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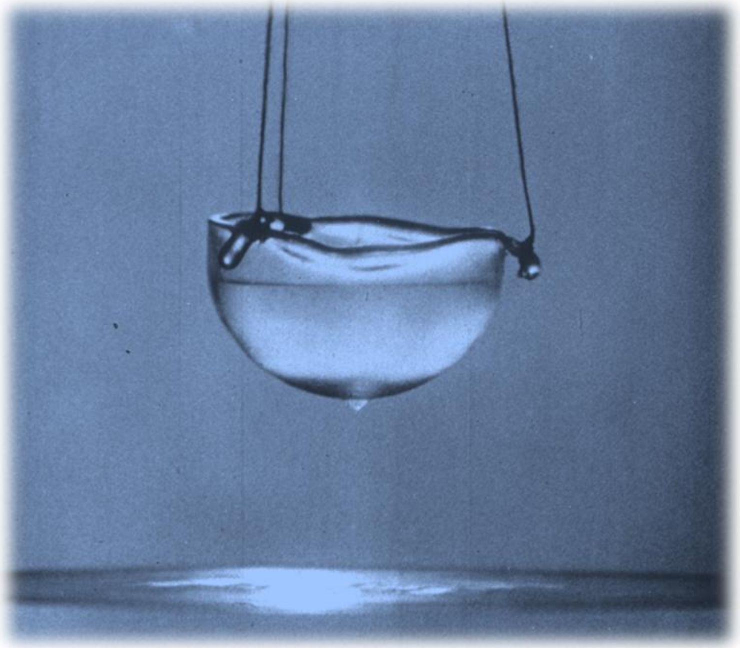
Superconductivity and Low Temperature Physics II



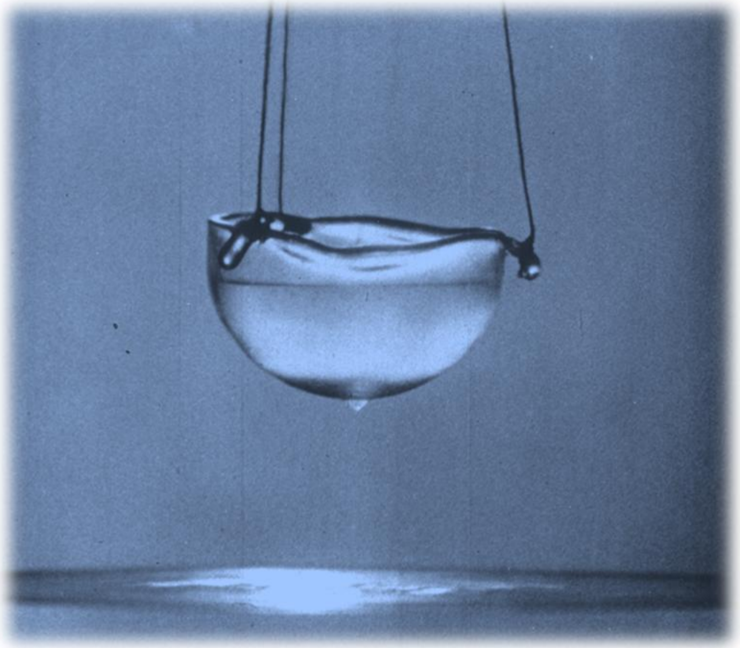
Lecture Notes
Summer Semester 2024

R. Gross

Chapter 1

Quantum Gases & Liquids

Superconductivity
and
Low Temperature
Physics II



Lecture No. 1

R. Gross

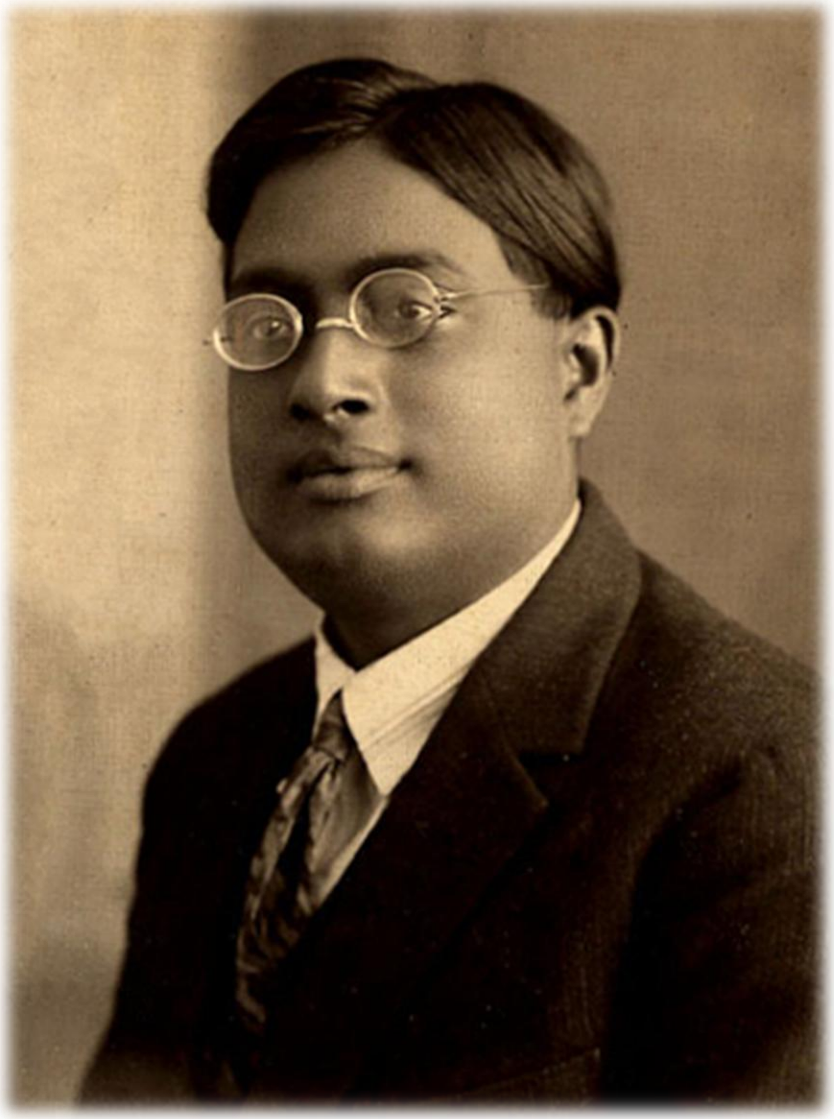
Contents Part I: Quantum Gases & Liquids

Contents:

I.1 Foundations and General Properties

- **Satyendranath Bose (1924)**
  - letter to Einstein: proposes a new method to derive Planck's law for spectral density of electromagnetic radiation emitted by a black body
  - Bose describes the radiation emitted by a black body in thermal equilibrium by a gas on non-interacting identical particles
    → ***photons***
  - description is in perfect agreement with Einstein‘s explanation of the photoelectric effect
  - Einstein immediately recognized that this description can be transferred to many other systems:
    (i) gases of identical particles with finite mass,
    (ii) phonons
  - Einstein strongly supported the publication of Bose‘s results, he translated the paper into German

Satyendranath Bose: *Plancks Gesetz und Lichtquantenhypothese*.
In: *Zeitschrift für Physik.* 26 (1924), S. 178-181.



Satyendranath Bose (around 1926)

I.1 Foundations and General Properties

What happens if you cool down a gas of non-interacting bosons ?

Below a certain temperature the particles feel each other due to their finite de Broglie wavelength, although they are non-interacting !

phase transition driven by statistics of particles (not by their interactions)

Bose-Einstein Condensation

thermodynamic phase transition into a gas of non-interacting particles:
driven by statistics of particles and not by interactions

I.1 Foundations and General Properties

- 1924: prediction of **Bose-Einstein Condensation (BEC**) by Einstein
- 1995: first observation of BEC on alkali atoms by

Eric A. Cornell **Wolfgang Ketterle** **Carl E. Wieman**

Nobel Prize in Physics 2001

"for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates"

→ long period between theoretical prediction and experimental realization:
high density, ultra-cold gas of non-interacting atoms required
→ problem: gas liquefies or solidifies due to finite interactions

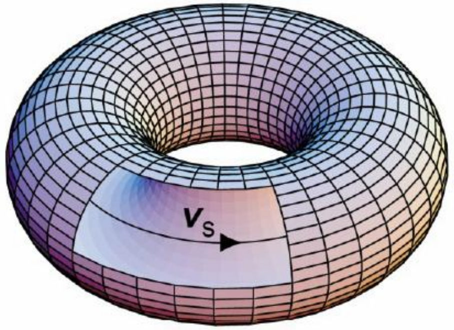
→ easier to realize:
interacting quantum gas, e.g. superfluid ^{4}He

I.1 Foundations and General Properties

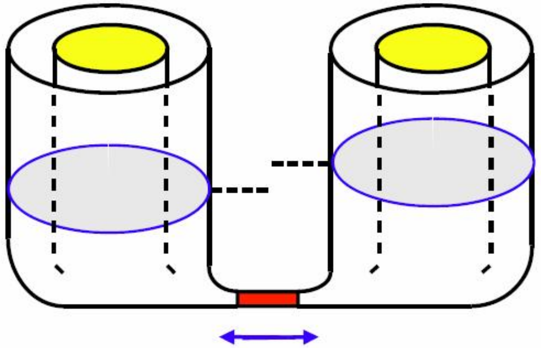
- **nomenclature**

• system of non-interacting particles:	**classical gas**	e.g. classical ideal gas
• if quantum fluctuations dominate:	**quantum gas**	e.g. Bose-Einstein Condensate, Fermi gas
• system of interacting particles:	**classical liquid**	e.g. water, liquid helium-4
• if quantum fluctuations dominate:	**quantum liquid**	e.g. superfluid helium-4, Fermi liquid

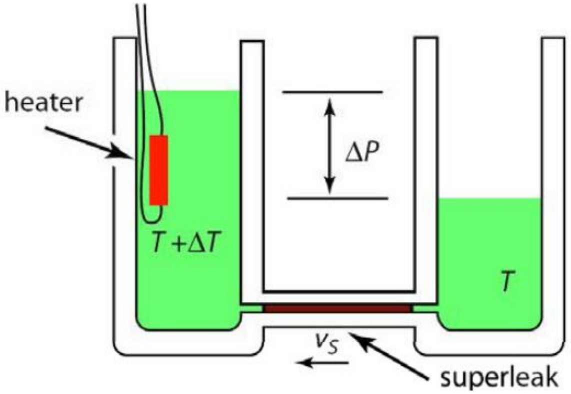
I.1.1 Quantum Gases & Liquids



supercurrents

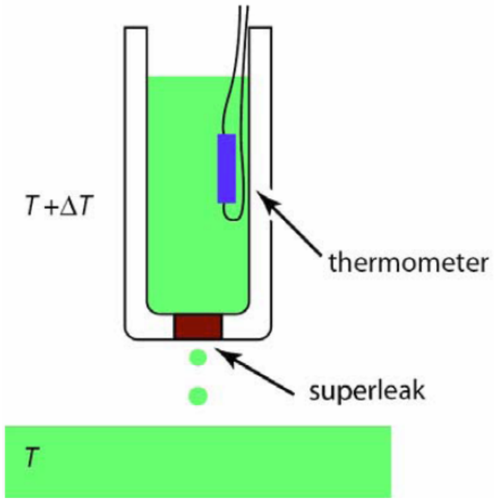


U-tube oscillation

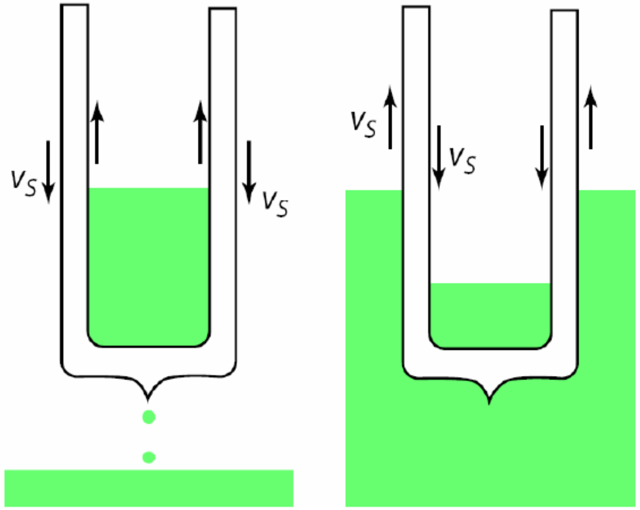


thermomech. effects

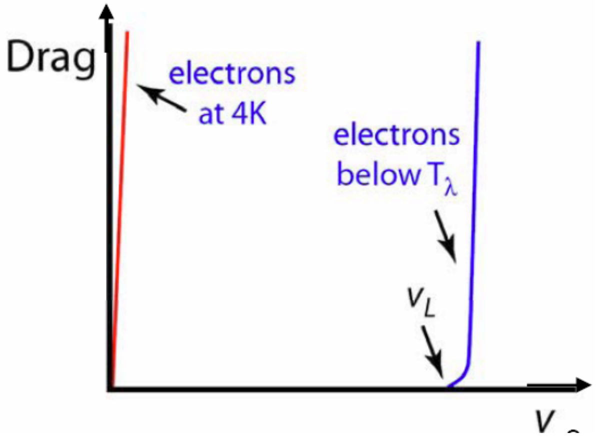
quantum liquids & condensates show fascinating properties → discussed in this lecture



superleaks



beaker film flow



critical velocity

Source: D.I. Bradley, Lancaster Univ.

I.1.1 Quantum Gases & Liquids

- **classical liquids and solids – characteristic energy scales:**

characteristic energies: (1) average interaction energy: $\langle V \rangle$ or $\langle E_{\mathrm{pot}} \rangle$
(2) thermal (kinetic) energy: $k_{\mathrm{B}}T$ or $\langle E_{\mathrm{kin}} \rangle$

criterion for liquid/solid: ***liquid***: $k_{\mathrm{B}}T \gg \langle V \rangle$
solid: $k_{\mathrm{B}}T \ll \langle V \rangle$

- **characteristic length scales in a gas:**

- particle distance: $d = n^{-1/3}$

- interaction range: $r_0 \simeq a$ (scattering length a)

- de Broglie wavelength:

$$\lambda_{\mathrm{T}} = \sqrt{\frac{h^2}{2\pi m k_{\mathrm{B}} T}} = \sqrt{\frac{2\pi\hbar^2}{m k_{\mathrm{B}} T}}$$

$$\frac{\hbar^2 k_{\mathrm{T}}^2}{2m} \simeq k_{\mathrm{B}}T \qquad k_{\mathrm{T}} = \frac{2\pi}{\lambda_{\mathrm{T}}}$$

$$\frac{h^2}{2m\lambda_{\mathrm{T}}^2} \simeq k_{\mathrm{B}}T \rightarrow \lambda_{\mathrm{T}} \simeq \left(\frac{h^2}{2mk_{\mathrm{B}}T}\right)^{1/2}$$

classical non-interacting gas: $d \gg r_0, \lambda_{\mathrm{T}}$, since atoms interact only during collisions

I.1.1 Quantum Gases & Liquids

- **calculation of the average thermal wavelength of a statistical ensemble (1)**

we consider a classical gas of N particles:
→ probability for a specific particular configuration at temperature T is given by Boltzmann distribution

$$P(\mathbf{p}_1, \dots, \mathbf{p}_N, \mathbf{r}_1, \dots, \mathbf{r}_N) = \frac{1}{Z_N} \exp(-\beta \mathcal{H})$$

$\beta = 1/k_\mathrm{B}T, \mathcal{H} =$ Hamiltonian

$$Z_N = \frac{1}{N!} \int \mathrm{d}^3 p_1, \dots, \mathrm{d}^3 p_N \mathrm{d}^3 r_1, \dots, \mathrm{d}^3 r_N \exp(-\beta \mathcal{H})$$

classical partition function, $\frac{1}{N!}$ results from fact that particles cannot be distinguished

partition function can be factorized into two terms (related to kinetic and potential energy)

$$Z_N = \frac{1}{N!} \underbrace{\prod_i \int \mathrm{d}^3 p_i \exp\left(-p_i^2/2mk_\mathrm{B}T\right)}_{\text{kinetic energy contribution}} \cdot \underbrace{\int \mathrm{d}^3 r_1, \dots, \mathrm{d}^3 r_N \exp\left(-\frac{1}{2}\beta \sum_{i \neq j} V(\mathbf{r}_i - \mathbf{r}_j)\right)}_{\text{potential energy contribution}}$$

product of independent factors

$$\left[(2\pi m k_\mathrm{B} T)^{3/2}\right]^N$$

(momenta of particles are statistically independent)

I.1.1 Quantum Gases & Liquids

- **calculation of the average thermal wavelength of a statistical ensemble (2)**

probability that particle has momentum in volume d^3p of phase space

$$P(\mathbf{p})\mathrm{d}^3p = (2\pi m k_\mathrm{B}T)^{-3/2}\exp\left(-p_i^2/2mk_\mathrm{B}T\right)\mathrm{d}^3p$$

probability that particle has momentum between p and $p + \mathrm{d}p$ $\left(\int_{-\infty}^{\infty}\mathrm{d}^3p = \int_{-\infty}^{\infty}p^2\mathrm{d}p\int_0^{\pi}\sin\theta\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi = 4\pi\int_{-\infty}^{\infty}p^2\mathrm{d}p\right)$

$$P_\mathrm{MB}(p)\mathrm{d}p = \frac{4\pi p^2}{\underbrace{(2\pi m k_\mathrm{B}T)^{3/2}}_{p_\mathrm{T}^3}}\exp(-p^2/2mk_\mathrm{B}T)\,\mathrm{d}p$$

Maxwell-Boltzmann distribution

the most likely momentum of the particles is $p = \sqrt{2mk_\mathrm{B}T}$, the average momentum is $\bar{p} = \sqrt{8mk_\mathrm{B}T/\pi}$

➔ **thermal de Broglie wavelength**

$$\lambda_\mathrm{T} = \frac{h}{p_\mathrm{T}} = \sqrt{\frac{h^2}{2\pi m k_\mathrm{B}T}} = \sqrt{\frac{2\pi\hbar^2}{mk_\mathrm{B}T}}$$

I.1.1 Quantum Gases & Liquids

- **cooling down a classical gas** → $\lambda_{\mathrm{T}} = \sqrt{\frac{h^2}{2\pi m k_{\mathrm{B}} T}}$ increases and we can distinguish three regimes

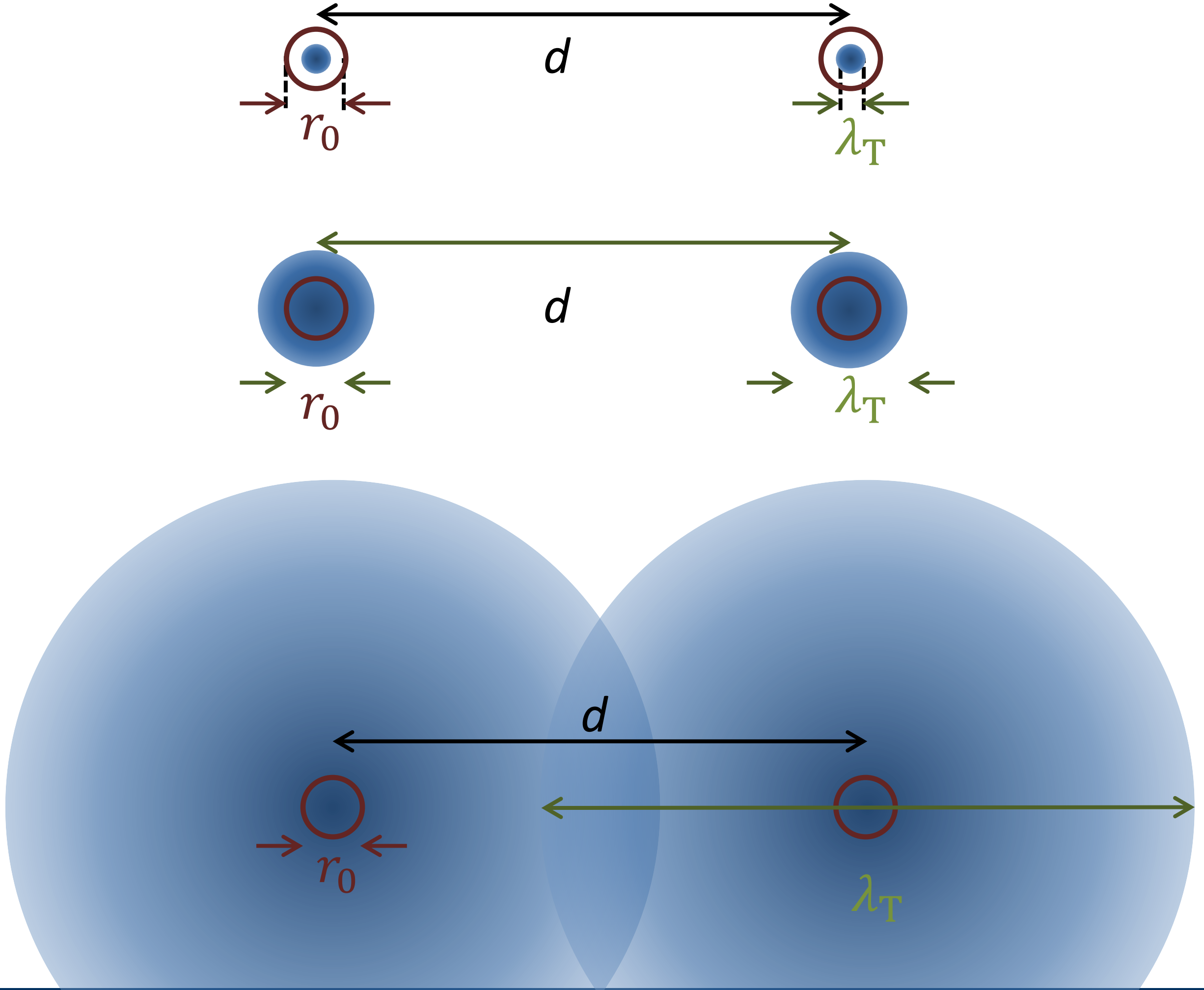
(1) classical regime: $\lambda_{\mathrm{T}} \ll r_0 \ll d$

(2) quantum collisions: $r_0 \ll \lambda_{\mathrm{T}} \ll d$

collissions can no longer be treated classically

(3) quantum degeneracy: $r_0 \ll d \le \lambda_{\mathrm{T}}$

all degrees of freedom of the gas must be treated quantum mechanically



I.1.1 Quantum Gases & Liquids

- **define new energy scale related to particle distance d**

(3) **zero-point energy E_{zp}**

with - particle density $n = N/V, \quad d = n^{-1/3}$

→ momentum uncertainty $\Delta p = \hbar \Delta k = \frac{h}{d} = h n^{1/3}$

➔ *zero-point energy*

$$E_{\mathrm{zp}} = \frac{\Delta p^2}{2m} = \frac{h^2}{2md^2} = \frac{h^2 n^{2/3}}{2m} = \pi k_{\mathrm{B}} T_{\mathrm{zp}}$$

characteristic temperature

for an ideal gas the energy $E = \pi k_{\mathrm{B}} T$ is assumed; the factor π results from the partition function

I.1.1 Quantum Gases & Liquids

- **classification of liquids** **(interacting systems with $\langle V \rangle > 0$)** :

(a) classical liquid: $T_{\text{zp}} \ll T > T_{\text{melting}}$ (quantum fluctuation play no role, thermal fluctuations dominate)

(b) quantum liquid: $T_{\text{zp}} \gg T > T_{\text{melting}}$ (quantum fluctuation dominate)

proportional to $\langle V \rangle$ or $\langle E_{\text{pot}} \rangle$

quantum parameter:

$$\Lambda \equiv \frac{\langle E_{\text{zp}} \rangle}{\langle E_{\text{pot}} \rangle}$$

quantum liquid:

$$\boldsymbol{\Lambda > 1}$$

classical liquid:

$$\boldsymbol{\Lambda < 1}$$

I.1.1 Quantum Gases & Liquids

- **classification of gases** **(non-interacting systems: $\langle V \rangle = 0$)** :

(a) classical gas: $T_{\mathrm{zp}} < T$

(b) quantum gas: $T < T_{\mathrm{zp}}$

$$T_{\mathrm{zp}} = \frac{h^2 n^{2/3}}{2\pi m k_{\mathrm{B}}} = \frac{h^2}{2\pi m k_{\mathrm{B}} d^2} \Longleftrightarrow T = \frac{h^2}{2\pi m k_{\mathrm{B}} \lambda_{\mathrm{T}}^2}$$

(a) classical gas: $\lambda_{\mathrm{T}} \leq n^{-1/3} \simeq d \rightarrow n \cdot \lambda_{\mathrm{T}}^3 \leq 1$

(b) quantum gas: $\lambda_{\mathrm{T}} \geq n^{-1/3} \simeq d \rightarrow n \cdot \lambda_{\mathrm{T}}^3 \geq 1$

average particle distance

e.g. electrons @ 300 K: $\lambda_{\mathrm{T}} = 4.2$ nm, $d \simeq 0.2$ nm for Cu ($n \simeq 8.5 \times 10^{28}$ m^{-3})

➔ particles become indistinguishable → degenerate quantum gas

I.1.1 Quantum Gases & Liquids

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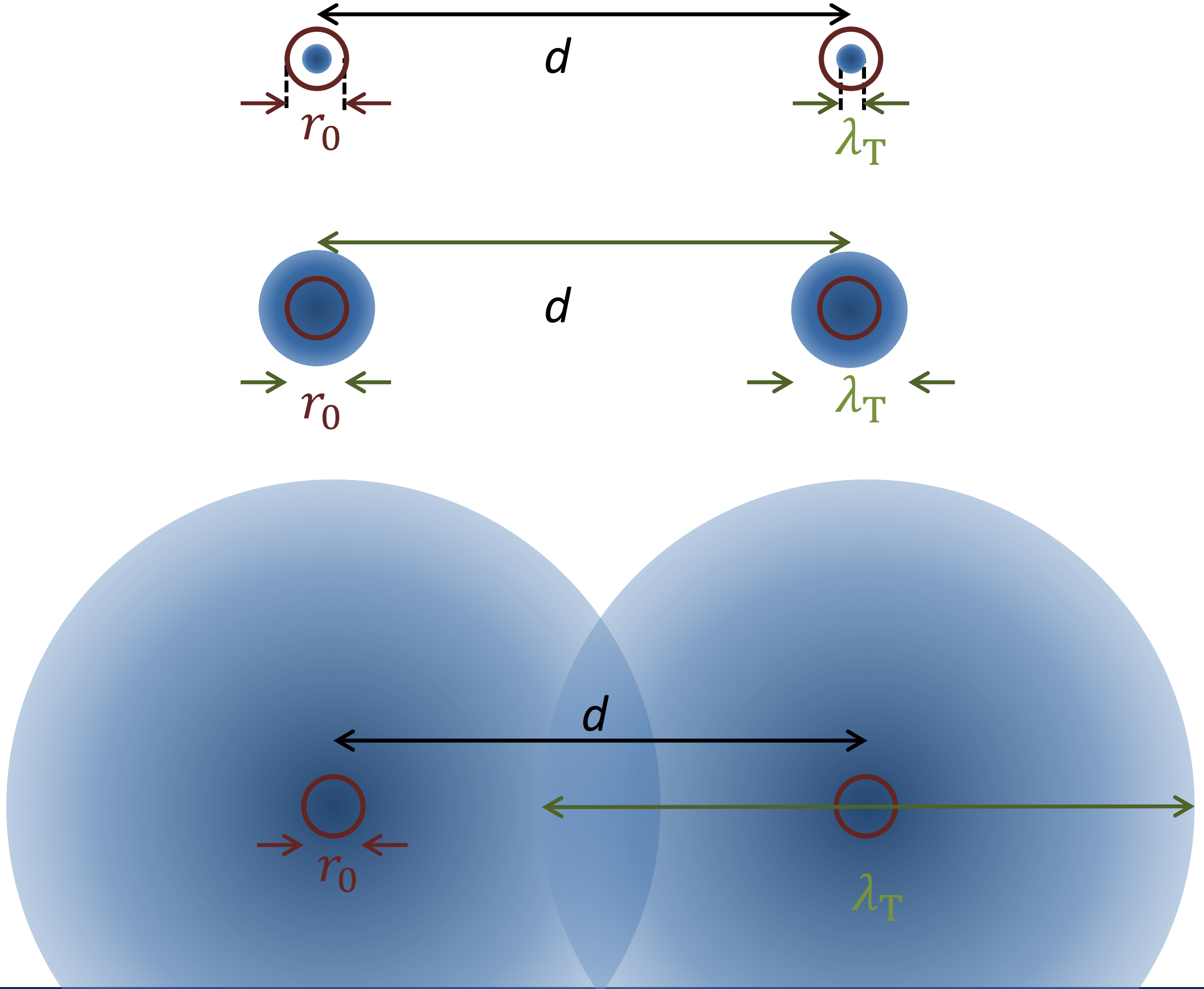
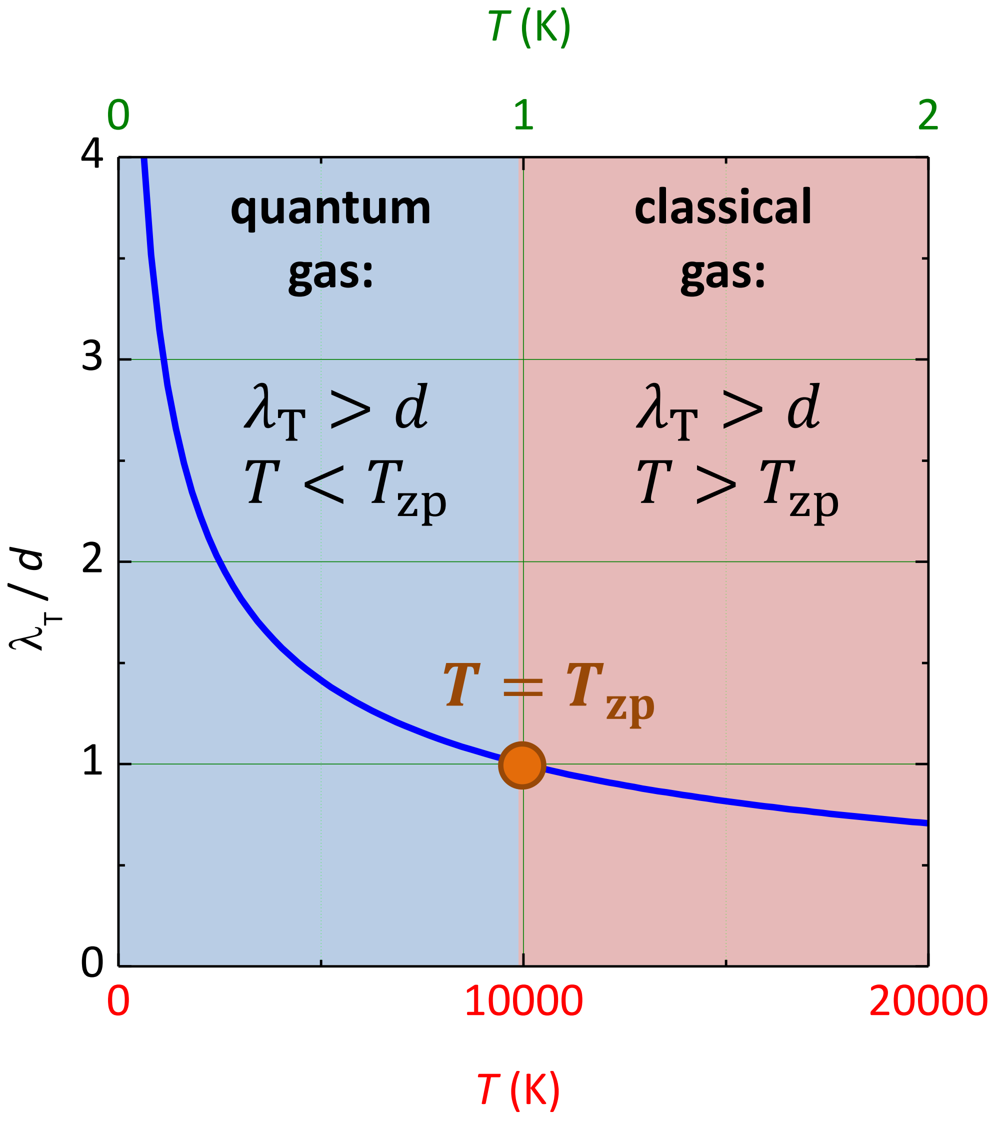


I.1.1 Quantum Gases & Liquids



^{4}He: $m = 6.65 \times 10^{-27}$ kg

$n \simeq 4 \times 10^{27}$ m^{-3}

$$\lambda_T = \sqrt{\frac{h^2}{2\pi m k_B T}} \propto \frac{1}{\sqrt{T}}$$

e$^-$: $m = 9.11 \times 10^{-31}$ kg

$n \simeq 3.6 \times 10^{27}$ m^{-3}

degenerate:
indistinguishable particles

non-degenerate:
distinguishable particles

I.1.1 Quantum Gases & Liquids

- **quantum statistics:**

photons, mesons, ^{4}He-atoms, ...	integral spin	$S = 0, \hbar, 2\hbar, 3\hbar, \ldots$
electrons, nucleons, ^{3}He-atoms, ...	half-integral spin	$S = \hbar/2, 3\hbar/2, 5\hbar/2, \ldots$

indistinguishability of identical particles:

symmetry of many-particle wavefunction with respect to particle exchange

$$\wp\, \Psi(1,2) \equiv \Psi(2,1) = \mathrm{e}^{\imath\alpha}\ \Psi(1,2)$$
$$\wp\, \Psi(2,1) \equiv \Psi(1,2) = (\mathrm{e}^{\imath\alpha})^2\, \Psi(1,2)$$

$$\Rightarrow \quad \mathrm{e}^{\imath\alpha} = \pm 1$$

- **bosons** (integral spin)

$$\Psi_{\mathrm{AB}}(1,2) = \frac{1}{\sqrt{2}}[\Psi_{\mathrm{A}}(1)\Psi_{\mathrm{B}}(2) + \Psi_{\mathrm{A}}(2)\Psi_{\mathrm{B}}(1)]$$

- **fermions** (semi-integral spin)

$$\Psi_{\mathrm{AB}}(1,2) = \frac{1}{\sqrt{2}}[\Psi_{\mathrm{A}}(1)\Psi_{\mathrm{B}}(2) - \Psi_{\mathrm{A}}(2)\Psi_{\mathrm{B}}(1)]$$

- **Pauli principle**: ***two fermions cannot occupy the same quantum state***

I.1.1 Quantum Gases & Liquids

- **example: two particles (1,2) in quantum states A and B:**

classical:
4 possibilities
double-occupancy ratio: **1/2**

$$A(1)A(2) \qquad B(1)B(2)$$
$$A(1)B(2) \qquad B(1)A(2)$$

bosons:
3 possibilities
double-occupancy ratio: **2/3**

$$A(1)A(2)$$
$$\frac{1}{\sqrt{2}}\{A(1)B(2)+A(2)B(1)\}$$
$$B(1)B(2)$$

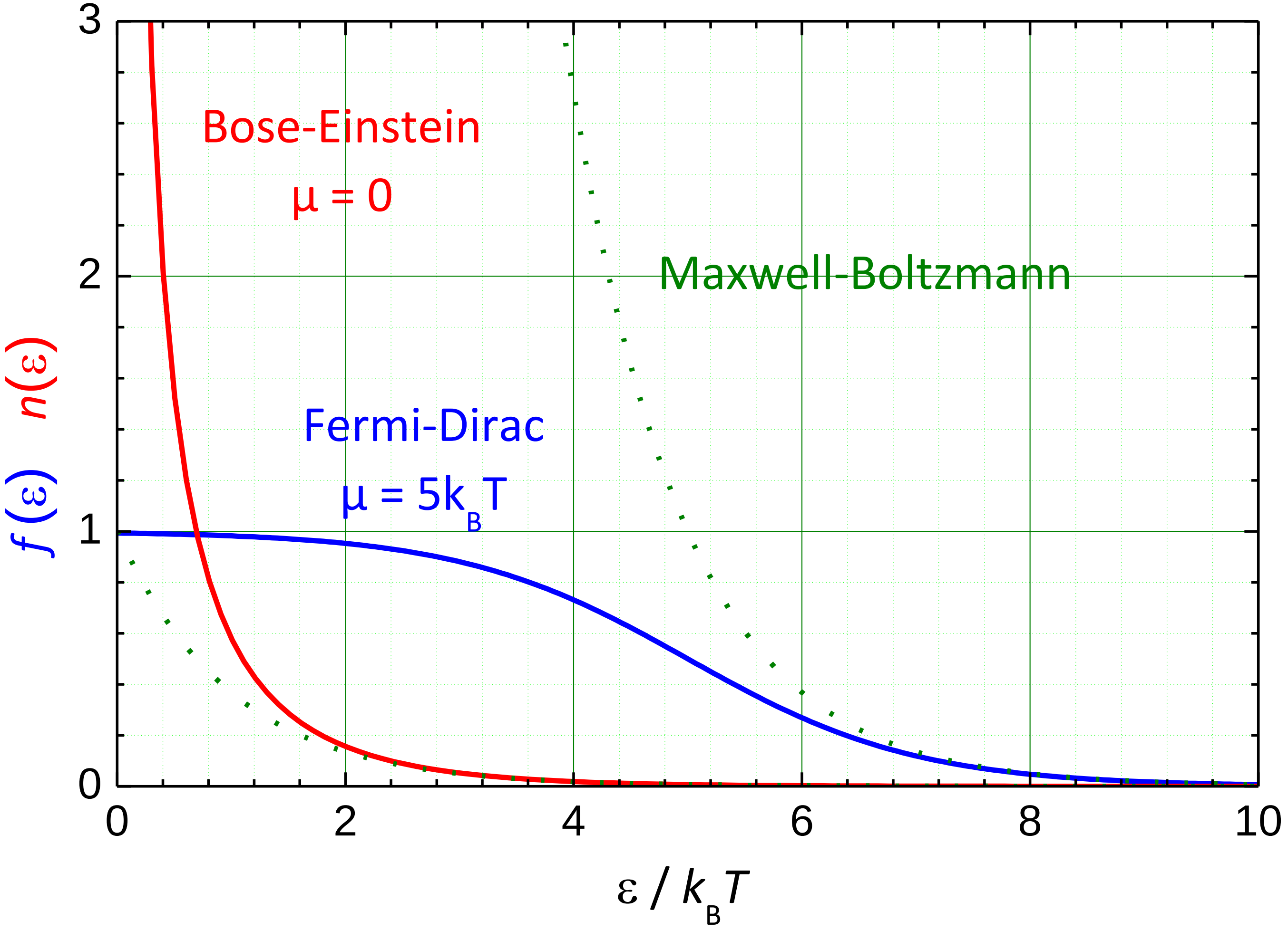
fermions:
1 possibility
double-occupancy ratio: **0**

$$\frac{1}{\sqrt{2}}\{A(1)B(2)-A(2)B(1)\}$$

bosons: tendency to occupy the same quantum state

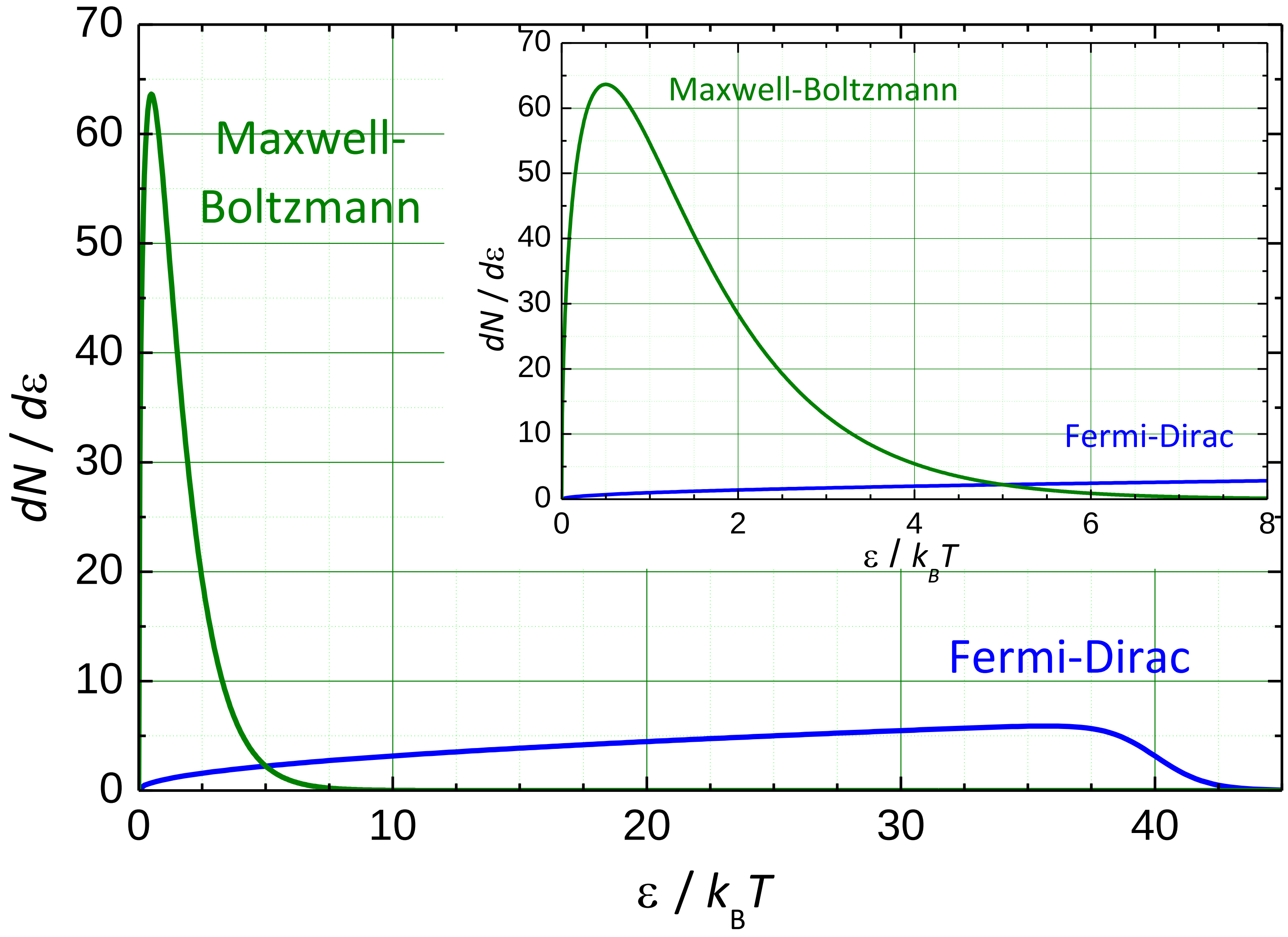
fermions: multiple occupancy of the same quantum state is forbidden

I.1.1 Quantum Gases & Liquids



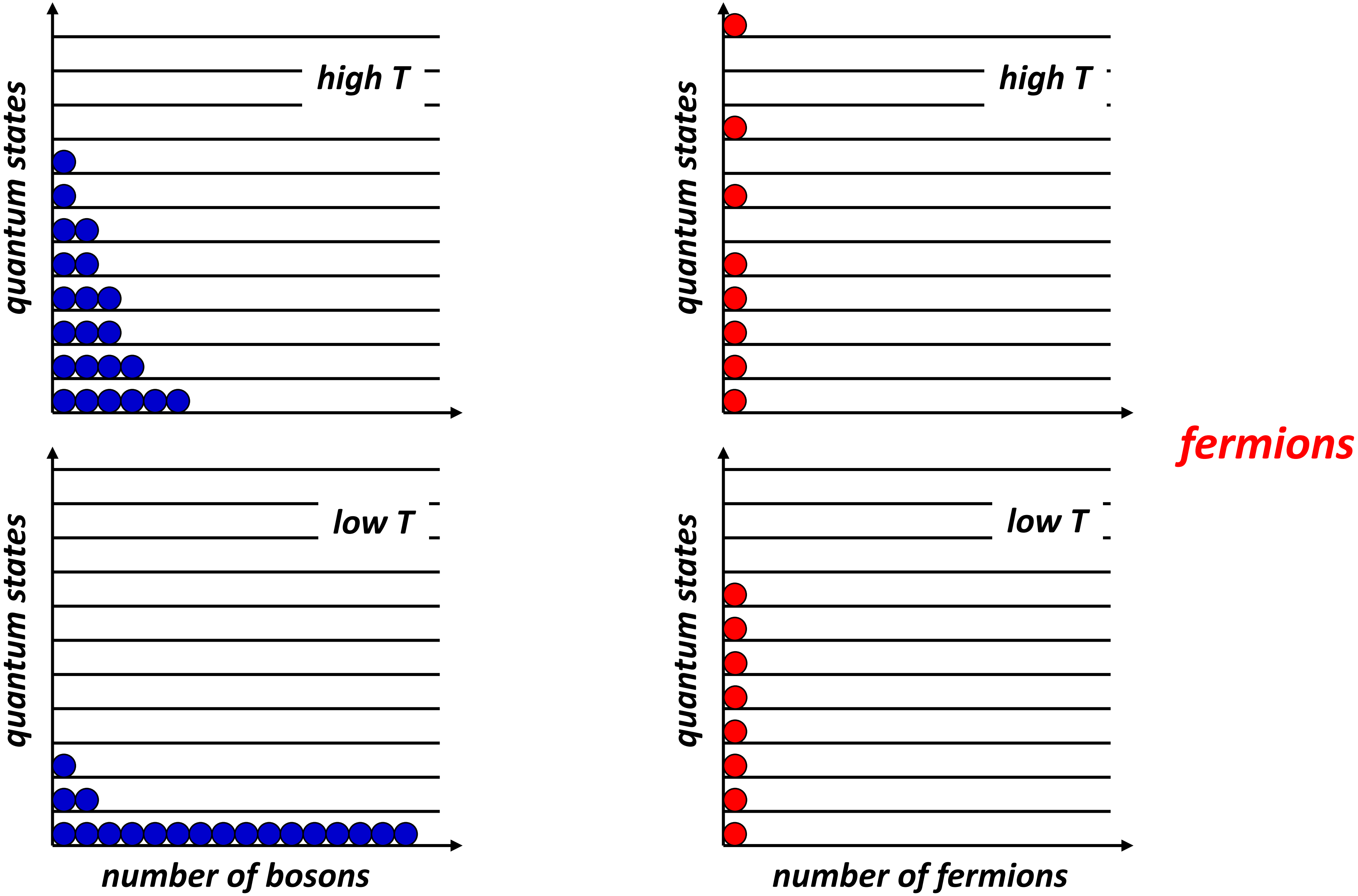
Bose-Einstein, **Fermi-Dirac** and **Maxwell-Boltzmann distributions** as a function of normalized energy. For the Bose-Einstein distribution we assumed $\mu = 0$, for the Fermi-Dirac distribution $\mu = 5k_\mathrm{B}T$ (for a metal μ is several eV corresponding to temperatures of several 10 000 K). For the Maxwell-Boltzmann distributions we also assumed $\mu = 0$ und $5k_\mathrm{B}T$ so that the classical distribution coincides with the quantum mechanical distributions for large $\varepsilon \gg k_\mathrm{B}T$.

I.1.1 Quantum Gases & Liquids



particle number per energy interval: $$\frac{dN}{d\varepsilon} = D(\varepsilon) \cdot f(\varepsilon)$$

I.1.1 Quantum Gases & Liquids



I.1.1 Quantum Gases & Liquids

- **well-known quantum gases/liquids:**

1. liquid helium:
(interacting systems)

- small mass and *d* → high$T_{\rm zp}$ (a few K)
- small $\langle V \rangle$ → liquid down to $T = 0$
- ^{4}He: spin 0 → Bose liquid
- ^{3}He: spin ½ → Fermi liquid

2. electrons in metals:

- "non-interacting" electron gas (in the simplest approximation)
- very small electron mass, higher density as for He
  → very high $T_{\rm zp}$ (a few 10^4 K)
  → $T_{\rm zp} \gg T_{\rm melting}$
- spin ½ → Fermi gas (Fermi liquid due to finite interactions)

3. nuclear matter:

- "non-interacting" fermions in nuclei, neutron stars, … (in the simplest approximation)
- very high density → very high $T_{\rm zp}$ (a few 10^{12} K)
- spin ½ → Fermi gas (Fermi liquid due to finite interactions)

I.1.1 Quantum Gases & Liquids

	mass (kg)	n (m^{-3})	T_{zp} (K)	statistics
^{3}He	5.01×10^{-27}	1.63×10^{28}	6.50	Fermi
^{4}He	6.65×10^{-27}	2.18×10^{28}	5.93	Bose
electrons in Na	9.11×10^{-31}	2.65×10^{28}	4.94×10^{4}	Fermi
nuclear matter	1.67×10^{-27}	1.95×10^{38}	1×10^{12}	Fermi

$$E_{\rm zp} = \frac{\Delta p^2}{2m} = \frac{h^2}{2md^2} = \frac{h^2 n^{2/3}}{2m} = \pi k_{\rm B} T_{\rm zp}$$

Contents Part I:

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I.1 Foundations and General Properties

I.2 ^{4}He as an Ideal Bose Gas

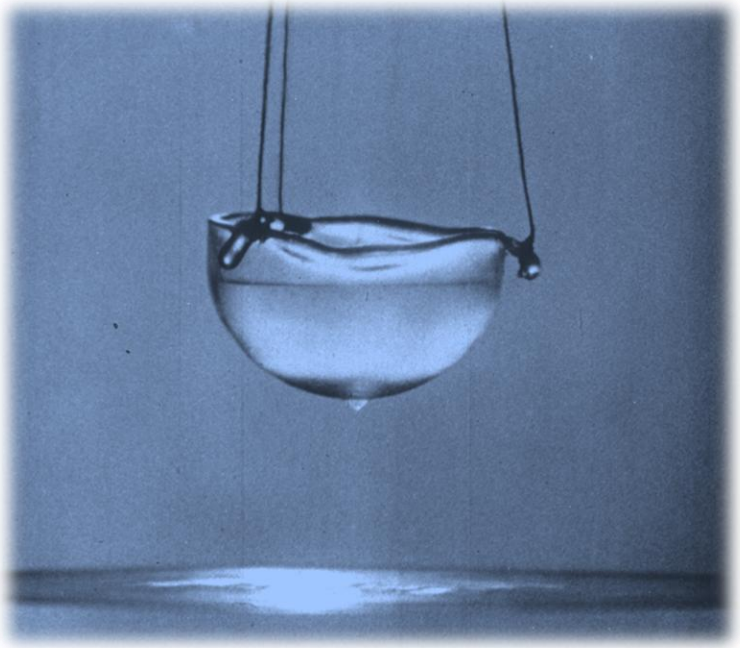
I.3 Superfluid ^{4}He

I.4 Vortices

I.5 ^{3}He

I.6 ^{3}He / ^{4}He mixtures

Superconductivity and Low Temperature Physics II



Lecture No. 2

R. Gross

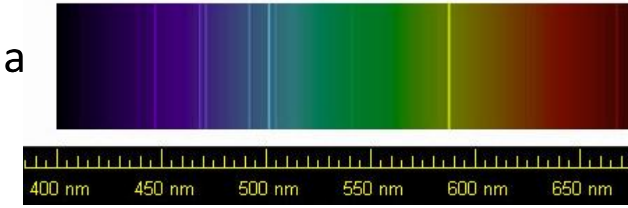
I.1.2 Helium

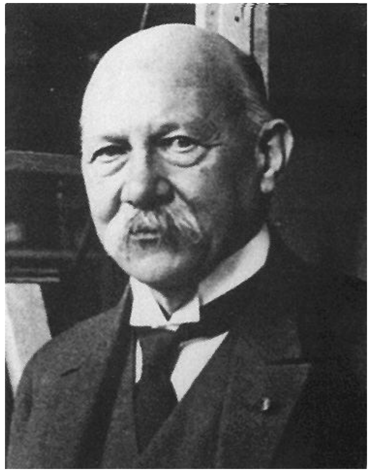
- *history:* ^{4}*He*

application of ^{4}***He in low temperature physics***

- soon after discovery on earth, race for the liquefaction of He
  → main opponents: ***Heike Kamerlingh Onnes*** and ***James Dewar***
  → Onnes extracted 360 l of He gas out of Monazit sand from North Carolina
- **1908**: first liquefaction: July 10, 1908
- **1925**: first liquefaction of He in Germany (March 7, 1925: 4.2 K, 200 ml) by ***Walther Meißner***, 3rd system world-wide besides Leiden and Toronto
- **1932**: Linde supplies first industrial He liequfier to University of Charkov, Ukraine
- **1947**: first commercial liquefier by ***S.C. Collins***

Helium was first detected as an unknown, yellow spectral line signature in sunlight during a solar eclipse in 1868 by Georges Rayet *et al.* and was subsequently confirmed by French astronomer Jules Janssen.





Heike Kamerlingh Onnes
* 21. 09. 1853
† 21. 02. 1926

James Dewar
* 20. 09. 1842
† 27.03.1923

I.1.2 Helium


SIR JAMES DEWAR

Sir James Dewar, (1842-1923) was a chemist and physicist, best known for his work with low-temperature phenomena. Dewar was born in Kincardine, Scotland, and educated at the University of Edinburgh. He was professor of experimental natural philosophy at the University of Cambridge, England, in 1875 and professor of chemistry at the Royal Institution of Great Britain in 1877, where he was appointed director of the Davy-Faraday Research Laboratory.

Dewar developed structural formulas for benzene (1867). He studied the specific heat of hydrogen and was the first person to produce hydrogen in liquid form (1898) and to solidify it (1899). He constructed a machine for producing liquid oxygen in quantity (1891). He invented the Dewar flask or thermos (1892) and co-invented cordite (1889), a smokeless gunpowder, with Sir Frederick Abel. His discovery (1905) that cooled charcoal can be used to help create high vacuums later proved useful in atomic physics. Dewar was knighted in 1904.

The Dewar flask or vacuum flask/bottle is a container for storing hot or cold substances, i.e. liquid air. It consists of two flasks, one inside the other, separated by a vacuum. The vacuum greatly reduces the transfer of heat, preventing a temperature change. The walls are usually made of glass because it is a poor conductor of heat; its surfaces are usually lined with a reflective metal to reduce the transfer of heat by radiation. Dewar used silver. The whole fragile flask rests on a shock-absorbing spring within a metal or plastic container, and the

air between the flask and the container provides further insulation. The common thermos bottle is an adaptation of the Dewar flask. Dewar invented the Dewar flask in 1892 to aid him in his work with liquid gases.

The vacuum flask was not manufactured for commercial/home use until 1904, when two German glass blowers formed Thermos GmbH. They held a contest to rename the vacuum flask and a resident of Munich submitted "Thermos", which came from the Greek word "Therme" meaning "hot."

