepsilon_F}{\hbar\Omega}\right) + 1}$$

E.C. Kemble, 1935

II.2.1 Electron Waves and Waveguides

- **modelling of nanostructures as complex waveguides**
  - example: ***quantum point contact***



net current:

$$I = I_l - I_r$$

spin

$$I_l = T\,\frac{2}{2\pi}\int dk_x\, ev_x\; f_l(k_x)$$

$$I_r = T\,\frac{2}{2\pi}\int dk_x\, ev_x\; f_r(k_x)$$

$$v_x = \frac{1}{\hbar}\frac{\partial\varepsilon}{\partial k_x}$$

open channel: $T = 1$

closed channel: $T = 0$

$$I = \frac{2e}{2\pi\hbar}\sum_{\text{open ch}} \underbrace{\int \mathrm{d}\varepsilon\,[f_l(\varepsilon) - f_r(\varepsilon)]}_{=\mu_l-\mu_r} = \frac{2e}{2\pi\hbar}\,N_{\text{open}}\,\underbrace{(\mu_l-\mu_r)}_{=eV} = 2\,\underbrace{\frac{e^2}{h}}_{=G_Q}\,N_{\text{open}}\,V$$

quantized conductance !!

II.2.1 Electron Waves and Waveguides

- **what is the meaning of the quantity** $G = \frac{I}{V} = 2\frac{e^2}{h} N_{\text{open}} = 2\, G_Q\, N_{\text{open}}$

  - ➢ for ballistic transport and reflectionless contacts ($T = 1$) there should not be any resistance!

  - ➢ where does the resistance come from ?

    → contact resistance from the interface between the ballistic conductor and the contact pads

    → resistance is denoted as ***contact resistance***

$$\boxed{G_c^{-1} = \frac{h}{e^2}\frac{1}{2N_{\text{open}}} = G_Q^{-1}\frac{1}{2N_{\text{open}}}}$$

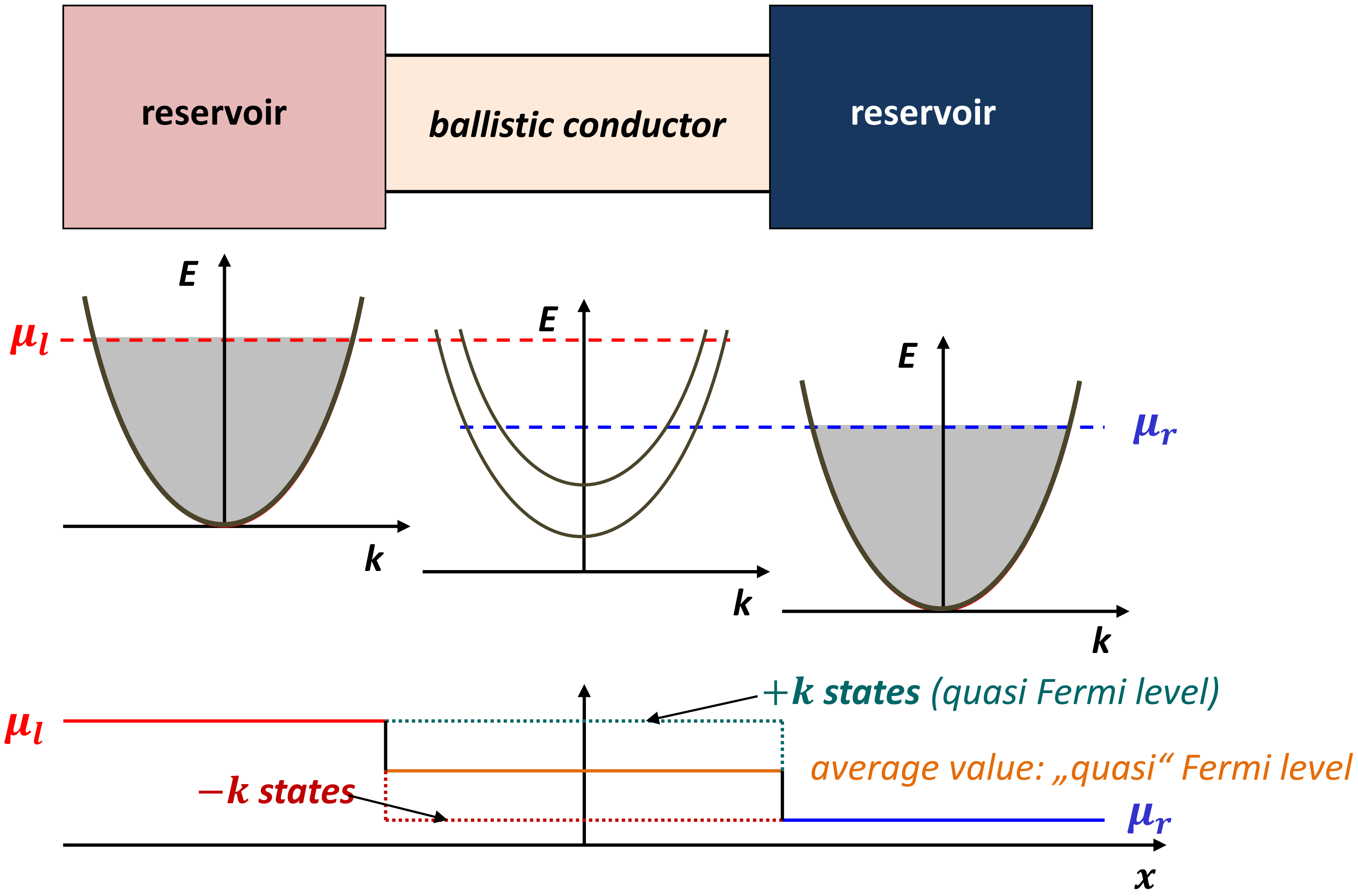
number of available modes

quantum resistance

25 812.807 Ω = 1 Klitzing

  - ➢ G_Q determined by fundamental constants, does not depend on materials properties, geometry or size of nanostructure

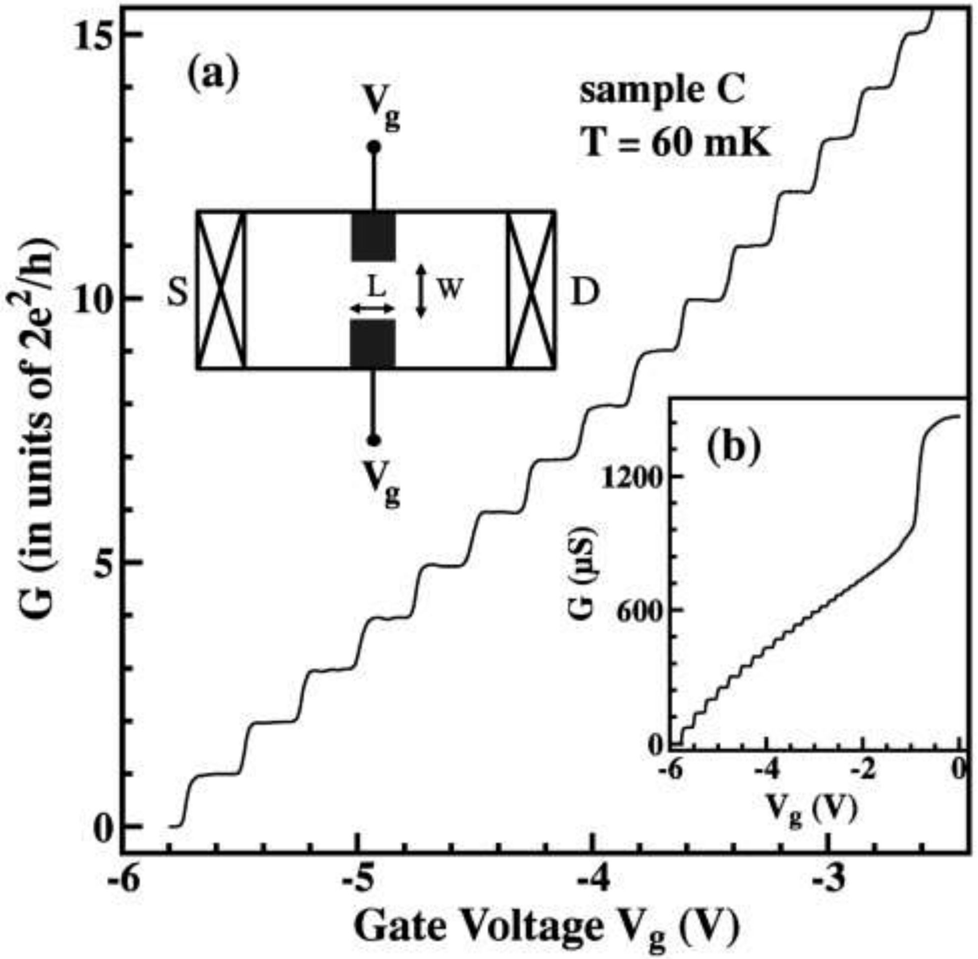
II.2.1 Electron Waves and Waveguides



➔ *voltage drop at interfaces (contact resistance) !!*

II.2.1 Electron Waves and Waveguides

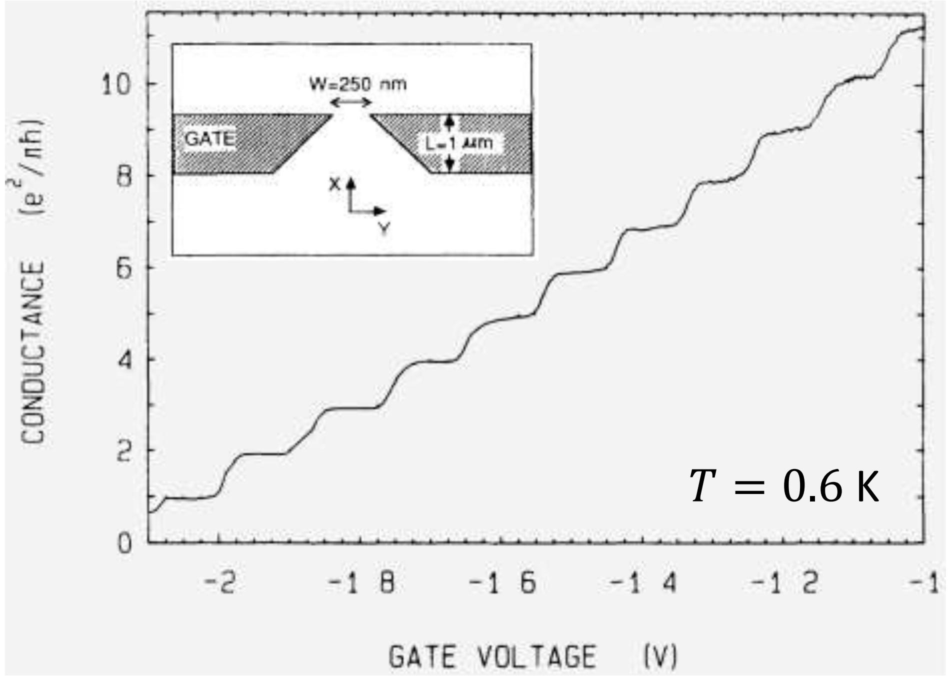
- **quantum point contact: experimental results**

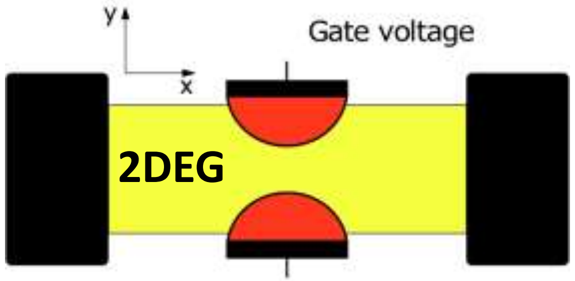


K.J. Thomas *et al.*, Phys. Rev. B **58**, 4846 (1998)

first experiment by

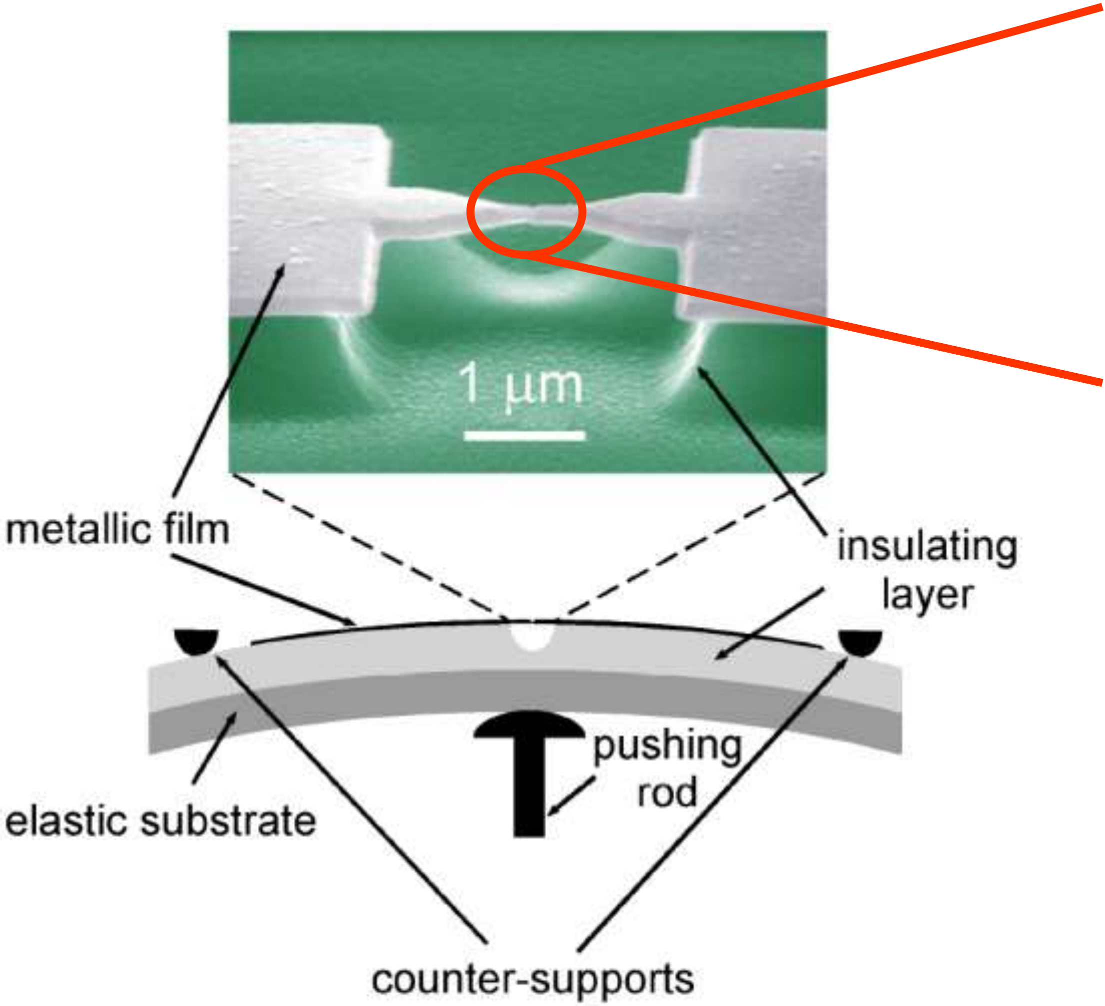
Van Wees et al., Phys. Rev. Lett. **60**, 848 (1988)

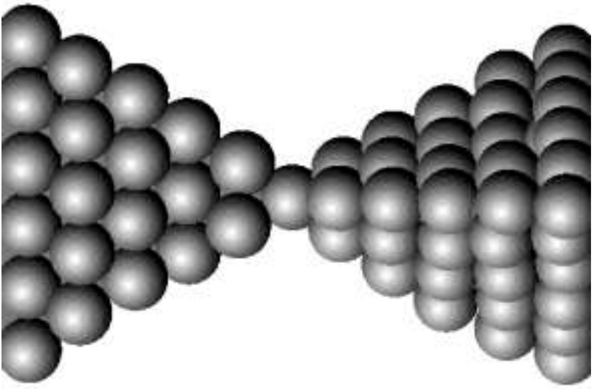


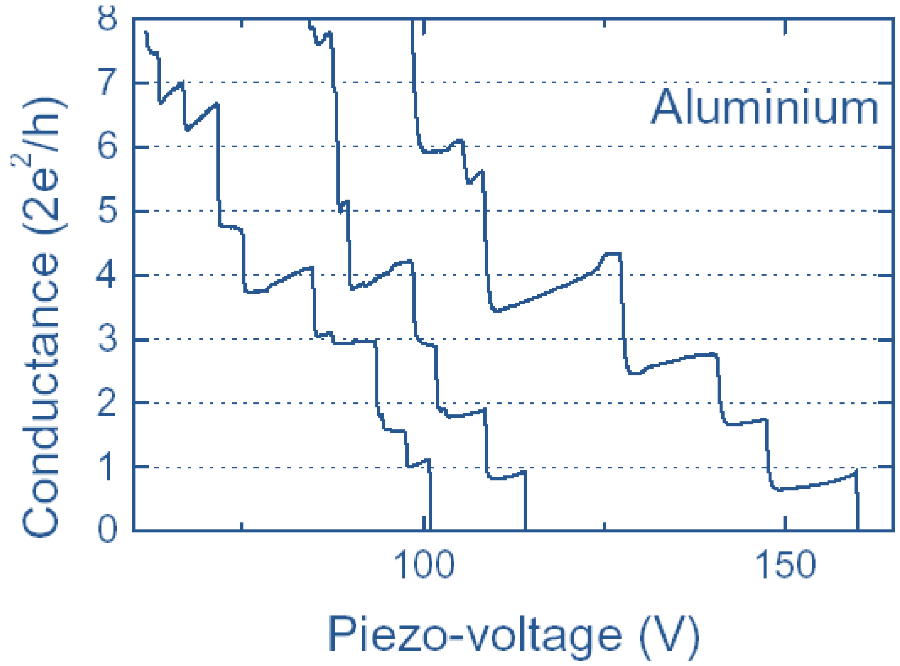


increasing gate voltage narrows channel
→ reduction of N_{open}

II.2.1 Electron Waves and Waveguides





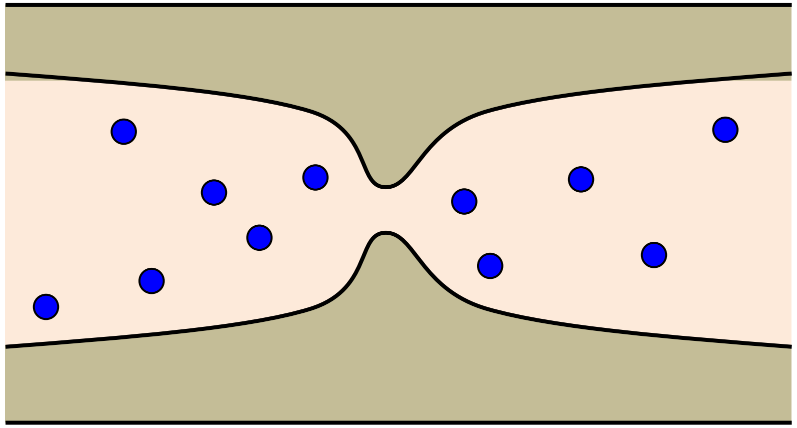


conduction through a single atom !

(Elke Scheer, Univ. Konstanz)

II.2.2 Landauer Formalism

- considered examples have been too simple: ***T* only 1 (open) or 0 (closed)**
- more complicated situation: ***ideal sample + scattering sites***

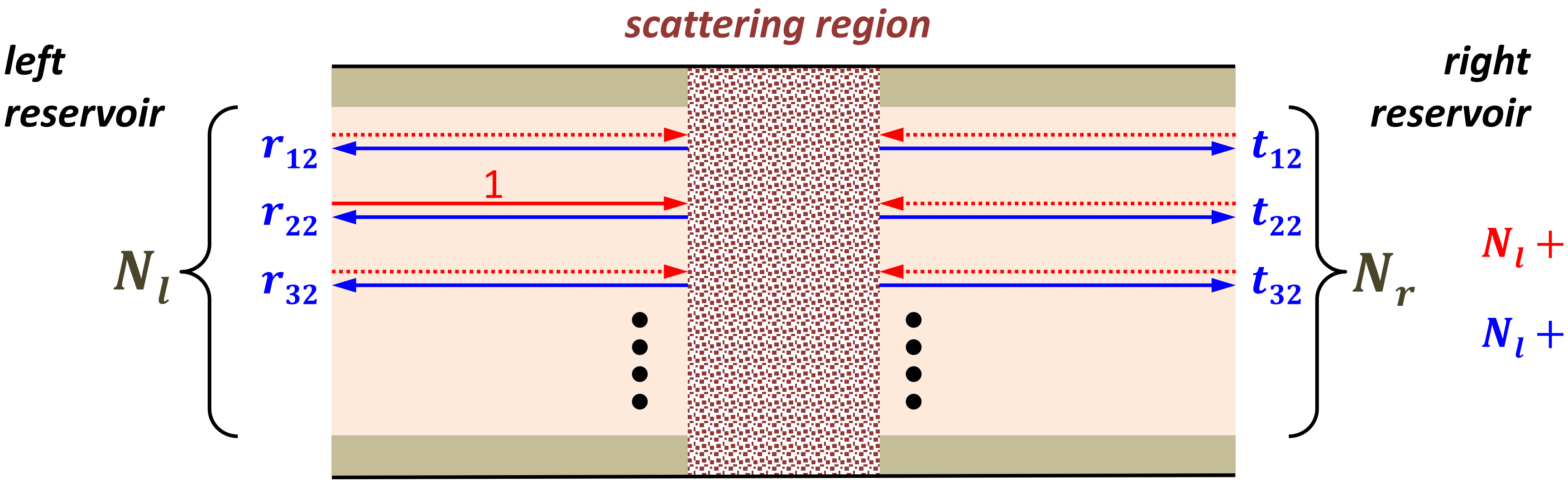


transmission probability
of the different modes
will no longer be only 0 or 1
➔
„dusty waveguide“

$$\mathbf{0 \leq \boldsymbol{T} \leq 1}$$

- T represents the **average probability** that an electron injected at one end will be **transmitted** to the other end
- treatment of the situation by a ***scattering matrix***

II.2.2 Landauer Formalism



$N_l + N_r$ *incoming amplitudes* a_l, a_r

$N_l + N_r$ *outgoing amplitudes* b_l, b_r

$N_l \times N_l$ reflection matrix $\hat{r}$ $N_l \times N_r$ transmission matrix $\hat{t}$

$$\mathbf{b} = \hat{s}\,\mathbf{a} \Rightarrow \begin{bmatrix} \mathbf{b}_l \\ \mathbf{b}_r \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{s}_{ll} & \hat{s}_{lr} \\ \hat{s}_{rl} & \hat{s}_{rr} \end{bmatrix}}_{\text{scattering matrix}} \begin{bmatrix} \mathbf{a}_l \\ \mathbf{a}_r \end{bmatrix} = \begin{bmatrix} \hat{r} & \hat{t}' \\ \hat{t} & \hat{r}' \end{bmatrix} \begin{bmatrix} \mathbf{a}_l \\ \mathbf{a}_r \end{bmatrix}$$

scattering matrix

➔ *relates amplitudes of outgoing waves with those of incoming waves*

transfer matrix $\widehat{M}$:

$$\begin{bmatrix} \mathbf{b}_r \\ \mathbf{a}_r \end{bmatrix} = \widehat{M} \begin{bmatrix} \mathbf{a}_l \\ \mathbf{b}_l \end{bmatrix} = \begin{bmatrix} \hat{t} - \hat{r}'\hat{t}'^{-1}\hat{r} & \hat{r}'\hat{t}'^{-1} \\ -\hat{t}'^{-1}\hat{r}' & \hat{t}'^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{a}_l \\ \mathbf{b}_l \end{bmatrix}$$

relates amplitudes of waves right of the scatterer with those left of the scatterer

➔ **"transfers" states across the scatterer**

II.2.2 Landauer Formalism

- **properties of the scattering matrix**

$$\hat{s} = \begin{bmatrix} \hat{r} & \hat{t}' \\ \hat{t} & \hat{r}' \end{bmatrix}$$

  - for given time reversal symmetry: $\hat{t}^T = \hat{t}'$ ⇨ $\hat{s}^T = \hat{s}$ ***symmetric matrix***

  - electrons do not disappear:

$$\underbrace{\sum_{n'} |r_{nn'}|^2}_{R_n} + \underbrace{\sum_m |t_{mn}|^2}_{T_n = 1 - R_n} = \left(\hat{s}^\dagger \hat{s}\right)_{nn} = 1$$

$$R_n = \left(\hat{r}^\dagger\, \hat{r}\right)_{nn} \qquad T_n = \left(t^\dagger\, \hat{t}\right)_{nn}$$

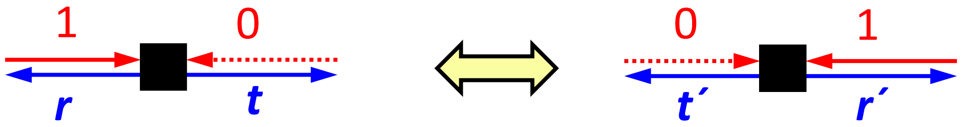
conjugate transpose of $\hat{s}$

⇨ $\hat{s}^\dagger \hat{s} = \hat{1}$ ***unitary matrix***

II.2.2 Landauer Formalism

- **properties of the scattering matrix**

  – example: *one channel scatterer*



$$\begin{bmatrix} \mathbf{b}_l \\ \mathbf{b}_r \end{bmatrix} = \begin{bmatrix} r & t' \\ t & r' \end{bmatrix} \begin{bmatrix} \mathbf{a}_l \\ \mathbf{a}_r \end{bmatrix}$$

r, t, r', t' are *complex numbers*

condition of unitarity

→ *only three independent parameters*

$$\hat{s} = \begin{bmatrix} r & t' \\ t & r' \end{bmatrix} = \begin{bmatrix} \sqrt{R}\, \mathrm{e}^{\imath\theta} & \sqrt{T}\, \mathrm{e}^{\imath\eta} \\ \sqrt{T}\, \mathrm{e}^{\imath\eta} & -\sqrt{R}\, \mathrm{e}^{\imath(2\eta-\theta)} \end{bmatrix}$$

$$R = |r|^2 \; = 1 - |t|^2 = \; 1 \; - \; T$$

follows from condition of unitarity

  – the phases θ and η do not manisfest themselves in transport across a single scatterer
    ➔ lead to **quantum interference effects** in multi-scatterer configurations

II.2.2 Landauer Formalism

- **properties of the scattering matrix:** condition of unitarity: $\hat{S}^{\dagger}\hat{S} = \hat{1}$

$$\begin{bmatrix} \hat{r}^{\star} & \hat{t}'^{\star} \\ \hat{t}^{\star} & \hat{r}'^{\star} \end{bmatrix} \cdot \begin{bmatrix} \hat{r} & \hat{t} \\ \hat{t}' & \hat{r}' \end{bmatrix} = \begin{bmatrix} \overbrace{|r|^2 + |t'|^2}^{=1} & \overbrace{r^{\star}t + t'^{\star}r'}^{=0} \\ \underbrace{t^{\star}r + r'^{\star}t'}_{=0} & \underbrace{|t|^2 + |r'|^2}_{=1} \end{bmatrix} = \hat{1}$$

$$\hat{s} = \begin{bmatrix} r & t' \\ t & r' \end{bmatrix} = \begin{bmatrix} \sqrt{R}\,\mathrm{e}^{\imath\theta} & \sqrt{T}\,\mathrm{e}^{\imath\eta} \\ \sqrt{T}\,\mathrm{e}^{\imath\eta} & -\sqrt{R}\,\mathrm{e}^{\imath(2\eta-\theta)} \end{bmatrix}$$

(i) $r^{\star}t + t'^{\star}r' = 0$

$$\sqrt{R}e^{-i\theta}\sqrt{T}\mathrm{e}^{i\eta} - \sqrt{T}\mathrm{e}^{-i\eta}\sqrt{R}\mathrm{e}^{i(2\eta-\theta)} =$$

$$\sqrt{T\,R}\mathrm{e}^{-i(\theta-\eta)} - \sqrt{T\,R}\,\mathrm{e}^{-i(\theta-\eta)} = 0 \quad !!$$

(ii) $t^{\star}r + r'^{\star}t' = 0$

$$\sqrt{T}\,\mathrm{e}^{-i\eta} \cdot \sqrt{R}\,\mathrm{e}^{i\theta} - \sqrt{R}\,\mathrm{e}^{-i(2\eta-\theta)} \cdot \sqrt{T}\mathrm{e}^{i\eta} =$$

$$\sqrt{T\,R}\,\mathrm{e}^{i(\theta-\eta)} - \sqrt{T\,R}\,\mathrm{e}^{i(\theta-\eta)} = 0 \quad !!$$

II.2.2 Landauer Formalism

- **description of transport properties by scattering matrix**

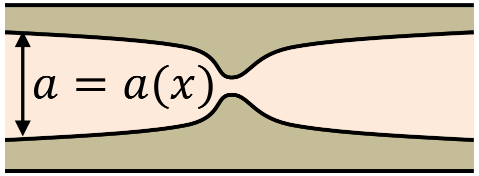
  - expression for the current:

$$I = 2\,e \sum_n \sum_{k_x} v_x(k_x)\, f_n(k_x) = 2e \sum_n \int_{\infty}^{\infty} \frac{dk_x}{2\pi} v_x(k_x)\, f_n(k_x)$$

spin

sum over transport channels

ocuppation probability



  - occupation probabilities for right- and left-moving electrons (for current in the left waveguide):

i. $k_x > 0$: $f_l(\varepsilon)$ (electrons moving to the right)

ii. $k_x < 0$: $R_n f_l(\varepsilon) + (1 - R_n) f_r(\varepsilon)$ (electrons moving to the left)

$$I = 2e \sum_n \left\{ \int_0^\infty \frac{\mathrm{d}k_x}{2\pi} v_x(k_x)\, f_l(\varepsilon) + \int_{-\infty}^{0} \frac{\mathrm{d}k_x}{2\pi} v_x(k_x) \left[R_n f_l(\varepsilon) + (1 - R_n) f_r(\varepsilon)\right] \right\}$$

$$1 - R_n = \sum_m |t_{mn}|^2 = T_n = \left(t^\dagger\, \hat{t}\right)_{nn}$$

$$\sum_n \left(t^\dagger\, \hat{t}\right)_{nn} = \mathrm{Tr}\left[t^\dagger\, \hat{t}\right]$$

$$I = 2e \sum_n \int_0^\infty \frac{\mathrm{d}k_x}{2\pi} v_x(k_x) (1 - R_n)\left[f_l(\varepsilon) - f_r(\varepsilon)\right] \underbrace{=}_{\mathrm{d}k_x = \mathrm{d}\varepsilon/\hbar v_x} \frac{2e}{2\pi\hbar} \int_0^\infty \mathrm{d}\varepsilon\, \mathrm{Tr}\left[t^\dagger\, \hat{t}\right]\left[f_l(\varepsilon) - f_r(\varepsilon)\right]$$

II.2.2 Landauer Formalism

- **description of transport properties by scattering matrix**
  - $\mathrm{Tr}\left[t^\dagger\,\hat{t}\right]$ can be represented by sum of ‚transmission‘ eigenvalues T_p of Hermitian matrix $\hat{t}^\dagger\hat{t}$ (for each energy ε)
  - expression for the current:

$$I = \frac{2e}{2\pi\hbar}\sum_p \int \mathrm{d}\varepsilon\; T_p(\varepsilon)\cdot\left[f_l(\varepsilon)-f_r(\varepsilon)\right] = 2G_Q \underbrace{\sum_p T_p}\cdot V$$

Landauer fomula

can usually assumed to be independent of ε

this gives just the number of open channels, if T_p is either 0 or 1

Rolf Wilhelm (William) Landauer
born 4. February 1927 in Stuttgart
† 27. April 1999 in Briarcliff Manor, N.Y.

Einstein relation ↔ Landauer formula

$$\sigma = 2e^2 N(\varepsilon_\mathrm{F}) D \quad \Leftrightarrow \quad G = 2\frac{e^2}{h} N\, T$$

single spin DOS

diffusion constant

number of modes

transmission probability

Landauer formula → ‘mesoscopic version’ of Einstein relation

II.2.2 Landauer Formalism

- **description of transport properties by scattering matrix: plausibility consideration**
  - consider a conductor with a single conduction channel
  - reservoir biased at V sends out the following number of electrons:

$$N(t) = \underbrace{Z(k)\Delta \mathrm{k}}_{\text{number}} \cdot \underbrace{\frac{1}{\hbar}\frac{\Delta\varepsilon}{\Delta k}}_{\text{velocity}} \cdot \underbrace{t}_{\text{time}} = \frac{\overbrace{2}^{\text{spin}}}{2\pi}\,\Delta k \cdot \frac{eV}{\hbar\Delta k} \cdot t = \frac{2eV}{h} \cdot t$$

emission frequency

  - the chance to pass is T_0, then the passed charge is just $Q(t) = eT_0 N(t)$
  - the average current is charge per time: $I = \frac{Q}{t} = 2\frac{e^2}{h}\,T_0\,V$
  - many channels: just sum up to obtain

$$\boxed{I = 2G_Q \sum_p T_p\,V}$$

II.2.2 Landauer Formalism

- **description of transport properties by scattering matrix: limitations and restrictions**

  - ***restrictions:***

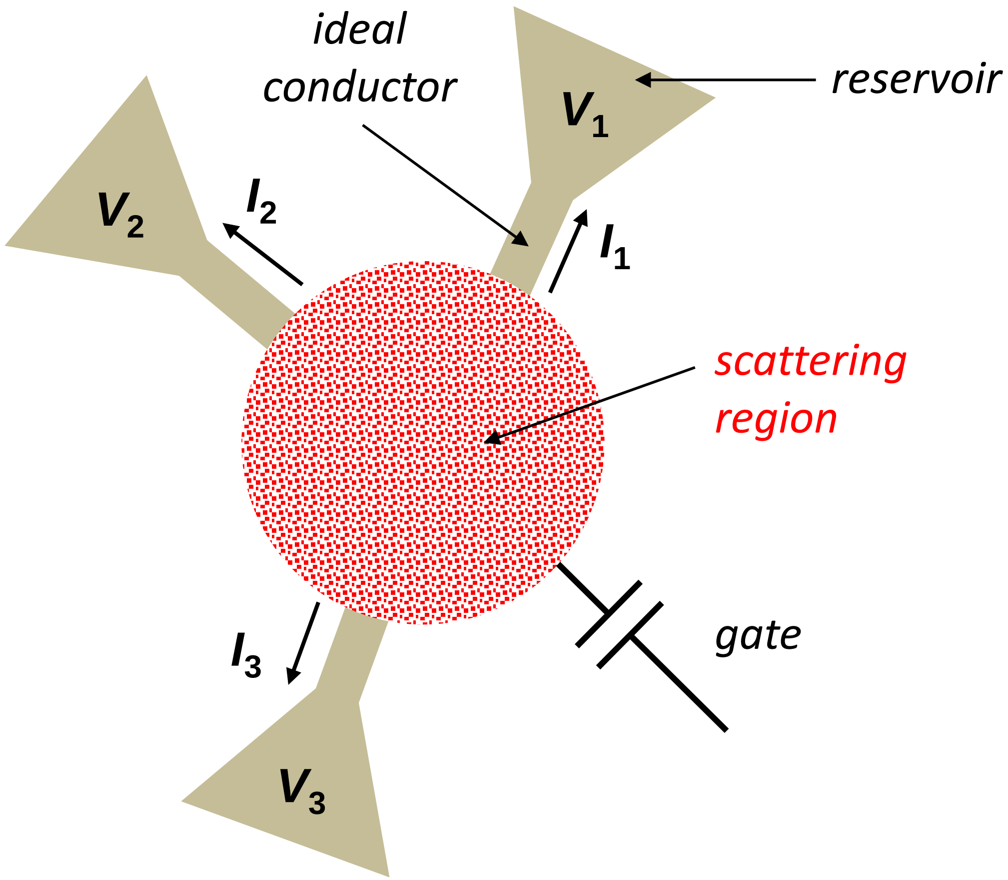
    → only ***elastic*** scattering (electrons pass the conductor at constant energy)
    → ***no interactions*** between electrons

  - ***limitations:***

    → low temperatures and low voltages
    → short conductors (shorter then inelastic scattering length)

II.2.3 Multi-terminal Conductors

- **Landauer formalism: multi-terminal conductors**
  - so far discussion of two-terminal systems, extension to multi-terminal conductors?



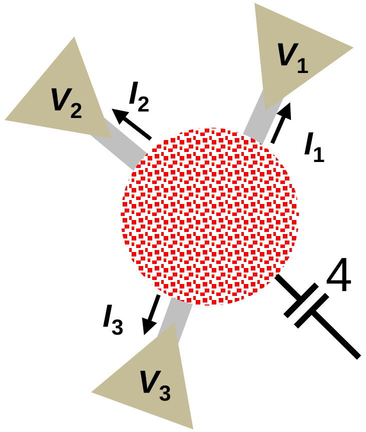
how to express currents in terms of voltages using the Landauer formalism ?

