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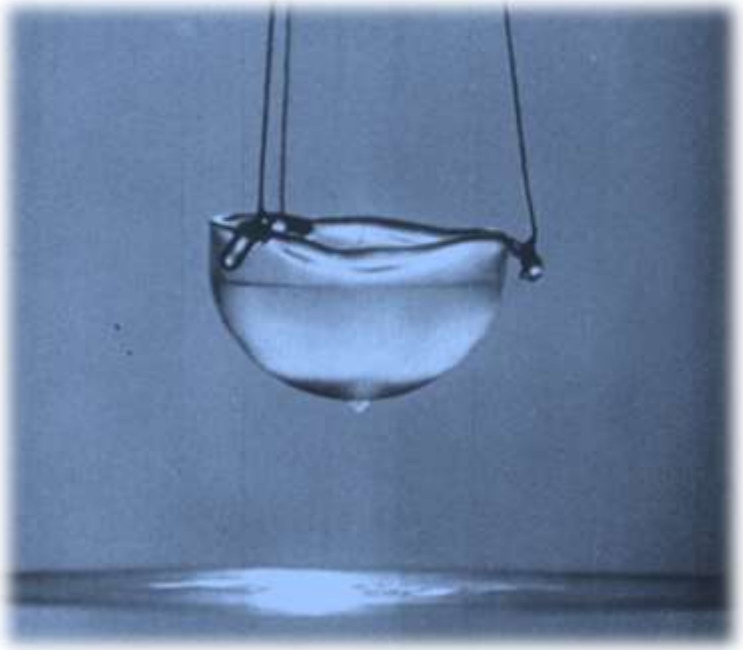
Technische Universität München

Superconductivity and Low Temperature Physics II



Lecture Notes
Summer Semester 2024

R. Gross

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Chapter 1

Quantum Liquids

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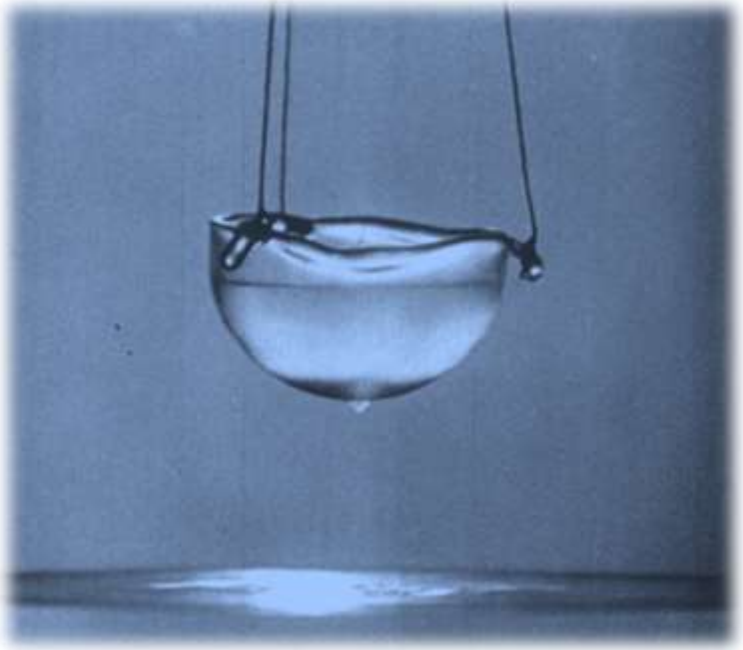

Lecture No. 1

R. Gross

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Contents Part I: Quantum Liquids

Contents:

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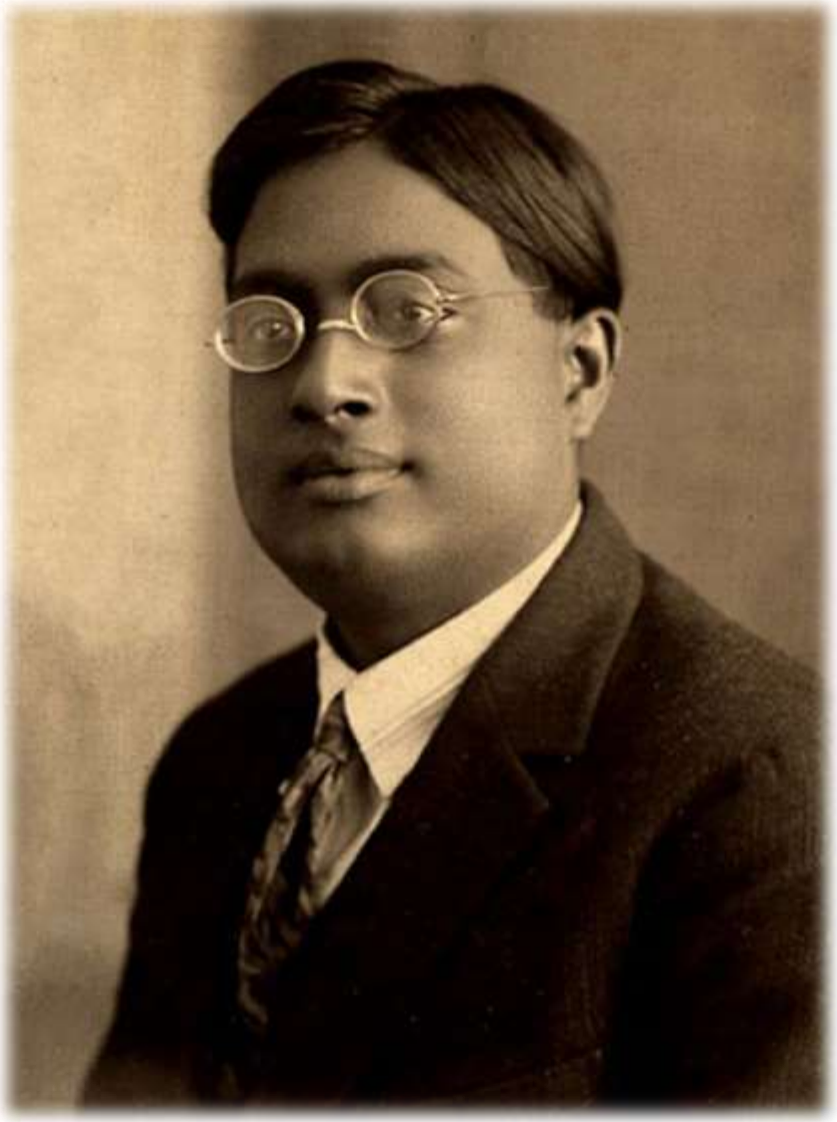
- **Satyendranath Bose (1924)**
  - letter to Einstein: proposes a new method to derive Planck's law for spectral density of electromagnetic radiation emitted by a black body
  - Bose describes the radiation emitted by a black body in thermal equilibrium by a gas on non-interacting identical particles
    → ***photons***
  - description is in perfect agreement with Einstein‘s explanation of the photoelectric effect
  - Einstein immediately recognized that this description can be transferred to many other systems:
    (i) gases of identical particles with finite mass,
    (ii) phonons
  - Einstein strongly supported the publication of Bose‘s results, he translated the paper into German