I.1.2 Helium

- ***history: ^{3}He***

  - **1933**: discovery by ***Oliphant, Kinsey*** and ***Rutherford***
    (they believed that ^{3}He is unstable, disproven by ***Alvarez*** and ***Cornog*** in 1939)
  - **1947**: first experiments with gaseous ^{3}He
  - **1949**: first condensation of ^{3}He gas (***Sydorniak and Co.***)
  - **1956**: theory of Fermi liquids (***L.D. Landau***, **Nobel Prize in Physics 1962**)
  - **1960s**: first experiments with the Fermi liquid ^{3}He (***various groups***)
  - **1963**: first theory of superfluid ^{3}He (***Anderson and Morel, Balian and Werthamer***)
  - **1971**: discovery of superfluid phase transition at 2 mK (***Lee, Osheroff, Richardson,*** **Nobel Prize in Physics 1996**)
  - **1975**: extension of BCS theory to triplet pairing (***Leggett***, **Nobel Prize in Physics 2003**)
  - **1977**: first experiments on superfluid ^{3}He in Germany at WMI (***Eska, Schuberth, Uhlig***)

I.1.2 Helium

- ***He isotops:*** ^{4}***He*** and ^{3}***He***

  - ^{4}***He:*** - obtained out of natural gas (e.g. sources in USA, Poland, Katar, Algeria, Russia contain up to 10%)

    - $L = 0$, $S = 0$, nuclear spin $\boldsymbol{I = 0}$ ➔ ***boson***

  - ^{3}***He:*** - natural amount of ^{3}He in ^{4}He is only 0.14 ppm

    - artificial generation of ^{3}He by nuclear reaction

    $$ {}^6_3\mathrm{Li} + {}^1\mathrm{n} \rightarrow {}^3_1\mathrm{T} + {}^4_2\mathrm{He} + 4.78\ \mathrm{MeV} \qquad {}^3_1\mathrm{T} \rightarrow {}^3_2\mathrm{He} + e^- + \bar{\nu}_e $$

    (12.5 years)

    - ^{3}He is very expensive, several 1000 € per liter

    - $L = 0, S = 0$, nuclear spin $\boldsymbol{I = ½}$ ➔ ***fermion***

- despite their simple electronic structure, the liquids of both He isotopes show
  - ***a rich variety of exotic phenomena***
  - ***pronounced differences due to different quantum statistics***

Contents Part I:

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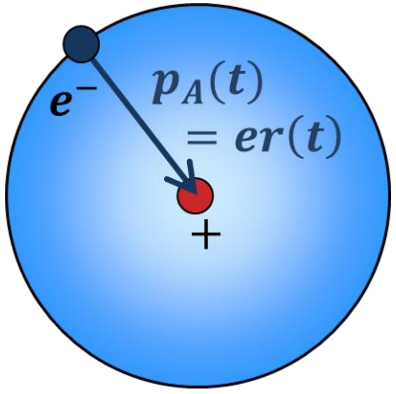
I.1.3 Van der Waals Bonding

- **neutral atom**, average dipole moment vanishes: $\langle \mathbf{p}_\mathrm{A} \rangle = \mathbf{0}$



$\langle \mathbf{p}_\mathrm{A} \rangle = \mathbf{0}$

→ dipole moment induced by electric field: $\mathbf{p}_\mathrm{A}^\mathrm{ind} \propto \mathbf{E}$

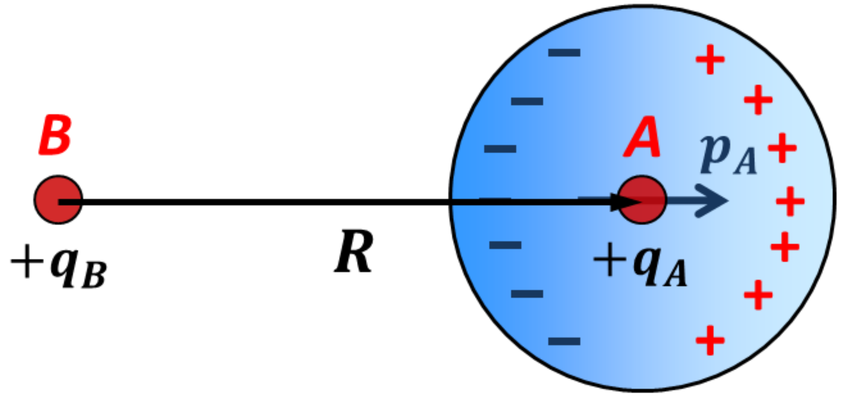
proportionality constant is **polarizability α_A** of atom A: $\boxed{\mathbf{p}_\mathrm{A}^\mathrm{ind} = \alpha_A\,\mathbf{E}}$

- **assumption**:
  $\mathbf{E}$-field is generated by charge q_B of ion B at distance R

$$\mathbf{p}_\mathrm{A}^\mathrm{ind} = \alpha_\mathrm{A}\mathbf{E}_\mathrm{B,ion} = \alpha_\mathrm{A}\frac{q_B}{4\pi\epsilon_0 R}\,\widehat{\mathbf{R}}$$

- potential energy of atom A

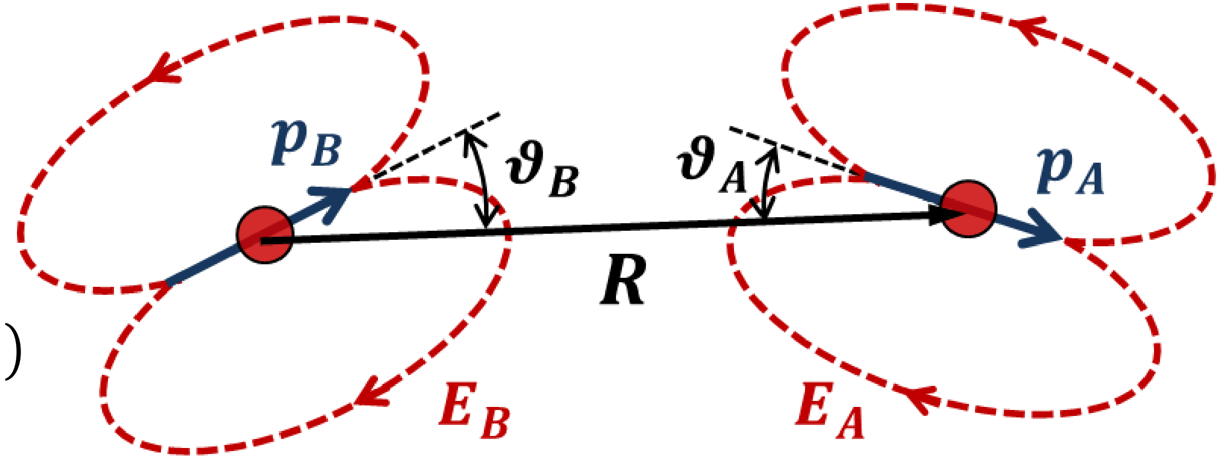
$$E_\mathrm{pot} = -\mathbf{p}_\mathrm{A}^\mathrm{ind}\cdot\mathbf{E}_\mathrm{B,ion} = -\left(\alpha_A\mathbf{E}_\mathrm{B,ion}\right)\cdot\mathbf{E}_\mathrm{B,ion} \propto \mathbf{E}_\mathrm{B,ion}^2$$



I.1.3 Van der Waals Bonding

- **assumption**:
  - field $\mathbf{E}(\mathbf{p}_B)$ at position of atom A is generated by atom B with dipole moment $\mathbf{p}_\mathrm{B}$
  - $\mathbf{E}(\mathbf{p}_B)$ induces dipole moment $\mathbf{p}_\mathrm{A}$ of atom A, which in turn generates field $\mathbf{E}(\mathbf{p}_A)$ at position of atom B

$$\mathbf{E}(\mathbf{p}_B) = \frac{1}{4\pi\epsilon_0 R^3}\left(3p_B\cos\vartheta_B\ \widehat{\mathbf{R}} - \mathbf{p}_B\right)$$



$$\mathbf{E}(\mathbf{p}_A) = \frac{1}{4\pi\epsilon_0 R^3}\left(3p_A\cos\vartheta_A\ \widehat{\mathbf{R}} - \mathbf{p}_A\right)$$

- **Interaction energy** (same atoms: $p_\mathrm{A} = p_\mathrm{B} = p,\ \alpha_\mathrm{A} = \alpha_\mathrm{B} = \alpha$)

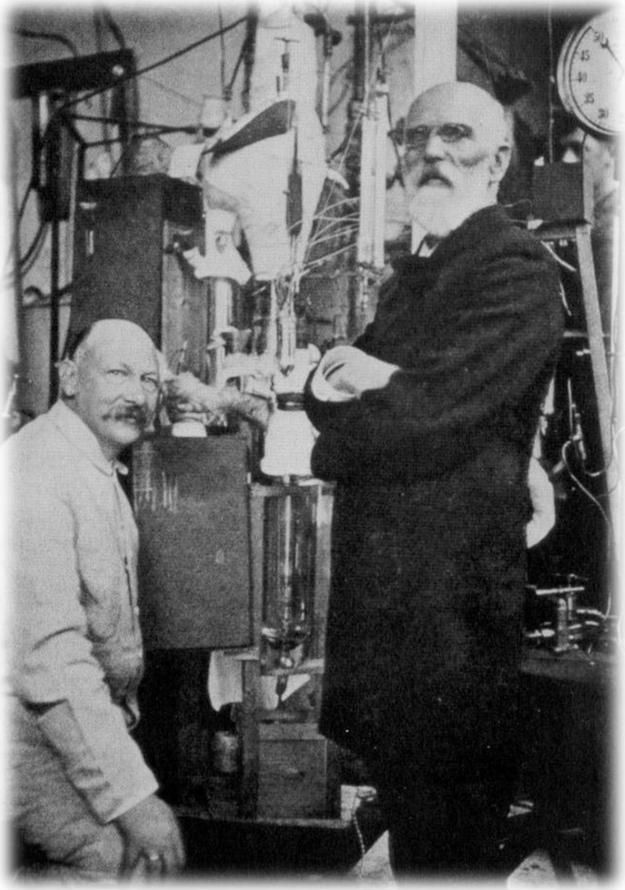
$$E_\mathrm{pot}^\mathrm{dd}(R) = \frac{1}{4\pi\epsilon_0}\frac{p_A p_B - 3\left(\widehat{\mathbf{R}}\cdot\mathbf{p}_A\right)\left(\widehat{\mathbf{R}}\cdot\mathbf{p}_B\right)}{R^3} \Rightarrow E_\mathrm{pot}^\mathrm{dd}(R) \propto -\frac{p_A\,p_B}{R^3} \propto -\frac{p}{R^3}\,\frac{\alpha\,p}{R^3} \propto -\frac{\alpha}{R^6}$$

$$\boxed{E_\mathrm{pot}^\mathrm{dd}(R) = -C\,\frac{\alpha}{R^6}}$$

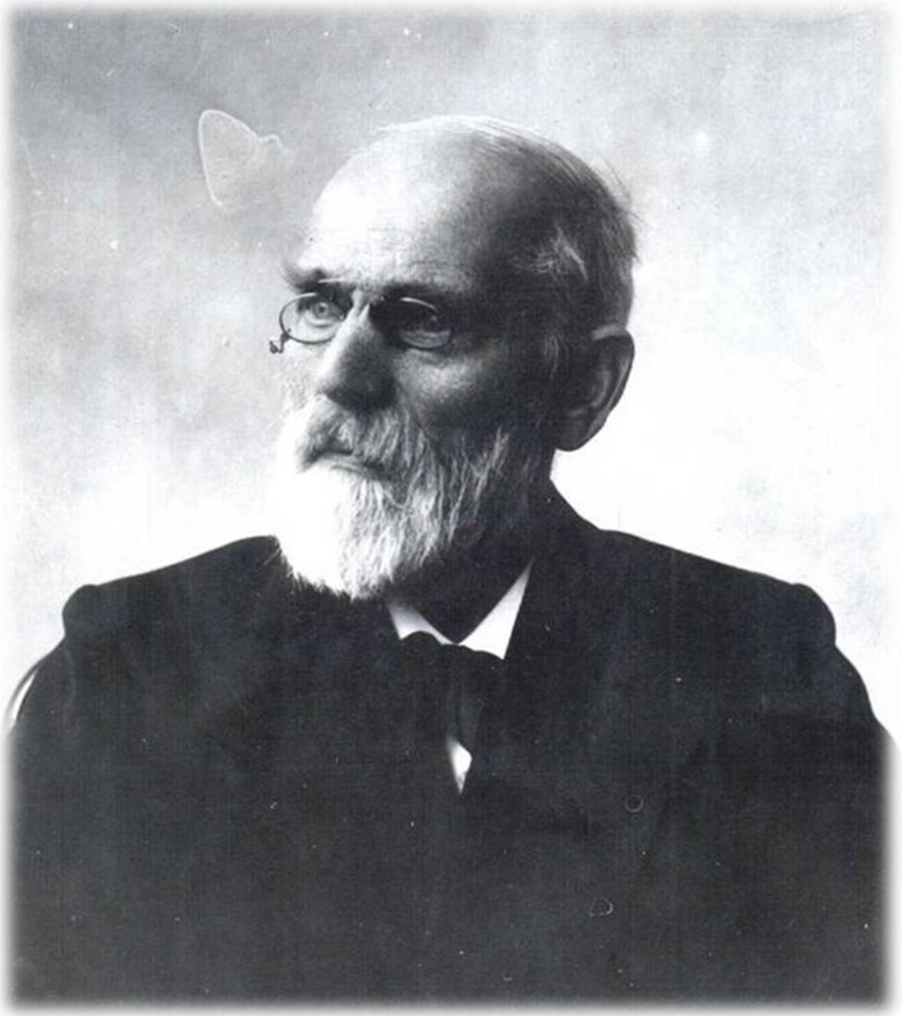
induced dipole-dipole interaction

- **attractive interaction**
- weak
- short range

I.1.3 Van der Waals Bonding



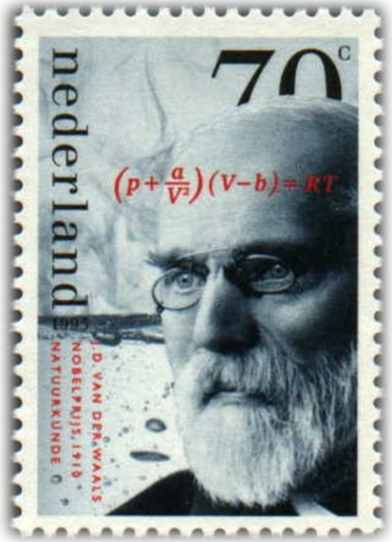
Van der Waals (right) and Heike Kamerlingh Onnes in Leiden with the helium 'liquefactor' (1908)



Johannes Diderik Van der Waals (1837 - 1923)

Nobel Prize in Physics (1910)

"for his work on the equation of state for gases and liquids"

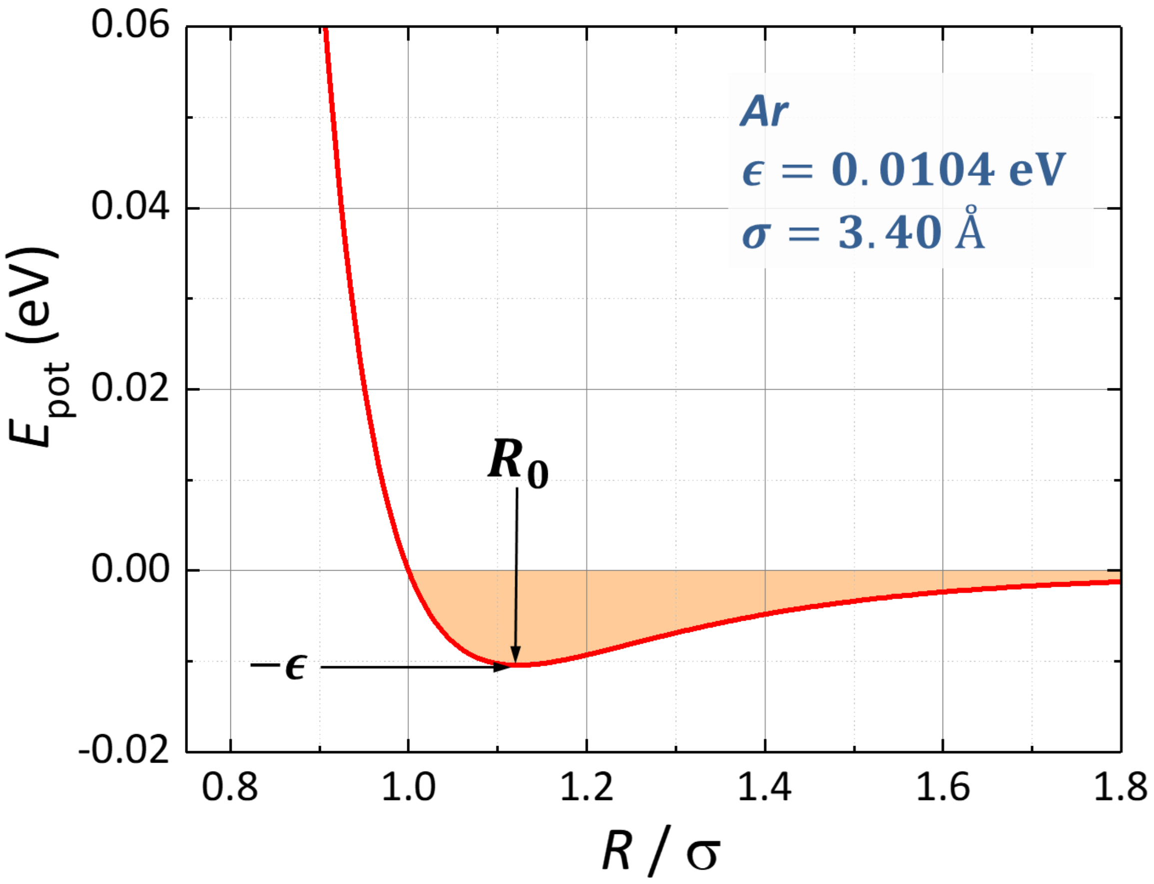


I.1.3 Van der Waals Bonding

- ***interaction potential*** with short range repulsion due to Pauli principle:

$$E_{\mathrm{pot}}(R) = 4\epsilon\left[\left(\frac{\sigma}{R}\right)^{12} - \left(\frac{\sigma}{R}\right)^{6}\right]$$

Lennard-Jones potential (two-particle interaction)



	Ne	Ar	Kr	Xe
ϵ (eV)	0.0031	0.0104	0.0141	0.0200
σ (Å)	2.74	3.40	3.65	3.98
R_0/σ	1.14	1.11	1.10	1.09

11.6 K ⇔ 1 meV

➔ ***Van der Waals bonding is very weak !!***

I.1.3 Van der Waals Bonding

determination of ***equilibrium distance*** of atoms in lattice: ➔ minimize **total potential energy**

$$U_{\text{tot}} = \frac{N}{2}\sum_{i\neq j} 4\epsilon\left[\left(\frac{\sigma}{r_{ij}}\right)^{12} - \left(\frac{\sigma}{r_{ij}}\right)^{6}\right]$$

$\sum$ = binding energy of a single atom j with all other atoms i.
Total binding energy is obtained by multiplication with $N/2$.

we rewrite

$$U_{\text{tot}}(R) = 2N\epsilon\left[A_{12}\left(\frac{\sigma}{R}\right)^{12} - A_6\left(\frac{\sigma}{R}\right)^{6}\right]$$

R: nearest neighbor distance

with

$$A_k = \sum_{i\neq j}\frac{1}{\alpha_{ij}^k}, \qquad k = 6, 12$$

(lattice sum)

$r_{ij} = \alpha_{ij}R$
k: number of nearest neighbors

equilibrium distance: $\partial U_{\text{tot}}(R)/\partial R = 0$

$$R_0 = \left(2\frac{A_{12}}{A_6}\right)^{1/6}\cdot\sigma$$

$$R_0 = 1.09\cdot\sigma$$

Kristallstruktur	A_6	A_{12}
fcc	14.4539	12.1319
hcp	14.4549	12.1323
bcc	12.253	9.114

$$U_{\text{tot}}(R_0) = -N\epsilon\begin{cases}8.61016 & \text{fcc} - \text{structure}\\ 8.61107 & \text{hcp} - \text{structure}\end{cases}$$

hcp has lowest energy but fcc is found experimentally for nobel gases

→ **effect of zero-point fluctuations**

→ **more detailed band structure calculation is required**

(modification of Van der Waals binding energy due to overlap of excited atomic states with neighboring atoms in crystal structure)

I.1.4 Zero Point Fluctuations

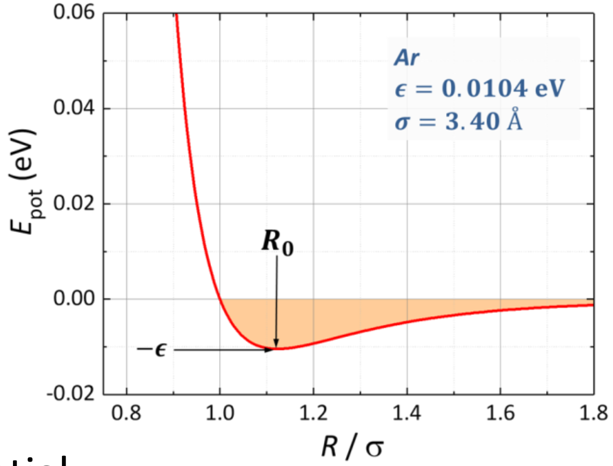
- estimate of ***zero-point fluctuation amplitude*** for harmonic oscillator:

$$E_{\mathrm{tot}} = E_{\mathrm{kin}} + E_{\mathrm{pot}}$$

- for $x = x_{\mathrm{max}}$:

$$E_{\mathrm{kin}} = 0, \qquad E_{\mathrm{pot}} = \frac{1}{2} k x_{\mathrm{max}}^2 = \frac{1}{2} M \omega^2 x_{\mathrm{max}}^2 \qquad\qquad \omega^2 = k/M$$

spring constant k is given by second derivative of the interaction potential



- by equating $\frac{1}{2} k x_{\mathrm{max}}^2$ with the ground state energy $\frac{1}{2}\hbar\omega$ of the harmonic oscillator, we obtain

$$x_{\mathrm{max}}^2 = \frac{\hbar}{M\omega} = \frac{\hbar}{\sqrt{kM}}$$

for He: $x_{\mathrm{max}} \approx 0.3 - 0.4$ times the lattice constant a

$a \approx \left(\frac{V_{\mathrm{mol}}}{N_{\mathrm{A}}}\right)^{1/3}$ $\quad V_{\mathrm{mol}}$ = mole volume, N_{A} = Avogadro number

(***Lindeman*** criterion for melting of solid: $\boldsymbol{x_{\mathrm{max}} \approx 0.3 \cdot a}$)

➔ Helium stays liquid even at $T = 0$ due to large zero-point fluctuations caused by
(i) weak Van der Waals forces (small k)
(ii) small mass M

I.1.4 Zero Point Fluctuations

- we can use uncertainty relation to estimate the ***kinetic energy*** due to zero-point fluctuations:

$$\Delta p \cdot \Delta x \simeq \hbar \quad \Rightarrow \quad \Delta p \sim \frac{\hbar}{\Delta x} \approx \frac{\hbar}{0.3\,a} \approx \frac{h}{a}$$

$a =$ lattice constant

$$\boxed{E_{\mathrm{zp}} = \frac{\Delta p^2}{2m} \approx \frac{h^2}{2Ma^2} \approx 18\ \mathrm{K}}$$

➔ large quantum mechanical zero-point energy due to small mass M

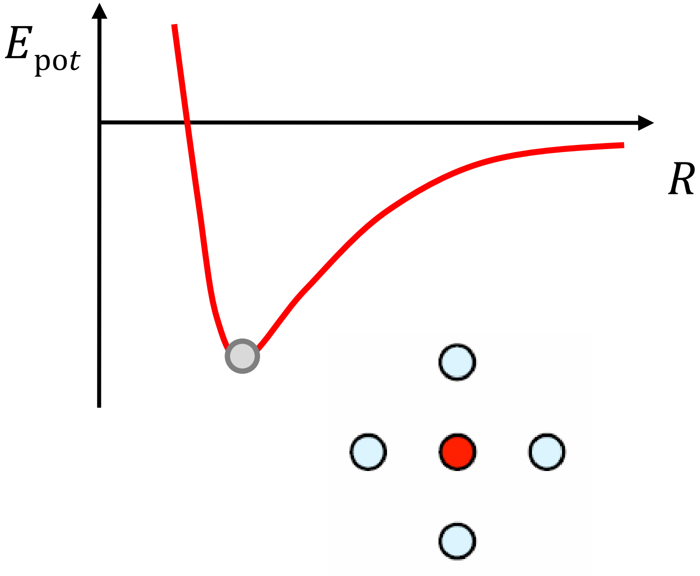
➔ kinetic energy due to zero-point fluctuations is comparable to potential energy due to Van der Waals interaction

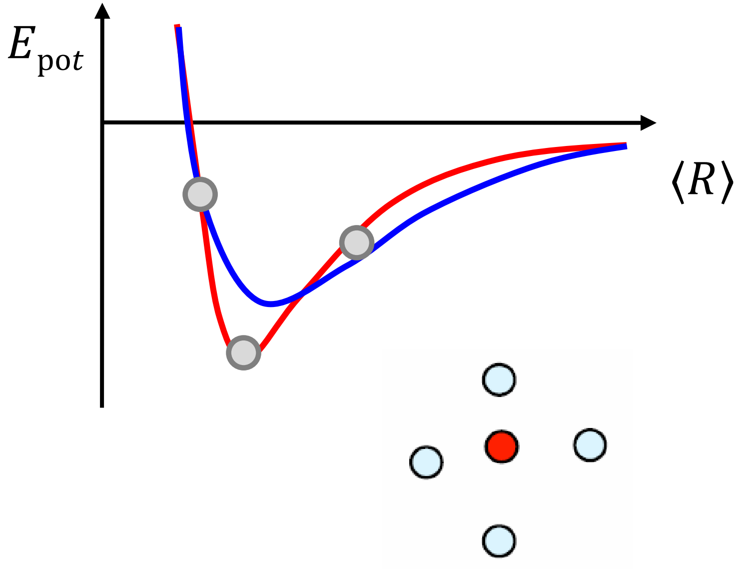
due to strong influence of quantum fluctuations, helium stays liquid down to temperatures, where $\lambda_{\mathrm{T}} \geq d \simeq a$

➔ ***Helium becomes a quantum liquid***

I.1.4 Zero Point Fluctuations

- **total potential energy for solid and liquid He**





solid: regular lattice

→ same distance between atoms
→ $E_{\text{pot}}\,(R) \;\approx$ potential $U_{\text{tot}}\,(R)$

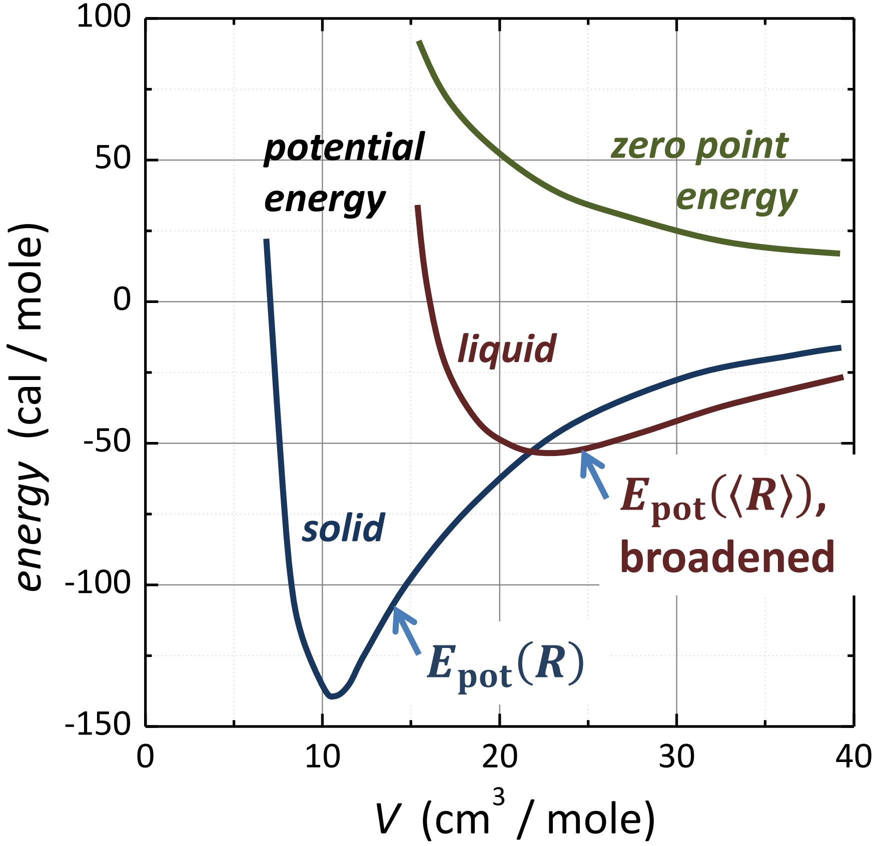
liquid: disordered system

→ varying distance between atoms
→ E_{pot} is smeared out due to averaging over various distances
→ reduced depth and increased width of $E_{\text{pot}}(\langle R \rangle)$

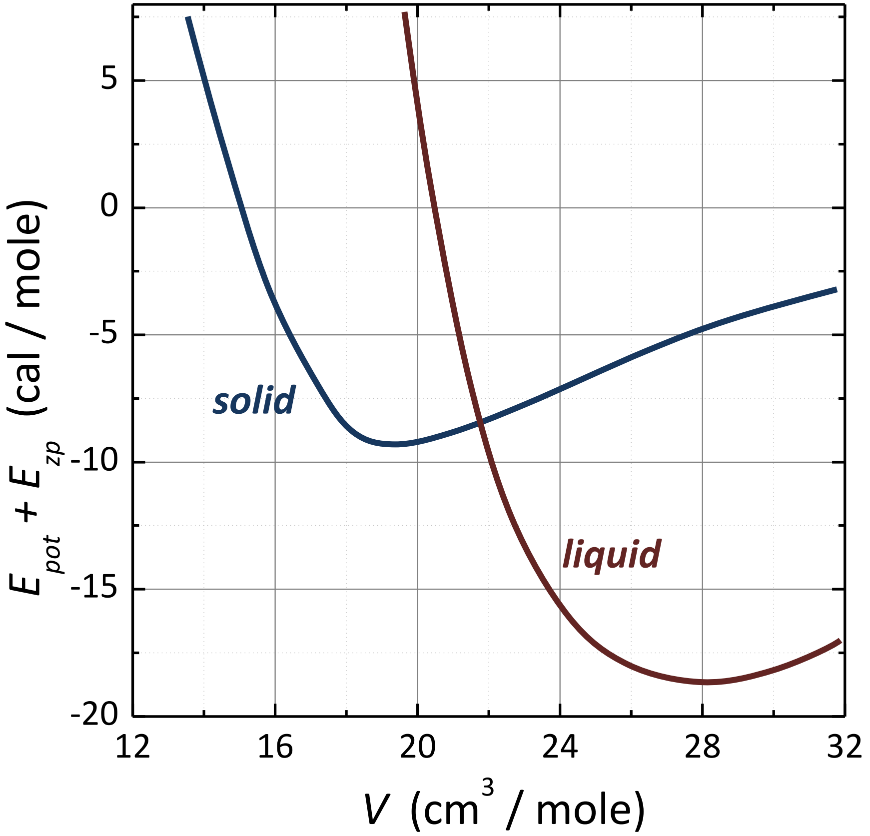
Why does He not become solid at low T ??

I.1.4 Zero Point Fluctuations

- **zero-point fluctuations play a crucial role**



for liquid helium:
contribution of E_{zp} smaller due to broadened E_{pot}

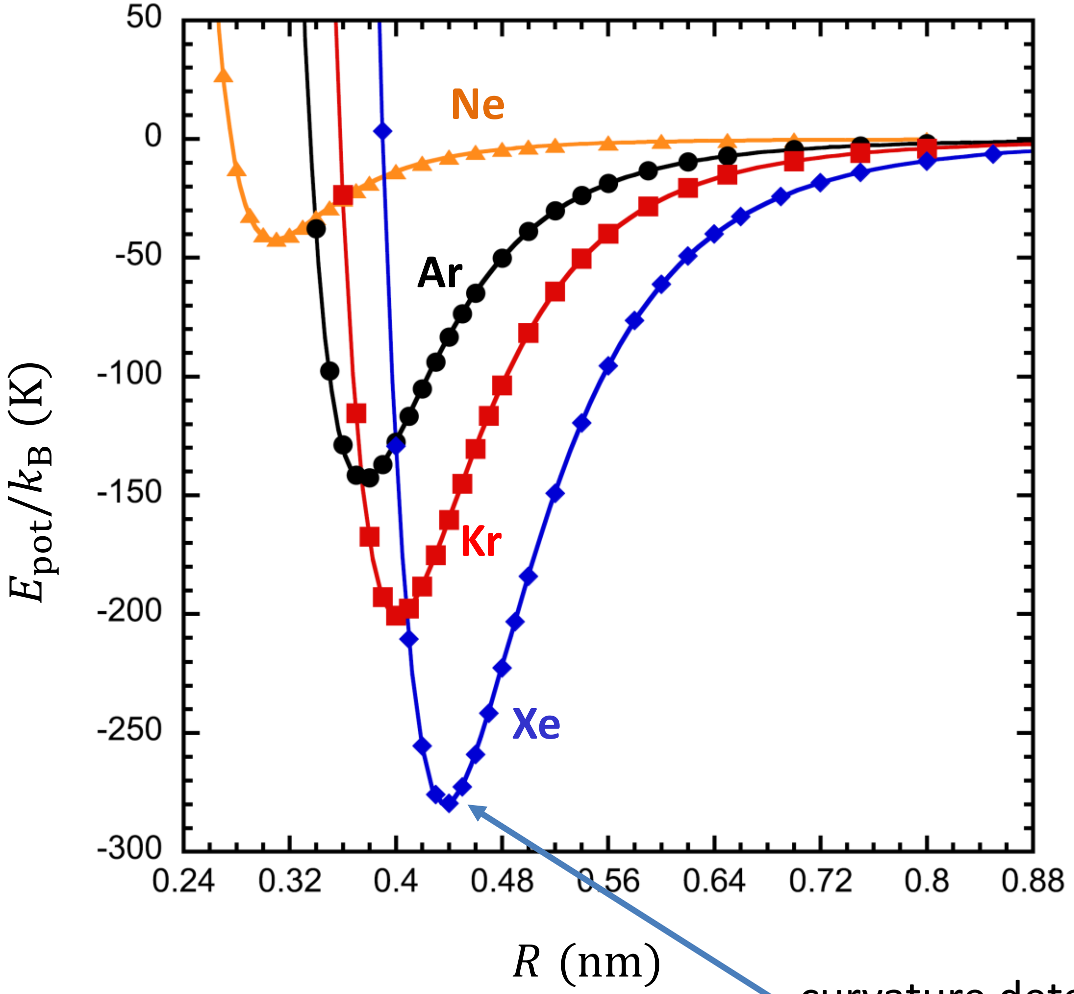


- ***without quantum fluctuations:***
  ^{4}He would be solid at a molar volume of **10 cm³**

- ***with quantum fluctuations:***
  ^{4}He stays liquid with molar volume of **28 cm³**

I.1.4 Zero Point Fluctuations

R. Gross © Walther-Meißner-Institut (2004 - 2024)



Comparison of *ab initio* potential energies for Ne, Ar, Kr, Xe (symbols) with SAAP calculations (solid lines)

U. K. Deiters et al., J. Chem. Phys. **150**, 134504 (2019)
https://doi.org/10.1063/1.5085420

comparison of ^{4}He and Ne:

	He	Ne
potential minimum ε	1.03 meV	3.94 meV
interatomic distance R_0	0.265 nm	0.296 nm

1 meV = 11.6 K

curvature determines spring constant: $k = \frac{1}{2} \frac{\partial^2 E_{\mathrm{pot}}(R)}{\partial R^2}\Big|_{R=R_0} = \frac{36\varepsilon}{R_0^2}$ (for fcc)

I.1.4 Zero Point Fluctuations

quantum parameter: $$\Lambda \equiv \frac{\langle E_{\mathrm{zp}} \rangle}{\langle E_{\mathrm{pot}} \rangle}$$

quantum liquid:

$$\boldsymbol{\Lambda \gtrsim 1}$$

liquid	Xe	Kr	Ar	Ne	H_2	^{4}He	^{3}He
Λ	0.06	0.10	0.19	0.59	1.73	2.64	3.05

large E_{zp}, but also large E_{pot}

identical E_{pot}, but different E_{zp}

mass of ^{3}He smaller than of ^{4}He

➔ even stronger effect of quantum fluctuations

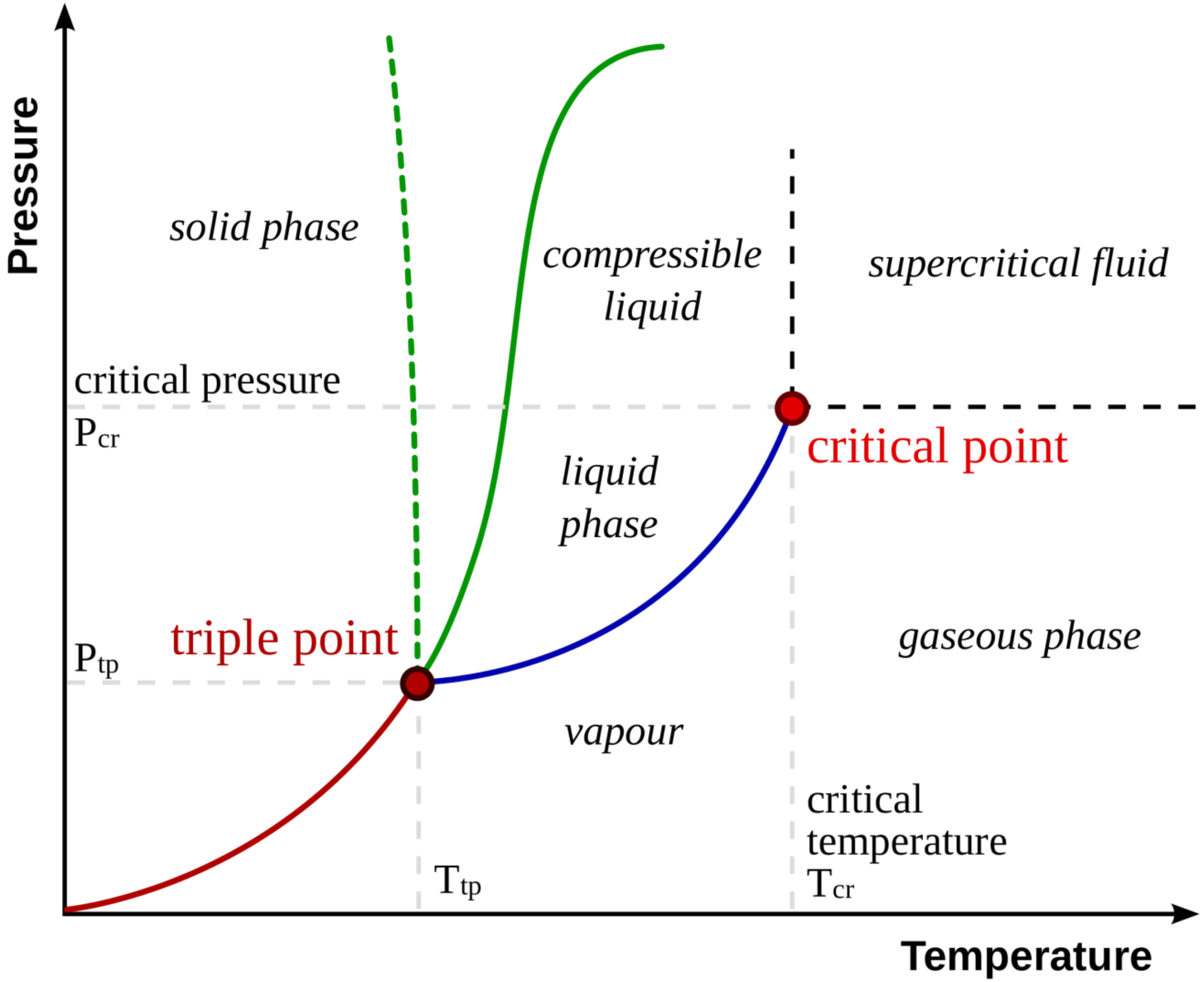
➔ larger quantum parameter

➔ larger molar volume

^{3}He: **40 cm³/mole**, ^{4}He: **28 cm³/mole**

I.1.6 pT phase diagram of ^{4}He

- **general nomenclature, definitions**



https://commons.wikimedia.org/wiki/File:Phase-diag2.svg

I.1.5 Helium under Pressure

- ***enthalpy:*** $H = U + pV$

  $H = $ ***min ?***

- use of isenthalpy lines:

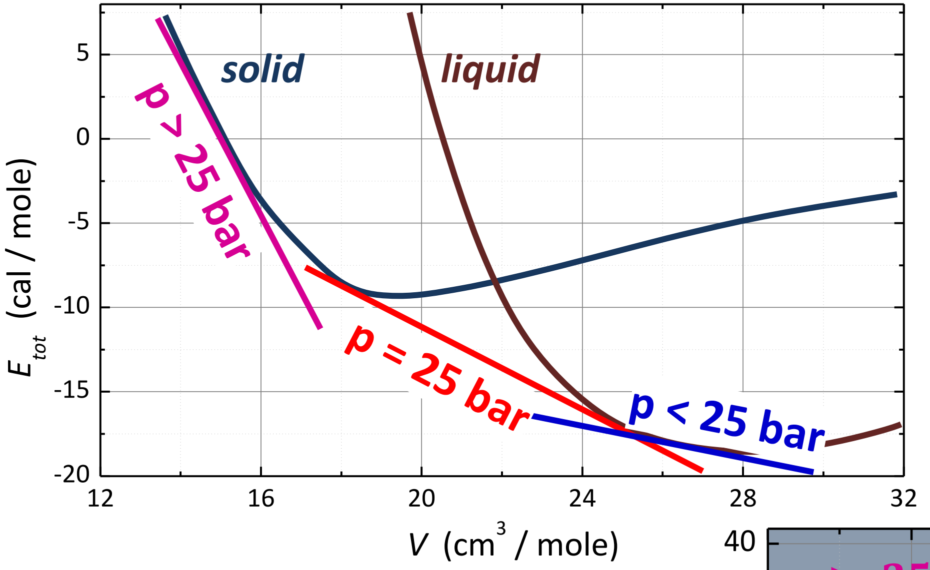
  $H = $ ***const.***

- with $U = H - pV$

  ➔ straight lines with slope $-p$

- ***lowest isenthalpy:***

  ➔ tangent with slope $-p$

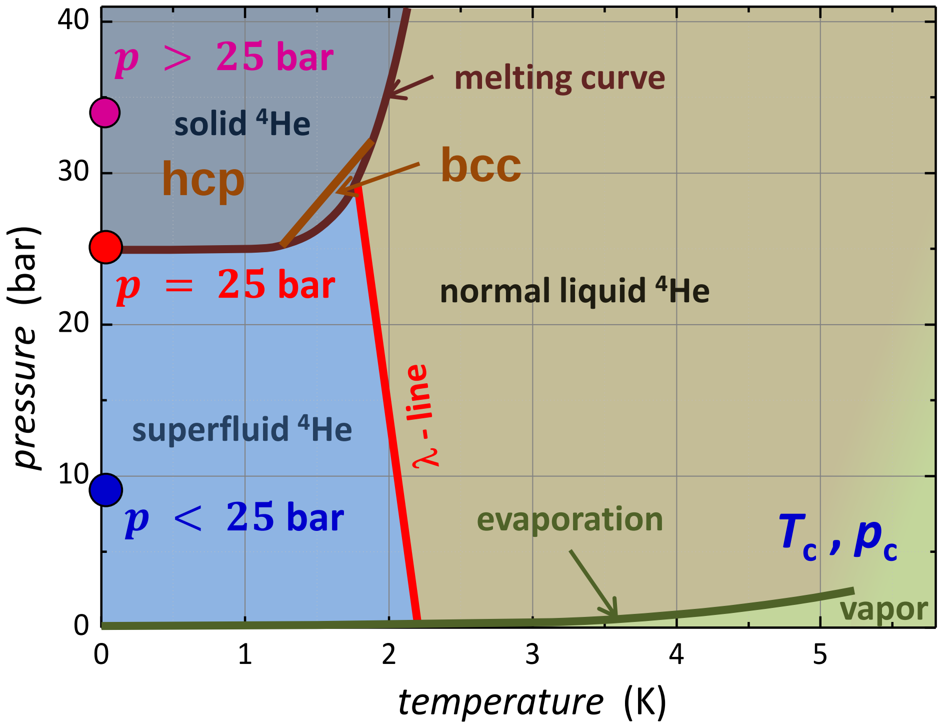


$$H = U + pV$$
$$dH = \underbrace{dU}_{TdS - pdV} + pdV + Vdp$$
$$dH = TdS + Vdp$$

$p > 25$ **bar: solid**

$p = 25$ **bar: coexistence**

$p < 25$ **bar: liquid**



I.1.6 pT phase diagram of ^{4}He

- ***Clausius-Clapeyron:***

$$\boxed{\frac{\partial p_m}{\partial T} = \frac{\Delta S_m}{\Delta V_m}}$$

$$\Delta S_m = S_{\text{liquid}} - S_{\text{solid}}$$

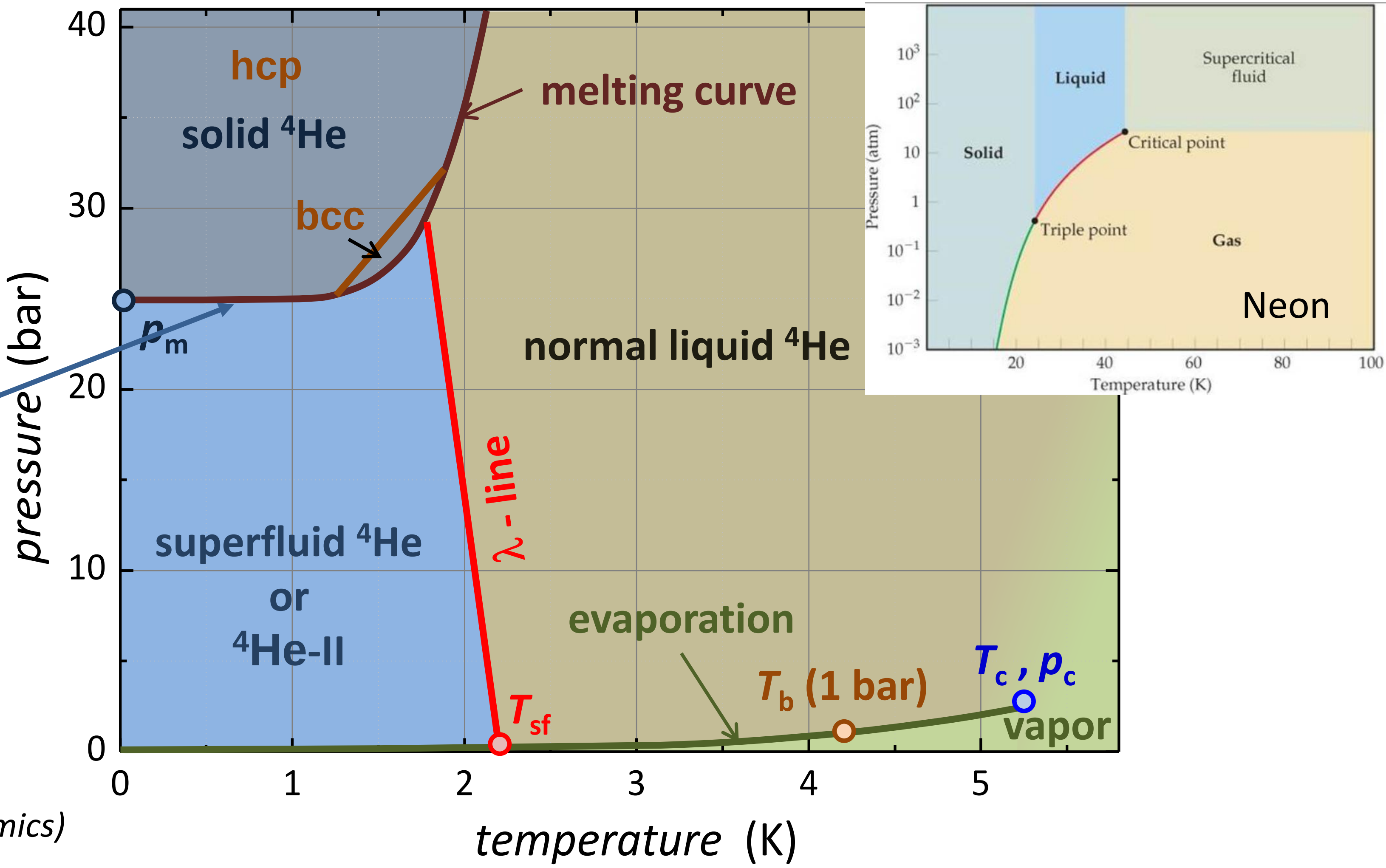
$$\Delta V_m = V_{\text{liquid}} - V_{\text{solid}}$$

$$\frac{\partial p_m}{\partial T} = \frac{\Delta S_m}{\Delta V_m} > 0$$

$$\Delta S_m \rightarrow 0 \quad \text{for} \quad T \rightarrow 0$$

$$\Rightarrow \frac{\partial p_m}{\partial T} \rightarrow 0 \quad \text{for} \quad T \rightarrow 0$$

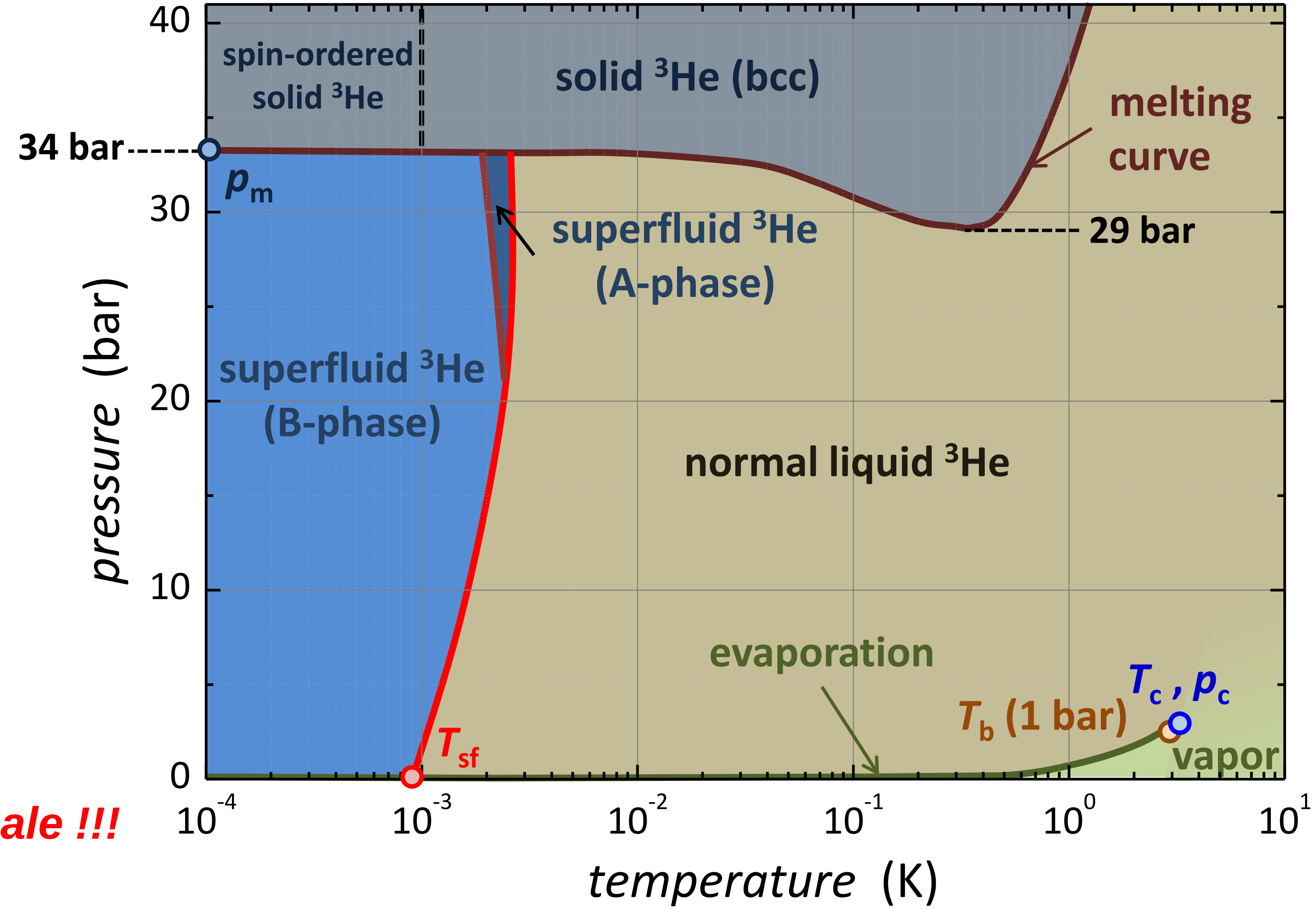
(agrees with 3rd Law of thermodynamics)



Phase diagram of ^{4}He at low temperatures. ^{4}He remains liquid at zero temperature if the pressure is below 2.5 MPa (approximately 25 atmospheres) → ***no triple point***.

The liquid has a phase transition to a ***superfluid phase***, also known as He-II, at the temperature of 2.17 K (at vapor pressure). The solid phase has either hexagonal close packed (hcp) or body centered cubic (bcc) symmetry.

I.1.6 pT phase diagram of ^{3}He



log T scale !!!

big difference in value of T_{sf}

- normal/superfluid transition dominated by quantum effects
- quantum statistics matters!! (bosons vs. Fermions)
- **nature of superfluid phase in ^{4}He and ^{3}He is different**

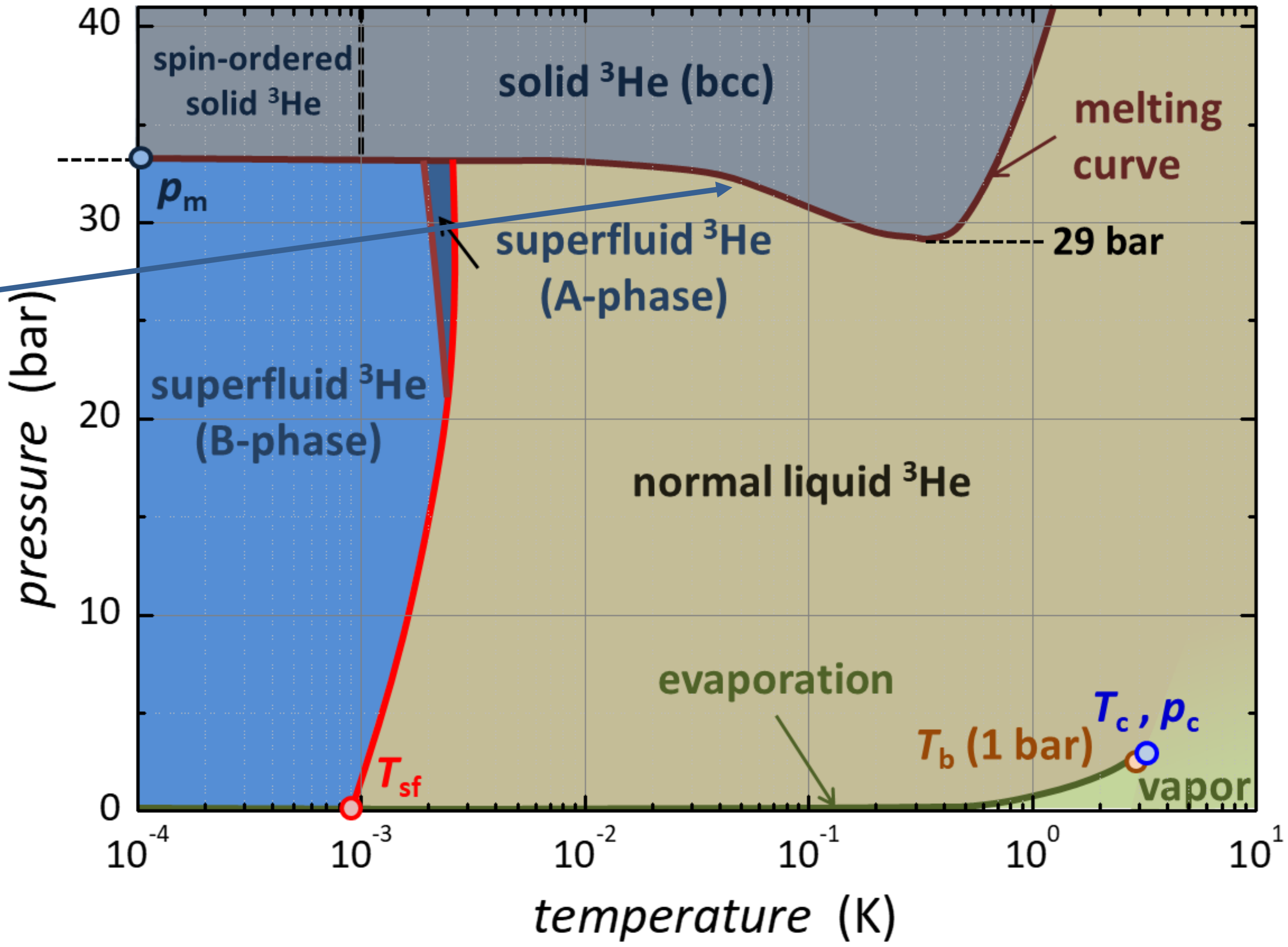
- phase diagram of ^{3}He: note the logarithmic temperature scale. There are ***two superfluid phases of ^{3}He, A and B***.
- the line within the solid phase indicates a transition between spin-ordered and spin disordered structures (at low and high temperatures, respectively).