II.2.3 Multi-terminal Conductors

- **Landauer formalism: multi-terminal conductors**

– conduction matrix G_{kl}

$$\begin{pmatrix} I_1 \\ \vdots \\ I_n \end{pmatrix} = \begin{pmatrix} G_{11} & \cdots & G_{1n} \\ \vdots & \ddots & \vdots \\ G_{n1} & \cdots & G_{nn} \end{pmatrix} \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix}$$



$$I_k = \sum_l G_{kl} V_l$$

– properties of conduction matrix:

→ *current conservation (Kirchhoff's law):*

$$\sum_{k=1}^{n} I_k = 0 \quad \Rightarrow \sum_{k=1}^{n} G_{kl} = 0$$

sum of conduction coefficients in each column must be zero

→ *no current, if potential is shifted by the same amount in all leads*

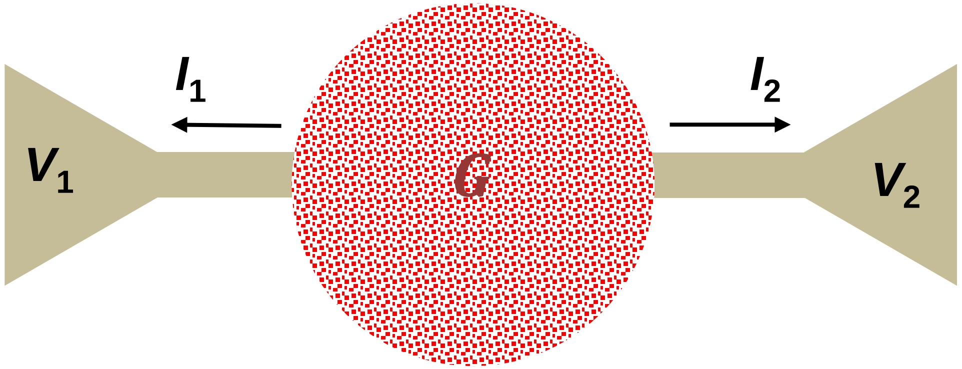
$$\sum_{l=1}^{n} G_{kl} = 0$$

sum of conduction coefficients in each row must be zero

– consequence of the sum rules: currents I_k voltage differences

II.2.3 Multi-terminal Conductors

- **Landauer formalism: multi-terminal conductors**
  - simplest case: two-terminal conductor



  - the conduction matrix only has a ***single independent element***:

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} -G & G \\ G & -G \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$I_1 = G(V_2 - V_1)$$
$$I_2 = G(V_1 - V_2)$$

II.2.3 Multi-terminal Conductors

- **Landauer formalism: multi-terminal conductors**

  – scattering matrix for ***multi-terminal*** conductors

  – number of modes: $N \; = \; N_1 + N_2 + N_3 + \cdots$
    → scattering matrix is $N \times N$ matrix

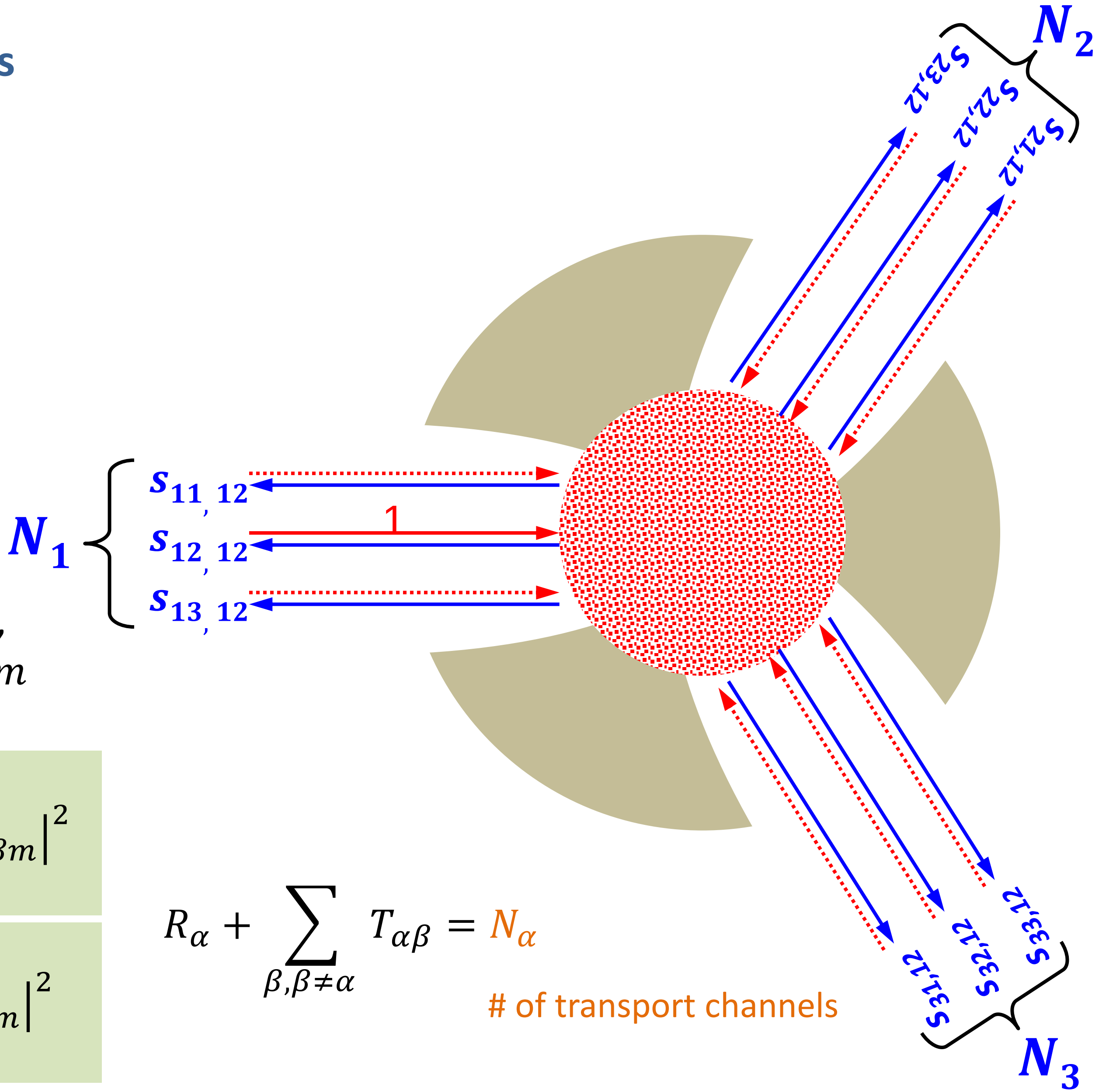
  – meaning of $s_{\beta m,\alpha n}$: $b_{\beta m} = s_{\beta m,\alpha n} \, a_{\alpha n}$
    → propagation amplitude
    from terminal α, transport channel n,
    to the terminal β, transport channel m

  – transmission probability
    from lead α to β:

$$T_{\alpha\beta} = \sum_{n=1}^{N_\alpha} \sum_{m=1}^{N_\beta} \left| s_{\alpha n,\beta m} \right|^2$$

  – reflection probability
    from lead α into α:

$$R_\alpha = \sum_{n=1}^{N_\alpha} \sum_{m=1}^{N_\alpha} \left| s_{\alpha n,\alpha m} \right|^2$$



$$R_\alpha + \sum_{\beta,\beta\neq\alpha} T_{\alpha\beta} = N_\alpha$$

# of transport channels

II.2.3 Multi-terminal Conductors

- **Landauer formalism: multi-terminal conductors**

  – properties of scattering matrix:

    → reflection back into same lead α: $s_{\alpha n,\alpha m}$

    → transmission from lead β to lead α: $s_{\alpha n,\beta m}$

  – current conservation requires $\boxed{\hat{s}^{\dagger}\hat{s} = \hat{1}}$ *(unitary matrix)*

$$\sum_{\alpha n} s^{*}_{\alpha n,\gamma l}\; s_{\alpha n,\beta m} = \delta_{\gamma\beta}\delta_{lm}$$

  – time reversibility relation
    - we know: if $\Psi(\mathbf{r},\mathbf{B})$ solves Schrödinger equation then also $\Psi^{*}(\mathbf{r},-\mathbf{B})$
    - application to asymptotic scattering states: taking complex conjugate of scattering state b, then the incoming state a becomes the complex conjugate of b^{*} ➔ corresponds to reversal of time direction

$$\left.\begin{array}{l} b = s(\mathbf{B})a \quad \Rightarrow \quad b^{*} = s^{*}(\mathbf{B})\, a^{*} \\ a^{*} = s(-\mathbf{B})b^{*} \quad \Rightarrow s^{-1}(-\mathbf{B})a^{*} = s^{-1}(-\mathbf{B})s(-\mathbf{B})b^{*} \;\Rightarrow s^{-1}(-\mathbf{B})a^{*} = b^{*} \end{array}\right\} \Rightarrow \; s^{-1}(-\mathbf{B}) = s^{*}(\mathbf{B}) = s^{\dagger}(\mathbf{B})$$

due to unitarity

$$\boxed{s_{\alpha n,\beta m}(\mathbf{B}) = s_{\beta m,\alpha n}(-\mathbf{B})}$$

II.2.3 Multi-terminal Conductors

- **Landauer formalism: multi-terminal conductors**
  - sum rules:

$$R_\alpha + \sum_{\beta,\beta\neq\alpha} T_{\alpha\beta} = N_\alpha$$

\# of transport channels in lead α

$$R_\beta + \sum_{\alpha,\alpha\neq\beta} T_{\beta\alpha} = N_\beta$$

\# of transport channels in lead β

$$T_{\alpha\beta} = \sum_{n=1}^{N_\alpha} \sum_{m=1}^{N_\beta} \left|s_{\alpha n,\beta m}\right|^2$$

$$R_\alpha = \sum_{n=1}^{N_\alpha} \sum_{m=1}^{N_\alpha} \left|s_{\alpha n,\alpha m}\right|^2$$

  - example: ***two-terminal conductor***

	$\beta = 1$	$\beta = 2$	$\sum =$
$\alpha = 1$	R_1	T_{12}	N_1
$\alpha = 2$	T_{21}	R_2	N_2
$\sum =$	N_1	N_2	

$$R_1 + T_{12} = R_1 + T_{21} \quad \Rightarrow \quad T_{12} = T_{21}$$

transmission function is reciprocal !
→ time reversal symmetry

II.2.3 Multi-terminal Conductors

- **Landauer formalism: multi-terminal conductors**
  - multi-terminal expression of Landauer formula relates currents to voltages via a scattering matrix (cf. page 42)

$$I_\alpha = 2e \sum_n \left\{ \int_0^\infty \frac{\mathrm{d}k_x}{2\pi} v_x(k_x)\, f_\alpha(\varepsilon) + \int_{-\infty}^{0} \frac{\mathrm{d}k_x}{2\pi} v_x(k_x) \sum_{\beta m} \left|s_{\alpha n,\beta m}\right|^2 f_\beta(\varepsilon) \right\}$$

$$I_\alpha = 2e \sum_n \int_0^\infty \frac{\mathrm{d}k_x}{2\pi} v_x(k_x) \sum_{\beta m} \left\{ \left|s_{\alpha n,\beta m}\right|^2 - \delta_{\alpha\beta}\delta_{mn} \right\} f_\beta(\varepsilon) \underbrace{=}_{\mathrm{d}k_x = \mathrm{d}\varepsilon/\hbar v_x} \underbrace{\frac{2e}{2\pi\hbar}}_{=G_Q/e} \int_0^\infty \mathrm{d}\varepsilon \sum_{\beta mn} \left\{ \left|s_{\alpha n,\beta m}\right|^2 - \delta_{\alpha\beta}\delta_{mn} \right\} f_\beta(\varepsilon)$$

  - probability for transmission from α to β:

$$I_\alpha = -\frac{G_Q}{e} \int_0^\infty \mathrm{d}\varepsilon \sum_\beta \mathrm{Tr} \left\{ \delta_{\alpha\beta}\delta_{mn} - \hat{s}_{\alpha\beta}^\dagger \hat{s}_{\alpha\beta} \right\} f_\beta(\varepsilon)$$

- trace includes all possible transport channels

- if all $f_\beta(\varepsilon)$ are the same, e.g. in thermal equilibrium and no voltages applied, then $\sum_\beta = 0$ (current conservation, *follows from unitarity*)

  - we apply voltage V_γ to terminal γ and keep all other at ε_F ➔ the only surviving term in $\sum_\beta$ is the one for $\beta = \gamma$ and the integral yields eV_γ

$$I_\alpha = -\frac{G_Q}{e} \mathrm{Tr} \left\{ \delta_{\alpha\gamma}\delta_{mn} - \hat{s}_{\alpha\gamma}^\dagger \hat{s}_{\alpha\gamma} \right\} eV_\gamma = G_{\alpha\gamma}\, V_\gamma \quad \Longrightarrow \quad G_{\alpha\gamma} = -G_Q\, \mathrm{Tr} \left\{ \delta_{\alpha\gamma}\delta_{mn} - \hat{s}_{\alpha\gamma}^\dagger \hat{s}_{\alpha\gamma} \right\}$$

multi-terminal Landauer formula

II.2.3 Multi-terminal Conductors

- **Summary: Landauer formalism: multi-terminal conductors**

  - linear transport regime:

$$G_{\alpha\gamma} = -G_Q \,\mathrm{Tr}\left\{\delta_{\alpha\gamma}\delta_{mn} - \hat{s}_{\alpha\gamma}^{\dagger}\hat{s}_{\alpha\gamma}\right\}$$

  - relation to two-terminal expression: $\alpha, \gamma = l, r$

$$G_{lr} = G_Q \,\mathrm{Tr}\left\{\hat{s}_{lr}^{\dagger}\hat{s}_{lr}\right\} = G_Q \mathrm{Tr}\left[\hat{t}^{\dagger}\,\hat{t}\right]$$

  - time reversal symmetry:

$$G_{\alpha\gamma}(\mathbf{B}) = G_{\gamma\alpha}(-\mathbf{B})$$

this is in agreement with Onsager symmetry relations !

II.2.3 Multi-terminal Conductors

- **Landauer formalism: multi-terminal conductors**

  – example: **three-terminal scattering element**

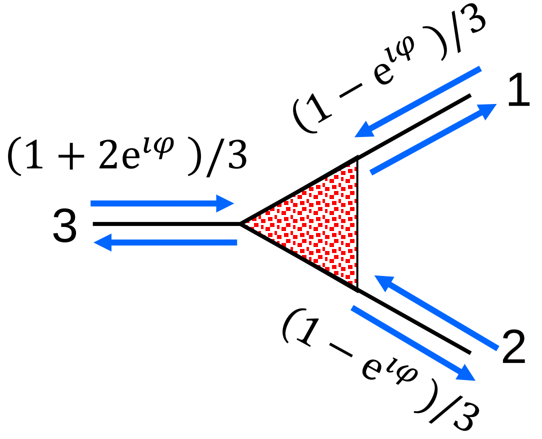
  scattering matrix for fully symmetric beam splitter:

$$\hat{s}_{\mathrm{BS}} = \frac{1}{3}\begin{pmatrix} 1+2\mathrm{e}^{\imath\varphi} & 1-\mathrm{e}^{\imath\varphi} & 1-\mathrm{e}^{\imath\varphi} \\ 1-\mathrm{e}^{\imath\varphi} & 1+2\mathrm{e}^{\imath\varphi} & 1-\mathrm{e}^{\imath\varphi} \\ 1-\mathrm{e}^{\imath\varphi} & 1-\mathrm{e}^{\imath\varphi} & 1+2\mathrm{e}^{\imath\varphi} \end{pmatrix}$$

diagonal elements: $R = |1+2\mathrm{e}^{\imath\varphi}|^2 = [5+4\cos\varphi]/9$

$R = 1$ for $\varphi = 0$ (total reflection)

$R = 1/3$ for $\varphi = \pi$ (equal division)



fully symmetric ideal beam splitter

II.2.3 Multi-terminal Conductors

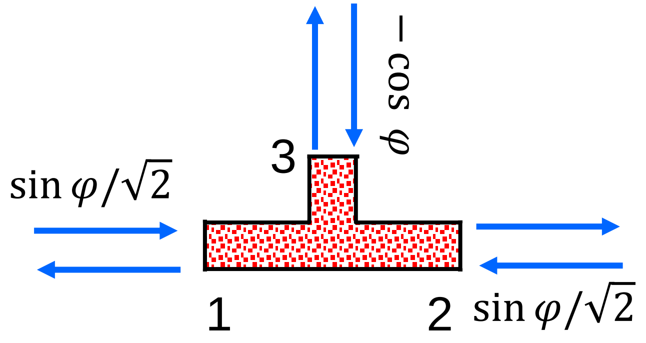
- **Landauer formalism: multi-terminal conductors**

  – example: three-terminal scattering element

scattering matrix:

$$\hat{s}_{\mathrm{BS}} = \begin{pmatrix} -\sin^2\left(\frac{\varphi}{2}\right) & \cos^2\left(\frac{\varphi}{2}\right) & \sin\left(\frac{\varphi}{2}\right) \\ \cos^2\left(\frac{\varphi}{2}\right) & -\sin^2\left(\frac{\varphi}{2}\right) & \sin\left(\frac{\varphi}{2}\right) \\ \sin\left(\frac{\varphi}{2}\right) & \sin\left(\frac{\varphi}{2}\right) & -\cos\varphi \end{pmatrix}$$

T-type symmetric ideal beam splitter



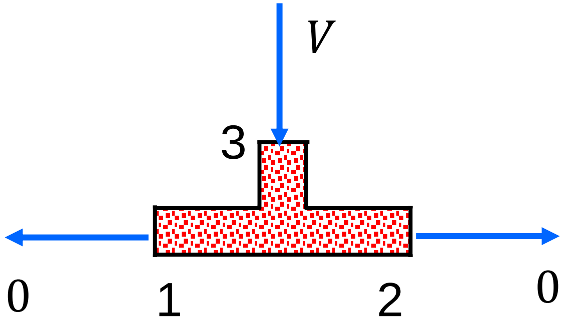
for $\varphi = \pi/2$:

$$\hat{s}_{\mathrm{BS}} = \begin{pmatrix} -1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & -1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}$$

$$G_{\alpha\beta} = -G_Q \,\mathrm{Tr}\left\{\delta_{\alpha\beta}\delta_{mn} - \hat{s}^{\dagger}_{\alpha\beta}\hat{s}_{\alpha\beta}\right\}$$

conductance matrix:
(for $\varphi = \pi/2$)

$$G_{\alpha\beta} = G_Q \begin{pmatrix} -3/4 & 1/4 & 1/2 \\ 1/4 & -3/4 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$$



example: $I_3 = G_Q V, \;\; I_1 = I_2 = -G_Q V/2$

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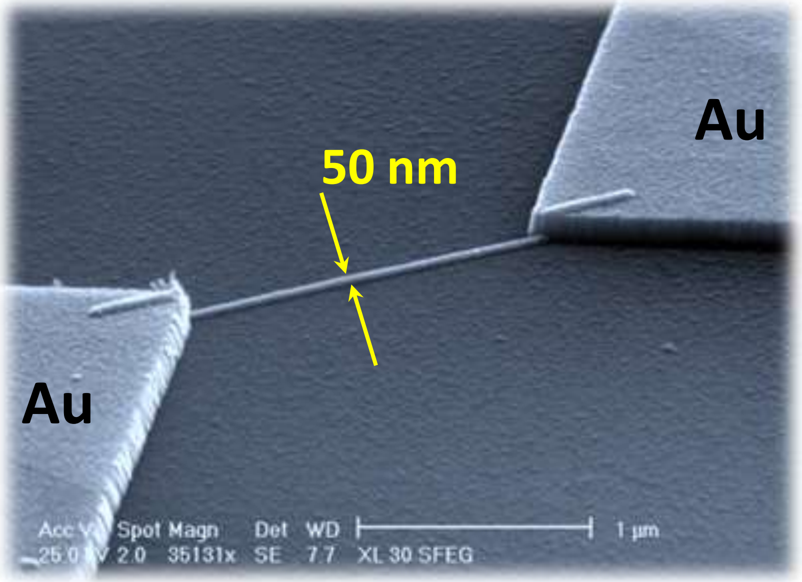


Lecture No. 9

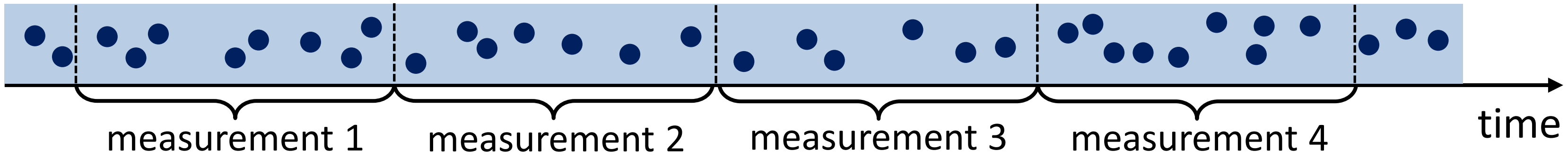
R. Gross

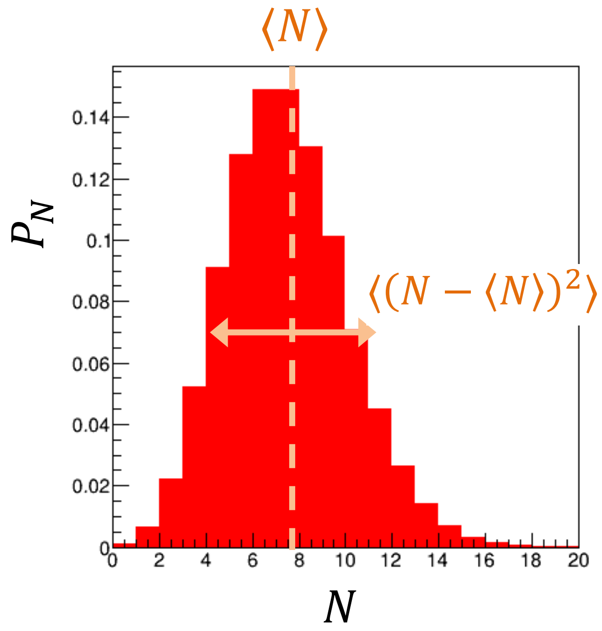
II.2.4 Statistics of Charge Transport

- **Landauer formalism: counting electrons**

  – electron transfer is stochastic process

  ➔ ***measured number of electrons transferred in time interval Δt is random***





  – **important aspects**:

  i. averaging allows to get rid of fluctuations of individual measurements

  ii. study of statistics provides additional information on transport through nanostructure

  – probability P_N to count N electrons:

$$\sum_N P_N = 1$$

normalization of distribution

$$\langle N \rangle = \sum_N N \, P_N$$

average number (1st cumulant)

$$\langle (N - \langle N \rangle)^2 \rangle = \sum_N N^2 \, P_N - \left(\sum_N N \, P_N \right)^2$$

variance (2nd cumulant)

II.2.4 Statistics of Charge Transport

- **cumulant generation function of random variable N**

$$K_N(t) = \ln\langle \mathrm{e}^{tN}\rangle = \sum_{n=1}^{\infty} \kappa_n \frac{t^n}{n!} = \kappa_1\, t + \kappa_2^2 \frac{t^2}{2} + \cdots$$

$\kappa_1 = \mu = \langle N\rangle$ average value

$\kappa_2^2 = \sigma^2 = \langle (N - \langle N\rangle)^2\rangle$ variance

$$\langle \mathrm{e}^{\imath tN}\rangle = \sum_N P_N\, \mathrm{e}^{\imath tN}$$

(Fourier transform of the probability density function)

k^{th} cumulant: differentiate expansion k-times with respect to t and evaluate result at $t = 0$

$$\kappa_n = \frac{\partial^n}{\partial t^n} K_N(t)\,\Big|_{t=0}$$

example: 1st cumulant

$$\frac{\partial}{\partial t} K_N(t)\,\Big|_{t=0} = \kappa_1 + \kappa_2^2 t + \cdots \Big|_{t=0} \underbrace{=}_{t=0} \kappa_1 = \langle N\rangle$$

- **characteristic function of random variable N**

$$H(t) = \ln\langle \mathrm{e}^{\imath tN}\rangle = \sum_{n=1}^{\infty} \kappa_n \frac{(\imath t)^n}{n!} = \kappa_1\, \imath t \; - \kappa_2^2 \frac{t^2}{2} + \cdots$$

$\kappa_1 = \mu = \langle N\rangle$ average value

$\kappa_2 = \sigma^2 = \langle (N - \langle N\rangle)^2\rangle$ variance

$$\langle \mathrm{e}^{\imath tN}\rangle = \sum_N P_N\, \mathrm{e}^{\imath tN}$$

(Fourier transform of the probability density function)

k^{th} cumulant: differentiate expansion k-times with respect to $\imath t$ and evaluate result at $t = 0$

$$\kappa_n = \frac{1}{\imath^n} \frac{\partial^n}{\partial t^n} H_N(t)\,\Big|_{t=0}$$

example: 1st cumulant

$$\frac{1}{\imath} \frac{\partial}{\partial t} H_N(t)\,\Big|_{t=0} = \kappa_1 + \imath \kappa_2^2 t + \cdots \Big|_{t=0} \underbrace{=}_{t=0} \kappa_1 = \langle N\rangle$$

II.2.4 Statistics of Charge Transport

- **characteristic function (1)**

k^{th} cumulant: differentiate expansion k-times with respect to $\imath t$ and evaluate result at $t = 0$

$$\kappa_n = \frac{1}{\imath^n}\frac{\partial^n}{\partial t^n} H_N(t)\,\Big|_{t=0}$$

example: 1st cumulant

$$\frac{1}{\imath}\frac{\partial}{\partial t} H_N(t)\,\Big|_{t=0} = \kappa_1 + \imath\kappa_2^2 t + \cdots\Big|_{t=0} \underbrace{=}_{t=0} \kappa_1 = \langle N\rangle$$

  - use of **characteristic function** $H_N(t) = \ln\langle \mathrm{e}^{\imath tN}\rangle = \ln \sum_N P_N\, \mathrm{e}^{\imath tN}$

    ($\langle \mathrm{e}^{\imath tN}\rangle$ = Fourier transform of the probability density function)

  - application to statistics of electron transfer:
    - we assume large measurement time Δt so that $\langle Q\rangle = \langle I\rangle \Delta t \gg e$
    - we divide Δt into very small intervals $\mathrm{d}t$ so that $\langle Q\rangle = \langle I\rangle \mathrm{d}t \ll e$

      ➔ probability to transfer one electron within $\mathrm{d}t$: **$\Gamma\,\mathrm{d}t \ll 1$ (Γ = transfer rate)**

      ➔ probability to transfer no electron within $\mathrm{d}t$: $1 - \Gamma\,\mathrm{d}t$
    - we assume that all electrons move in the same direction
    - we neglect probability to transfer two (or more) electrons within $\mathrm{d}t$ ($\mathcal{O}(\Gamma \mathrm{d}t)^2$))

$$\langle \mathrm{e}^{\imath tN}\rangle_{N,\mathrm{d}t} = \textstyle\sum_N P_N\,\mathrm{e}^{\imath tN} = \underbrace{(1-\Gamma\,\mathrm{d}t\,)}_{N=0} + \underbrace{(\Gamma\,\mathrm{d}t)\mathrm{e}^{\imath t}}_{N=1} + \cdots = 1 + \underbrace{\Gamma\,\mathrm{d}t(\mathrm{e}^{\imath t}-1)}_{\ll 1} + \cdots \simeq \exp[\Gamma\,\mathrm{d}t(\mathrm{e}^{\imath t}-1)]$$

$$\langle \mathrm{e}^{\imath tN}\rangle_{N,\Delta t} \underbrace{=}_{\text{independent events}} \left[\Pi_{N,\mathrm{d}t}(t)\right]^{\Delta t/\mathrm{d}t} = \{\exp[\Gamma\,\mathrm{d}t(\mathrm{e}^{\imath t}-1)]\}^{\Delta t/\mathrm{d}t} = \exp\left[\underbrace{\Gamma\,\Delta t}_{=\bar{N}}(\mathrm{e}^{\imath t}-1)\right] = \exp[\bar{N}(\mathrm{e}^{\imath t}-1)]$$

$$P_N = \int_0^{2\pi} \frac{\mathrm{d}t}{2\pi}\langle \mathrm{e}^{\imath tN}\rangle_{N,\Delta t}\,\mathrm{e}^{-\imath Nt} \simeq \int_0^{2\pi}\frac{\mathrm{d}t}{2\pi}\,\mathrm{e}^{[\bar{N}(\mathrm{e}^{\imath t}-1)]}\mathrm{e}^{-\imath Nt} = \frac{\bar{N}^N}{N!}\mathrm{e}^{-\bar{N}\Delta t}$$

Poisson distribution

individual transfer processes are not correlated since $\Gamma\,\mathrm{d}t \ll 1$

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II.2.4 Statistics of Charge Transport

- **characteristic function (2)**

  - opposite example: ideally transmitting channel
    - since there is no scattering, the total momentum of all electrons does not change ➔ current does not fluctuate

$$P_N = \delta(N - \bar{N}) \qquad \Rightarrow \langle \mathrm{e}^{\imath t N} \rangle = \sum_N P_N\, \mathrm{e}^{\imath t N} = \mathrm{e}^{\imath t (N - \bar{N})}$$

  - intermediate case: $0 < T_p < 1$:
    - the transmitted electrons are correlated, but not fully
    - characteristic function is given by **Levitov formula**

L. S. Levitov and G. B. Lesovik, JETP Lett. **58**, 230 (1993)

$$\ln\langle \mathrm{e}^{\imath t N} \rangle = 2\,\Delta t \int \frac{\mathrm{d}\varepsilon}{2\pi\hbar} \sum_p \ln\left\{1 + T_p(\mathrm{e}^{\imath t} - 1)\, f_l(\varepsilon)[1 - f_r(\varepsilon)] + T_p(\mathrm{e}^{-\imath t} - 1) f_r(\varepsilon)[1 - f_l(\varepsilon)]\right\}$$

note:

- the transfer processes from left to right and vice versa are correlated
- for $f_l(\varepsilon) = f_r(\varepsilon) = 1$, the total current is zero
  ➔ if there are no correlations, there would be current fluctuations
  ➔ electrons moving left are blocked by electrons filling the state and vice versa

  - limiting case: $k_\mathrm{B} T \ll eV$: integral over energy gives eV

$$\ln\langle \mathrm{e}^{\imath t N} \rangle = \pm \frac{2eV\Delta t}{2\pi\hbar} \sum_p \ln\left\{1 + T_p(\mathrm{e}^{\pm \imath t} - 1)\right\}$$

± for different sign of voltage

II.2.4 Statistics of Charge Transport

- **calculation of cumulants**
  - starting point is Levitov formula

$$\ln\langle \mathrm{e}^{\imath t N}\rangle = 2\,\Delta t \int \frac{\mathrm{d}\varepsilon}{2\pi\hbar} \sum_p \ln\{1 + T_p(\mathrm{e}^{\imath t} - 1)\, f_l(\varepsilon)[1 - f_r(\varepsilon)] + T_p(\mathrm{e}^{-\imath t} - 1) f_r(\varepsilon)[1 - f_l(\varepsilon)]\}$$

  - **1st cumulant:** $\langle N\rangle = \left.\frac{\partial \ln\langle \mathrm{e}^{\imath t N}\rangle}{\partial(\imath t)}\right|_{t=0} = \frac{2eV\Delta t}{2\pi\hbar} \sum_p \int \mathrm{d}\varepsilon\, T_p(\varepsilon)[f_l(\varepsilon) - f_r(\varepsilon)]$

$$I = e\frac{\langle N\rangle}{\Delta t} = \frac{2eV}{2\pi\hbar} \sum_p \int \mathrm{d}\varepsilon\, T_p(\varepsilon)[f_l(\varepsilon) - f_r(\varepsilon)]$$

Landauer fomula

  - **2nd cumulant:** $\langle (N - \langle N\rangle)^2\rangle = \left.\frac{\partial^2 \ln\langle \mathrm{e}^{\imath t N}\rangle}{\partial(\imath t)^2}\right|_{t=0}$

$$\langle (N - \langle N\rangle)^2\rangle = \frac{2e\Delta t}{2\pi\hbar} \sum_p \int \mathrm{d}\varepsilon \left\{T_p(\varepsilon)\left[f_l(\varepsilon)\big(1 - f_l(\varepsilon)\big) + f_r(\varepsilon)\big(1 - f_r(\varepsilon)\big)\right] + T_p(\varepsilon)\left(1 - T_p(\varepsilon)\right)\left(f_l(\varepsilon) - f_r(\varepsilon)\right)^2\right\}$$

Case 1: equilibrium: $V = 0\ \left(f_l(\varepsilon) = f_r(\varepsilon)\right)$:

$$\langle (Q - \langle Q\rangle)^2\rangle_{\mathrm{eq}} = \underbrace{\frac{2e^2\Delta t}{2\pi\hbar}}_{=2G_Q\Delta t} k_\mathrm{B}T \underbrace{\sum_p T_p}_{=1} = 2G_Q k_\mathrm{B} T\Delta t$$

II.2.4 Statistics of Charge Transport

$$S(\omega) = 2\int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{e}^{-\imath\omega\tau}\langle I(t)I(t+\tau)\rangle$$

- **Nyquist-Johnson noise**
  - interpretation of result $\langle(Q-\langle Q\rangle)^2\rangle_{\mathrm{eq}} = 2G_Q k_\mathrm{B} T\Delta t$
    - ➢ if Δt is large enough, variance of the transmitted charge can be interpreted as zero-frequency current noise with $\Delta I = \Delta Q/\Delta t$ we obtain the current fluctuation $\Delta I^2 = \langle(Q-\langle Q\rangle)^2\rangle_{\mathrm{eq}}/\Delta t^2 = 2G_Q k_\mathrm{B} T/\Delta t$
    - ➢ with the current noise power spectral density $S_I(0) = \Delta I^2 2\Delta t = \frac{\Delta I^2}{\mathrm{BW}}$, we obtain

$$S_I(0) = 4G_Q k_\mathrm{B} T$$

Nyquist-Johnson noise

  - **Wiener–Khinchin theorem**: relates the autocorrelation function $AC_I(\tau)$ to the power spectral density $S_I(\omega)$

$$AC_I(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_I(\omega)\,\mathrm{e}^{\imath\omega\tau}\,\mathrm{d}f \qquad S_I(\omega) = \int_{-\infty}^{\infty} AC_I(\tau)\,\mathrm{e}^{-\imath\omega\tau}\,\mathrm{d}\tau$$

$$AC_I(\tau) = \langle I(t)\hat{I}^*(t+\tau)\rangle = \lim_{\Delta t\to\infty}\int_{-\Delta t}^{+\Delta t} \mathrm{d}t\, I(t)\hat{I}^*(t+\tau)$$

II.2.4 Statistics of Charge Transport

- **shot noise**

$$\langle (N-\langle N\rangle)^2\rangle = \frac{2e\Delta t}{2\pi\hbar}\sum_p \int \mathrm{d}\varepsilon \left\{T_p(\varepsilon)[f_l(\varepsilon)(1-f_l(\varepsilon)) + f_r(\varepsilon)(1-f_r(\varepsilon))] + T_p(\varepsilon)\left(1-T_p(\varepsilon)\right)\left(f_l(\varepsilon)-f_r(\varepsilon)\right)^2\right\}$$

Case 2: $\mathrm{e}V \gg k_\mathrm{B}T$ ➔ only 2nd term on rhs survises (we assume $T_p(\varepsilon) = const.$)

$$\langle (Q-\langle Q\rangle)^2\rangle_{\mathrm{e}V\gg k_\mathrm{B}T} = 4eG_Q V\Delta t \sum_p T_p(1-T_p)$$

with $S_I = \langle (Q-\langle Q\rangle)^2\rangle_\mathrm{eq}/\Delta t^2$

$$S_I(\omega) = 4eG_Q V \sum_p T_p(1-T_p)$$

with $\langle I\rangle = 2G_Q V \sum_p T_p$

$$S_I(\omega) = 2e\langle I\rangle \left[\frac{\sum_p T_p(1-T_p)}{\sum_p T_p}\right]$$

Schottky expression

$F = 1$

no correlations in transmission: Poisson process

Fano factor

$\mathbf{0 \le F \le 1}$

takes into account correlations in the transmission processes

ideal quantum point contact:
only open ($T_p = 1$) or closed ($T_p = 0$) channels
➔ no shot noise !

Contents Part II: Quantum Transport in Nanostructures

Contents:

II.3 Quantum Interference Effects

charge carriers are phase coherent if $L_\varphi > L$

- low temperatures (→ L_φ gets large), nanoscale samples (*L* gets small)
  - interference of multiply scattered charge carriers
  - *corrections to the classical conductance*

- *macroscopic and mesoscopic* samples:

  weak localization (WL)

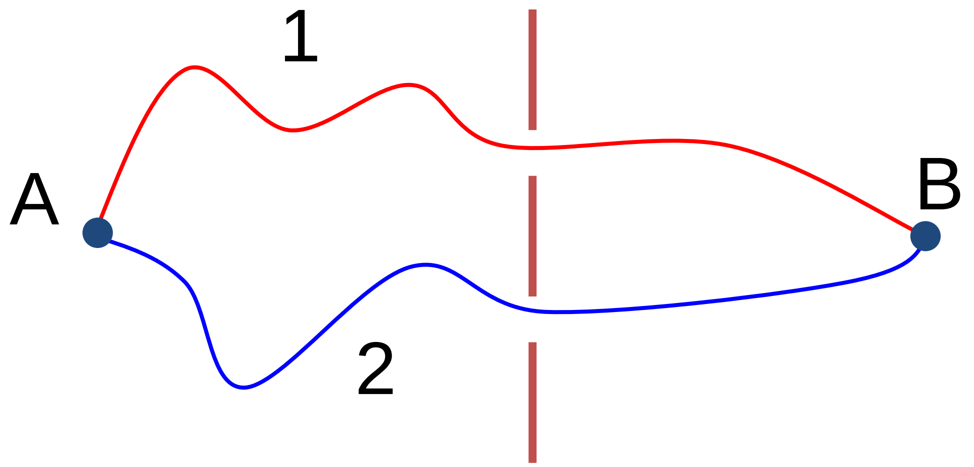
- *mesoscopic* samples:

  Aharonov-Bohm (AB) oscillations

  Universal Conductance Fluctuations (UCFs)

II.3.1 Double Slit Experiment

- **effect of quantum coherence: transmission through double slit**



- basic quantum mechanics: ***double slit experiment***
- probability of propagation from point A to point B:

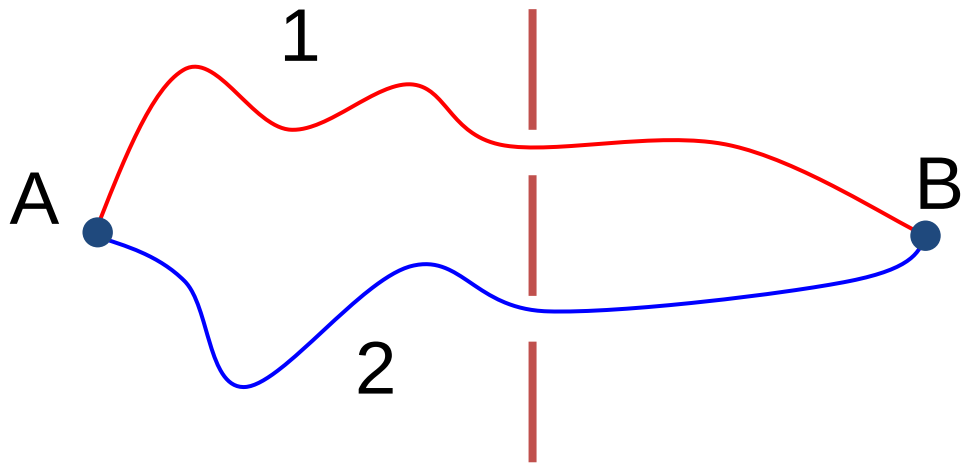
$$P_{\mathrm{AB}} = |A_1 + A_2|^2 = \underbrace{|A_1|^2}_{=P_1} + \underbrace{|A_2|^2}_{=P_2} + \underbrace{A_1 A_2^* + A_1^* A_2}_{2\mathrm{Re}\,[A_1 A_2^*]}$$

classical result

interference term: quantum mechanical

II.3.1 Double Slit Experiment

- **effect of quantum coherence: transmission through double slit**



$$P_{\mathrm{AB}} = P_{\mathrm{classical}} + 2\sqrt{P_1 P_2}\cos\varphi$$

interference terms may be
destructive or ***constructive***

→ depends on phase shift φ

$P_1 = P_2$: $\quad P_{\mathrm{AB}} = 2P + 2P\cos\varphi = 2P(1+\cos\varphi)$

problem:
calculate phase shift φ as a function of geometry, electric potential, magnetic field, …

II.3.1 Double Slit Experiment

- phase shifts
  - **geometric phase**:

definition: geometric phase = phase difference acquired when a system is subjected to cyclic adiabatic processes, which results from the geometrical properties of the parameter space of the Hamiltonian
➔ geometric phase occurs when system parameters are changed very slowly (adiabatically), and eventually brought back to the initial configuration

example: in quantum mechanics, this could involve rotations but also translations of particles, which are apparently undone at the end

important:
one might expect that the waves in the system return to the initial state (characterized by amplitude and phase). However, if the parameter excursions correspond to a loop instead of a self-retracing back-and-forth variation, then it is possible that the initial and final states differ in their phases. This phase difference is the geometric phase, and its occurrence typically indicates that the system's parameter dependence is singular (its state is undefined) for some combination of parameters.