Satyendranath Bose: *Plancks Gesetz und Lichtquantenhypothese*.
In: *Zeitschrift für Physik.* 26 (1924), S. 178-181.



Satyendranath Bose (around 1926)

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What happens if you cool down a gas of non-interacting bosons ?

Below a certain temperature the particles feel each other due to their finite de Broglie wavelength, although they are non-interacting !

phase transition driven by statistics of particles (not by their interactions)

Bose-Einstein Condensation

thermodynamic phase transition into a gas of non-interacting particles:
driven by statistics of particles and not by interactions

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I.1 Foundations and General Properties

- 1924: prediction of **Bose-Einstein Condensation (BEC**) by Einstein
- 1995: first observation of BEC on alkali atoms by

Eric A. Cornell **Wolfgang Ketterle** **Carl E. Wieman**

Nobel Prize in Physics 2001

"for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates"

→ long period between theoretical prediction and experimental realization:
high density, ultra-cold gas of non-interacting atoms required
→ problem: gas liquefies or solidifies due to finite interactions

→ more simple to realize:
interacting quantum gas, e.g. superfluid ^{4}He

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- **nomenclature**

• system of non-interacting particles:	**classical gas**	e.g. classical ideal gas
• if quantum fluctuations dominate:	**quantum gas**	e.g. Bose-Einstein Condensate, Fermi gas
• system of interacting particles:	**classical liquid**	e.g. water, liquid helium-4
• if quantum fluctuations dominate:	**quantum liquid**	e.g. superfluid helium-4, Fermi liquid

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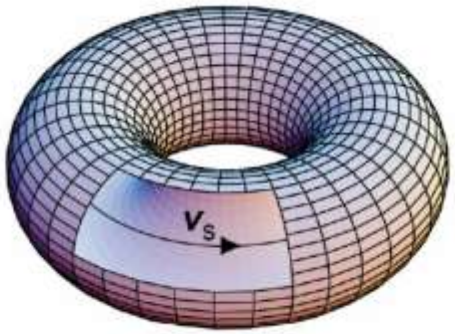
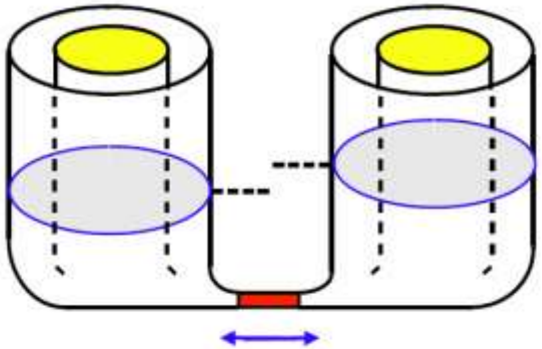
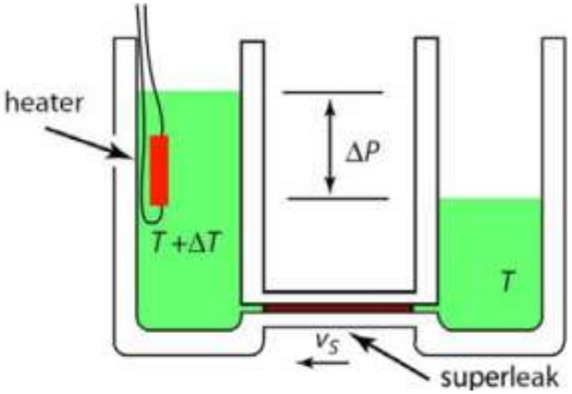
I.1.1 Quantum Liquids