I.1.6 pT phase diagram of ^{3}He

- ***Clausius-Clapeyron equation:***

$$\frac{\partial p_m}{\partial T} = \frac{\Delta S_m}{\Delta V_m}$$

$$\frac{\partial p_m}{\partial T} = \frac{\Delta S_m}{\Delta V_m} < 0$$



since volume of liquid state is larger, the entropy of the liquid state must be smaller

➔ more ordered liquid state!?
➔ explanation follows later:
ordering in momentum space in Fermi liquid

I.1.7 Characteristic Properties of Helium Isotopes

	^{3}He	^{4}He
boiling point T_b (K) (@ p = 1 bar)	**3.19**	**4.21**
critical temperature T_c (K)	**3.32**	**5.20**
critical pressure p_c (bar)	**1.16**	**2.29**
superfluid transition temperature T_s (K)	**0.0025**	**2.177**
density ρ (g/cm³) (for T → 0 K)	**0.076**	**0.145**
density ρ (g/cm³) (@ boiling point)	**0.055**	**0.125**
classical molar volume V_m (cm³ /mole) (@ saturated vapor pressure and T = 0 K)	**12**	**12**
actual molar volume V_m (cm³ /mole) (@ saturated vapor pressure and T = 0 K)	**36.84**	**27.58**
melting pressure p_m (bar) (@ T = 0 K)	**34.39**	**25.36**

strong difference between superfluid properties of ^{4}He and ^{3}He

→ different nuclear spin → different quantum statistics

I.1.7 Characteristic Properties of Helium Isotopes

^{4}He	$p^+\uparrow \; p^+\downarrow$	$n\uparrow \; n\downarrow$	$e^-\uparrow \; e^-\downarrow$	**boson**
^{3}He	$p^+\uparrow \; p^+\downarrow$	$n\uparrow$	$e^-\uparrow \; e^-\downarrow$	**fermion**

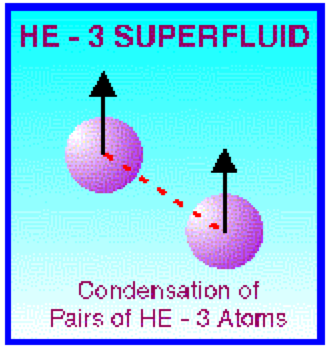
different nuclear spin:
^{4}He: $I = 0$
^{3}He: $I = 1/2$

same chemical properties

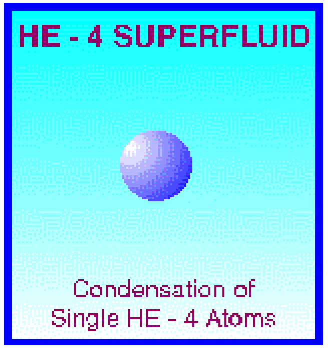
different statistics

^{3}He: *pairing similar to superconductivity in metals*
weak attractive interaction between Fermions → „Cooper pairs“

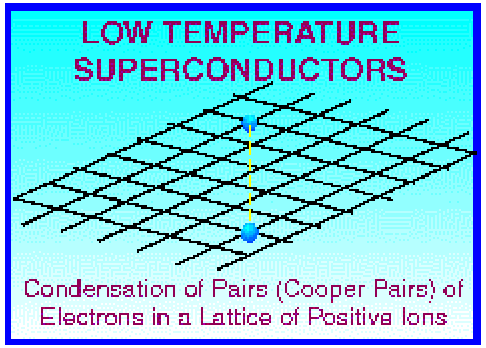
^{4}He: *Bose-Einstein condensation of Bosons without „pairing interaction“*
→ no attractive interaction required



Condensation of Pairs of HE - 3 Atoms



Condensation of Single HE - 4 Atoms

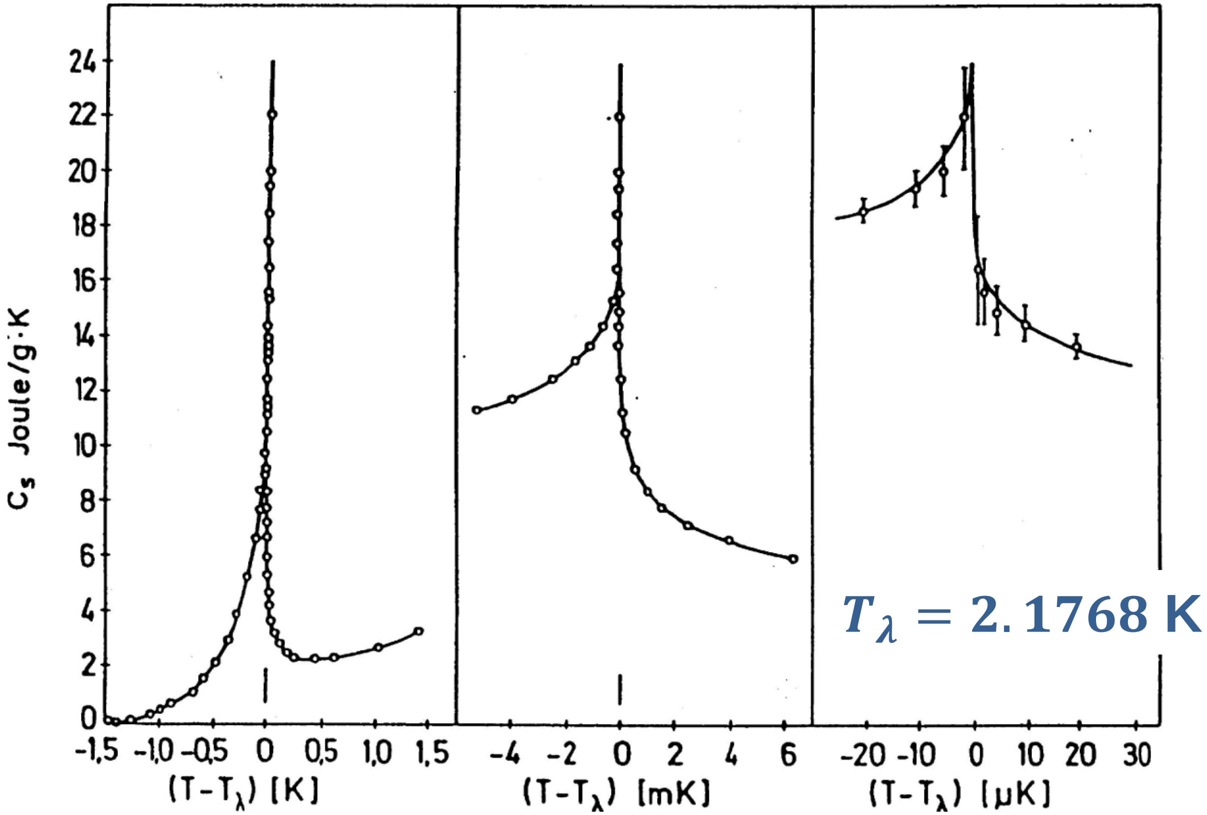


Condensation of Pairs (Cooper Pairs) of Electrons in a Lattice of Positive Ions

I.1.8 Specific Heat of ^{4}He – Lambda Point

- first measurements by ***Dana*** and ***Kamerlingh Onnes*** in 1923
  - anomalous increase of C_p at about $T_\lambda = 2.177$ K
  - in publication (1926) some data points were omitted, since they were believed to result from exp. artefacts
- confirmation by ***Clusius*** and ***Keesom*** in 1932 (introduced denotion **He I** and **He II** → **phase transition**)
  - ***He I: normal fluid, He II: superfluid***

R. Gross © Walther-Meißner-Institut (2004 - 2024)



Buckingham, M. J. and Fairbank, W. M.,
"The Nature of the Lambda Transition", in Progress in Low Temperature Physics III, 1961.

specific heat:

$$C_V \equiv \left(\frac{\partial U}{\partial T}\right)_V$$

observation: specific heat diverges at $T = T_\lambda$
→ **1st order phase transition ??**
→ shape: ***Lambda point***

free energy F: *continuous, but shows a kink*

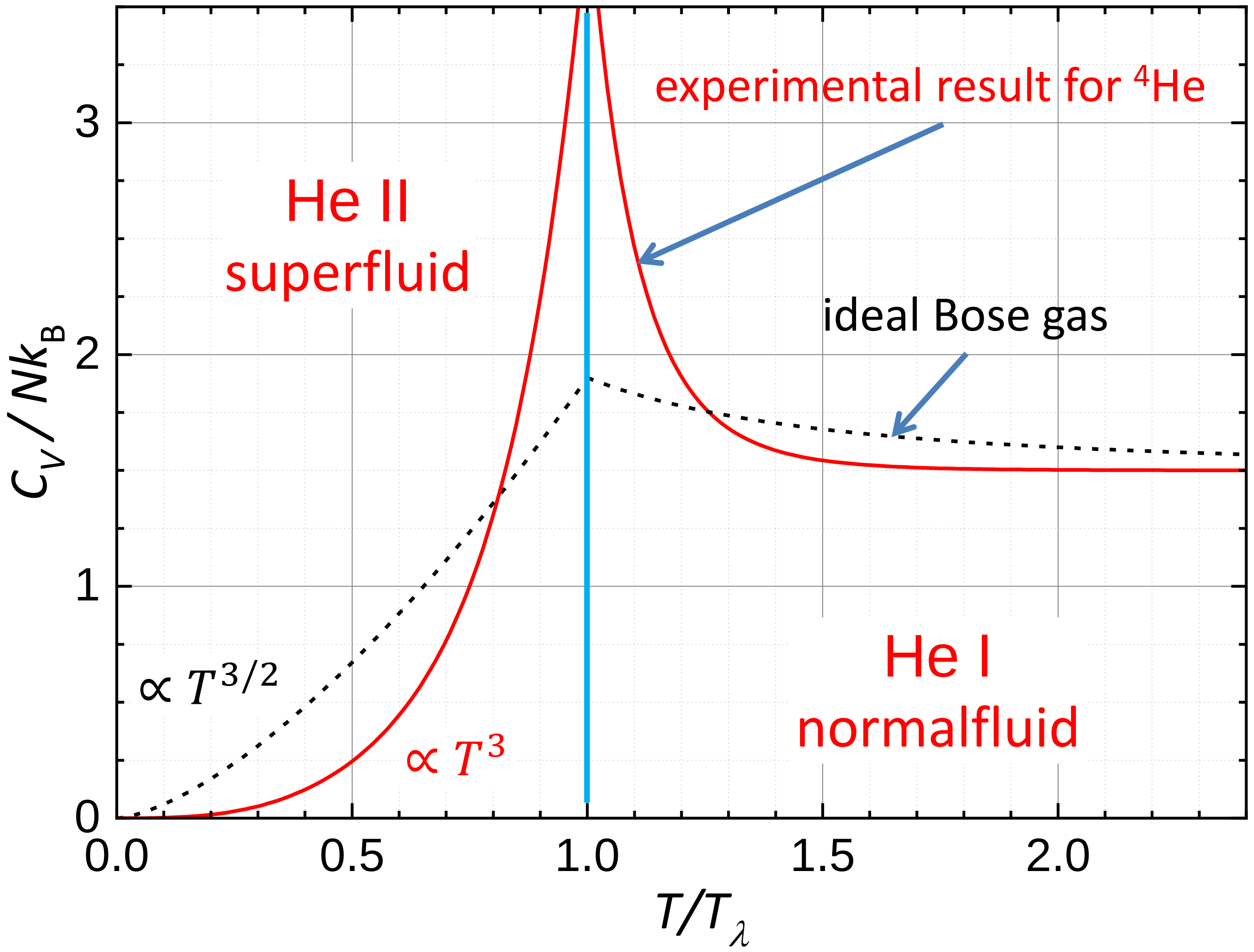
$(F = U - TS \Rightarrow dF = -SdT - pdV + \mu dN$ with $dU = TdS - pdV + \mu dN)$

entropy $S = \left(-\frac{\partial F}{\partial T}\right)_{V,N}$: *shows a jump*

spec. heat $C_V = T\left(\frac{\partial S}{\partial T}\right)_{V,N} = T\left(-\frac{\partial^2 F}{\partial T^2}\right)_{V,N}$: *shows divergence*

I.1.8 Specific Heat of ^{4}He – Lambda Point

- **comparison of specific heat of ^{4}He and that of an ideal Bose-Einstein condensate**



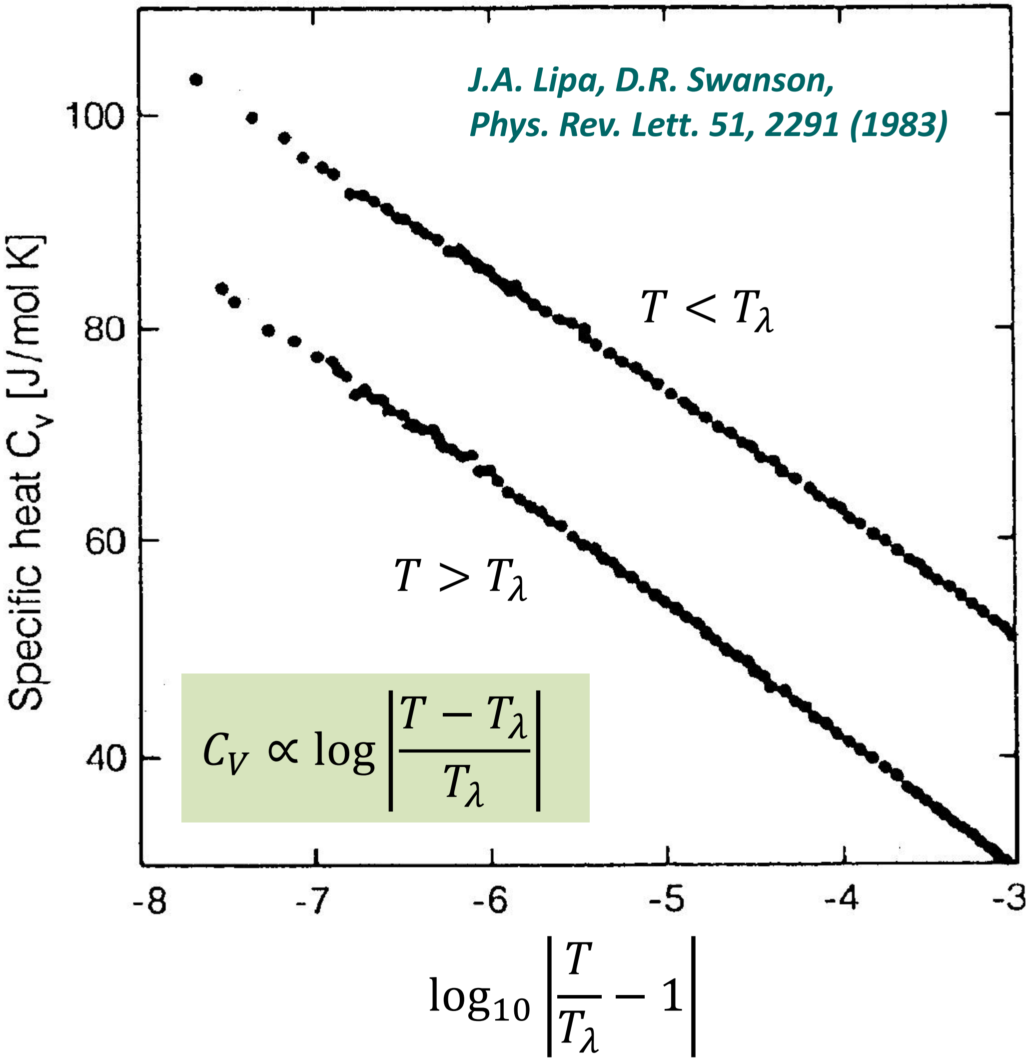
2nd order phase transition is expected for ideal Bose-Einstein condensate

➔ no divergence of C_V at $T = T_\lambda$

measured behavior of ^{4}He near T_λ is more complicated

(detailed discussion follows later)

I.1.8 Specific Heat of ^{4}He – Precision Measurement

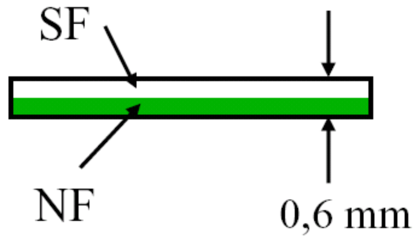


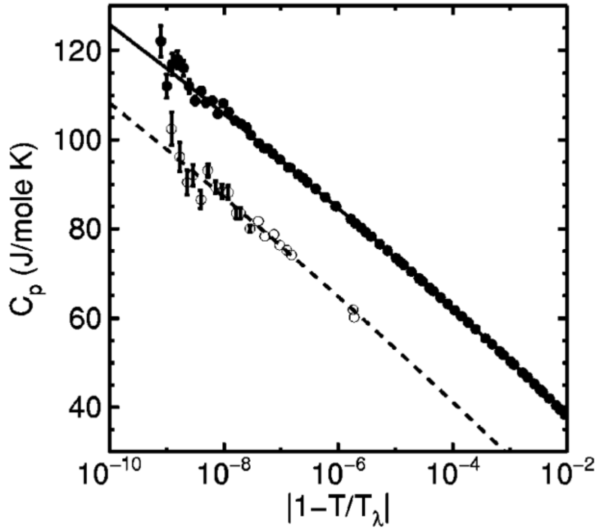
J.A. Lipa, D.R. Swanson, Phys. Rev. Lett. 51, 2291 (1983)

$$C_V \propto \log\left|\frac{T - T_\lambda}{T_\lambda}\right|$$

- specific heat is measured with very high temperature resolution (≈ nK)
- extreme purity of liquid He is important

  → ***model system for testing physics of phase transitions***
- experimental problem:
  → hydrostatic pressure
  → space experiments

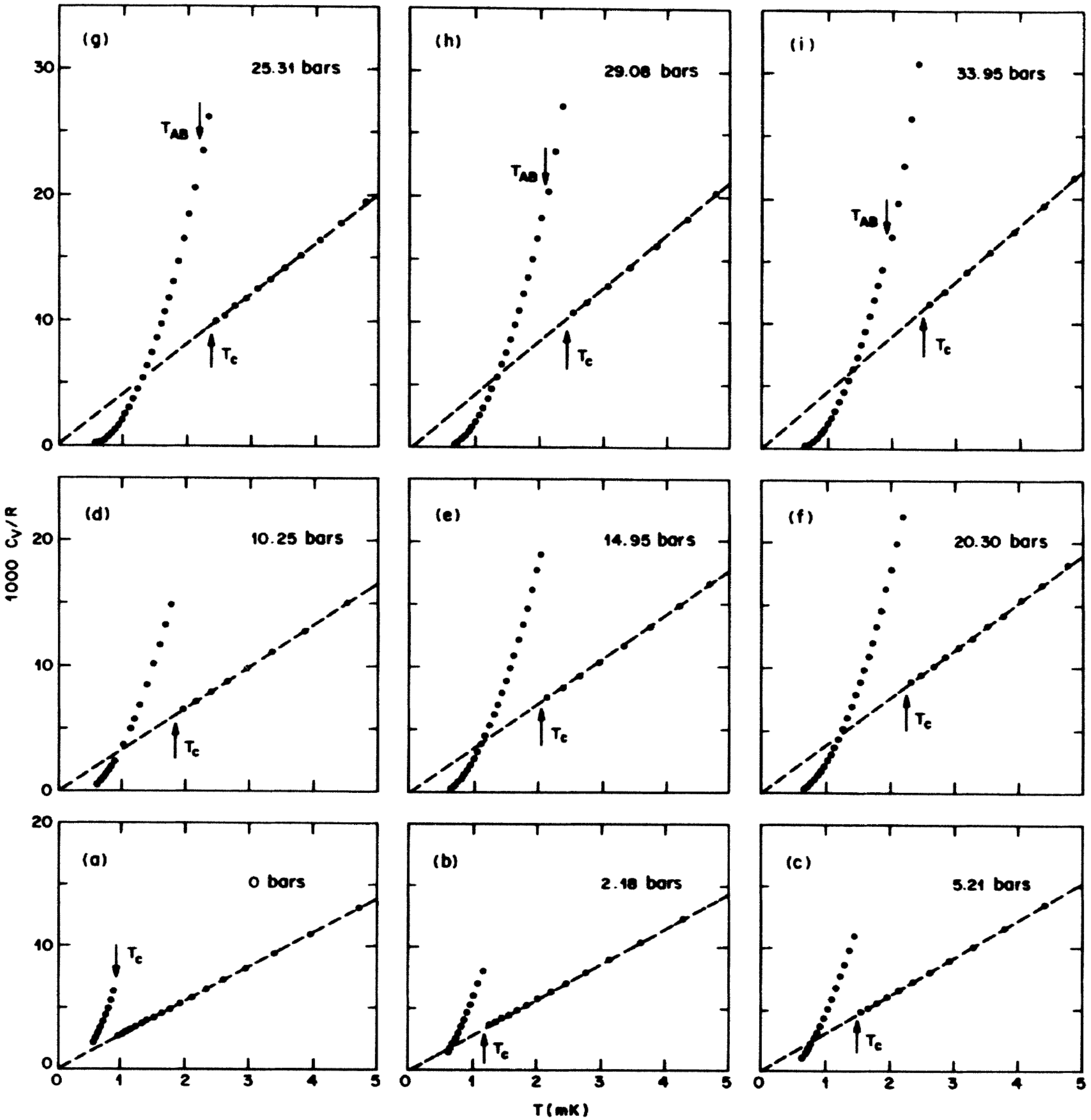




Specific heat of liquid helium in zero gravity very near the lambda point
J. A. Lipa et al., Phys. Rev. B 68, 174518 (2003)

➔ key result: second order phase transition with logarithmic singularity

I.1.8 Specific Heat of ^{3}He



$C_V(T)$ similar to a metallic superconductor

- $T > T_c$:
  linear $C_V(T)$ of Fermi liquid

$$C_V(T) = \gamma T$$

$$\gamma = \frac{3}{2}\frac{n}{E_F}\frac{\pi^2}{3}k_B^2$$

(Sommerfeld, 1927)

- $T = T_c$:
  jump due to superfluid transition
  (as for SCs and predicted by BCS theory)
- $T < T_c$:
  reduction of C_V due to condensation of Fermions (similar to superconductors)

Source:
Greywall, Phys. Rev. B 33, 7520 (1986)
^{3}He specific heat and thermometry at millikelvin temperatures

I.1 Summary of Experimental Observations

- specific heat of ^{4}He is large: $C_V \approx$ **1 J/g K** (compared to about 10^{-5} J/g K of Cu, $C_V \propto n \cdot T/T_\mathrm{F}$)

- latent heat of evaporation of ^{4}He is small (**20.59 J/g**), but large compared to specific heat of other materials
  → ***application as cooling liquid***

- strong anomaly of specific heat at T_λ, origin unknown for more than 30 years
  → originates from ***Bose-Einstein condensation*** (bosonic character of ^{4}He, interacting bosonic system)

- shape of anomaly resembles greek letter λ → ***Lambda point***

- below T_λ entropy and specific heat decrease rapidly with decreasing T
  → ***condensation into superfluid***

- for ^{4}He temperature variation of C_V below T_λ determined by collective excitations in superfluid
  → ***phonons, rotons*** (detailed discussion later)

I.1 Summary of Experimental Observations

- for ^{4}He phase transition at T_λ → **Lambda point** is due to **bosonic character of ^{4}He**

  ➔ **no such transition was expected for ^{3}He due to nuclear spin $I = 1/2$** (**fermionic character of ^{3}He**)

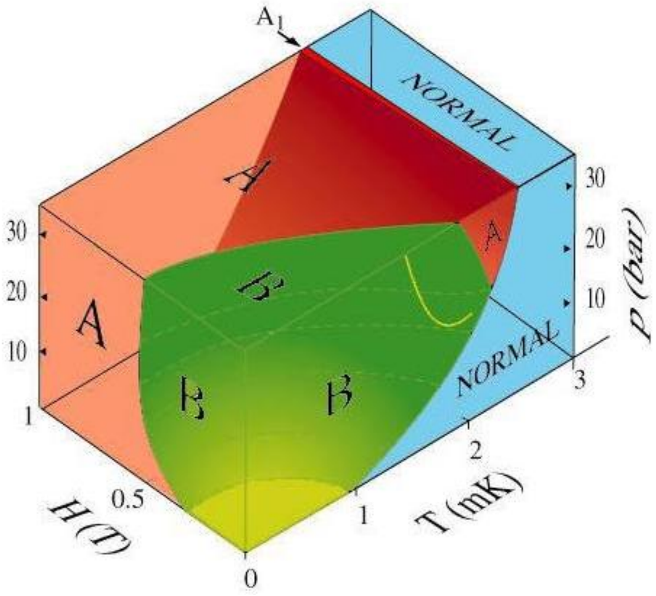
- only after the development of BCS theory in 1957 (condensation of Cooper pairs into macroscopic quantum state), the **search for the superfluid phase of ^{3}He** has been intensified

- 1963: **Theory of superfluid ^{3}He** (Anderson and Morel, Balian and Werthamer)

- 1971: after the temperature regime below 5 mK has been accessed, the **superfluid phase of ^{3}He has been found** by Osheroff, Richardsen and Lee in NMR experiment

  D.D. Osheroff, R.C. Richardsen, D.M. Lee,

  Phys. Rev. Lett. 28, 885 (1972)



- there is not only a single superfluid phase, but three, depending on temperature, pressure and applied magnetic field

- 1977: first experiment with ^{3}He in Germany at WMI (Eska, Schuberth, Uhlig, ...)

Contents Part I: Quantum Gases & Liquids

Contents:

I.2 ^{4}He as Ideal Bose Gas

- **basic assumption**

^{4}He is treated as a non-interacting gas of bosons

we already have learnt that there are Van der Waals interactions

➔ treatment of ^{4}He as an ideal bosonic gas is only a ***first order approximation***

➔ ***interactions*** lead to deviations from behavior of ideal gas of non-interacting bosons

I.2 ^{4}He as Ideal Bose Gas

- **high temperatures**:

average distance $\bar{R}$ of atoms in gas is large compared to ***de Broglie wavelength*** $\boldsymbol{\lambda_T = h/p \propto 1/\sqrt{T}}$

➔ ***classical gas with Maxwell-Boltzmann statistics***

thermal de Broglie wave length:

$$\lambda_{\mathrm{T}} = \sqrt{\frac{h^2}{2\pi m k_{\mathrm{B}} T}} = \sqrt{\frac{2\pi\hbar^2}{m k_{\mathrm{B}} T}}$$

compare I.1.1

- **low temperatures**:

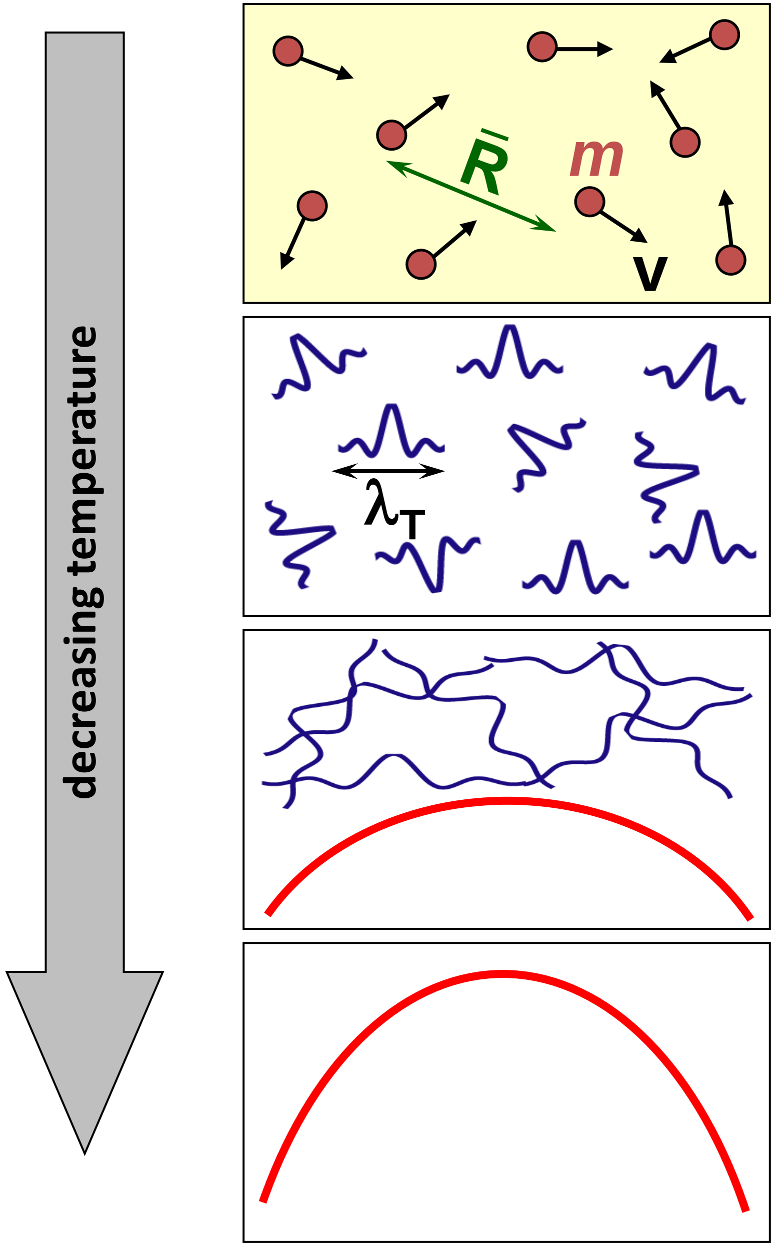
- quantum mechanical description, as soon as average particle distance $\bar{R} = \lambda_{\mathrm{T}}$

- classical gas of distiguishable particles turns into quantum soup of indistinguishable identical particles

➔ ***quantum gas with Bose statistics for ^{4}He (or Fermi statistics for ^{3}He)***

➔ ***Bose-Einstein condensation for bosons (^{4}He)***

I.2.1 Bose-Einstein Condensation



high temperature:

thermal velocity v_T

density $\frac{N}{V} = n \simeq \frac{1}{\bar{R}^3}$

billiard balls

low temperature:

de Broglie wavelength λ_T

$\lambda_T = h/mv_T \propto 1/\sqrt{T}$

wave packets

$T = T_{BEC}$:

Bose-Einstein condensation

$\lambda_T \approx \bar{R}$

wave packets overlap

$T \rightarrow 0$:

pure Bose-Einstein condensate

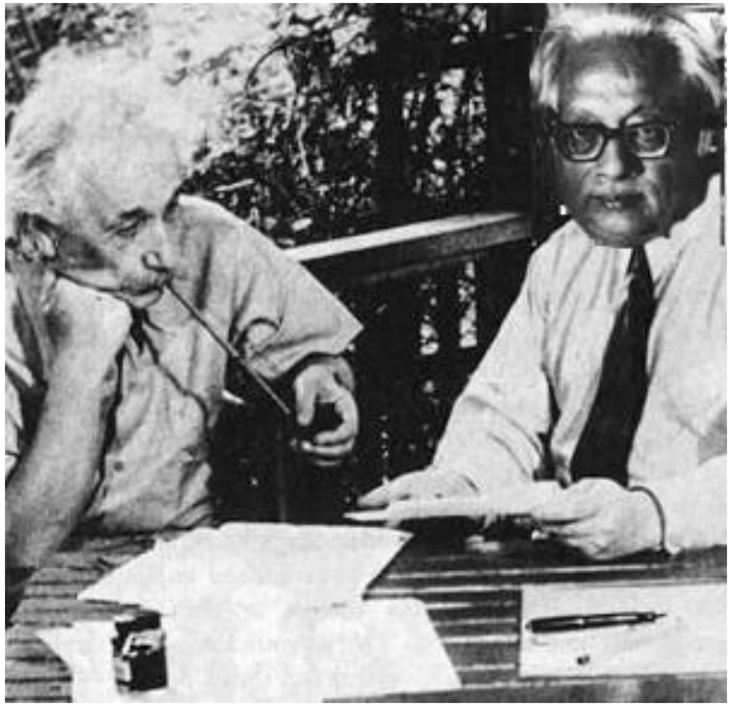
„Super“-mater wave

I.2.1 Bose-Einstein Condensation

Bose -
Einstein
Condensation

I.2.1 Bose-Einstein Condensation

- phase transition occurs for ***interactionless bosonic particles***
- originates from the ***indistinguishability of identical particles***

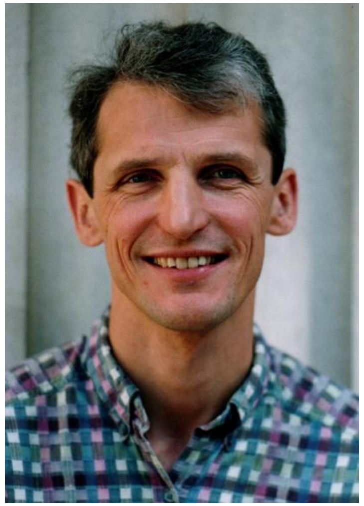


Albert Einstein und Satyendranath Bose

proposed in 1924 that below a characteristic temperature T_B a macroscopic number of bosons can condense into the state of lowest energy

experimental confirmation in 1995

Eric A. Cornell



Wolfgang Ketterle

Carl E. Wieman

Kungliga
Svenska Vetenskapsakademien
har den 9 oktober 2001 beslutat
att med det
NOBELPRIS
som detta år tillerkännes den som inom
fysikens område gjort den viktigaste
upptäckten eller uppfinningen
gemensamt belöna
Wolfgang Ketterle
Eric A Cornell och Carl E Wieman
för uppnåendet av Bose-Einsteinkondensation
i förtunnade gaser av alkaliatomer samt
för tidiga fundamentala studier av
kondensatens egenskaper

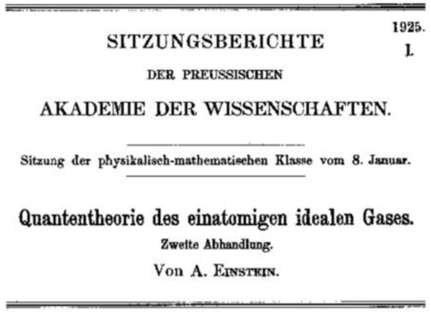
STOCKHOLM DEN 10 DECEMBER 2001

Erling Norrby

Nobel Prize in Physics 2001

"for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates"

I.2.1 Bose-Einstein Condensation



1925.
I.

SITZUNGSBERICHTE
DER PREUSSISCHEN
AKADEMIE DER WISSENSCHAFTEN.

Sitzung der physikalisch-mathematischen Klasse vom 8. Januar.

Quantentheorie des einatomigen idealen Gases.
Zweite Abhandlung.
Von A. Einstein.

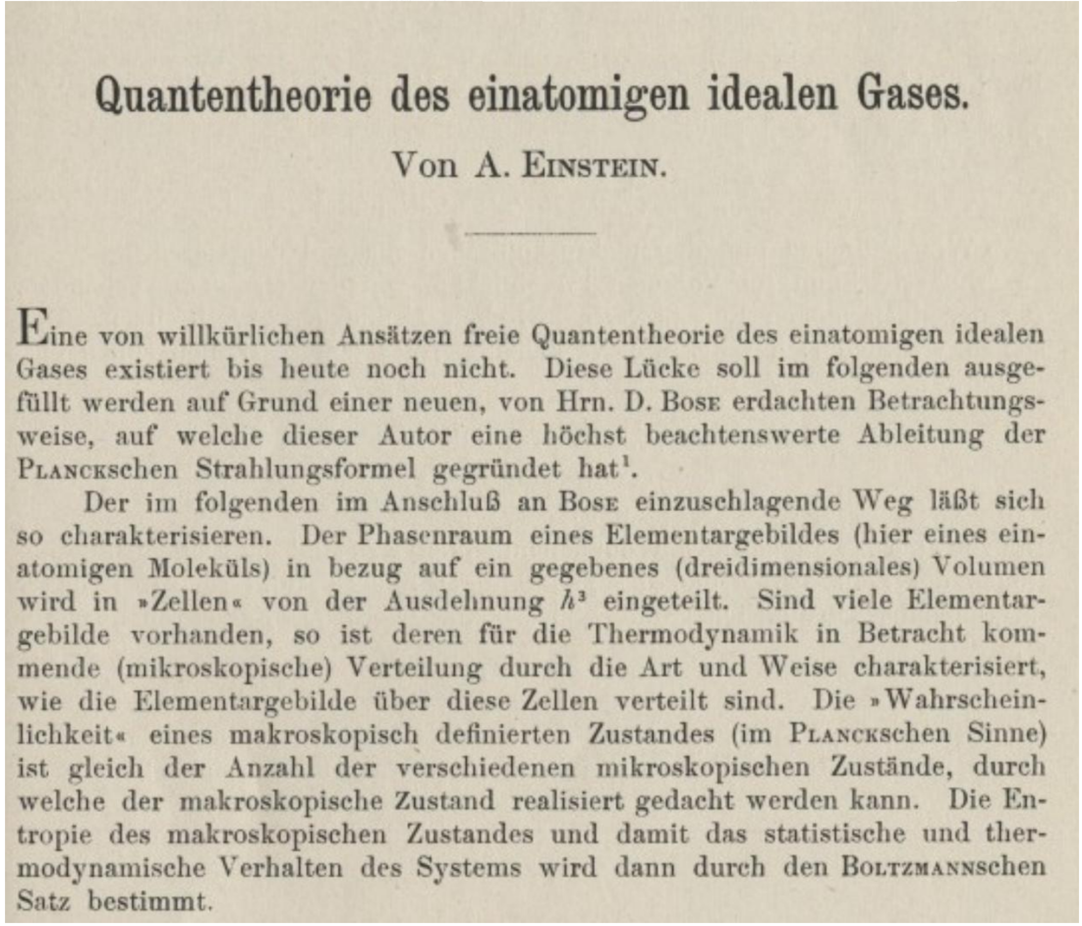
1925.
III.

SITZUNGSBERICHTE
DER PREUSSISCHEN
AKADEMIE DER WISSENSCHAFTEN.

Sitzung der physikalisch-mathematischen Klasse vom 29. Januar.

Zur Quantentheorie des idealen Gases.
Von A. Einstein.

A. Einstein.
Quantentheorie des einatomigen Gases.
SITZUNGSBERICHTE DER PREUSSICHEN AKADEMIE DER WISSENSCHAFTEN
PHYSIKALISCH-MATHEMATISCHE KLASSE 3, 1925.

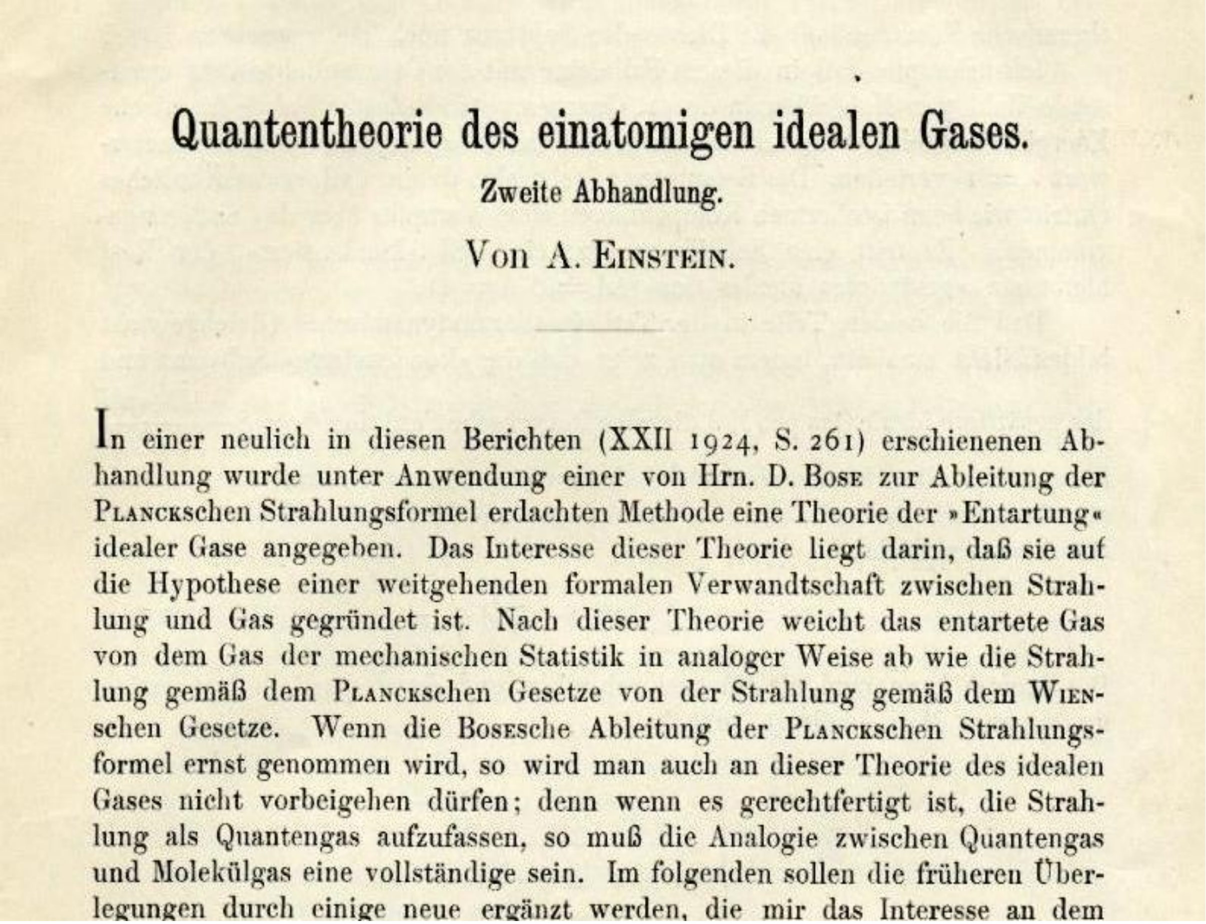


Quantentheorie des einatomigen idealen Gases.

Von A. Einstein.

Eine von willkürlichen Ansätzen freie Quantentheorie des einatomigen idealen Gases existiert bis heute noch nicht. Diese Lücke soll im folgenden ausgefüllt werden auf Grund einer neuen, von Hrn. D. Bose erdachten Betrachtungsweise, auf welche dieser Autor eine höchst beachtenswerte Ableitung der Planckschen Strahlungsformel gegründet hat[1].

Der im folgenden im Anschluß an Bose einzuschlagende Weg läßt sich so charakterisieren. Der Phasenraum eines Elementargebildes (hier eines einatomigen Moleküls) in bezug auf ein gegebenes (dreidimensionales) Volumen wird in »Zellen« von der Ausdehnung h^3 eingeteilt. Sind viele Elementargebilde vorhanden, so ist deren für die Thermodynamik in Betracht kommende (mikroskopische) Verteilung durch die Art und Weise charakterisiert, wie die Elementargebilde über diese Zellen verteilt sind. Die »Wahrscheinlichkeit« eines makroskopisch definierten Zustandes (im Planckschen Sinne) ist gleich der Anzahl der verschiedenen mikroskopischen Zustände, durch welche der makroskopische Zustand realisiert gedacht werden kann. Die Entropie des makroskopischen Zustandes und damit das statistische und thermodynamische Verhalten des Systems wird dann durch den Boltzmannschen Satz bestimmt.



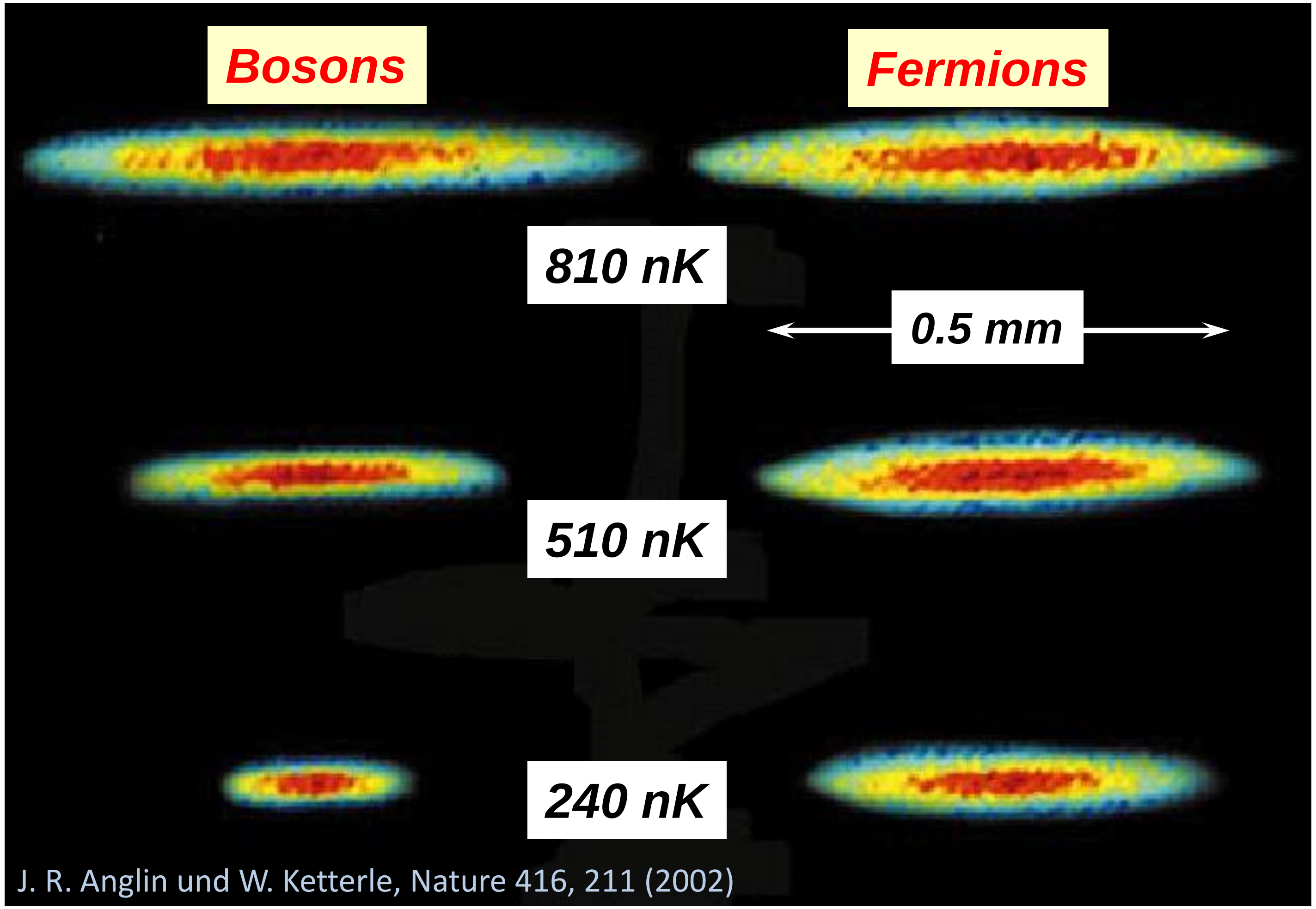
Quantentheorie des einatomigen idealen Gases.

Zweite Abhandlung.

Von A. Einstein.

In einer neulich in diesen Berichten (XXII 1924, S. 261) erschienenen Abhandlung wurde unter Anwendung einer von Hrn. D. Bose zur Ableitung der Planckschen Strahlungsformel erdachten Methode eine Theorie der »Entartung« idealer Gase angegeben. Das Interesse dieser Theorie liegt darin, daß sie auf die Hypothese einer weitgehenden formalen Verwandtschaft zwischen Strahlung und Gas gegründet ist. Nach dieser Theorie weicht das entartete Gas von dem Gas der mechanischen Statistik in analoger Weise ab wie die Strahlung gemäß dem Planckschen Gesetze von der Strahlung gemäß dem Wienschen Gesetze. Wenn die Bosesche Ableitung der Planckschen Strahlungsformel ernst genommen wird, so wird man auch an dieser Theorie des idealen Gases nicht vorbeigehen dürfen; denn wenn es gerechtfertigt ist, die Strahlung als Quantengas aufzufassen, so muß die Analogie zwischen Quantengas und Molekülgas eine vollständige sein. Im folgenden sollen die früheren Überlegungen durch einige neue ergänzt werden, die mir das Interesse an dem

I.2.1 Bose-Einstein Condensation



J. R. Anglin und W. Ketterle, Nature 416, 211 (2002)

Pauli exclusion principle
does not allow further
compression of fermions in
momentum space
at low T

I.2.1 Bose-Einstein Condensation

- **degenerate multiparticle system: Bosons and Fermions (cf. I.1.1)**

  - at given density *n*:
    particle occupies cube with $d = 1/n^{1/3}$

  - ***classical gas (liquid)***: $\lambda_\mathrm{T} \leq d$

    particles are distinguishable

  - ***quantum gas (liquid)***: $\lambda_\mathrm{T} \geq d$

    particles are indistinguishable
    ➔ *degenerate quantum gas (liquid)*

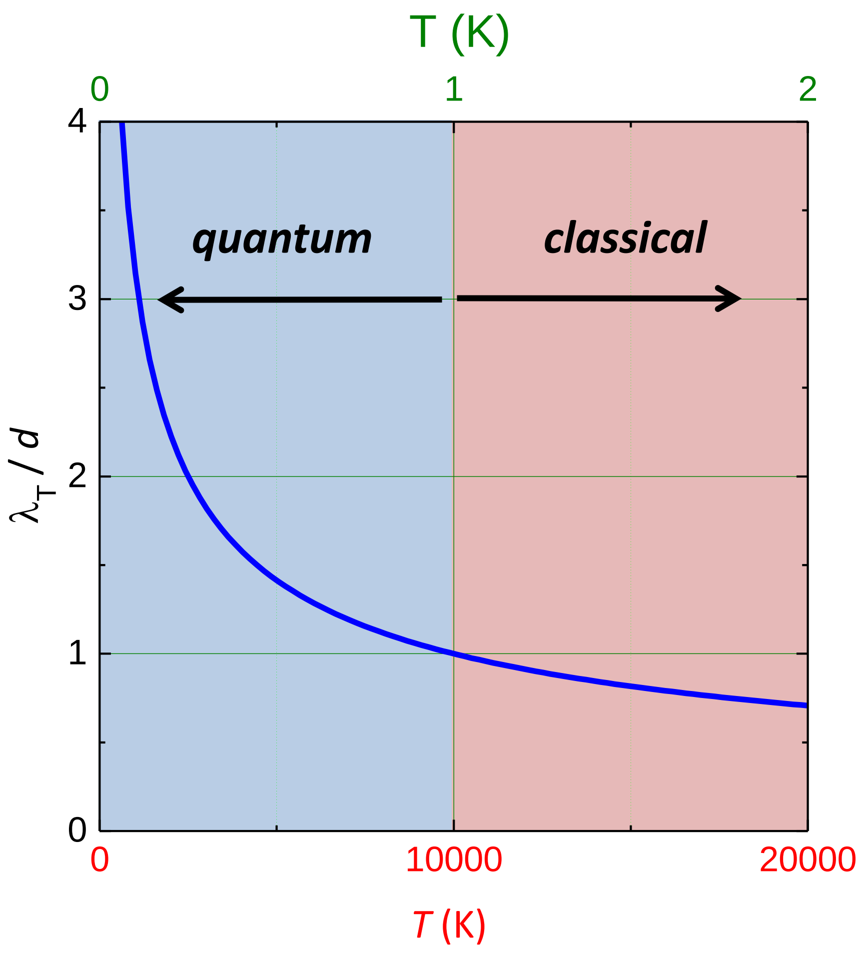
  - ***Bosons***: (^{4}He, ….)

    ➔ *Bose-Einstein condensation*

  - ***Fermions***: (^{3}He, e, p, n, ….)

    ➔ *Fermi sea*

^{4}He: $n = 4 \cdot 10^{21}$ cm^{-3},



e$^-$: $n = 3.6 \cdot 10^{21}$ cm^{-3}

I.2.1 Bose-Einstein Condensation

- cooling down atoms

λ_T^3/d^3

	temperature	density (1/cm³)	phase space density
atomic oven	500 K	10^{-1}	10^{-13}
Laser cooling	50 µK	10^{11}	10^{-6}
evaporation cooling	500 nK	10^{14}	2.612
Bose-Einstein condensation		10^{15}	10^{7}

$$\lambda_\mathrm{T} = \sqrt{\frac{h^2}{2\pi m k_\mathrm{B} T}} = \sqrt{\frac{2\pi\hbar^2}{m k_\mathrm{B} T}} \propto \frac{1}{\sqrt{mT}}$$

I.2.1.1 Laser Cooling

- **Laser cooling:**
  techniques in which atomic and molecular samples are cooled down to near absolute zero **through the interaction with one or more laser fields**

- **Laser cooling techniques**

  - Doppler cooling (Schawlow, Hänsch)

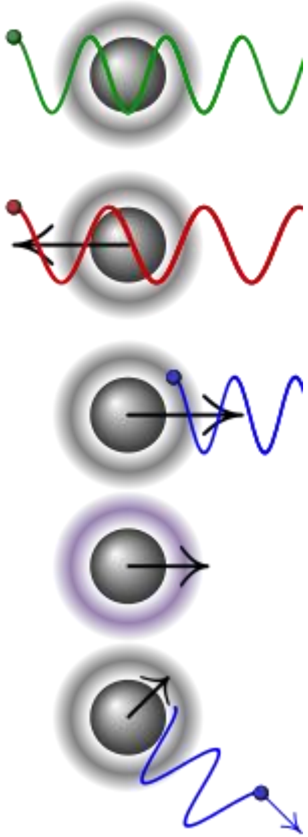
    T.W. Hänsch, and A.L. Schawlow, *Cooling of gases by laser radiation*, Optics Communications, 13, 68 (1975)
  - other techniques
    - Sisyphus cooling
    - Resolved sideband cooling
    - Velocity selective coherent population trapping (VSCPT)
    - Anti-Stokes inelastic light scattering (typically in the form of fluorescence or Raman scattering)
    - Cavity mediated cooling
    - Sympathetic cooling

- **Nobel Price in Physics 1997:**

  „***for the development of methods to cool and trap atoms with laser light***"
  for **Steven Chu**, **Claude Cohen-Tannoudji** und **William D. Phillips**

I.2.1.1 Laser Cooling – Doppler Cooling

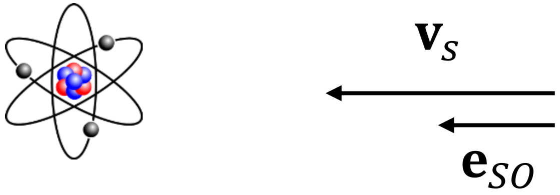
- in **Doppler cooling**, the frequency of ***light is tuned slightly below an electronic transition in the atom***
- because the light is detuned to the "red" (i.e., at lower frequency) of the transition, the atoms will absorb more photons if they move towards the light source, due to the Doppler effect



- ***a stationary atom sees the laser neither red- nor blue-shifted and does not absorb the photon***
- ***an atom moving away from the laser sees it red-shifted and does not absorb the photon***
- ***an atom moving towards the laser sees it blue-shifted and absorbs the photon, slowing the atom***
- ***the photon excites the atom, moving an electron to a higher quantum state.***
- ***the atom re-emits a photon,***
  as its direction is random, there is no net change in momentum over many absorption-emission cycles

I.2.1.1 Laser Cooling – Doppler Cooling

- ***Doppler shift*** (how does the moving atom see the laser frequency)



atom = observer (O)
fixed position

laser = signal source (S) with frequency f_S
moving at volocity v_S towards atom

$$f_O = \frac{f_S}{1 - \frac{\mathbf{v}_S \cdot \mathbf{e}_{SO}}{c}}$$

laser frequency seen by the atom is blue-shifted,
if laser and atom move towards each other

➔ red detuned laser light becomes resonant if atom moves towards the laser

- ***momentum/velocity change of atom due to photon absorption***

$$\frac{\Delta p}{p} = \frac{\hbar k_{ph}}{mv} = \frac{\Delta v}{v} \quad\Rightarrow\quad \Delta v = \frac{\hbar k_{ph}}{m}$$

I.2.1.1 Laser Cooling – Doppler Cooling

- ***atom-laser detuning with atom at rest:***

$$\delta_0 = \omega_{atom} - \omega_{laser}$$

- ***atom-laser detuning with atom moving at velocity v:***

$$\delta_{\text{Doppler}} = 2\pi(f_O - f_S) = 2\pi f_O \frac{\mathbf{v}\cdot\mathbf{e}_{SO}}{c} = 2\pi f_O \frac{\hbar\mathbf{k}\cdot\mathbf{v}}{\hbar\omega_O} = \mathbf{k}\cdot\mathbf{v}$$

$$f_O = \frac{f_S}{1 - \frac{\mathbf{v}_S\cdot\mathbf{e}_{SO}}{c}}$$

effective detuning between atom and laser

$$\delta_{\text{eff}} = \delta_0 - \delta_{\text{Doppler}} = \delta_0 - \mathbf{k}\cdot\mathbf{v}$$

we use:

$$p = \frac{E}{c}, \qquad c = \frac{E}{p} = \frac{\hbar\omega}{mv} = \frac{\hbar\omega}{\hbar k}$$

- ***force acting on the atom:***

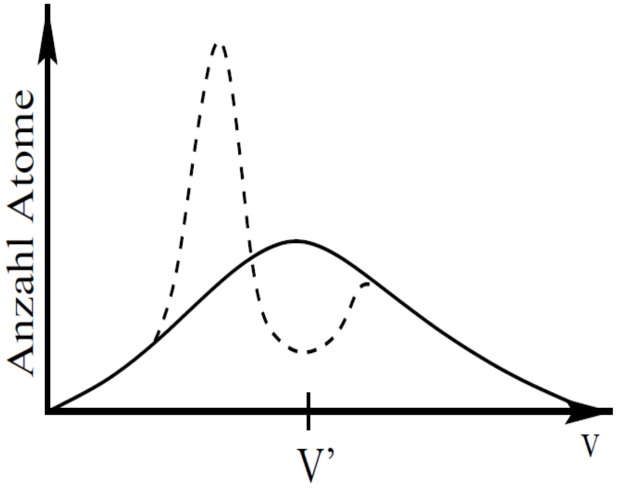
$$\mathbf{F} = \hbar\mathbf{k}\,\gamma = \hbar\mathbf{k}\frac{\Gamma}{2}\frac{(I/I_0)}{1 + \frac{I}{I_0} + \left(\frac{2\delta_{\text{eff}}}{\Gamma}\right)^2}$$

γ = absorption rate
Γ = line width of atom transition,
spontaneous emisson rate
I_0 = saturation intensity

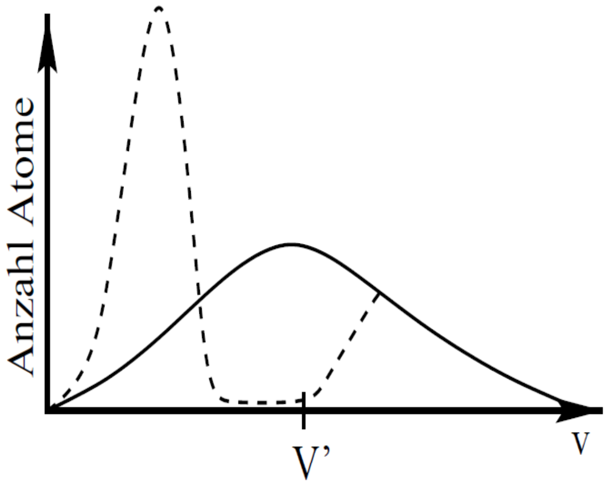
large intensity $I \gg I_0$:

$$\mathbf{F} \rightarrow \hbar\mathbf{k}\frac{\Gamma}{2}$$

I.2.1.1 Laser Cooling – Doppler Cooling



- velocity distribution of an atom beam before and after (dashed line) the interaction with a laser beam of fixed frequency



- velocity distribution of an atom beam before and after (dashed line) the interaction with a laser beam of continuously decreasing frequency

I.2.1.1 Laser Cooling – Doppler Cooling

- ***achievable minimum temperature in Doppler cooling:***

- spontaneous absorption and emission are random processes

→ the average force acting on the atoms is statistically fluctuating
→ Brownian motion in phase space

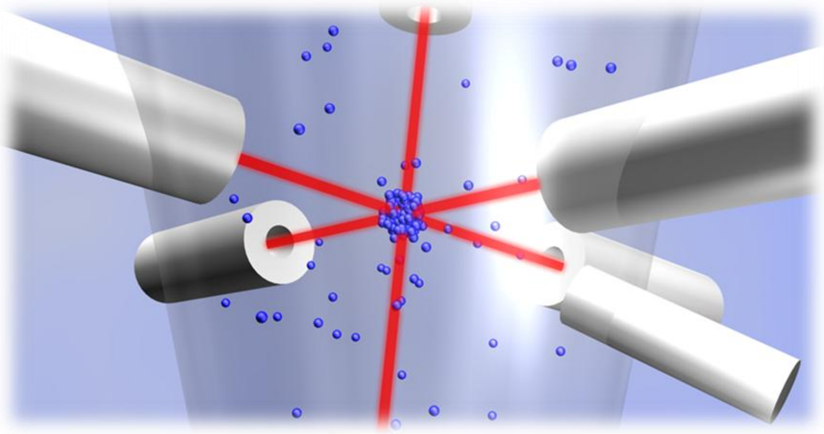
- minimum temperature is given by the compensation of heating effect due to the random absorption and emission processes and maximum cooling rate

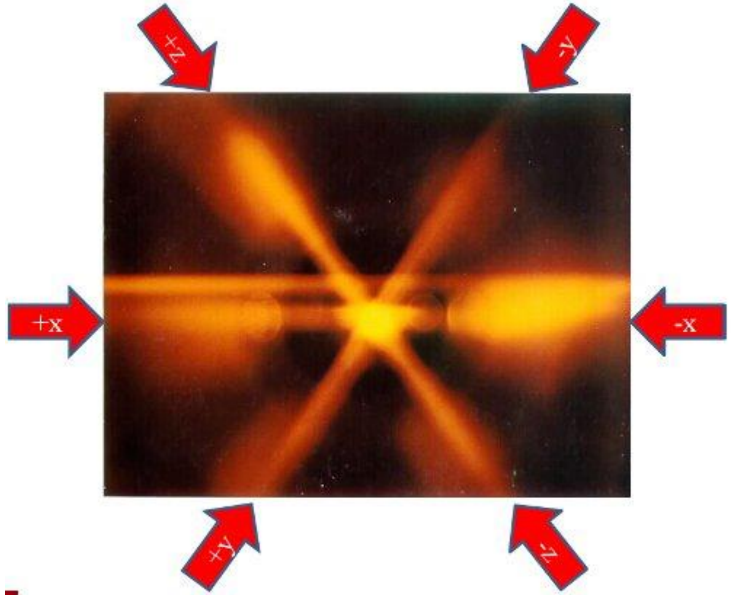
$$\boldsymbol{k_\mathrm{B} T_\mathrm{Doppler} = \frac{\hbar \Gamma}{2}}$$

- e.g. $T_\mathrm{Doppler} = 240\ \mu\mathrm{K}$ for Na corresponding to average velocity of 1 km/h
$T_\mathrm{Doppler} = 140\ \mu\mathrm{K}$ for Rb

I.2.1.1 Laser Cooling – Optical Molasses

- **optical molasses** (Steven Chu, 1985)
- an **optical molasses** consists of 3 pairs of counter-propagating circularly polarized laser beams intersecting in the region where the atoms are present
- while for simple Doppler cooling the minimum temperature for Na atoms is about 250 µK, optical molasses can cool the atoms down to 40 µK, i.e. an order of magnitude colder.