- we consider eigenstate $\psi(t_0) = |m[\mathbf{R}(t_0)]\rangle$ with energy $\varepsilon_m(t_0)$ and change system parameter $\mathbf{R}$ adiabatically along path Γ

$$\boxed{\gamma_m(t) = \imath \int_{t_0,\Gamma}^{t} \underbrace{\langle m[\mathbf{R}(t)]|\nabla_{\mathbf{R}}|m[\mathbf{R}(t)]\rangle}_{\substack{\mathbf{A}_m(\mathbf{R}) \\ \text{Mead}-\text{Berry Vector Potential}}} \cdot \mathrm{d}\mathbf{R}}$$

geometric phase or Berry phase

II.3.1 Double Slit Experiment

- **phase shifts**
  - **dynamical phase** (phase shift due to energy or potential):

we consider wave function of electron in semi-classical approximation

local wave vector at position x

$$\psi(x) = \exp[\imath\varphi(x)] = \exp[\imath k(x)x] \Longrightarrow \frac{\mathrm{d}\varphi}{\mathrm{d}x} = k(x) = \sqrt{2m[\varepsilon - V(x)]}\,/\hbar$$

(i) $V(x) = const.$

$$\Delta\varphi = \int_A^B \frac{\mathrm{d}\varphi}{\mathrm{d}x}\,\mathrm{d}x = \int_A^B k(x)\,\mathrm{d}x = \varphi(B) - \varphi(A) \underbrace{=}_{V(x)=const} kL$$

usually, absolute value of phase is not interesting, but the relative phase shift between different paths

(ii) $V(x) \neq const.$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}k} = \frac{\mathrm{d}}{\mathrm{d}k}\int \frac{\mathrm{d}\varphi}{\mathrm{d}x}\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}k}\int k\,\mathrm{d}x = \int \mathrm{d}x$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\varepsilon} = \frac{\mathrm{d}\varphi}{\mathrm{d}k}\underbrace{\frac{\mathrm{d}k}{\mathrm{d}\varepsilon}}_{v=\frac{1}{\hbar}\frac{\mathrm{d}\varepsilon}{\mathrm{d}k}} = \frac{\mathrm{d}\varphi}{\mathrm{d}k}\frac{1}{\hbar v(x)} = \int_A^B \frac{\mathrm{d}x}{\hbar v(x)} = \int_{t_A}^{t_B} \frac{\mathrm{d}t}{\hbar} = \frac{\tau}{\hbar}$$

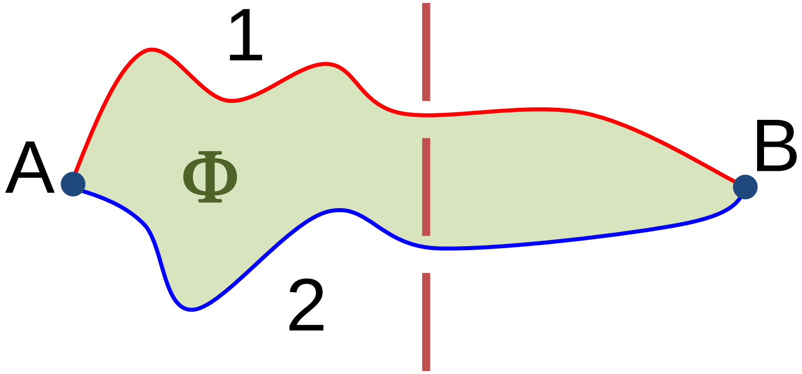
time of flight between points A and B at energy ε

$$\Delta\varphi = \frac{\mathrm{d}\varphi}{\mathrm{d}\varepsilon}\Delta\varepsilon = \int_A^B \varepsilon(x)\,\frac{\mathrm{d}x}{\hbar v(x)} = \int_{t_A}^{t_B} \varepsilon\big(x(t)\big)\,\frac{\mathrm{d}t}{\hbar} \underbrace{=}_{V(x)=const} \frac{\varepsilon}{\hbar}\tau$$

depends on the state's energy and the time it takes the system to propagate from A to B

II.3.1 Double Slit Experiment

- **phase shifts**
  - **Aharonov-Bohm phase** (charged particle in magnetic field)



➢ canonical momentum: $\mathbf{p} = m\mathbf{v} + q\mathbf{A}$

$$\mathbf{k}(x) \to \mathbf{k}(x) - \underbrace{\frac{q}{\hbar}\mathbf{A}(x)}$$

results in phase shift $\varphi_{\rm mag}$ due to vector potential $\mathbf{A}(x)$

$$\varphi_{\rm mag} = \frac{e}{\hbar}\int_A^B \mathbf{A}\cdot \mathrm{d}\mathbf{x} = \frac{e}{\hbar}\int_{t_A}^{t_B} \mathbf{A}\cdot \mathbf{v}(t)\,\mathrm{d}t \qquad (q=-e)$$

opposite phase shift for time-reversed path

note: $\varphi_{\rm mag}$ depends on gauge $\mathbf{A} \to \mathbf{A} + \nabla\chi(x)$ and is therefore unphysical and not observable

➢ gauge invariant quantity is the phase accumulated along a closed path (electron returns to the same point):

Aharonov-Bohm phase

$$\varphi_{\rm AB} = \frac{e}{\hbar}\oint \mathbf{A}\cdot \mathrm{d}\mathbf{x} \underbrace{=}_{\text{Stokes theorem}} \frac{e}{\hbar}\int \mathbf{B}\cdot \mathrm{d}\mathbf{F} = 2\pi\frac{\Phi}{\Phi_0}$$

$\Phi_0 = \frac{h}{e}$ („normal" flux quantum)

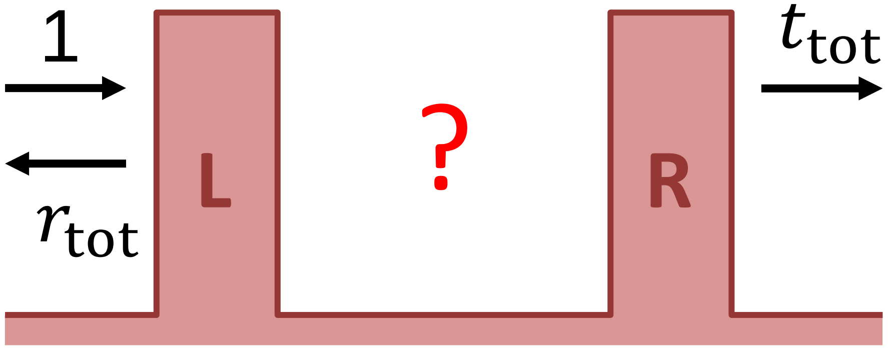
(in superconductors we have $q_s = -2e$ and therefore $\Phi_0 = h/2e$)

phase difference $\Delta\varphi = \varphi_1 - \varphi_2$ between 1 and 2 corresponds to $\varphi_{\rm AB}$ due to opposite sign of phase shift on time-reversed path

II.3.2 Double Tunnel Junction

- **quantum interference effect in double tunnel junction**

  - we consider only a single conductance channel
  - no magnetic field



  - ***„classical“ expectation***:

    (tunneling) resistances are added

    multiplication of transmission probabilities $T_{\mathrm{L}} \cdot T_{\mathrm{R}}$

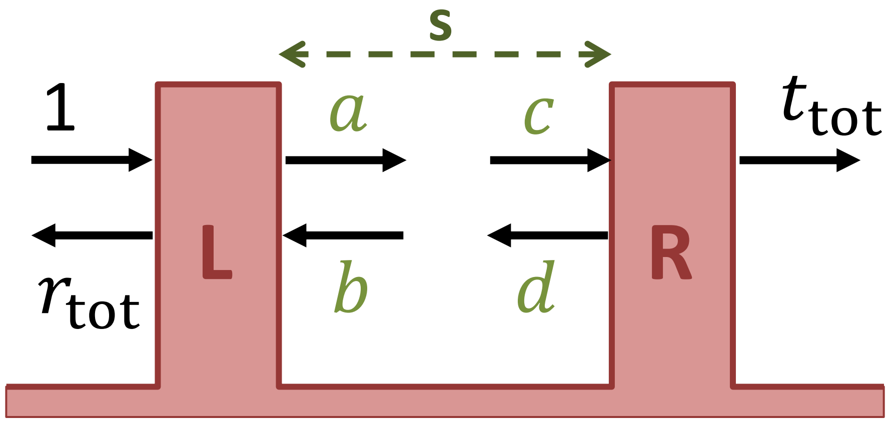
Ohm‘s law: $R = R_1 + R_2$

$$G = \frac{G_1 G_2}{G_1 + G_2}$$

  - what is the role of **quantum interference**?
  - how do individual ***scattering matrices*** have to be ***combined***?

II.3.2 Double Tunnel Junction

- **quantum interference effect in double tunnel junction**



acquired phase during propagation between barriers

$$\boldsymbol{\varphi = k \cdot s}$$

$$\begin{bmatrix} \hat{r}_L & \hat{t}'_L \\ \hat{t}_L & \hat{r}'_L \end{bmatrix} \begin{bmatrix} \mathrm{e}^{i\varphi} & 0 \\ 0 & \mathrm{e}^{i\varphi} \end{bmatrix} \begin{bmatrix} \hat{r}_R & \hat{t}'_R \\ \hat{t}_R & \hat{r}'_R \end{bmatrix}$$

scattering matrix of left barrier

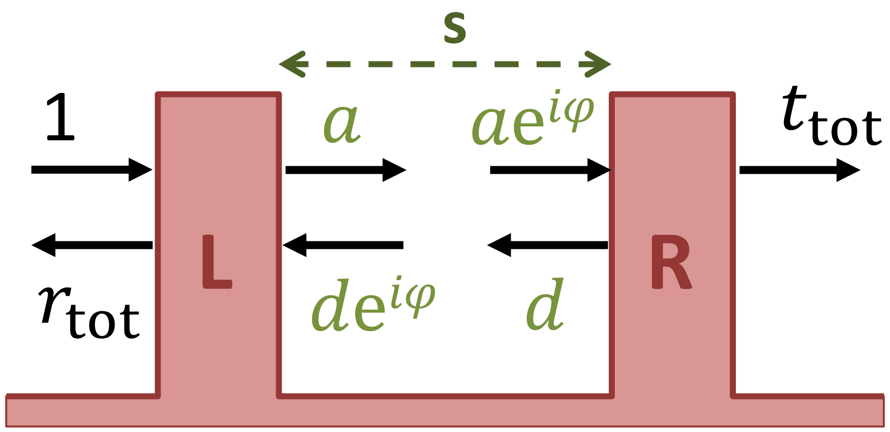
propagation between barriers

scattering matrix of right barrier

R. Gross © Walther-Meißner-Institut (2004 - 2023)

II.3.2 Double Tunnel Junction

- **quantum interference effect in double tunnel junction**



$$\begin{bmatrix} r_{\mathrm{tot}} \\ a \end{bmatrix} = \begin{bmatrix} \hat{r}_L & \hat{t}'_L \\ \hat{t}_L & \hat{r}'_L \end{bmatrix} \begin{bmatrix} 1 \\ de^{i\varphi} \end{bmatrix}$$

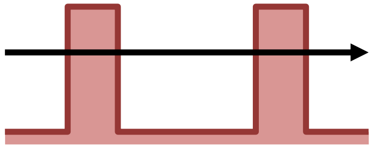
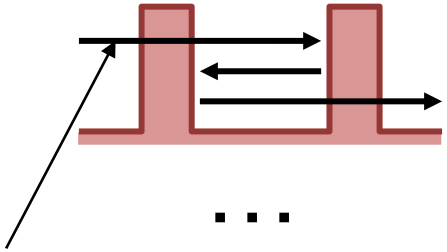
outgoing modes *incoming modes*

$$\begin{bmatrix} d \\ t_{\mathrm{tot}} \end{bmatrix} = \begin{bmatrix} \hat{r}_R & \hat{t}'_R \\ \hat{t}_R & \hat{r}'_R \end{bmatrix} \begin{bmatrix} ae^{i\varphi} \\ 0 \end{bmatrix}$$

outgoing modes *incoming modes*

II.3.2 Double Tunnel Junction

- **quantum interference effect in double tunnel junction**

process	**amplitude**	**probability**
	$t_L t_R \mathrm{e}^{\imath\varphi}$	$T_L T_R$
	$t_L t_R r_L' r_R \mathrm{e}^{\imath 3\varphi}$	$T_L T_R R_L R_R$
…	…	…

path can be viewed as Feynman path

sum of all amplitudes:

sum of all probabilities:

coherent

$$t_{\mathrm{tot}} = \frac{t_L t_R}{1 - r_L' r_R \mathrm{e}^{\imath 2\varphi}}$$

$$T_{\mathrm{classical}} = \frac{T_L T_R}{1 - R_L R_R}$$

incoherent

$$T_{\mathrm{tot}} = |t_{\mathrm{tot}}|^2$$

$$T_{\mathrm{tot}} = |t_{\mathrm{tot}}|^2 = \frac{T_L T_R}{1 + R_L R_R - 2\sqrt{R_L R_R}\cos\chi}$$

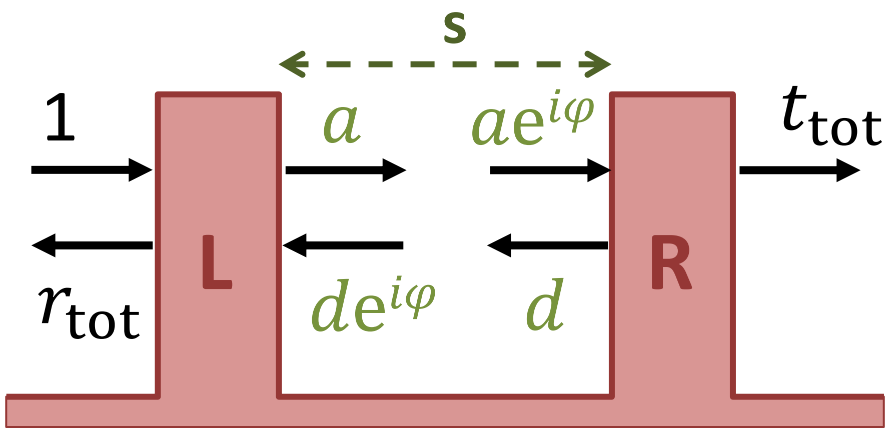
for $T_{L,R} \ll 1$:

$$G_{\mathrm{class}} = G_Q T_{\mathrm{class}} = \frac{G_L G_R}{G_Q[1-(1-T_L)(1-T_R)]}$$

$$G_{\mathrm{class}} \simeq \frac{G_L G_R}{G_Q[T_L + T_R]} = \frac{G_L G_R}{G_L + G_R}$$

II.3.2 Double Tunnel Junction

- **quantum interference effect in double tunnel junction**



$$t_{\text{tot}} = \frac{t_L t_R}{1 - r_L' r_R e^{\imath 2\varphi}}$$

$$T_{\text{tot}}(\varepsilon) = |t_{\text{tot}}|^2 = \frac{T_L T_R}{1 + R_L R_R - 2\sqrt{R_L R_R}\cos\chi(\varepsilon)}$$

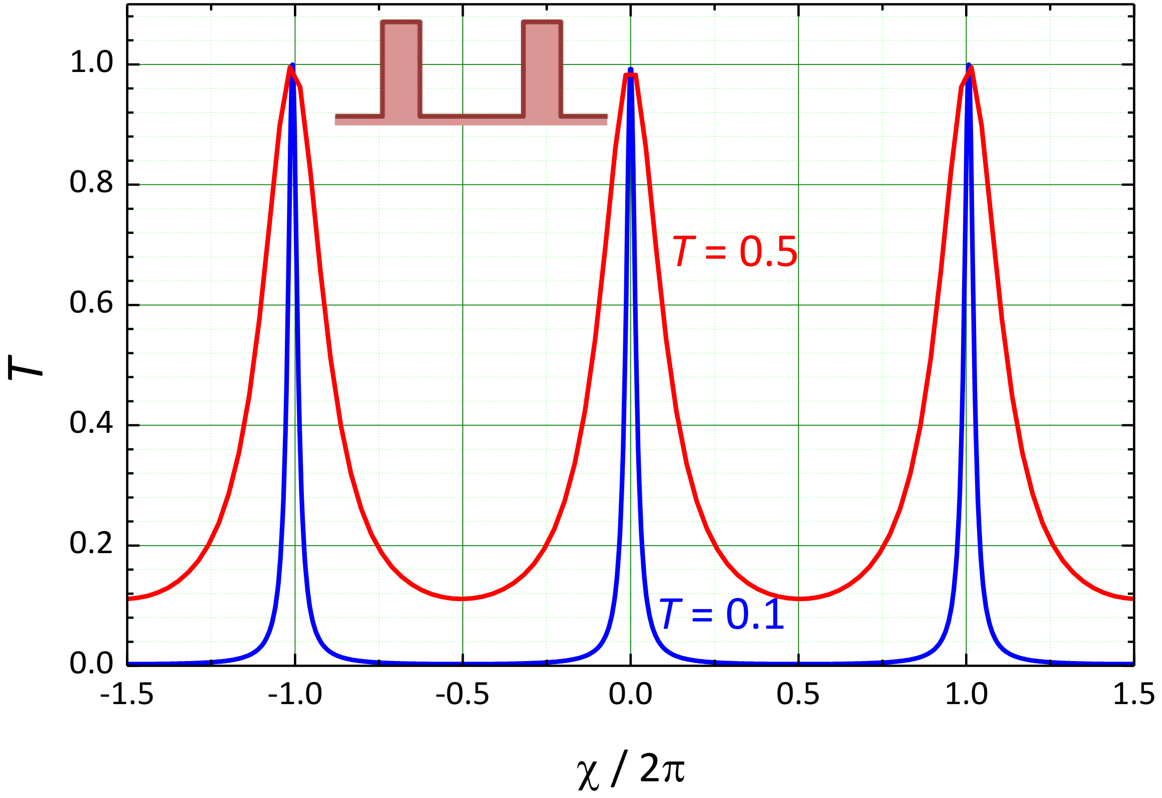
phase accumulated during the round trip

$$\chi(\varepsilon) = 2\varphi(\varepsilon) = 2k(\varepsilon)s$$

$$k(\varepsilon) = \sqrt{\frac{2m\varepsilon}{\hbar^2}}$$

II.3.2 Double Tunnel Junction

- **quantum interference effect in double tunnel junction**
  - transmission coefficient ***depends on energy***



$$T_{\text{tot}}(\varepsilon) = |t_{\text{tot}}|^2 = \frac{T_L T_R}{1 + R_L R_R - 2\sqrt{R_L R_R}\cos[2k(\varepsilon)s]}$$

assume $T_L = T_R = T \ll 1,$
$R_L = R_R = R \simeq 1$

between peaks: $T(\varepsilon) \approx T^2$

maximum value: $T_{\text{max}} = \frac{T_L T_R}{\left(1-\sqrt{R_L R_R}\right)^2} \simeq 1$ @ $\chi = n \cdot 2\pi$

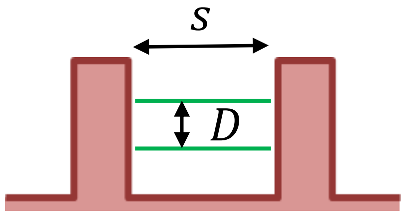
maximum value: $T_{\text{min}} = \frac{T_L T_R}{\left(1+\sqrt{R_L R_R}\right)^2} \ll 1$ @ $\chi = \left(n + \frac{1}{2}\right) \cdot 2\pi$

➔ ***resonant tunneling***
(or Fabry-Perot resonances)

→ *double barrier structure behaves as an* ***optical interferometer***
→ *resonant tunneling is* ***quantum interference effect***

II.3.2 Double Tunnel Junction

- **quantum interference effect in double tunnel junction**



  – how does the transmission $T(\varepsilon)$ look like close to the transmission resonances?

$$\cos\chi = \cos(2ks) \simeq 1 - \frac{1}{2}(2ks)^2 \quad \text{for} \quad \chi \ll 1$$

$$\cos\chi \simeq 1 - \frac{\varepsilon - \varepsilon_{\text{res}}}{2D} \qquad (2ks)^2 = \frac{8ms^2(\varepsilon - \varepsilon_{\text{res}})}{\hbar^2} = \frac{\varepsilon - \varepsilon_{\text{res}}}{D}$$

D = level spacing in potential well of width s

  – after some math:

$$T(\varepsilon) = \frac{T_L T_R}{\left(\frac{T_L + T_R}{2}\right)^2 + \left(\frac{\varepsilon - \varepsilon_{\text{res}}}{D}\right)^2}$$

transmission assumes Lorentzian shape

$$T(\varepsilon) = \frac{D^2\, T_L T_R}{\left(\frac{D(T_L + T_R)}{2}\right)^2 + (\varepsilon - \varepsilon_{\text{res}})^2}$$

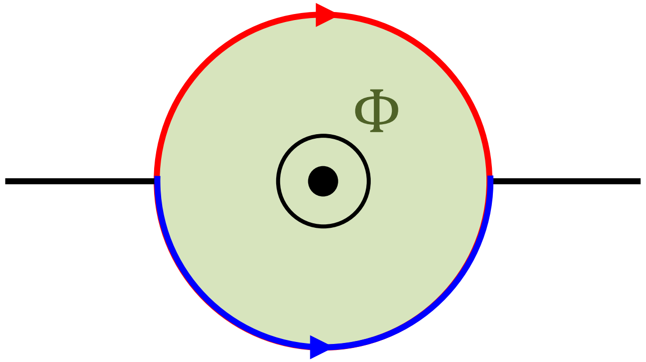
energy width of transmission resonance: $d = D\,(T_L + T_R)$

- ➢ interpretation in terms of a particle that moves back and forth between the two potential wells and escapes at a certain tunneling rates Γ_L and Γ_R
- ➢ with $d = \hbar(\Gamma_L + \Gamma_R)$ according to uncertainty relation we obtain well-known **Breit-Wigner formula**

II.3.3 Aharonov-Bohm Effect

- **quantum interference effects in multiply connected conductors, e.g. rings**
  - phase shift due to magnetic field

two trajectories enclosing magnetic flux



accumulated phase with vector potential: $\mathbf{k}(x) \to \mathbf{k}(x) - \frac{q}{\hbar}\mathbf{A}(x)$

$$\varphi_{1,2} = kL_{1,2} + \frac{e}{\hbar}\int_{1,2} \mathbf{A}\cdot \mathrm{d}\mathbf{x} \qquad (q = -e)$$

$$\varphi_2 - \varphi_1 = k(L_2 - L_1) + \frac{e}{\hbar}\oint \mathbf{A}\cdot \mathrm{d}\mathbf{x}$$

$$\varphi_{\mathrm{AB}} = \frac{e}{\hbar}\oint \mathbf{A}\cdot \mathrm{d}\mathbf{x} \underbrace{=}_{\text{Stokes theorem}} \frac{e}{\hbar}\int \mathbf{B}\cdot \mathrm{d}\mathbf{F} = 2\pi\frac{\Phi}{\Phi_0}$$

all quantities are periodic in Φ/Φ_0, even if there is NO magnetic field at the trajectories!

$\Phi_0 = \frac{h}{e}$ („normal“ flux quantum)

(in superconductors we have $q_s = -2e$ and therefore $\Phi_0 = h/2e$)

II.3.3 Aharonov-Bohm Effect

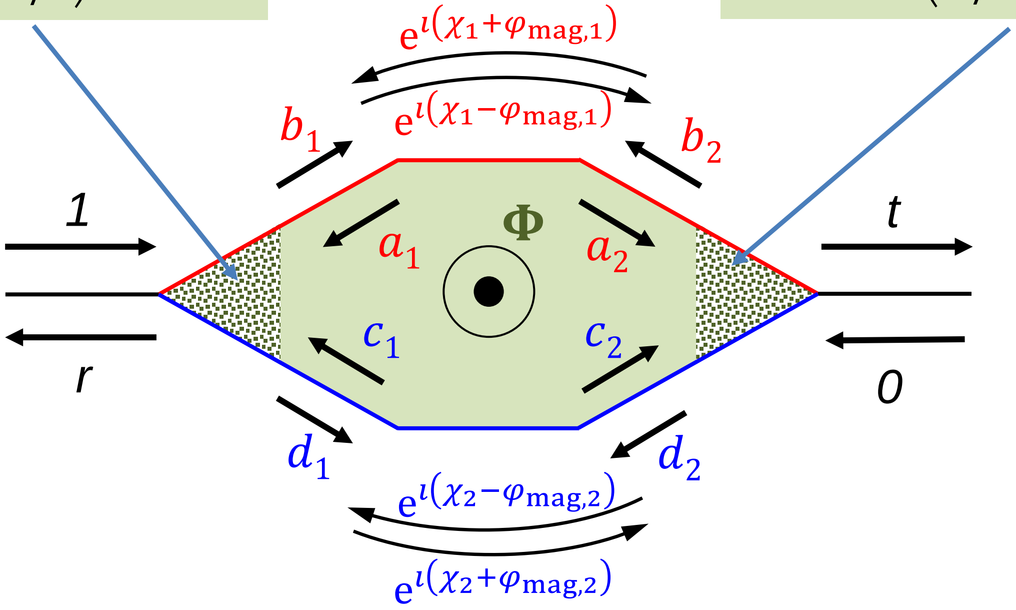
- **description of Aharonov-Bohm ring by two beam splitters and loop**

$$\begin{pmatrix} r \\ b_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/2 & 1/2 \\ 1/\sqrt{2} & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 \\ a_1 \\ c_1 \end{pmatrix}$$

left beam splitter

$$\begin{pmatrix} t \\ b_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/2 & 1/2 \\ 1/\sqrt{2} & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 0 \\ a_2 \\ c_2 \end{pmatrix}$$

right beam splitter



dynamical phase:

$$\chi_{1,2} = kL_{1,2}$$

magnetic phase:

$$\varphi_{\mathrm{mag},1} + \varphi_{\mathrm{mag},2} = \varphi_{\mathrm{AB}} = 2\pi \frac{\Phi}{\Phi_0}$$

$$\begin{pmatrix} 0 & e^{\iota(\chi_1+\varphi_{\mathrm{mag},1})} \\ e^{\iota(\chi_1-\varphi_{\mathrm{mag},1})} & 0 \end{pmatrix}$$

upper arm (1)

$$\begin{pmatrix} 0 & e^{\iota(\chi_2-\varphi_{\mathrm{mag},2})} \\ e^{\iota(\chi_2+\varphi_{\mathrm{mag},2})} & 0 \end{pmatrix}$$

lower arm (2)

II.3.3 Aharonov-Bohm Effect

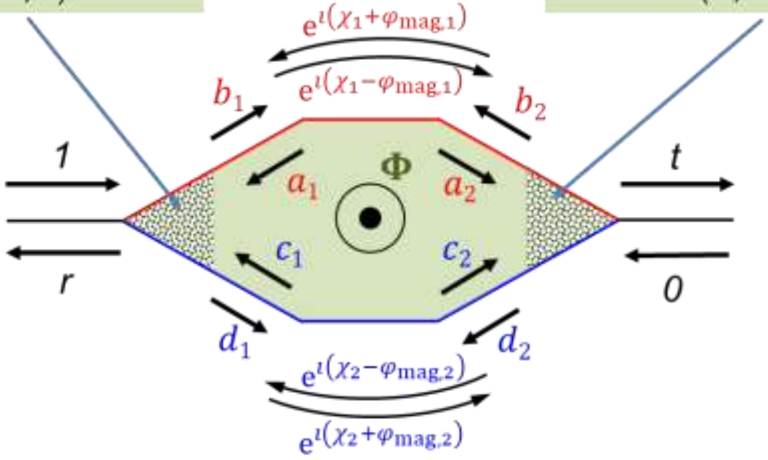
- **description of Aharonov-Bohm ring by two beam splitters and loop**

$$\begin{pmatrix} r \\ b_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/2 & 1/2 \\ 1/\sqrt{2} & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 \\ a_1 \\ c_1 \end{pmatrix}$$

left beam splitter

$$\begin{pmatrix} t \\ b_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/2 & 1/2 \\ 1/\sqrt{2} & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 0 \\ a_2 \\ c_2 \end{pmatrix}$$

right beam splitter



$$\chi_{1,2} = kL_{1,2}$$

$$\varphi_{\mathrm{mag},1} + \varphi_{\mathrm{mag},2} = \varphi_{\mathrm{AB}} = 2\pi\frac{\Phi}{\Phi_0}$$

- **example 1**: electron enters from left, takes lower path and goes out to the right:

$$t_1 = \frac{1}{\sqrt{2}} e^{\imath(\chi_2+\varphi_{\mathrm{mag},2})} \frac{1}{\sqrt{2}} = \frac{1}{2} e^{\imath(\chi_2+\varphi_{\mathrm{mag},2})}$$

- **example 2**: electron enters from left, takes upper path and goes out to the right:

$$t_2 = \frac{1}{\sqrt{2}} e^{\imath(\chi_1-\varphi_{\mathrm{mag},1})} \frac{1}{\sqrt{2}} = \frac{1}{2} e^{\imath(\chi_1-\varphi_{\mathrm{mag},1})}$$

➔ phase difference in two paths: $\chi_2 - \chi_1 + \varphi_{\mathrm{mag},2} + \varphi_{\mathrm{mag},1} = \chi_2 - \chi_1 + \varphi_{\mathrm{AB}}$ (depends on dynamical phases χ_2, χ_1)

II.3.3 Aharonov-Bohm Effect

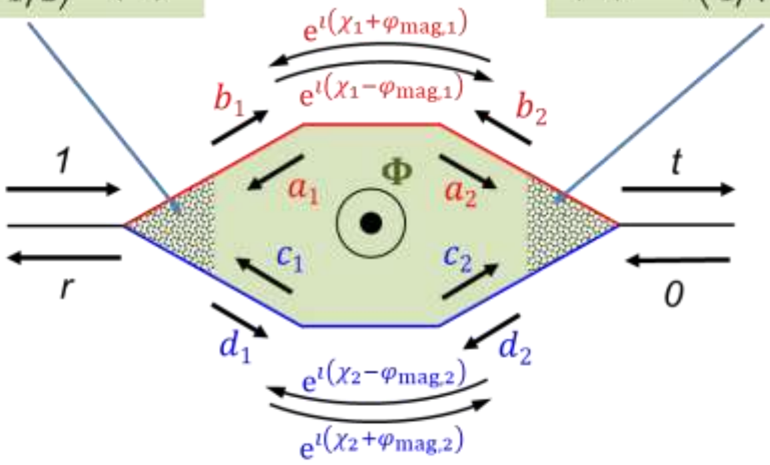
- **description of Aharonov-Bohm ring by two beam splitters and loop**

$$\begin{pmatrix} r \\ b_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/2 & 1/2 \\ 1/\sqrt{2} & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 \\ a_1 \\ c_1 \end{pmatrix}$$

left beam splitter

$$\begin{pmatrix} t \\ b_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/2 & 1/2 \\ 1/\sqrt{2} & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 0 \\ a_2 \\ c_2 \end{pmatrix}$$

right beam splitter



$$\chi_{1,2} = kL_{1,2}$$

$$\varphi_{\mathrm{mag},1} + \varphi_{\mathrm{mag},2} = \varphi_{\mathrm{AB}} = 2\pi\frac{\Phi}{\Phi_0}$$

➢ **example 3**: electron takes upper path + full clockwise turn:

$$t_3 = \frac{1}{\sqrt{2}} e^{\iota(\chi_1-\varphi_{\mathrm{mag},1})} \frac{1}{2} e^{\iota(\chi_2-\varphi_{\mathrm{mag},2})} \frac{1}{2} e^{\iota(\chi_1-\varphi_{\mathrm{mag},1})} \frac{1}{\sqrt{2}} = \frac{1}{8} e^{\iota(2\chi_1+\chi_2-2\varphi_{\mathrm{mag},1}-\varphi_{\mathrm{mag},2})}$$

➢ **example 4**: electron takes upper path + full counter-clockwise turn (time-reversed path):

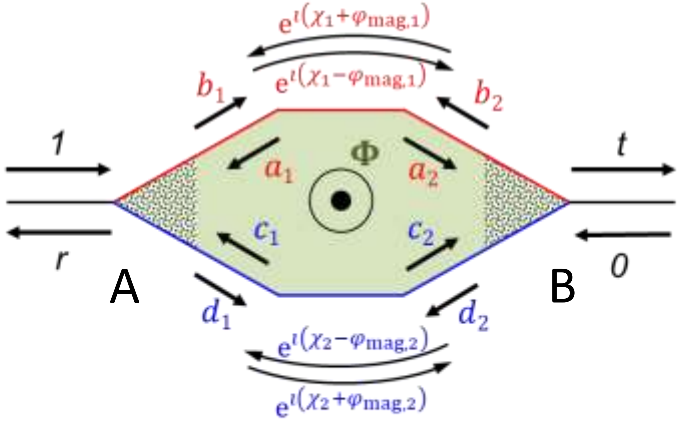
$$t_4 = \frac{1}{\sqrt{2}} e^{\iota(\chi_1-\varphi_{\mathrm{mag},1})} \left(-\frac{1}{2}\right) e^{\iota(\chi_1+\varphi_{\mathrm{mag},1})} \frac{1}{2} e^{\iota(\chi_2+\varphi_{\mathrm{mag},2})} \frac{1}{\sqrt{2}} = -\frac{1}{8} e^{\iota(2\chi_1+\chi_2+\varphi_{\mathrm{mag},2})}$$

➔ phase difference in two paths: $2\varphi_{\mathrm{mag},2} + 2\varphi_{\mathrm{mag},1} = 2\varphi_{\mathrm{AB}}$ (independent of dynamical phases χ_2, χ_1)

II.3.3 Aharonov-Bohm Effect

- **description of Aharonov-Bohm ring by two beam splitters and loop**

$$P_{\mathrm{AB}} = P_{\mathrm{classical}} + 2\sqrt{P_1 P_2}\cos\Delta\varphi$$



$$\chi_{1,2} = kL_{1,2}$$

$$\varphi_{\mathrm{mag},1} + \varphi_{\mathrm{mag},2} = \varphi_{\mathrm{AB}} = 2\pi\frac{\Phi}{\Phi_0}$$

$$\Delta\varphi = \chi_2 - \chi_1 + \varphi_{\mathrm{AB}}$$

$$P_{\mathrm{AB}} \propto \cos(\chi_2 - \chi_1 + \varphi_{\mathrm{AB}})$$

universal conductance fluctuations

P_{AB} depends on dynamical phases
→ configuration of scattering sites matters
→ removed by ensemble averaging
→ $\cos(2\pi\,\Phi/\Phi_0)$: flux period $\Phi_0 = h/e$

$$\Delta\varphi = 2\varphi_{\mathrm{AB}}$$

$$P_{\mathrm{AB}} \propto \cos(2\varphi_{\mathrm{AB}})$$

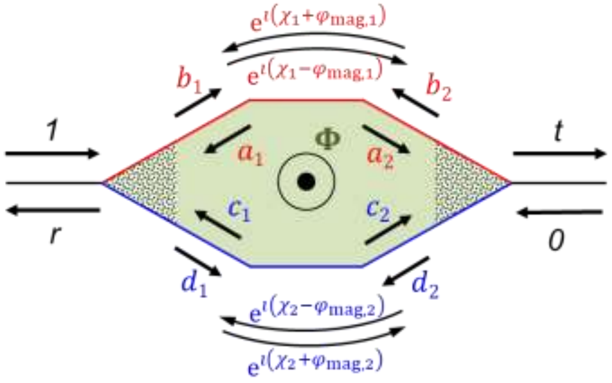
Altshuler-Aronov-Spivak oscillations

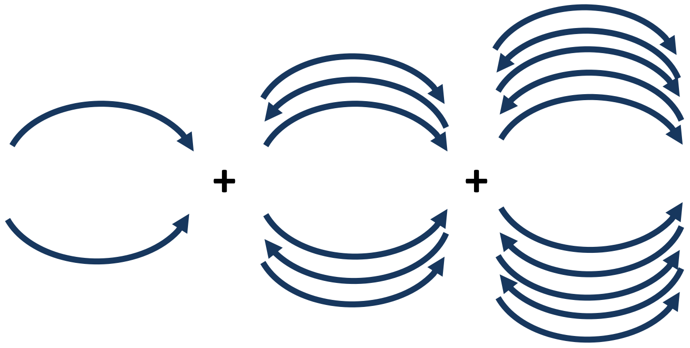
P_{AB} independent of dynamical phases
→ configuration of scattering sites does not matter
→ survives ensemble averaging
→ $\cos(4\pi\,\Phi/\Phi_0)$: flux period $\Phi_0/2 = h/2e$

+ many other trajectories

II.3.3 Aharonov-Bohm Effect

- **description of Aharonov-Bohm ring by two beam splitters and loop**
  - summing up (without closed loops):

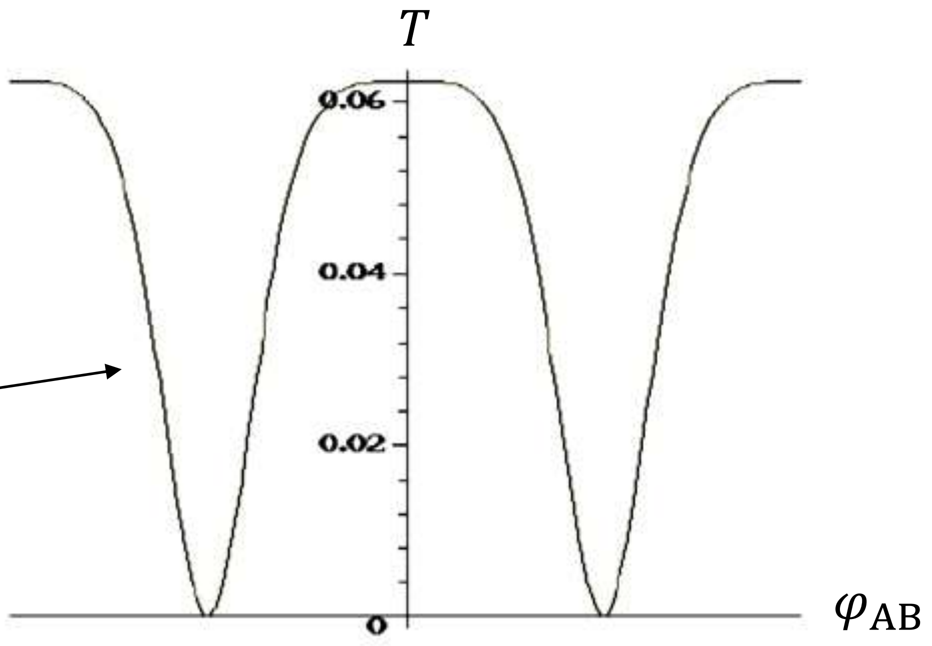




$$\chi_1 = \chi_2 = \chi = kL_{1,2} = kL$$
$$\varphi_{\mathrm{mag},1} = \varphi_{\mathrm{mag},2} = \varphi_{\mathrm{mag}}$$
$$\varphi_{\mathrm{mag},1} + \varphi_{\mathrm{mag},2} = \varphi_{\mathrm{AB}} = 2\pi\frac{\Phi}{\Phi_0}$$

$$T = \frac{(1-\cos 2\chi)(1+\cos^2\varphi_{\mathrm{AB}})}{\sin^2 2\chi + \left[\cos 2\chi - \frac{1}{2}(1+\cos\varphi_{\mathrm{AB}})\right]^2}$$

$$2\chi = \frac{\pi}{4} \Rightarrow \quad T = \frac{(1+\cos^2\varphi_{\mathrm{AB}})}{1+\frac{1}{4}[1-\cos\varphi_{\mathrm{AB}}]^2}$$



Aharonov-Bohm effect: flux dependent transmission

II.3.4 Weak Localization

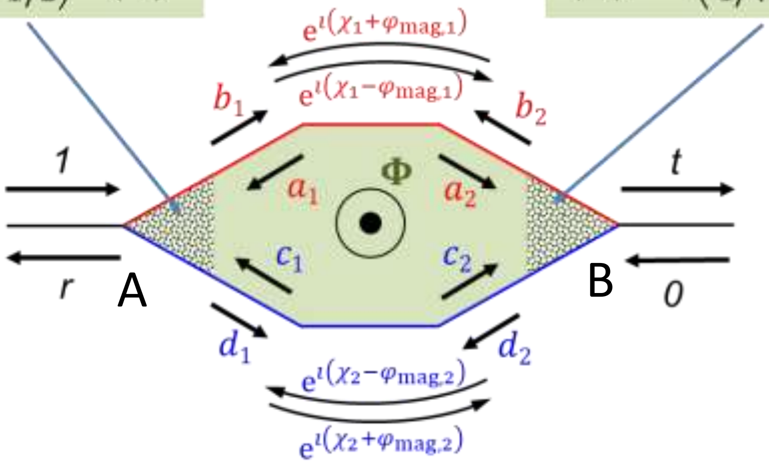
- **description of Aharonov-Bohm ring by two beam splitters and loop**

$$\begin{pmatrix} r\\ b_1\\ d_1\end{pmatrix} = \begin{pmatrix}0 & 1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/2 & 1/2\\ 1/\sqrt{2} & 1/2 & -1/2\end{pmatrix}\begin{pmatrix}1\\ a_1\\ c_1\end{pmatrix}$$

left beam splitter

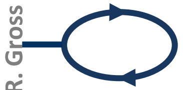
$$\begin{pmatrix} t\\ b_2\\ d_2\end{pmatrix} = \begin{pmatrix}0 & 1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/2 & 1/2\\ 1/\sqrt{2} & 1/2 & -1/2\end{pmatrix}\begin{pmatrix}0\\ a_2\\ c_2\end{pmatrix}$$

right beam splitter



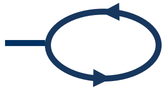
$$\chi_{1,2} = kL_{1,2}$$

$$\varphi_{\mathrm{mag},1} + \varphi_{\mathrm{mag},2} = \varphi_{\mathrm{AB}} = 2\pi\frac{\Phi}{\Phi_0}$$