supercurrents

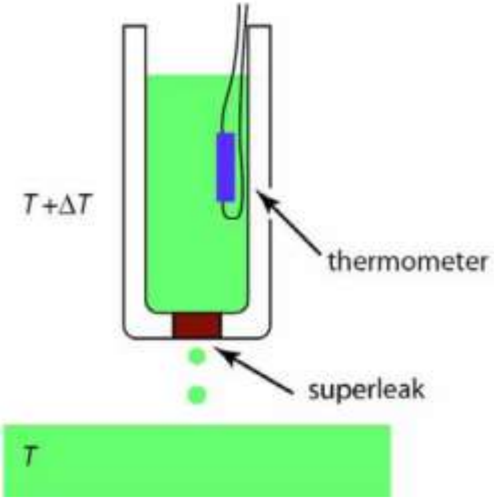


U-tube oscillation

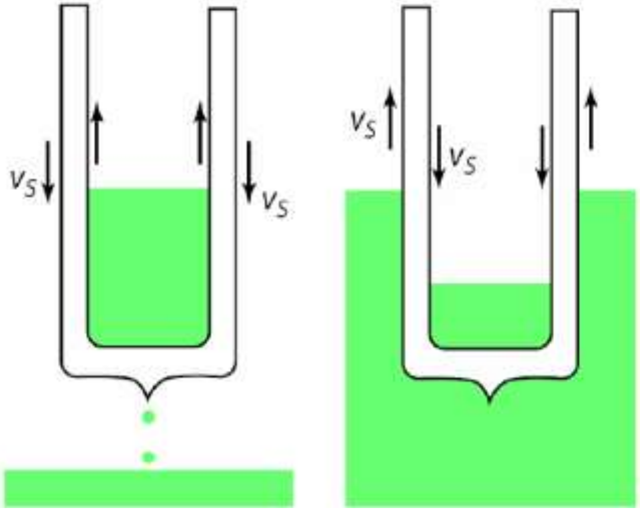


thermomech. effects

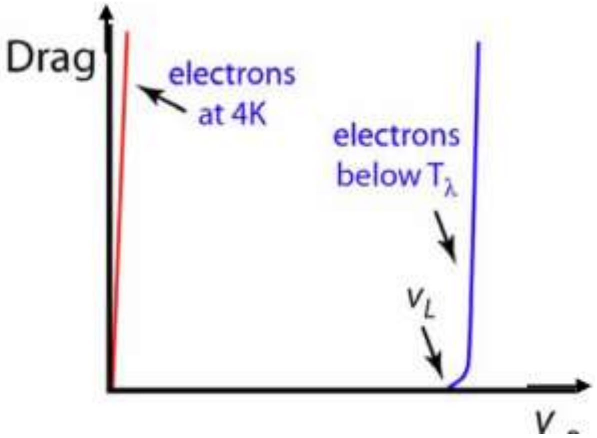
quantum liquids & condensates show fascinating properties → discussed in this lecture



superleaks



beaker film flow



critical velocity

Source: D.I. Bradley, Lancaster Univ.

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I.1.1 Quantum Liquids

- **classical liquids and solids – characteristic energy scales:**

characteristic energies: (1) average interaction energy: $\langle V \rangle$ or $\langle E_{\mathrm{pot}} \rangle$
(2) thermal (kinetic) energy: $k_{\mathrm{B}}T$ or $\langle E_{\mathrm{kin}} \rangle$

criterion for liquid/solid: ***liquid***: $k_{\mathrm{B}}T \gg \langle V \rangle$
solid: $k_{\mathrm{B}}T \ll \langle V \rangle$

- **characteristic length scales in a gas:**

- particle distance: $d = n^{-1/3}$
- interaction range: $r_0 \simeq a$ (scattering length a)
- de Broglie wavelength:

$$\lambda_{\mathrm{T}} = \sqrt{\frac{h^2}{2\pi m k_{\mathrm{B}}T}} = \sqrt{\frac{2\pi\hbar^2}{m k_{\mathrm{B}}T}}$$

$$\frac{\hbar^2 k_{\mathrm{T}}^2}{2m} \simeq k_{\mathrm{B}}T \qquad k_{\mathrm{T}} = \frac{2\pi}{\lambda_{\mathrm{T}}}$$

$$\frac{h^2}{2m\lambda_{\mathrm{T}}^2} \simeq k_{\mathrm{B}}T \rightarrow \lambda_{\mathrm{T}} \simeq \left(\frac{h^2}{2mk_{\mathrm{B}}T}\right)^{1/2}$$

classical non-interacting gas: $d \gg r_0, \lambda_{\mathrm{T}}$, since atoms interact only during collisions

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I.1.1 Quantum Liquids

• calculation of the average thermal wavelength of a statistical ensemble (1)

we consider a classical gas of N particles:
→ probability for a specific particular configuration at temperature T is given by Boltzmann distribution

$$P(\mathbf{p}_1, \dots, \mathbf{p}_N, \mathbf{r}_1, \dots, \mathbf{r}_N) = \frac{1}{Z_N} \exp(-\beta\mathcal{H})$$

$\beta = 1/k_\mathrm{B}T$, $\mathcal{H}$ = Hamiltonian

$$Z_N = \frac{1}{N!} \int \mathrm{d}^3p_1, \dots, \mathrm{d}^3p_N \mathrm{d}^3r_1, \dots, \mathrm{d}^3r_N \exp(-\beta\mathcal{H})$$

classical partition function, $\frac{1}{N!}$ results from fact that particles cannot be distinguished

partition function can be factorized into two terms (related to kinetic and potential energy)

$$Z_N = \frac{1}{N!} \underbrace{\prod_i \int \mathrm{d}^3p_i \exp\left(-p_i^2/2mk_\mathrm{B}T\right)}_{\text{kinetic energy contribution}} \cdot \underbrace{\int \mathrm{d}^3r_1, \dots, \mathrm{d}^3r_N \exp\left(-\frac{1}{2}\beta \sum_{i \neq j} V\left(\mathbf{r}_i - \mathbf{r}_j\right)\right)}_{\text{potential energy contribution}}$$

product of independent factors

$$\left[(2\pi m k_\mathrm{B}T)^{3/2}\right]^N$$

(momenta of particles are statistically independent)

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- **calculation of the average thermal wavelength of a statistical ensemble (2)**

probability that particle has momentum in volume $\mathrm{d}^3 p$ of phase space

$$P(\mathbf{p})\mathrm{d}^3 p = (2\pi m k_\mathrm{B} T)^{-3/2} \exp\left(-p_i^2/2mk_\mathrm{B}T\right)\mathrm{d}^3 p$$

probability that particle has momentum between p and $p + \mathrm{d}p$ $\left(\int_{-\infty}^{\infty} \mathrm{d}^3 p = \int_{-\infty}^{\infty} p^2 \mathrm{d}p \int_0^{\pi} \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi = 4\pi \int_{-\infty}^{\infty} p^2 \mathrm{d}p\right)$

$$P_\mathrm{MB}(p)\mathrm{d}p = \frac{4\pi p^2}{\underbrace{(2\pi m k_\mathrm{B} T)^{3/2}}_{p_\mathrm{T}^3}} \exp(-p^2/2mk_\mathrm{B}T)\,\mathrm{d}p$$

Maxwell-Boltzmann distribution

the most likely momentum of the particles is $p = \sqrt{2mk_\mathrm{B}T}$, the average momentum is $\bar{p} = \sqrt{8mk_\mathrm{B}T/\pi}$

➔ **thermal de Broglie wavelength**

$$\lambda_\mathrm{T} = \frac{h}{p_\mathrm{T}} = \sqrt{\frac{h^2}{2\pi m k_\mathrm{B} T}} = \sqrt{\frac{2\pi\hbar^2}{mk_\mathrm{B}T}}$$

I.1.1 Quantum Liquids

- **cooling down a classical gas** → $\lambda_T = \sqrt{\dfrac{h^2}{2\pi m k_B T}}$ increases and we can distinguish three regimes