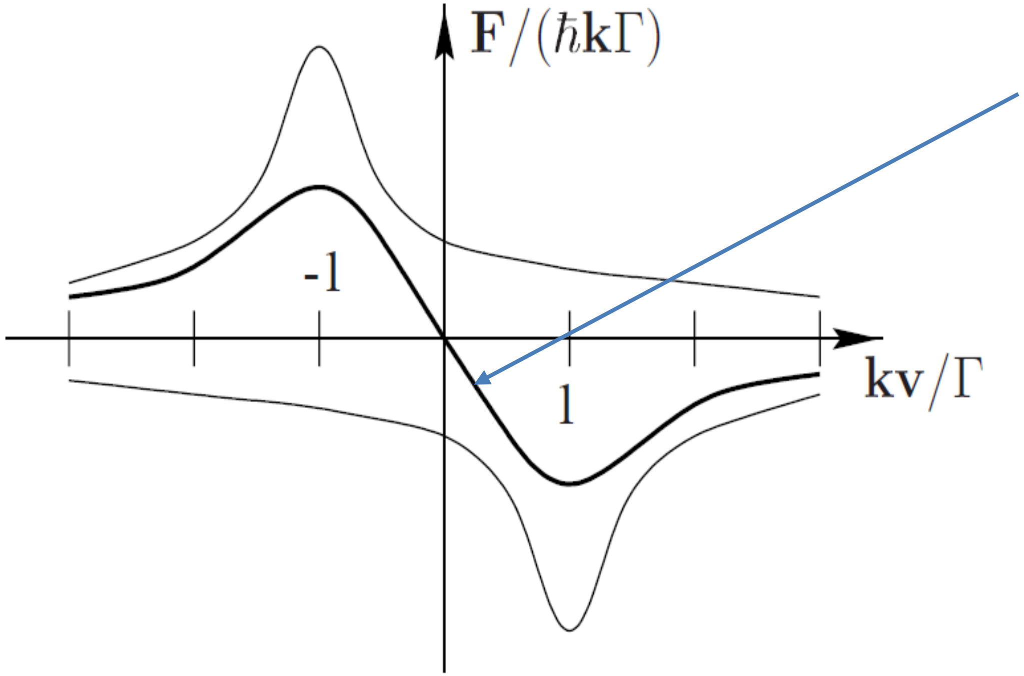




I.2.1.1 Laser Cooling – Optical Molasses

- force acting by two laser with opposite directions

$$\mathbf{F} = +\hbar\mathbf{k}\frac{\Gamma}{2}\frac{(I/I_0)}{1+\frac{I}{I_0}+\left(\frac{2(\delta_0-\mathbf{k}\cdot\mathbf{v})}{\Gamma}\right)^2} - \hbar\mathbf{k}\frac{\Gamma}{2}\frac{(I/I_0)}{1+\frac{I}{I_0}+\left(\frac{2(\delta_0+\mathbf{k}\cdot\mathbf{v})}{\Gamma}\right)^2}$$



total force

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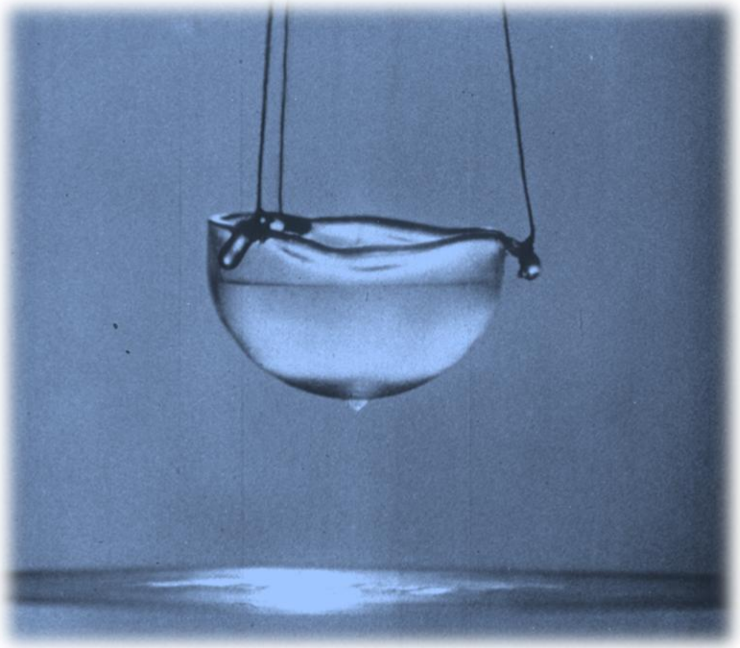
Superconductivity and Low Temperature Physics II



Lecture No. 3

R. Gross
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Contents Part I: Quantum Gases & Liquids

Contents:

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I.2.2 Ideal Bose Gas

- **consider a gas of N ($\approx 10^{22}$) non-interacting bosons of mass m in a cube of volume $V = L^3$ ($\approx$ 1 cm³),**

canonical ensemble (given T, V, N)

- eigen energies: $$\varepsilon_k = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n^2$$ with $n^2 = n_x^2 + n_y^2 + n_z^2$

- $\Delta\varepsilon / k_B T \leq 10^{-14}$

→ level spacing small compared to thermal energy (e.g. He @ 10 K)
→ no influence on experimental result

- occupation probability: $$n_k = \frac{1}{\exp\left(\frac{\varepsilon_k - \mu}{k_B T}\right) - 1}$$

Bose-Einstein distribution

μ = chemical potential

important note:
- $\mu \leq \varepsilon_k$ for all k to avoid negative occupation probabilities
- usually one sets $\varepsilon_0 = 0$ for the lowest energy state

- energy and particle number: $$U(T, V, \mu) = \sum_k \varepsilon_k \, n_k$$ $$N(T, V, \mu) = \sum_k n_k$$

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I.2.2 Ideal Bose Gas

- **derivation of Bose-Einstein distribution (1)**
  - we use the single spin density of states of free particles in 3D: $D(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon}\,, \qquad N(\varepsilon) = \frac{D(\varepsilon)}{V}$
  - total entropy of the particle gas: $S = k_\mathrm{B} \ln \Omega$ $\quad \Omega =$ number of the available states at given total energy E
  - Ω is determined by the number of possibilities to distribute the N particles on the available states
    - we divide the k-space into spherical shells with the shell s containing M_s states
    - the number of possibilities to put N_s particles on M_s states is given by

$$\Omega_s = \frac{(N_s + M_s - 1)!}{N_s!\,(M_s - 1)!}$$

(we use $M_s = D(\varepsilon)\delta\varepsilon_s$ for the number of availble states in the shell s)

    - the total number Ω of possibilities is obtained by multiplying the numbers Ω_s

$$\Omega = \prod_s \Omega_s = \prod_s \frac{(N_s + M_s - 1)!}{N_s!\,(M_s - 1)!}$$

I.2.2 Ideal Bose Gas

- **derivation of Bose-Einstein distribution (2)**

  – we use the Stirling approximation $\ln N! = N\ln N - N$ and for $N_s, \Omega_s \gg 1$ we obtain

$$S = k_{\mathrm{B}} \ln \Omega = k_{\mathrm{B}} \sum_s [(N_s + M_s)\ln(N_s + M_s) - N_s \ln N_s - M_s \ln M_s]$$

  – in thermal equilibrium the particles are distributed on the shells to maximize S under the boundary conditions $N = \sum_s N_s = const.$ and $U = \sum_s \varepsilon_s N_s = const.$

  – we maximize S for $N = const.$ and $U = const.$ we obtain $(dU = TdS + \mu dN)$

$$\frac{\partial S}{\partial N_s} - \frac{1}{T}\frac{\partial U}{\partial N_s} + \frac{\mu}{T}\frac{\partial N}{\partial N_s} = 0 = \frac{\partial S}{\partial N_s} - k_{\mathrm{B}}\beta \frac{\partial U}{\partial N_s} + k_{\mathrm{B}}\beta\mu \frac{\partial N}{\partial N_s} \qquad \beta = 1/k_{\mathrm{B}}T$$

(Lagrangian multipliers)

  – performing the differentiation we obtain

$$\ln(N_s + M_s) - \ln N_s - \beta\varepsilon_s + \beta\mu = 0 \quad \Rightarrow \quad N_s = \frac{1}{\exp(\beta(\varepsilon_s - \mu)) - 1} M_s$$

  – average particle number occupying state in shell s

$$\boxed{n_{\mathrm{BE}} = \left[\exp\left(\frac{\varepsilon_s - \mu}{k_{\mathrm{B}}T}\right) - 1\right]^{-1}}$$

I.2.2 Ideal Bose Gas

- how to get rid of the summation over k ?

$$\sum_k \ldots \Rightarrow \int Z(k) d^3k = \frac{V}{(2\pi)^3} \int d^3k$$

$$\sum_k \ldots \Rightarrow \int D(\varepsilon) d\varepsilon = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int \sqrt{\varepsilon}\, d\varepsilon$$

for free bosons with dispersion $\varepsilon_k = \frac{\hbar^2 k^2}{2m}$:

$$Z(k) = \frac{V}{(2\pi)^3}$$

$$D(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\varepsilon}$$

$$U(T,V,\mu) = \sum_k \varepsilon_k\, n_k \quad \Rightarrow \quad \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int \frac{\varepsilon^{3/2}}{\exp\left(\frac{\varepsilon-\mu}{k_\mathrm{B}T}\right) - 1} d\varepsilon$$

$$N(T,V,\mu) = \sum_k n_k \quad \Rightarrow \quad \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int \frac{\varepsilon^{1/2}}{\exp\left(\frac{\varepsilon-\mu}{k_\mathrm{B}T}\right) - 1} d\varepsilon$$

Note:
we assume that the states are not degenerate, otherwise we have to multiply with degeneracy g_k

➔ we assume $g_k = 1$

crucial point: the integral is bounded!!
If the particle number N is too large and the temperature too low, the equation can no longer be satisfied

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I.2.2 Ideal Bose Gas

- **evaluation of the equation for the particle number**

$$N(T,V,\mu) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int\limits_0^\infty \frac{\varepsilon^{1/2}}{\exp\left(\frac{\varepsilon-\mu}{k_\mathrm{B}T}\right)-1}\,d\varepsilon = \frac{V}{4\pi^2}\left(\frac{2mk_\mathrm{B}T}{\hbar^2}\right)^{3/2}\int\limits_0^\infty \frac{\gamma \mathrm{e}^{-x}}{1-\gamma\,\mathrm{e}^{-x}}\,x^{1/2}\,dx$$

with $x = \varepsilon/k_\mathrm{B}T = \beta\varepsilon$ and ***fugacity*** $\gamma = \exp\left(\frac{\mu}{k_\mathrm{B}T}\right)$

Gamma function: $\Gamma(t) = \int_0^\infty \mathrm{e}^{-y}y^{t-1}\,dy$

- to solve the integral we use the series expansion

$$\frac{\gamma\mathrm{e}^{-x}}{1-\gamma\,\mathrm{e}^{-x}} = \gamma\mathrm{e}^{-x}(1+\gamma\mathrm{e}^{-x}+\gamma^2\mathrm{e}^{-2x}+\cdots) = \sum_{p=1}^{\infty}\gamma^p\mathrm{e}^{-px}$$

- plugging in this expression we can perform the integration by using

$$\int\limits_0^\infty \mathrm{e}^{-px}x^{1/2}\,dx \underbrace{=}_{y=px} \frac{1}{p^{3/2}}\underbrace{\int\limits_0^\infty \mathrm{e}^{-y}y^{1/2}\,dy}_{\text{Gamma function: }\Gamma(t)=\int_0^\infty \mathrm{e}^{-y}y^{t-1}\,dy} = \frac{1}{p^{3/2}}\underbrace{\frac{\sqrt{\pi}}{2}}_{\Gamma(3/2)} \quad\Rightarrow\quad \int\limits_0^\infty \frac{\gamma\mathrm{e}^{-x}}{1-\gamma\,\mathrm{e}^{-x}}\,x^{1/2}\,dx = \frac{\sqrt{\pi}}{2}\underbrace{\sum_{p=1}^{\infty}\frac{\gamma^p}{p^{3/2}}}_{g_{3/2}(\gamma)}$$

⇒ $$N(T,V,\mu) = V\underbrace{\left(\frac{mk_\mathrm{B}T}{2\pi\hbar^2}\right)^{3/2}}_{1/\lambda_T^3} g_{3/2}(\gamma) = \frac{V}{\lambda_T^3}\,g_{3/2}(\gamma)$$

⇒ $$n(T,\mu)\,\lambda_T^3 = g_{3/2}(\gamma)$$

I.2.2 Ideal Bose Gas

- analysis of the function $g_{3/2}(\gamma)$

$$g_{3/2}(\gamma) = \sum_{p=1}^{\infty} \frac{\gamma^p}{p^{3/2}}$$

i. $|\gamma| < 1$: the series converges

ii. $|\gamma| > 1$: the series diverges

iii. $|\gamma| = 1$: the series converges and has the value $g_{3/2}(1)$

$$g_{3/2}(1) = \sum_{p=1}^{\infty} \frac{1}{p^{3/2}} = \zeta\left(\frac{3}{2}\right) = 2.612$$

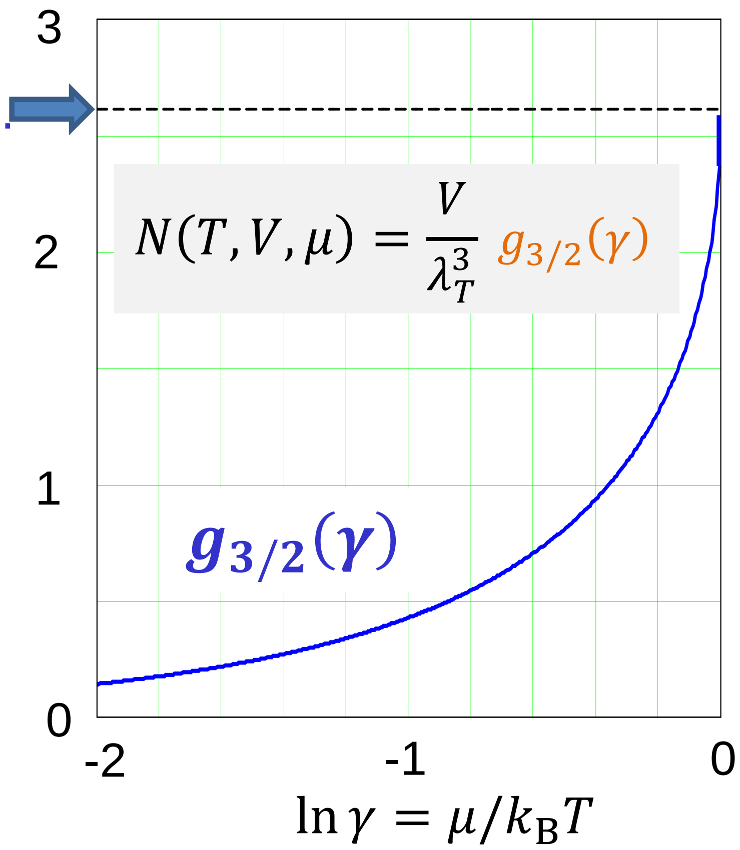
$$\zeta(s) = \sum_{p=1}^{\infty} \frac{1}{p^s}$$

(Rieman Zeta function)

$$\frac{dg_{3/2}(z)}{dz} = \frac{1}{z}\sum_{p=1}^{\infty} \frac{\gamma^p}{p^{1/2}}$$

slope diverges for $\gamma = 1$

$\zeta(3/2) = 2.61237 \ldots$



$$N(T,V,\mu) = \frac{V}{\lambda_T^3}\, g_{3/2}(\gamma)$$

important fact: $g_{3/2}(\gamma)$ has maximum value 2.61 for $\gamma \to 1$, resp. $\ln\gamma \to 0$
particle number N at given T is bounded to maximum value !!

I.2.2 Ideal Bose Gas

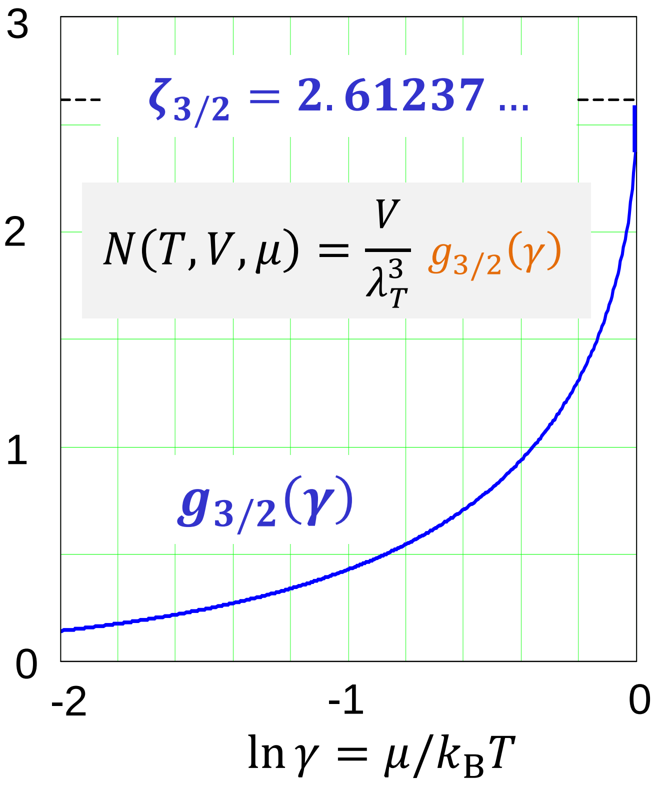
- **what happens with decreasing T ?**

– **λ_T increases $\propto 1/\sqrt{T}$** ➔ $g_{3/2}(\gamma)$ has to increase to keep $N(T,V,\mu)$ constant

- **problem:**
  $g_{3/2}(\gamma)$ has maximum value $g_{3/2}(1) = \zeta(3/2) = 2.61$ for $\gamma = 1$, resp. $\mu = 0$,
  ➔ $g_{3/2}(\gamma = 1)$ is related to **maximum value $\lambda_T = \lambda_{T,\mathrm{BEC}}$** ,
  resp. to **minimum temperature T_BEC**

$$\lambda^3_{T,\mathrm{BEC}} \cdot n = g_{3/2}(\gamma = 1) = \zeta(3/2) = 2.61237 \ldots$$

defines the characteristic temperature T_BEC:
for $T < T_\mathrm{BEC}$ it is not possible to satisfy the equation $N(T,\mu) = \frac{V}{\lambda_T^3}\, g_{3/2}(\gamma)$



$$N(T,V,\mu) = \frac{V}{\lambda_T^3}\, g_{3/2}(\gamma)$$

- with $\lambda_{T,\mathrm{BEC}} = h/\sqrt{2\pi m k_\mathrm{B} T_\mathrm{BEC}}$, we obtain:

$$n = 2.612 \left(\frac{m k_\mathrm{B} T_\mathrm{BEC}}{2\pi\hbar^2}\right)^{3/2} \Rightarrow \quad T_\mathrm{BEC} = \frac{2\pi\hbar^2}{m k_\mathrm{B}} \left(\frac{n}{2.612}\right)^{2/3}$$

Bose-Einstein condensation temperature

➢ $\lambda^3_{T,\mathrm{BEC}} \cdot n \simeq O(1)$ → *criterion for the importance of quantum effects or degeneracy*

I.2.2 Ideal Bose Gas

- **how to solve the problem ?**
  - physical solution of the problem that we cannot satisfy the equation $N(T,V,\mu) = \frac{V}{\lambda_T^3}\, g_{3/2}(\gamma)$ below T_{BEC}

➔ we have to put a significant number N_0 of the particles into the ground state $k = 0$

➔ this is in agreement with the Bose-Einstein distribution

➔ setting $\varepsilon(k=0) = \varepsilon_0 = 0$, the number of bosons in the ground state is

$$N_0(T,\mu) = \frac{1}{\exp\left(\frac{0-\mu}{k_\mathrm{B}T}\right) - 1} \simeq \frac{1}{1 - \frac{\mu}{k_\mathrm{B}T} - 1} = \frac{k_\mathrm{B}T}{|\mu|} \qquad \Rightarrow N_0(T,\mu) \rightarrow \infty \ \text{ for } \ |\mu| = |\varepsilon_0 - \mu| \rightarrow 0$$

I.2.2 Ideal Bose Gas

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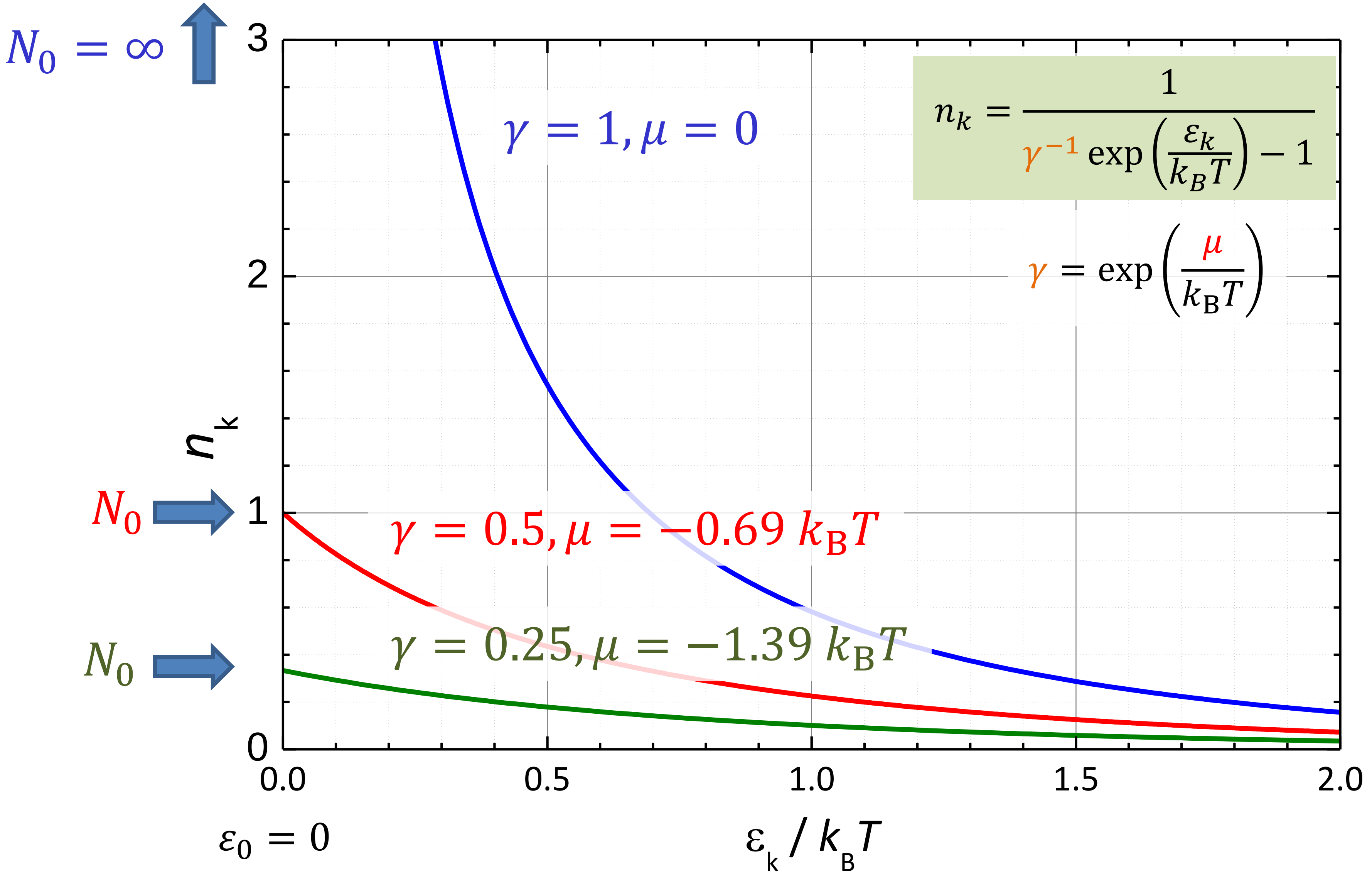
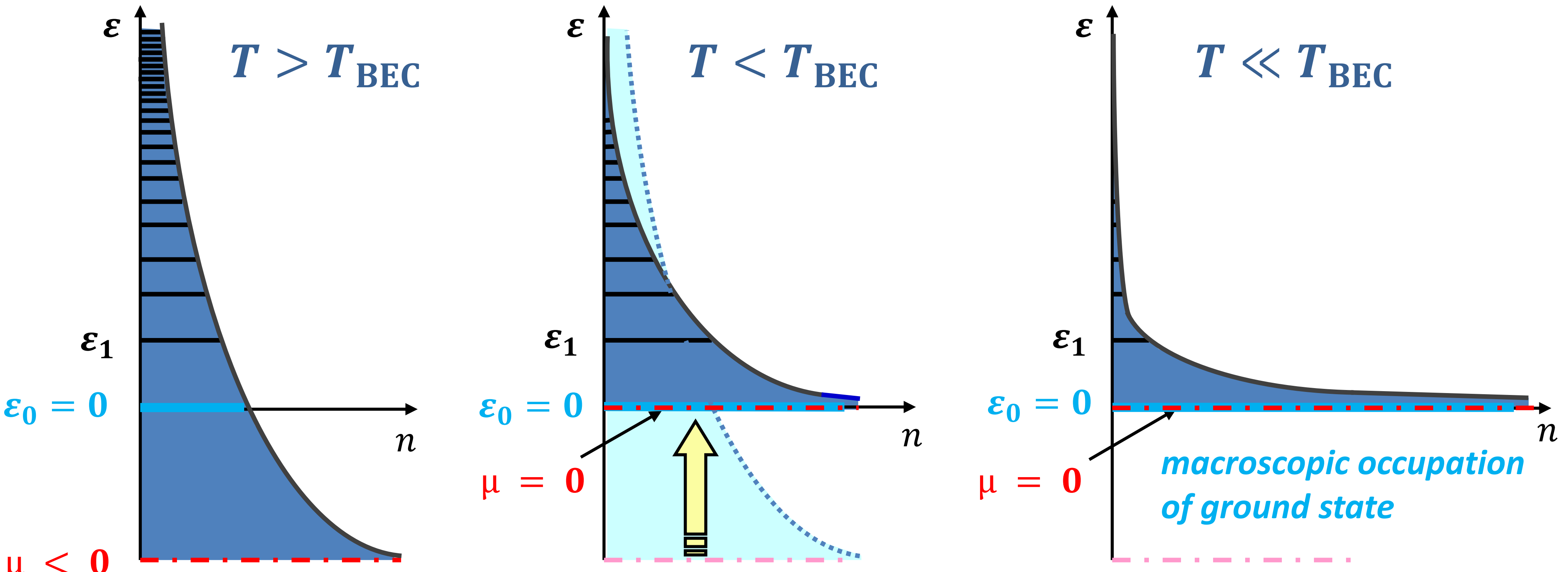


$$n_k = \frac{1}{\gamma^{-1} \exp\left(\frac{\varepsilon_k}{k_B T}\right) - 1}$$

$$\gamma = \exp\left(\frac{\mu}{k_B T}\right)$$

if $|\varepsilon_0 - \mu| = |0 - \mu| = |\mu| \to 0$, the occupation number $N_0 = n(k = 0)$ of the ground state $\varepsilon_0 = 0$ strongly increases

I.2.2 Ideal Bose Gas



macroscopic occupation of ground state

- calculation of $N_0(T) = N(\varepsilon_0, T)$ for $(\varepsilon_0 - \mu) = (0 - \mu) = -\mu \to 0$:

$$N_0(T,\mu) = \frac{1}{\exp\left(\frac{\varepsilon_0 - \mu}{k_B T}\right) - 1} \simeq \frac{1}{1 + \frac{\varepsilon_0 - \mu}{k_B T} - 1} = \frac{k_B T}{\varepsilon_0 - \mu} \to \infty \quad \text{for} \quad \varepsilon_0 - \mu \to 0$$

➔ $N_0(T,\mu)$ grows dramtically for $(\varepsilon_0 - \mu) \to 0$, that is, for $\mu \to 0$

Note:

$$\varepsilon_0 - \mu = k_B T \ln\left(1 + \frac{1}{N_0}\right) \simeq \frac{k_B T}{N_0}$$

➔ difference between ground state and chemical potential is much smaller than $k_B T$

I.2.2 Ideal Bose Gas

- **detailed analysis of the chemical potential**

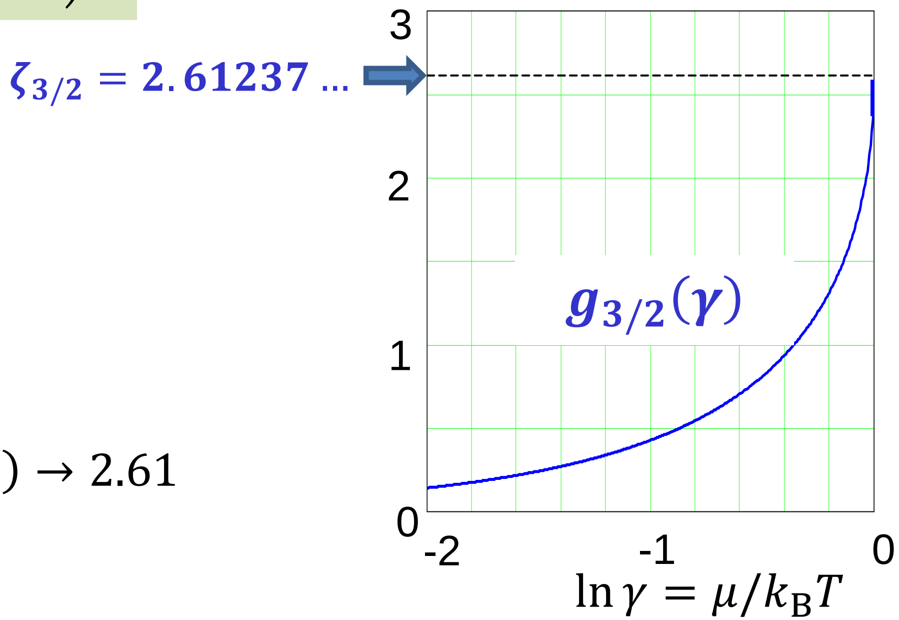
$$n(T,\mu)=\frac{N(T,V,\mu)}{V}=\left(\frac{mk_\mathrm{B}T}{2\pi\hbar^2}\right)^{3/2} g_{3/2}(\gamma)=\frac{1}{\lambda_T^3}\,g_{3/2}(\gamma)\;\Rightarrow\; g_{3/2}(\gamma)=\left(\frac{2\pi\hbar^2}{mk_\mathrm{B}T}\right)^{3/2} n(T,\mu)$$

– for high temperatures T and/or small particle density n we can use the approximation $g_{3/2}(\gamma)\simeq\gamma+\cdots\simeq \mathrm{e}^{\beta\mu}+\cdots$

$$\Rightarrow \mathrm{e}^{\beta\mu}=\left(\frac{2\pi\hbar^2}{mk_\mathrm{B}T}\right)^{3/2} n(T,\mu) \;\Rightarrow\; \boxed{\mu\simeq-\frac{3}{2}k_\mathrm{B}T\ln\left(\frac{mk_\mathrm{B}T}{2\pi\hbar^2 n^{2/3}}\right)}$$

- **for $T\to\infty$**:
  $\lambda_T\to 0,\ \mu\to-\infty,\gamma\to 0,\ln\gamma\to-\infty$ and therefore $g_{3/2}(\gamma)\to 0$

- **for decreasing T**:
  λ_T increaes, $\mu/k_\mathrm{B}T$ increases → 0, γ increases → 1 and therefore $g_{3/2}(\gamma)\to 2.61$



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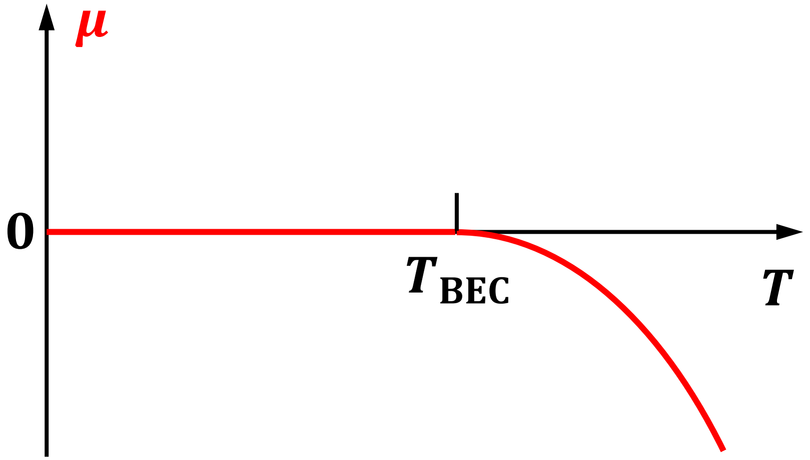
I.2.2 Ideal Bose Gas

- **what happens if we decrease T to values below T_{BEC}?**
  - for $T < T_{\mathrm{BEC}}$, there is no solution of equation $n(T,\mu) = \frac{1}{\lambda_T^3}\, g_{3/2}(\gamma)$ as $g_{3/2}(\gamma)$ has maximum value $g_{3/2}(1)$ for $\gamma \to 1$

    ➔ for $T < T_{\mathrm{BEC}}$, the particle number N_0in the ground quantum state ($\varepsilon_0 = 0$) must become large (first recognized by Einstein)

    occupation of the ground state $\varepsilon_0 = 0$: $$N_0 = \frac{1}{\exp(-\mu/k_{\mathrm{B}}T) - 1} \quad \Rightarrow \quad \mu = -k_{\mathrm{B}}T \ln\left(1 + \frac{1}{N_0}\right) \simeq -k_{\mathrm{B}}T \frac{1}{N_0}$$
  - from the total particle number N a macroscopically large number N_0 has to be placed in ground state ε_0
  - macroscopically large means that N_0 is proportional to system volume V
  - if the fraction of particles in the ground state becomes finite at $T = T_{\mathrm{BEC}}$, then in the thermodynamic limit $V \to \infty$ the particle number $N_0 \to \infty$ and hence the chemical potential $\mu \to 0$

    ➔ $\mu = 0$ for $T \leq T_{\mathrm{BEC}}$



as μ corresponds to the average energy required to add a particle to the systems, the result is evident: we are adding particles only to the ground state $\varepsilon_0 = 0$

I.2.2 Ideal Bose Gas

- **for $T \leq T_{\mathrm{BEC}}$: how are the N particles distributed among the available states ?**

  - $N = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\sqrt{\varepsilon}}{\exp\left(\frac{\varepsilon-\mu}{k_B T}\right)-1}\, d\varepsilon$ yields $\mu \to 0$ for $T \to T_{\mathrm{BEC}}$
  - for $\mu \to 0$ ($T \to T_{\mathrm{BEC}}$), we obtain $N_0 = \frac{1}{\exp\left(\frac{\varepsilon_0-\mu}{k_B T}\right)-1} \simeq \frac{k_B T}{|\mu|} \to \infty$

$$\varepsilon_0 = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 \propto \frac{1}{V^{2/3}} \propto \frac{1}{N^{2/3}} \to 0 \quad \text{for} \quad N \to \infty \quad \text{(thermodynamic limit)}$$

below T_{BEC}: ***macroscopic occupation $N_0 \gg 1$ of the ground state is required***

- **key result:**

  - at $T \leq T_{\mathrm{BEC}}$ a finite fraction of all particles occupies the ground state
  - this phenomenon is called ***Bose-Einstein condensation***
  - the name „***condensation***“ is used since in analogy to the gaseous-liquid transition ***the condensed particles do no longer contribute to the pressure*** (see discussion below)

I.2.2 Ideal Bose Gas

- **for $T \leq T_{\mathrm{BEC}}$: occupation of the first excited state**

for $\mu \rightarrow 0$ and $T \sim 1$ K: level spacing for 10^{22} atoms in a box with $L = 1$ cm: $\Delta\varepsilon/k_{\mathrm{B}}T = \varepsilon_1/k_{\mathrm{B}}T \leq 10^{-14}$

$$\Rightarrow \quad n_1(T) = \frac{1}{1+\exp\left(\frac{\epsilon_1-\mu}{k_{\mathrm{B}}T}\right)-1} \simeq \frac{1}{1+\frac{\varepsilon_1-\mu}{k_{\mathrm{B}}T}-1} \simeq \frac{k_{\mathrm{B}}T}{\varepsilon_1-\mu} \approx 10^{14}$$

n_1 and all $n_{k>1}$ are small compared to N_0 and the total particle number $N = 10^{22}$

(justifies why only the ground state was singled out in the calculation of T_{BEC})

– more detailed calculation: $n_1/N_0 \propto N^{-1/3}$, i.e. $n_1/N_0 \rightarrow 0$ for $N \rightarrow \infty$

$$\frac{n_1}{N_0} = \frac{k_{\mathrm{B}}T}{\varepsilon_1-\mu}\cdot\frac{\varepsilon_0-\mu}{k_{\mathrm{B}}T} = -\frac{\mu}{\varepsilon_1-\mu} \simeq -\frac{\mu}{\varepsilon_1} \simeq \frac{k_{\mathrm{B}}T}{N_0\varepsilon_1}$$

we use $N_0(T,\mu) \simeq \frac{k_{\mathrm{B}}T}{\varepsilon_0-\mu} = \frac{k_{\mathrm{B}}T}{-\mu}$

for $T < T_{\mathrm{BEC}}/2$ we can use the crude approximations $N_0 \simeq N,\ T \simeq T_{\mathrm{BEC}} \propto N^{2/3}$:

$$\frac{n_1}{N_0} \propto \frac{1}{N^{1/3}} \propto \frac{1}{V^{1/3}}$$

➔ already at $T \simeq T_{\mathrm{BEC}}/2$, we have $N_0 \gg n_1, n_{k>1}$

– for $T \rightarrow 0$: $\varepsilon_{k>0}/k_{\mathrm{B}}T \rightarrow \infty$ and hence $n_{k>0} \rightarrow 0$, $\mu/k_{\mathrm{B}}T = 1/N_0 \simeq 0$: ***all bosons occupy ground state***

I.2.2 Ideal Bose Gas

- **for $T < T_{\mathrm{BEC}}$ we have to treat the ground state ε_0 separately**

$$N = \sum_{\mathbf{k}} \frac{1}{\exp\left(\frac{\varepsilon_{\mathbf{k}}-\mu}{k_B T}\right)-1} = N_0 + N_{ex} = N_0 + \sum_{\mathbf{k}\neq 0}\left[\exp\left(\frac{\varepsilon_{\mathbf{k}}}{k_B T}\right)-1\right]^{-1} = N_0 + \int_0^\infty D(\varepsilon)n(\varepsilon)\,d\varepsilon$$

particles in ground state

particles in excited states

- we obtain for **free bosons**:

$$N_{ex} = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{\varepsilon^{1/2}}{\exp\left(\frac{\varepsilon-\mu}{k_B T}\right)-1}\,d\varepsilon = \frac{V}{4\pi^2}\left(\frac{2mk_\mathrm{B}T}{\hbar^2}\right)^{3/2}\underbrace{\int_0^\infty \frac{\gamma\,\mathrm{e}^{-x}}{1-\gamma\,\mathrm{e}^{-x}}\,x^{1/2}\,dx}_{=\Gamma(3/2)\,\zeta(3/2)}$$

with $x = \beta\varepsilon$ and

fugacity $\gamma = \exp\left(\frac{\mu}{k_\mathrm{B}T}\right) = 1$

⇨ $$n = n_0 + n_{ex} = n_0 + 2.612\left(\frac{mk_\mathrm{B}T}{2\pi\hbar^2}\right)^{3/2}$$ ⇨ $$\frac{n_0}{n} = 1 - \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2}$$

n_0: condensate density

n_{ex}: normal fluid density

$$T_{\mathrm{BEC}} = \frac{2\pi\hbar^2}{mk_\mathrm{B}}\left(\frac{n}{2.612}\right)^{2/3} \Rightarrow n = 2.612\left(\frac{2\pi\hbar^2}{mk_\mathrm{B}T_{\mathrm{BEC}}}\right)^{3/2}$$

I.2.2 Ideal Bose Gas

we assume constant particle number:

$N = N_0(T) + N_{ex}(T) = \text{const.}$

- **Summary: temperature dependence of N_0 and N_{ex}**

  - $T > T_{\mathrm{BEC}}$: $N_0 = O(1)$ ➔ can be neglected

    $$N = N_{ex} = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{\sqrt{\varepsilon}}{\exp\left(\frac{\varepsilon-\mu}{k_\mathrm{B}T}\right)-1}\,d\varepsilon$$

    is valid

  - $T = T_{\mathrm{BEC}}$: $\mu \to 0$, $\lambda^3_{T,\mathrm{BEC}} \cdot n = 2.612$ ➔ $N = N_{ex} = 2.612 \cdot \frac{V}{\lambda^3_{T,\mathrm{BEC}}}$

  - $T \leq T_{\mathrm{BEC}}$: $\mu = 0$,

    $$N = N_0 + N_{ex} = N_0 + 2.612 \cdot \frac{V}{\lambda_T^3} = N_0 + 2.612 \cdot \frac{V}{\lambda_T^3} \cdot \frac{\lambda^3_{T,\mathrm{BEC}}}{\lambda^3_{T,\mathrm{BEC}}}$$

    $$N = N_0 + 2.612 \cdot \frac{V}{\lambda^3_{T,\mathrm{BEC}}} \cdot \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2} = N_0 + N \cdot \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2}$$

➔ $$\frac{N_0}{N} = 1 - \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2} \qquad \frac{N_{ex}}{N} = \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2}$$

I.2.2 Ideal Bose Gas

- normal fluid density ($T < T_{\mathrm{BEC}}$)

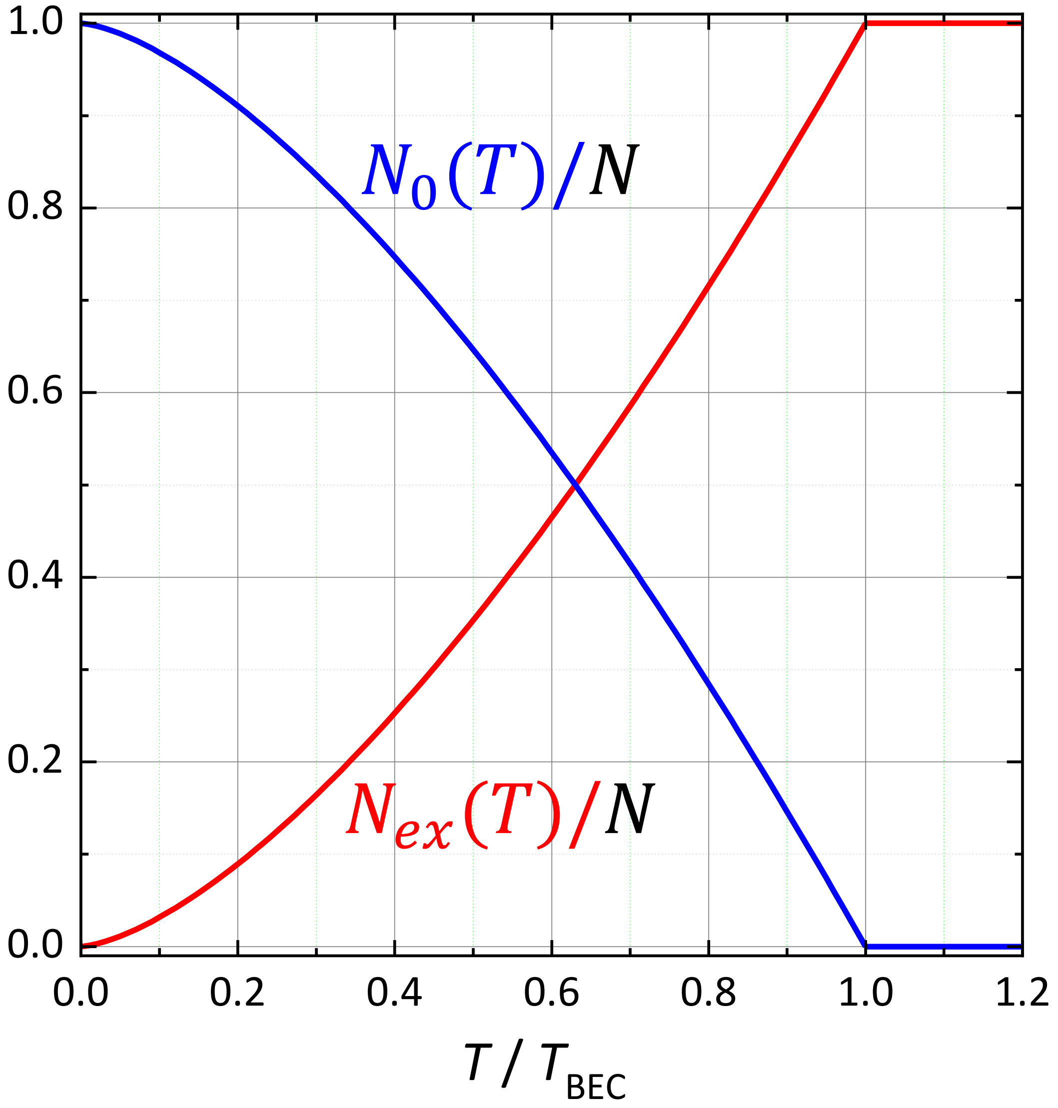
$$\frac{N_{ex}}{N} = \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2}$$

- superfluid or condensate density

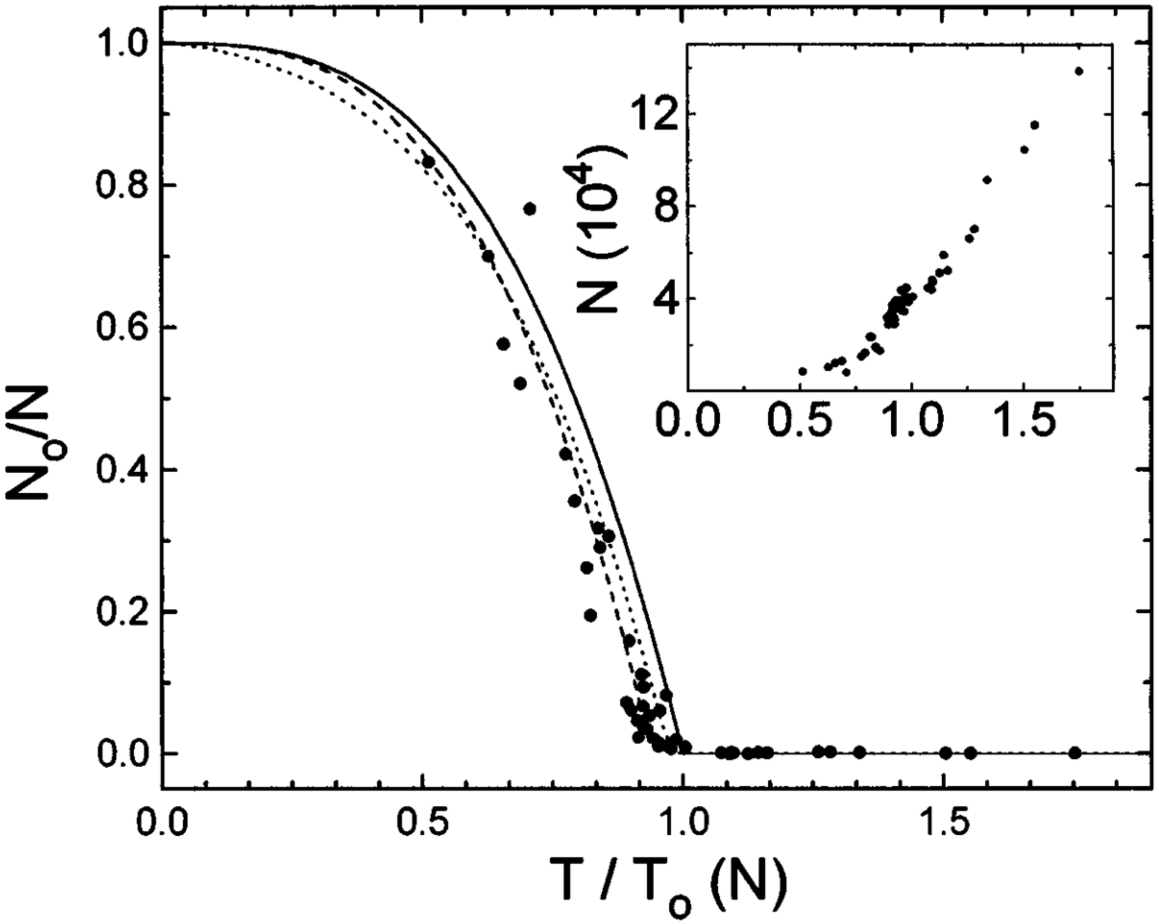
$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2}$$

$$T_{\mathrm{BEC}} = \frac{2\pi\hbar^2}{mk_B}\left(\frac{n}{2.612}\right)^{2/3}$$

$$\lambda_T^3 = \left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2}$$



I.2.2 Ideal Bose Gas

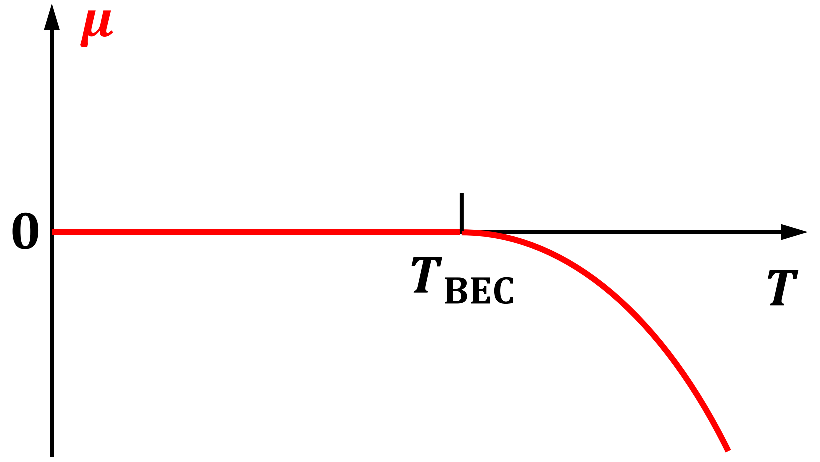


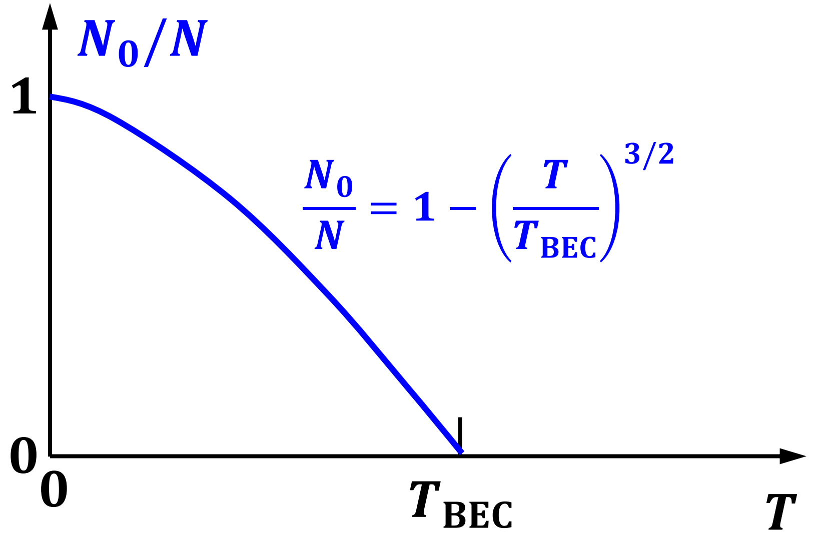
Total number N (inset) and ground-state fraction N_0/N as a function of T/T_{BEC}.

The solid (dotted) line shows the infinite (finite) N theory curves. At the transition, the cloud consists of 40 000 $^{87}\mathrm{Rb}$ atoms at 280 nK.