- **example 5**: electron takes full clockwise turn:

$$r_3 = \frac{1}{\sqrt{2}}\mathrm{e}^{\iota(\chi_1-\varphi_{\mathrm{mag},1})}\frac{1}{2}\mathrm{e}^{\iota(\chi_2-\varphi_{\mathrm{mag},2})}\frac{1}{\sqrt{2}} = \frac{1}{4}\mathrm{e}^{\iota(\chi_1+\chi_2-\varphi_{\mathrm{mag},1}-\varphi_{\mathrm{mag},2})}$$



- **example 6**: electron takes full counter-clockwise turn (time-reversed path):

$$r_4 = \frac{1}{\sqrt{2}}\mathrm{e}^{\iota(\chi_2+\varphi_{\mathrm{mag},2})}\frac{1}{2}\mathrm{e}^{\iota(\chi_1+\varphi_{\mathrm{mag},1})}\frac{1}{\sqrt{2}} = \frac{1}{4}\mathrm{e}^{\iota(\chi_1+\chi_2+\varphi_{\mathrm{mag},1}+\varphi_{\mathrm{mag},2})}$$

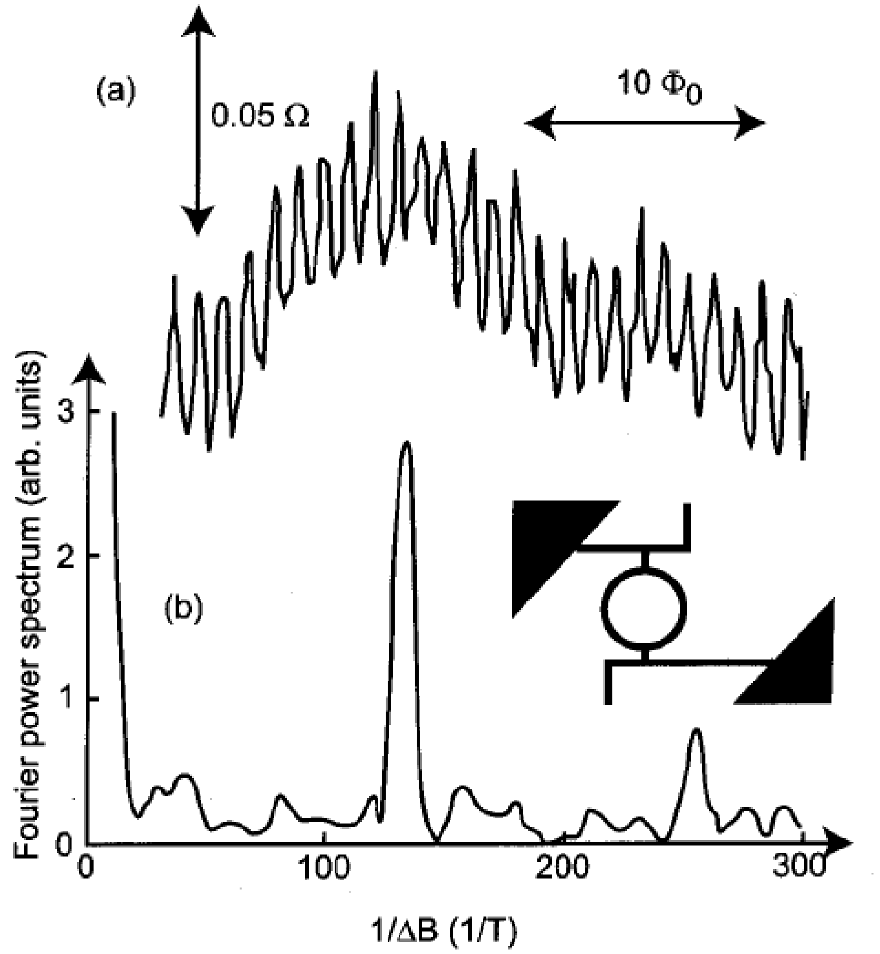
$$P_{\mathrm{AA}} \propto \cos\Delta\varphi \propto \cos\left(4\pi\frac{\Phi}{\Phi_0}\right)$$

@ $B = 0$: enhanced back-scattering due to time-reversed paths
➔ weak localization

➔ phase difference in two paths: $2\varphi_{\mathrm{mag},2} + 2\varphi_{\mathrm{mag},1} = 2\varphi_{\mathrm{AB}}$ (indepent of dynamical phases χ_2, χ_1)

II.3.3 Aharonov-Bohm Effect

- **Aharonov-Bohm effect: experiments**



R. Webb et al, PRL **54**, 2696 (1985)

Aharonov-Bohm (AB) oscillations:

- period: $\Phi = \Phi_0 = h/e$
- amplitude: $G_\mathrm{Q} = 2e^2/h$
- one channel in Landauer model

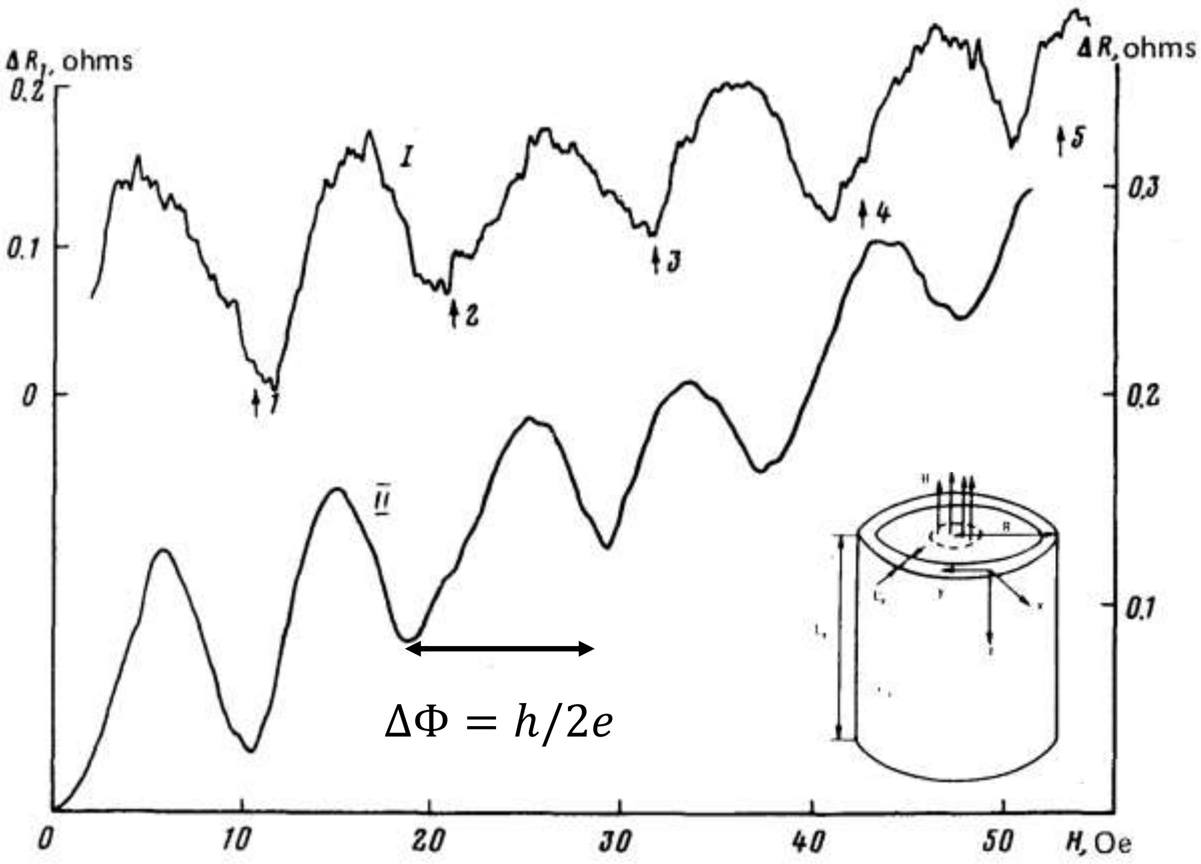
Fourier analysis shows that there are also weak oscillations with half period

→ higher order interferences:
Altshuler-Aronov-Spivak (AAS) oscillations

- period: $\Phi = \Phi_0/2 = h/2e$
- Interference of time-reversed traces
- constructive interference for $B = 0$
- coherent backscattering

II.3.3 Aharonov-Bohm Effect

- **Aharonov-Bohm effect: experiments**

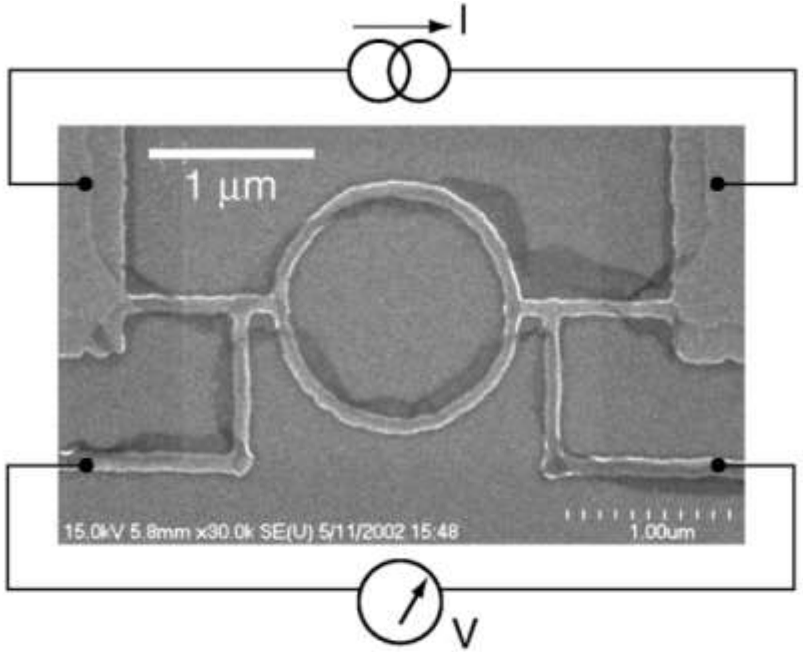


Aharonov–Bohm like magneto-conductance oscillations (***Altshuler-Aronov-Spivak (AAS) oscillations***) observed in normally conducting Mg cylinders of diameter 1.5 µm. Left and right resistance scales correspond to samples 1 and 2, respectively. The periodicity of the oscillations corresponds to $\Delta\Phi = h/2e$.

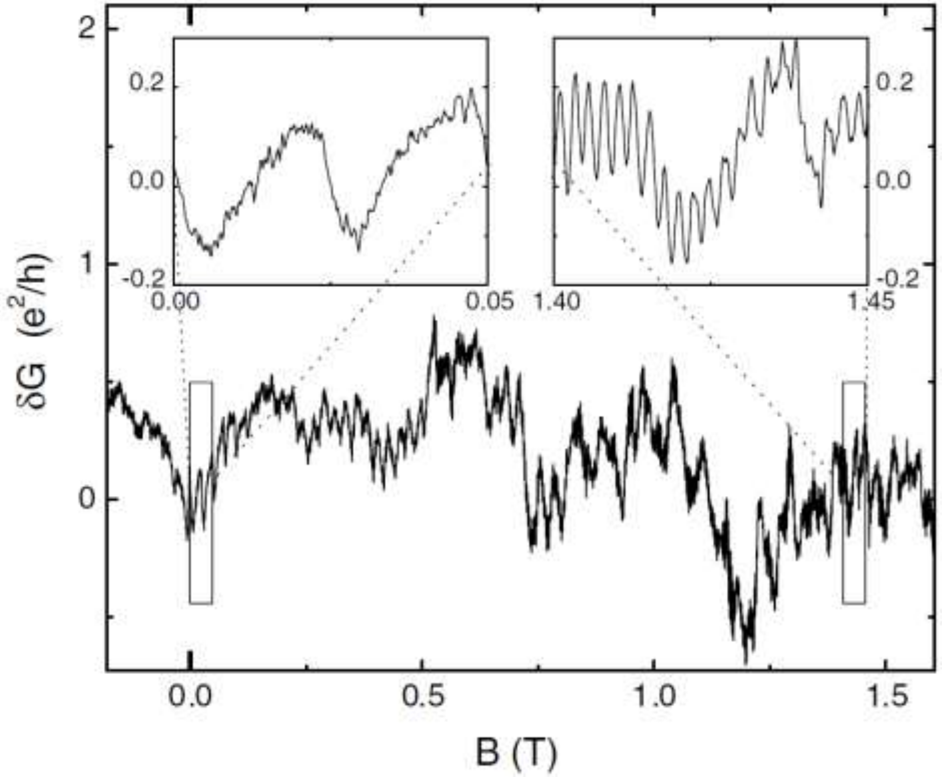
D.Y. Sharvin, Y.V. Sharvin, *Sov. Phys. JETP Lett.* **34**, 272 (1981).

II.3.3 Aharonov-Bohm Effect

- **Aharonov-Bohm effect: experiments**



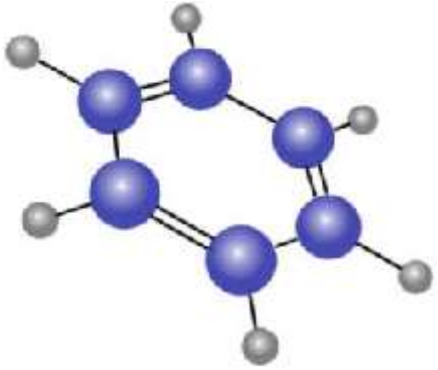
Cu ring on Si, width 80 nm



F. Pierre et. al., PRL **89**, 206804 (2002)

- conductance of a Cu ring in units of $G_\mathrm{Q} = e^2/h$, as a function of magnetic field at $T \ = 100$ mK.
- narrow AB oscillations $\Delta B \approx 2.5$ mT are superimposed on larger and broader ***universal conductance fluctuations***.

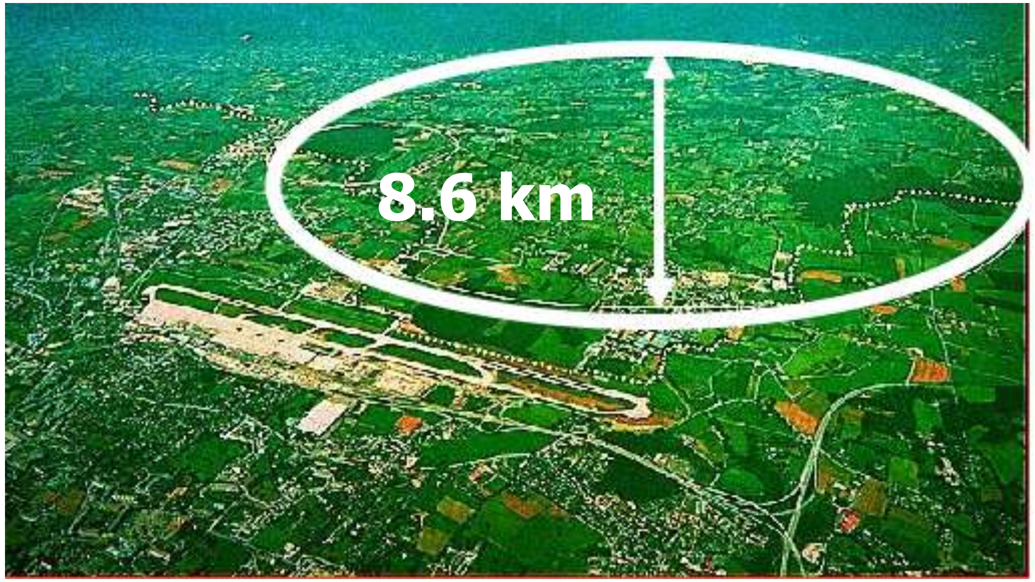
- **Aharonov-Bohm effect: experiments**



Benzene ring

0.5 nm

Large **E**lectron **P**ositron Collider at CERN (Geneva)



ring accelerator

10^{13}

AB effect: one flux quantum (h/e) through ring area:

$$\frac{h/e}{\pi r^2} = 5000\,T$$

$$\frac{h/e}{\pi r^2} = 7 \times 10^{-23}\,T$$

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DER
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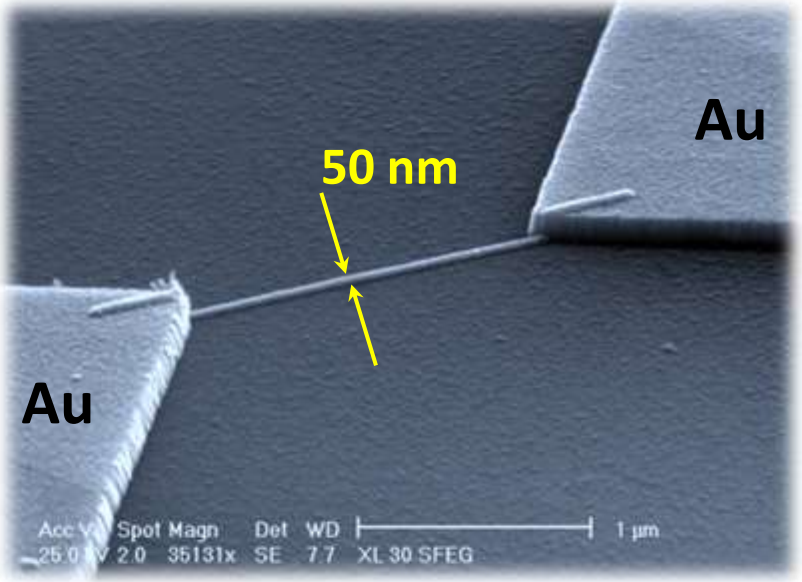
Technische
Universität
München

Superconductivity and Low Temperature Physics II



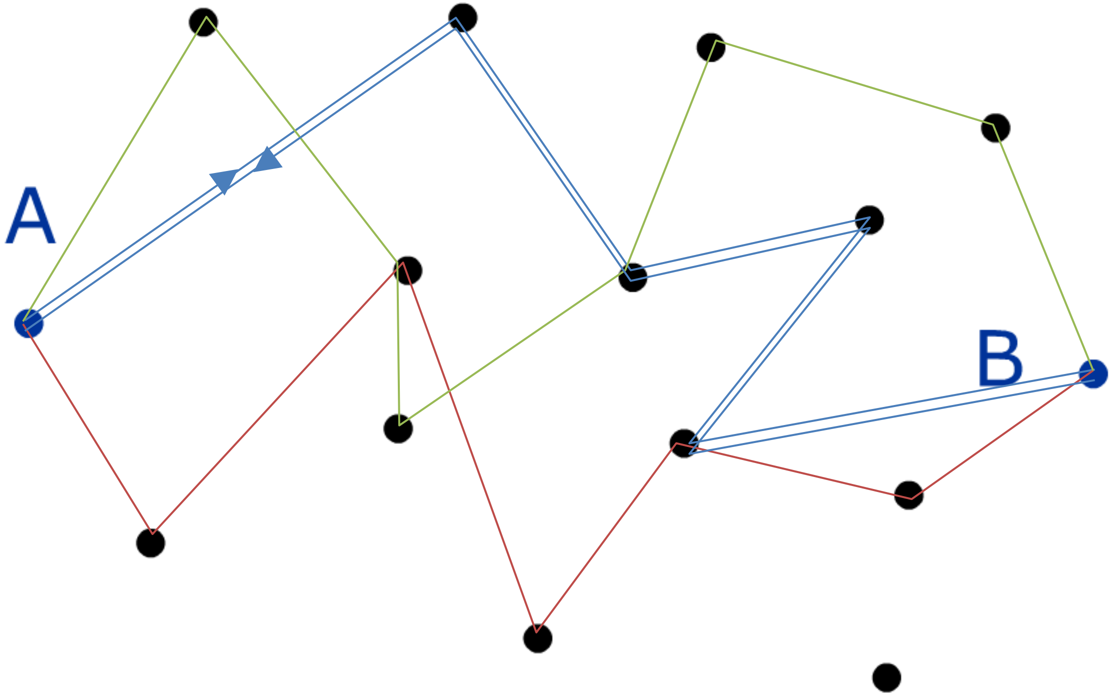
Lecture No. 10

R. Gross

II.3.4 Weak Localization

Weak localization:

interference of time reversed electron paths



II.3.4 Weak Localization

- **quantum interference of time-reversed trajectories**

$$P_{\mathrm{AB}} = |A_1 + A_2|^2 = \underbrace{\underbrace{|A_1|^2}_{=P_1} + \underbrace{|A_2|^2}_{=P_2}}_{\text{classical result}} + \underbrace{\underbrace{A_1 A_2^* + A_1^* A_2}_{2\mathrm{Re}\,[A_1 A_2^*]}}_{\text{interference term: quantum mechanical}} \longrightarrow 2|A_1 A_2| \cos\Delta\varphi$$

$$\langle \cos\Delta\varphi \rangle = 0 \ \ !?$$

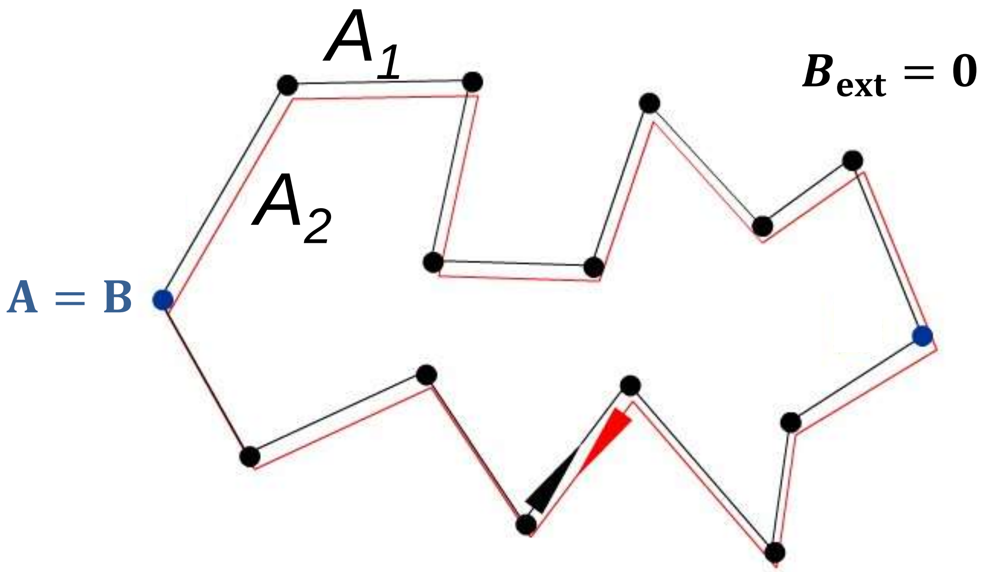
does averaging over many paths destroy interference effects in diffusive conductor ?

time-reversed trajectories:

we consider a closed loop with A = B

➔ the amplitude A_2 is just a time reversal of A_1

$$|A_1 + A_2|^2 = |A_1 + A_1^*|^2 = 4\,|A_1|^2$$

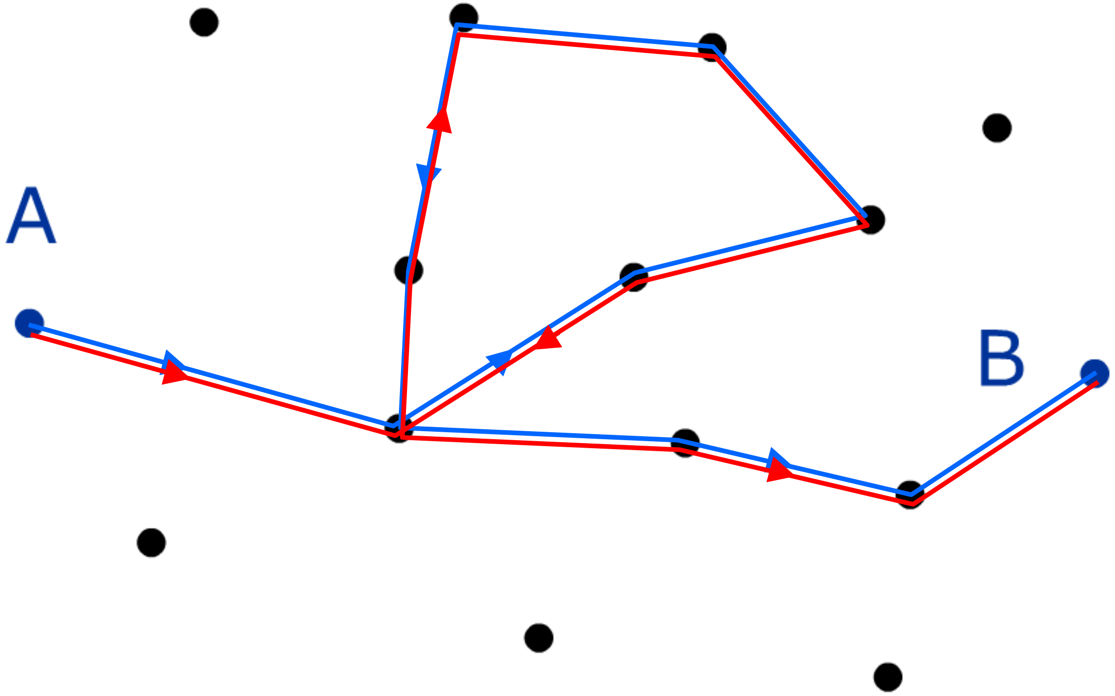


– the backscattering probability is enhanced by factor **2** for all time-reversed paths!!!

– this is a predecessor of **localization**

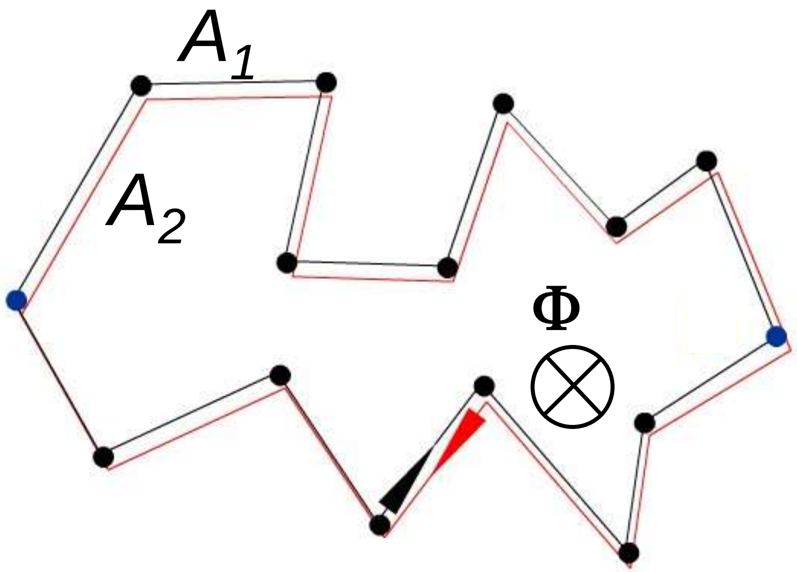
II.3.4 Weak Localization

- **quantum interference of time-reversed trajectories**
  - increased backscattering probability to original position makes **self-intersecting scattering paths** important
  - interference effects make it more likely that a charge carrier is doing closed paths than without any interference
    ➔ **increased net resistivity**
  - applied magnetic field reduces backscattering probability
    ➔ **decrease of resistivity with increasing field**



II.3.4 Weak Localization

- **magnetic field dependence of weak localization**



  - calculate phase difference of time reversed paths:

$$\varphi_{\mathrm{mag},A_2} - \varphi_{\mathrm{mag},A_1} = \frac{2e}{h} \oint \mathbf{A} \cdot \mathrm{d}\mathbf{s} = 2\varphi_{\mathrm{AB}}$$

  - loss of constructive interference due to additional φ_{AB}

$$\varphi_{\mathrm{mag},A_2} - \varphi_{\mathrm{mag},A_1} = \frac{2e}{h} \oint \mathbf{A} \cdot \mathrm{d}\mathbf{s} = 2\varphi_{\mathrm{AB}} = 4\pi \frac{\Phi}{\Phi_0}$$

$\Phi = B_{\mathrm{ext}} \cdot F$ = flux enclosed in the loop
F = area of the enclosed loop

  - characteristic field defined by $\varphi_{\mathrm{mag},A_2} - \varphi_{\mathrm{mag},A_1} = 2\pi$ (complete dephasing):

$$B^* = \frac{\Phi_0}{2F} = \frac{\Phi_0}{2\pi L_\varphi^2} = \frac{\hbar}{2eL_\varphi^2}$$

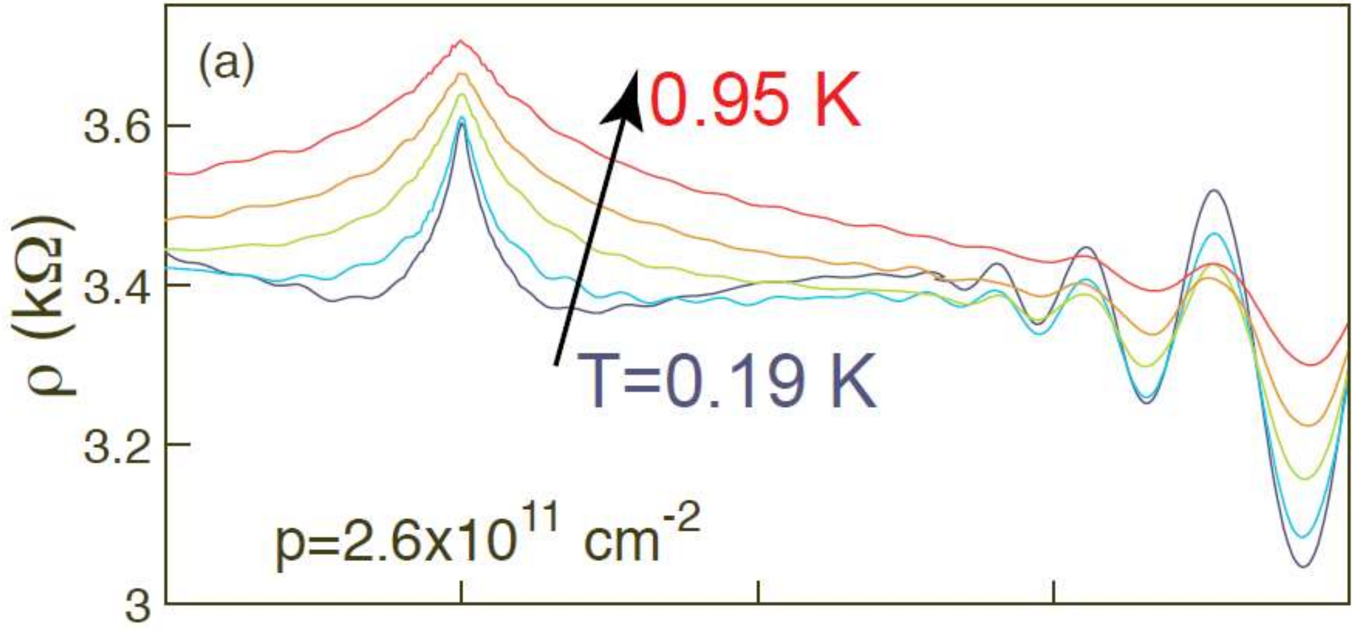
II.3.4 Weak Localization

- **weak localization: important facts**

  - coherent backscattering: called the **weak localization** (the relative number of contributing closed loops is small)
  - effect is important, since it is sensitive to weak magnetic fields:
    - **small fields**: contributions of large rings oscillate rapidly, phase difference in small rings almost unchanged
    - the **larger** the **field**, the fewer loops/rings contribute to constructive backscattering
    - resistance drops to classical value for large fields, if phase shift in smallest rings is about 2π
  - weak localization has to be distinguished from strong localization (due to strong disorder)

II.3.4 Weak Localization

- **weak localization: experiments**



weak localization
in SiGe 2-dimensional quantum well
with hole gas

V. Senz, Ph.D. Thesis, ETH Zürich (2002)

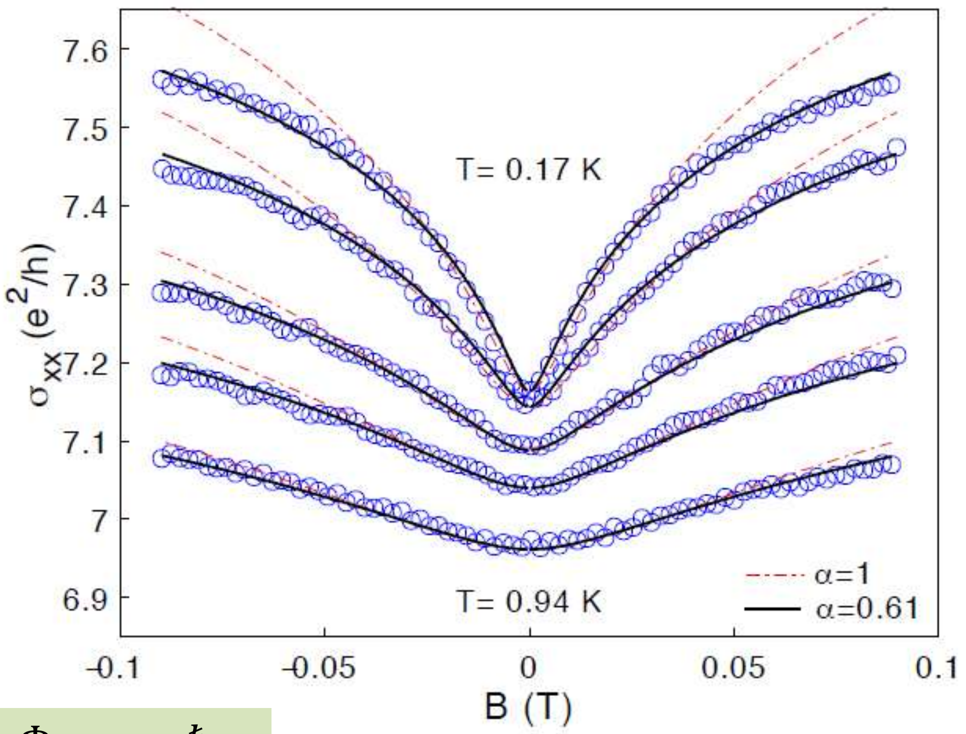
- **requirement:**

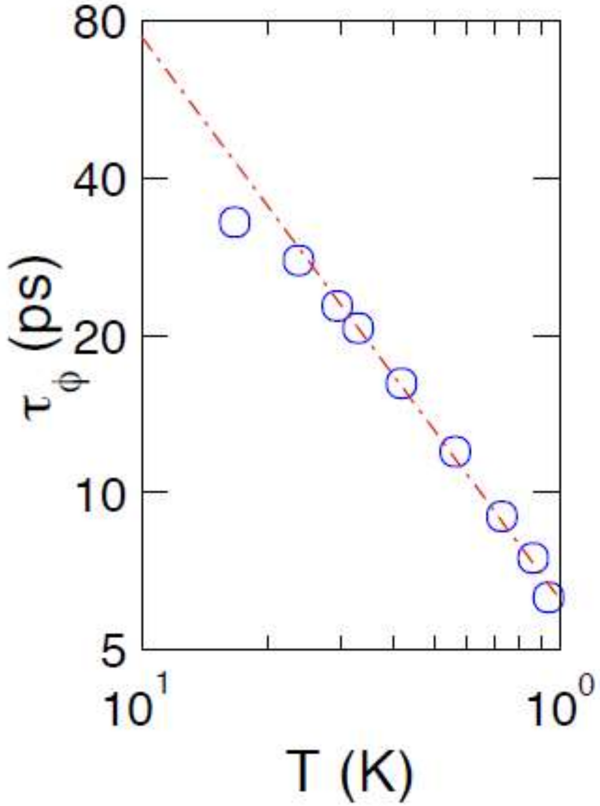
  sample larger than elastic scattering length: $L > \ell$ (diffusive transport)
- **observations**:
  - conductivity is reduced by $\approx 2e^2/h$ for $B_{\text{ext}} = 0$
  - large B_{ext}: Shubnikov de-Haas oscillations

II.3.4 Weak Localization

- **weak localization: measurement of phase coherence time**
  - as dependence of magnitude of WL on the coherence time is known to be $\tau_\varphi \simeq L_\varphi^2/D$

    ➔ ***weak localization experiments can be used to determine τ_φ***





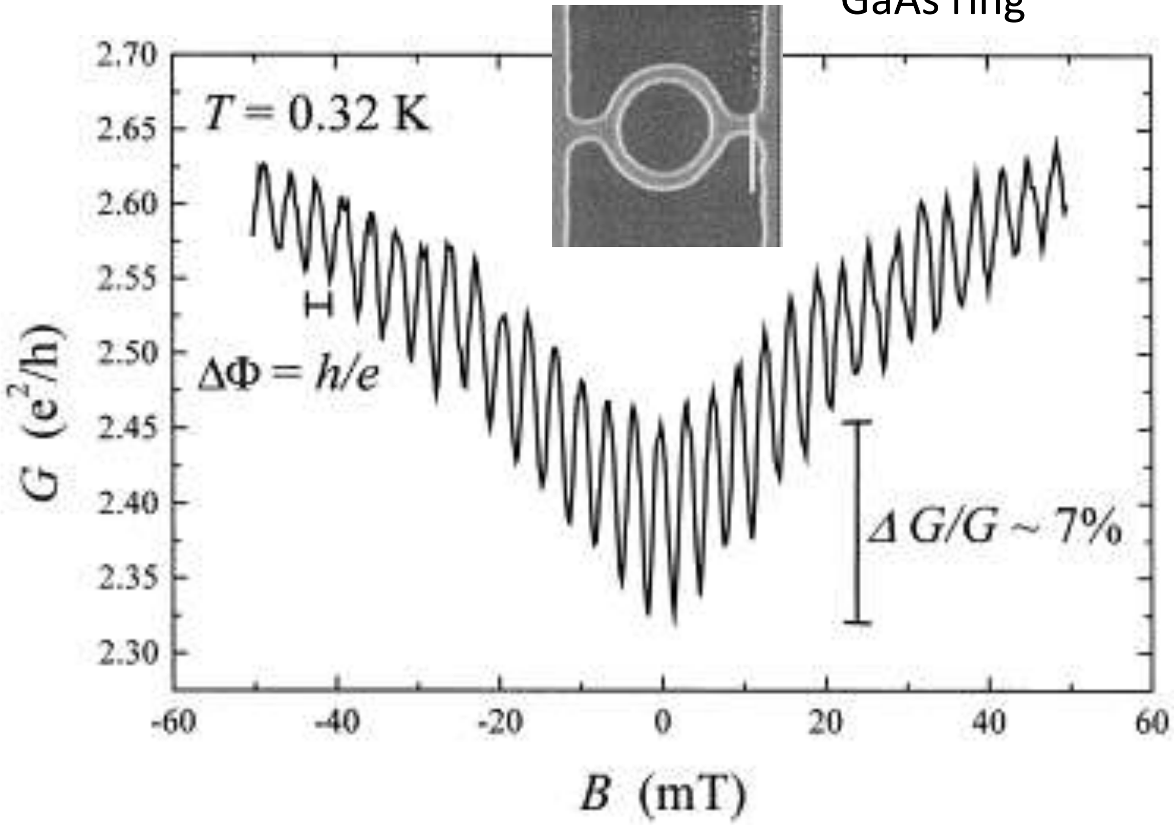
$$B^* = \frac{\Phi_0}{2F} = \frac{\Phi_0}{2\pi L_\varphi^2} = \frac{\hbar}{2eL_\varphi^2}$$