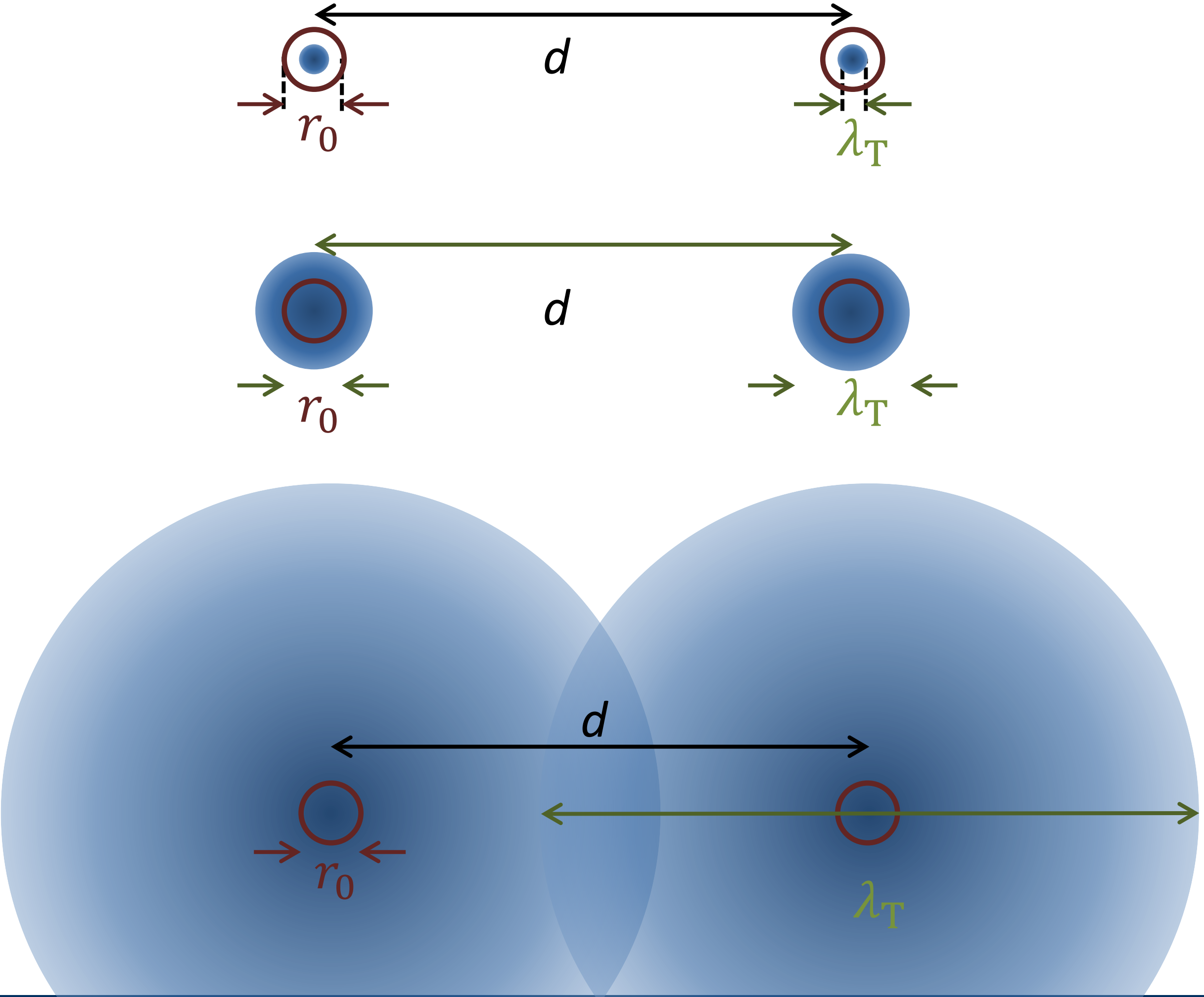
(1) classical regime: $\lambda_T \ll r_0 \ll d$

(2) quantum collisions: $r_0 \ll \lambda_T \ll d$

collissions can no longer be treated classically

(3) quantum degeneracy: $r_0 \ll d \leq \lambda_T$

all degrees of freedom of the gas must be treated quantum mechanically



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I.1.1 Quantum Liquids

- **define new energy scale related to particle distance d**

(3) **zero-point energy E_{zp}**

with - particle density $n = N/V, \quad d = n^{-1/3}$

→ momentum uncertainty $\Delta p = \hbar \Delta k = \frac{h}{d} = h n^{1/3}$

➔ ***zero-point energy***

$$E_{\mathrm{zp}} = \frac{\Delta p^2}{2m} = \frac{h^2}{2md^2} = \frac{h^2 n^{2/3}}{2m} = \pi k_{\mathrm{B}} T_{\mathrm{zp}}$$

characteristic temperature

for an ideal gas the energy $E = \pi k_{\mathrm{B}} T$ is assumed; the factor π results from the partition function

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I.1.1 Quantum Liquids

- **classification of liquids** (interacting liquids with $\langle V \rangle > 0$) :

(a) classical liquid: $T_{\mathrm{zp}} \ll T_{\mathrm{melting}} \ll T$ (quantum fluctuation play no role)

(b) quantum liquid: $T_{\mathrm{melting}} < T < T_{\mathrm{zp}}$ (quantum fluctuation dominate)

proportional to $\langle V \rangle$ or $\langle E_{\mathrm{pot}} \rangle$

quantum parameter:

$$\Lambda \equiv \frac{\langle E_{\mathrm{zp}} \rangle}{\langle E_{\mathrm{pot}} \rangle}$$

quantum liquid:

$$\boldsymbol{\Lambda > 1}$$

classical liquid:

$$\boldsymbol{\Lambda < 1}$$

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I.1.1 Quantum Liquids

- **classification of gases** **(non-interacting systems: $\langle V \rangle = 0$) :**

(a) classical gas: $T_{zp} < T$

(b) quantum gas: $T < T_{zp}$

$$T_{zp} = \frac{h^2 n^{2/3}}{2\pi m k_B} = \frac{h^2}{2\pi m k_B d^2} \quad \Leftrightarrow \quad T = \frac{h^2}{2\pi m k_B \lambda_T^2}$$