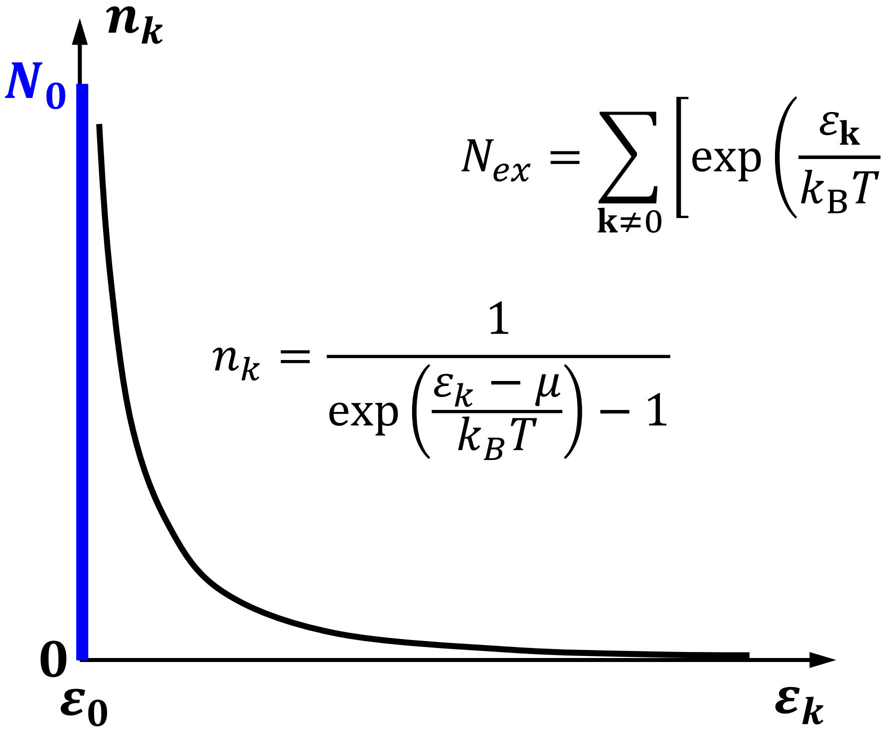
Bose-Einstein Condensation in a Dilute Gas: Measurement of Energy and Ground-State Occupation
J. R. Ensher, D. S. Jin, M. R. Matthews, C. E. Wieman, and E. A. Cornell
Phys. Rev. Lett. 77, 4984 (1996)

I.2.2 Ideal Bose Gas





$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2}$$



$$N_{ex} = \sum_{\mathbf{k}\neq 0} \left[\exp\left(\frac{\varepsilon_{\mathbf{k}}}{k_{\mathrm{B}}T}\right) - 1\right]^{-1}$$

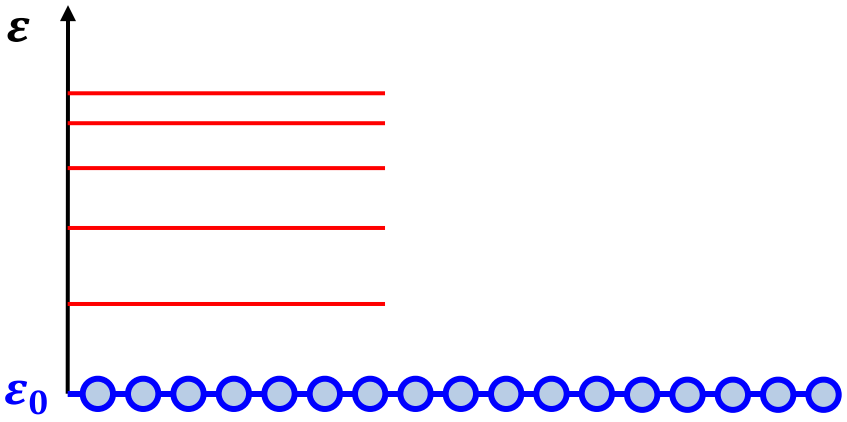
$$n_k = \frac{1}{\exp\left(\frac{\varepsilon_k - \mu}{k_B T}\right) - 1}$$

I.2.2 Ideal Bose Gas

- Bose-Einstein condensation can be viewed as condensation of gas in momentum space

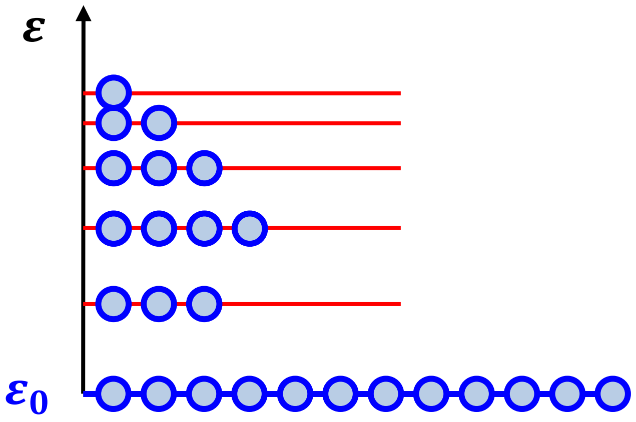
➔ *all atoms have the same momentum*

- Bose-Einstein condensation corresponds to order-disorder phase transition (first order phase transition)



$T = 0$

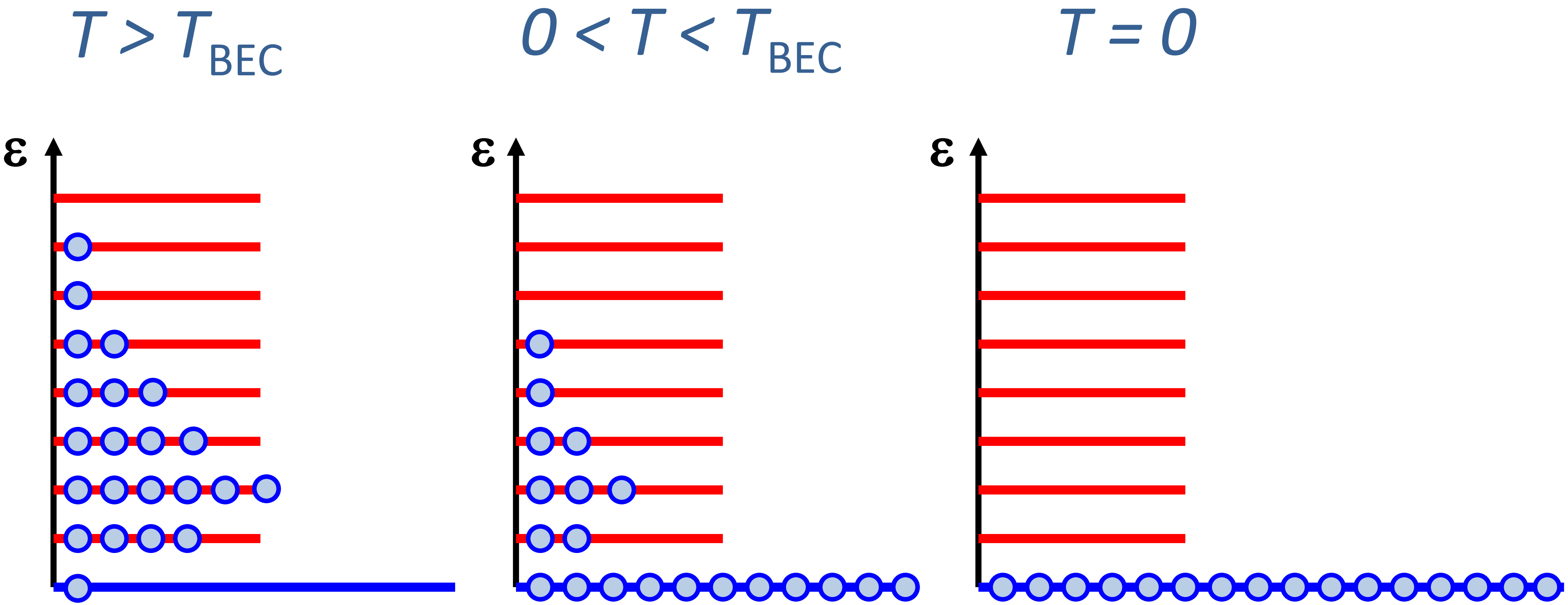
only ground state
is occupied



$0 < T < T_{\mathrm{BEC}}$

still macroscopic
occupation of the
ground state

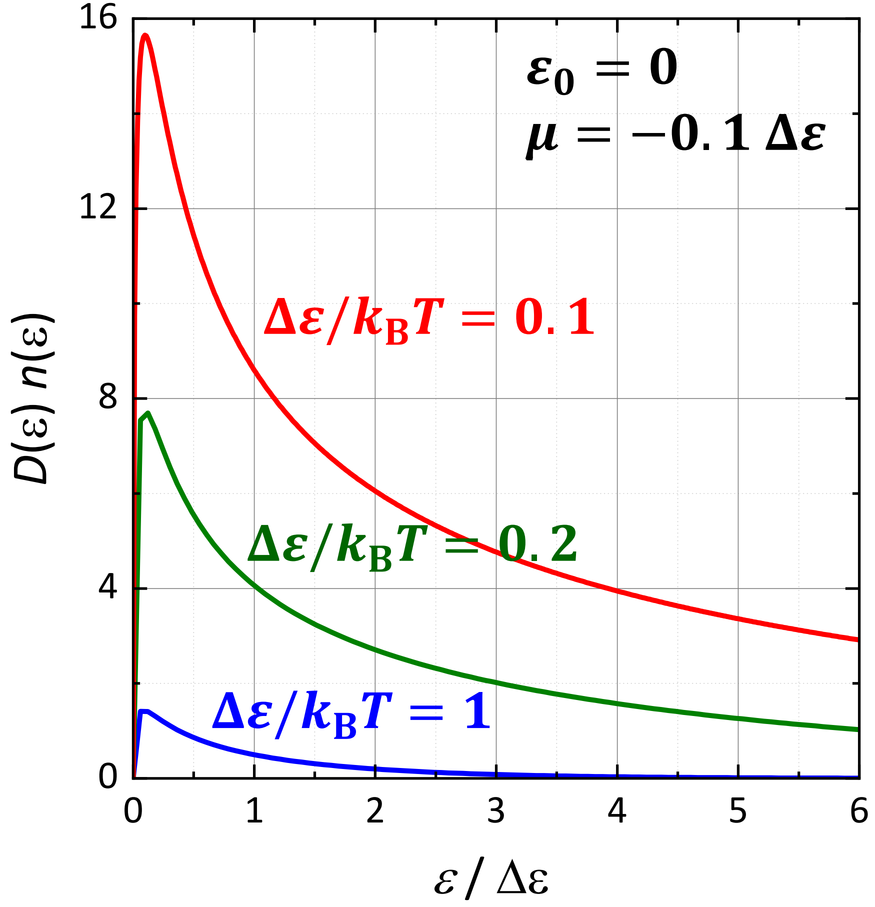
I.2.2 Ideal Bose Gas



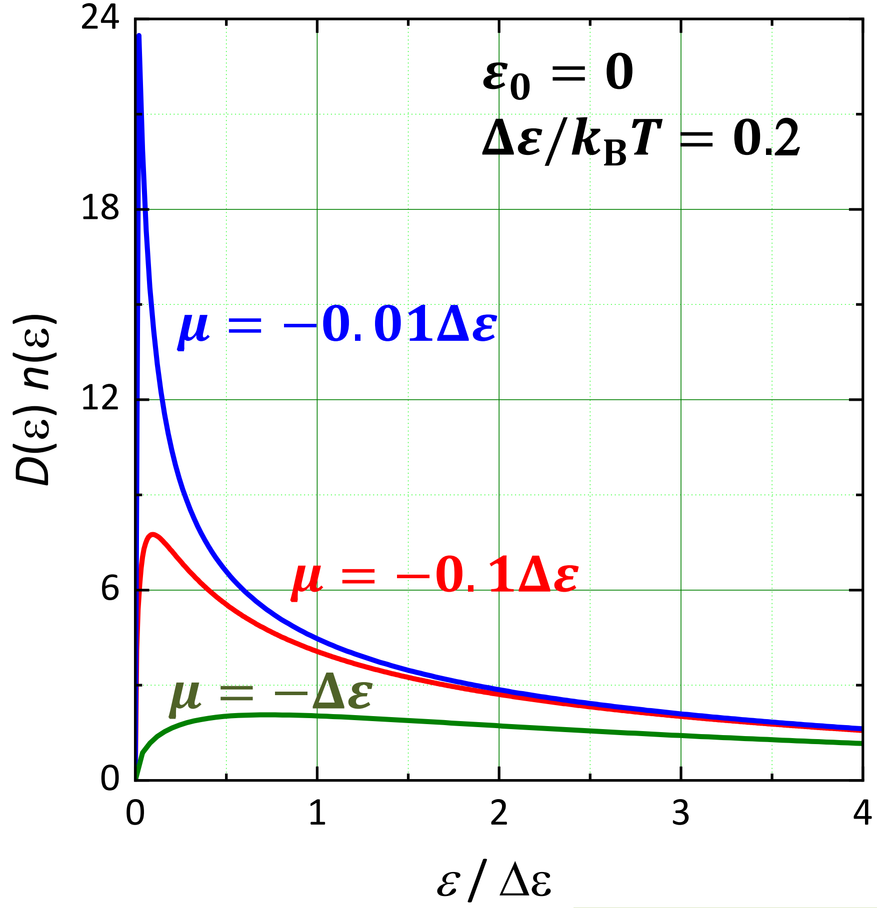
excitations

condensate

I.2.2 Ideal Bose Gas



decrease T at $\mu = const.$



decrease μ at $T = const.$

we defined the Bose-Einstein temperature $T_{\rm BEC}$ as the temperature, where $N_{ex}/N = 1$:

$$T_{\rm BEC} = \frac{2\pi\hbar^2}{mk_{\rm B}}\left(\frac{n}{2.612}\right)^{2/3}$$

I.2.2 Ideal Bose Gas

- **thermodynamic properties of ideal Bose gas: inner energy *U***

free bosons:

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m}$$

$$D(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon}$$

$$U = \sum_k \varepsilon_k\, n_k \quad \Rightarrow \quad U = \int D(\varepsilon)\varepsilon\, d\varepsilon = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int \frac{\varepsilon^{3/2}}{\exp\left(\frac{\varepsilon-\mu}{k_\mathrm{B}T}\right)-1}\, d\varepsilon$$

$$U = \frac{V}{4\pi^2} k_\mathrm{B}T \left(\frac{2mk_\mathrm{B}T}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\gamma \mathrm{e}^{-x}}{1-\gamma\, \mathrm{e}^{-x}} x^{3/2}\, dx$$

with $x = \varepsilon/k_\mathrm{B}T = \beta\varepsilon$ and ***fugacity*** $\gamma = \exp\left(\frac{\mu}{k_\mathrm{B}T}\right)$

we use again $\frac{\gamma \mathrm{e}^{-x}}{1-\gamma\, \mathrm{e}^{-x}} = \gamma \mathrm{e}^{-x}(1+\gamma \mathrm{e}^{-x} + \gamma^2 \mathrm{e}^{-2x} + \cdots) = \sum_{p=1}^\infty \gamma^p \mathrm{e}^{-px}$ and $\int_0^\infty \mathrm{e}^{-px} x^{3/2}\, dx = \frac{1}{p^{5/2}} \underbrace{\int_0^\infty \mathrm{e}^{-y} y^{3/2}\, dy}_{\Gamma(t)=\int_0^\infty \mathrm{e}^{-y} y^{t-1}\, dy} = \frac{1}{p^{5/2}} \underbrace{\frac{3\sqrt{\pi}}{4}}_{\Gamma(5/2)}$

$$\Rightarrow \int_0^\infty \frac{\gamma \mathrm{e}^{-x}}{1-\gamma\, \mathrm{e}^{-x}} x^{3/2}\, dx = \underbrace{\frac{3\sqrt{\pi}}{4}}_{\Gamma(5/2)} \underbrace{\sum_{p=1}^\infty \frac{\gamma^p}{p^{5/2}}}_{g_{5/2}(\gamma)} = \Gamma(5/2) g_{5/2}(\gamma) \quad \Rightarrow U = V\, \frac{3}{2} k_\mathrm{B}T \underbrace{\left(\frac{mk_\mathrm{B}T}{2\pi\hbar^2}\right)^{3/2}}_{1/\lambda_T^3} g_{5/2}(\gamma)$$

with $V = N\lambda_T^3 / g_{3/2}(\gamma)$, resp. $V = N\lambda_{T,\mathrm{BEC}}^3 / g_{3/2}(1)$ we obtain

$$U = \frac{3}{2} N k_\mathrm{B}T \frac{g_{5/2}(\gamma)}{g_{3/2}(\gamma)} \qquad U = \frac{3}{2} N k_\mathrm{B}T \left(\frac{T}{T_\mathrm{BEC}}\right)^{3/2} \frac{g_{5/2}(\gamma)}{g_{3/2}(1)}$$

I.2.2 Ideal Bose Gas

- thermodynamic properties of ideal Bose gas: **inner energy U**

α	$\Gamma(\alpha)$	$\zeta_\alpha(\gamma=1)$
1	1	∞
1.5	$\sqrt{\pi}/2 = 0.886$	2.612
2	1	$\pi^2/6 = 1.645$
2.5	$3\sqrt{\pi}/4 = 1.329$	1.341
3	2	1.202
3.5	$15\sqrt{\pi}/8 = 3.323$	1.127
4	6	$\pi^4/90 = 1.082$

Γ = Gamma function
ζ = Riemann Zeta function

i. $T > T_{\mathrm{BEC}}$:

$$U(T) = \frac{3}{2} N k_{\mathrm{B}} T \frac{g_{5/2}(\gamma)}{g_{3/2}(\gamma)}$$

ii. $T \gg T_{\mathrm{BEC}}$: it is sufficient to use only the first term in the series evolution of g:

$$g_{5/2}(\gamma) = \sum_{p=1}^{\infty} \frac{\gamma^p}{p^{5/2}} \simeq \gamma; \qquad g_{3/2}(\gamma) = \sum_{p=1}^{\infty} \frac{\gamma^p}{p^{3/2}} \simeq \gamma \quad \text{since } \gamma \to 1$$

$$U(T) = \frac{3}{2} N k_{\mathrm{B}} T$$

result for gas of classical non-interacting particles
(quantum statistics does not play any role at high T)

iii. $T < T_{\mathrm{BEC}}$:

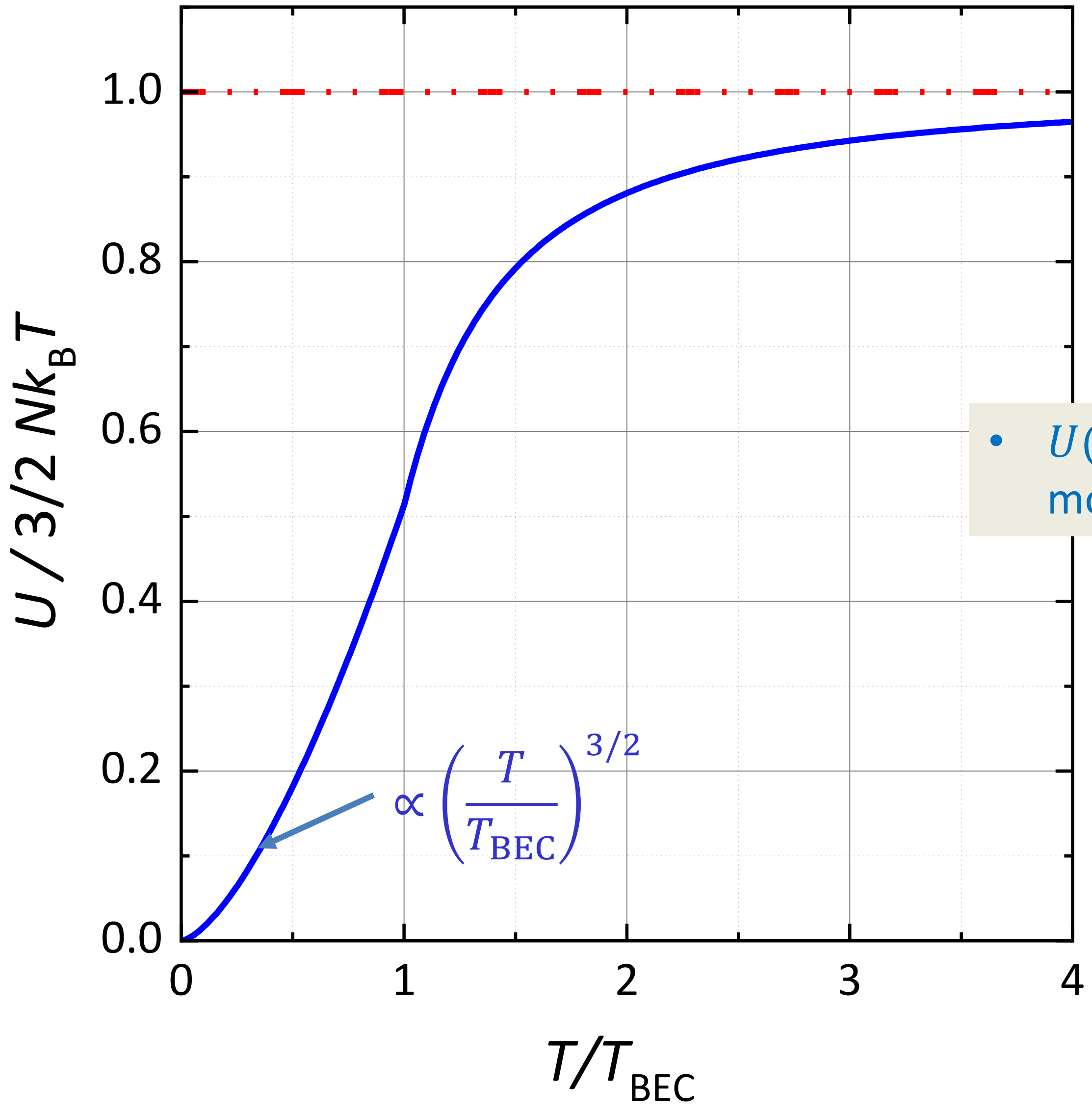
$$U(T) = \frac{3}{2} N k_{\mathrm{B}} T \frac{g_{5/2}(1)}{g_{3/2}(1)}$$

$$g_{5/2}(1) = \zeta\left(\frac{5}{2}\right) = 1.341, \qquad g_{3/2}(1) = \zeta\left(\frac{3}{2}\right) = 2.612$$

$$U(T) = \frac{3}{2} N_{ex} k_{\mathrm{B}} T \frac{g_{5/2}(1)}{g_{3/2}(1)} = \frac{3}{2} N \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2} k_{\mathrm{B}} T \frac{g_{5/2}(1)}{g_{3/2}(1)} = 0.770\; N \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2} k_{\mathrm{B}} T$$

we use $\frac{N_0}{N} = 1 - \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{\frac{3}{2}} \Rightarrow N_{ex} = N - N_0 = N \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2}$ particles in ground state $\varepsilon_0 = 0$ do not contribute!

I.2.2 Ideal Bose Gas



$U(T) = \frac{3}{2} N k_B T$ classical result

- $U(T)$ decreases towards lower T since more and more particles condense into the ground state ε_0

I.2.2 Ideal Bose Gas

- **thermodynamic properties of ideal Bose gas: specific heat** $C_V(T) = \left(\frac{\partial U}{\partial T}\right)_{V,N}$

i. $\boldsymbol{T > T_{\mathrm{BEC}}}$: $\mu < 0, \gamma < 1$, $g_{5/2}(\gamma)$ and $g_{3/2}(\gamma)$ are functions of T and $n = N/V$

$$U(T) = \frac{3}{2} N k_{\mathrm{B}} T \frac{g_{5/2}(\gamma)}{g_{3/2}(\gamma)}$$

$$g_{3/2}(1) = \zeta(3/2)$$

with $V = N\lambda_T^3/\, g_{3/2}(\gamma)$, resp. $V = N\lambda_{T,\mathrm{BEC}}^3/\, g_{3/2}(1)$ we obtain

$$U(T) = \frac{3}{2} N k_{\mathrm{B}} T \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2} \frac{g_{5/2}(\gamma)}{g_{3/2}(1)} = \frac{3}{2} N k_{\mathrm{B}} T \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2} \frac{g_{5/2}(\gamma)}{\zeta(3/2)}$$

$$C_V(T) = \left(\frac{\partial U}{\partial T}\right)_{V,N} = \frac{15}{4} N k_{\mathrm{B}} \frac{g_{5/2}(\gamma)}{\zeta(3/2)} \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2} + \frac{3}{2} N k_{\mathrm{B}} \frac{T}{\zeta(3/2)} \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2} \underbrace{\frac{\partial g_{5/2}(\gamma)}{\partial \gamma}}_{\gamma^{-1} g_{3/2}(\gamma)} \left(\frac{\partial \gamma}{\partial T}\right)_N$$

to derive $\left(\frac{\partial \gamma}{\partial T}\right)_N$ we use $\lambda_T^3 \cdot \frac{N}{V} = \lambda_T^3 \cdot n = g_{3/2}(\gamma) \quad \Rightarrow 1 = \frac{g_{3/2}(\gamma)}{\lambda_T^3 \cdot n} = \frac{g_{3/2}(\gamma)}{\zeta(3/2)} \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2}$

temperature derivative yields: $0 = \frac{3}{2T} \frac{g_{3/2}(\gamma)}{\zeta(3/2)} \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2} + \frac{1}{\zeta(3/2)} \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2} \underbrace{\frac{\partial g_{3/2}(\gamma)}{\partial \gamma}}_{\gamma^{-1} g_{1/2}(\gamma)} \left(\frac{\partial \gamma}{\partial T}\right)_N = \frac{3}{2T} g_{3/2}(\gamma) + \gamma^{-1} g_{1/2}(\gamma) \left(\frac{\partial \gamma}{\partial T}\right)_N$

$$\rightarrow \left(\frac{\partial \gamma}{\partial T}\right)_N = -\frac{3\gamma}{2T} \frac{g_{3/2}(\gamma)}{g_{1/2}(\gamma)}$$

I.2.2 Ideal Bose Gas

- **thermodynamic properties of ideal Bose gas: specific heat** $C_V(T) = \left(\frac{\partial U}{\partial T}\right)_{V,N}$

i. $T > T_{\mathrm{BEC}}$:

$$C_V(T) = \left(\frac{\partial U}{\partial T}\right)_{V,N} = \frac{15}{4}Nk_{\mathrm{B}}\,\frac{g_{5/2}(\gamma)}{\zeta(3/2)}\left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2} + \frac{3}{2}Nk_{\mathrm{B}}\,\frac{T}{\zeta(3/2)}\left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2}\underbrace{\frac{\partial g_{5/2}(\gamma)}{\partial \gamma}}_{\gamma^{-1}g_{3/2}(\gamma)}\left(\frac{\partial \gamma}{\partial T}\right)_N$$

$$\left(\frac{\partial \gamma}{\partial T}\right)_N = -\frac{3\gamma}{2T}\,\frac{g_{3/2}(\gamma)}{g_{1/2}(\gamma)}$$

$$C_V(T) = \frac{15}{4}Nk_{\mathrm{B}}\,\frac{g_{5/2}(\gamma)}{\zeta(3/2)}\left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2} + \frac{3}{2}Nk_{\mathrm{B}}\,\frac{T}{\zeta(3/2)}\left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2}\frac{g_{3/2}(\gamma)}{\gamma}\left(-\frac{3\gamma}{2T}\,\frac{g_{3/2}(\gamma)}{g_{1/2}(\gamma)}\right)$$

$$C_V(T) = Nk_{\mathrm{B}}\,\frac{g_{5/2}(\gamma)}{\zeta(3/2)}\left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2}\left[\frac{15}{4} - \frac{9}{4}\,\frac{g_{3/2}(\gamma)}{g_{5/2}(\gamma)}\,\frac{g_{3/2}(\gamma)}{g_{1/2}(\gamma)}\right]$$

$$C_V(T) = Nk_{\mathrm{B}}\left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2}\left[\frac{15}{4}\,\frac{g_{5/2}(\gamma)}{\zeta(3/2)} - \frac{9}{4}\,\underbrace{\frac{g_{3/2}(\gamma)}{\zeta(3/2)}}_{\left(\frac{T_{\mathrm{BEC}}}{T}\right)^{3/2}}\,\frac{g_{3/2}(\gamma)}{g_{1/2}(\gamma)}\right]$$

we use $\lambda_T^3 \cdot \frac{N}{V} = \lambda_T^3 \cdot n = g_{3/2}(\gamma) \quad \Rightarrow 1 = \frac{g_{3/2}(\gamma)}{\lambda_T^3 \cdot n} = \frac{g_{3/2}(\gamma)}{\zeta(3/2)}\left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2}$

$$C_V(T) = \frac{15}{4}Nk_{\mathrm{B}}\left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2}\frac{g_{5/2}(\gamma)}{\zeta(3/2)} - \frac{9}{4}Nk_{\mathrm{B}}\,\frac{g_{3/2}(\gamma)}{g_{1/2}(\gamma)}$$

I.2.2 Ideal Bose Gas

- **thermodynamic properties of ideal Bose gas: specific heat** $C_V(T) = \left(\frac{\partial U}{\partial T}\right)_{V,N}$

ii. $\boldsymbol{T \gg T_{\rm BEC}}$:

$$C_V(T) = \frac{3}{2} N k_{\rm B}, \qquad C_V^{\rm mol}(T) = \frac{3}{2} N_A k_{\rm B} = \frac{3}{2} R$$

result for classical gas of non-interacting particles

(quantum statistics does not play any role at high T)

$$U(T) = \frac{3}{2} N k_{\rm B} T$$

iii. $\boldsymbol{T < T_{\rm BEC}}$:

$$C_V(T) = \frac{15}{4} N k_{\rm B} \frac{\zeta_{5/2}(1)}{\zeta_{3/2}(1)} \left(\frac{T}{T_{\rm BEC}}\right)^{3/2} = 1.925 \cdot N k_{\rm B} \left(\frac{T}{T_{\rm BEC}}\right)^{3/2}$$

$$U(T) = \frac{3}{2} N \left(\frac{T}{T_{\rm BEC}}\right)^{3/2} k_{\rm B} T \; \frac{g_{5/2}(1)}{g_{3/2}(1)}$$

$$g_{5/2}(1) = \zeta(5/2) \qquad g_{3/2}(1) = \zeta(3/2)$$

interpretation: all $N_{\rm ex} = N\left(\frac{T}{T_{\rm BEC}}\right)^{3/2}$ non-condensed particles contribute $O(k_{\rm B})$ to the specific heat

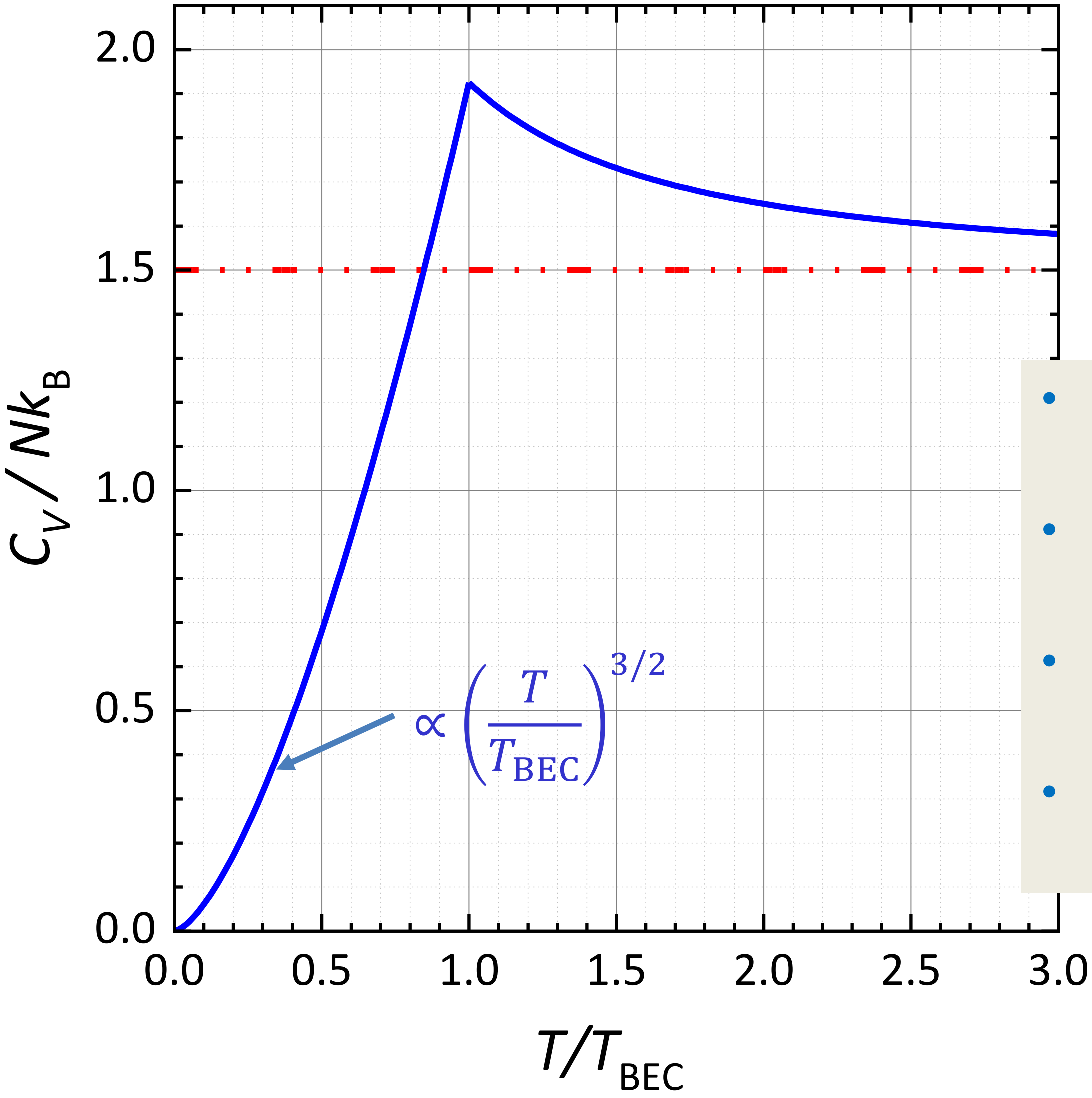
$$C_V(T) = \frac{5}{2}\frac{U(T)}{T} \quad \Longleftrightarrow \quad S(T) = \frac{5}{3}\frac{U(T)}{T} \quad \Longleftrightarrow \quad F(T) = -\frac{2}{3} U(T)$$

$C_V = T\, \partial S / \partial T$

$F = U - TS$

I.2.2 Ideal Bose Gas

- **thermodynamic properties of ideal Bose gas: specific heat** $C_V(T) = \left(\frac{\partial U}{\partial T}\right)_{V,N}$



$C_V(T) = \frac{3}{2} N k_B$ classical result

- $C_V(T)$ decreases towards lower T since more and more particles condense into the ground state ε_0
- temperature T_{BEC} represents phase transition temperature
- $C_V(T)$ shows cusp at T_{BEC}, indicating that the free energy is not analytical at T_{BEC}
- phase transition is usually interpreted as 2nd order phase transition

I.2.2 Ideal Bose Gas

$$F = U - TS$$
$$dF = TdS - pdV + \mu dN - TdS - SdT$$
$$dF = -pdV + \mu dN - SdT$$
$$\left(\frac{\partial F}{\partial V}\right)_{N,T} = -p, \left(\frac{\partial F}{\partial N}\right)_{V,T} = \mu, \left(\frac{\partial F}{\partial T}\right)_{N,V} = -S$$

- **thermodynamic properties of ideal Bose gas: pressure** $p = \left(\frac{\partial F}{\partial V}\right)_{N,T}$
  - we discuss only the case $\boldsymbol{T < T_{\mathrm{BEC}}}$:

$$p = -\left(\frac{\partial F}{\partial V}\right)_{N,T} = \frac{2}{3}\left(\frac{\partial U}{\partial V}\right)_{N,T}$$

$$U(T) = \frac{3}{2} N k_\mathrm{B} T \frac{g_{5/2}(1)}{g_{3/2}(1)}$$

$$F(T) = -\frac{2}{3} U(T)$$

with $V = N\lambda^3_{T,\mathrm{BEC}}/g_{3/2}(1)$ we obtain

$$U = \frac{3}{2} N k_\mathrm{B} T \frac{g_{5/2}(1)}{g_{3/2}(1)} = \frac{3}{2}\frac{V}{\lambda^3_{T,\mathrm{BEC}}} g_{3/2}(1)\, k_\mathrm{B} T \frac{g_{5/2}(1)}{g_{3/2}(1)} = \frac{3}{2}\frac{V}{\lambda^3_{T,\mathrm{BEC}}} k_\mathrm{B} T\, g_{5/2}(1)$$

$$p = \frac{2}{3}\left(\frac{\partial U}{\partial V}\right)_{N,T} = \frac{k_\mathrm{B} T}{\lambda_T^3} g_{5/2}(1)$$

pressure of the gas does not depend on V in the BEC regime
➔ compressibility of the BEC phase is infinite

this pathological behavior is removed by the inclusion of finite interactions

I.2.2 Ideal Bose Gas

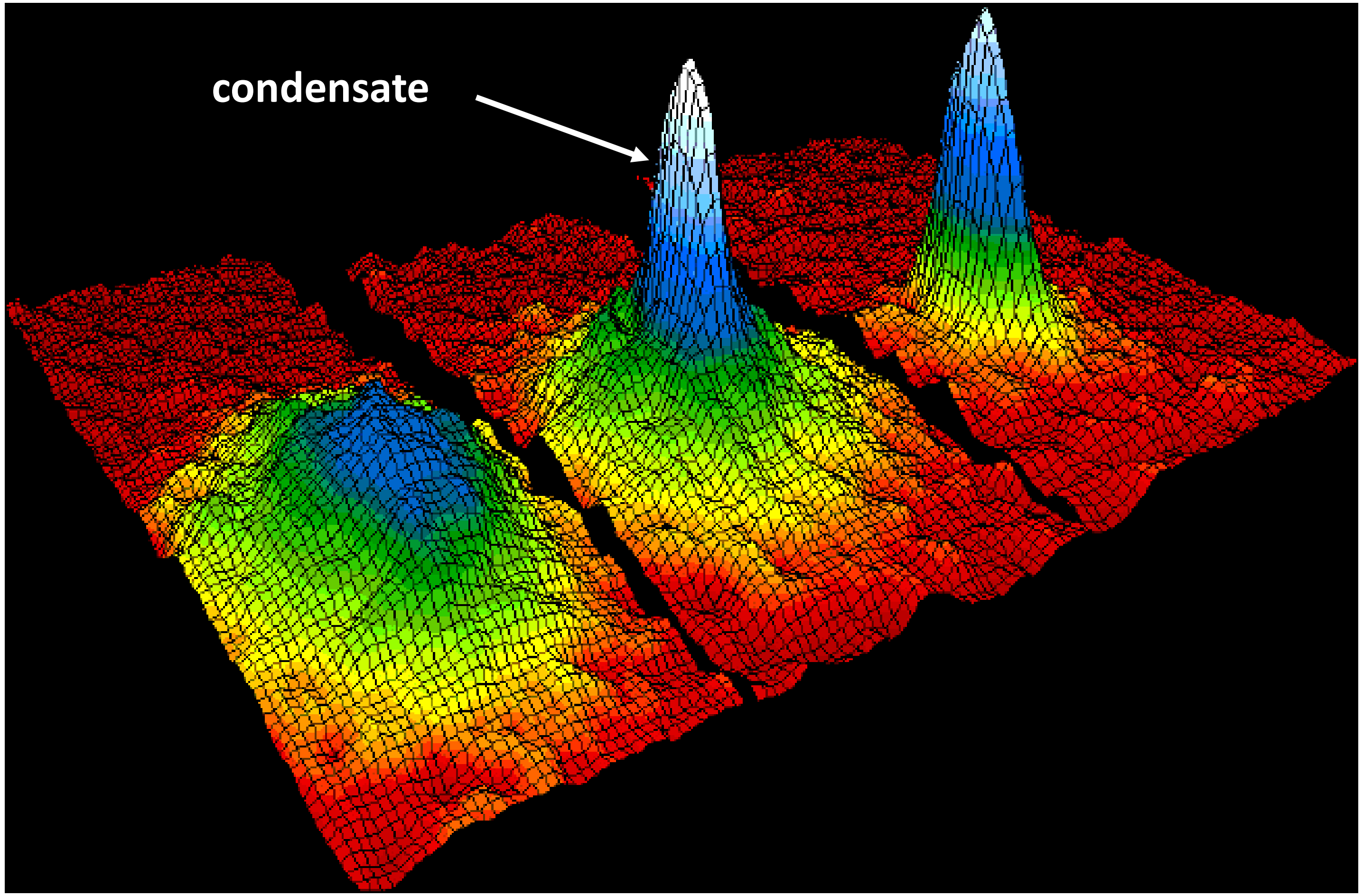
- **experimental realization with ultra-cold atomic gases**
  - first realization in 1995 with trapped alkali metal atoms, e.g. ^{87}Rb (Eric A. Cornell, Carl E. Wieman, Wolfgang Ketterle)
  - typical densities: $n \simeq 10^{11} - 10^{15}\ \mathrm{cm}^{-3}$ ➔ $T_\mathrm{BEC} \simeq 10\ \mathrm{nK}\ - 1\ \mu\mathrm{K}$ due to large atomics mass (compared to ^{4}He)
  - ^{87}Rb: electron configuration: (Kr)$5s^1$ $\Rightarrow J = \frac{1}{2}$, nuclear spin $I = \frac{3}{2}$ $\Rightarrow$ total spin is even → **boson**
    similar behavior: ^{7}Li, ^{23}Na
  - **important**: alkali metal atoms interact (short-range repulsion, attractive Van der Waals interaction)
    - ➢ atoms do not cluster since most interactions are elastic two-particle interactions:
      → particles cannot bind since they cannot get rid of their kinetic energy
    - ➢ rate for three-particle interactions is very small
    - ➢ description of pairwise interactions: $V(\mathbf{r}_1 - \mathbf{r}_2) \simeq g\,\delta(\mathbf{r}_1 - \mathbf{r}_2)$

g = strength of the scatterer

$a_s = \frac{mg}{4\pi\hbar^2}$ = s-wave scattering length

(interactions and effect of trapping potential will be discussed later)

I.2.2 Ideal Bose Gas



Anderson, M. H.; Ensher, J. R.; Matthews, M. R.; Wieman, C. E.; Cornell, E. A.
Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor
Science **269,** 198-201 (1995).

velocity distribution for a gas of rubidium atoms, $n = 2.5 \times 10^{12}$ cm^{-3}

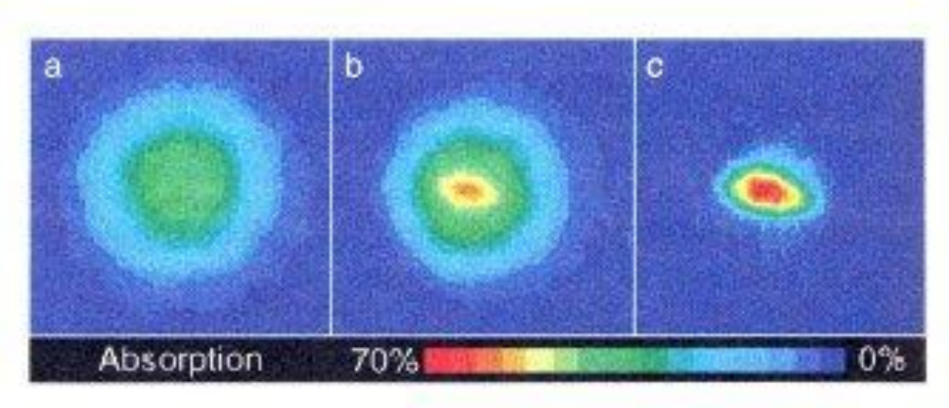
the measured distribution confirmed the existence of a Bose-Einstein condensate.

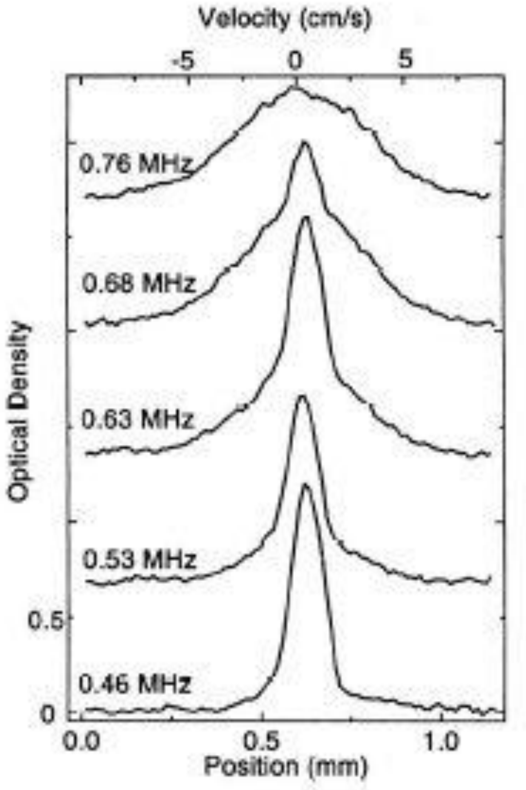
Left: $T > T_{\mathrm{BEC}} \simeq 170$ nK
Center: $T < T_{\mathrm{BEC}}$
Right: nearly pure condensate

experimental technique:

- switch-off the trap potential and let the condensate freely expand
- image the position of the atoms after a specific time by absorption of laser light by the atom cloud
- measured spatial distribution is proportional to velocity distribution in condensate

I.2.2 Ideal Bose Gas





velocity distribution for a gas of sodium atoms, $n \sim 10^{14}$ cm^{-3}

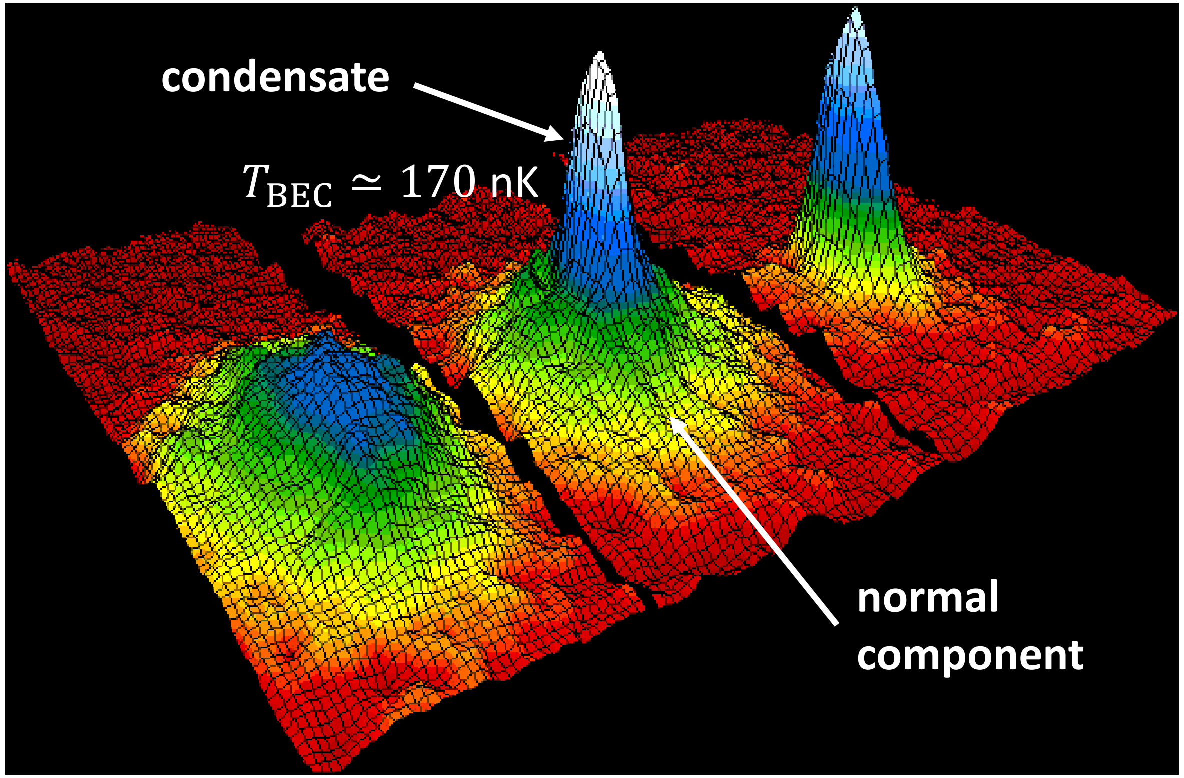
the measured distribution confirmed the existence of a Bose-Einstein condensate.

Left: $T > T_{\mathrm{BEC}} \simeq 2$ µK
Center: $T < T_{\mathrm{BEC}}$
Right: after further evaporation cooling

K. B. Davis, M. -O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle
Bose-Einstein Condensation in a Gas of Sodium Atoms
Phys. Rev. Lett. **75**, 3969 (1995)

I.2.2 Ideal Bose Gas

- **what determines the width of the central peak associated with the condensate**



Anderson, M. H.; Ensher, J. R.; Matthews, M. R.; Wieman, C. E.; Cornell, E. A.
Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor
Science **269,** 198-201 (1995).

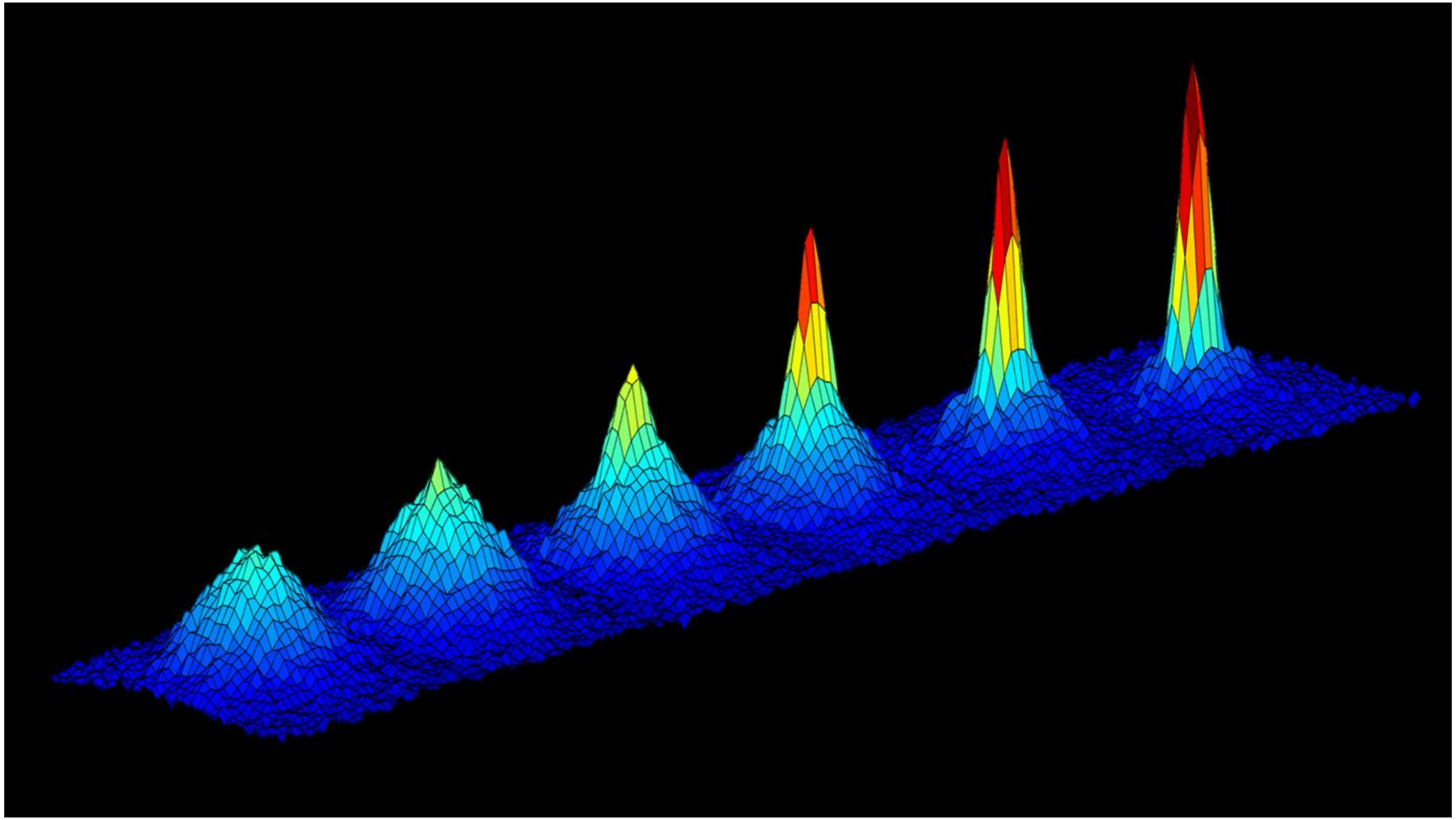
expected:

- delta function distribution since condensate atoms occupy the same state

observed:

- finite width Δx of the condensate due to trapping of condensate in trapping potential of finite width Δx
- resulting finite momentum distribution:
  $\Delta p \simeq \hbar/\Delta x \quad \Rightarrow \quad \Delta v \simeq \hbar/m\Delta x$
- width of the normal component:
  $\frac{1}{2} m \langle v_x^2 \rangle = \frac{1}{2} k_{\mathrm{B}} T \;\Rightarrow\; \Delta v \simeq \sqrt{k_{\mathrm{B}} T/m}$

I.2.2 Ideal Bose Gas



Space Station Experiment (2018)

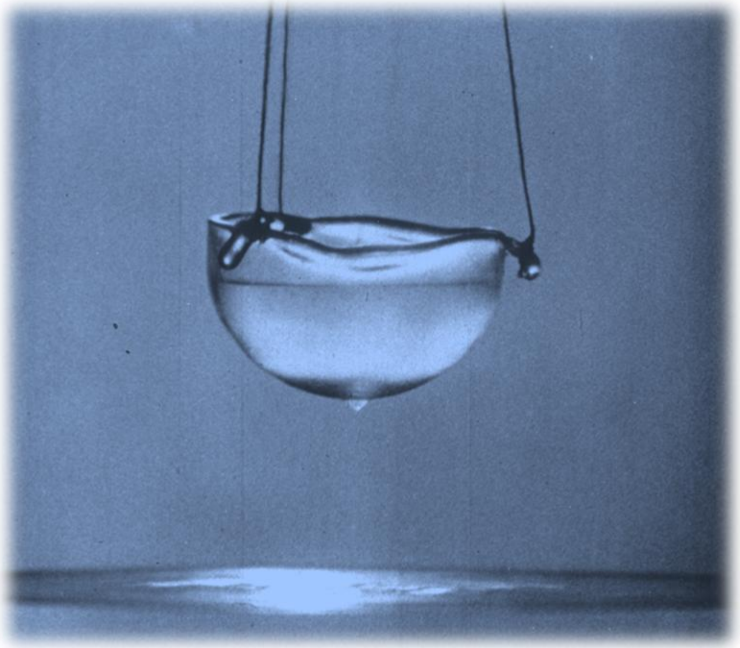
Changing density of a cloud of rubidium atoms as it is cooled to lower and lower temperatures (going from left to right) approaching absolute zero. The emergence of a sharp peak in the later graphs confirms the formation of a Bose-Einstein condensate occurring here at about $T_{\mathrm{BEC}} = 130$ nK

I.2.2 Ideal Bose Gas

- **concluding remarks**

  - until today, Bose-Einstein condensates have been realized with many different atoms
  - ideal model system, as most parameters can be well controlled and changed
  - it became evident that the finite interactions between the particles has an important impact
    - ➢ ideal **non-interacting Bose gas is not a superfluid** that can show frictionless flow
      - ➔ no persistent current of the condensate
      - ➔ no quantized vortices
    - ➢ we will see later that this is due to the fact that the so-called **Landau critical velocity** of the ideal BEC is zero
    - ➢ **finite Landau critical velocity is obtained only for interacting Bose gas**

Superconductivity and Low Temperature Physics II



Lecture No. 4

R. Gross
© Walther-Meißner-Institut

Contents Part I: Quantum Gases & Liquids

Contents:

I.2.3 Bose Gas with Interactions

- **How to treat the finite interaction between particles ?**

  - description by **pairwise interaction**: $V(\mathbf{r}_1 - \mathbf{r}_2) \simeq g\,\delta(\mathbf{r}_1 - \mathbf{r}_2)$

    $g =$ strength of the interaction

    $a_s = \frac{mg}{4\pi\hbar^2} =$ s-wave scattering length

  - modeling of pairwise interactions by **average „mean field“ potential** seen by every particle due to interaction with all others

    $$V_{\text{eff}}(\mathbf{r}) \simeq g\,n(\mathbf{r})$$

  - in addition, particles may see finite **trapping potential** $V_{\text{trap}}(\mathbf{r})$ of e.g. a magnetic trap

  - Schrödinger equation for particles in trap:

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{trap}}(\mathbf{r}) + V_{\text{eff}}(\mathbf{r})\right)\psi_k(\mathbf{r}) = \varepsilon_k \psi_k(\mathbf{r})$$

important:
Schrödinger equation is nonlinear since $V_{\text{eff}}(\mathbf{r})$ depends on $n(\mathbf{r})$ and hence on wave function $\psi_k(\mathbf{r})$ itself (see below)
➔ **Gross-Pitaevskii equation**

particle density and number:

$$n(\mathbf{r}) = \sum_k \left[\exp\left(\frac{(\varepsilon_k - \mu)}{k_B T}\right) - 1\right]^{-1} |\psi_k(\mathbf{r})|^2$$

$$N = \int n(\mathbf{r}) d^3r = \sum_k \left[\exp\left(\frac{(\varepsilon_k - \mu)}{k_B T}\right) - 1\right]^{-1}$$

I.2.3 Bose Gas with Interactions

- **Now: more detailed discussion of finite interactions between particles**

assumptions: (i) ***weak interaction*** (allows perturbative treatment)
(ii) ***repulsive interaction*** (attractive interaction may lead to instability)

we will show: **repulsive interaction stabilizes the Bose-Einstein condensate**
(counter-intuitive, as a repulsive interaction is expected to push particles out of a single-particle ground state)

- **we start with a simple Gedanken experiment:**
  we consider two bosons and show that in the presence of a repulsive interaction, it is more favorable to put them into the same eigenstate

I.2.3 Bose Gas with Interactions

- **We first consider the case without any interaction:**
  - bosons = particles in a box described by normalized wave functions $\phi(\mathbf{r}) = \exp(\imath\mathbf{k}\cdot\mathbf{r})$, $\tilde{\phi}(\mathbf{r}) = \exp\left(\imath\tilde{\mathbf{k}}\cdot\mathbf{r}\right)$,

➢ **two bosons in the same eigenstate**
$(\mathbf{k} = \tilde{\mathbf{k}}, \phi = \tilde{\phi}, |\phi(\mathbf{r}_1)|^2 = |\phi(\mathbf{r}_2)|^2 = 1)$

$$\psi(\mathbf{r}_1,\mathbf{r}_2) = \phi(\mathbf{r}_1)\phi(\mathbf{r}_2) = \exp\left(\imath\mathbf{k}\cdot(\mathbf{r}_1+\mathbf{r}_2)\right)$$

symmetric wave function: $\psi(\mathbf{r}_1,\mathbf{r}_2) = \psi(\mathbf{r}_2,\mathbf{r}_1)$

$$\int |\psi(\mathbf{r}_1,\mathbf{r}_2)|^2 \, \mathrm{d}^3 r_1 \mathrm{d}^3 r_2 = 1$$

➢ **two bosons in different eigenstates**:

$(\mathbf{k} \neq \tilde{\mathbf{k}}, \phi \neq \tilde{\phi}, |\phi(\mathbf{r}_1)|^2 = \left|\tilde{\phi}(\mathbf{r}_1)\right|^2 = 1)$

$$\psi(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{\sqrt{2}}\left[\phi(\mathbf{r}_1)\tilde{\phi}(\mathbf{r}_2) + \phi(\mathbf{r}_2)\tilde{\phi}(\mathbf{r}_1)\right]$$

symmetric superposition of the two possibilities $\phi(\mathbf{r}_1)\tilde{\phi}(\mathbf{r}_2)$ and $\phi(\mathbf{r}_2)\tilde{\phi}(\mathbf{r_2})$

$$\int |\psi(\mathbf{r}_1,\mathbf{r}_2)|^2 \, \mathrm{d}^3 r_1 \mathrm{d}^3 r_2 = 1$$

I.2.3 Bose Gas with Interactions

- **what happens if we add a repulsive potential $V(\mathbf{r}_1 - \mathbf{r}_2)$ between the two bosons?**

  - **energy change for the two bosons in the same eigenstate**: $\psi(\mathbf{r}_1, \mathbf{r}_2) = \phi(\mathbf{r}_1)\phi(\mathbf{r}_2) = \exp\big(\imath\mathbf{k}\cdot(\mathbf{r}_1 + \mathbf{r}_2)\big)$

    $(\mathbf{k} = \tilde{\mathbf{k}}, \phi = \tilde{\phi}, |\phi(\mathbf{r}_1)|^2 = |\phi(\mathbf{r}_2)|^2 = 1)$

$$\langle E_{\mathrm{int}}\rangle = \int \mathrm{d}^3r_1\mathrm{d}^3r_2\, |\psi(\mathbf{r}_1, \mathbf{r}_2)|^2 V(\mathbf{r}_1 - \mathbf{r}_2) = \int \mathrm{d}^3r_1\mathrm{d}^3r_2\, V(\mathbf{r}_1 - \mathbf{r}_2) = \int \mathrm{d}^3r\, V(\mathbf{r}) = E_0 > 0$$

we use $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ and $\mathrm{d}^3r_1\mathrm{d}^3r_2 = \mathrm{d}^3R\mathrm{d}^3r$

note that $\int \mathrm{d}^3r\, V(\mathbf{r}) = E_0 > 0$ for a purely repulsive potential

I.2.3 Bose Gas with Interactions

- **energy change for the two bosons in different eigenstates**: $\psi(\mathbf{r}_1,\mathbf{r}_2)=\frac{1}{\sqrt{2}}\left[\phi(\mathbf{r}_1)\tilde{\phi}(\mathbf{r}_2)+\phi(\mathbf{r}_2)\tilde{\phi}(\mathbf{r}_1)\right]$

$$\delta E=\int \mathrm{d}^3r_1\mathrm{d}^3r_2\ |\psi(\mathbf{r}_1,\mathbf{r}_2)|^2 V(\mathbf{r}_1-\mathbf{r}_2) \qquad (\mathbf{k}\neq\tilde{\mathbf{k}},\phi\neq\tilde{\phi},|\phi(\mathbf{r}_1)|^2=|\tilde{\phi}(\mathbf{r}_1)|^2=1)$$

$$=\frac{1}{2}\int \mathrm{d}^3r_1\mathrm{d}^3r_2\ \left[|\phi(\mathbf{r}_1)|^2|\tilde{\phi}(\mathbf{r}_2)|^2+|\phi(\mathbf{r}_2)|^2|\tilde{\phi}(\mathbf{r}_1)|^2+\phi(\mathbf{r}_1)\tilde{\phi}(\mathbf{r}_2)\phi^*(\mathbf{r}_2)\tilde{\phi}^*(\mathbf{r}_1)+c.c.\right]V(\mathbf{r}_1-\mathbf{r}_2)$$