Senz et al., PRB **61**, 5082 (2000)

II.3.3 Aharonov-Bohm Effect

- **weak localization: in combination with Aharonov-Bohm effect**

GaAs ring

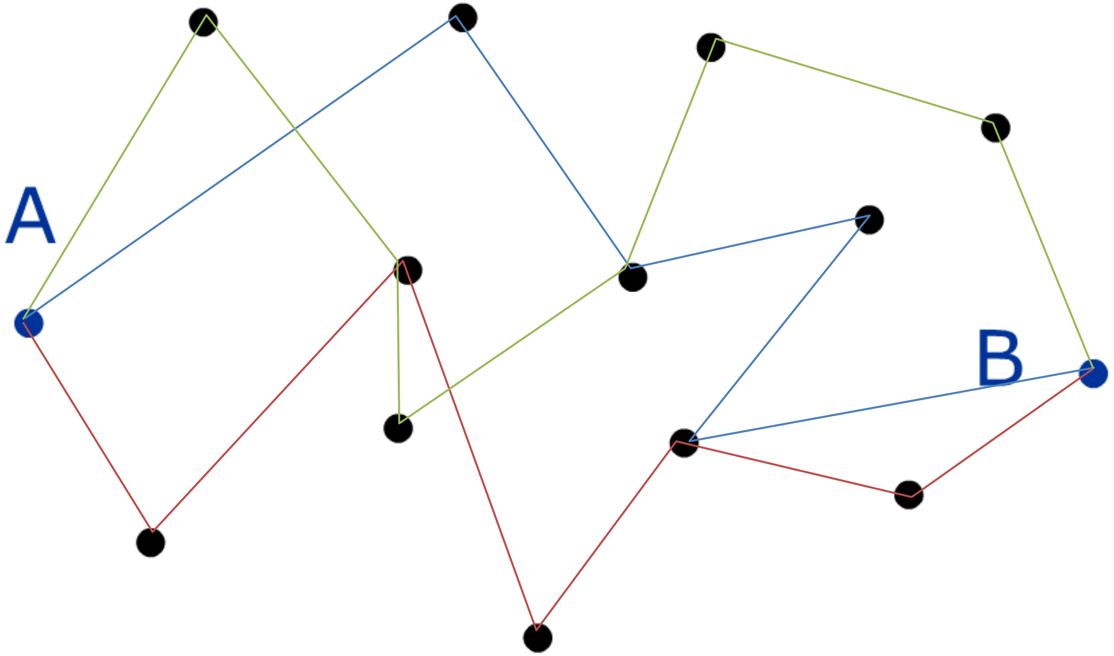


S. Pedersen, A.E. Hansen, A. Kristensen, C.B. Sørensen, P.E. Lindelof, Aharonov–Bohm effect in GaAs/GaAlAs ring interferometers, Materials Science and Engineering: B **74**, 234-238 (2000)

II.3.5 Universal Conductance Fluctuations

Universal Conductance Fluctuations:

fluctuation of conductance due to different configuration of scatters



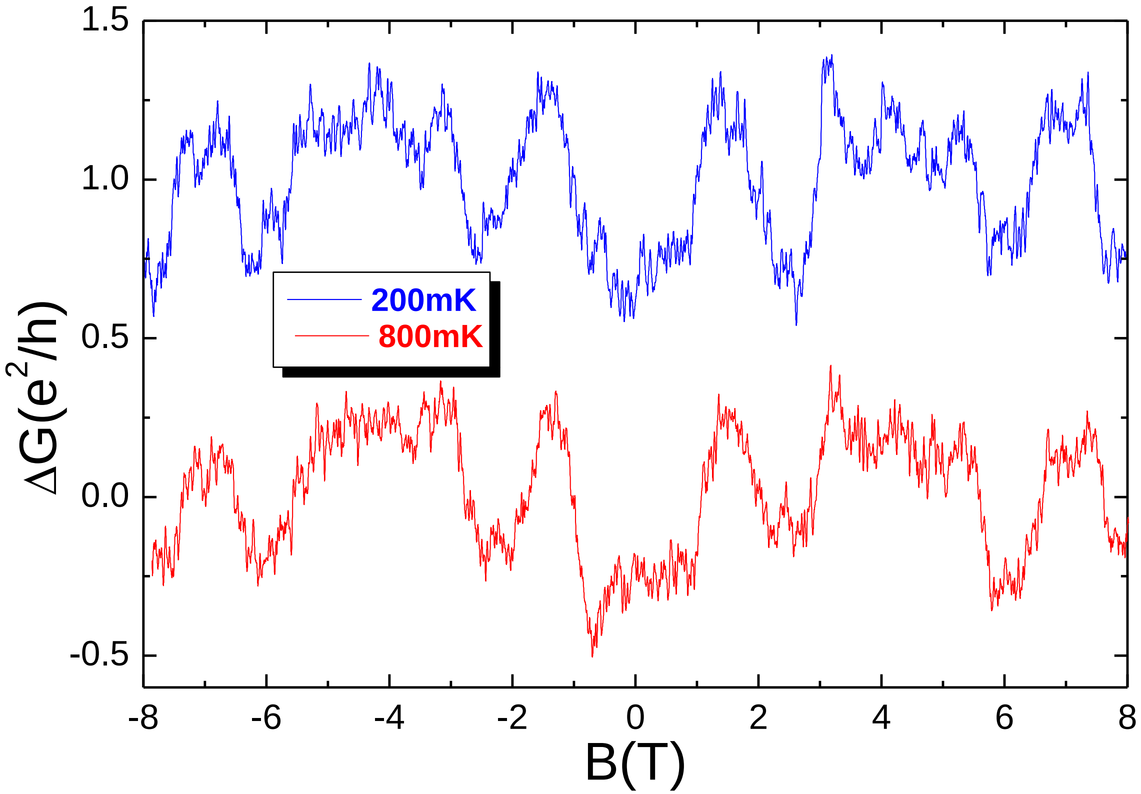
$$P_{\mathrm{AB}} = \left|\sum_p A_p \mathrm{e}^{\imath\chi_p}\right|^2 = \sum_p A_p^2 + \left|\sum_{p\neq p'} A_p A_{p'} \mathrm{e}^{\imath\left(\chi_p-\chi_{p'}\right)}\right|$$

phases χ_p depend on specific configuration of scatters in each sample

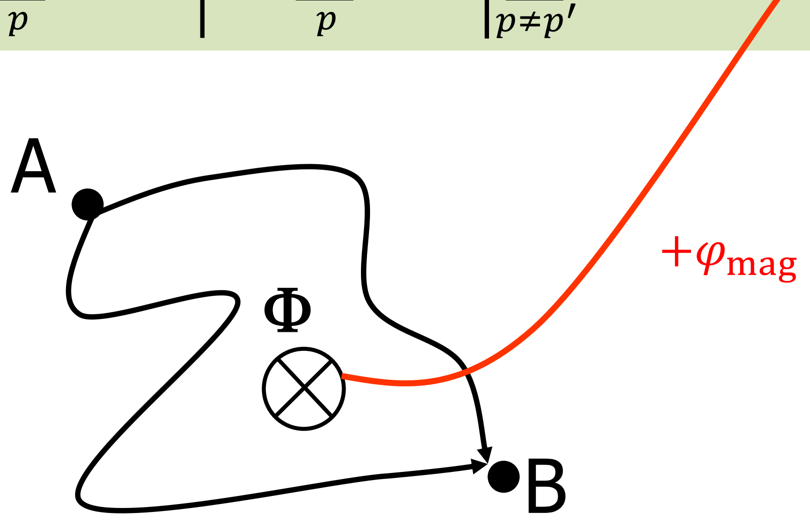
➔ P_{AB} and hence conductance is fingerprint of this configuration

II.3.5 Universal Conductance Fluctuations

- **experimental study of universal conductance fluctuations**
  - would require fabrication of many samples with different (random) configuration of scatters
  - **ergodicity theorem**: same result is obtained for a single sample measured at different applied magnetic fields



$$P_{\mathrm{AB}} = \left|\sum_p A_p \mathrm{e}^{\iota\chi_p}\right|^2 = \sum_p A_p^2 + \left|\sum_{p\neq p'} A_p A_{p'} \mathrm{e}^{\iota\left(\chi_p - \chi_{p'}\right)}\right|$$



➔ random phase shifts
➔ position of scatters becomes important

II.3.5 Universal Conductance Fluctuations

- experimental observations and facts
  - **irregular conductance variations** as a function of field (B), carrier density (n), and voltage (V)
  - **conductance variations are symmetric** with respect to $\boldsymbol{B}$ (2 probe setup)
  - different in each individual sample (”**magnetic fingerprint**”), fluctuations characterize **impurity configuration**
  - caused by **quantum interference**
  - **amplitude of conductance variations** is of the order $\boldsymbol{e^2/h}$**,** ***not*** noise
  - theory based on ergodicity theorem

variance of ensemble conductance:

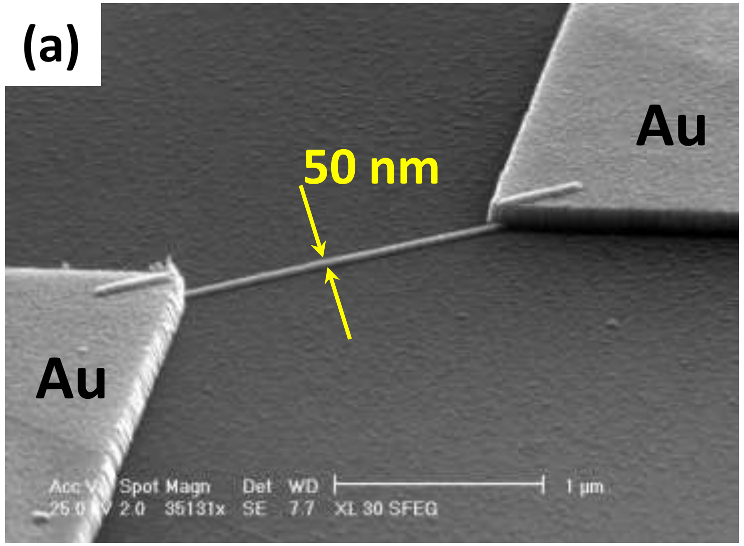
consider an ensemble of **macroscopically identical** but **microscopically different samples** (different configurations of scattering centers)

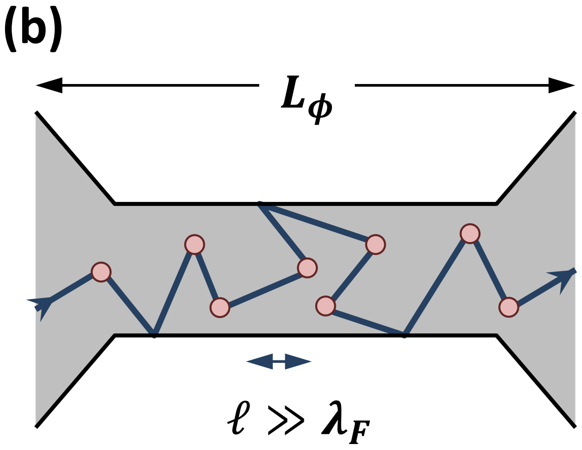
$$\langle (G-\langle G\rangle)^2\rangle = \frac{e^4}{h^2}\left\langle \left(\sum_{mn} T_{mn} - \sum_{mn}\langle T_{mn}\rangle\right)^2\right\rangle \qquad T_{mn} = |t_{mn}|^2$$

→ complicated calculation

II.3.5 Universal Conductance Fluctuations

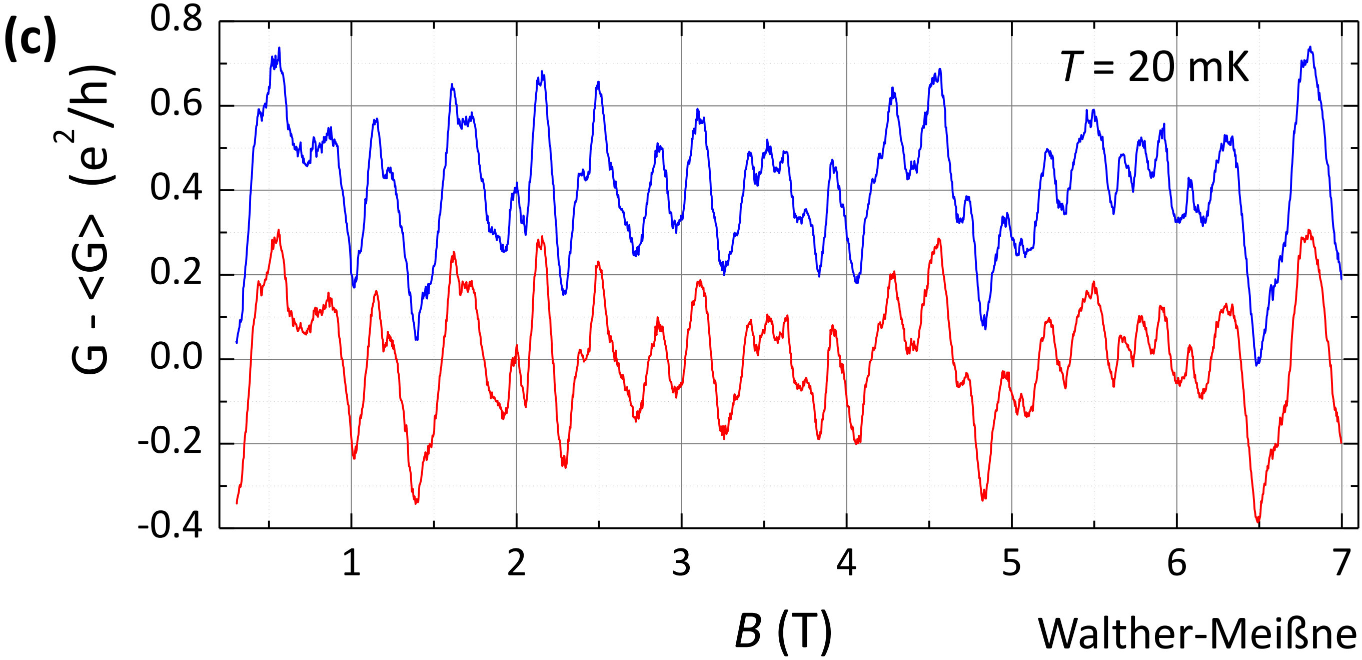
- **UCFs in Au wires**





red and blue curve taken at different days without warming up the sample

➔ no noise effect !!



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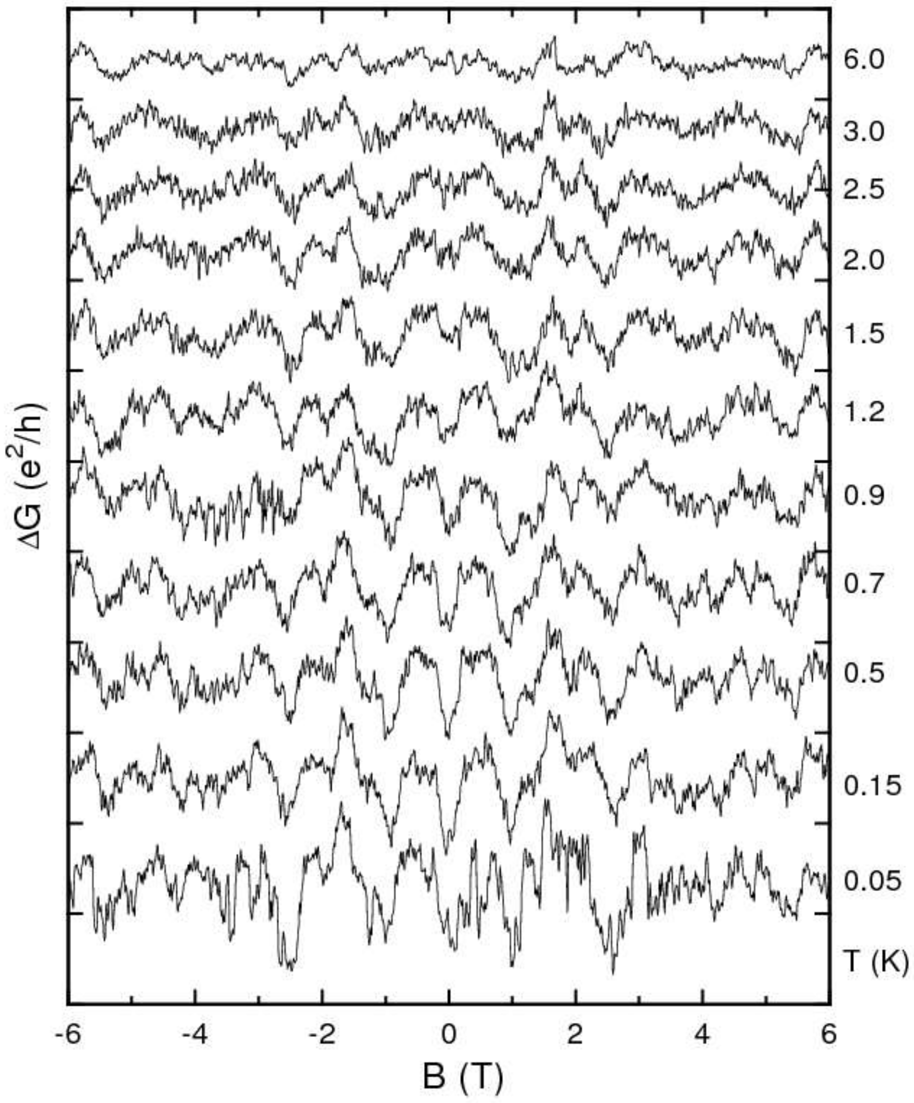
II.3.5 Universal Conductance Fluctuations

- **UCFs in Au wires**



UCF in gold nanowire

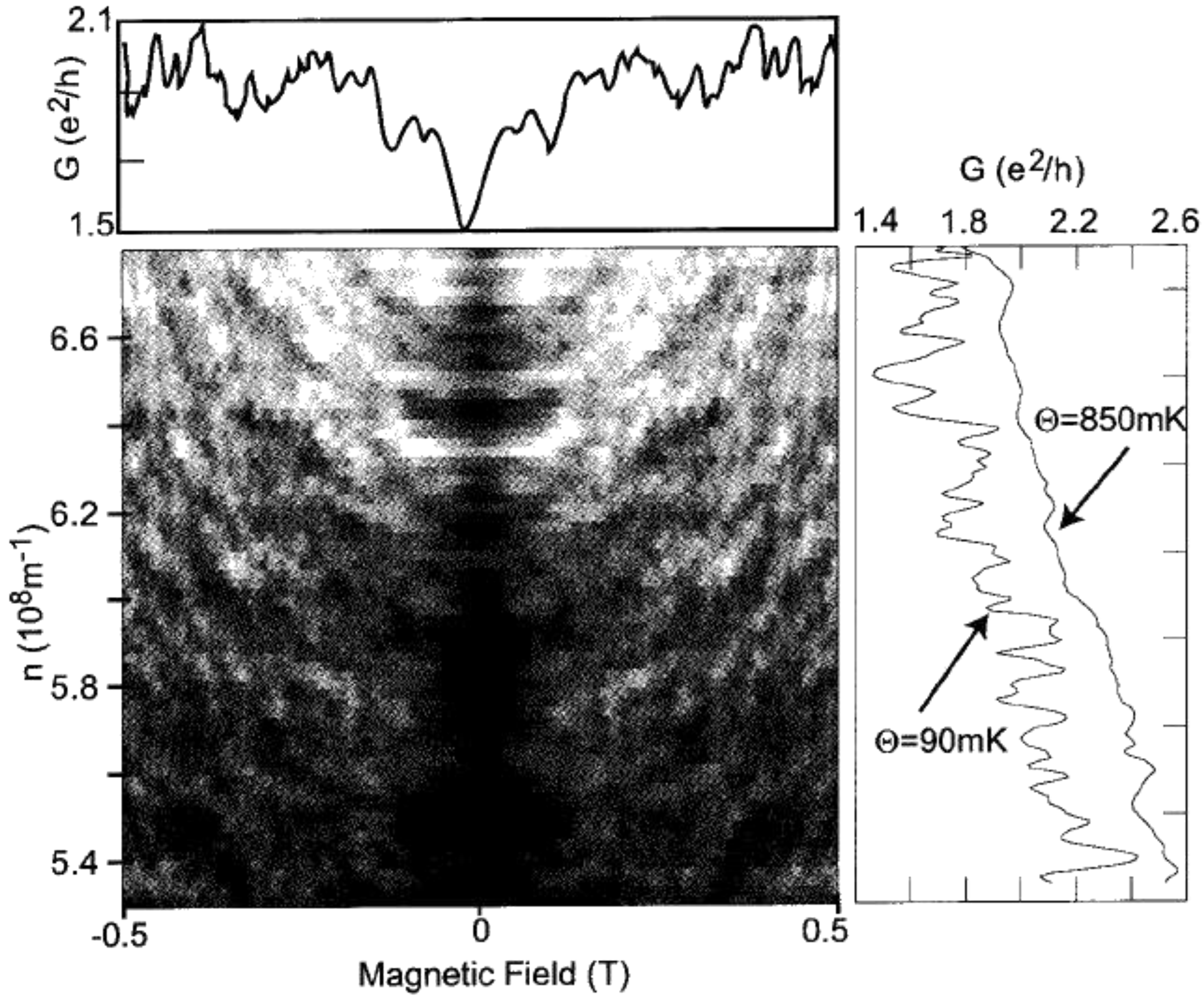
$L = 600$ nm

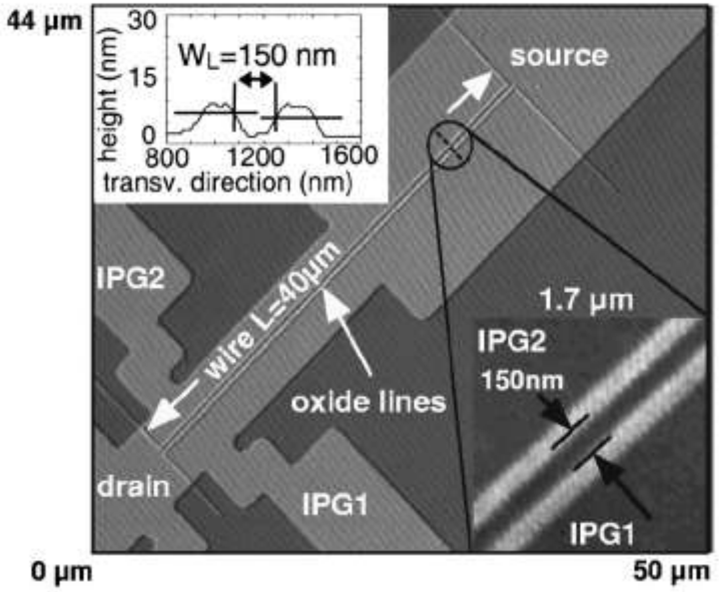
$W = 60$ nm

UCF amplitude decreases with increasing T as phase coherence length becomes smaller than sample length

H. Hegger, Ph.D. Thesis, Universität zu Köln (1997)

- **UCFs in GaAs quantum wire**





data from Heinzel (2003)

Contents Part II: Quantum Transport in Nanostructures

Contents:

II.4 From Quantum Mechanics to Ohm‘s Law

- **two different points of view:**

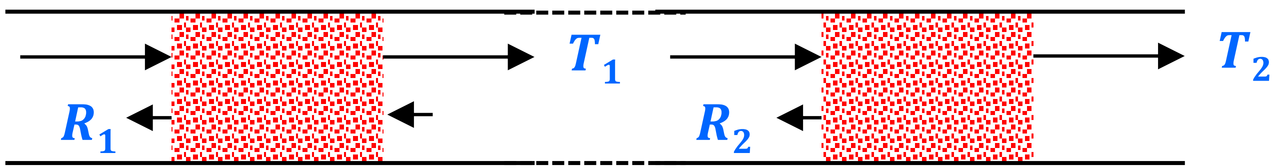
  ➔ quantum transport
  (electron waves, scattering/transfer matrix)

  ➔ classical transport
  (electric currents, charged particles, friction due to scattering, Ohm‘s law)

What is the bridge between these limiting cases ??

II.4 From Quantum Mechanics to Ohm‘s Law

- **consider two conductors with transmission probabilities T_1 and T_2 connected in series**



– what is the transmission probability T_{12} ?

– if $T_{12} = T_1 T_2$, then for a chain of scatterers we would expect the transmission probability to drop exponentially with the length of the chain:

$$T(L) = \exp(-L/L_0) \qquad \text{as } e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$$

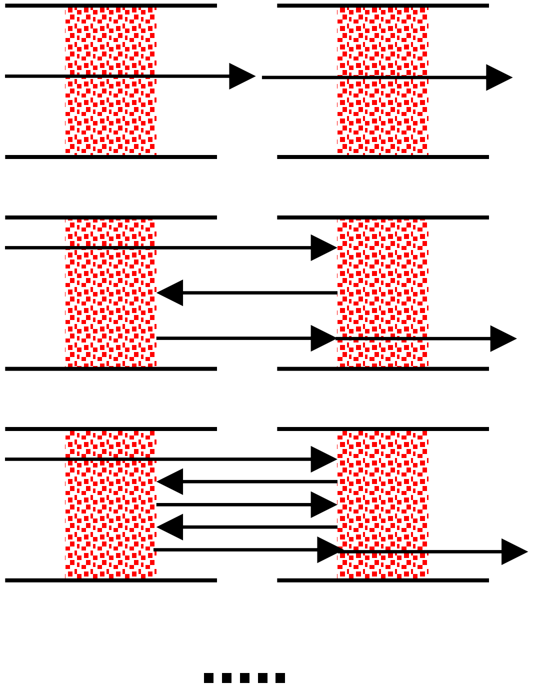
→ no Ohm‘s law

– ***problem***: if we assume $T_{12} = T_1 T_2$, then we do not take into account multiple reflections

→ to obtain the correct result we have to add the probabilities of multiply reflected paths

II.4 From Quantum Mechanics to Ohm‘s Law

- **two scatterers in series**



$$T_1 T_2$$

$$+$$

$$T_1 T_2 R_1 R_2$$

$$+$$

$$T_1 T_2 R_1^2 R_2^2$$

$$+$$

.....

transmission probabilities

$$T_{12} = \frac{T_1 T_2}{1 - R_1 R_2}$$

incoherent processes

with $T_1 = 1 - R_1$ and $T_2 = 1 - R_2$

$$\frac{1 - T_{12}}{T_{12}} = \frac{1 - T_1}{T_1} + \frac{1 - T_2}{T_2}$$

additive property

II.4 From Quantum Mechanics to Ohm‘s Law

- ***N* scatterers in series**

$$\frac{1-T(N)}{T(N)} = N\,\frac{1-T}{T} \quad \Longrightarrow \quad \boxed{T(N) = \frac{T}{N(1-T)+T}}$$

– number of scatterers in conductor of length L can be written as $N \;=\; n\,L$, where n is the linear density

$$\Longrightarrow \quad \boxed{T(L) = \frac{L_0}{L+L_0}} \quad \text{with} \quad L_0 = \frac{T}{n(1-T)}$$

– L_0 is of the order of the mean free path ℓ

$$\ell = \frac{1}{n(1-T)} \quad \Longrightarrow \quad \ell = \frac{1}{n(1-T)} \simeq \frac{T}{n(1-T)} = L_0 \qquad \text{(for } T \text{ close to 1)}$$

linear density of scatterers

scattering probability

II.4 From Quantum Mechanics to Ohm‘s Law

- **quantum conductance for N channels**

  - wide conductor with $M \approx k_\mathrm{F} W/\pi$ modes: $$G \approx 2G_Q M\,T = 2\frac{e^2}{h} M\,T \approx \frac{e^2 W}{\pi}\,T\,\frac{2k_\mathrm{F}}{h}$$

  - 2D density of tranverse modes: $$n_{2D} = \frac{1}{2\pi}\frac{2m}{\hbar^2} \Longrightarrow n_{2\mathrm{D}} v_\mathrm{F} = \frac{1}{2\pi}\frac{2m}{\hbar^2}\frac{\hbar k_\mathrm{F}}{m} = \frac{2k_\mathrm{F}}{h}$$

$$G \approx \frac{e^2 W}{\pi}\,T\,\frac{2k_\mathrm{F}}{h} = \frac{e^2 W}{\pi}\,T\,n_{2\mathrm{D}} v_\mathrm{F}$$

  - using $T(L) = \frac{L_0}{L+L_0}$ yields: $$G \approx \frac{W}{L+L_0}\,\underbrace{e^2 n_{2\mathrm{D}}\,\underbrace{v_\mathrm{F}\,L_0\pi}_{\approx\ \textit{diffusion constant } D}}_{\approx\ \sigma\ \textit{(Einstein relation)}}$$

$$\Longrightarrow \boxed{G \approx \frac{W}{L+L_0}\,\sigma}$$

or

$$\boxed{R = \frac{1}{G} \approx \frac{L+L_0}{W}\frac{1}{\sigma} = \frac{L}{\sigma W} + \frac{L_0}{\sigma W}}$$

resistance obeying Ohm‘s law

length independent interface resistance

II.4 From Quantum Mechanics to Ohm‘s Law

- conclusions

  – **Ohm‘s law is obtained from the expression for the quantum conductance**

  → by summing up ***probabilities of multiply reflected paths***

  → note that by summing up probabilities ***coherence effects are neglected***
  (of course these are not contained in Ohm‘s law, incoherent transport)

  – sample size $L \gg$ phase coherence length L_φ: large phase shifts (also affected by disorder)

  formally identical samples: - very different phase shifts,
  - but same ohmic resistance, since interference effects average out for $L \gg L_\varphi$

  – $L < L_\varphi$: interference effects play important role
  → deviation from Ohm‘s law
  → different resistance for formally identical samples due to different impurity configurations

II.4 From Quantum Mechanics to Ohm‘s Law

- **Where is the resistance ??**
  - expression for quantum conductance: $G = 2\frac{e^2}{h} M\, T$

    → scatterers give rise to resistance by reducing T

  - example: waveguide with M modes and a single scatterer

$$\frac{1}{G} = \underbrace{\frac{h}{2e^2}\frac{1}{M}}_{\text{„interface“ resistance}} + \underbrace{\frac{h}{2e^2}\frac{1}{M}\frac{1-T}{T}}_{\text{„scatterer“ resistance}}$$

→ scatterer resistance determined by properties of scatterer via its transmissivity

  - remaining questions:

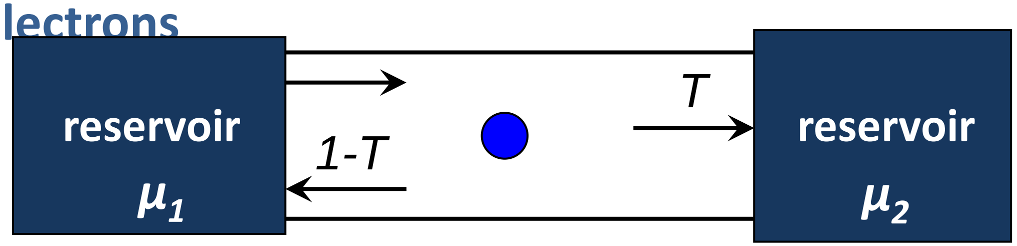
    → can we associate a resistance with the scatterer ?

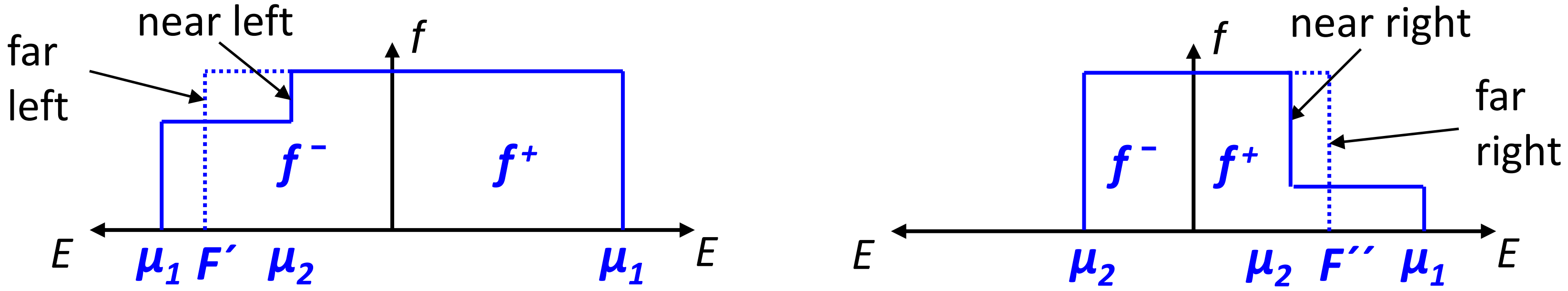
    → what about the potential drop ? Does it occur across the scatterer ?

    → what about Joule heating ? Dissipation at the scatterer ?

II.4 From Quantum Mechanics to Ohm‘s Law

- **energy distribution of the electrons**





- reservoirs: $f^+ = \vartheta(\mu_1 - E)$ $f^- = \vartheta(\mu_2 - E)$

step functions

- near left and right:

  (non-equilibrium)

$$f^+ = \vartheta(\mu_1 - E) + T\{\vartheta(\mu_1 - E) - \vartheta(\mu_2 - E)\}$$

$$f^- = \vartheta(\mu_2 - E) + (1 - T)\{\vartheta(\mu_1 - E) - \vartheta(\mu_2 - E)\}$$

partial filling
of states for
$\mu_2 < E < \mu_1$

- far left and right: $f^+ = \vartheta(F' - E)$ $f^- = \vartheta(F'' - E)$

  (equilibrium)

$$F' = \mu_2 + (1 - T)\{\mu_1 - \mu_2\} \qquad F'' = \mu_2 + T\{\mu_1 - \mu_2\}$$

(follows from the conservation of the number of electrons)

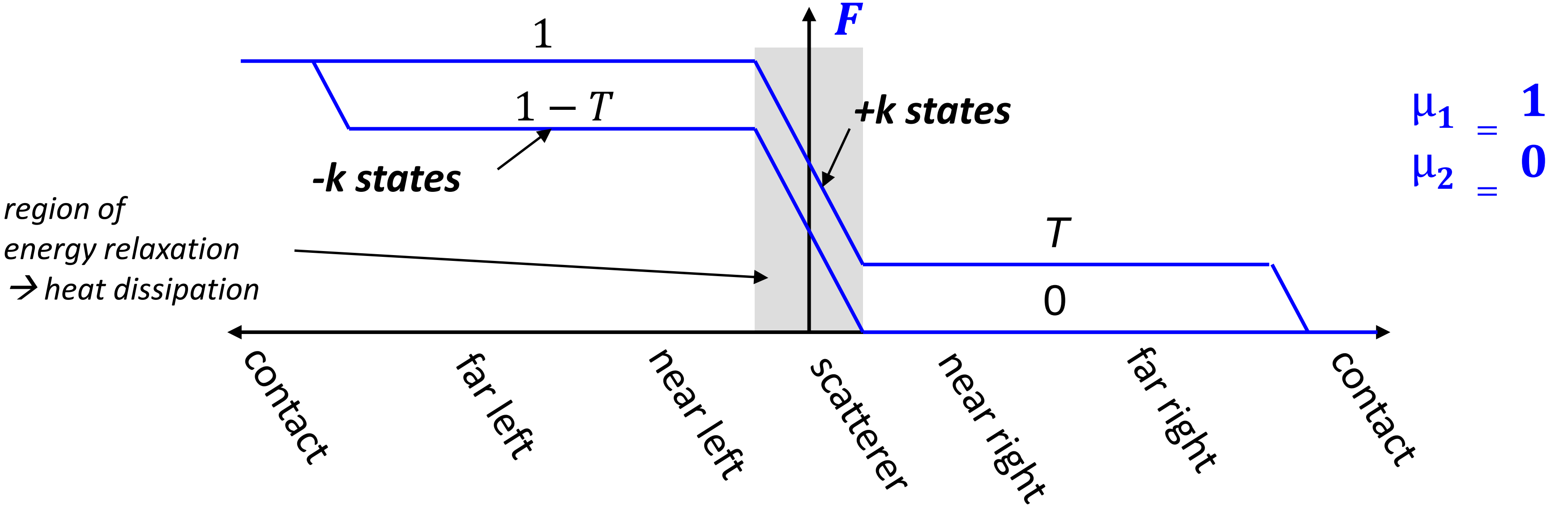
II.4 From Quantum Mechanics to Ohm‘s Law

- **spatial variation of the electrochemical potential**

- left and right to the scatterer *(after energy relaxation):*

$$F^+ = \mu_1 \qquad \text{(left)} \qquad\qquad F^- = \mu_2 + (1-T)\{\mu_1 - \mu_2\} \qquad \text{(left)}$$
$$F^+ = \mu_2 + T\{\mu_1 - \mu_2\} \qquad \text{(right)} \qquad\qquad F^- = \mu_2 \qquad \text{(right)}$$

- close to scatterer *(nonequilibrium distribution, F can be defined via the number of electrons)*



- drop of electrochemical potential across scatterer → localized „scatterer“ resistance
- drop close to contact → contact resistance

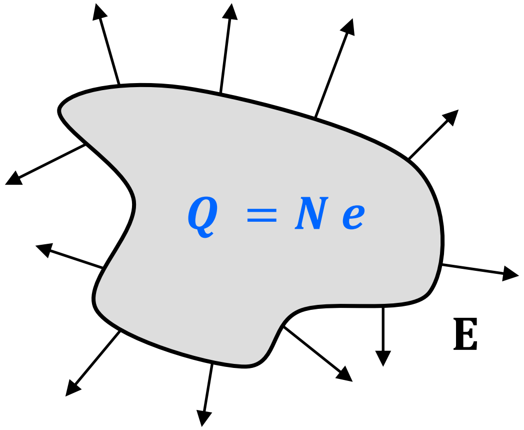
Contents Part II: Quantum Transport in Nanostructures

Contents:

II.5 Coulomb Blockade

- **charge quantization and charging energy**

  – *electric charge* is *quantized* for an isolated island



  – charging energy:

$$\varepsilon = \frac{Q^2}{2C} = \frac{N^2 e^2}{2C} = n^2 E_c \qquad \text{with} \quad \varepsilon_c = \frac{e^2}{2C}$$

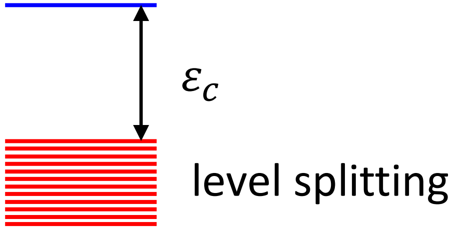
  – how large is ε_c for island of size L *(bring charge e from ∞ to island)*

$$\varepsilon_c \simeq \frac{e^2}{\epsilon_0 L} \approx \frac{10\ \text{eV}}{L\ [\text{nm}]}$$

typically in *meV regime* for 100 nm-sized samples

  – level splitting in nm-sized island:

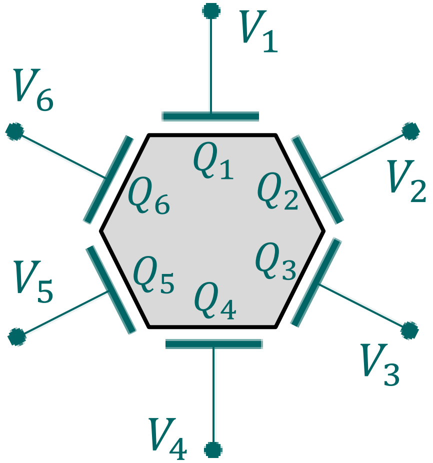
$$\delta\varepsilon \simeq \frac{\varepsilon_\text{F}}{N_\text{atom}} \approx \frac{1\ \text{eV}}{L^3\ [\text{nm}^3]}$$



typically *in µeV regime* for 100 nm-sized samples

II.5 Coulomb Blockade

- capacitance model for metallic island



– charge on the island $Q_0 = \sum_{i=1}^{k} C_i V_i + \bar{Q}_0$ ← charge for all $V_i = 0$ „background charge“

– potential V_0 of the island is not known, but its charge Q_0 is known to be Ne
➔ electrostatic potential of the island:

$$V_0(Q_0) = \frac{Q_0 - \bar{Q}_0}{C_\Sigma} - \sum_{i=1}^{k} \frac{C_i}{C_\Sigma} V_i \quad \text{with} \quad C_\Sigma = \sum_{i=1}^{k} C_i$$

– electrostatic energy needed to put additional charge $\Delta Q = Ne$ on island

$$\varepsilon_{\mathrm{el}}(N) = \int_{\bar{Q}_0}^{\bar{Q}_0+Ne} V_0(Q_0)\, dQ_0 = \frac{(Ne)^2}{2C_\Sigma} - eN \sum_{i=1}^{k} \frac{C_i}{C_\Sigma} V_i$$

– energy needed to charge the island with one additional charge $\Delta Q = e$

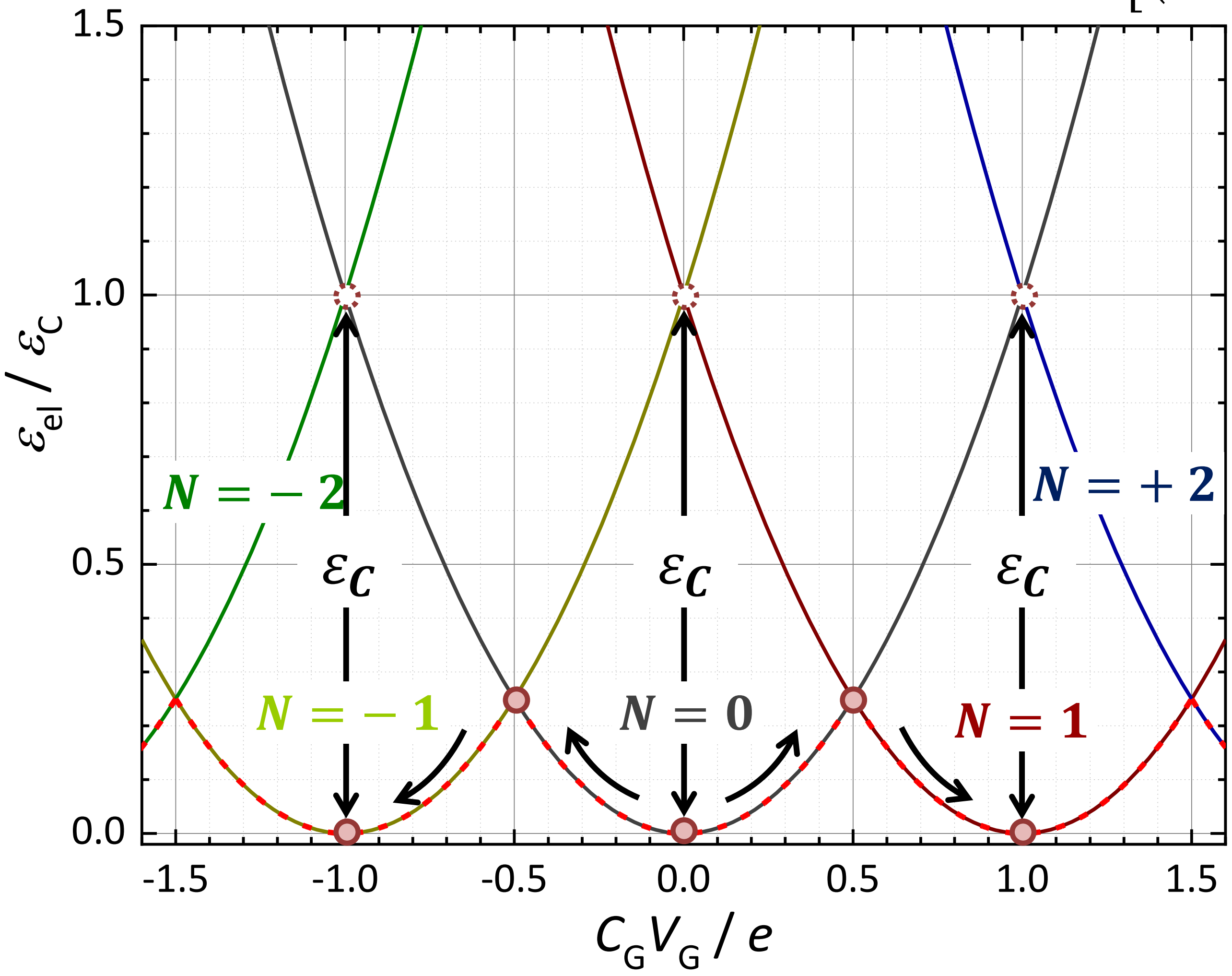
$$\varepsilon_{\mathrm{el}}(N+1) - \varepsilon_{\mathrm{el}}(N) = \frac{e^2}{C_\Sigma}\left(N + \frac{1}{2}\right) - e \sum_{i=1}^{k} \frac{C_i}{C_\Sigma} V_i$$

II.5 Coulomb Blockade

$$\varepsilon_{\mathrm{el}}(\Delta Q = Ne) = \frac{(Ne)^2}{2C_\Sigma} - eN\sum_{i=1}^{k}\frac{C_i}{C_\Sigma}V_i$$

- **capacitance model for metallic island – only a single capacitance ($C_\Sigma = C$)**
  - electrostatic energy $\varepsilon_{\mathrm{el}}(\Delta Q = Ne) = \frac{(Ne)^2}{2C} - eNV = \frac{e^2}{2C}\left[\left(N - \frac{CV}{e}\right)^2 - \left(\frac{CV}{2}\right)^2\right]$

omitted as independent of N



$$\varepsilon_{\mathrm{el}}(\Delta Q) = \frac{e^2}{2C}\left(N - \frac{CV}{e}\right)^2 = E_C\left(N - \frac{CV}{e}\right)^2$$

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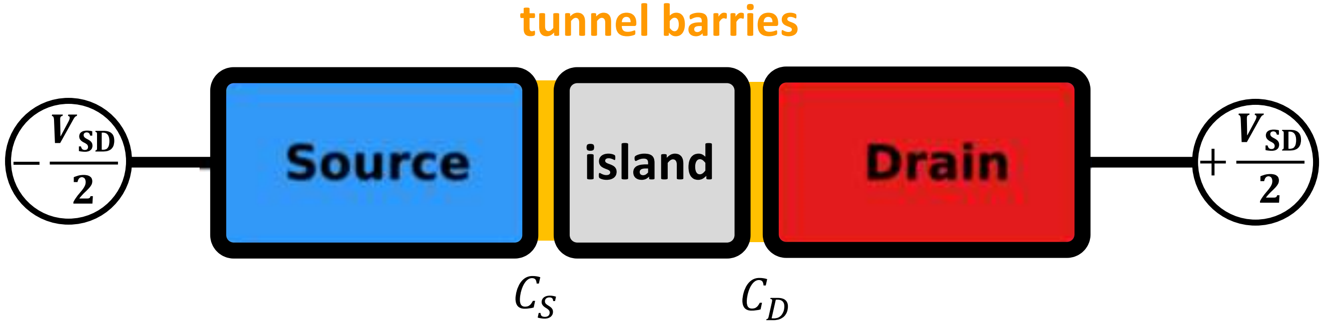
II.5 Coulomb Blockade

- **capacitance model for 2-terminal device**



$$\varepsilon_{\mathrm{el}}(\Delta Q) = \frac{(Ne)^2}{2C_\Sigma} - eN\sum_{i=1}^{k}\frac{C_i}{C_\Sigma}V_i$$

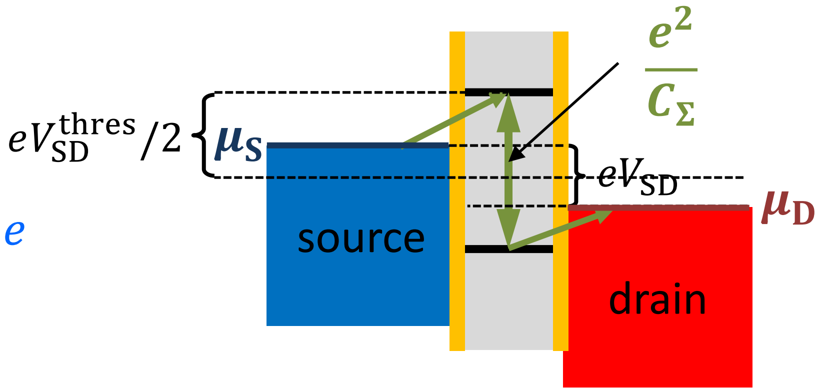
$$C_\Sigma = C_S + C_D$$

– electrostatic energy barrier for removing one electron to drain: $\Delta Q = +e$ on island

$$\varepsilon_{\mathrm{el}}(\Delta Q) = \frac{e^2}{2C_\Sigma} + e\frac{C_{\mathrm{S}}}{C_\Sigma}\frac{(-V_{\mathrm{SD}})}{2} - e\frac{C_{\mathrm{D}}}{C_\Sigma}\frac{V_{\mathrm{SD}}}{2} \underbrace{=}_{C_S=C_D=C} \frac{e^2}{2C_\Sigma} - \frac{eV_{\mathrm{SD}}}{2}$$

– electrostatic energy barrier for adding an electron from source: $\Delta Q = -e$

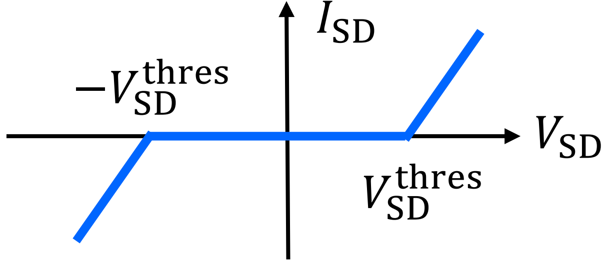
$$\varepsilon_{\mathrm{el}}(\Delta Q) = \frac{e^2}{2C_\Sigma} + e\frac{C_{\mathrm{S}}}{C_\Sigma}\frac{(-V_{\mathrm{SD}})}{2} - e\frac{C_{\mathrm{D}}}{C_\Sigma}\frac{V_{\mathrm{SD}}}{2} \underbrace{=}_{C_S=C_D=C} \frac{e^2}{2C_\Sigma} - \frac{eV_{\mathrm{SD}}}{2}$$



– at $T = 0$, current transport sets in if energy barrier is reduced to zero

threshold SD-voltage: $\left|V_{\mathrm{SD}}^{\mathrm{thres}}\right| = \frac{e}{C_\Sigma}$

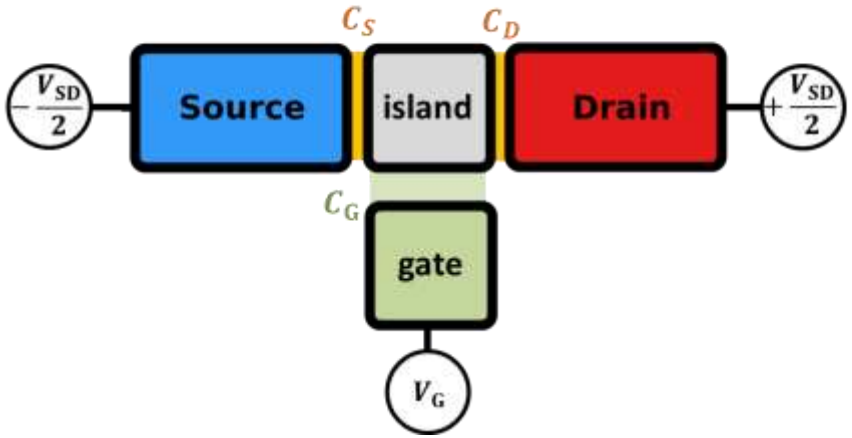
Coulomb blockade effect
for $|V_{\mathrm{SD}}| \leq \left|V_{\mathrm{SD}}^{\mathrm{thres}}\right|$



II.5 Coulomb Blockade

$$\varepsilon_{\mathrm{el}}(\Delta Q) = \frac{(Ne)^2}{2C_\Sigma} - eN\sum_{i=1}^{k}\frac{C_i}{C_\Sigma}V_i$$

- **capacitance model for SET: electrostatic energy**

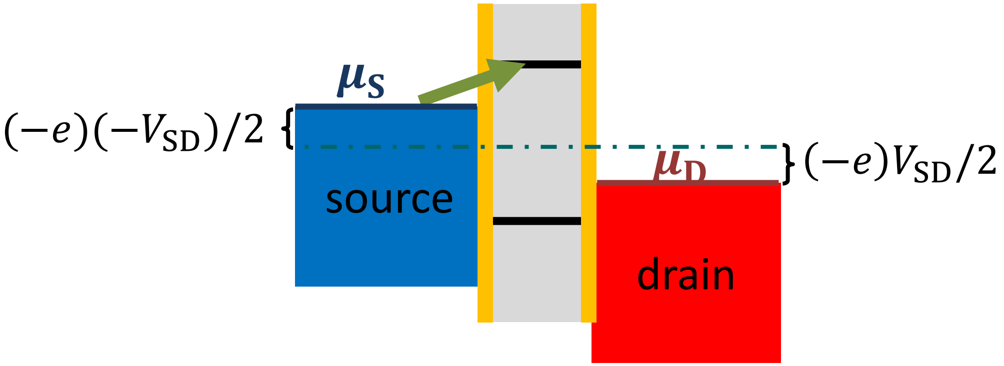


– charging the neutral island by $\Delta Q = -Ne$ from source at constant V_G

$$\varepsilon_{\mathrm{el}}(\Delta N) = \frac{(-Ne)^2}{2C_\Sigma} + Ne\frac{C_{\mathrm{S}}}{C_\Sigma}\frac{(-V_{\mathrm{SD}})}{2} - Ne\frac{C_{\mathrm{D}}}{C_\Sigma}\frac{V_{\mathrm{SD}}}{2} - Ne\frac{C_{\mathrm{G}}}{C_\Sigma}V_G \underset{C_S=C_D=C}{\underbrace{=}} \frac{(Ne)^2}{2C_\Sigma} - Ne\left(\frac{C}{C_\Sigma}V_{\mathrm{SD}} + \frac{C_{\mathrm{G}}}{C_\Sigma}V_{\mathrm{G}}\right)$$

– electrostatic energy difference between adding
$\Delta N = N + 1$ and $\Delta N = N$ electrons

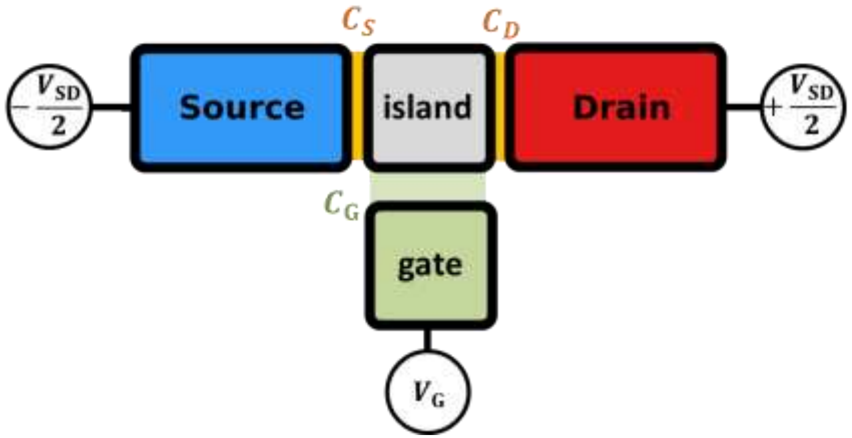
$$\varepsilon_{\mathrm{el}}(N+1) - \varepsilon_{\mathrm{el}}(N) = \left(N + \frac{1}{2}\right)\frac{e^2}{C_\Sigma} - e\left(\frac{C}{C_\Sigma}V_{\mathrm{SD}} + \frac{C_{\mathrm{G}}}{C_\Sigma}V_G\right)$$



II.5 Coulomb Blockade

$$\varepsilon_{\mathrm{el}}(\Delta Q) = \frac{(Ne)^2}{2C_\Sigma} - eN\sum_{i=1}^{k}\frac{C_i}{C_\Sigma}V_i$$

- **capacitance model for SET: electrostatic energy**



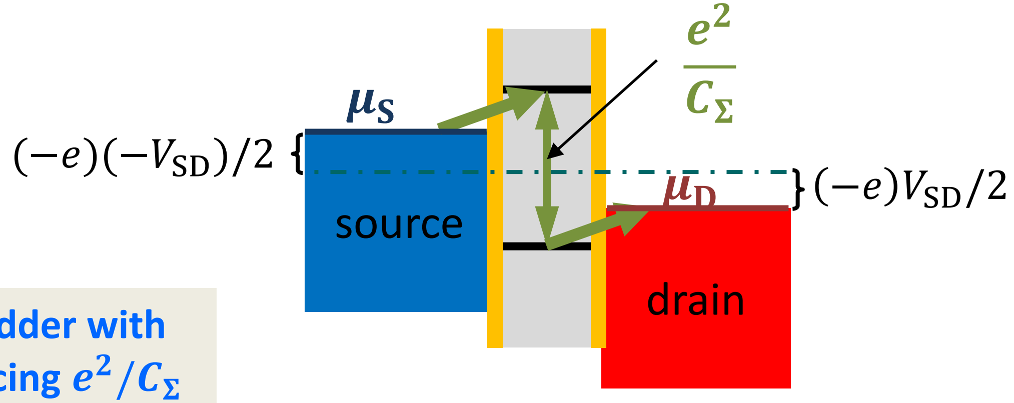
– charging the island by removing $\Delta Q = -Ne$ to drain at constant V_G (corresponds to adding $\Delta Q = +Ne$ to island)

$$\varepsilon_{\mathrm{el}}(\Delta N) = \frac{(\Delta Ne)^2}{2C_\Sigma} - N\,e\,\frac{C_{\mathrm{S}}}{C_\Sigma}\frac{-V_{\mathrm{SD}}}{2} + Ne\frac{C_{\mathrm{D}}}{C_\Sigma}\frac{V_{\mathrm{SD}}}{2} + Ne\frac{C_{\mathrm{G}}}{C_\Sigma}V_G \underbrace{=}_{C_S=C_D=C} \frac{(\Delta Ne)^2}{2C_\Sigma} + Ne\left(\frac{C}{C_\Sigma}V_{\mathrm{SD}} + \frac{C_{\mathrm{G}}}{C_\Sigma}V_{\mathrm{G}}\right)$$

– electrostatic energy difference between removing $\Delta N = N-1$ and $\Delta N = N$ electrons

$$\varepsilon_{\mathrm{el}}(N) - \varepsilon_{\mathrm{el}}(N-1) = \left(N - \frac{1}{2}\right)\frac{e^2}{C_\Sigma} + e\left(\frac{C}{C_\Sigma}V_{\mathrm{SD}} + \frac{C_{\mathrm{G}}}{C_\Sigma}V_G\right)$$

$$\varepsilon_{\mathrm{el}}(N+1) - \varepsilon_{\mathrm{el}}(N) - [\varepsilon_{\mathrm{el}}(N) - \varepsilon_{\mathrm{el}}(N-1)] = \frac{e^2}{C_\Sigma}$$

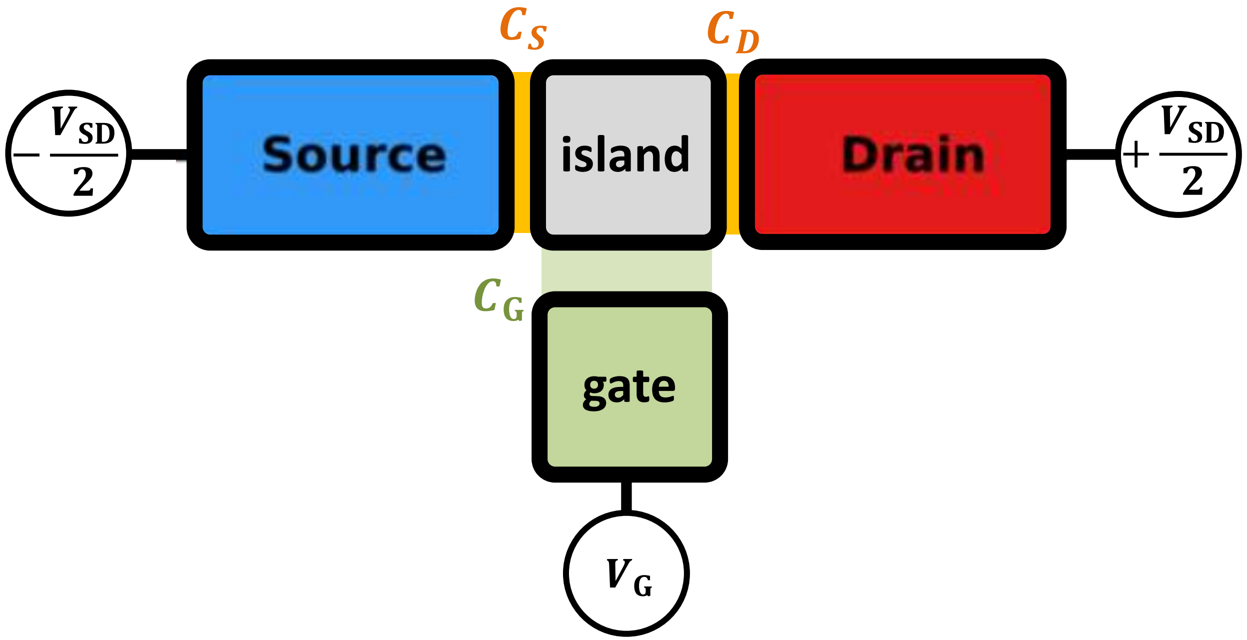


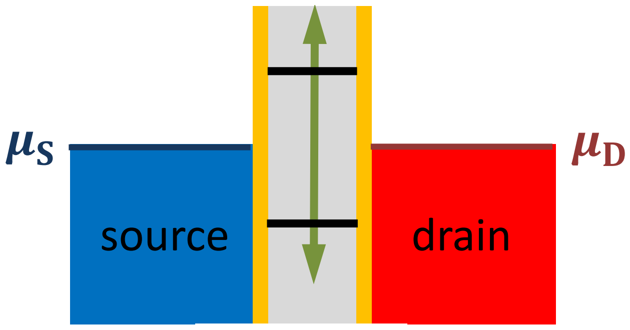
energy ladder with fixed spacing e^2/C_Σ

II.5 Coulomb Blockade

- capacitance model for 3-terminal device: single electron transistor (SET)

$$\varepsilon_{\mathrm{el}}(\Delta Q) = \frac{(Ne)^2}{2C_\Sigma} - eN\sum_{i=1}^{k}\frac{C_i}{C_\Sigma}V_i$$





additional V_G shifts potential energy of island

- electrostatic energy barrier for removing one electron to drain ($\Delta Q = +e$) [or adding one electron from source ($\Delta Q = -e$)] at finite V_{G}

$$\varepsilon_{\mathrm{el}}(\Delta Q) = \frac{e^2}{2C_\Sigma} + e\frac{C_{\mathrm{S}}}{C_\Sigma}\frac{-V_{\mathrm{SD}}}{2} - e\frac{C_{\mathrm{D}}}{C_\Sigma}\frac{V_{\mathrm{SD}}}{2} - e\frac{C_{\mathrm{G}}}{C_\Sigma}V_G \underbrace{=}_{C_S=C_D=C} \frac{e^2}{2C_\Sigma} - eV_{\mathrm{SD}} - e\frac{C_{\mathrm{G}}}{C_\Sigma}V_G \quad \text{with} \quad C_\Sigma = C_S + C_D + C_G$$

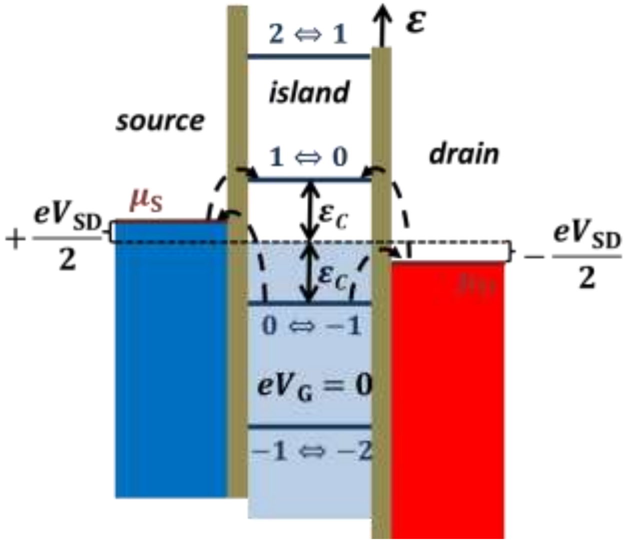
at $V_{\mathrm{SD}} \simeq 0$: $\varepsilon_{\mathrm{el}}(\Delta Q) = \frac{e^2}{2C_\Sigma} - \underbrace{eV_{\mathrm{SD}}}_{\simeq 0} - e\frac{C_{\mathrm{G}}}{C_\Sigma}V_{\mathrm{G}}$ ➔ transport allowed for $V_{\mathrm{G}}^{\mathrm{trans}} = \frac{e}{2C_{\mathrm{G}}}$

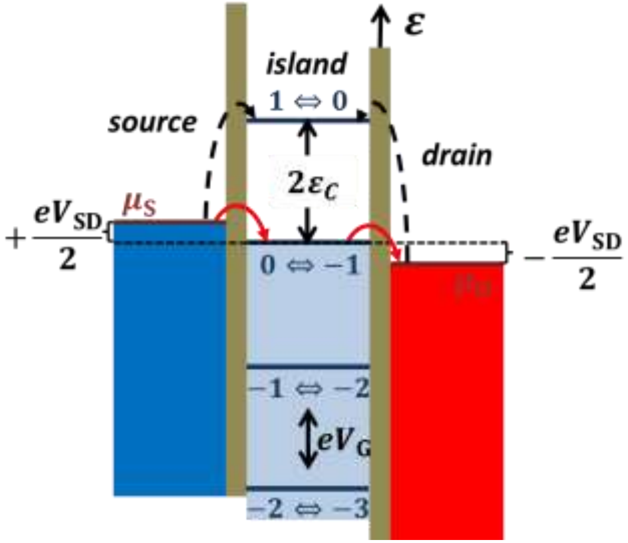
- analog result for adding one electron from source ($\Delta Q = -e$) at finite V_{G}

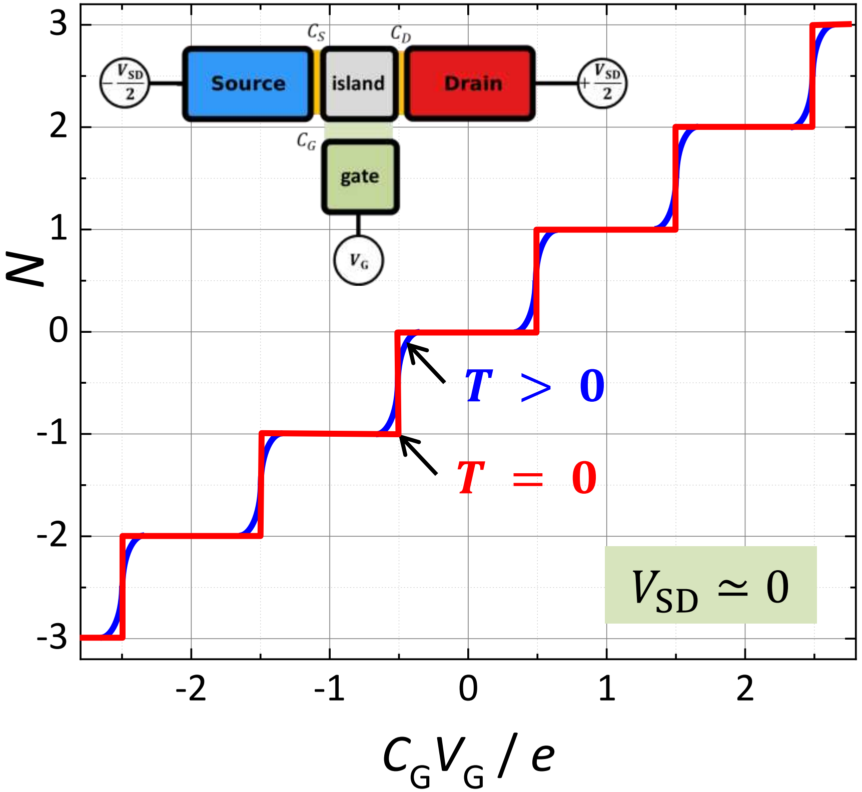
➔ periodic peaks in I_{SD} at $V_{\mathrm{G}} = N \cdot \frac{e}{2C_{\mathrm{G}}}$

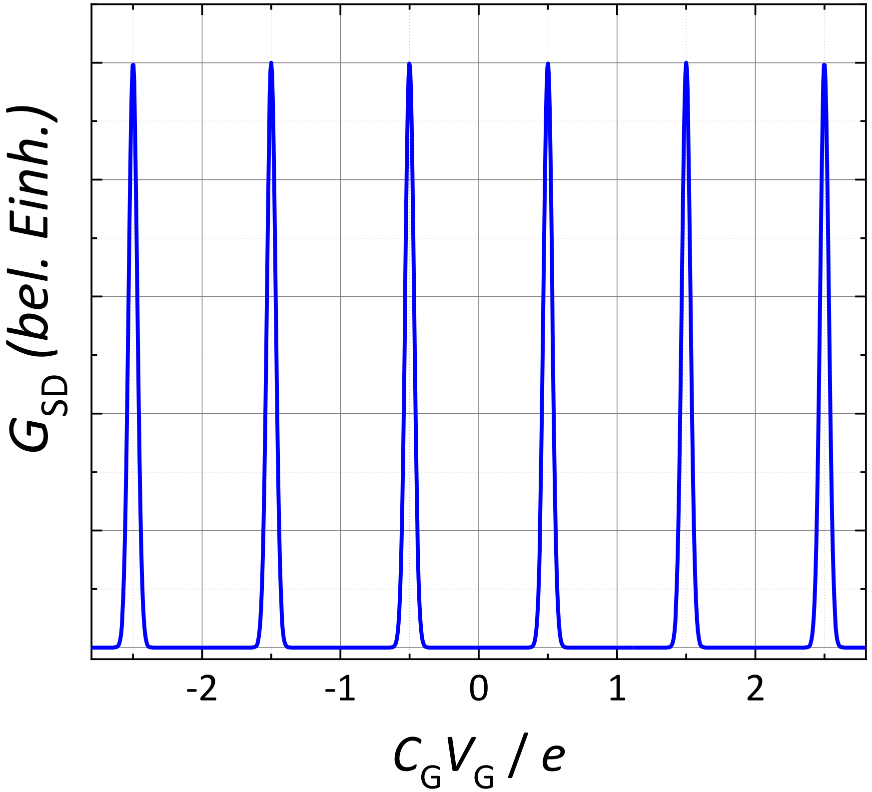
II.5 Coulomb Blockade

- **capacitance model for SET: current flow at $V_{\mathrm{SD}} \simeq 0$ as a function of V_{G}**







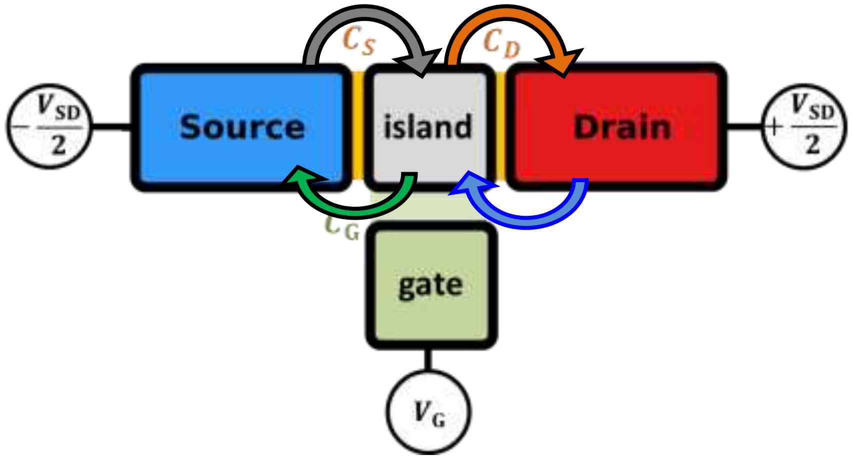


for $V_{\mathrm{SD}} \simeq 0$:

periodic peaks in SD-current I_{SD} at $V_{\mathrm{G}} = N \cdot \frac{e}{2C_{\mathrm{G}}}$

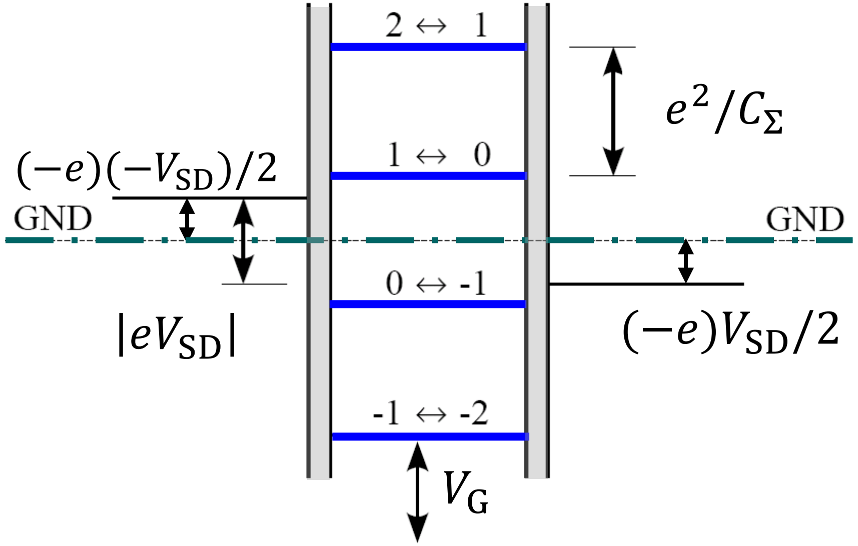
II.5 Coulomb Blockade

- **capacitance model for SET: current flow at finite $V_{\mathrm{SD}}, V_{\mathrm{G}}$**



at a given N on island, four different electron transfer processes are possible

1. **from the left**: $N \rightarrow N+1$ $\quad \Delta\varepsilon_{\mathrm{FL}}(N) = \varepsilon_{\mathrm{el}}(N+1) - \varepsilon_{\mathrm{el}}(N)$
2. **to the left**: $N \rightarrow N-1$ $\quad \Delta\varepsilon_{\mathrm{TL}}(N) = \varepsilon_{\mathrm{el}}(N-1) - \varepsilon_{\mathrm{el}}(N)$
3. **from the right**: $N \rightarrow N+1$ $\quad \Delta\varepsilon_{\mathrm{FR}}(N) = \varepsilon_{\mathrm{el}}(N+1) - \varepsilon_{\mathrm{el}}(N)$
4. **to the right**: $N \rightarrow N-1$ $\quad \Delta\varepsilon_{\mathrm{TR}}(N) = \varepsilon_{\mathrm{el}}(N-1) - \varepsilon_{\mathrm{el}}(N)$

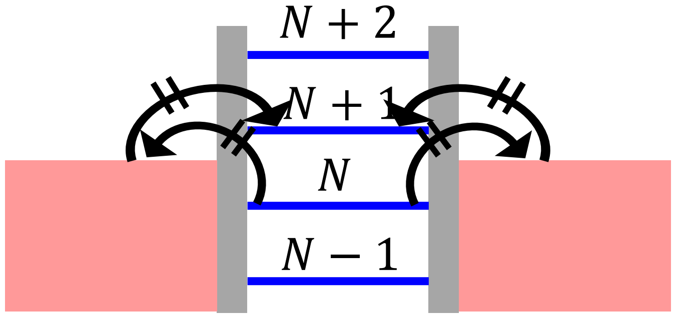


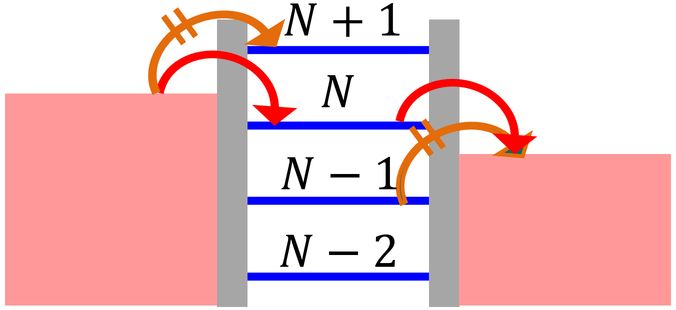
II.5 Coulomb Blockade

- capacitance model for SET: current flow at finite $V_{\text{SD}}, V_{\text{G}}$

allowed and forbidden electron transfer processes:

- ➢ $\boldsymbol{T > 0}$: all transfer processes are allowed (by thermal activation)
- ➢ $\boldsymbol{T = 0}$: only transfer processes with $\Delta\varepsilon < 0$ are allowed





Coulomb blockade

single electron tunneling

$$\boldsymbol{\Delta\varepsilon_{\text{FL,TL,FR,TR}}(N) > 0}$$

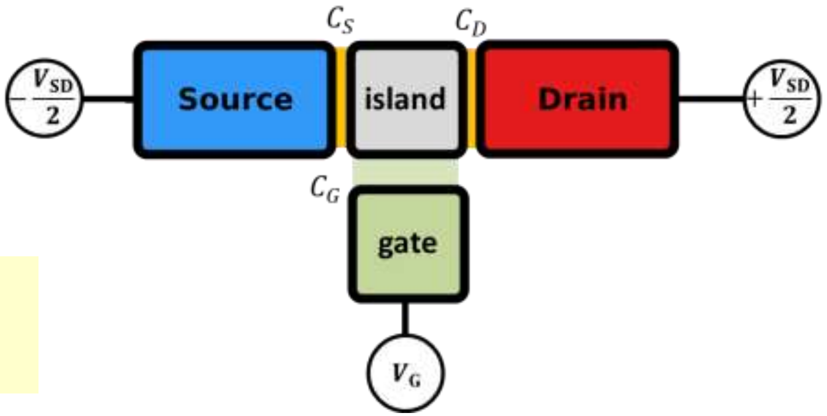
$$\boldsymbol{\Delta\varepsilon_{\text{FL}}(N) < 0 \qquad \Delta\varepsilon_{\text{TR}}(N) < 0}$$

$$\boldsymbol{\Delta\varepsilon_{\text{FL}}(N+1) > 0 \qquad \Delta\varepsilon_{\text{TR}}(N-1) > 0}$$

no second additional or missing electron on island !!