(a) classical gas: $\lambda_T \leq n^{-1/3} \simeq d \;\rightarrow\; n \cdot \lambda_T^3 \leq 1$

(b) quantum gas: $\lambda_T \geq n^{-1/3} \simeq d \;\rightarrow\; n \cdot \lambda_T^3 \geq 1$

average particle distance

e.g. electrons @ 300 K: $\lambda_T = 4.2$ nm, $d \simeq 0.2$ nm for Cu ($n \simeq 8.5 \times 10^{28}$ m^{-3})

➔ *particles become indistinguishable* → *degenerate quantum liquid*

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I.1.1 Quantum Liquids

- **cooling down a classical gas** → $\lambda_\mathrm{T} = \sqrt{\frac{h^2}{2\pi m k_\mathrm{B} T}}$ increases and we can distinguish three regimes

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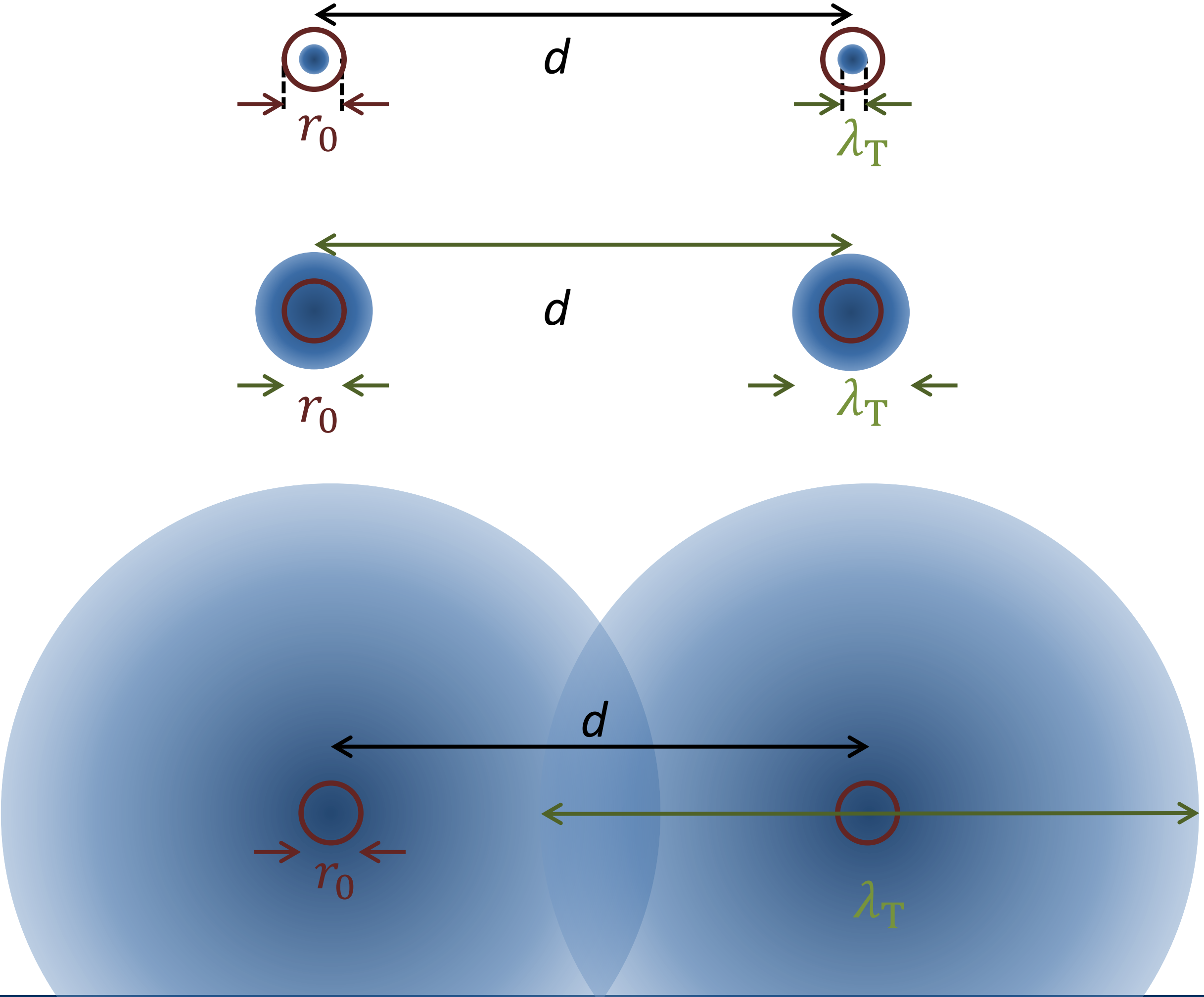