- since $|\phi(\mathbf{r}_1)|^2=1$ and $|\phi(\mathbf{r}_2)|^2=1$, the first two terms in [] yield the contribution $\int \mathrm{d}^3r\ V(\mathbf{r})=E_0$
- the last term in [] comes from the fact that the wave function must be symmetric with respect to the exchange of particles (bosons) since they are indistinguishable ➔ results in additional **exchange energy**
- since $\phi(\mathbf{r}_1)\tilde{\phi}(\mathbf{r}_2)\phi^*(\mathbf{r}_2)\tilde{\phi}^*(\mathbf{r}_1)=\exp\left[\imath(\mathbf{k}-\tilde{\mathbf{k}})\cdot(\mathbf{r}_1-\mathbf{r}_2)\right]$, last term in [] results in an energy change $V_{\mathbf{k}-\tilde{\mathbf{k}}}/2$ with

$$V_{\mathbf{k}-\tilde{\mathbf{k}}}=\int \mathrm{d}^3r\ \mathrm{e}^{\imath(\mathbf{k}-\tilde{\mathbf{k}})\cdot\mathbf{r}}\ V(\mathbf{r})$$

Fourier transform of $V(\mathbf{r})$
(we assume $V(\mathbf{r})=V(-\mathbf{r})$ so that the FT is real: $V_\mathbf{k}^*=V_\mathbf{k}$ ➔ the complex conjugate yields the same contribution $V_{\mathbf{k}-\tilde{\mathbf{k}}}/2$)

➔ $\langle E_\mathrm{int}\rangle=E_0+V_{\mathbf{k}-\tilde{\mathbf{k}}}$ (if we assume two bosons with different wave numbers)

➔ $\langle E_\mathrm{int}\rangle=E_0$ (if we assume two bosons with the same wave number)

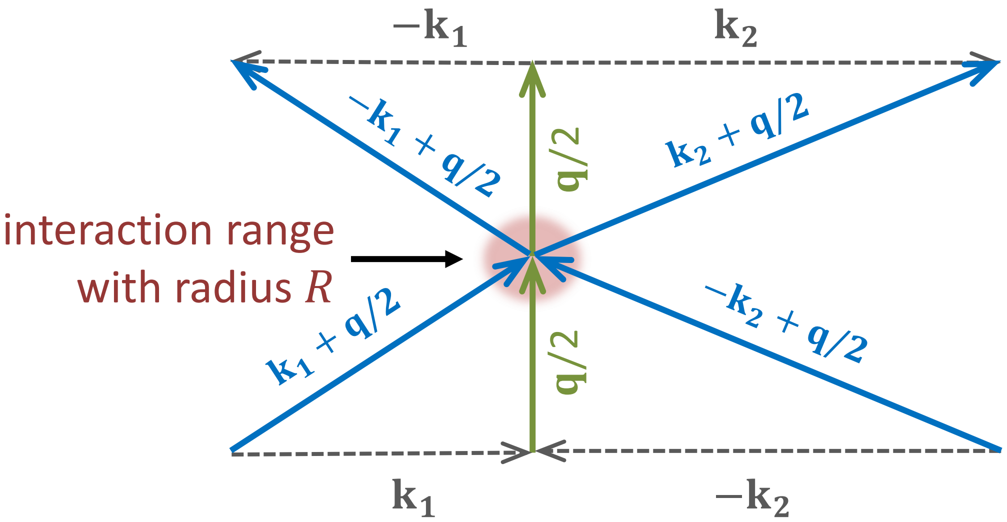
due to the bosonic nature of the particles, it is energetically favorable to put them into the same quantum state in the presence of a repulsive interaction potential

I.2.3 Bose Gas with Interactions

- **description of the finite interaction between particles within second quantization formalism**

$$\mathcal{H}_{\text{int}} = \frac{V_0}{2} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}}^{N} \hat{b}^\dagger_{\mathbf{k}_2+\mathbf{q}/2} \hat{b}^\dagger_{-\mathbf{k}_1+\mathbf{q}/2} \hat{b}_{-\mathbf{k}_2+\mathbf{q}/2} \hat{b}_{\mathbf{k}_1+\mathbf{q}/2}$$

$\hat{b}^\dagger_{\mathbf{k}}, \hat{b}_{\mathbf{q}}$: creation and annihilation operators
$V_0 > 0$: repulsive interaction



terms in $\sum$ describe the following scattering process:

particle $|\mathbf{k}_1 + \mathbf{q}/2\rangle$ is destroyed and $|-\mathbf{k}_1 + \mathbf{q}/2\rangle$ created and
particle $|-\mathbf{k}_2 + \mathbf{q}/2\rangle$ is destroyed and $|\mathbf{k}_2 + \mathbf{q}/2\rangle$ created

I.2.3 Bose Gas with Interactions

- **continuum limit**

  - quite frequently, we take the continuum limit, where $\mathbf{r}$ lives not on a lattice, but in a continuum
  - in this case, it is convenient to replace the creation and annhilation operators $\hat{b}_{\mathbf{k}}^{\dagger}, \hat{b}_{\mathbf{q}}$ with the continuum field operators $\hat{\psi}^{\dagger}(\mathbf{r}), \hat{\psi}(\mathbf{r})$

$$\hat{b}_{\mathbf{k}}^{\dagger} = \frac{1}{\sqrt{V}} \int d\mathbf{r}\, e^{\imath \mathbf{k}\cdot\mathbf{r}}\, \hat{\Psi}^{\dagger}(\mathbf{r}) \qquad \left[\hat{\psi}(\mathbf{r}), \hat{\psi}^{\dagger}\left(\mathbf{r}'\right),\right] = \delta(\mathbf{r}-\mathbf{r}') \quad \text{(commutation relation)}$$

  - any orbital basis $\{\phi_{\mathbf{k}}\}$ can be written in terms of the continuum field operators as

$$\hat{b}_{\phi_{\mathbf{k}}}^{\dagger} = \int d\mathbf{r}\;\; \phi_{\mathbf{k}}(\mathbf{r})\, \hat{\Psi}^{\dagger}(\mathbf{r}) \qquad \hat{\psi}^{\dagger}(\mathbf{r}) = \sum_{\mathbf{k}} \langle \phi_{\mathbf{k}} | \mathbf{r} \rangle\, \hat{b}_{\phi_{\mathbf{k}}}^{\dagger} = \sum_{\mathbf{k}} \phi_{\mathbf{k}}^{\star}(\mathbf{r})\, \hat{b}_{\phi_{\mathbf{k}}}^{\dagger}$$

- **non-interacting BEC as a coherent state**

  - in BEC we put many bosons into the same orbital state Ψ_0
  - coherent state for many bosons in the same orbital

$$|\alpha\rangle = e^{\alpha \hat{b}^\dagger_{\Psi_0}} |0\rangle \qquad \text{with } \langle N_0\rangle = |\alpha|^2 \quad \text{(mean particle number)}$$

  - effect of field operator $\widehat{\Psi}(\mathbf{r})$ on coherent state

$$\hat{\psi}(\mathbf{r})\,|\alpha\rangle = \sum_{\mathbf{k}} \phi_{\mathbf{k}}\,(\mathbf{r})\;\hat{b}_{\phi_{\mathbf{k}}}|\alpha\rangle = \alpha\;\Psi_0(\mathbf{r})|\alpha\rangle$$

➔ using a coherent state turns the field operator $\widehat{\Psi}(\mathbf{r})$ into a scalar

$$\hat{\psi}(\mathbf{r}) \;\rightarrow\; \psi(\mathbf{r}) = \alpha\,\Psi_0(\mathbf{r}) = \sqrt{N_0}\;\Psi_0(\mathbf{r})$$

I.2.3 Bose Gas with Interactions

- **description of the finite interaction between particles within second quantization formalism**

$$\mathcal{H}_{\text{int}} = \frac{V_0}{2}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}}^{N} \hat{b}^{\dagger}_{\mathbf{k}_2+\mathbf{q}/2}\hat{b}^{\dagger}_{-\mathbf{k}_1+\mathbf{q}/2}\hat{b}_{-\mathbf{k}_2+\mathbf{q}/2}\hat{b}_{\mathbf{k}_1+\mathbf{q}/2}$$

we have made the following simplifying assumptions:

- we assume a ***local interaction*** $V(\mathbf{r}_1 - \mathbf{r}_2)$ → Fourier transform $V_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}}$ does not depend on $\mathbf{k}_1$ and $\mathbf{k}_2$: $V_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}} \simeq V_{\mathbf{q}}$
- since we are at very low T, the involved velocities and hence wavevectors $\mathbf{q}$ are very small
- if the interaction range R of the potential is small, $qR \ll 1$, then the FT $V_{\mathbf{q}}$ is essentially independent of $\mathbf{q}$ and we can use the approximation $V_{\mathbf{q}} = V_0$ (valid for short-range repulsion)
- as above, we assume two bosons with the same wave number $\mathbf{k}_1 = \mathbf{k}_2$ and set it equal to zero (ground state)
- we describe the two states of the interacting bosons with the bosonic operators $\hat{b}$ and $\hat{\hat{b}}$

$$\mathcal{H}_{\text{int}} = \frac{V_0}{2}\left[\hat{b}^{\dagger}\,\hat{b}^{\dagger}\hat{b}\hat{b} + \hat{b}^{\dagger}\,\hat{\hat{b}}^{\dagger}\hat{b}\hat{\hat{b}} + \hat{\hat{b}}^{\dagger}\,\hat{b}^{\dagger}\hat{b}\hat{\hat{b}} + \hat{b}^{\dagger}\,\hat{\hat{b}}^{\dagger}\hat{\hat{b}}\hat{b} + \hat{\hat{b}}^{\dagger}\,\hat{b}^{\dagger}\hat{\hat{b}}\hat{b}\right]$$

we do not include terms like $\hat{\hat{b}}^{\dagger}\,\hat{\hat{b}}^{\dagger}\hat{\hat{b}}\hat{\hat{b}}$, as we do not consider states occupied with two $\hat{\hat{b}}$ bosons

I.2.3 Bose Gas with Interactions

- **relevant relations**
  - both states occupied with single boson: $|1{,}1\rangle = \hat{b}^\dagger \hat{\hat{b}}^\dagger\ |0\rangle$ with $|0\rangle$ = normalized vacuum state
  - $\langle 1{,}1|1{,}1\rangle = \left\langle 0\middle|\hat{\hat{b}}\hat{b}\ \hat{b}^\dagger\hat{\hat{b}}^\dagger\middle|0\right\rangle = 1,$ as $\hat{\hat{b}}$ and $\hat{b}$ commute and $\hat{b}\hat{b}^\dagger\ |0\rangle = 0$
  - one state occupied with two bosons and the other is empty: $|2{,}0\rangle = \hat{b}^\dagger\hat{b}^\dagger|0\rangle = \left(\hat{b}^\dagger\right)^2|0\rangle$
  - $\langle 2{,}0|2{,}0\rangle = \langle 0|bb\hat{b}^\dagger\hat{b}^\dagger|0\rangle = \langle 0|b\left(1+\hat{b}^\dagger b\right)\hat{b}^\dagger|0\rangle = 2,$ as $\left[b,\hat{b}^\dagger\right] = 1$ and $b\hat{b}^\dagger|0\rangle = 0$

    $\Rightarrow$ $|2{,}0\rangle = \frac{1}{\sqrt{2}}\left(\hat{b}^\dagger\right)^2|0\rangle$ is the correctly normalized state

– **interaction energy** for the states $|1{,}1\rangle$ and $|2{,}0\rangle$:

$|2{,}0\rangle$: only the first term in $\frac{V_0}{2}\left[\hat{b}^\dagger\ \hat{b}^\dagger\hat{b}\hat{b} + \hat{b}^\dagger\ \hat{\hat{b}}^\dagger\hat{b}\hat{\hat{b}} + \hat{\hat{b}}^\dagger\ \hat{b}^\dagger\hat{b}\hat{\hat{b}} + \hat{b}^\dagger\ \hat{\hat{b}}^\dagger\hat{\hat{b}}\hat{b} + \hat{\hat{b}}^\dagger\ \hat{b}^\dagger\hat{\hat{b}}\hat{b}\right]$ contributes

$$\langle 0|\hat{b}\hat{b}(\hat{b}^\dagger\ \hat{b}^\dagger\hat{b}\hat{b})\hat{b}^\dagger\hat{b}^\dagger|0\rangle = 2\langle 0|\hat{b}\hat{b}(\hat{b}^\dagger\ \hat{b}^\dagger)|0\rangle = 2 \qquad \to \boldsymbol{\langle 2,0|\mathcal{H}_{int}|2,0\rangle = V_0}$$

$|1{,}1\rangle$: all four last terms in $\frac{V_0}{2}\left[\hat{b}^\dagger\ \hat{b}^\dagger\hat{b}\hat{b} + \hat{b}^\dagger\ \hat{\hat{b}}^\dagger\hat{b}\hat{\hat{b}} + \hat{\hat{b}}^\dagger\ \hat{b}^\dagger\hat{b}\hat{\hat{b}} + \hat{b}^\dagger\ \hat{\hat{b}}^\dagger\hat{\hat{b}}\hat{b} + \hat{\hat{b}}^\dagger\ \hat{b}^\dagger\hat{\hat{b}}\hat{b}\right]$

contribute equally, since $\hat{\hat{b}}$ and $\hat{b}$ commute: $\left[\hat{\hat{b}},\hat{b}\right] = \left[\hat{\hat{b}}^\dagger,\hat{b}^\dagger\right] = 0$

$$4\left\langle 0\middle|\hat{\hat{b}}\hat{b}\left(\hat{b}^\dagger\hat{\hat{b}}^\dagger\hat{b}\hat{\hat{b}}\right)\hat{b}^\dagger\hat{\hat{b}}^\dagger\middle|0\right\rangle = 4\langle 0|\hat{b}\hat{b}^\dagger\hat{b}\hat{b}^\dagger|0\rangle\left\langle 0\middle|\hat{\hat{b}}\hat{\hat{b}}^\dagger\hat{\hat{b}}\hat{\hat{b}}^\dagger\middle|0\right\rangle = 4 \quad \to \boldsymbol{\langle 1,1|\mathcal{H}_{int}|1,1\rangle = 2V_0}$$

I.2.3 Bose Gas with Interactions

- **generalization to states with N bosons** (without calculation)

$$\langle N,1|\mathcal{H}_{int}|N,1\rangle - \langle N+1,0|\mathcal{H}_{int}|N+1,0\rangle = N\,V_0$$

- **it is energetically more favorable to put an additional boson into the same state as occupied by all the others rather than putting it into another state**
- **energy difference scales with the particle number N and is therefore an extensive variable**

I.2.3 Bose Gas with Interactions

- **Bogoliubov theory of weakly interacting Bose gas** (N.N Bogoliubov, J. Phys. USSR **11**, 23 (1947))
  - **we start by considering a set of ground states $|N\rangle$ with $N \gg 1$**
    - ➢ since ground state $k = 0$ is occupied by large particle number, we have to deal with large quantum numbers
      - ○ $\langle N|\hat{b}_0^\dagger \hat{b}_0|N\rangle = N$ and $\langle N|\hat{b}_0 \hat{b}_0^\dagger|N\rangle = N + 1$ ➔ since $N \gg 1$: $\langle \hat{b}_0^\dagger \hat{b}_0 \rangle \approx \langle \hat{b}_0 \hat{b}_0^\dagger \rangle$ and $\langle [\hat{b}_0, \hat{b}_0^\dagger] \rangle \ll N$

        → we can set $[\hat{b}_0, \hat{b}_0^\dagger] \simeq 0$, i.e. the operators commute like classical quantities

        → we can treat the creation and annihilation operators as classical (complex) numbers
      - ○ as $\hat{b}_0^\dagger$ and $\hat{b}_0$ are complex conjugate, we can take them real by eliminating an irrelevant phase factor:

        $N = \langle \hat{b}_0^\dagger \hat{b}_0 \rangle = \hat{b}_0^2$ and $\hat{b}_0^\dagger = \hat{b}_0 = \sqrt{N}$
  - **interaction energy** (lowest order in powers of $\hat{b}_0$):

$$\mathcal{H}_{\text{int}}^{(0)} = \frac{V_0}{2}\left[\underbrace{\hat{b}^\dagger\, \hat{b}^\dagger \hat{b}\hat{b}}_{\hat{b}_0^2 \hat{b}_0^2 = N_0^2} + \hat{b}^\dagger\, \hat{\hat{b}}^\dagger \hat{b}\hat{\hat{b}} + \hat{\hat{b}}^\dagger\, \hat{b}^\dagger \hat{b}\hat{\hat{b}} + \hat{b}^\dagger\, \hat{\hat{b}}^\dagger \hat{\hat{b}}\hat{b} + \hat{\hat{b}}^\dagger\, \hat{b}^\dagger \hat{\hat{b}}\hat{b}\right]$$

dominating term is the one with all wave numbers equal and zero

⇨ $$\boxed{E_{\text{int}}^{(0)} = \frac{V_0}{2} N_0^2}$$

I.2.3 Bose Gas with Interactions

- **interaction energy, next order term in powers of $\hat{b}_0$:**

→ we cannot have three $\hat{b}_0$ factors due to momentum conservation

→ we have to consider only terms with two $\hat{\hat{b}}$ factors in the next order

→ momentum conservation requires that we have factors $\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}$ (4 possibilities), $\hat{b}^\dagger_{\mathbf{k}}\hat{b}^\dagger_{-\mathbf{k}}$ (1 possibility) and $\hat{b}_{\mathbf{k}}\hat{b}_{-\mathbf{k}}$ (1 possibility)

$$\mathcal{H}^{(1)}_{\text{int}} = \frac{V_0}{2}\hat{b}_0^2 \sum_{\mathbf{k}\neq\mathbf{0}} \left(4\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} + \hat{b}^\dagger_{\mathbf{k}}\hat{b}^\dagger_{-\mathbf{k}} + \hat{b}_{\mathbf{k}}\hat{b}_{-\mathbf{k}}\right)$$

$\mathcal{H}^{(1)}_{\text{int}}$ contains a contribution $\hat{b}_0^2 \sum_{\mathbf{k}} \hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} = N_0 \sum_{\mathbf{k}} \hat{n}_{\mathbf{k}}$, where $\hat{n}_{\mathbf{k}}$ is the particle number operator for bosons in state $\mathbf{k}$

➔ we have to take into account the bosons that are not in the ground state $\mathbf{k} = \mathbf{0}$ (as already done for the non-interacting Bose gas)

$$N = N_0 + \sum_{\mathbf{k}\neq\mathbf{0}} \hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} \Rightarrow \frac{V_0}{2}\left(\hat{b}_0^2\right)^2 = \frac{V_0}{2}N_0^2 = \frac{V_0}{2}\left(N - \sum_{\mathbf{k}\neq\mathbf{0}} \hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}\right)^2 \simeq \frac{V_0}{2}\left(N^2 - 2N\sum_{\mathbf{k}\neq\mathbf{0}} \hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}\right) \simeq \underbrace{\frac{V_0}{2}N^2}_{\simeq E^{(0)}_{\text{int}} = \frac{V_0}{2}N_0^2} - \frac{V_0}{2}2N\sum_{\mathbf{k}\neq\mathbf{0}} \hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}$$

$$\mathcal{H} = \underbrace{\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}}\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}}_{\text{kinetic energy}} + \underbrace{N\frac{V_0}{2}\sum_{\mathbf{k}\neq\mathbf{0}}\left(2\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} + \hat{b}^\dagger_{\mathbf{k}}\hat{b}^\dagger_{-\mathbf{k}} + \hat{b}_{\mathbf{k}}\hat{b}_{-\mathbf{k}}\right)}_{\text{additional ground state energy}}$$

the last two terms do not conserve particle number:
additional states are excited out of the condensate or absorbed into the condensate

I.2.3 Bose Gas with Interactions

discussed in detail in lecture SC+LTP 1

- **diagonalization of the Hamiltonian by Bogoliubov transformation**

  - introduction of new operators being linear combinations of the original bosonic operators

spinors of Bogoliubov operators: $\mathcal{B}_{\mathbf{k}}^{\dagger} = (\alpha_{\mathbf{k}}^{\dagger}, \beta_{-\mathbf{k}}) = \underbrace{\mathcal{C}_{\mathbf{k}}^{\dagger}}_{(\hat{b}_{\mathbf{k}}^{\dagger}, \hat{b}_{-\mathbf{k}})} \; \mathcal{U}_{\mathbf{k}}$ $\qquad$ $\mathcal{B}_{\mathbf{k}} = (\alpha_{\mathbf{k}}, \beta_{-\mathbf{k}}^{\dagger}) = \mathcal{U}_{\mathbf{k}}^{\dagger} \; \underbrace{\mathcal{C}_{\mathbf{k}}}_{(\hat{b}_{\mathbf{k}}, \hat{b}_{-\mathbf{k}}^{\dagger})^{T}}$

$$\alpha_{\mathbf{k}}^{\dagger} = u_{\mathbf{k}} \hat{b}_{\mathbf{k}}^{\dagger} - v_{\mathbf{k}} \hat{b}_{-\mathbf{k}} \qquad \alpha_{\mathbf{k}} = u_{\mathbf{k}}^{\star} \hat{b}_{\mathbf{k}} - v_{\mathbf{k}}^{\star} \hat{b}_{-\mathbf{k}}^{\dagger}$$

$$\beta_{-\mathbf{k}} = v_{\mathbf{k}}^{\star} \hat{b}_{\mathbf{k}}^{\dagger} + u_{\mathbf{k}}^{\star} \hat{b}_{-\mathbf{k}} \qquad \beta_{-\mathbf{k}}^{\dagger} = v_{\mathbf{k}} \hat{b}_{\mathbf{k}} + u_{\mathbf{k}} \hat{b}_{-\mathbf{k}}^{\dagger}$$

  - appropriate unitary matrix to make transformed energy matrix $\tilde{\mathcal{E}}_{\mathbf{k}} = \mathcal{U}_{\mathbf{k}}^{\dagger} \, \mathcal{E}_{\mathbf{k}} \, \mathcal{U}_{\mathbf{k}}$ diagonal:

$$\mathcal{U}_{\mathbf{k}} = \begin{pmatrix} u_{\mathbf{k}} & v_{\mathbf{k}}^{\star} \\ -v_{\mathbf{k}} & u_{\mathbf{k}}^{\star} \end{pmatrix}$$

$$\mathcal{U}_{\mathbf{k}}^{\dagger} = \begin{pmatrix} u_{\mathbf{k}}^{\star} & -v_{\mathbf{k}}^{\star} \\ v_{\mathbf{k}} & u_{\mathbf{k}} \end{pmatrix}$$

$$\Longrightarrow \quad \mathcal{U}_{\mathbf{k}}^{\dagger} \, \mathcal{E}_{\mathbf{k}} \, \mathcal{U}_{\mathbf{k}} = \tilde{\mathcal{E}}_{\mathbf{k}} = \begin{pmatrix} E_{\mathbf{k}} & D_{\mathbf{k}} \\ -D_{\mathbf{k}} & -E_{\mathbf{k}} \end{pmatrix}$$

choose $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ such that off-diagonal terms vanish

I.2.3 Bose Gas with Interactions

- **diagonalization of the Hamiltonian by Bogoliubov transformation** (without detailed calculation)

$$\Rightarrow \quad u_{\mathbf{k}} = \sqrt{\frac{1}{2}\frac{(\varepsilon_{\mathbf{k}} + NV_0)}{E_{\mathbf{k}}} + 1}$$

$$v_{\mathbf{k}} = \sqrt{\frac{1}{2}\frac{(\varepsilon_{\mathbf{k}} + NV_0)}{E_{\mathbf{k}}} - 1}$$

$$u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 = 1$$

with

$$E_{\mathbf{k}} = \sqrt{(\varepsilon_{\mathbf{k}} + NV_0)^2 - (NV_0)^2} = \sqrt{\varepsilon_{\mathbf{k}}\,(\varepsilon_{\mathbf{k}} + 2NV_0)}$$

note:
the term NV_0 can be interpreted as the chemical potential of the condensate, as adding one more particle absorbs the interaction energy V_0 multiplied by the local number N of particles

resulting Hamiltonian

$$\mathcal{H} = \underbrace{\mathcal{H}_0}_{\text{ground state energy}} + \underbrace{\sum_{\mathbf{k}} \left\{E_{\mathbf{k}}\alpha_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}} + E_{\mathbf{k}}\beta_{-\mathbf{k}}^{\dagger}\beta_{-\mathbf{k}}\right\}}_{\text{elementary excitations}}$$

elementary excitations of the interacting Bose gas are non-interacting bosonic quasiparticles described by the creation and annihilation operators $\alpha_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}$ and $\beta_{-\mathbf{k}}^{\dagger}\beta_{-\mathbf{k}}$

I.2.3 Bose Gas with Interactions

- **nature of excitations**

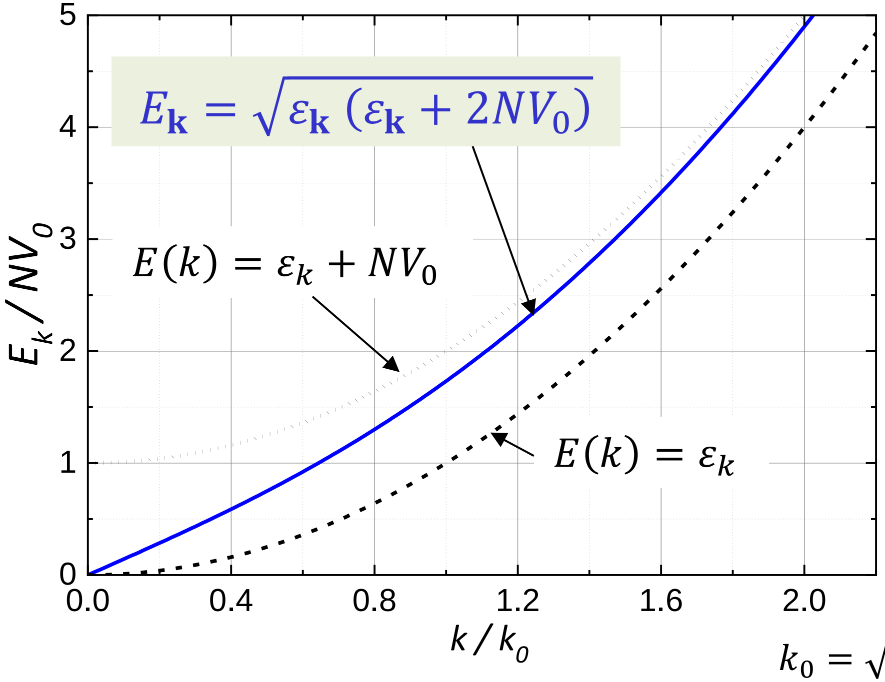
  – large $\mathbf{k}$: kinetic energy dominates: $\varepsilon_{\mathbf{k}} \gg NV_0 \quad \Rightarrow E_{\mathbf{k}} \simeq \varepsilon_{\mathbf{k}}, u_{\mathbf{k}} \simeq 1, v_{\mathbf{k}} \simeq 0,$

$$\alpha_{\mathbf{k}}^\dagger = u_{\mathbf{k}} \hat{\mathrm{b}}_{\mathbf{k}}^\dagger - v_{\mathbf{k}} \hat{\mathrm{b}}_{-\mathbf{k}} \simeq \hat{\mathrm{b}}_{\mathbf{k}}^\dagger$$

excitations are identical to bare bosons with quadratic dispersion

  – $\mathbf{k} \to \mathbf{0}$: kinetic energy vanishes: $\varepsilon_{\mathbf{k}} \ll NV_0 \quad \Rightarrow E_{\mathbf{k}} \simeq \sqrt{2NV_0\varepsilon_{\mathbf{k}}} = \sqrt{NV_0/m}\,\hbar k = \hbar k\, c_s$

c_s = sound velocity



$k_0 = \sqrt{2mNV_0/\hbar^2}$

quasiparticle spectrum that is linear in energy
➔ qp become "propagating sound waves" for interacting system, quantized sound waves: phonons

$u_{\mathbf{k}} \simeq v_{\mathbf{k}}$ with $u_{\mathbf{k}}^2 \simeq v_{\mathbf{k}}^2 = 1/2$

excitations are equal superpositions of $\hat{\mathrm{b}}_{\mathbf{k}}^\dagger$ and $\hat{\mathrm{b}}_{-\mathbf{k}}$ ($\hat{\mathrm{b}}_{-\mathbf{k}}^\dagger$ and $\hat{\mathrm{b}}_{\mathbf{k}}$) both adding (removing) a momentum $\hbar\mathbf{k}$ to the system
➔ consistent with excitation of quantized sound waves

I.2.3 Bose Gas with Interactions

- **topological excitations**

  - **Bogoliubov theory**:
    in first-order approximation, excitations of the superfluid can be viewed as non-interacting Bogoliubov quasiparticles
  - if we go to higher order approximations, we may obtain interactions among these quasiparticles
  - there is an additional type of excitation that cannot be described by going to higher order approximations:

topological excitations (e.g. vortices, see section I.4)

I.2.3 Bose Gas with Interactions

- **Gross-Pitaevskii equation** (E. P. Gross , Nuovo Cimento **20**, 454 (1961), L. P. Pitaevskii, Sov. Phys. JETP. **13**, 451 (1961))
  - the total wavefunction of a system of N bosons all in the same state Ψ is taken as the product of single-particle functions (trial wavefunction)

$$\psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \Psi(\mathbf{r}_1)\Psi(\mathbf{r}_2)\cdots\Psi(\mathbf{r}_N) = \prod_{i=1}^{N}\Psi(\mathbf{r}_i)$$

$\mathbf{r}_i$: coordinate of the *i*-th boson

  - at low T, the de Broglie wavelength is large compared to the interaction range

➔ scattering processes can be approximated by s-wave scattering (hard-sphere potential $\propto \delta(\mathbf{r}_i - \mathbf{r}_j)$)

$$\mathcal{H} = \sum_{i=1}^{N} -\frac{\hbar^2}{2m}\nabla_i^2 + V_{\text{trap}}(\mathbf{r}_i) + \sum_{i<j} \underbrace{\frac{4\pi\hbar^2 a_s}{m}}_{U_0} \delta(\mathbf{r}_i - \mathbf{r}_j)$$

$U_0 = \frac{4\pi\hbar^2 a_s}{m}$: effective interaction strength

  - expectation value of Hamiltonian for state $\psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$ in steady state with all N particles in state Ψ

$$E = N\int dV\left(-\frac{\hbar^2}{2m}|\nabla\Psi(\mathbf{r})|^2 + +V_{\text{trap}}(\mathbf{r})|\Psi(\mathbf{r})|^2 + \frac{N-1}{2}U_0|\Psi(\mathbf{r})|^4\right)$$

in the interaction term, $N(N-1)/2$ is the number of term:

➔ interaction term = number of ways of making pairs of bosons times the interaction energy $\int dV\, U_0|\Psi(\mathbf{r})|^4$ of two bosons in the state $\Psi(\mathbf{r})$

➔ $N(N-1)/2 \int dV\, U_0|\Psi(\mathbf{r})|^4 \simeq \frac{1}{2}N^2 U_0$ since $\int dV\, |\Psi(\mathbf{r})|^2 = 1$ and $N \gg 1$

I.2.3 Bose Gas with Interactions

- **Gross-Pitaevskii equation** (E. P. Gross , Nuovo Cimento **20**, 454 (1961), L. P. Pitaevskii, Sov. Phys. JETP. **13**, 451 (1961))
  - minimization of energy by variational method (variation of E with respect to Ψ and $\Psi^\star$) leads to **Gross-Pitaevskii equation (GPE)** for single-particle wavefunction

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{trap}}(\mathbf{r}) + U_0|\Psi(\mathbf{r})|^2\right)\Psi(\mathbf{r},\mathbf{t}) = \mu\,\Psi(\mathbf{r},\mathbf{t})$$

μ: chemical potential

$U = \frac{4\pi\hbar^2 a_s}{m}$: effective interaction strength

- ➢ GPE = classical model equation for the ground-state single-particle wavefunction in a BEC.
- ➢ GPE has a similar form as the Ginzburg–Landau equation and is sometimes referred to as the nonlinear Schrödinger equation
- ➢ eigenvalue of the GPE is the chemical potential and not the energy per particle
- ➢ for a non-interacting system with all particles in the same state the chemical potential would be equal to the energy per particle, but not for the interacting systems
- ➢ for a uniform Bose gas: $\mu = U_0|\Psi|^2 = U_0 n$, with $n = N/V$ being the particle density

same result is obtained if one uses a coherent state as the trial wave function

I.2.3 Bose Gas with Interactions

- **discussion of the Gross-Pitaevskii equation**
  - in analogy to the coherence length in a SC resulting from the 1st GL equation, the GPE yields a characteristic **healing length ξ** of the superfluid (changes of $\rho(r) = |\psi(r)|^2$ are impossible on a length scale smaller than ξ)

$$\boxed{\xi = \frac{\hbar}{\sqrt{2mU_0\rho}}}$$

$\xi \simeq 0.1$ nm in superfluid ^{4}He and therefore of the order of the interatomic distance

  - meaning of the phase of the wave function

by using $\psi\nabla^2\psi^\star - \psi^\star\nabla^2\psi = \boldsymbol{\nabla}\cdot[\psi^\star\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^\star]$ we obtain from the GPE

$$\frac{\mathrm{d}\rho(\mathbf{r},\mathrm{t})}{\mathrm{d}t} = \frac{\imath\hbar}{2m}\boldsymbol{\nabla}\cdot[\psi^\star(\mathbf{r},\mathrm{t})\boldsymbol{\nabla}\psi(\mathbf{r},\mathrm{t}) - \psi(\mathbf{r},\mathrm{t})\boldsymbol{\nabla}\psi^\star(\mathbf{r},\mathrm{t})]$$

  - by defining $\mathbf{J}(\mathbf{r},t) = -\frac{\imath\hbar}{2m}\boldsymbol{\nabla}\cdot[\psi^\star(\mathbf{r},\mathrm{t})\boldsymbol{\nabla}\psi(\mathbf{r},\mathrm{t}) - \psi(\mathbf{r},\mathrm{t})\boldsymbol{\nabla}\psi^\star(\mathbf{r},\mathrm{t})]$, we obtain the continuity equation of the particle density

$$\boxed{\frac{\mathrm{d}\rho(\mathbf{r})}{\mathrm{d}t} + \boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{r},t) = 0}$$

➔$\mathbf{J}(\mathbf{r},t) = \rho(\mathbf{r},t)\mathbf{v}_s(\mathbf{r},t)$ = particle current density

  - using the notation $\psi(\mathbf{r},\mathrm{t}) = \sqrt{\rho(\mathbf{r},t)}\,\mathrm{e}^{\imath\phi(\mathbf{r},t)}$ we obtain the superfluid velocity $\mathbf{v}(\mathbf{r},t) = \frac{\hbar}{m}\boldsymbol{\nabla}\phi(\mathbf{r},\mathrm{t})$

Contents Part I: Quantum Liquids

Contents:

I.2.4 Bose-Einstein Condensation of ^{4}He

- **What do we already know about ^{4}He?**

  – ^{4}He does not become solid at ambient pressure due to the weak Van der Waals interaction and the large zero-point fluctuations

  – ^{4}He is a boson and shows superfluidity
  discovered in 1937 by **Pyotr Kapitsa** *("Viscosity of Liquid Helium Below the λ-Point“,* Nature **141**, 74 (1938))
  and **John F. Allen**, and **Don Misener** *("Flow of Liquid Helium II“,* Nature **142**, 643 (1938))

  – the properties of superfluid ^{4}He often deviate from those of an ideal Bose-Einstein condensate
    - ➢ $C_V \propto T^3$ instead of $C_V \propto T^{3/2}$ at low T
    - ➢ behavior of C_V around T_{BEC} is different – sharp λ-shaped peak instead of "cusp": ^{4}He shows critical behavior consistent with 3D-XY universality class

  – description of superfluid ^{4}He by macroscopic wave function $\psi(\mathbf{r},t)$ (coherent state)

$$\psi(\mathbf{r},t) = \psi_0(\mathbf{r},t)e^{\imath\theta(\mathbf{r},t)}, \qquad |\psi_0(\mathbf{r},t)|^2 = n_0(\mathbf{r},t), \qquad N_0 = \int |\psi_0(\mathbf{r},t)|^2 \, \mathrm{d}^3 r$$

$n_0 = \frac{N_0}{V}$ = particle density in condensate

I.2.4 Bose-Einstein Condensation of ^{4}He

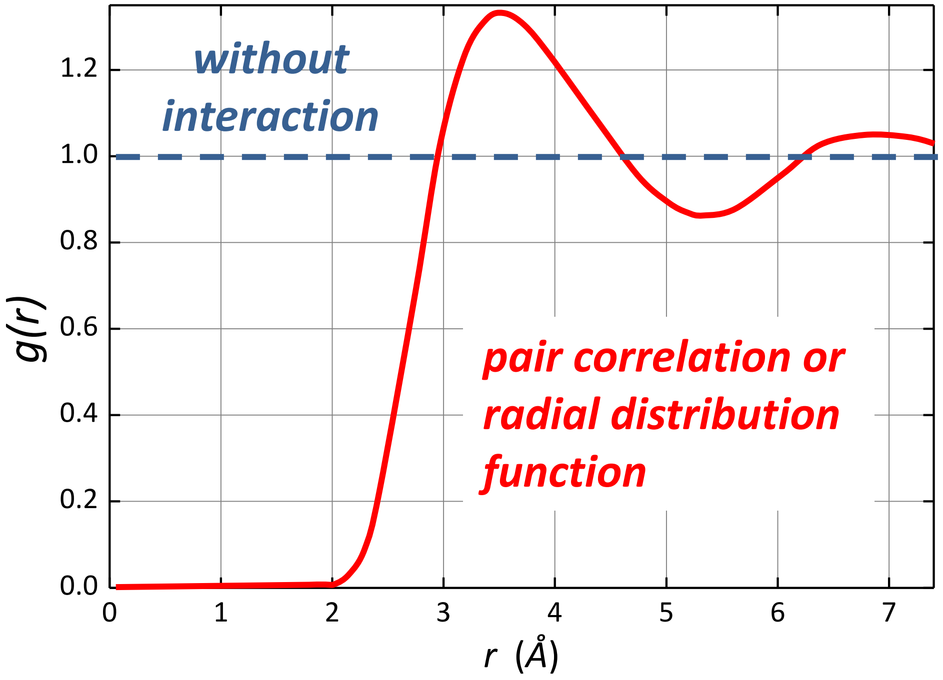
- **How well does the ideal Bose gas describe the normal-superfluid transition temperature?**

$$T_\mathrm{BEC} = \frac{2\pi\hbar^2}{mk_\mathrm{B}}\left(\frac{n}{2.612}\right)^{2/3}$$

- ➢ $T_\mathrm{BEC} \simeq 0.5$ K by using data of gaseous ^{4}He (is well below normal-superfluid transition temperature)
- ➢ $T_\mathrm{BEC} \simeq 3.1$ K by using data of liquid ^{4}He (agrees only qualitatively with measured $\boldsymbol{T_\lambda = 2.17}$ **K**)

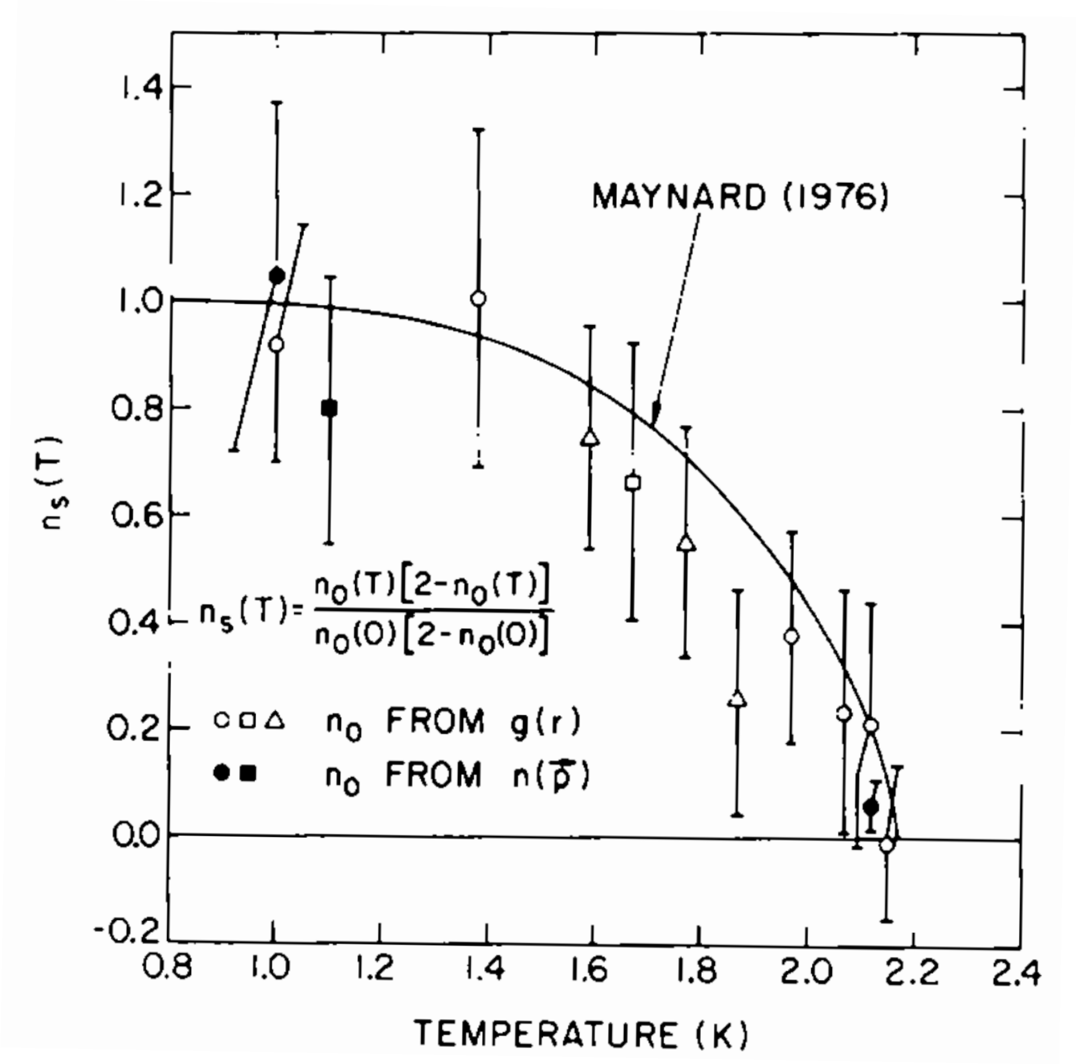
^{4}He is not an ideal Bose gas:
interaction of He atoms

- ➢ hard core repulsion
- ➢ admixture of higher $\mathbf{k}$-states (is equivalent to larger k-uncertainty)
- ➢ reduction of particle number in ground state



I.2.4 Bose-Einstein Condensation of ^{4}He

- **superfluid fraction of ^{4}He is only about 14% for $T \to 0$: $n_0(T=0) \simeq 0.14\, n$**



experimental observations

$T \ll T_\lambda$: $\quad n_s(T) \simeq n_s(0) - AT^\alpha$ with $\alpha \simeq 4$

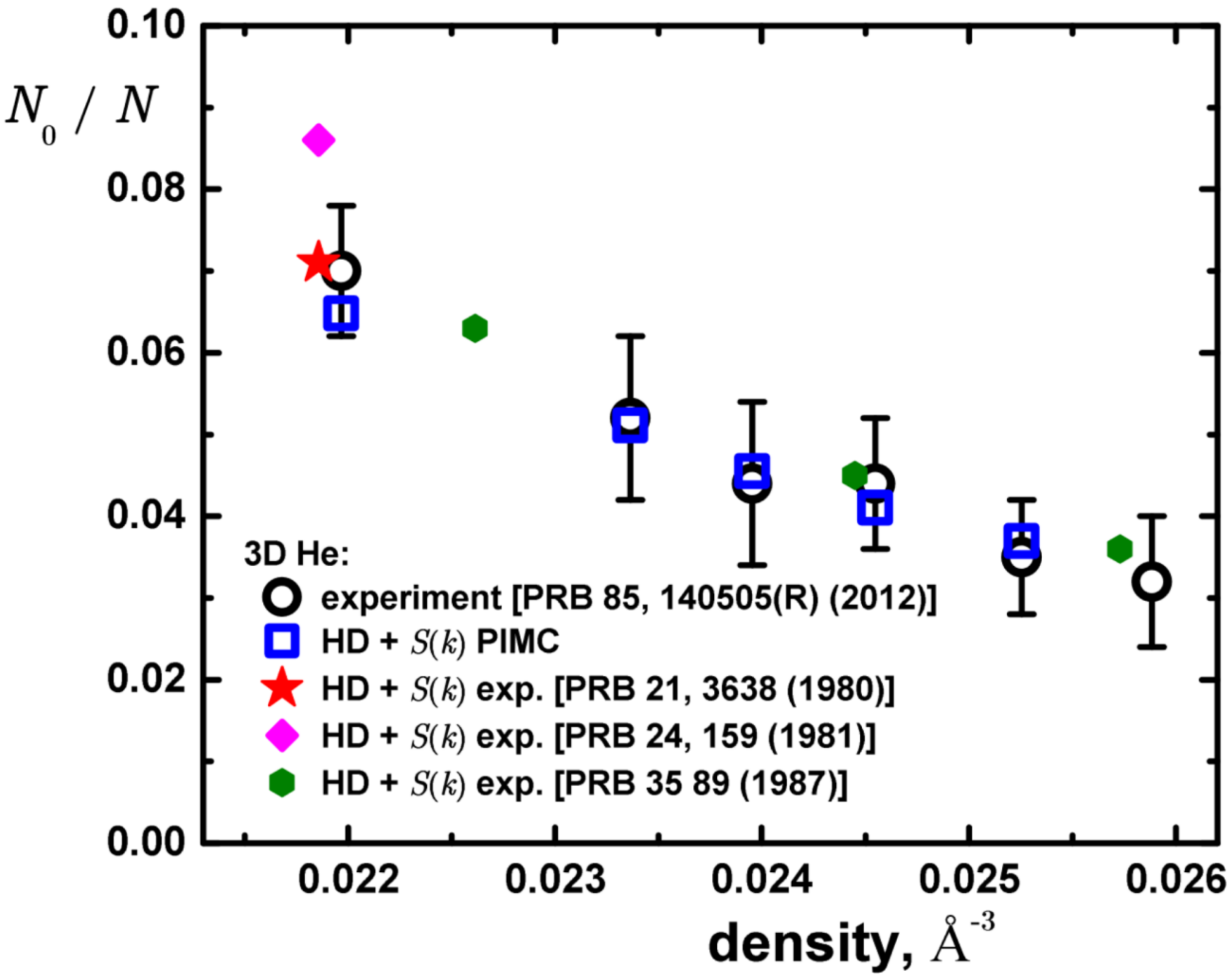
$$T \sim T_{\rm BEC}: \quad n_s(T) \simeq \begin{cases} B(T_\lambda - T)^\beta & \text{for} \quad T < T_\lambda \\ 0 & \text{for} \quad T > T_\lambda \end{cases}$$

data obtained from x-ray scattering and neutron scattering

J.S. Brooks and R.J. Donnelly, J. Phys. Chem. Ref. Data **6**, 51 (1977) .

I.2.4 Bose-Einstein Condensation of ^{4}He

- **superfluid fraction of ^{4}He**



condensate fraction of superfluid ^{4}He as a function of the density. Open circles are experimental results obtained from deep inelastic neutron scattering, squares are HD predictions at the same densities. The star diamond, and hexagonal points correspond to quantum-field hydrodynamics (HD) results using the experimental measures of the static structure factor S(**p**)

Lozovik, Yurii & Kurbakov, I. & Astrakharchik, G. & Boronat, J. (2020). *Universal relation between the static structure factor and the condensate fraction*.

I.2.4 Bose-Einstein Condensation of ^{4}He

- **superfluid fraction of ^{4}He – more detailed treatment (1)**
  - reduced superfluid fraction is caused by the fact that superfluid ^{4}He is a strongly interacting Bose gas described by

$$\mathcal{H} = \sum_{i=1}^{N} \frac{\hbar^2}{2m} \nabla_i^2 + \frac{1}{2} \sum_{i \neq j} V\left(\mathbf{r}_i - \mathbf{r}_j\right)$$

  - solutions are symmetric N-particle wave functions

$$\psi\left(\mathbf{r}_1, \dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots, \mathbf{r}_N\right) = \psi\left(\mathbf{r}_1, \dots, \mathbf{r}_j, \dots, \mathbf{r}_i, \dots, \mathbf{r}_N\right)$$

$$\mathcal{H}\psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = E_n \psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$$

  - occupation probability of eigenstate with energy E_n

$$P_n = \frac{1}{Z_N} \exp(-E_n/k_\mathrm{B}T) \quad \text{with partition function} \quad Z_N = \sum_n \exp(-E_n/k_\mathrm{B}T)$$

  - if particle number is not fixed (grand canonical ensemble): $Z_N = \sum_{n,N} \exp\left(-\frac{E_n^{(N)}}{k_\mathrm{B}T} - \mu N\right)$

I.2.4 Bose-Einstein Condensation of ^{4}He

- **superfluid fraction of ^{4}He – more detailed treatment (2)**
  - for grand canonical ensemble:

$$\Omega(T,\mu) = -k_\mathrm{B} T \ln Z$$

grand canonical potential

$$\langle N \rangle = k_\mathrm{B} T \frac{\partial \ln Z}{\partial \mu}$$

average particle number

$$U = \langle \mathcal{H} \rangle = \mu \langle N \rangle - k_\mathrm{B} T \frac{\partial \ln Z}{\partial \beta}$$

inner energy (with $\beta = 1/k_\mathrm{B}T$)

  - calculations have to be performed numerically, e.g. by perturbation theory (if interactions are weak) or quantum Monte Carlo simulations in the case of strong interactions (as for superfluid ^{4}He)

key result: superfluid ^{4}He forms a condensate of He atoms in the ground state, however, not all atoms occupy ground state at $T = 0$

see e.g. D.M. Ceperley, Rev. Mod. Phys. 67, 279-355 (1995)

I.2.4 Bose-Einstein Condensation of ^{4}He

- **superfluid fraction of ^{4}He – more detailed treatment (3)**

  – single particle density matrix: $$\rho(\mathbf{r}_1-\mathbf{r}_1') = N\int \psi_0^*(\mathbf{r}_1,\mathbf{r}_2,\dots,\mathbf{r}_N)\psi_0(\mathbf{r}_1',\mathbf{r}_2,\dots,\mathbf{r}_N)\mathrm{d}^3r_2\dots\mathrm{d}^3r_N$$

  - corresponds to correlation function between the many-body wave functions at the coordinates $(\mathbf{r}_1,\mathbf{r}_2,\dots,\mathbf{r}_N)$ and $(\mathbf{r}_1',\mathbf{r}_2,\dots,\mathbf{r}_N)$

  - by integrating over all coordinates despite $\mathbf{r}_1$ and $\mathbf{r}_1'$, we average over all configurations of the particles despite the first ones

  – properties of the single particle density matrix:

    - ➢ depends only on relative coordinate $\mathbf{r}_1-\mathbf{r}_1'$ due to translational invariance
    - ➢ for $\mathbf{r}_1=\mathbf{r}_1'$ we obtain: $\int \rho_1(0)\,\mathrm{d}^3r_1 = N \;\Rightarrow\; \rho_1(0)=\frac{N}{V}=n$

  – for ground state of an **ideal non-interacting Bose gas**, we have $\psi_0(\mathbf{r}_1,\mathbf{r}_2,\dots,\mathbf{r}_N)=\psi_0(\mathbf{r}_1)\psi_0(\mathbf{r}_2)\dots\psi_0(\mathbf{r}_N)$ since all particles occupy the same state ψ_0

$$\rho(\mathbf{r}_1-\mathbf{r}_1') = N\psi_0^*(\mathbf{r}_1)\psi_0(\mathbf{r}_1')\int|\psi_0(\mathbf{r}_1)|^2|\psi_0(\mathbf{r}_2)|^2\dots|\psi_0(\mathbf{r}_N)|^2\,\mathrm{d}^3r_2\dots\mathrm{d}^3r_N = N\psi_0^*(\mathbf{r}_1)\psi_0(\mathbf{r}_1')$$

$$\rho(\mathbf{r}_1-\mathbf{r}_1')=\frac{N}{V}=n \quad \text{since}\;\; \psi_0(\mathbf{r})=\frac{1}{\sqrt{V}}\exp(\imath\mathbf{k}\cdot\mathbf{r})=\frac{1}{\sqrt{V}}$$

for $\mathrm{k}=0$

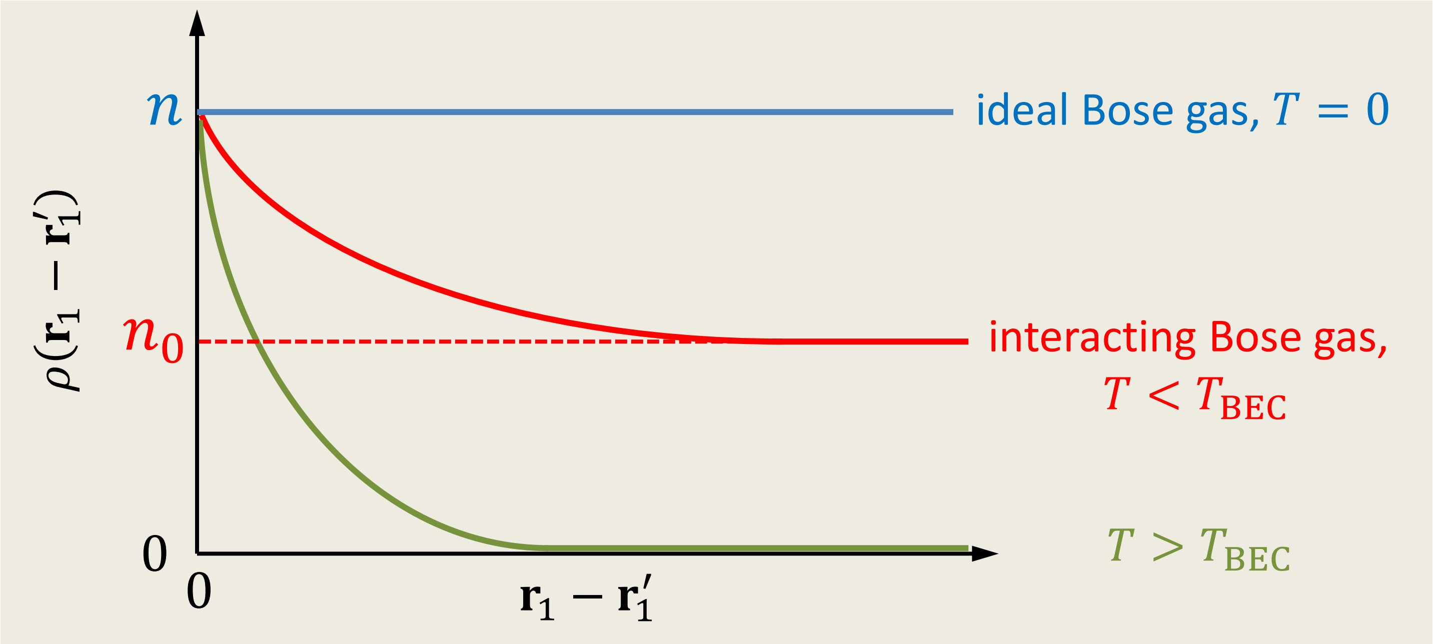
I.2.4 Bose-Einstein Condensation of ^{4}He

- **superfluid fraction of ^{4}He – more detailed treatment (4)**

  – for **interacting Bose gas** we obtain a more complicated result
    - ➢ $\rho(\mathbf{r}_1 - \mathbf{r}_1')$ is no longer independent of $\mathbf{r}_1 - \mathbf{r}_1'$
    - ➢ $\rho(\mathbf{r}_1 - \mathbf{r}_1')$ decreases with increasing $\mathbf{r}_1 - \mathbf{r}_1'$ and converges towards a constant value at large $\mathbf{r}_1 - \mathbf{r}_1'$

  – condensate density:

$$n_0 = \lim_{|\mathbf{r}_1 - \mathbf{r}_1'| \to \infty} \rho(\mathbf{r}_1 - \mathbf{r}_1')$$

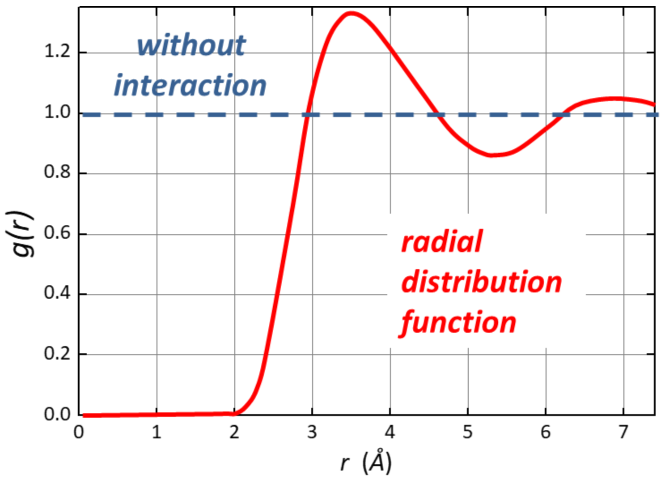


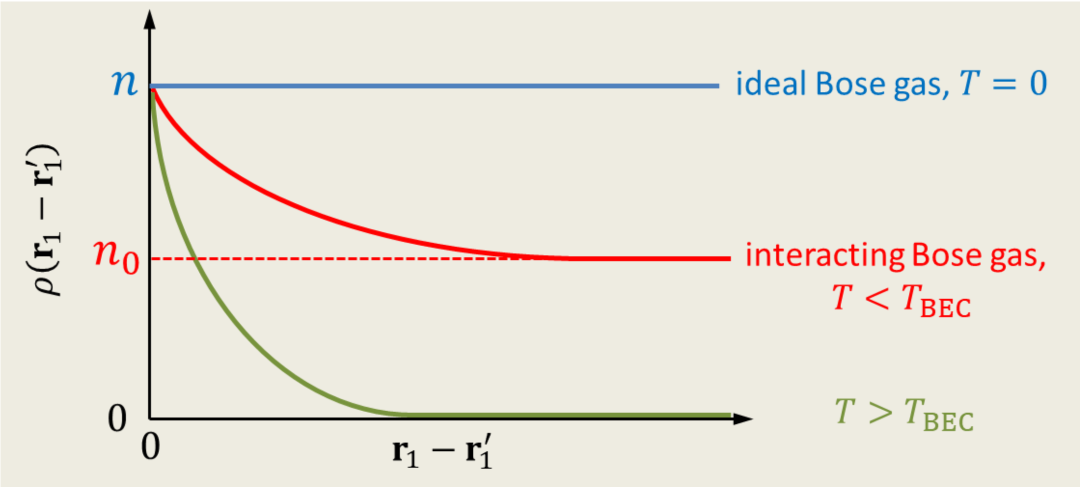
for superfluid ^{4}He:

$$\boldsymbol{n_0(T = 0) \simeq 0.14\, n}$$

I.2.4 Bose-Einstein Condensation of ^{4}He

- **superfluid fraction of ^{4}He – more detailed treatment (5)**
  - plausibility consideration by considering spatial distribution function





  - without interactions particles can be described by plane waves with $k = k_0$
    ➔ $g(\mathbf{r}_1 - \mathbf{r}_2) = 1$, all particles in ground state: $n_0 = n$
  - in the presence of interactions, $g(\mathbf{r}_1 - \mathbf{r}_2) \neq 1$ for short distances
    ➔ we have to admix states with higher k to describe the situation: spatial localization requires finite Δk
    ➔ some states with $k > k_0$ are occupied: $n_0 < n$
    ➔ n_0 becomes smaller, if $g(\mathbf{r}_1 - \mathbf{r}_2)$ deviates more strongly from $g(\mathbf{r}_1 - \mathbf{r}_2) = 1$, i.e. with increasing interactions

Contents Part I: Quantum Liquids

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I.1 Foundations and General Properties

I.2 ^{4}He as an Ideal Bose Gas

I.3 Superfluid ^{4}He

I.4 Vortices

I.5 ^{3}He

I.6 ^{3}He / ^{4}He mixtures

I.3 Superfluid ^{4}He

- **nomenclature and history**

  - superfluid ^{4}He can flow through tiny capillaries without any friction ➔ nomenclature ***superfluidity***
  - **discovery of superfluidity** in 1937 by two groups:
    (i) *P. Kapitza, Nature 141, 74 (1938)*
    (ii) *J.F. Allen, A.D. Misener, Nature 141, 75 (1938)*
  - **theory of superfluidity**:
    - ***F. London*** (1938): superfluidity caused by ordered state in momentum space
      F. London, Nature 141, 643 (1938); Phys. Rev. 54, 947 (1938).
    - ***L. Tisza*** (1938): two-fluid model: prediction of second sound
      L. Tisza, Nature 141, 913 (1938).
    - ***L.D. Landau*** (1941-47): two-fluid hydrodynamics of superfluid ^{4}He
      → *superfluidity as a consequence of the excitation spectrum*
      L.D. Landau, J. Phys. USSR 5, 71 (1941); ibid. 8, 1 (1944); ibid. 11, 91 (1947).
    - ***R.P. Feynman*** (1953): excitation spectrum postulated by Landau follows from quantum mechanics
      R.P. Feynman, Phys. Rev. 90, 1116 (1953).