II.5 Coulomb Blockade

- **capacitance model for SET: current flow at finite $V_{\mathrm{SD}}, V_{\mathrm{G}}$**
  - in which range of V_{SD} and V_{G} is the electron transport blocked ?
  - assumptions: $C_S = C_D = C$, symmetric SD voltage bias



A
$$\varepsilon_{\mathrm{el}}(N+1) - \varepsilon_{\mathrm{el}}(N) = \left(N + \frac{1}{2}\right)\frac{e^2}{C_\Sigma} - e\left(\frac{C}{C_\Sigma}V_{\mathrm{SD}} + \frac{C_{\mathrm{G}}}{C_\Sigma}V_G\right)$$

B
$$\varepsilon_{\mathrm{el}}(N) - \varepsilon_{\mathrm{el}}(N-1) = \left(N - \frac{1}{2}\right)\frac{e^2}{C_\Sigma} + e\left(\frac{C}{C_\Sigma}V_{\mathrm{SD}} + \frac{C_{\mathrm{G}}}{C_\Sigma}V_G\right)$$

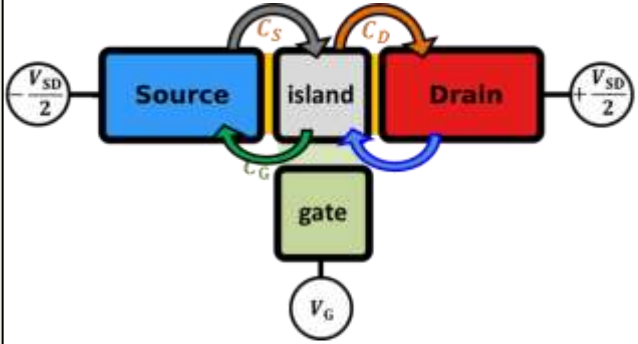
1. **from the left**: $N \rightarrow N+1$ $\quad \Delta\varepsilon_{\mathrm{FL}}(0) = \varepsilon(+1) - \varepsilon(0) = -\frac{1}{2}\frac{e^2}{C_\Sigma} - e\left(\frac{C}{C_\Sigma}V_{\mathrm{SD}} + \frac{C_{\mathrm{G}}}{C_\Sigma}V_G\right)$ $\quad (N=-1, N+1=0)$ A
2. **to the left**: $N \rightarrow N-1$ $\quad \Delta\varepsilon_{\mathrm{TL}}(0) = \varepsilon(-1) - \varepsilon(0) = -\frac{1}{2}\frac{e^2}{C_\Sigma} + e\left(\frac{C}{C_\Sigma}V_{\mathrm{SD}} + \frac{C_{\mathrm{G}}}{C_\Sigma}V_G\right)$ $\quad (N=+1, N-1=0)$ B
3. **from the right**: $N-1 \rightarrow N$ $\quad \Delta\varepsilon_{\mathrm{FR}}(0) = \varepsilon(0) - \varepsilon(-1) = +\frac{1}{2}\frac{e^2}{C_\Sigma} - e\left(\frac{C}{C_\Sigma}V_{\mathrm{SD}} + \frac{C_{\mathrm{G}}}{C_\Sigma}V_G\right)$ $\quad (N-1=-1, N=0)$ A
4. **to the right**: $N+1 \rightarrow N$ $\quad \Delta\varepsilon_{\mathrm{TR}}(0) = \varepsilon(0) - \varepsilon(+1) = +\frac{1}{2}\frac{e^2}{C_\Sigma} + e\left(\frac{C}{C_\Sigma}V_{\mathrm{SD}} + \frac{C_{\mathrm{G}}}{C_\Sigma}V_G\right)$ $\quad (N+1=1, N=0)$ B

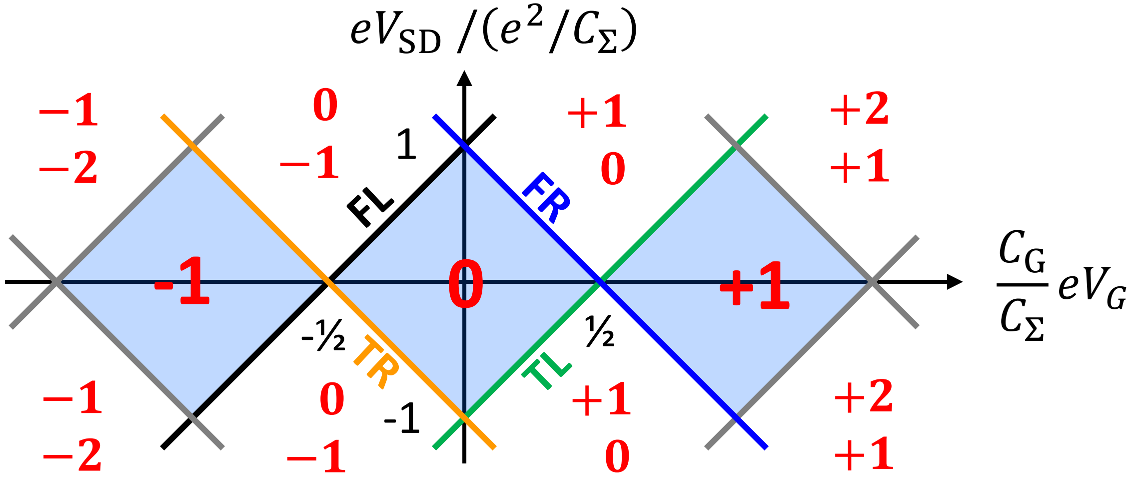
We consider processes where the final state is always the state with $N = 0$

II.5 Coulomb Blockade

- **capacitance model for SET: current flow at finite $V_{\mathrm{SD}}, V_{\mathrm{G}}$**

1. **from the left:** $N \rightarrow N+1$ $\quad \Delta\varepsilon_{\mathrm{FL}}(0) = \varepsilon(+1) - \varepsilon(0) = -\frac{1}{2}\frac{e^2}{C_\Sigma} - e\left(\frac{C}{C_\Sigma}V_{\mathrm{SD}} + \frac{C_{\mathrm{G}}}{C_\Sigma}V_G\right)$
2. **to the left:** $N \rightarrow N-1$ $\quad \Delta\varepsilon_{\mathrm{TL}}(0) = \varepsilon(-1) - \varepsilon(0) = -\frac{1}{2}\frac{e^2}{C_\Sigma} + e\left(\frac{C}{C_\Sigma}V_{\mathrm{SD}} + \frac{C_{\mathrm{G}}}{C_\Sigma}V_G\right)$
3. **from the right:** $N-1 \rightarrow N$ $\quad \Delta\varepsilon_{\mathrm{FR}}(0) = \varepsilon(0) - \varepsilon(-1) = +\frac{1}{2}\frac{e^2}{C_\Sigma} - e\left(\frac{C}{C_\Sigma}V_{\mathrm{SD}} + \frac{C_{\mathrm{G}}}{C_\Sigma}V_G\right)$
4. **to the right:** $N+1 \rightarrow N$ $\quad \Delta\varepsilon_{\mathrm{TR}}(0) = \varepsilon(0) - \varepsilon(+1) = +\frac{1}{2}\frac{e^2}{C_\Sigma} + e\left(\frac{C}{C_\Sigma}V_{\mathrm{SD}} + \frac{C_{\mathrm{G}}}{C_\Sigma}V_G\right)$

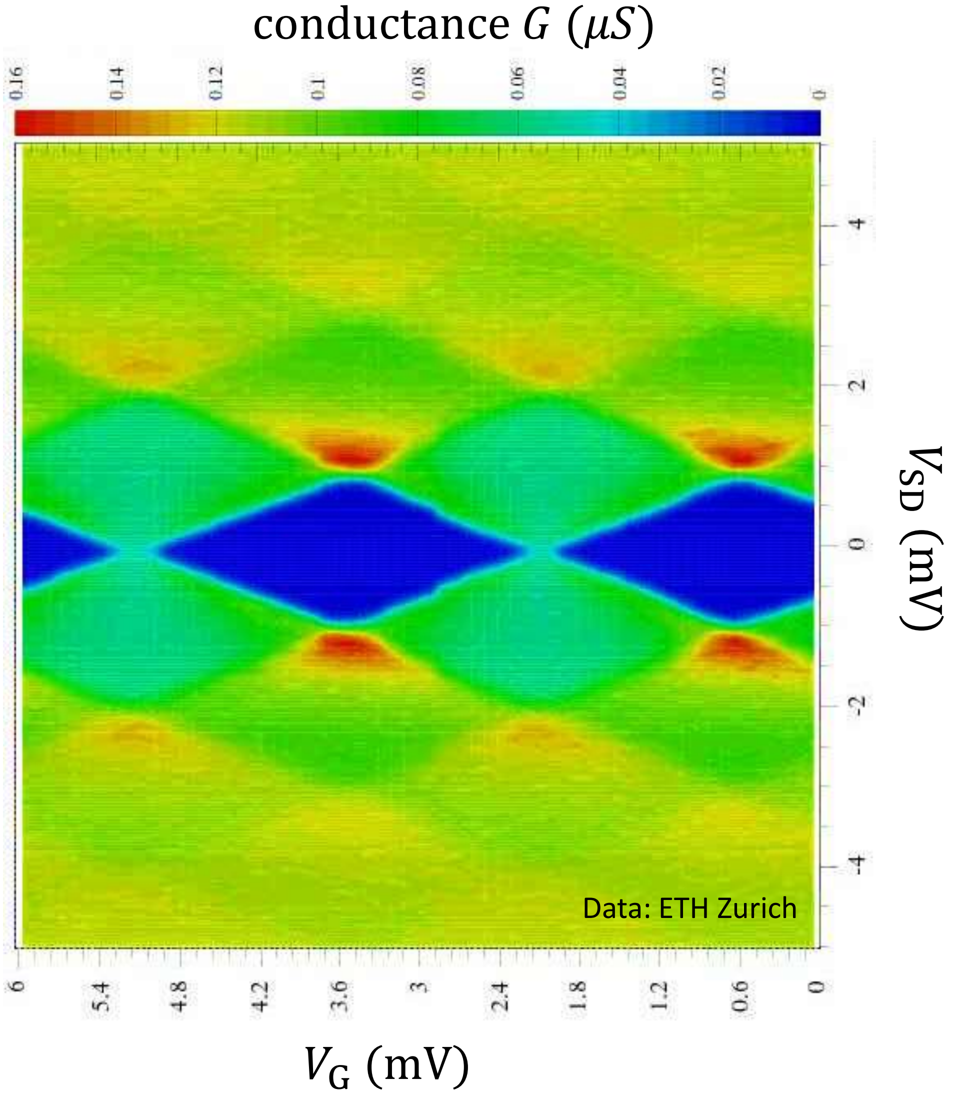




blue areas mark blockade regimes:
„***Coulomb diamonds***“

II.5 Coulomb Blockade

- **capacitance model for SET: current flow at finite V_{SD}, V_G**



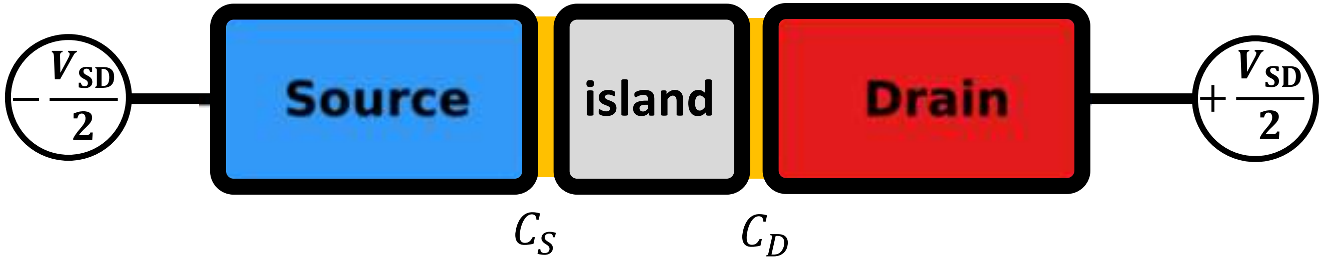
Data: ETH Zurich

Single Electron Transistor – Coulomb Diamonds:

blue regions of vanishing conductance correspond to the Coulomb blockade regime (no current flow)

II.5 Coulomb Blockade

- capacitance model for SET: current flow at finite $V_{\mathrm{SD}}, V_{\mathrm{G}}$



tunneling barries
(characterized by tunneling resistance R)

weak coupling of island to metallic leads (reservoirs)

→ too weak: no electron transfer

→ too strong: no conservation of charge number, no single electron effects

too little

just right

too much

II.5 Coulomb Blockade

- **requirements for the experimental observation of the Coulomb blockade:**

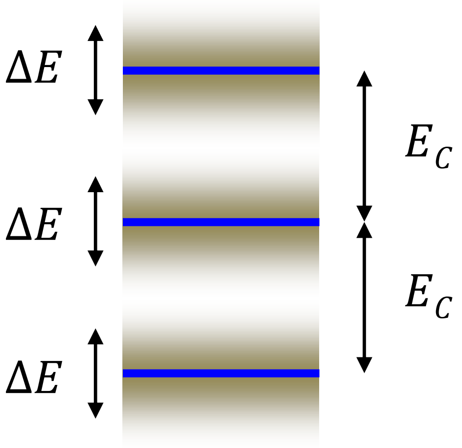
  – thermal fluctuations must be small enough:

$$E_c = \frac{e^2}{2C} > k_\mathrm{B}T \quad \Rightarrow \quad C < \frac{e^2}{2k_\mathrm{B}T} \approx 1\ \mathrm{fF}\ \ @\ 1\ \mathrm{K}$$

  – quantum fluctuations must be small enough:

$$E_c = \underbrace{\frac{\hbar}{\tau} \simeq \frac{\hbar}{RC}}_{\text{level broadening } \Delta E} \quad \Rightarrow \quad R > \frac{h}{e^2} = R_Q \simeq 25\ k\Omega$$

R_Q = *quantum resistance*



  – requirement for voltage:

$$E_c > eV \quad \Rightarrow \quad V < \frac{e}{2C} \approx 80\ \mu\mathrm{V}\ \ @\ 1\ \mathrm{fF}$$

II.5 Coulomb Blockade

- **SET: current-voltage characteristics**
  - facts: (i) charging state is determined by N
    (ii) no quantum coherence between different states
  - probability $p_N(t)$ to find system in state N at time t:
    → given by *Master equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,p_N(t) = -\underbrace{[\Gamma_F(N)+\Gamma_T(N)]p_N(t)}_{\substack{\text{tunneling from}\\ \text{and to island}\\ \text{with } N \text{ electrons}}} + \underbrace{\Gamma_T(N-1)p_{N-1}(t)}_{\substack{\text{tunneling to island}\\ \text{with } N-1 \text{ electrons}}} + \underbrace{\Gamma_F(N+1)p_{N+1}(t)}_{\substack{\text{tunneling from island}\\ \text{with } N+1 \text{ electrons}}}$$

with tunneling rates $\Gamma_F = \Gamma_{FL} + \Gamma_{FR}$ and $\Gamma_T = \Gamma_{TL} + \Gamma_{TR}$

  - if we know p_N for stationary state, we get currents as

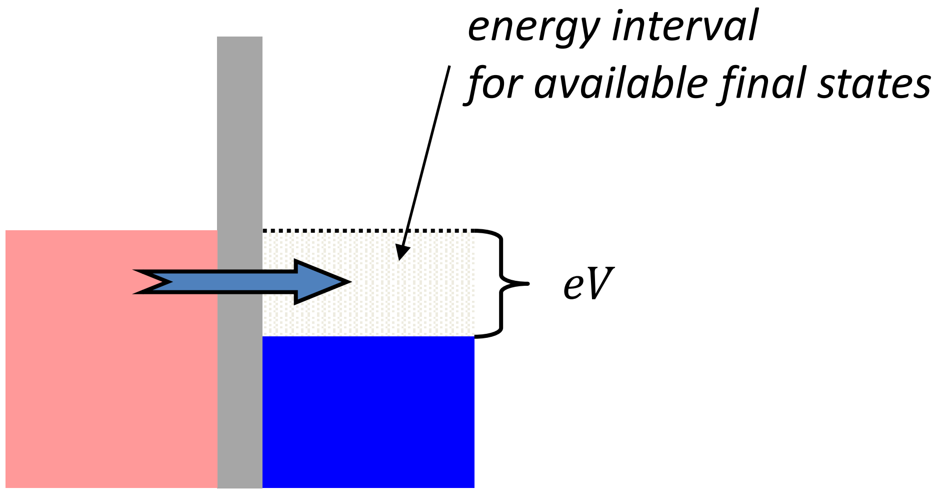
$$\left.\begin{aligned} I_L &= e\sum_N [\Gamma_{FL}(N) - \Gamma_{TL}(N)]\,p_N \\ I_L &= e\sum_N [\Gamma_{TR}(N) - \Gamma_{FR}(N)]\,p_N \end{aligned}\right\} \quad I = I_L + I_R$$

II.5 Coulomb Blockade

- **SET: current-voltage characteristics**

tunneling rates for single tunnel junction:

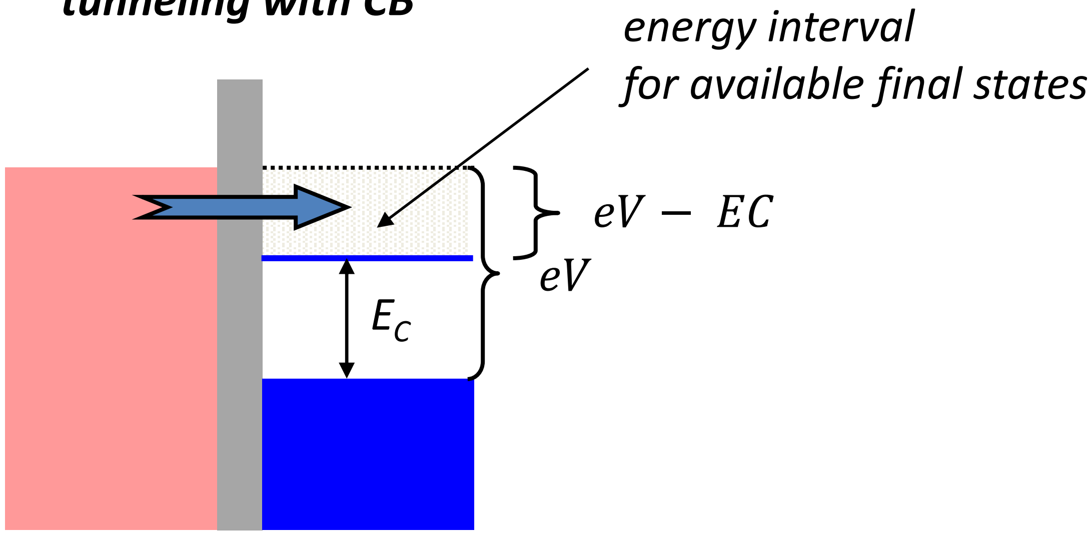
tunneling without CB



$$I = G_{\text{tun}} V$$

tunneling rate: $\Gamma_{\text{tun}} = \frac{I}{e} = \frac{G_{\text{tun}}}{e^2} eV$

tunneling with CB



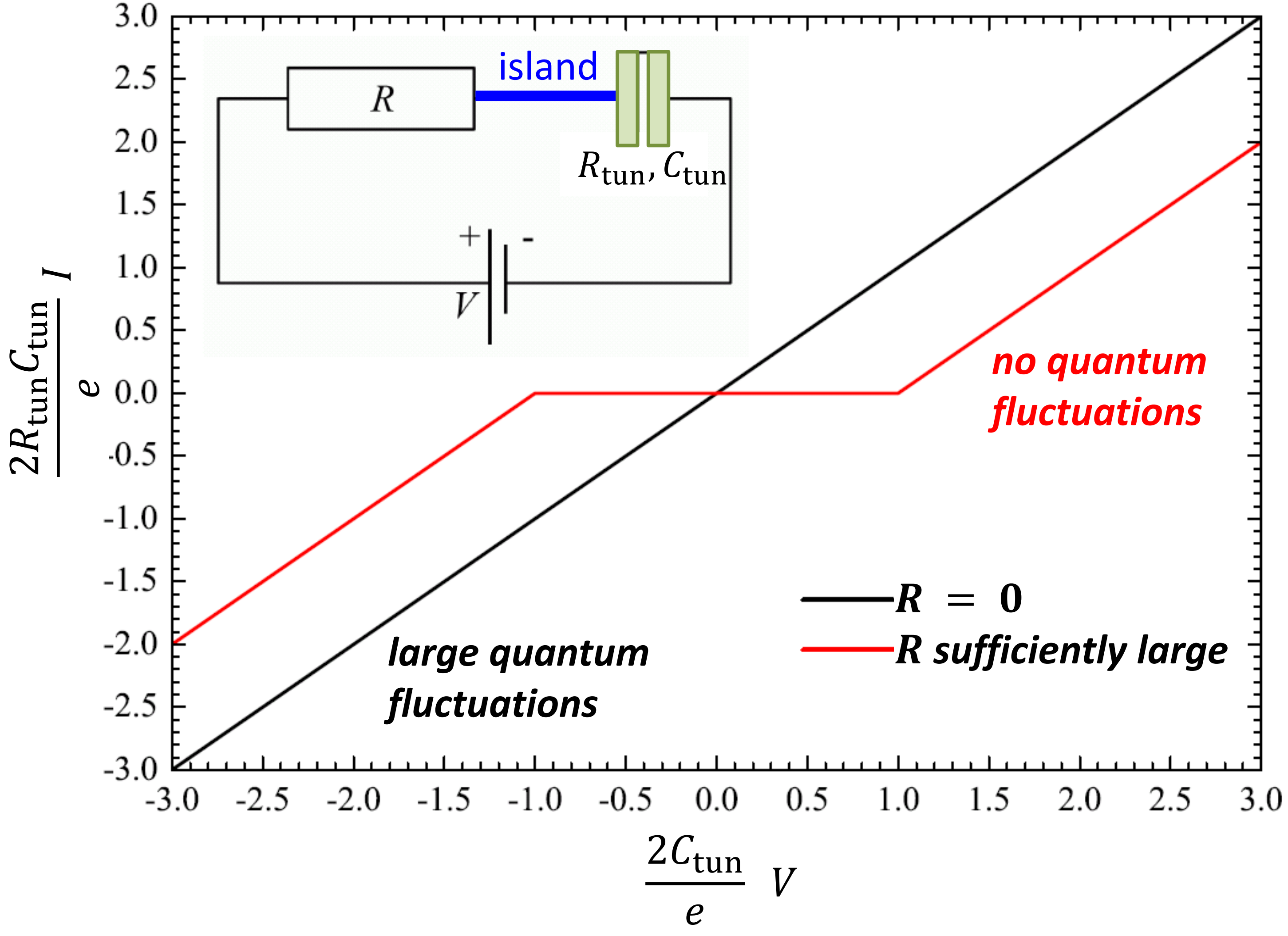
tunneling rate: $\Gamma_{\text{tun}} = 0 \qquad \text{for } eV < E_c$

blockade regime

$$\Gamma_{\text{tun}} = \frac{G_{\text{tun}}}{e^2}(eV - E_c) \quad \text{for } eV < E_c$$

II.5 Coulomb Blockade

- **SET: current-voltage characteristics**



IVC for tunneling with Coulomb blockade

R. Gross © Walther-Meißner-Institut (2004 - 2023) - additional topic

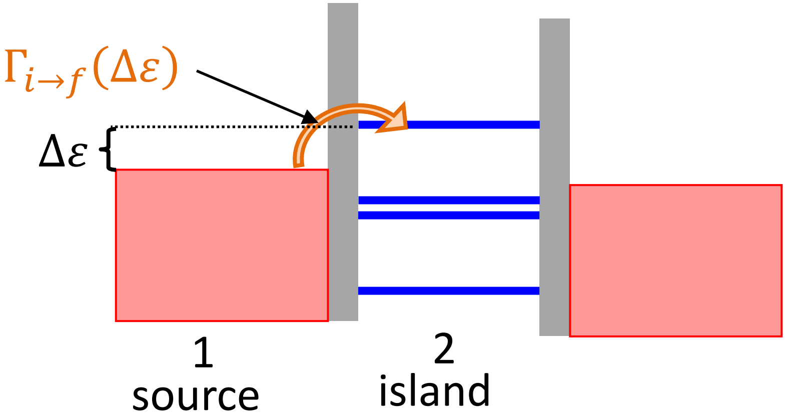
II.5 Coulomb Blockade

- **SET: tunneling rates and IVC**
  - electrostatic energy changes as electron tunnels
    → determine tunneling rate at electron energy change of $\Delta\varepsilon$:

$$\Gamma_{i\to f} = \frac{2\pi}{\hbar}|\langle i|H_{\text{tun}}|f\rangle|^2\,\delta\big(\varepsilon_f - \varepsilon_i - \Delta\varepsilon\big)$$

Fermi's Golden Rule

  - total transition rate from conductor 1 (source) to 2 (island):
    - tunneling rate proportional to density of states $D(\varepsilon)$
    - occupation probability given by Fermi functions $f(\varepsilon)$
    - integration over all energies



$$\Gamma_{i\to f}(\Delta\varepsilon) = \frac{2\pi}{\hbar}\int_{-\infty}^{\infty} \mathrm{d}\varepsilon\,|\langle i|H_{\text{tun}}|f\rangle|^2\,\underbrace{D_i(\varepsilon)f(\varepsilon)}_{\substack{\text{occupied}\\ \text{initial states}}}\;\underbrace{D_f(\varepsilon+\Delta\varepsilon)[1-f(\varepsilon+\Delta\varepsilon)]}_{\substack{\text{empty}\\ \text{final states}}}$$

II.5 Coulomb Blockade

- **SET: tunneling rates and IVC**

  – simplifying assumptions:

  → H_tun is energy independent
  → $D(\varepsilon)$ is energy independent

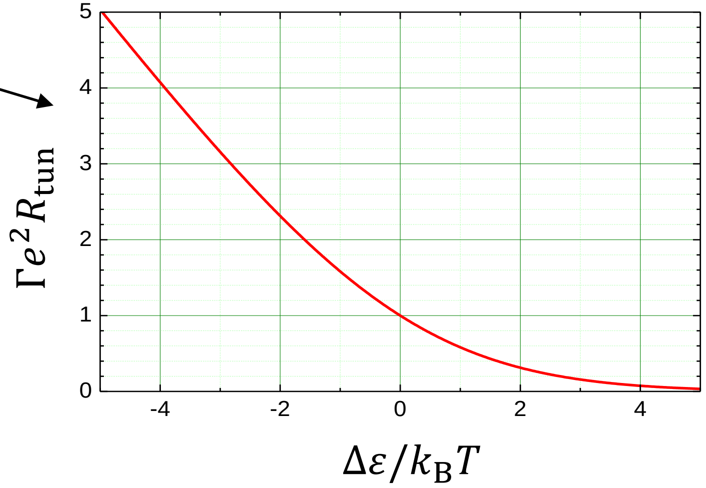
  ⇨ $$f(\varepsilon)[1-f(\varepsilon+\Delta\varepsilon)] = \frac{f(\varepsilon)-f(\varepsilon+\Delta\varepsilon)}{1-\exp(\Delta\varepsilon/k_\mathrm{B}T)}$$

  at low T: Fermi functions ≈ step functions

  ⇨ $$\Gamma_{i\to f}(\Delta\varepsilon) = \frac{1}{e^2 R_\text{tun}} \frac{\Delta\varepsilon}{\exp(\Delta\varepsilon/k_\mathrm{B}T)\ -1}$$

  with

  $$R_\text{tun} = \frac{\hbar}{2\pi e^2} D^2\, |\langle i|H_\text{tun}|f\rangle|^2$$



  – net current

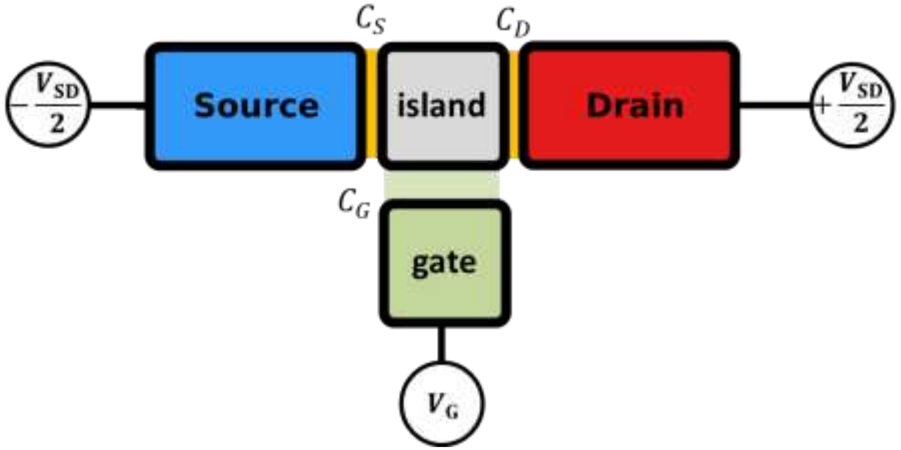
  $$I = e\,[\Gamma_{1\to2}\,(\Delta\varepsilon_{1\to2}) - \Gamma_{2\to1}\,(\Delta\varepsilon_{2\to1})]$$

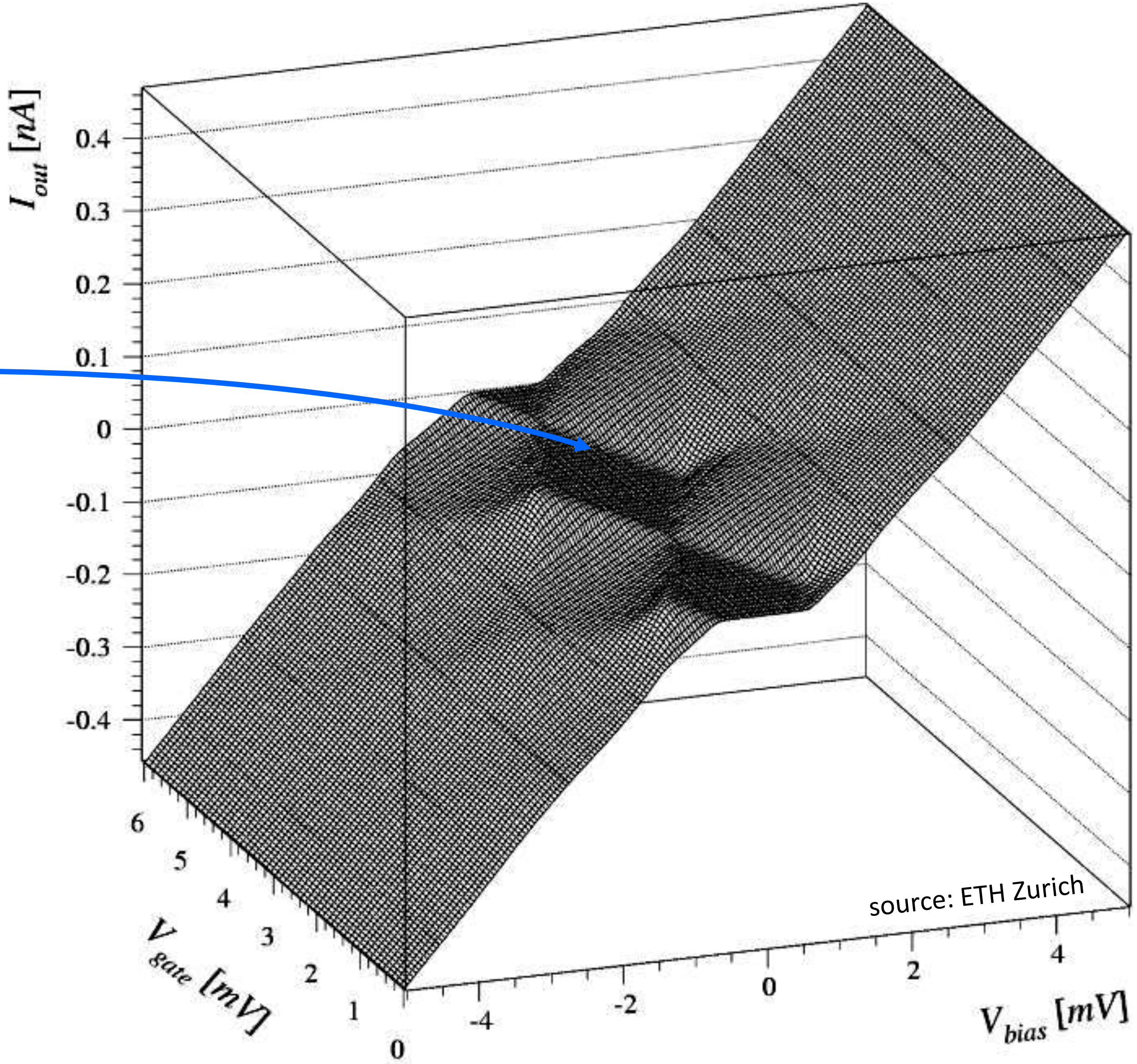
  – current from current source (1) to island (2) in steady state

  $$I = e\sum_N p(N)\,\{\Gamma_{1\to2}\,[\Delta\varepsilon_{1\to2}(N)] - \Gamma_{2\to1}\,[\Delta\varepsilon_{2\to1}(N)]\}$$

  (equivalent expression for current from island to drain)

- **SET: current-voltage characteristics**

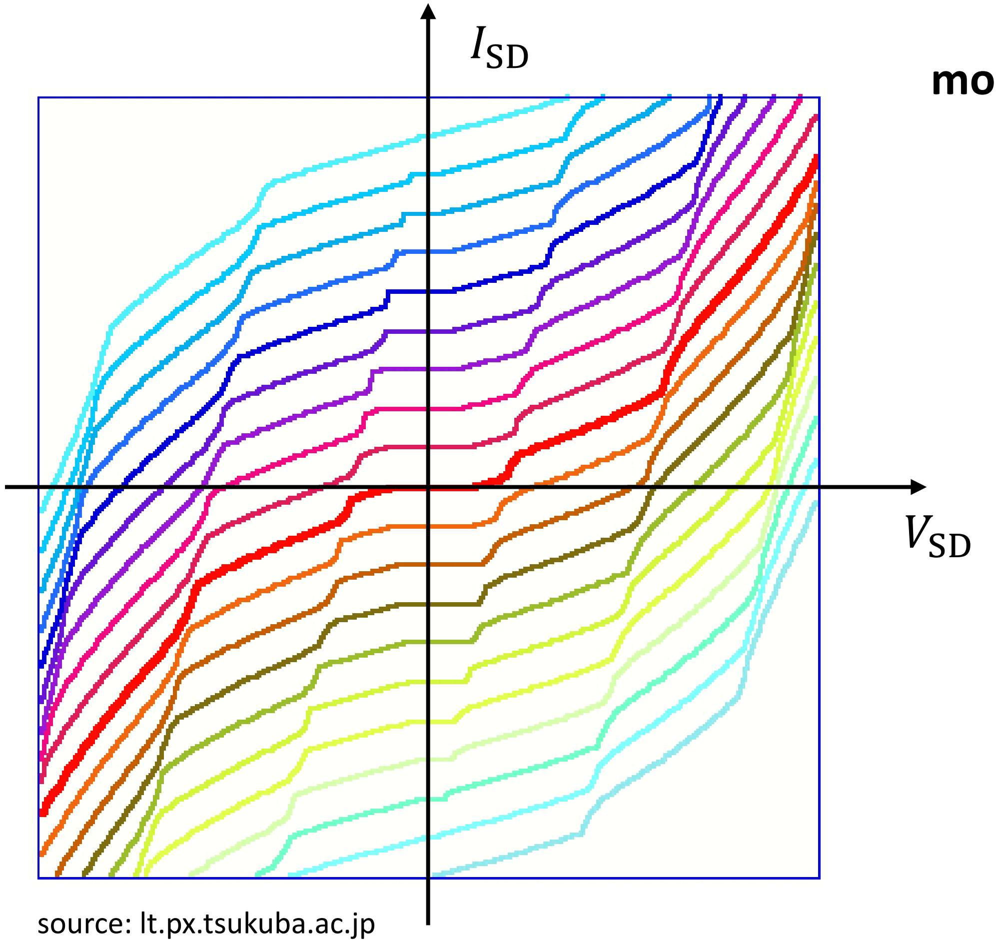




source: ETH Zurich

II.5 Coulomb Blockade

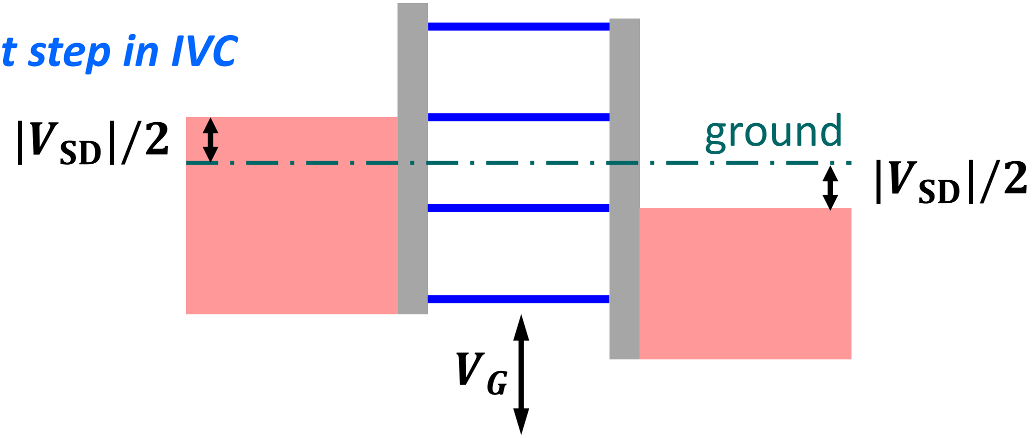
- **SET: current-voltage characteristics - Coulomb staircase**