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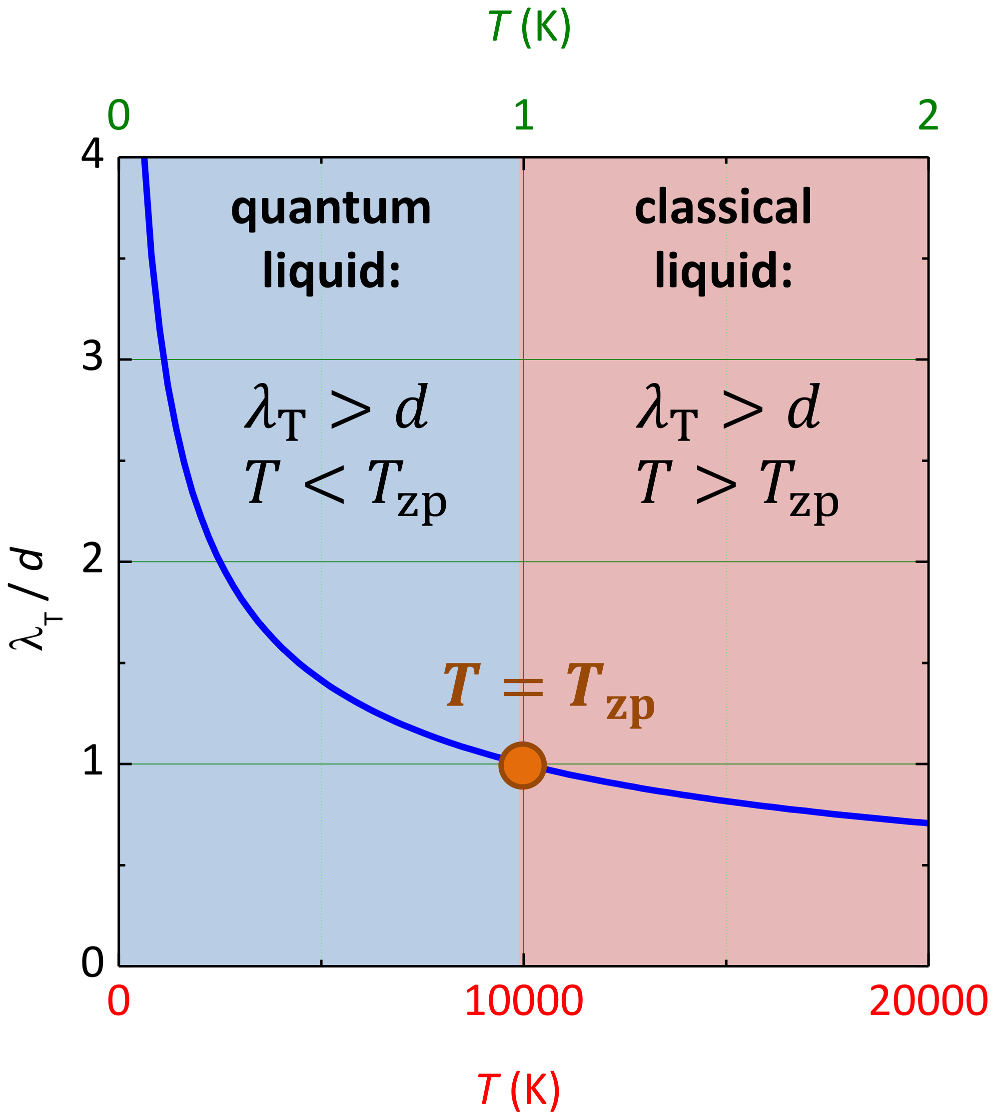

^{4}He: $m = 6.65 \times 10^{-27}$ kg, $n \simeq 4 \times 10^{27}$ m^{-3}

$$\lambda_T = \sqrt{\frac{h^2}{2\pi m k_B T}} \propto \frac{1}{\sqrt{T}}$$

e$^-$: $m = 9.11 \times 10^{-31}$ kg, $n \simeq 3.6 \times 10^{27}$ m^{-3}

degenerate: indistinguishable particles

non-degenerate: distinguishable particles

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- **quantum statistics:**

photons, mesons, ^{4}He-atoms, ...	integral spin	$S = 0, \hbar, 2\hbar, 3\hbar, \ldots$
electrons, nucleons, ^{3}He-atoms, ...	half-integral spin	$S = \hbar/2, 3\hbar/2, 5\hbar/2, \ldots$

indistinguishability of identical particles:

symmetry of many-particle wavefunction with respect to particle exchange

$$\wp\, \Psi(1,2) \equiv \Psi(2,1) = \mathrm{e}^{\imath\alpha}\ \Psi(1,2)$$
$$\wp\, \Psi(2,1) \equiv \Psi(1,2) = (\mathrm{e}^{\imath\alpha})^2\, \Psi(1,2)$$

$\Rightarrow$ $\mathrm{e}^{\imath\alpha} = \pm 1$

- **bosons** (integral spin) $$\Psi_{AB}(1,2) = \frac{1}{\sqrt{2}}[\Psi_A(1)\Psi_B(2) + \Psi_A(2)\Psi_B(1)]$$
- **fermions** (semi-integral spin) $$\Psi_{AB}(1,2) = \frac{1}{\sqrt{2}}[\Psi_A(1)\Psi_B(2) - \Psi_A(2)\Psi_B(1)]$$
- **Pauli principle**: ***two fermions cannot occupy the same quantum state***