I.3.1 Two-Fluid Model

- ***two-fluid model allows phenomenological description of superfluid ^{4}He***

microscopic description is difficult → treatment using hydrodynamics

  - already in 1938: anomalous properties of superfluid ^{4}He can be well described by two-fluid model: ***mixture of two non-interacting fluid components*** *(I. Tisza, J. de Phys. et Radium* ***1****, 164 (1938))*

  - formal description of superfluid ^{4}He as the sum of a normal and a superfluid component

$$\rho = \rho_s + \rho_n$$

ρ_s, ρ_n = superfluid and normal fluid densities

  - $\boldsymbol{T = 0}$**:** $\rho_s = \rho$ and $\rho_n = 0$

  - $\boldsymbol{T = T_\lambda}$**:** $\rho_s = 0$ and $\rho_n = \rho$

➢ **superfluid component**: ***no entropy:*** $\boldsymbol{S_s = 0}$ ***zero viscosity:*** $\boldsymbol{\eta_s = 0}$

➢ **normal fluid component**: ***carries total entropy:*** $\boldsymbol{S_n = S,}$ ***finite viscosity:*** $\boldsymbol{\eta_n = \eta}$

I.3.1 Two-Fluid Model

- **superfluid component**

  – description of superfluid component by macroscopic wave function

  as done in superconductivity for electronic system: charged superfluid

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r})e^{\imath\theta(\mathbf{r})}, \qquad |\psi_0(\mathbf{r})|^2 = n_0(\mathbf{r}), \qquad N_0 = \int |\psi_0(\mathbf{r})|^2 \, \mathrm{d}^3 r$$

$n_0 = \frac{N_0}{V}$ = particle density in condensate

  – use quantum mechanics to derive expression for the particle current density (probability current density in 3D)

$$J_\rho = \frac{\hbar}{2m\imath}\left[\psi^*(\mathbf{r})\boldsymbol{\nabla}\psi(\mathbf{r}) - \psi(\mathbf{r})\boldsymbol{\nabla}\psi^*(\mathbf{r})\right]$$

$$J_\rho = \frac{\hbar}{2m\imath}\Bigg[\underbrace{\sqrt{n_0}\,\mathrm{e}^{-\imath\theta(\mathbf{r})}}_{\psi^*(\mathbf{r})}\underbrace{\left(\mathrm{e}^{\imath\theta(\mathbf{r})}\boldsymbol{\nabla}\sqrt{n_0} + \imath\sqrt{n_0}\mathrm{e}^{\imath\theta(\mathbf{r})}\boldsymbol{\nabla}\theta(\mathbf{r})\right)}_{\nabla\psi(\mathbf{r})} - \underbrace{\sqrt{n_0}\,\mathrm{e}^{\imath\theta(\mathbf{r})}}_{\psi(\mathbf{r})}\underbrace{\Big(\mathrm{e}^{-\imath\theta(\mathbf{r})}\boldsymbol{\nabla}\sqrt{n_0} - \imath\sqrt{n_0}\mathrm{e}^{-\imath\theta(\mathbf{r})}\boldsymbol{\nabla}\theta(\mathbf{r})}_{\nabla\psi^*(\mathbf{r})}$$

$$\boxed{J_\rho(\mathbf{r}) = n_0 \underbrace{\frac{\hbar}{m}\,\nabla\theta(\mathbf{r})}_{v_s} = n_0 v_s(\mathbf{r})}$$

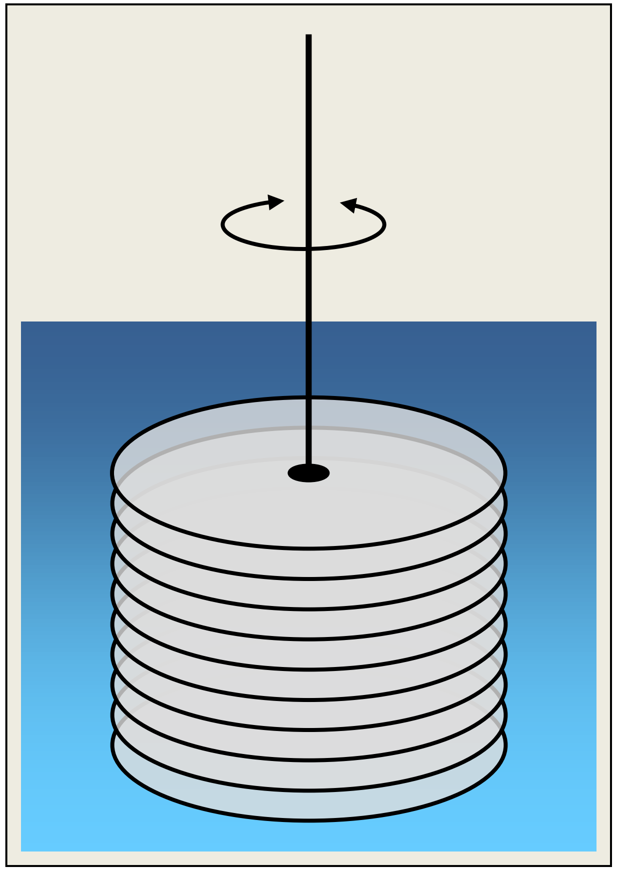
current density – phase relation (as in superconductivity)

v_s = superfluid velocity

I.3.1 Two-Fluid Model

- **what is the ratio of the normal und superfluid component? How does it depend on temperature?**
  - torsional pendulum experiment: 50 Al disks, thickness: 13 µm, diameter: 3.5 cm, disk separation: 210 µm

(E.L. Andronikashvili, Zh. Eksperim. i. Teor. Fiz. 18, 424 (1948))



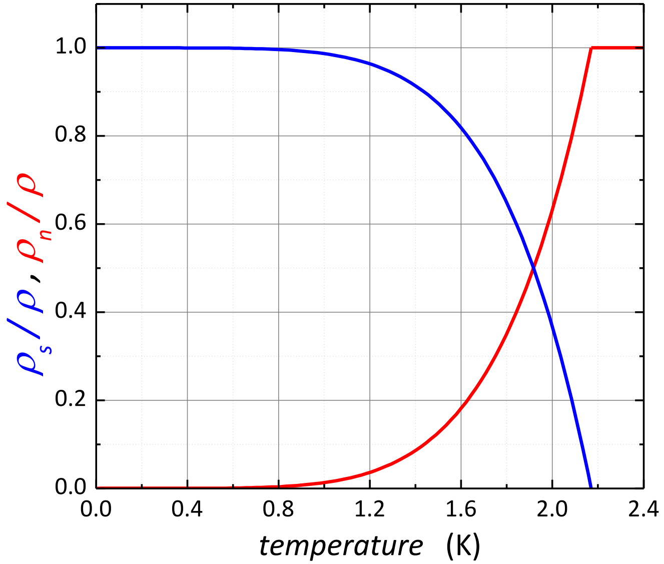
- *only normal component is moved during rotation*
- *skin depth of viscous wave:*

$$\sqrt{2\eta_n/\rho_n\omega}$$

→ *large compared to disk separation below* T_λ
→ *complete normal component is moved*

- *moved mass of normal component results in change of moment of inertia*

→ *change of oscillation frequency:* $\omega \propto 1/\text{mass}$



empirical relation by fitting data:

$$\frac{\rho_n}{\rho(T_\lambda)} = \left(\frac{T}{T_\lambda}\right)^{5.6}$$

cf. normal fluid density of ideal BEC:

$$\frac{\rho_n}{\rho} = \left(\frac{T}{T_{\mathrm{BEC}}}\right)^{3/2}$$

I.3.1 Two-Fluid Model

- **physical consequence of macroscopic wave function description**

$$J_\rho(\mathbf{r}) = n_0 \frac{\hbar}{m} \nabla\theta(\mathbf{r}) = n_0 v_s(\mathbf{r}) \quad \Rightarrow \quad v_s(\mathbf{r}) = \frac{\hbar}{m} \nabla\theta(\mathbf{r})$$

  - since the curl of the gradient of a scalar function vanishes, we obtain

$$\nabla \times v_s(\mathbf{r}) = 0$$

$v_s(\mathbf{r})$ cannot take the form $\Omega \times \mathbf{r}$ corresponding to a rotating liquid

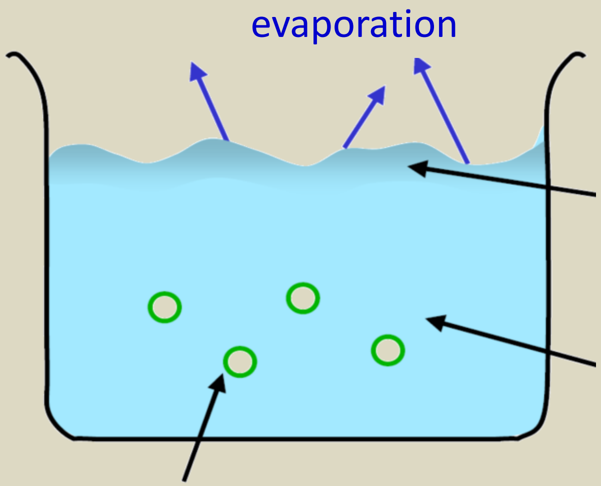
➔ **it is impossible to set the superfluid into rotation**

note:
we will see later that this is true only for slow enough rotations of the vessel containing the superfluid.
At large rotation speed, we are generating vortices (topological excitations) in the superfluid
➔ the critical rotation speed corresponds to the lower critical field in superconductors!!

I.3.2 Experimental Observations

- **boiling of liquid helium**
  - $T > T_\lambda$: - liquid He is boiling
    - latent heat of evaporation is removed from bath
    - temperature is decreasing
  - $T \simeq T_\lambda$: - temperature is decreasing only slowly → large specific heat
    - temperature does not stay constant → no latent heat, 2. order phase transition
    - *boiling suddenly stops !!!*

boiling of „normal" liquid:



lower T at surface:
vapor pressure decreases

higher T within liquid:
vapor pressure > hydrostatic pressure
➔ generation of vapor bubbles

vapor bubbles in liquid

boiling of superfluid helium:

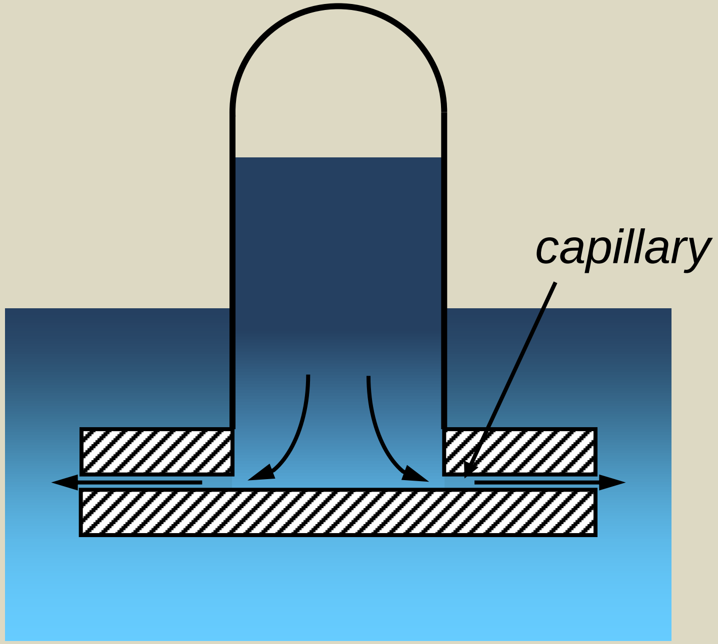
- ➢ *very high thermal conductivity*
- ➢ low T not only at surface, but also within liquid
- ➢ no vapor bubbles in liquid

➔ ***evaporation only from surface, no boiling***

I.3.2 Experimental Observations

- viscosity η of superfluid ^{4}He: ***two different measuring methods***

flow through narrow capillary



$\eta < 10^{-11}$ poise

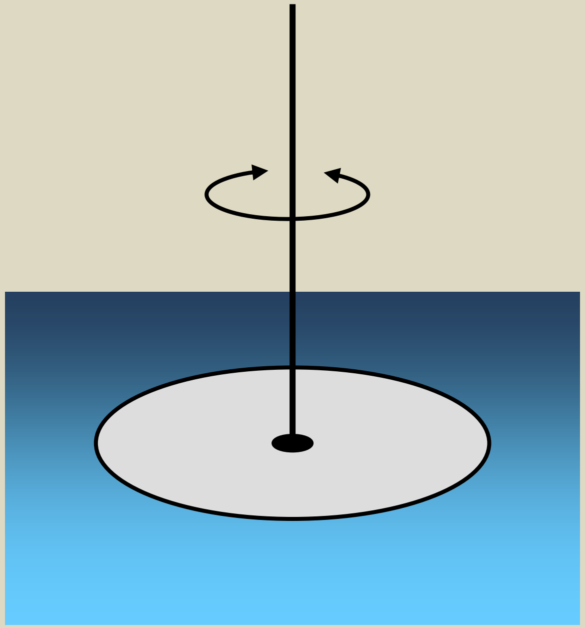
(cf. water: 8.90 × 10^{-5} poise @ 25°C)

- flow without measurable friction
- viscosity $\eta \approx 0$

➔ ***superfluid component flows out***

(1 Poise = 1 g/cm s = 0.1 Pa · s)

torsional pendulum



$\eta \approx 10^{-5}$ poise

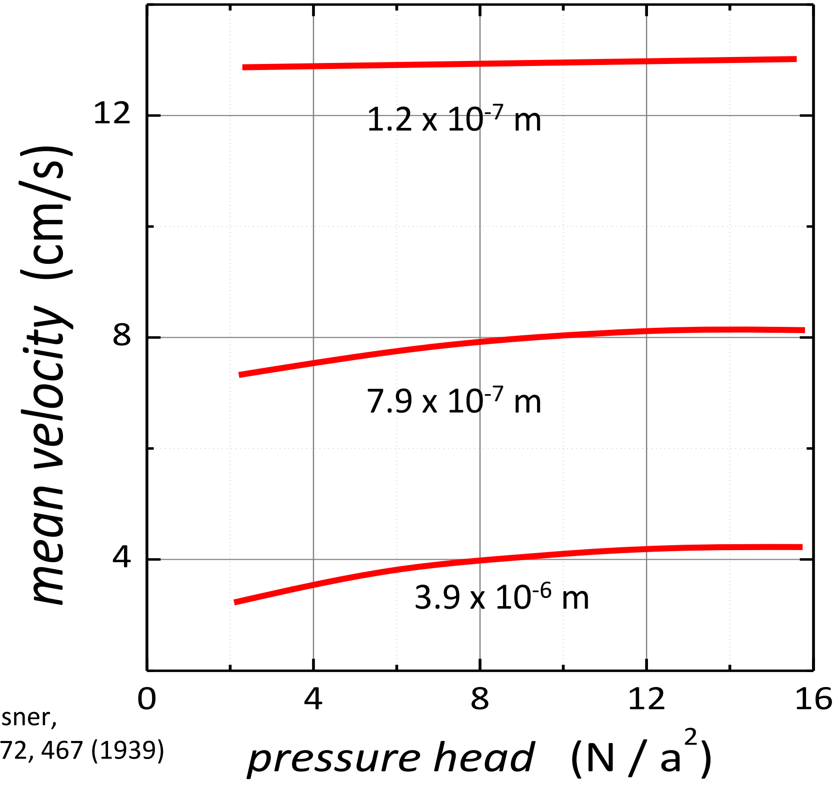
- damping of oscillation: $\eta > 0$
- scattering of **phonons** and **rotons**

➔ ***normal component picks up momentum***

I.3.2 Experimental Observations

- viscosity η of superfluid ^{4}He: ***experimental results***

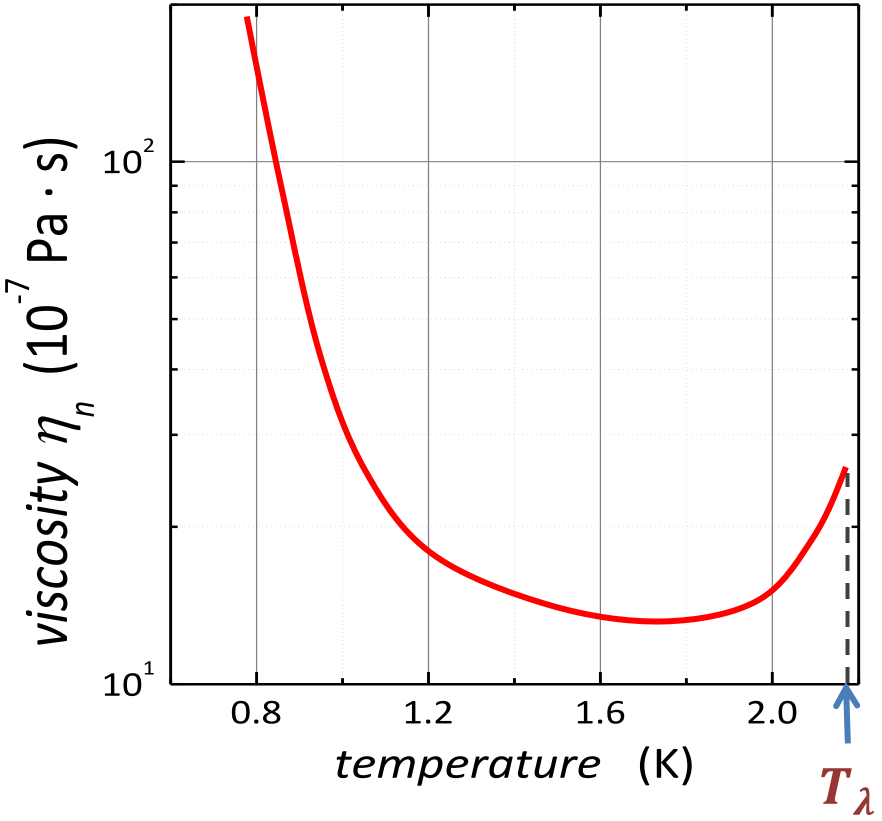
➢ **flow through narrow capillary**



J.F. Allen, A.D. Miesner,
Proc. Roy. Soc. A 172, 467 (1939)

experimental result depends on measuring method !!
➔
explanation by two-fluid model

➢ **torsional pendulum**

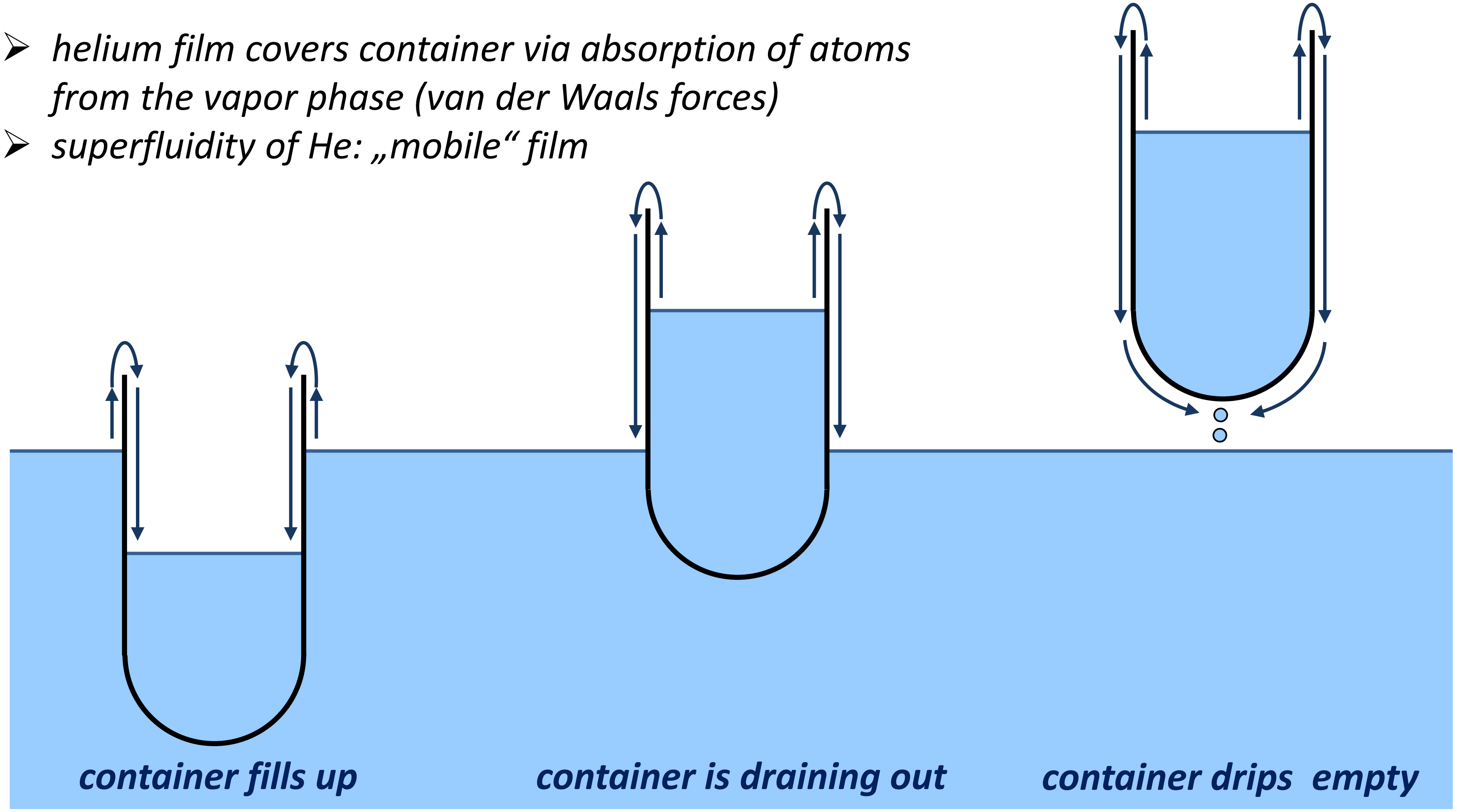


- flow velocity v about independent of pressure
- flow velocity increases v with decreasing diameter a of capillary

expected for viscous flow through tube of length L and diameter a: $\Delta p/L \sim \eta(v/a^4)$ (laminar flow, Hagen-Poiseuille law)
experimental result of Kapitza: $\Delta p \simeq 0$ independent of flow velocity v ➔ $\eta \simeq 0$

I.3.2 Experimental Observations

- **superfluid film flow**



- ➢ *helium film covers container via absorption of atoms from the vapor phase (van der Waals forces)*
- ➢ *superfluidity of He: „mobile“ film*

overshooting during fill up and draining out ➔ ***oscillation of filling height***

I.3.2 Experimental Observations

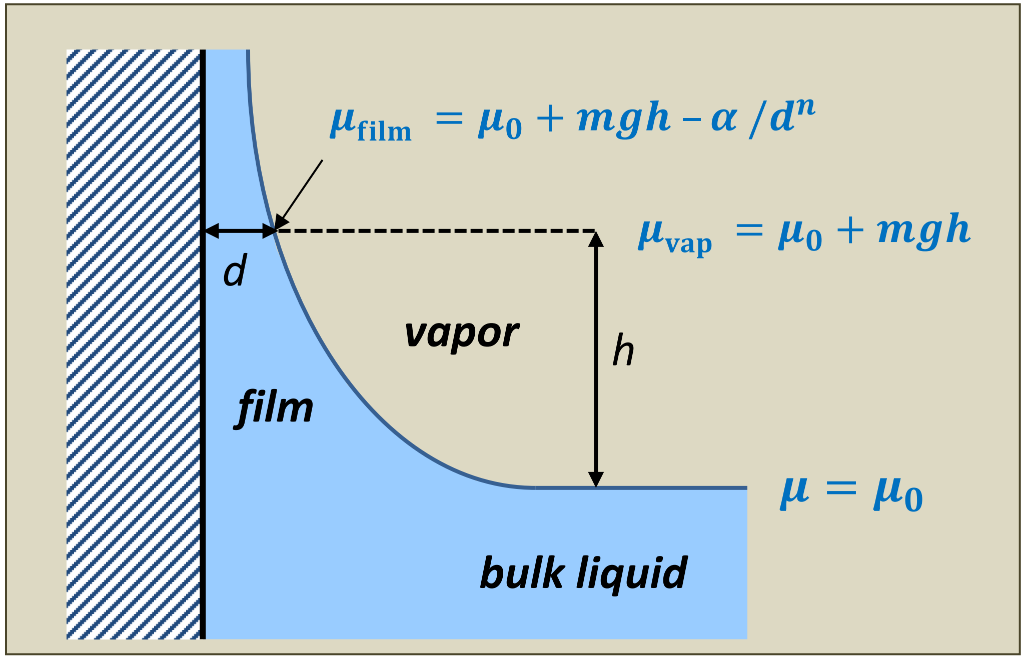
- **superfluid film flow: *estimate of superfluid film thickness***

  - in equilibrium ***chemical potentials*** are identical: $\mu_{\mathrm{film}} = \mu_{\mathrm{vap}} = \mu_0$ (*for film in saturated vapor*)

    - ➢ for film: $\mu_{\mathrm{film}} = \mu_0 + \underbrace{mgh - \alpha/d^n}_{=0}$ ($n \approx 3$ for $d < 5$ nm, α = Hamaker constant, α is determined by dielectric properties of wall and He atoms)

      bulk liquid *gravitation* *van der Waals potential*

    - ➢ for vapor: $\mu_{\mathrm{vap}} = \mu_0 + mgh$



equilibrium condition: $\mu_{\mathrm{film}} = \mu_0$

$$d = \left(\frac{\alpha}{mgh}\right)^{1/3}$$

typical values:

$d\ (\mathrm{nm}) \approx 30/[h\ (\mathrm{cm})]^{1/3}$

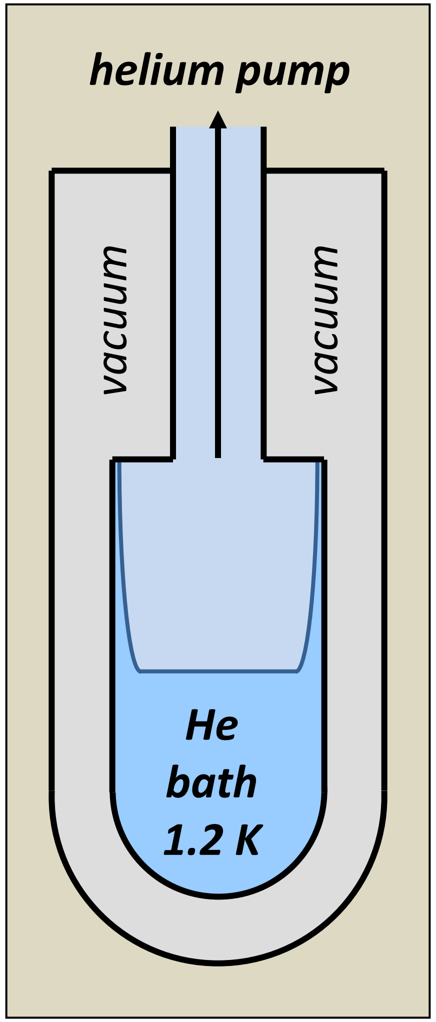
$h = 5$ cm → $d = 20$ nm

normal fluid blocked in thin film

➔ ***superleak***

I.3.2 Experimental Observations

- **superfluid film flow: *acting as a heat leak***



- film creeps up along the container walls and pumping line
- film creeps up to the position, where $T = 2.18$ K
- evaporation from film dominates, since vapor pressure at height h is much lower
- estimate of film thickness for „***unsaturated film***“

➔ d depends on gas pressure p (kinetic gas theory)

$$p(h) = p_{\mathrm{sat}} \exp\left(-\frac{mgh}{k_{\mathrm{B}}T}\right) \quad \Rightarrow \quad h = -\frac{k_{\mathrm{B}}T}{mg} \ln\left(\frac{p(h)}{p_{\mathrm{sat}}}\right)$$

$$d = \left(\frac{\alpha}{mgh}\right)^{1/3} \quad \Rightarrow \quad d = \left(\frac{-\alpha}{k_{\mathrm{B}}T \ln(p/p_{\mathrm{sat}})}\right)^{1/3}$$

- ➢ estimate of the flow of liquid for a critical superfluid velocity of $v_s = 30$ cm/s:

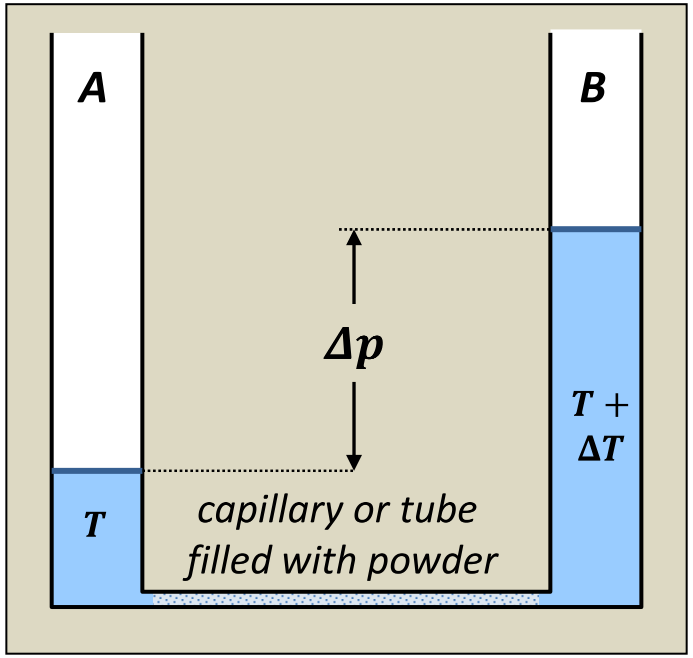
$$\dot{V}_{\mathrm{liq}} = 2\pi R d v_s \approx 1\ \mathrm{cm^3/h}\ \text{ for }\ R = 5\ \mathrm{mm} \ ➔\ \dot{V}_{\mathrm{gas}} \approx 1\ \mathrm{l/h}$$

- ➢ min. pressure achieved by vacuum pump with $\dot{V}_{\mathrm{pump}} \approx 10^4$ l/h

$$p_{\min} = p\, \dot{V}_{\mathrm{gas}}/\dot{V}_{\mathrm{pump}} \approx 10^{-4}\ \mathrm{bar} \quad \Longrightarrow \quad \boldsymbol{T_{\min} \approx 1}\ \mathrm{K}$$

I.3.2 Experimental Observations

- thermomechanical effect: ***generate ΔT by applying Δp***



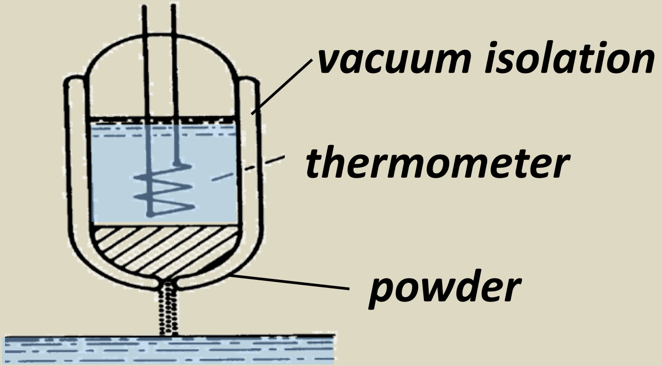
- starting point: same filling level and temperature T of container A and B
- apply Δp ➔ only superfluid component flows through capillary from A to B, but cannot transport heat from A to B
- ***temperature decreases in B and increases in A, since normal fluid fraction increases in A and decreases in B***
- remove Δp ➔ original state is recovered
- note: mass flow is not accompanied by heat flow since only superfluid component can flow

only superfluid can flow from A to B through narrow capillary

increase/decrease of normal fluid concentration in A/B

similar experiment:

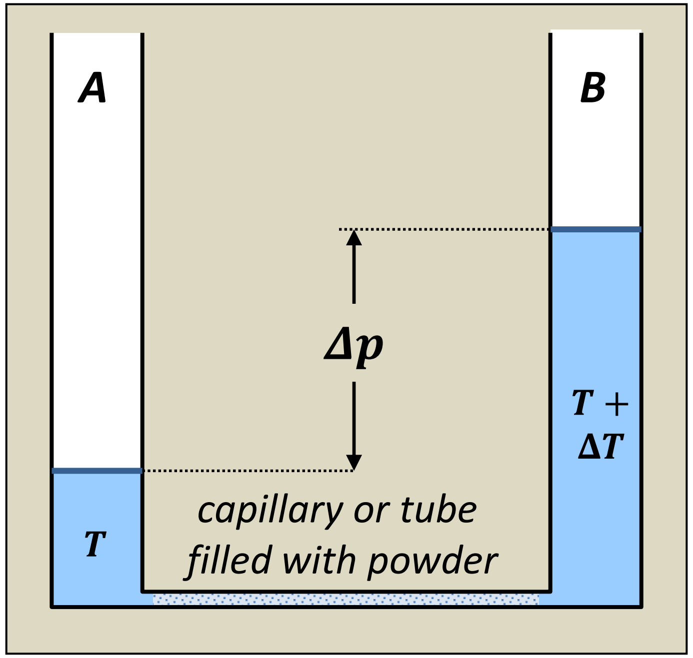
- only superfluid flows through powder
- increase of normal fluid concentration
- increase of *T* in container



I.3.2 Experimental Observations

$$G = U - TS + pV$$
$$dG = TdS - pdV + \mu dN - TdS - SdT + pdV + Vdp$$
$$dG = -SdT + Vdp + \mu dN$$

- **thermomechanical effect:** ***generate ΔT by applying Δp***



only superfluid can flow from A to B through narrow capillary

- $S = S_n, \quad \sigma_m = \frac{S}{m} = \frac{S_n}{m}$ (entropy per mass)

$$\Delta p = \frac{S_n}{V}\,\Delta T = \frac{S_n}{m}\frac{m}{V}\,\Delta T = \sigma_m\,\rho_n\,\Delta T$$

$$\boxed{\Delta p = \sigma_m\,\rho_n\,\Delta T}$$ ➔ $\Delta p \to 0$ for $\rho_n \to 0$

- the configuration can be considered as two systems A and B in equilibrium:

  ⇒ $\mu_A = \mu_B = \mu$ and particle number is conserved: $N = N_A + N_B$

- **Gibbs-Duhem relation**:

$$G(T, p, \mu_A, \mu_B) = N_A\mu_A + N_B\mu_B = N\mu$$

$$\Rightarrow dG = \mu dN + N d\mu \qquad (dG = 0 \;\rightarrow \mu dN = -N d\mu)$$

- comparison with $dG = -SdT + Vdp + \mu dN$ yields

$$-SdT + Vdp - Nd\mu = 0$$

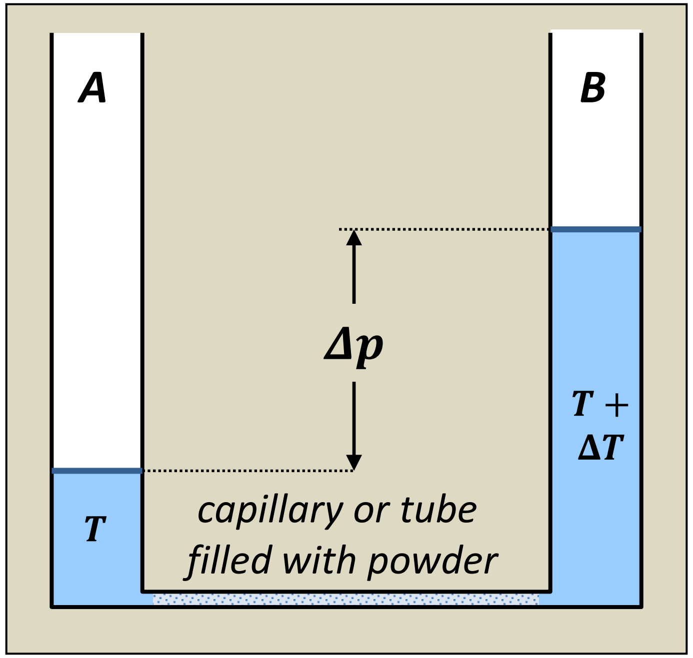
$$-S(T_B - T_A) + V(p_B - p_A) - N\,\underbrace{(\mu_B - \mu_A)}_{=0} = 0$$

$$\Rightarrow \quad \underbrace{V\,(p_B - p_A)}_{\text{work associated with mass flow}} = \underbrace{S\,(T_B - T_A)}_{\text{compensating heat change}}$$

- since superfluid flow does not carry heat, there is no mechanism to equilibrate T_A and T_B
- nevertheless, the particle flow results in $\mu_A = \mu_B = \mu$

I.3.2 Experimental Observations

- **thermomechanical effect:** ***generate ΔT by applying Δp***

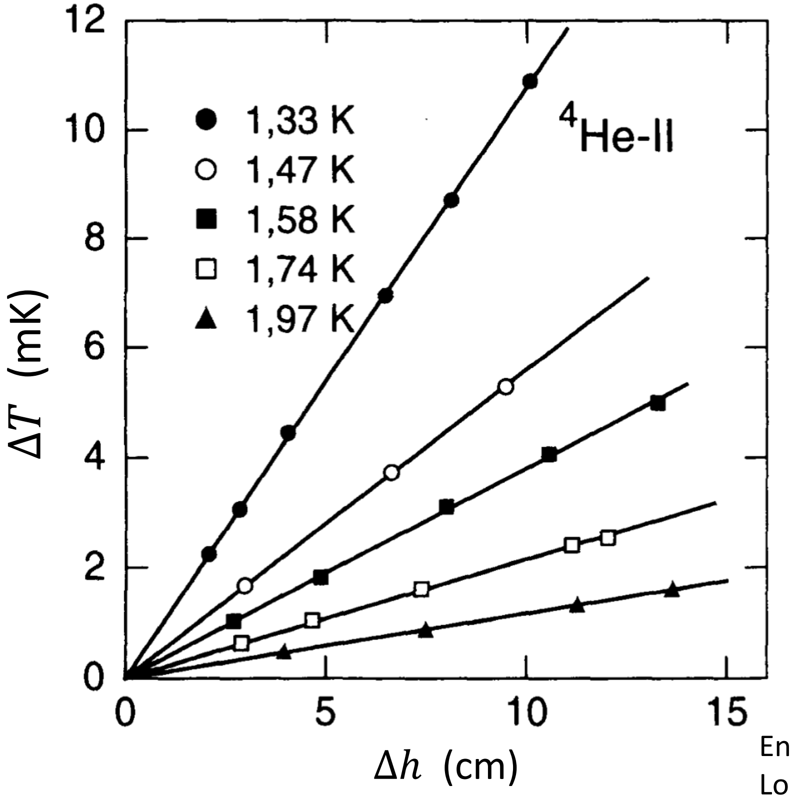


➢ $S = S_n, \quad \sigma_m = \frac{S}{m} = \frac{S_n}{m}$ (entropy per mass)

$$\Delta p = \frac{S_n}{V}\,\Delta T = \frac{S_n}{m}\frac{m}{V}\,\Delta T = \sigma_m\,\rho_n\,\Delta T$$

$$\Delta p = \sigma_m\,\rho_n\,\Delta T$$

➔ $\Delta p \to 0$ for $\rho_n \to 0$

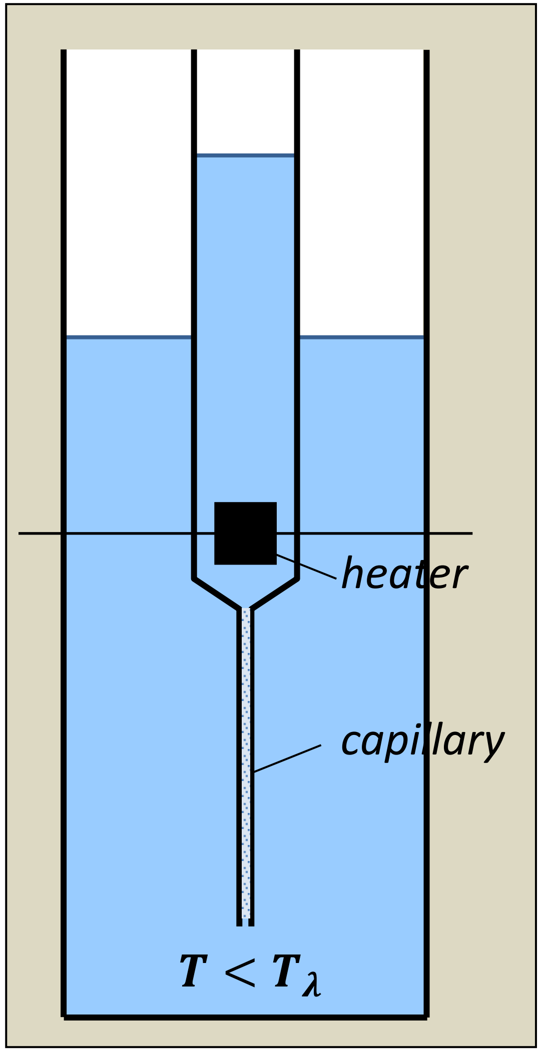


Enns, Hunklinger,
Low Temperature Physics, Springer (2000)

I.3.2 Experimental Observations

- **inverse thermomechanical effect:** ***generate Δp by applying ΔT***

$$\boxed{\Delta p = \sigma_n \, \rho_n \, \Delta T}$$



– heating generates increase of normal fluid concentration

– superfluid components flows through capillary to establish balance

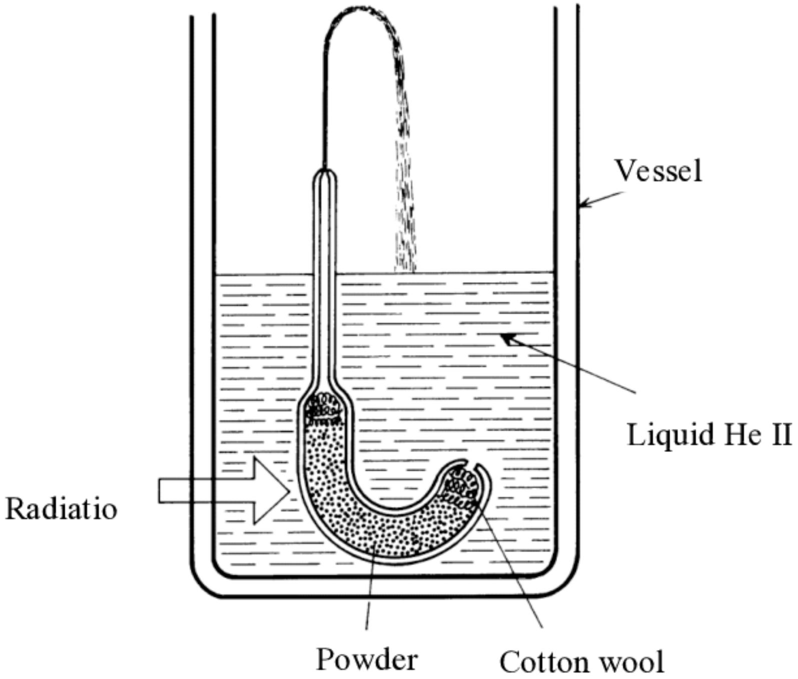
– analoguous to osmotic pressure

– superfluid flows very rapidly through capillary so that a fountain pressure can be generated

➔ ***fountain effect***

I.3.2 Experimental Observations

- **inverse thermomechanical effect: *fountain effect***



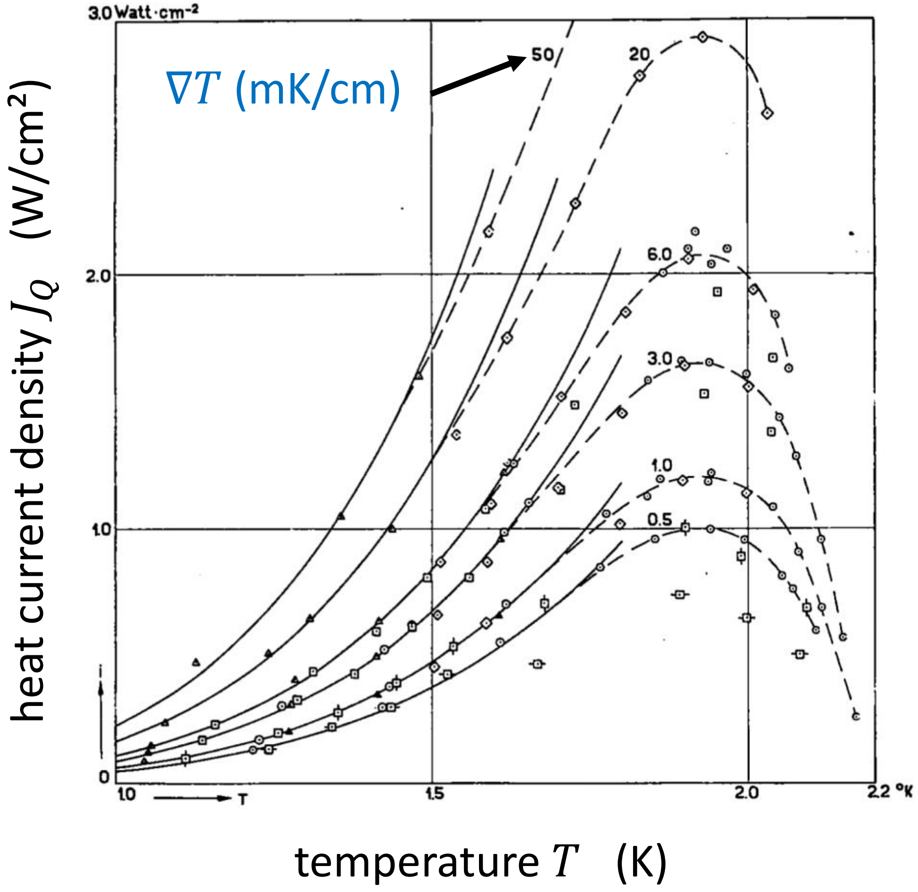
- light is absorbed by emery powder → heating
- lower opening is closed by cotton wool → only superfluid can enter
- a He fountain is generated by the thermomechanical effect
  (typical height up to 30 cm)

I.3.2 Experimental Observations

- **thermal conductivity**
  - thermal conductivity of superfluid ^{4}He is by more than five orders of magnitude larger than that of normal liquid ^{4}He
  - explains disappearance of boiling below T_λ
  - heat conduction through narrow capillaries has maximum at ≈ 1.8 K
  - heat current density varies as $\boldsymbol{J_Q \propto \nabla T^{1/3}}$
    (if we would use $\boldsymbol{J_Q \simeq -\lambda\, \nabla T}$, then $\boldsymbol{\lambda \propto \nabla T^{-2/3}}$ and would diverge for small ∇T)
  - linear $\boldsymbol{J_Q \propto \nabla T}$ dependence only for small heat current densities

➔ ***no Umklapp (U) processes possible in superfluid ^{4}He***
(no periodic lattice)

➔ ***excitations (phonons, rotons) move ballistically, only surface scattering***

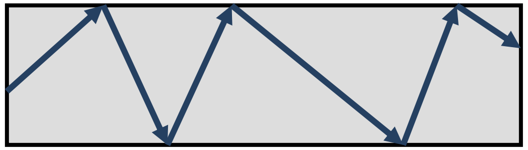


W.H. Keesom, B.F. Saris, L. Meyer, Physica **7**, 817 (1940)

I.3.2 Experimental Observations

- **thermal conductivity: comparison to thermal conductivity of solids**

solids at low *T*:



➔ *ballistic phonons*
➔ *N processes do not cause momentum relaxation*
➔ only surface scattering

kinetic gas theory:

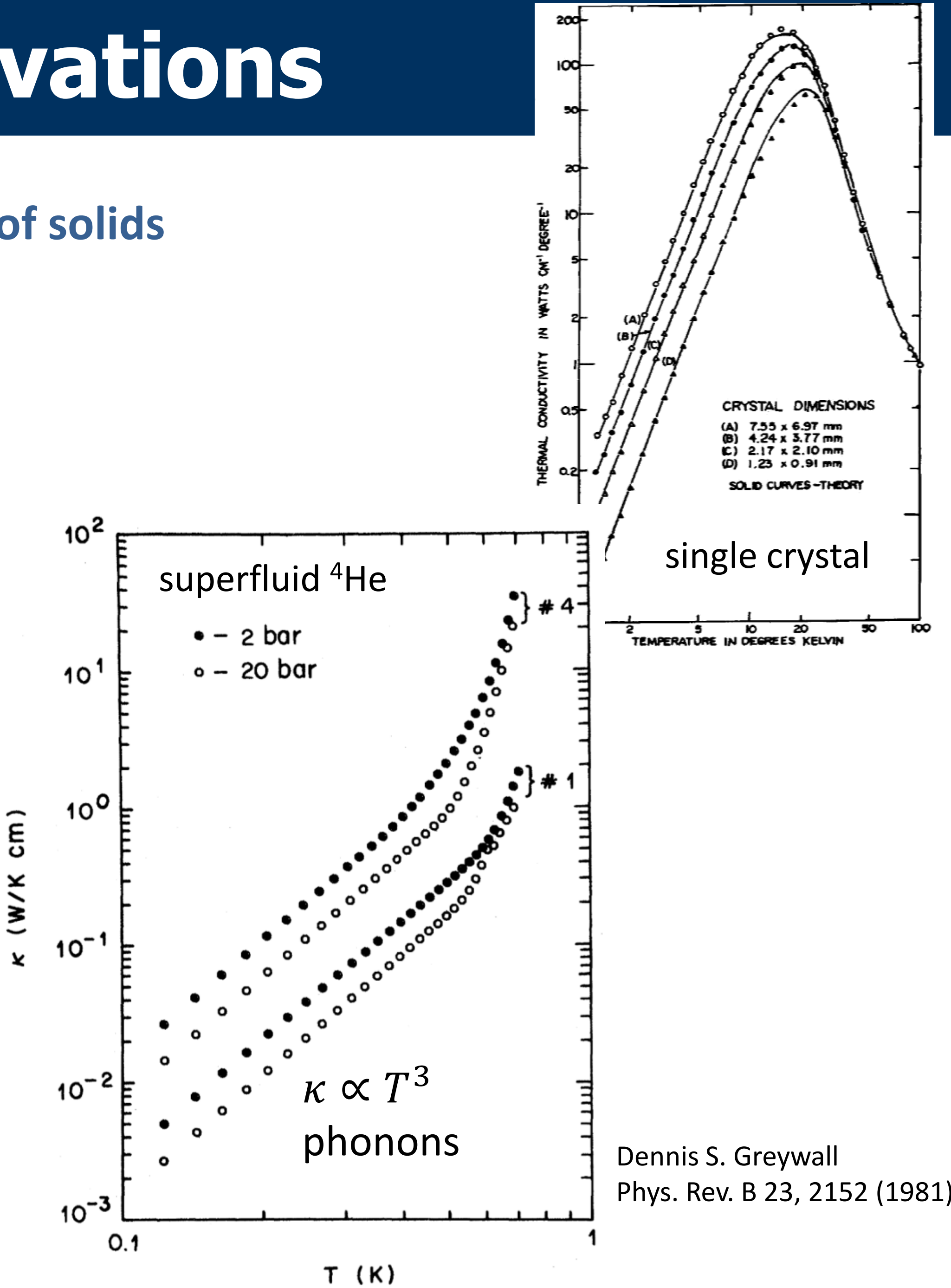
$$\Rightarrow \quad \kappa = \frac{1}{3}\,\langle v\rangle\,\ell\,C_V$$

function of d
(e.g. Hagen-Poiseuille)

$n \cdot c_V^\star$

thermal conductivity κ is proportional to

- *mean velocity of the particles*
- *mean free path*
- *density of the particles*
- *specific heat carried by particles*



Dennis S. Greywall
Phys. Rev. B 23, 2152 (1981)

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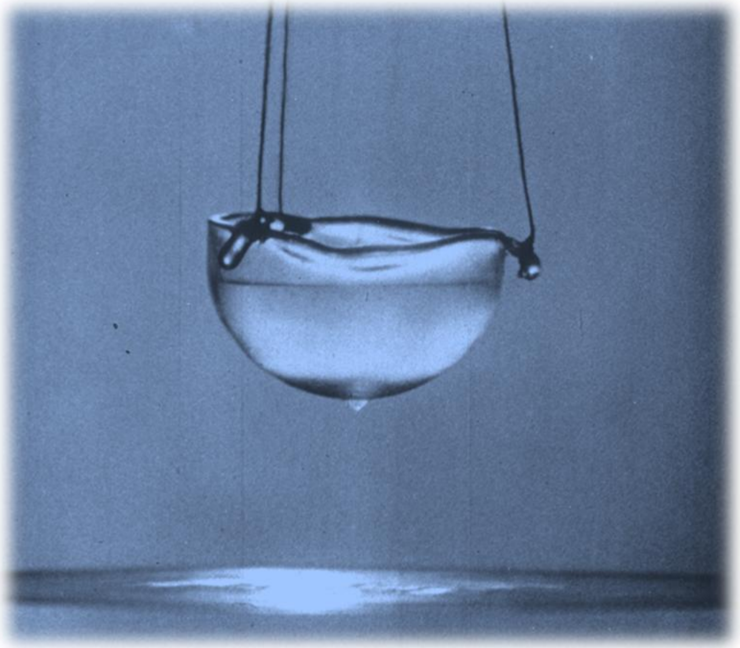
Superconductivity and Low Temperature Physics II



Lecture No. 5

R. Gross

Contents Part I: Quantum Liquids

Contents:

I.3.3 Two-Fluid Hydrodynamics

- **two-fluid hydrodynamics**

  – starting point for hydrodynamic description are **Navier-Stokes equations**:

➢ **normal fluid**:

entropy per mass

$$\underbrace{\rho_n \frac{\mathrm{D}\mathbf{v}_n}{\mathrm{D}t}}_{\text{inertia}} = -\underbrace{\frac{\rho_n}{\rho}\boldsymbol{\nabla} p}_{\text{pressure gradient}} - \underbrace{\rho_n \sigma_m \boldsymbol{\nabla} T}_{\text{temperature gradient}} - \underbrace{\frac{\rho_s \rho_n}{2\rho}\boldsymbol{\nabla}(\mathbf{v_n}-\mathbf{v_s})^2}_{\text{add. term due to compressibility}} + \underbrace{\eta_n \nabla^2 \mathbf{v}_n}_{\text{viscosity}}$$

ρ_s, ρ_n = superfluid and normal fluid densities
$\mathbf{v}_s, \mathbf{v}_n$ = superfluid and normal fluid velocities
η_n = normal fluid viscosity

terms have unit force/volume

with the definition of the material derivative $\mathrm{D}\mathbf{v}/\mathrm{D}t$:

$$\frac{\mathrm{D}\mathbf{v}_n}{\mathrm{D}t} = \frac{\partial \mathbf{v}_n}{\partial t} + \underbrace{\left(\frac{\partial \mathbf{v}_n}{\partial \mathbf{x}}\right)\cdot \mathbf{v}_n}_{\approx 0}$$

and

$$\left(\frac{\partial \mathbf{v}_\mathrm{n}}{\partial \mathbf{x}}\right) = \boldsymbol{\nabla}\left(\frac{\|\mathbf{v_n}\|^2}{\mathbf{2}}\right) + \underbrace{(\boldsymbol{\nabla}\times\mathbf{v_n})\times\mathbf{v_n}}$$