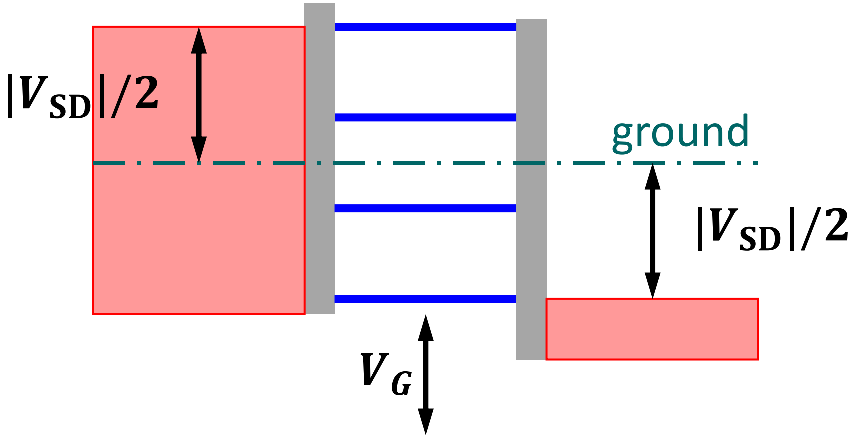
source: lt.px.tsukuba.ac.jp

movie shows variation of IVC with varying gate voltage

1st step in IVC

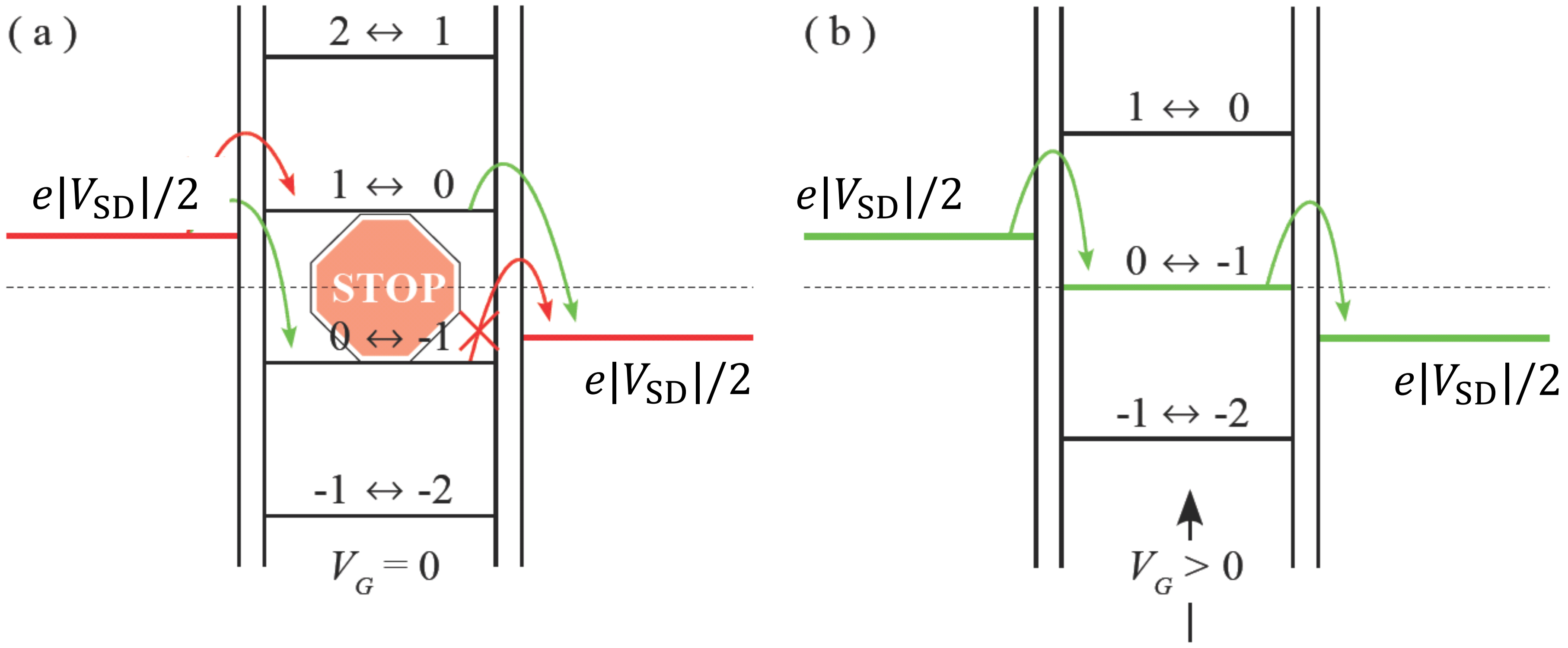


2nd step in IVC



II.5 Coulomb Blockade

- **SET: variation of the gate voltage – Coulomb oscillations:**



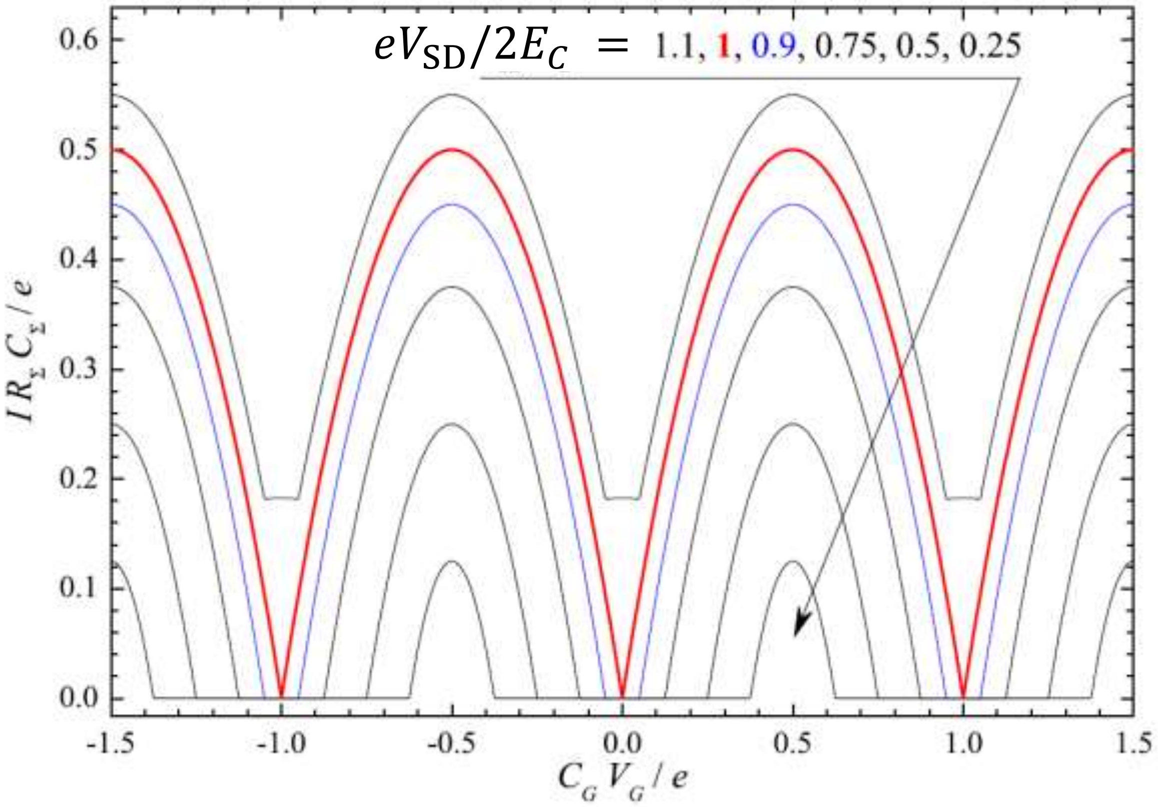
  - gate voltage shifts up and down the energy levels of the island
  - at small SD-voltages: *conductance can be varied considerably by gate voltage*

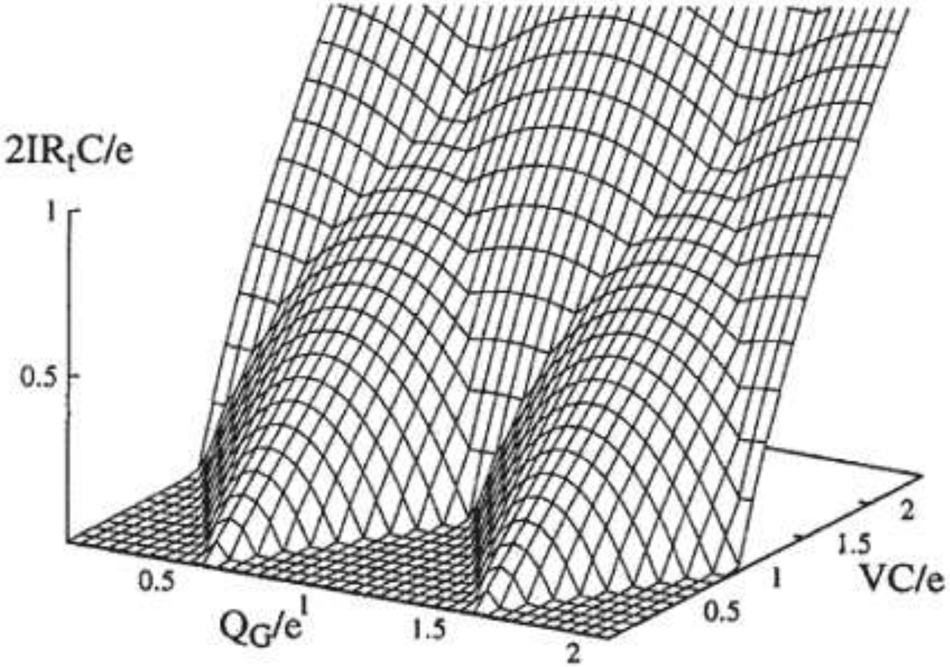
➔ ***Coulomb Oscillations***

R. Gross © Walther-Meißner-Institut (2004 - 2023)

II.5 Coulomb Blockade

- **SET: variation of the gate voltage – Coulomb oscillations:**



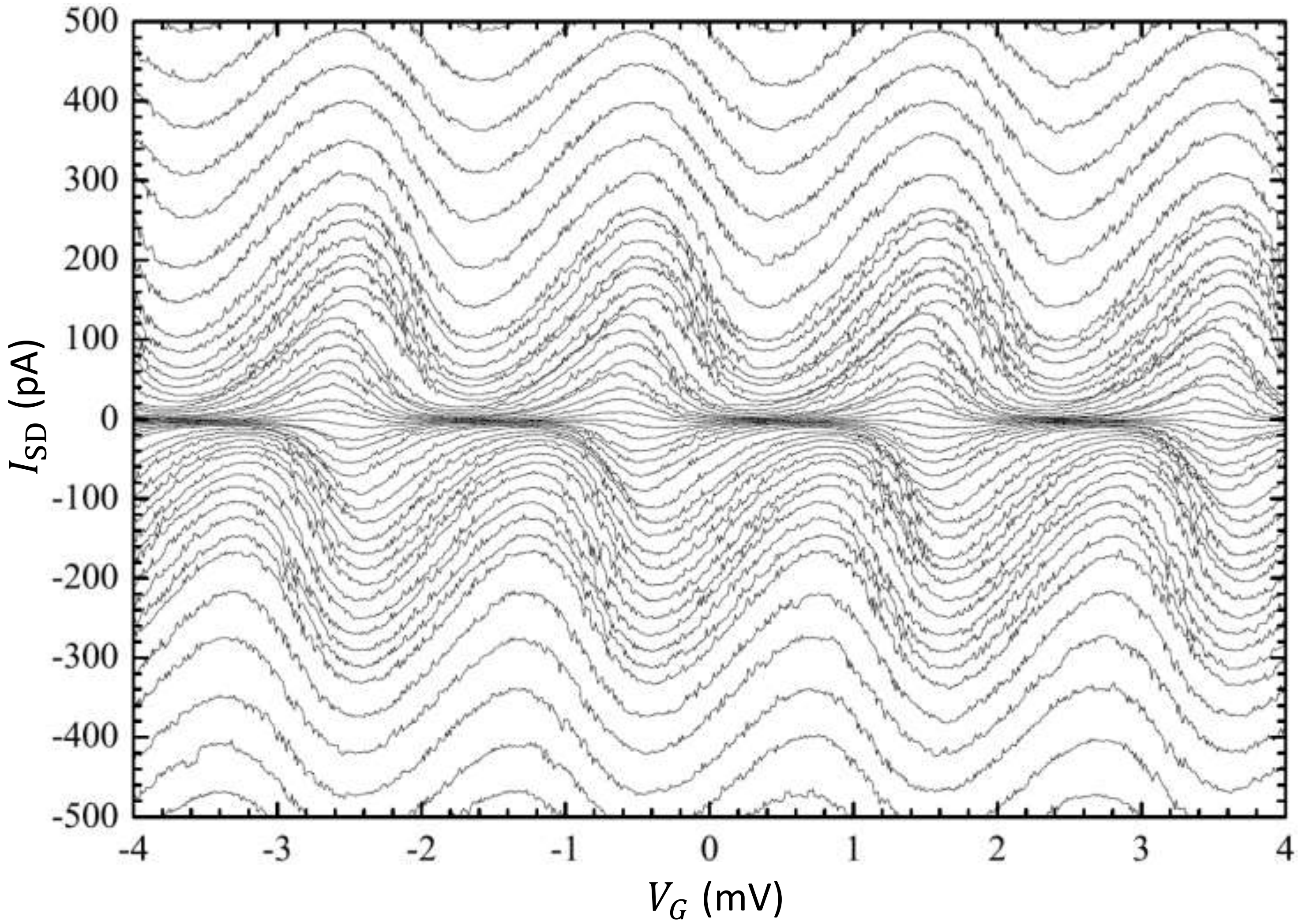


note:
- large dI_{SD} / dV_G

➔ *use as ultra-sensitive electrometer*

II.5 Coulomb Blockade

- **SET: variation of the gate voltage – Coulomb oscillations:**



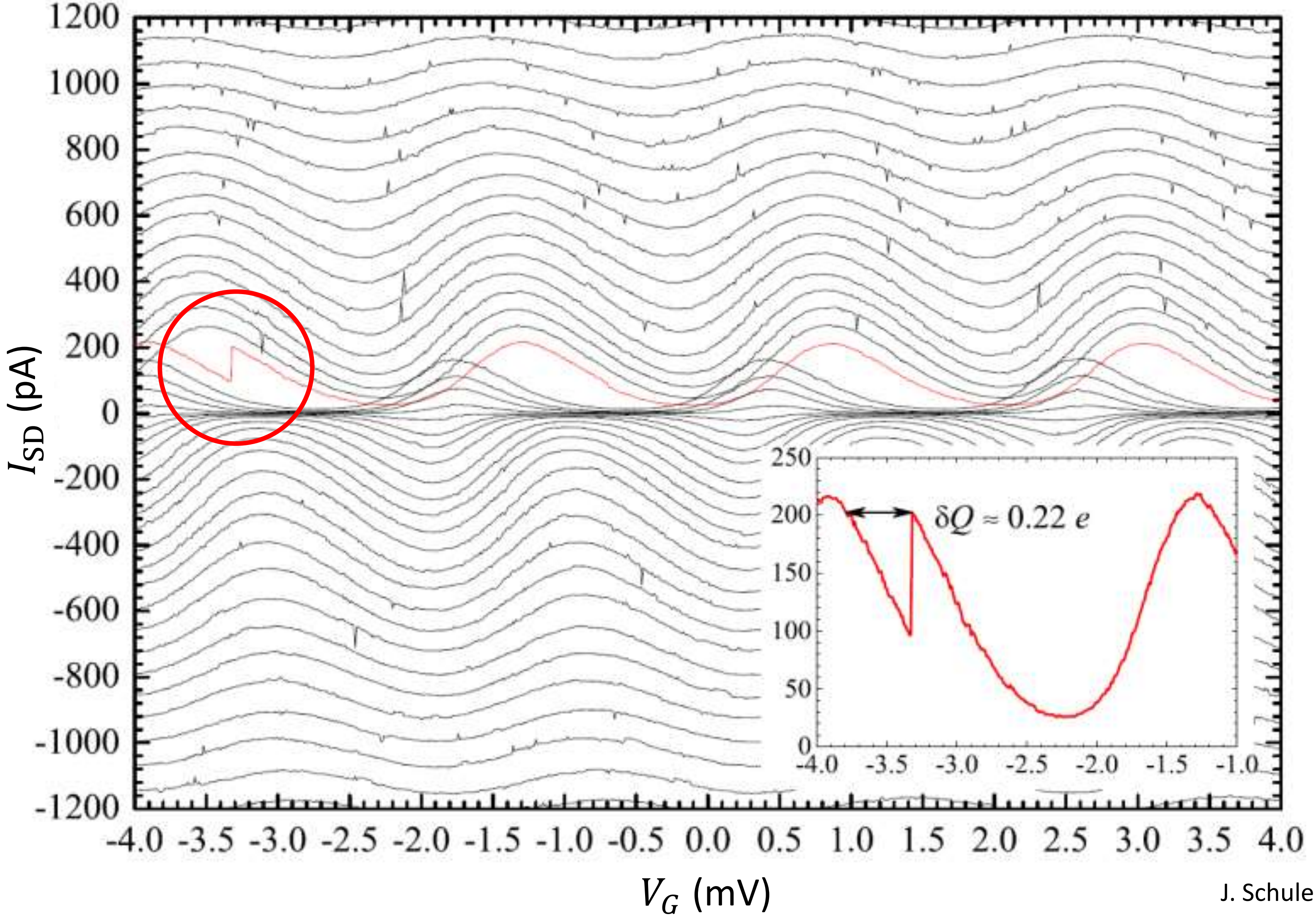
experimental data on
Al/AlO$_x$/Al/AlO$_x$/Al - SET

V_{SD} is varied for different curves

J. Schuler, Ph.D. Thesis (WMI 2005)

II.5 Coulomb Blockade

- SET Coulomb oscillations - effect of single fluctuating background charges



Al/AlO$_x$/ Al/AlO$_x$/Al - SET

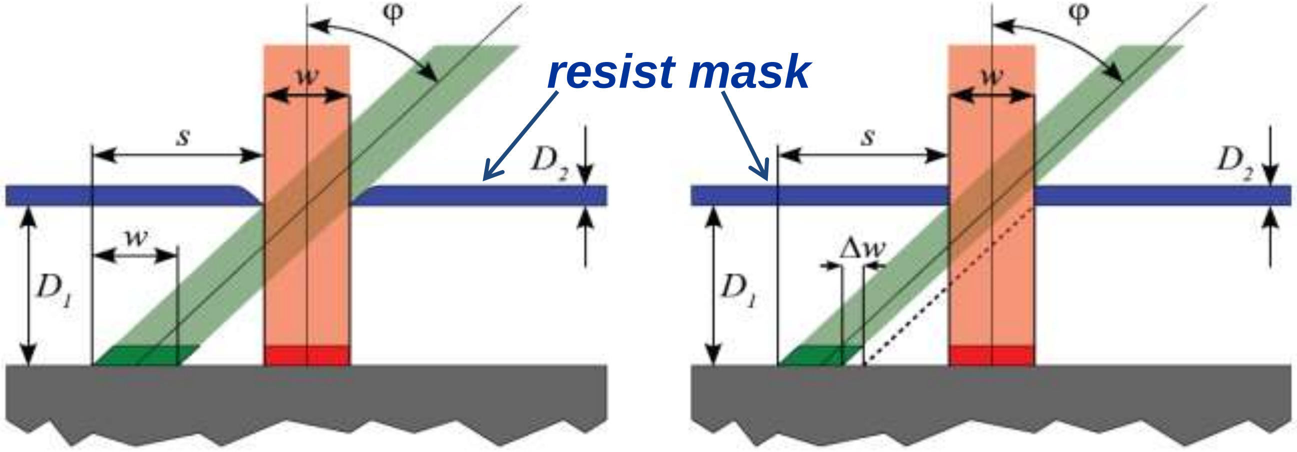
shift of $I_{SD}(V_G)$ curve due to fluctuating background charge

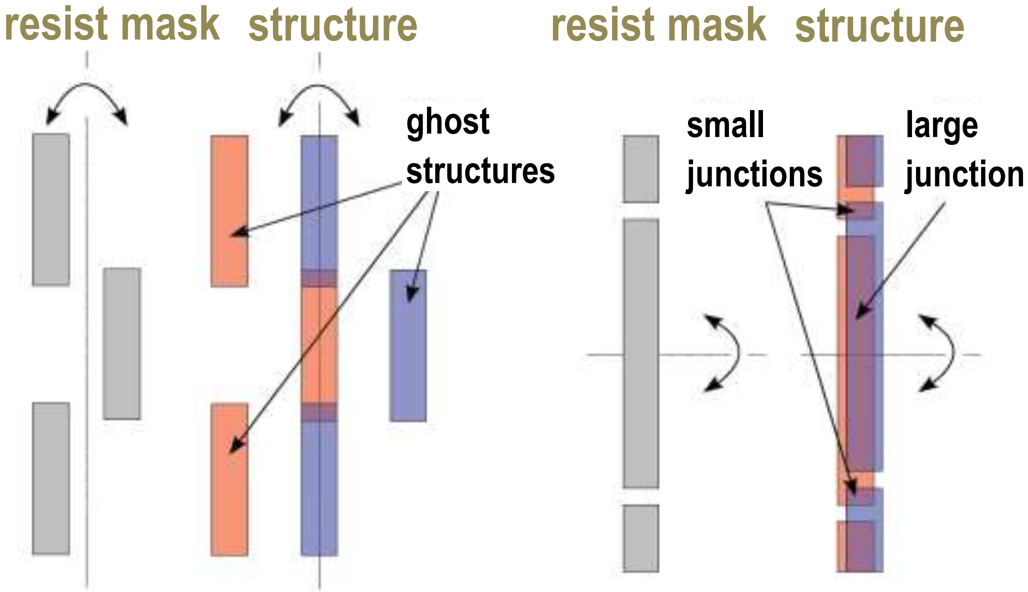
J. Schuler, Ph.D. Thesis (WMI 2005)

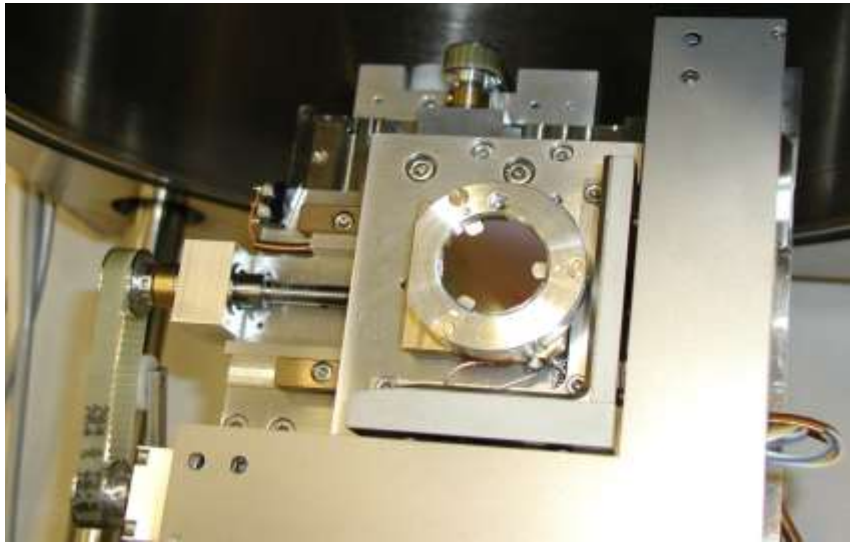
II.5 Coulomb Blockade

- **SET fabrication by two-angle shadow evaporation**

fabrication of sub-µm Josephson Junctions by shadow vaporation technique







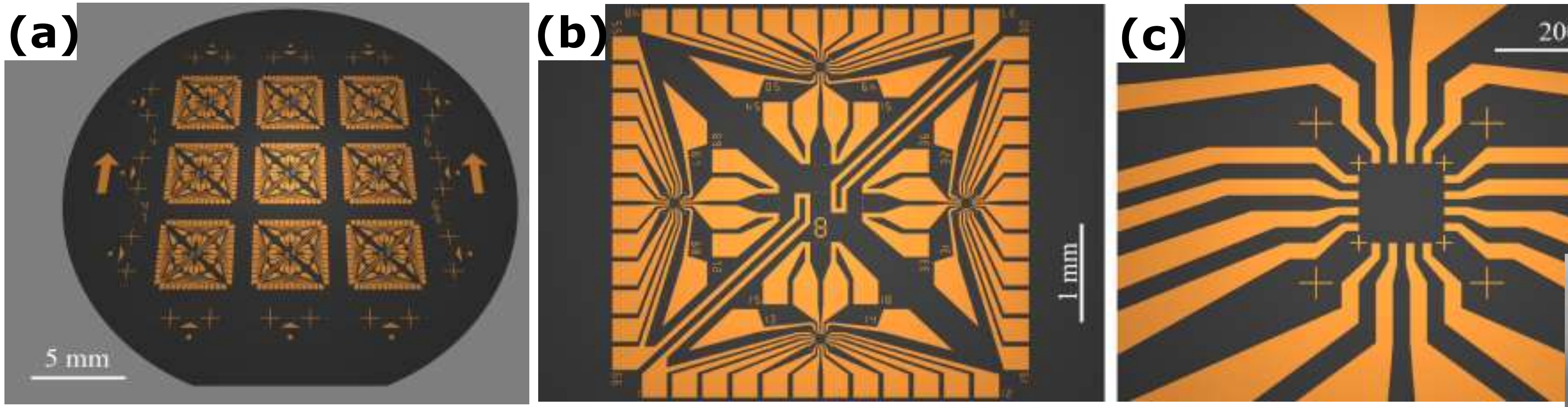
⬜ **resist mask** 🟦 first layer 🟧 second layer 🟪 tunnel junction

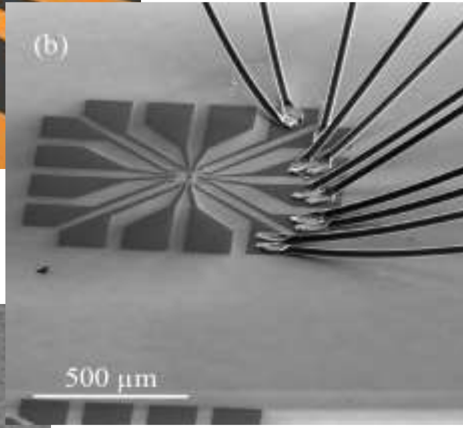
J. Schuler, Ph.D Thesis (2005)

II.5 Coulomb Blockade

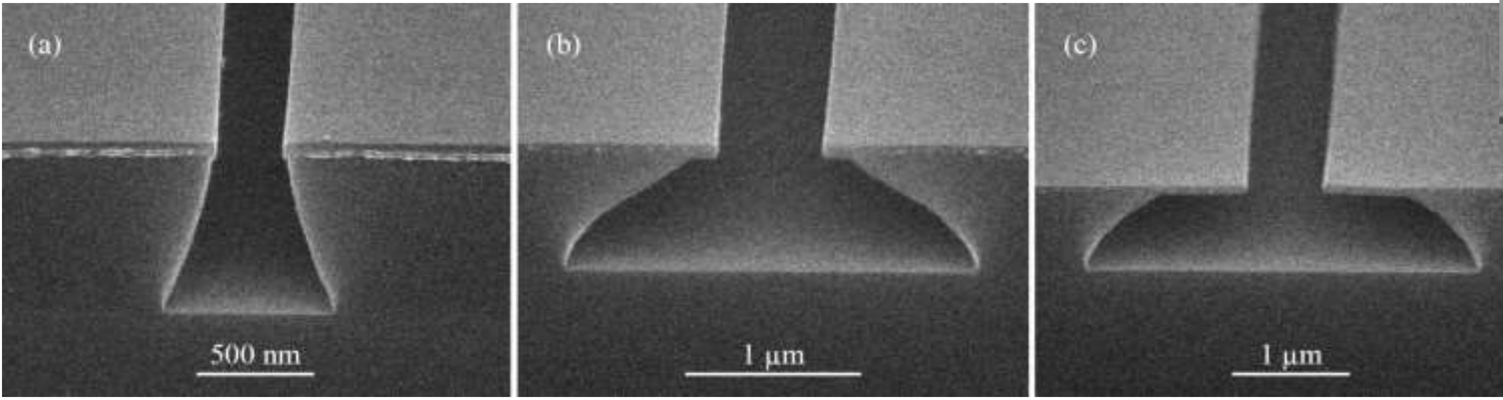
- **SET fabrication by two-angle shadow evaporation**

Optical Lithography





Electron Beam Lithography

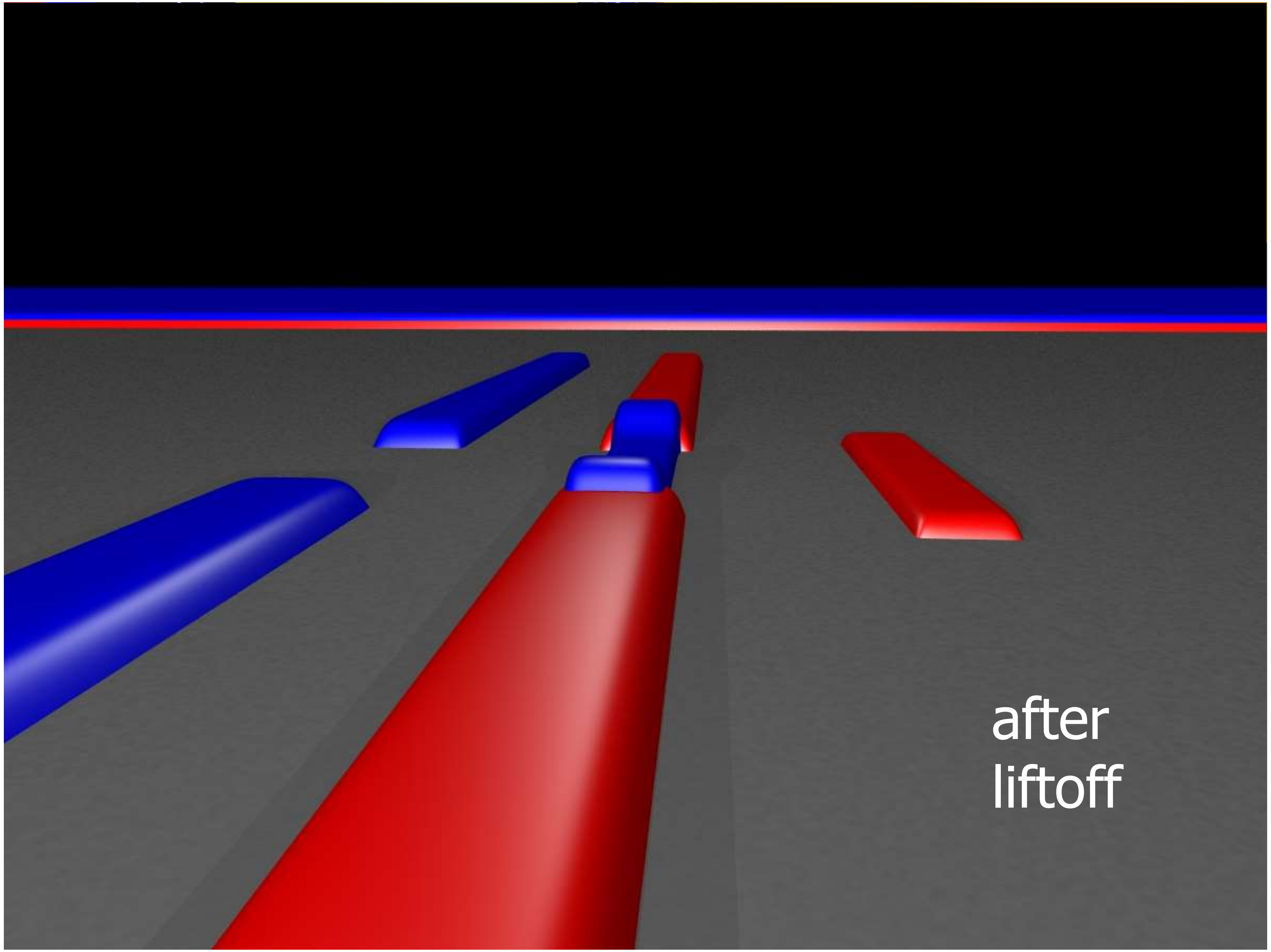


two-layer e-beam resist

Si substrate

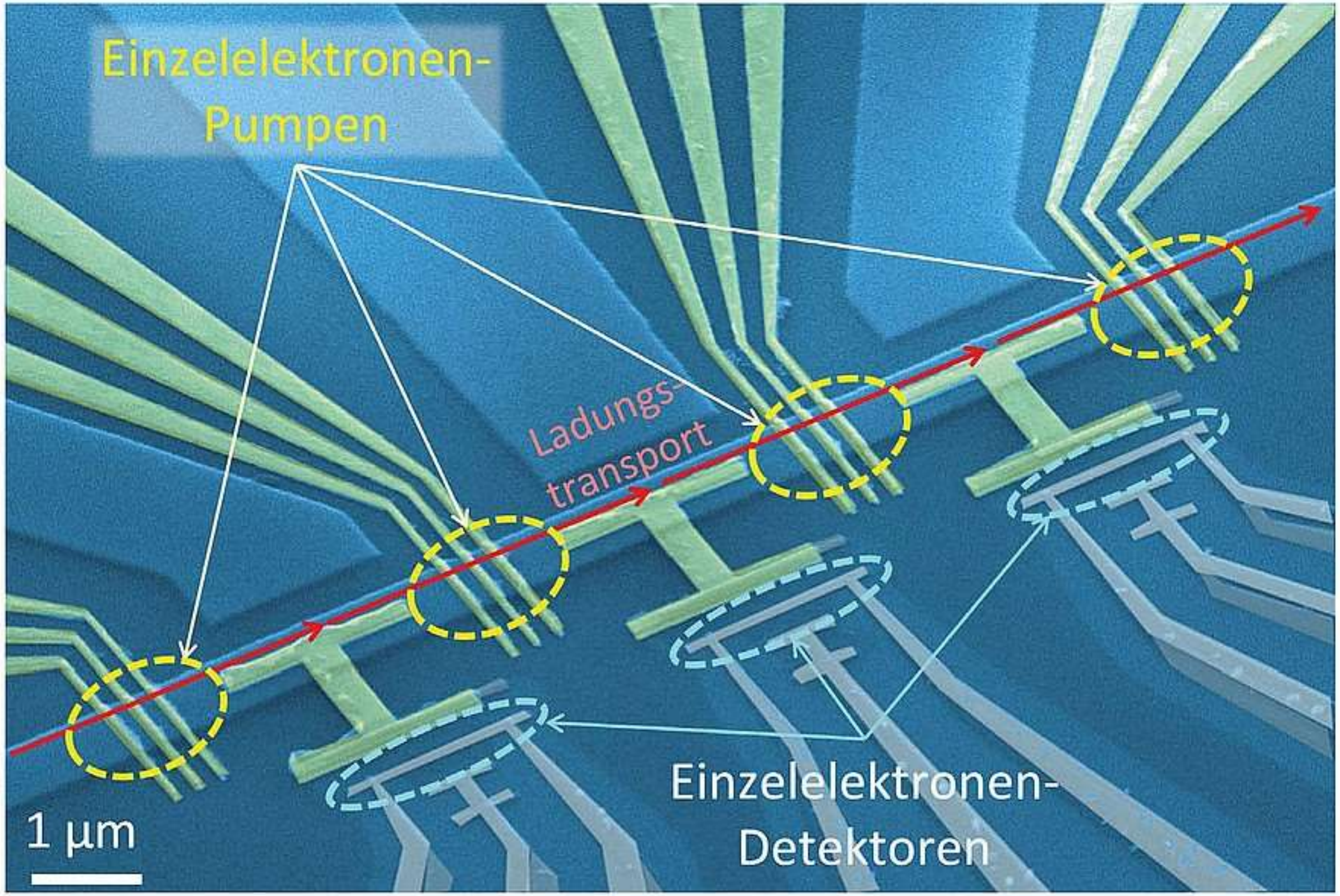
II.5 Coulomb Blockade

- **SET fabrication by two-angle shadow evaporation**



II.5 Coulomb Blockade

- **SET application: single electron detector**

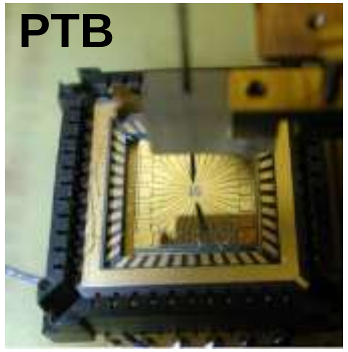


prototype of a self-referenced quantum current source developed at PTB with four semiconductor single-electron current sources ("single-electron pumps") connected in series and three **metallic single electron detectors**

II.5 Coulomb Blockade

- **SET applications**

  – sensitive electrometers: $\frac{\Delta Q}{Q} \simeq 10^{-5}\, e$

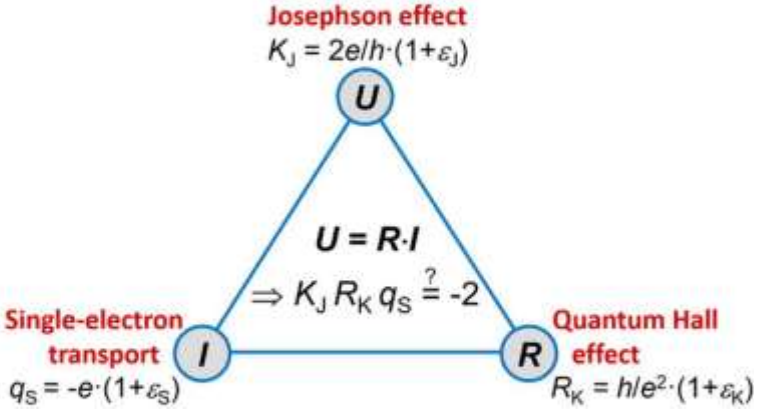


  – electron pumps

  → transporting electrons one by one:
  counting of electrons

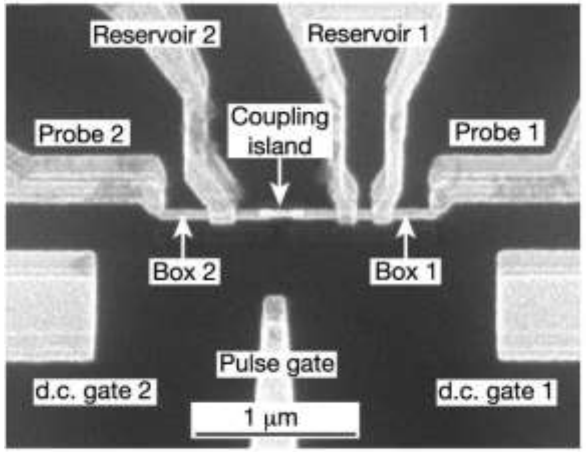
  → current standard: $I = e \cdot f$

  → application of oscillating gate voltage



  – charge Qubits

  → basic element for quantum information systems

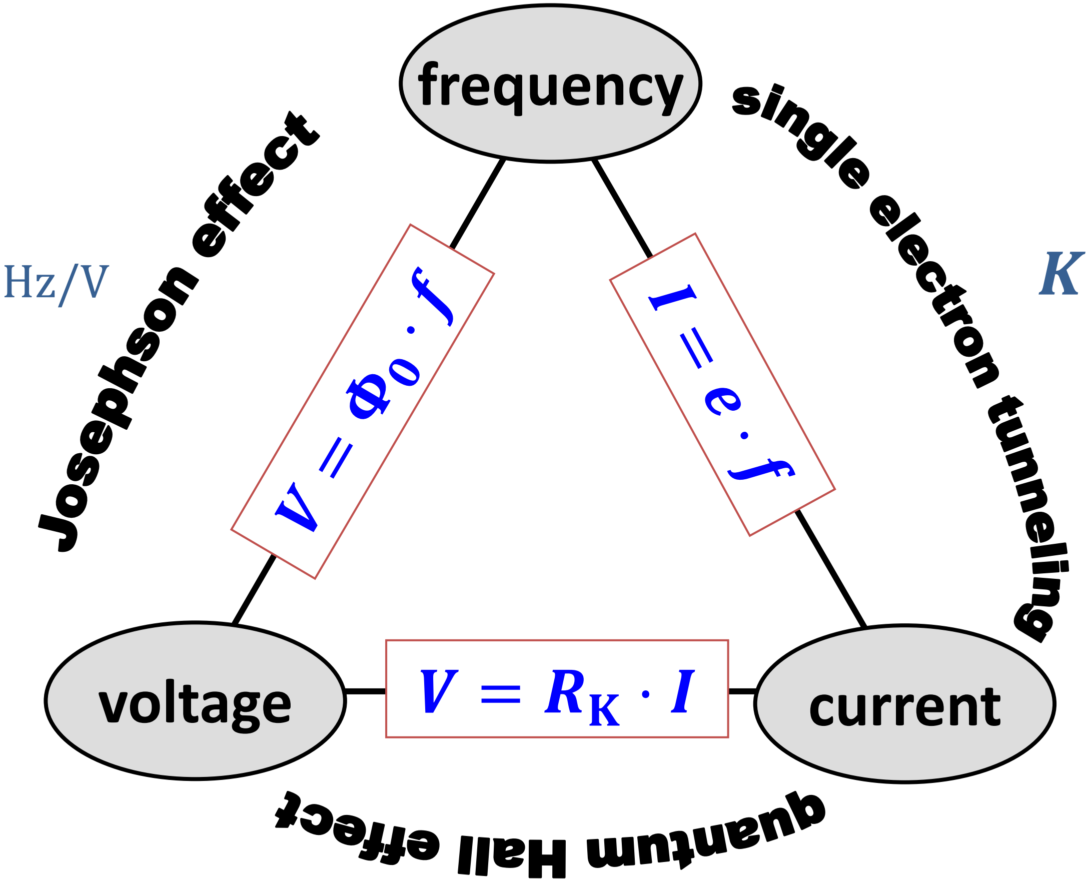


Quantum oscillations in two coupled charge qubits
Yu. A. Pashkin, T. Yamamoto, O. Astafiev, Y. Nakamura, D. V. Averin and J. S. Tsai
Nature 421, 823-826(20 February 2003)

II.5 Coulomb Blockade

- **SET applications – the quantum metrology triangle**

$$K_J = \frac{2e}{h} = \frac{1}{\Phi_0}$$
$$= 483\,597.848\,4\,\ldots\ \mathrm{GHz/V}$$



$K_I = 1/e$ not yet availabe

$$R_K = \frac{h}{e^2} = 25\,812.807\,45\,\ldots\,\Omega$$