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I.1.1 Quantum Liquids

- **example: two particles (1,2) in quantum states A and B:**

classical: 4 possibilities
double-occupancy ratio: **1/2**

$$A(1)A(2) \qquad B(1)B(2)$$
$$A(1)B(2) \qquad B(1)A(2)$$

bosons: 3 possibilities
double-occupancy ratio: **2/3**

$$A(1)A(2)$$
$$\frac{1}{\sqrt{2}}\{A(1)B(2)+A(2)B(1)\}$$
$$B(1)B(2)$$

fermions: 1 possibility
double-occupancy ratio: **0**

$$\frac{1}{\sqrt{2}}\{A(1)B(2)-A(2)B(1)\}$$

bosons: tendency to occupy the same quantum state

fermions: multiple occupancy of the same quantum state is forbidden

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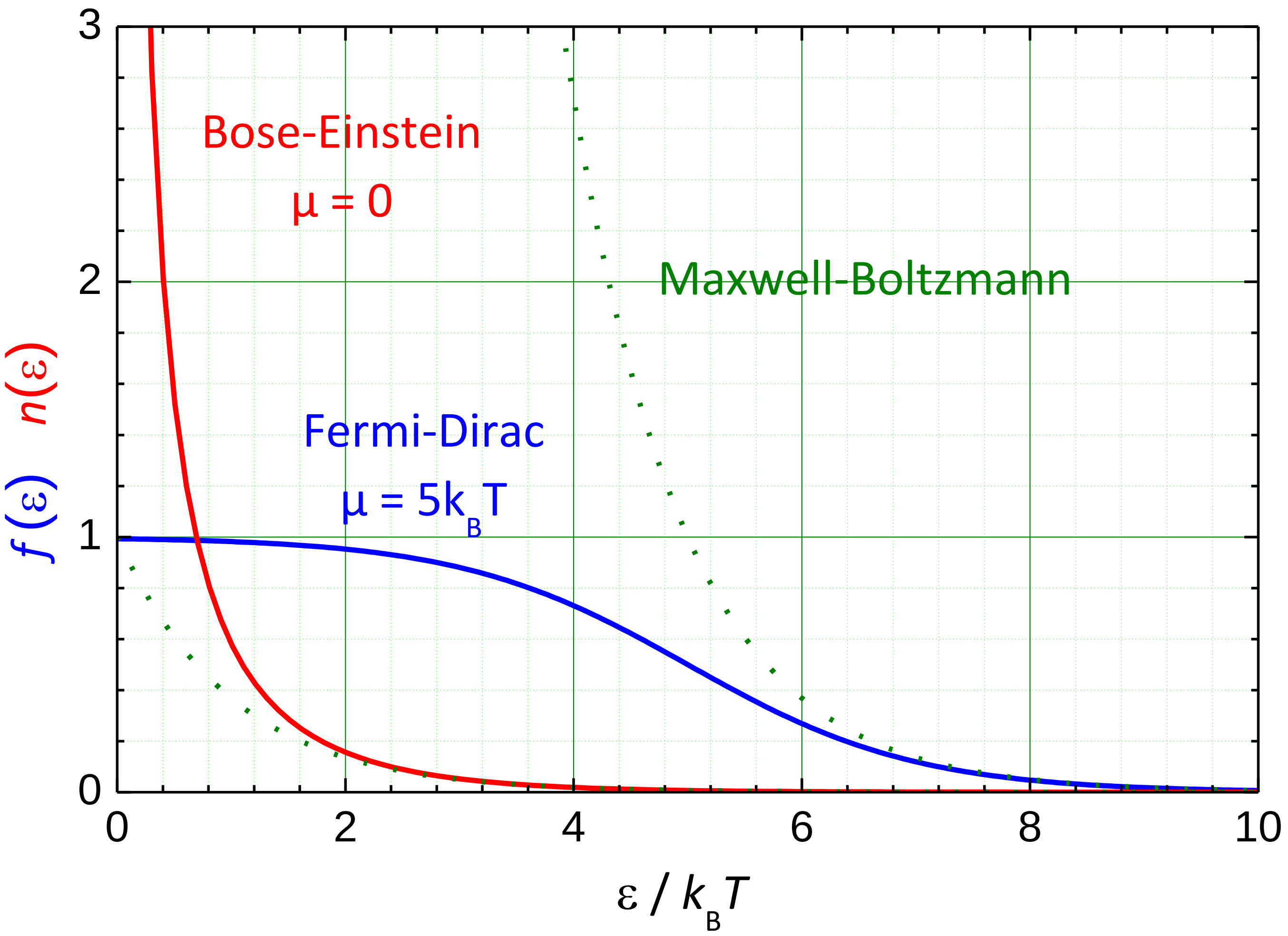
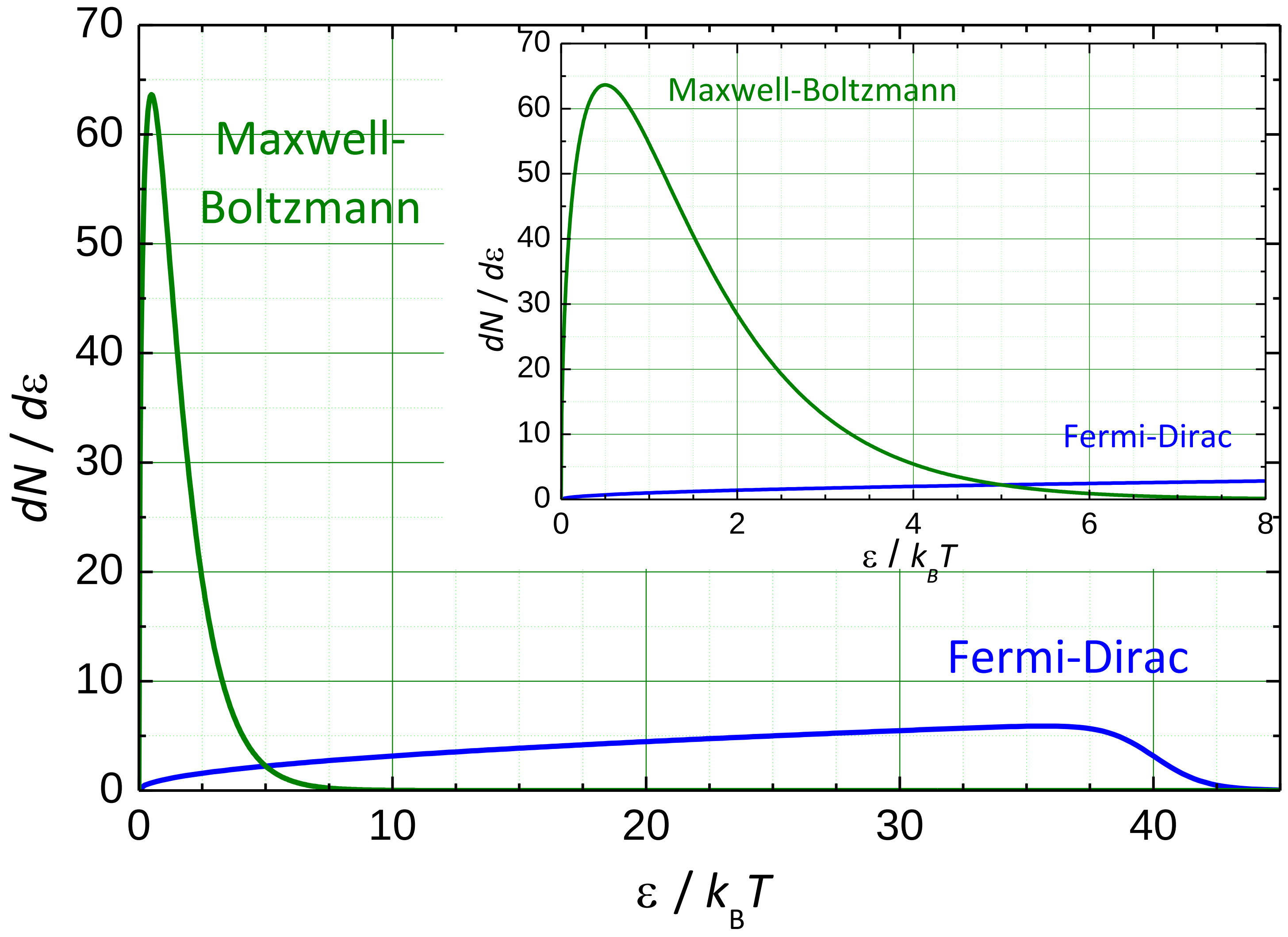


Bose-Einstein, **Fermi-Dirac** and **Maxwell-Boltzmann distributions** as a function of normalized energy. For the Bose-Einstein distribution we assumed $\mu = 0$, for the Fermi-Dirac distribution $\mu = 5k_\mathrm{B}T$ (for a metal μ is several eV corresponding to temperatures of several 10 000 K). For the Maxwell-Boltzmann distributions we also assumed $\mu = 0$ und $5k_\mathrm{B}T$ so that the classical distribution coincides with the quantum mechanical distributions for large $\varepsilon \gg k_\mathrm{B}T$.



particle number per energy interval: $$\frac{dN}{d\varepsilon} = D(\varepsilon) \cdot f(\varepsilon)$$

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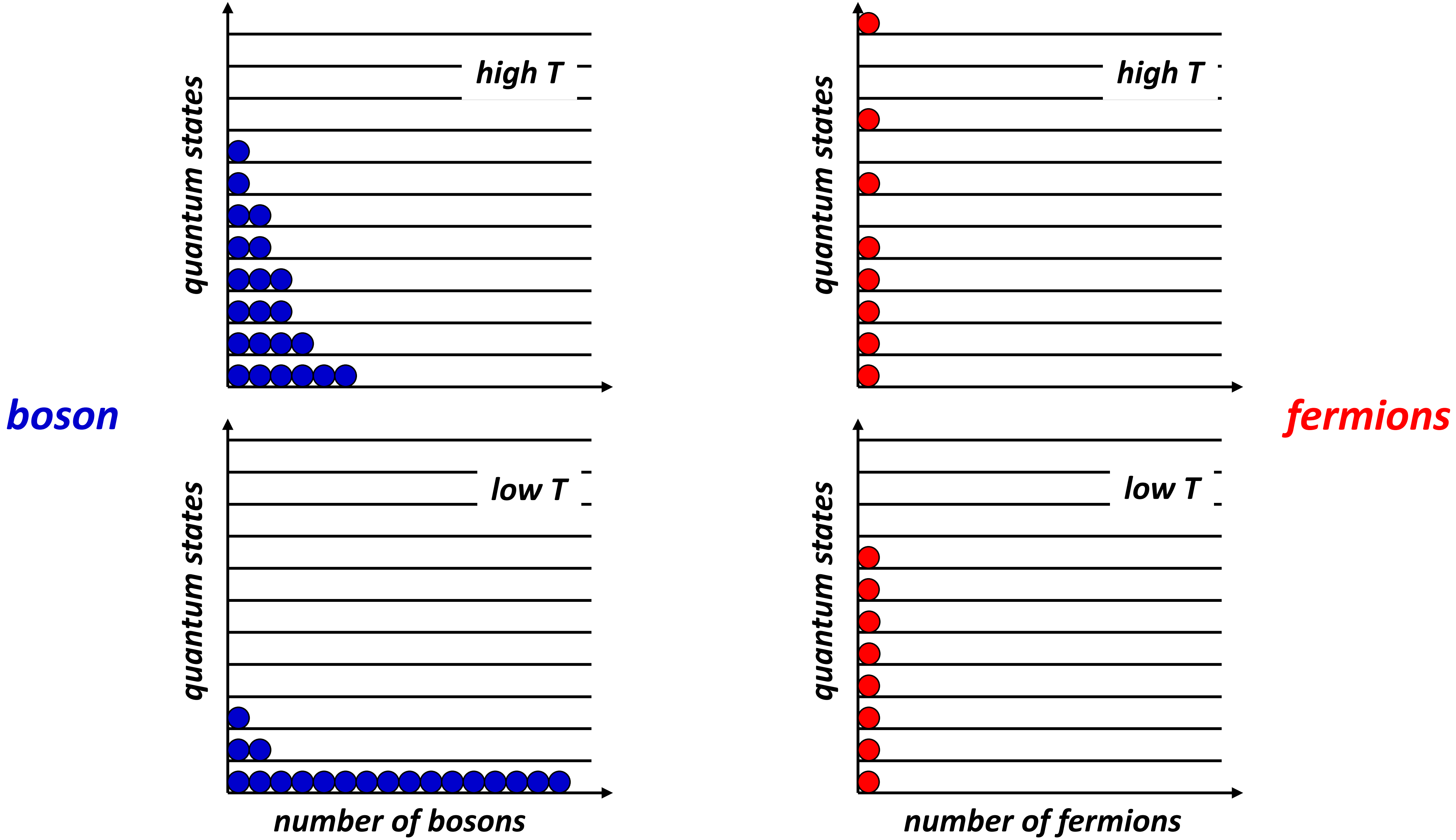
I.1.1 Quantum Liquids



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I.1.1 Quantum Liquids

- **well-known quantum liquids:**

1. *liquid helium:*

- small mass and *d* → high T_{zp} (a few K)
- small $\langle V \rangle$ → liquid down to $T = 0$
- ^{4}He: spin 0 → Bose liquid
- ^{3}He: spin ½ → Fermi liquid

2. *electrons in metals:*

- "non-interacting" electron gas
- very small electron mass, higher density as for He
  → very high T_{zp} (a few 10^4 K)
  → $T_{\mathrm{zp}} \gg T_{\mathrm{melting}}$
- spin ½ → Fermi gas (Fermi liquid due to finite interactions)

3. *nuclear matter:*

- fermions in nuclei, neutron stars, …
- very high density → very high T_{zp} (a few 10^{12} K)
- spin ½ → Fermi liquid

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I.1.1 Quantum Liquids

	mass (kg)	n (m^{-3})	T_{zp} (K)	statistics
^{3}He	5.01 x 10^{-27}	1.63 x 10^{28}	6.50	Fermi
^{4}He	6.65 x 10^{-27}	2.18 x 10^{28}	5.93	Bose
electrons in Na	9.11 x 10^{-31}	2.65 x 10^{28}	4.94 x 10^4	Fermi
nuclear matter	1.67 x 10^{-27}	1.95 x 10^{38}	1 x 10^{12}	Fermi

$$E_{\rm zp} = \frac{\Delta p^2}{2m} = \frac{h^2}{2md^2} = \frac{h^2 n^{2/3}}{2m} = \pi k_{\rm B} T_{\rm zp}$$

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Contents Part I: Quantum Liquids

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