→ if $\mathbf{v}$ is small, quadratic terms in $\mathbf{v}$ can be neglected

curl of the velocity (called vorticity) disappears for irrotational flow (no vortices)

the tensor derivative of the velocity vector

I.3.3 Two-Fluid Hydrodynamics

- **two-fluid hydrodynamics**

➢ **superfluid (no viscous friction)**:

$$\underbrace{\rho_s \frac{\mathrm{D}\mathbf{v}_s}{\mathrm{D}t}}_{\text{inertia}} = -\underbrace{\frac{\rho_s}{\rho}\boldsymbol{\nabla}p}_{\text{pressure gradient}} - \underbrace{\rho_s \sigma_s\, \boldsymbol{\nabla}T}_{=0 \text{ as superfluid carries no entropy}} - \underbrace{\frac{\rho_s\rho_n}{2\rho}\,\boldsymbol{\nabla}(\mathbf{v_n}-\mathbf{v_s})^2}_{\text{add. term due to compressibility}} + \underbrace{\eta_s \nabla^2 \mathbf{v}_s}_{=0 \text{ as viscosity } \eta_s = 0}$$

ρ_s, ρ_n = superfluid and normal fluid densities
$\mathbf{v}_s, \mathbf{v}_n$ = superfluid and normal fluid velocities
η_n = normal fluid viscosity

terms have unit force/volume

with the definition of the material derivative $\mathrm{D}\mathbf{v}/\mathrm{D}t$:

$$\frac{\mathrm{D}\mathbf{v}_s}{\mathrm{D}t} = \frac{\partial \mathbf{v}_s}{\partial t} + \underbrace{\left(\frac{\partial \mathbf{v}_s}{\partial \mathbf{x}}\right)\cdot \mathbf{v}_s}_{\approx 0} \quad \text{and} \quad \left(\frac{\partial \mathbf{v}_s}{\partial \mathbf{x}}\right) = \boldsymbol{\nabla}\left(\frac{\|\mathbf{v}_s\|^2}{2}\right) + \underbrace{(\boldsymbol{\nabla}\times\mathbf{v}_s)\times\mathbf{v}_s}$$

the tensor derivative of the velocity vector

curl of the velocity (called vorticity)
disappears for irrotational flow (no vortices)

➔ **Euler type equation for superfluid** (if there are no vortices)

➔ detailed treatment including vortices: **Gross-Pitaevskii equation**

I.3.3 Two-Fluid Hydrodynamics

- **basics on the hydrodynamics of a *normal liquid*:**

  – state of local thermodynamic equilibrium is defined by five variables

    i. pressure $p(\mathbf{r},t)$
    ii. temperature $T(\mathbf{r},t)$
    iii. three components of the velocity field $\mathbf{v}(\mathbf{r},t)$

  – to determine these five variables, we have five equations (each linear in time)

    i. two scalar continuity equations for mass density ρ and entropy density $\sigma = \sigma_m \rho$ (σ_m = entropy per mass)

$$\frac{\partial \rho(\mathbf{r},t)}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{J}_{\boldsymbol{\rho}}(\mathbf{r},t) \qquad \frac{\partial \sigma(\mathbf{r},t)}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{J}_{\boldsymbol{\sigma}}(\mathbf{r},t)$$

$$\mathbf{J}_{\boldsymbol{\rho}}(\mathbf{r},t) = \rho(\mathbf{r},t)\,\mathbf{v}(\mathbf{r},t)$$
$$\mathbf{J}_{\boldsymbol{\sigma}}(\mathbf{r},t) = \sigma(\mathbf{r},t)\,\mathbf{v}(\mathbf{r},t)$$

    ii. three components of Euler‘s equation for the mass current

$$\frac{\partial \mathbf{J}_{\rho}(\mathbf{r},t)}{\partial t} = -\boldsymbol{\nabla}\, p(\mathbf{r},t)$$

I.3.3 Two-Fluid Hydrodynamics

- **basics on the hydrodynamics of a fluid composed of *normal fluid* and *superfluid*:**
  - state of local thermodynamic equilibrium is defined by eight variables
    1. pressure $p(\mathbf{r},t)$
    2. temperature $T(\mathbf{r},t)$
    3. three components of the velocity field $\mathbf{v}_s(\mathbf{r},t)$ plus three components of the velocity field $\mathbf{v}_n(\mathbf{r},t)$
  - to determine these eight variables, we have eight equations (each linear in time)
    1. two scalar continuity equations for mass density ρ and entropy density $\sigma = \sigma_m \rho$ (σ_m = entropy per mass)

$$\frac{\partial \rho(\mathbf{r},t)}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{J}_{\boldsymbol{\rho}}(\mathbf{r},t)$$

$$\frac{\partial \sigma(\mathbf{r},t)}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{J}_{\sigma}(\mathbf{r},t)$$

$$\mathbf{J}_{\boldsymbol{\rho}}(\mathbf{r},t) = \rho_s(\mathbf{r},t)\ \mathbf{v}_s(\mathbf{r},t) + \rho_n(\mathbf{r},t)\ \mathbf{v}_n(\mathbf{r},t)$$

$$\mathbf{J}_{\boldsymbol{\sigma}}(\mathbf{r},t) = \sigma(\mathbf{r},t)\mathbf{v}_n(\mathbf{r},t) = [\rho_n(\mathbf{r},t)\sigma_m]\ \mathbf{v}_n(\mathbf{r},t$$

(superfluid does not carry entropy, as condensate is associated with a single one-particle state)

    2. two vector equations for $\mathbf{J}_{\boldsymbol{\rho}}(\mathbf{r},t)$ and $\mathbf{v}_s(\mathbf{r},t)$

$$\frac{\partial \mathbf{J}_{\rho}(\mathbf{r},t)}{\partial t} = -\boldsymbol{\nabla}\, p(\mathbf{r},t)$$

$$\frac{\partial \mathbf{v}_s(\mathbf{r},t)}{\partial t} = -\frac{1}{m}\ \boldsymbol{\nabla}\, \mu(\mathbf{r},t)$$

μ = chemical potential, depends on $p(\mathbf{r},t)$ and $T(\mathbf{r},t)$

I.3.3 Two-Fluid Hydrodynamics

- **derivation of expression of equation of motion for $\mathbf{v}_s$ using $\frac{\partial \mathbf{v}_s(\mathbf{r},t)}{\partial t} = -\frac{1}{m}\,\boldsymbol{\nabla}\,\mu(\mathbf{r},t)$**
  - exact derivation of equation of motion of normal and superfluid phase is difficult
    → we use approach following „Gedanken“ experiment of Landau
  - we are adding superfluid component at constant volume and consider change of inner energy
    → *the resulting operation only changes the mass of the superfluid component, no entropy and volume change)*

$$dU = TdS - pdV + \mu dN = \underbrace{TdS}_{=0} - \underbrace{pdV}_{=0} + \underbrace{\mu_m}_{\text{potential energy per mass}} dm \qquad \Rightarrow -\boldsymbol{\nabla}(\mu_m dm) = \mathbf{F} = dm\frac{d\mathbf{v}_s}{dt}$$

  - since adding particles to the superfluid does not change the energy, we have

$$0 = TdS - pdV + \mu_m dm \qquad \Rightarrow \frac{d\mathbf{v}_s}{dt} = \frac{\Delta S}{\Delta m}\,\boldsymbol{\nabla}T - \frac{\Delta V}{\Delta m}\,\boldsymbol{\nabla}p = \sigma_m\,\boldsymbol{\nabla}T - \frac{1}{\rho}\,\boldsymbol{\nabla}p$$

  - with $\frac{d\mathbf{v}_s}{dt} = \frac{\partial \mathbf{v}_s}{\partial t} + v_s \cdot \nabla v_s \simeq \frac{\partial \mathbf{v}_s}{\partial t}$ (neglecting quadratic terms in v_s) we obtain

$$\boxed{\frac{\partial \mathbf{v}_s}{\partial t} = \sigma_m\,\boldsymbol{\nabla}T - \frac{1}{\rho}\,\boldsymbol{\nabla}p} \qquad \Rightarrow \quad m\frac{\partial \mathbf{v}_s}{\partial t} = S\,\boldsymbol{\nabla}T - V\,\boldsymbol{\nabla}p$$

I.3.3 Two-Fluid Hydrodynamics

- **derivation of expression of equation of motion for $\mathbf{v}_n$**

  – for the normal fluid component we obtain

$$\frac{\partial \mathbf{J}_\rho}{\partial t} = \frac{\partial(\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s)}{\partial t} = -\boldsymbol{\nabla}\mathbf{p} \qquad \Rightarrow \frac{\partial \mathbf{v}_n}{\partial t} = -\frac{1}{\rho_n}\boldsymbol{\nabla}\mathbf{p} - \frac{\rho_s}{\rho_n}\frac{\partial \mathbf{v}_s}{\partial t}$$

$$\frac{\partial \mathbf{v}_s}{\partial t} = \sigma_m \, \boldsymbol{\nabla} T - \frac{1}{\rho}\, \boldsymbol{\nabla} p$$

with $\frac{\partial \mathbf{v}_s}{\partial t} = \sigma_m \, \boldsymbol{\nabla} T - \frac{1}{\rho}\, \boldsymbol{\nabla} p$ we then obtain

$$\frac{\partial \mathbf{v}_n}{\partial t} = -\frac{1}{\rho_n}\boldsymbol{\nabla}\mathbf{p} - \frac{\rho_s}{\rho_n}\sigma_m \, \boldsymbol{\nabla} T + \frac{\rho_s}{\rho_n}\frac{1}{\rho}\, \boldsymbol{\nabla} p = -\frac{\rho_s}{\rho_n}\sigma_m \, \boldsymbol{\nabla} T + \frac{\rho_s - \rho}{\rho_n \rho}\boldsymbol{\nabla} p = -\frac{\rho_s}{\rho_n}\sigma_m \, \boldsymbol{\nabla} T - \frac{1}{\rho}\boldsymbol{\nabla} p$$

$$\frac{\partial \mathbf{v}_n}{\partial t} = -\frac{\rho_s}{\rho_n}\sigma_m \, \boldsymbol{\nabla} T - \frac{1}{\rho}\boldsymbol{\nabla} p \qquad \Rightarrow \quad m\frac{\partial \mathbf{v}_n}{\partial t} =- -\frac{\rho_s}{\rho_n} S \, \boldsymbol{\nabla} T - V \, \boldsymbol{\nabla} p$$

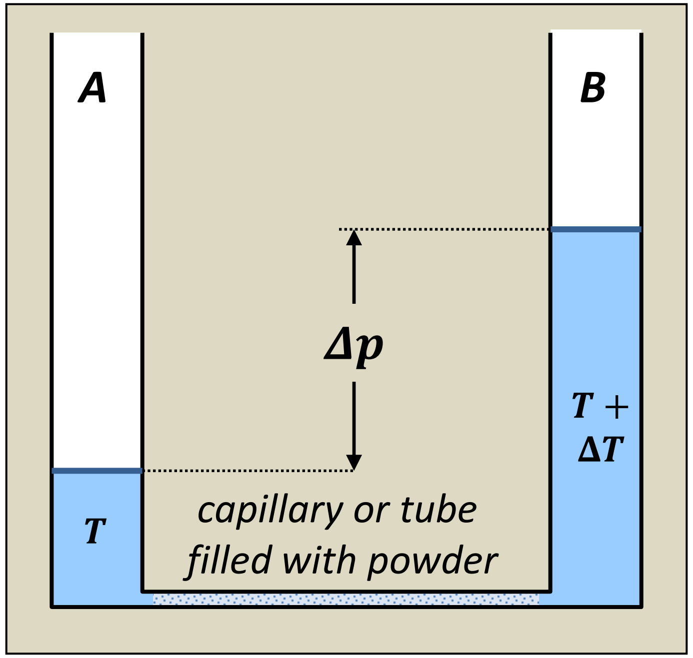
  – **equations of motion**

$$\frac{\partial \mathbf{v}_s}{\partial t} = \sigma_m \, \boldsymbol{\nabla} T - \frac{1}{\rho}\, \boldsymbol{\nabla} p \qquad \frac{\partial \mathbf{v}_n}{\partial t} = -\frac{\rho_s}{\rho_n}\sigma_m \, \boldsymbol{\nabla} T - \frac{1}{\rho}\boldsymbol{\nabla} p$$

note:
the equations of motion are valid only for the linear regime !

I.3.3 Two-Fluid Hydrodynamics

- **example: explanation of thermomechanical effect:**



$$\frac{\partial \mathbf{v}_s}{\partial t} = \sigma_m\,\nabla T - \frac{1}{\rho}\,\nabla p$$

- in steady state: $\frac{\partial \mathbf{v}_s}{\partial t} = 0$

$$0 = \sigma_m\,\nabla T - \frac{1}{\rho}\,\nabla p$$

$$\Delta p = \sigma_m \rho\,\Delta T = \sigma\,\Delta T$$

I.3.3 Two-Fluid Model

- **two-fluid hydrodynamics can be used to understand sound propagation in superfluid helium**

  - ***first sound:***
    normal and superfluid component move ***in-phase:*** $v_n = v_s$ $v_1 = 238$ m/s for $T \to 0$

  - ***second sound:***
    normal and superfluid component move ***out-of-phase:*** $\rho_n v_n + \rho_s v_s = 0$ $v_2 = \frac{v_1}{\sqrt{3}} \simeq 137$ m/s for $T \to 0$
    (Peshkov 1944)

  - ***third sound:*** ***waves propagating in thin He films***

  - ***fourth sound:*** ***compression wave propagating in a super leak***

    (super leak: very small opening or capillary, in which normal fluid cannot move because of its finite viscosity)

use two-fluid hydrodynamics to understand this behavior

I.3.3 Two-Fluid Hydrodynamics

- **two-fluid hydrodynamics – sound propagation (1)**

– take time derivative of $\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{J}_{\boldsymbol{\rho}}$ and insert into $\frac{\partial \mathbf{J}}{\partial t} + \underbrace{\rho \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v}}_{\approx 0} = -\boldsymbol{\nabla} \mathbf{p}$

$$\frac{\partial^2 \rho}{\partial t^2} = -\boldsymbol{\nabla} \cdot \frac{\partial \mathbf{J}_{\boldsymbol{\rho}}}{\partial t} = -\boldsymbol{\nabla} \cdot (-\boldsymbol{\nabla} \mathbf{p}) \quad \Rightarrow \quad \boxed{\frac{\partial^2 \rho}{\partial t^2} = \nabla^2 p}$$

– elimination of terms with $\mathbf{v}_n$ and $\mathbf{v}_s$ in $\frac{\partial S}{\partial t} = \frac{\partial(\rho \sigma_m)}{\partial t} = -\boldsymbol{\nabla} \cdot (\rho \sigma_m)\, \mathbf{v_n}$ and $\frac{\partial \mathbf{v}_s}{\partial t} = \sigma_m \, \boldsymbol{\nabla} T - \frac{1}{\rho} \, \boldsymbol{\nabla} p$ (cannot be observed experimentally

→ after some steps and neglecting higher order terms one obtains

$$\frac{\partial \mathbf{v}_s}{\partial t} = \sigma_m \, \boldsymbol{\nabla} T - \frac{1}{\rho} \, \boldsymbol{\nabla} p$$

$$\frac{\partial \mathbf{v}_n}{\partial t} = -\frac{\rho_s}{\rho_n} \sigma_m \, \boldsymbol{\nabla} T - \frac{1}{\rho} \boldsymbol{\nabla} p$$

$$\boxed{\frac{\partial^2 \sigma_m}{\partial t^2} = \frac{\rho_s}{\rho_n} \sigma_m^{\mathbf{2}} \, \nabla^2 T}$$

– in the two equations we have four variables: ρ, σ, p and T → only two of them are independent, we choose ρ and σ

– dependence of p and T on ρ and σ can be expressed as

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_{\sigma} \delta \rho + \left(\frac{\partial p}{\partial \sigma_m}\right)_{\rho} \delta \sigma_m \qquad \delta T = \left(\frac{\partial T}{\partial \rho}\right)_{\sigma} \delta \rho + \left(\frac{\partial T}{\partial \sigma_m}\right)_{\rho} \delta \sigma_m$$

we insert relations into expression for $\frac{\partial^2 \rho}{\partial t^2}$ and $\frac{\partial^2 \sigma_m}{\partial t^2}$

I.3.3 Two-Fluid Hydrodynamics

- **two-fluid hydrodynamics – sound propagation (2)**
  - insertion yields

$$\frac{\partial^2\rho}{\partial t^2} = \left(\frac{\partial p}{\partial\rho}\right)_\sigma \nabla^2\rho + \left(\frac{\partial p}{\partial\sigma_m}\right)_\rho \nabla^2\sigma_m$$

$$\frac{\partial^2\sigma}{\partial t^2} = \frac{\rho_s}{\rho_n}\sigma_m^2 \left[\left(\frac{\partial T}{\partial\rho}\right)_\sigma \nabla^2\rho + \left(\frac{\partial T}{\partial\sigma_m}\right)_\rho \nabla^2\sigma_m\right]$$

2nd oder partial differential equations

  - to solve these equations we use the ansatz (1D for simplicity)

$$\rho = \rho_0 + \rho_1 \exp\left[\imath\omega\left(t - \frac{x}{v}\right)\right] \qquad \sigma_m = \sigma_0 + \sigma_1 \exp\left[\imath\omega\left(t - \frac{x}{v}\right)\right]$$

  - inserting the ansatz yields

linear system of equations

$$\left[\left(\frac{v}{v_1}\right)^2 - 1\right]\rho_1 + \left(\frac{\partial p}{\partial\sigma_m}\right)_\rho \left(\frac{\partial\rho}{\partial p}\right)_\sigma \sigma_1 = 0$$

$$\left(\frac{\partial T}{\partial\rho}\right)_\sigma \left(\frac{\partial\sigma_m}{\partial T}\right)_\rho \rho_1 + \left[\left(\frac{v}{v_2}\right)^2 - 1\right]\sigma_1 = 0$$

with abbreviations

$$v_1^2 = \left(\frac{\partial p}{\partial\rho}\right)_{\sigma_m}$$

$$v_2^2 = \frac{\rho_s}{\rho_n}\sigma_m^2\left(\frac{\partial T}{\partial\sigma_m}\right)_\rho$$

I.3.3 Two-Fluid Hydrodynamics

- **two-fluid hydrodynamics – sound propagation (3)**
  - constraint equation for coefficients

$$\left[\left(\frac{v}{v_1}\right)^2 - 1\right] \cdot \left[\left(\frac{v}{v_2}\right)^2 - 1\right] = \left(\frac{\partial p}{\partial \sigma_m}\right)_\rho \left(\frac{\partial \rho}{\partial p}\right)_\sigma \left(\frac{\partial T}{\partial \rho}\right)_\sigma \left(\frac{\partial \sigma_m}{\partial T}\right)_\rho$$

  - the r.h.s. can be rewritten by using known thermodynamic relations

$$\left[\left(\frac{v}{v_1}\right)^2 - 1\right] \cdot \left[\left(\frac{v}{v_2}\right)^2 - 1\right] = \frac{C_p - C_V}{C_p}$$

C_p, C_V: specific heat at constant pressure/volume

  - for superfluid helium the difference $C_p - C_V$ is very small → we can use the approximation

$$\boxed{\left[\left(\frac{v}{v_1}\right)^2 - 1\right] \cdot \left[\left(\frac{v}{v_2}\right)^2 - 1\right] \simeq 0}$$

$$v_1^2 = \left(\frac{\partial p}{\partial \rho}\right)_{\sigma_m} \qquad v_2^2 = \frac{\rho_s}{\rho_n} \sigma^2 \left(\frac{\partial T}{\partial \sigma_m}\right)_\rho$$

I.3.3 Two-Fluid Hydrodynamics

- **two-fluid hydrodynamics: first sound**
  - propagation of usual sound waves with velocity $v = v_1$ is given by

$$\left[\left(\frac{v}{v_1}\right)^2 - 1\right]\rho_1 + \left(\frac{\partial p}{\partial \sigma}\right)_\rho \left(\frac{\partial \rho}{\partial p}\right)_\sigma \sigma_1 = 0$$

if we set $\rho_1 \neq 0$ and $\sigma_1 = 0$

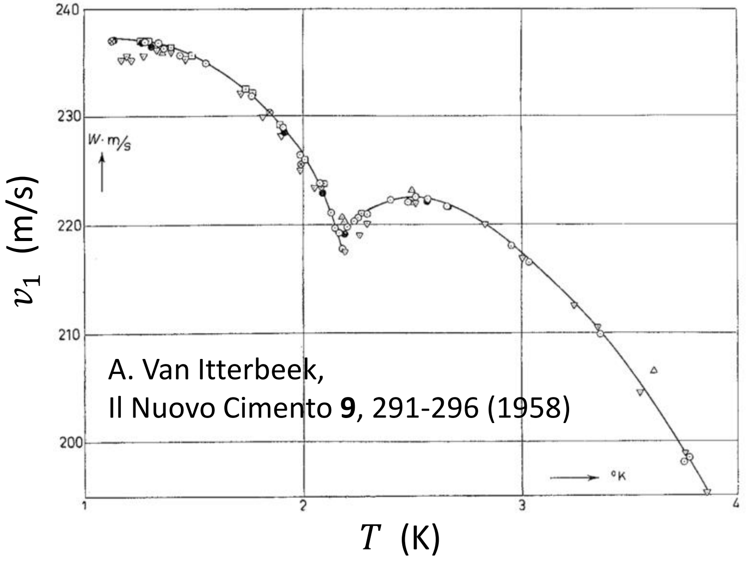
under these conditions we have $\nabla T = 0$ as for a usual sound waves (***first sound***)

  - from $\frac{\partial \mathbf{J}_\rho}{\partial t} = -\boldsymbol{\nabla}\mathbf{p}$ and $\frac{\partial \mathbf{v}_s}{\partial t} = \sigma_m \boldsymbol{\nabla} T - \frac{1}{\rho}\boldsymbol{\nabla} p$ we obtain

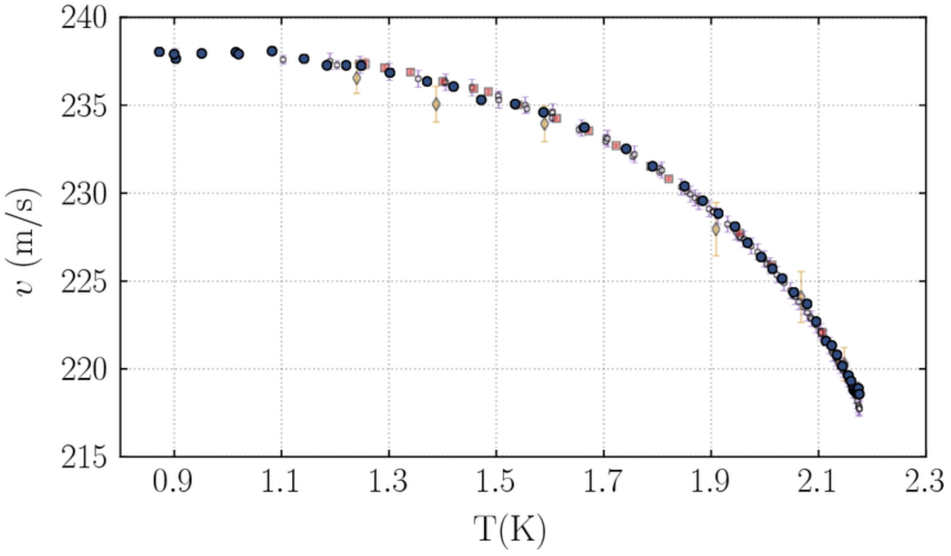
$v_n = v_s$ (acoustic mode)

➔ ***normal and superfluid component move in phase (first sound)***

- ➢ density changes are adiabatic
- ➢ superfluid He behaves as usual liquid
- ➢ sound velocity $v = v_1 = 238\ m/s$ for $T \to 0$
- ➢ $v_1(T)$ shows anomaly at $T = T_\lambda$



A. Van Itterbeek,
Il Nuovo Cimento **9**, 291-296 (1958)



L. Djadaojee, A. Douillet, J. Grucker, Jules, J. Low Temp. Phys. 203 (2021)

I.3.3 Two-Fluid Hydrodynamics

- **two-fluid hydrodynamics: second sound**
  - if we set $\rho_1 = 0$ and $\sigma_1 \neq 0$ we obtain temperature wave propagating with velocity $v = v_2$

$$\left(\frac{\partial T}{\partial \rho}\right)_\sigma \left(\frac{\partial \sigma}{\partial T}\right)_\rho \rho_1 + \left[\left(\frac{v}{v_2}\right)^2 - 1\right] \sigma_1 = 0$$

  - from $\frac{\partial^2 \rho}{\partial t^2} = \nabla^2 p$ we obtain $\nabla p = 0$
  - together with Euler's equation $\frac{\partial \mathbf{J}_\rho}{\partial t} + \underbrace{\rho \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v}}_{\approx 0} = -\boldsymbol{\nabla} \mathbf{p}$ we obtain: $\frac{\partial \mathbf{J}_\rho}{\partial t} = \frac{\partial(\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s)}{\partial t} = 0$
  - since $\mathbf{J}_\rho$ cannot be constant in a closed container → $\mathbf{J}_\rho = \mathbf{0}$

$$\rho_n v_n + \rho_s v_s = 0$$

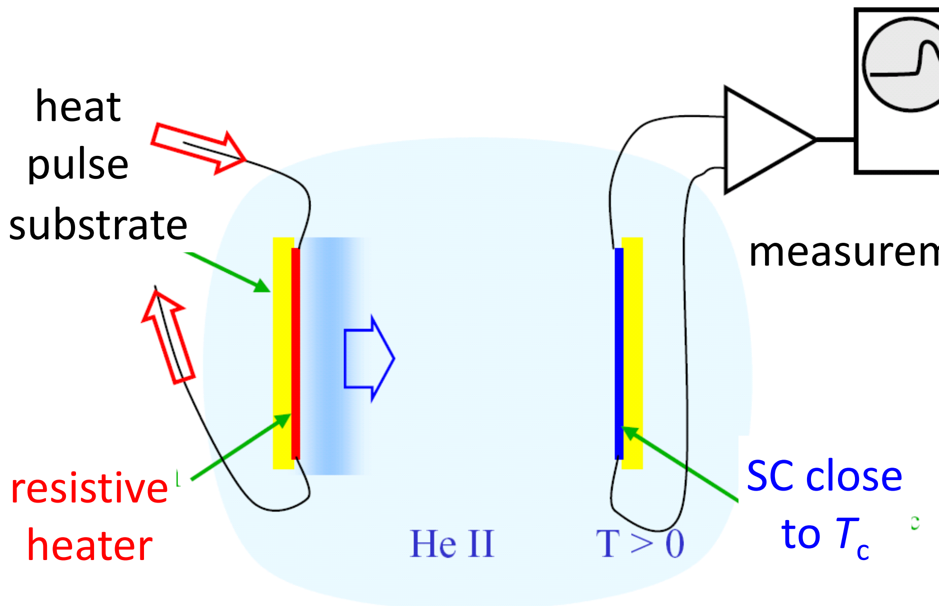
(optical mode)

  - velocity of the temperature wave

$$v_2^2 = \frac{\rho_s}{\rho_n} \sigma_m^{\mathbf{2}} \left(\frac{\partial T}{\partial \sigma_m}\right)_\rho = \frac{\rho_s}{\rho_n} \sigma_m^2 \frac{T}{C_p}, \qquad v_2 = \frac{v_1}{\sqrt{3}} \simeq 137 \text{ m/s for } T \to 0$$

I.3.3 Two-Fluid Hydrodynamics

- **two-fluid hydrodynamics: second sound – experiment**



heat pulse

substrate

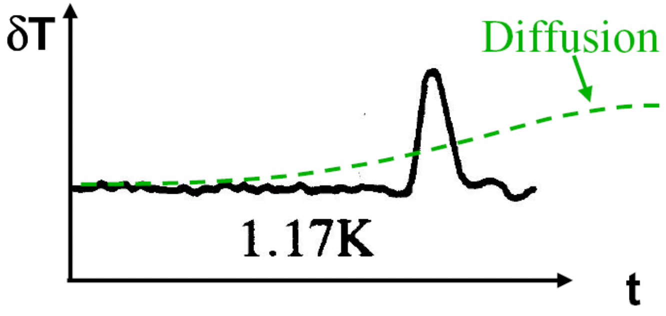
resistive heater

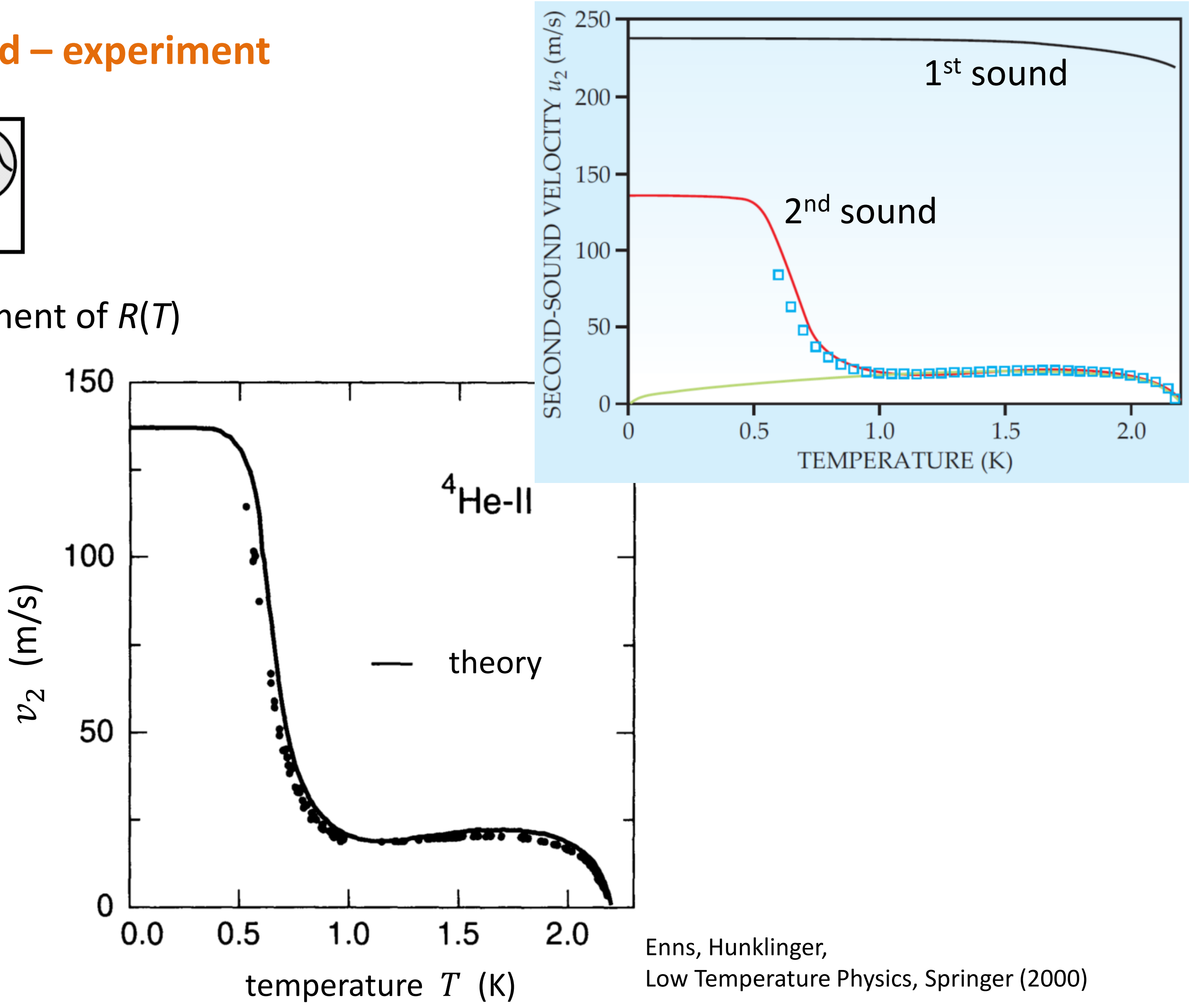
He II

T > 0

SC close to T_c

measurement of $R(T)$



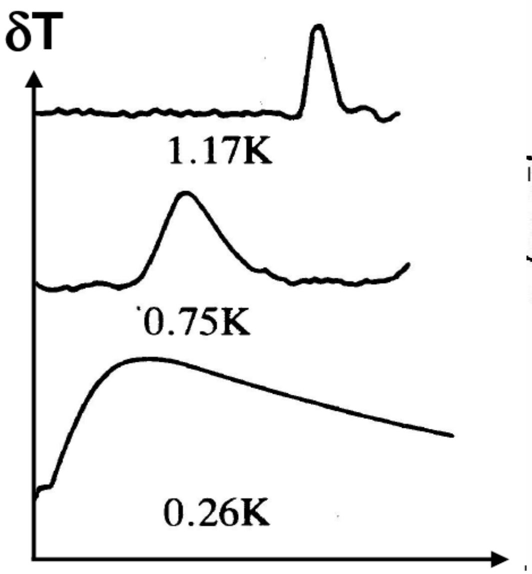


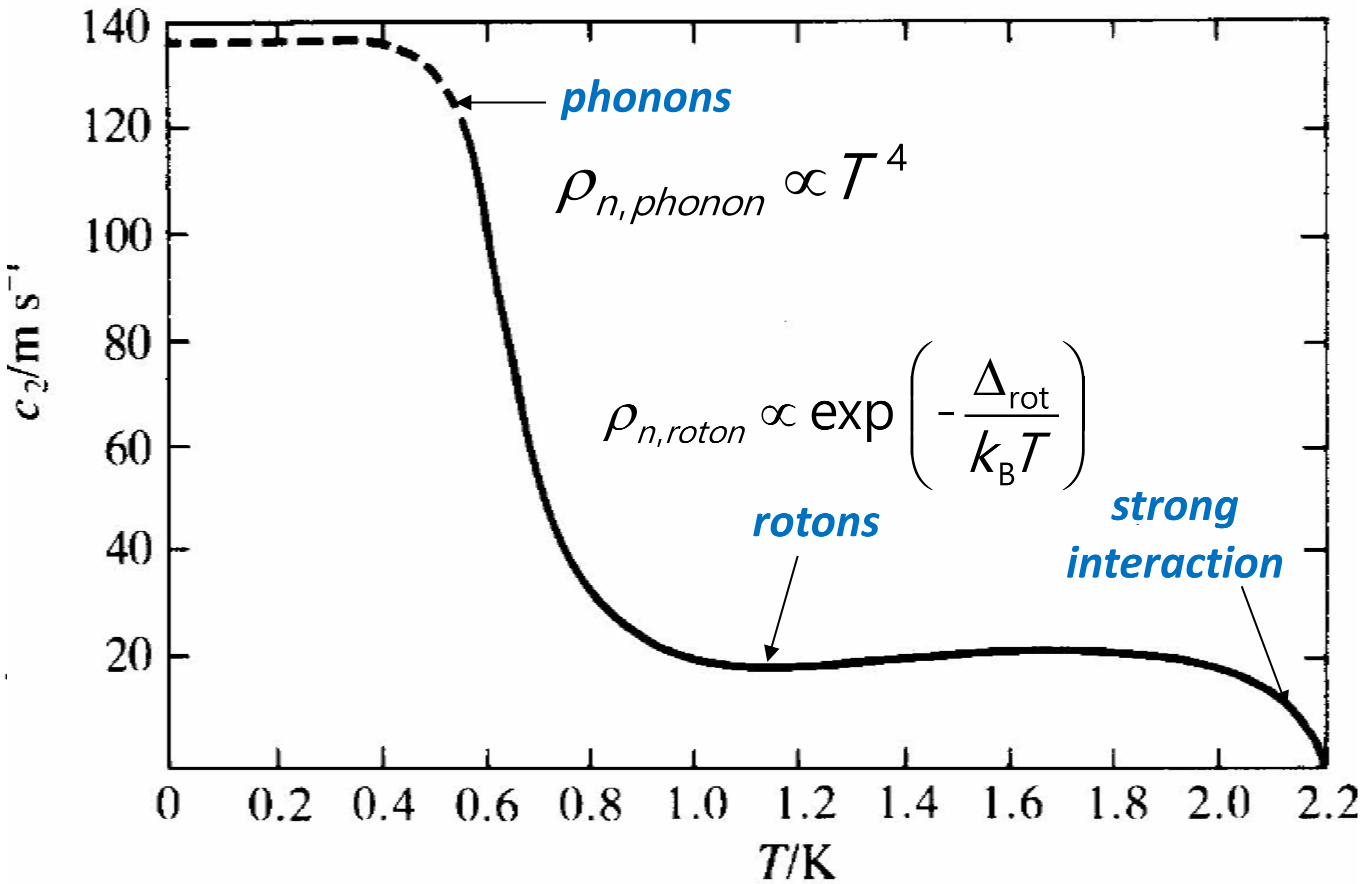
Enns, Hunklinger,
Low Temperature Physics, Springer (2000)

I.3.3 Two-Fluid Hydrodynamics

- **two-fluid hydrodynamics: second sound – experiment**

pulse experiment





$$\rho_{n,phonon} \propto T^4$$

$$\rho_{n,roton} \propto \exp\left(-\frac{\Delta_{rot}}{k_B T}\right)$$

$T \to T_\lambda$:

→ collision probability increases
→ picture of elementary excitations no longer valid
→ vortex rings

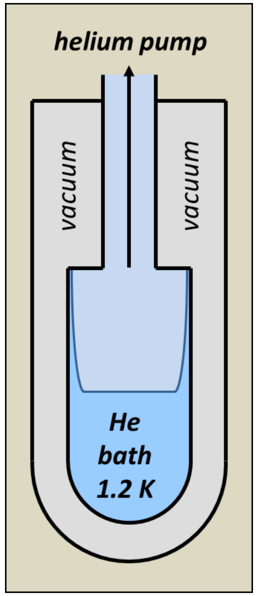
I.3.3 Two-Fluid Hydrodynamics

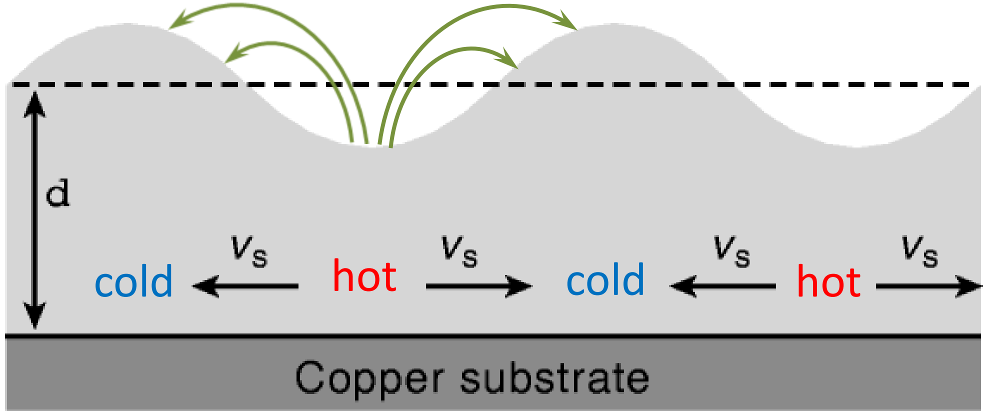
- **two-fluid hydrodynamics: third sound – thickness waves in thin helium films**
  - we already discussed the thickness of helium films on surfaces

$$d = \left(\frac{-\alpha}{k_B T \ln(p/p_{\mathrm{sat}})}\right)^{1/3} = \left(\frac{\alpha}{k_B T(\ln p_{\mathrm{sat}} - \ln p)}\right)^{1/3}$$

➔ film thickness depends on pressure and temperature

➔ local periodic heating causes thickness variation propagating along surface





- ➢ viscous normal component is fixed at surface
- ➢ superfluid component oscillates freely parallel to surface

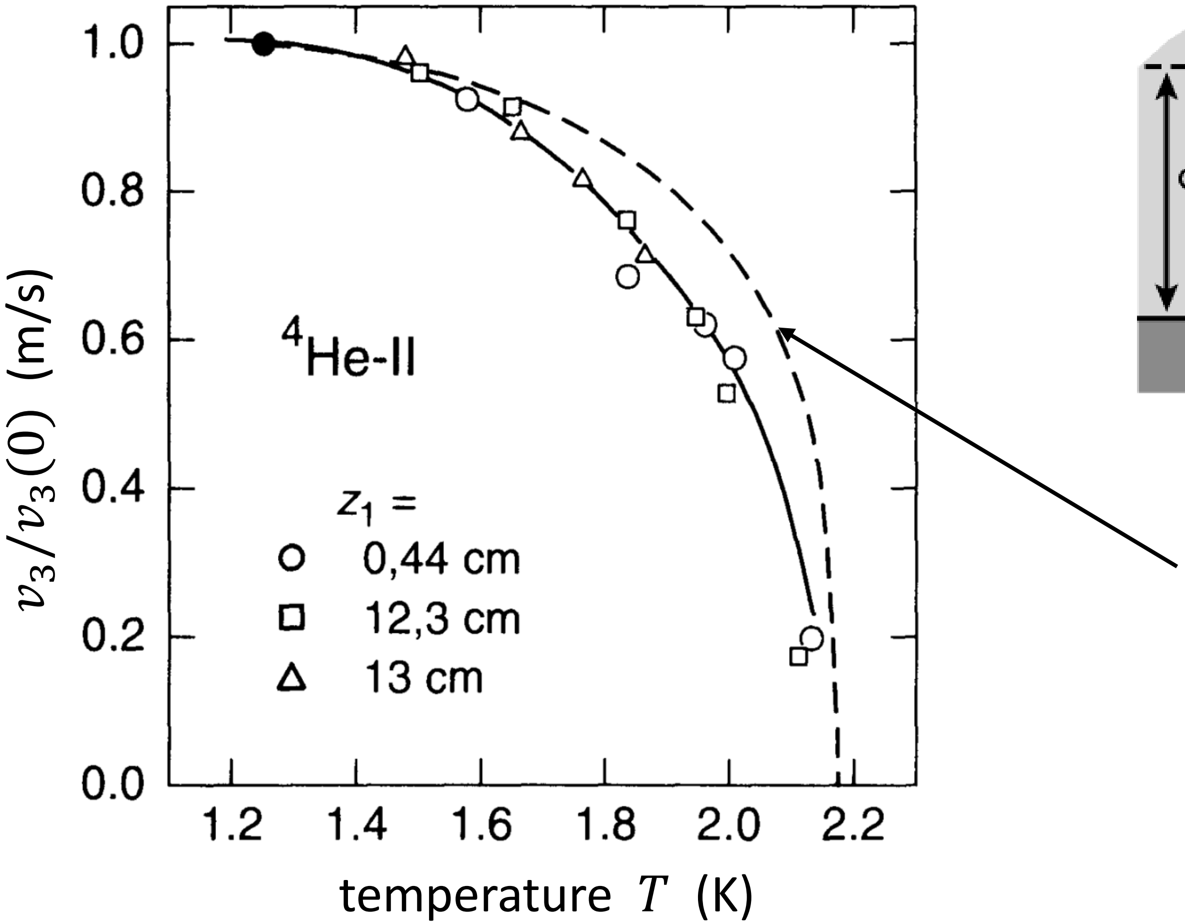
wave crest: surplus of superfluid → **cold**
wave valley: deficit of superfluid → **hot**

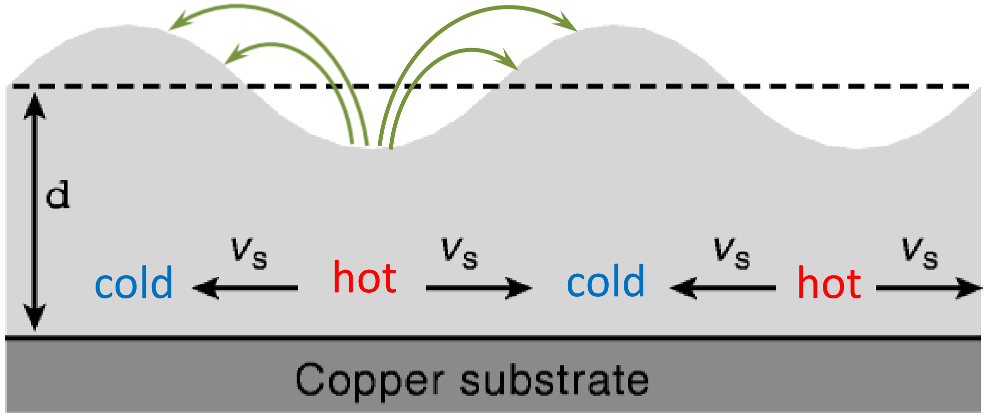
sound velocity: $v_3 \sim 10 \ldots 100$ m/s

detailed discussion complicated due to additional evaporation and condensation at surface

I.3.3 Two-Fluid Hydrodynamics

- **two-fluid hydrodynamics: third sound – thickness waves in thin helium films**





$$v_3^2 = \frac{3\alpha}{d^3}\frac{\rho_s}{\rho}\left(1+\frac{\sigma T}{L}\right)$$

L = latent heat

Enns, Hunklinger,
Low Temperature Physics, Springer (2000)

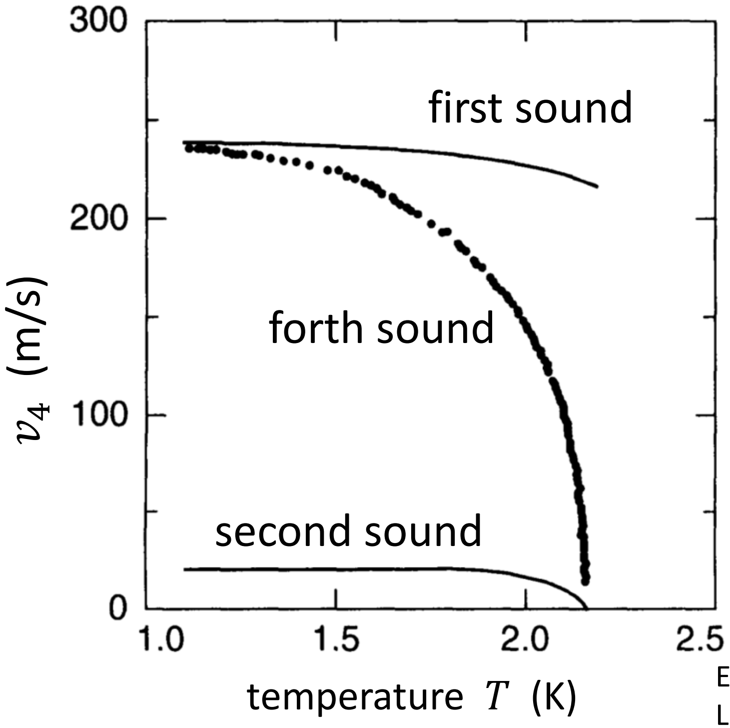
I.3.3 Two-Fluid Hydrodynamics

- **two-fluid hydrodynamics: forth sound – compression waves in superleaks**

(super leak = very small opening or capillary, in which normal fluid cannot move because of its finite viscosity)

  - only superfluid can move, whereas normal fluid is blocked due to its finite viscosity
  - compression waves are not only density oscillations, but also oscillations of the superfluid density ρ_s/ρ_n, the pressure p, the temperature T, and the entropy σ
  - sound velocity (without derivation)

$$v_4^2 \simeq \frac{\rho_s}{\rho} v_1^2 + \frac{\rho_n}{\rho} v_2^2$$



Enns, Hunklinger,
Low Temperature Physics, Springer (2000)

I.3.3 Two-Fluid Hydrodynamics

- **two-fluid hydrodynamics: heat transport**
  - example: mass flow $\dot{V}_n$ through capillary of diameter d:
    - the normal fluid component flows from hot to cold side
    - this mass flow is compensated by opposite flow of superfluid component
    - due to different entropies of both components this results in a finite heat flow (limited by η_n)
  - mass flow $\dot{V}_n$ of normal component according to classical hydrodynamics

$$\dot{V}_n = \frac{\beta}{\eta_n} \frac{\Delta p}{L}$$

η_n: viscosity of normal component
L: length of capillary
Δp: pressure difference

  - the parameter β depends on the geometry of the capillary: e.g. $\beta \propto r^4$ according to Hagen-Poiseuille law
  - heat flow = mass flow times heat/mass: $\dot{Q} = \rho_n \dot{V}_n \cdot \sigma_m T = \dot{V}_n \sigma T$

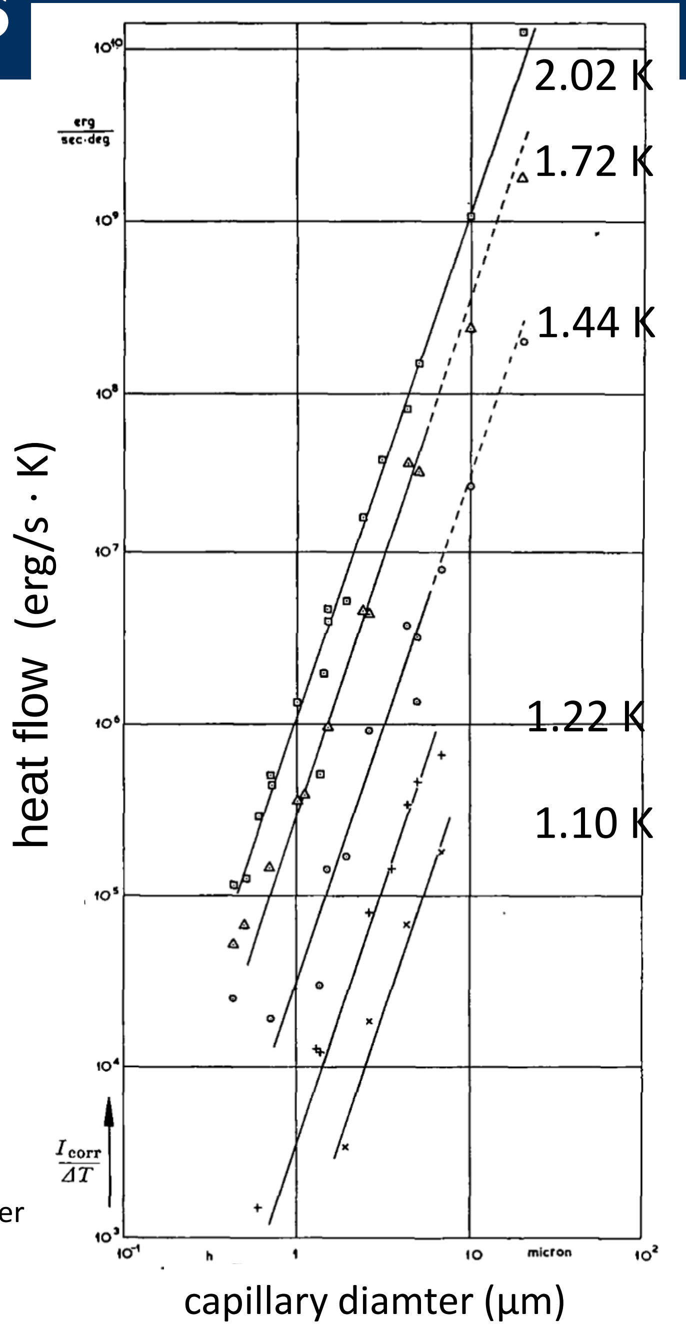
$$\dot{Q}_n = \rho_n \underbrace{\dot{V}_n}_{\frac{\beta}{\eta_n}\frac{\Delta p}{L}} \cdot \sigma_m T = \rho_n \frac{\beta}{\eta_n} \frac{\Delta p}{L} \sigma_m T \quad \text{with } \Delta p = \rho_n \sigma_m \Delta T \Rightarrow \quad \boxed{\dot{Q}_n = \frac{\beta T}{\eta_n L} (\rho_n \sigma_m)^2 \Delta T}$$

I.3.3 Two-Fluid Hydrodynamics

- **two-fluid hydrodynamics: heat transport**

$$\dot{Q}_n = \frac{\beta T}{\eta_n L}(\rho_n \sigma_m)^2 \Delta T$$

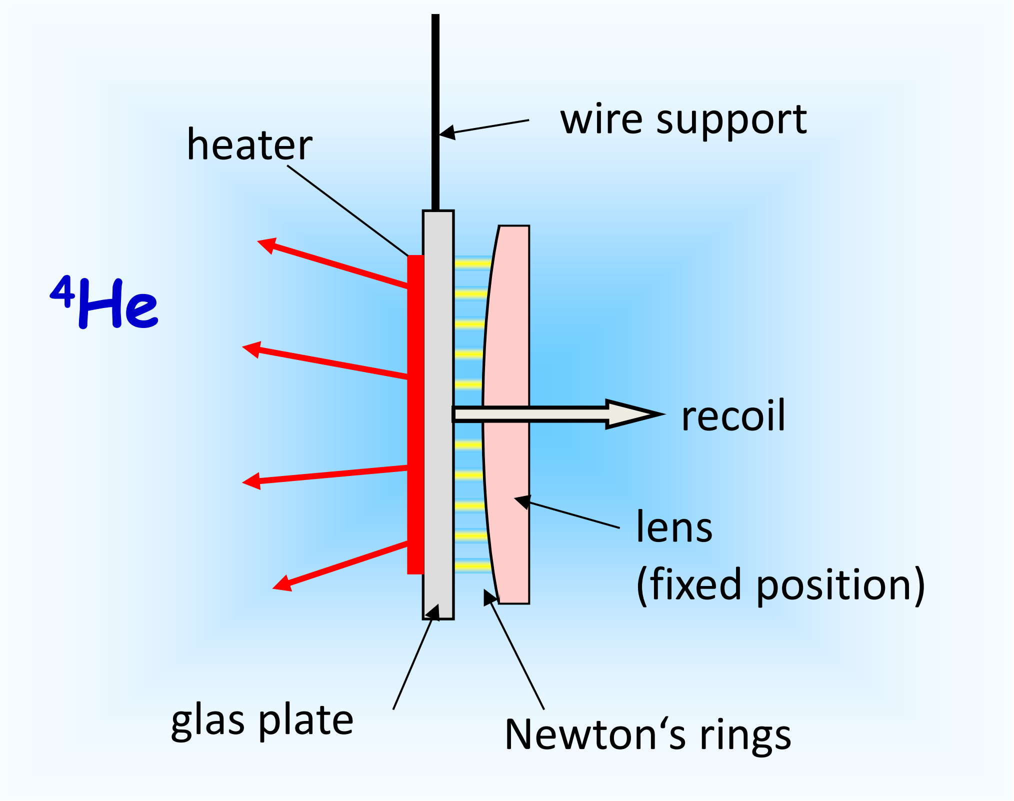
  - **experiment**:
    heat flow is found to be proportional to $\beta \propto d^3$ (for slit of width d)



A. Broese Van Groenou, J.D. Poll, A.M.G. Delsing, C.J. Gorter
Physica 22, 905-910 (1956)

I.3.3 Two-Fluid Hydrodynamics

- **two-fluid hydrodynamics: momentum flow associated with heat flow**



- ➢ heater generates normal fluid: phonons and rotons
- ➢ radiation to the left
- ➢ recoil to the right
- ➢ measurement of displacement of the glas plate via Newton‘s rings between glas plate and lens

- – momentum flow (momentum / area and time): $\rho\,\mathbf{v}\cdot\mathbf{v}$
- – resulting pressure on heat source:

$$p = \rho_n v_n^2 + \rho_s v_s^2$$

- – with heat flow per area:

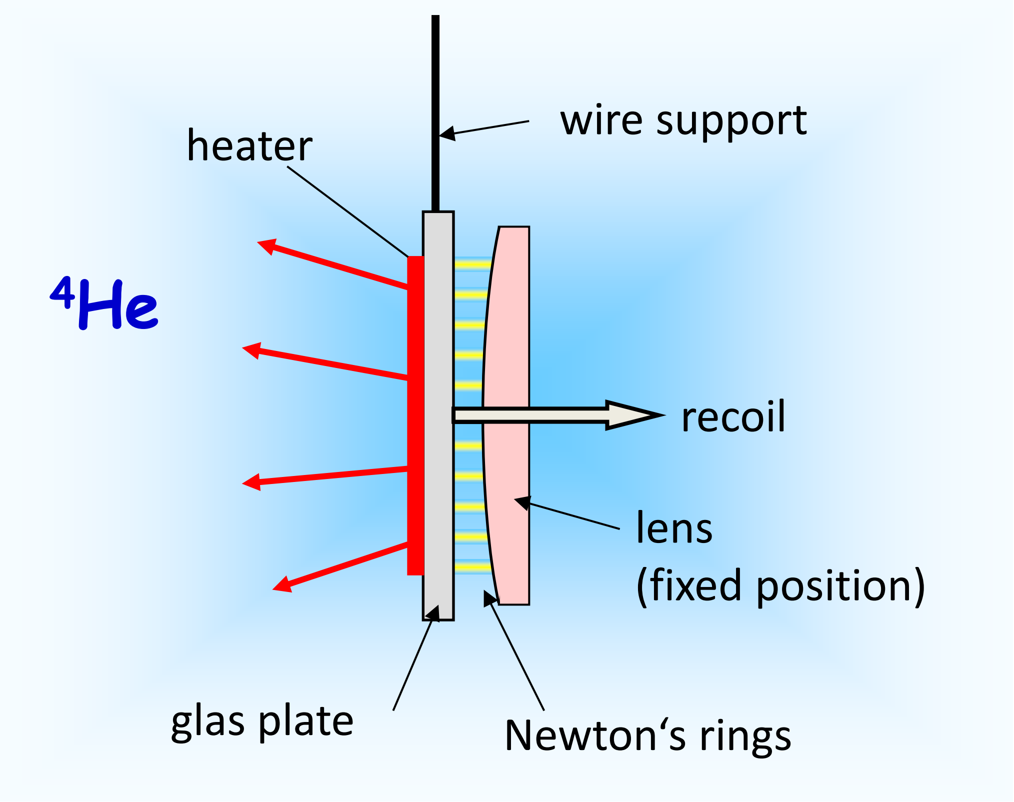
$$J_h = \rho\sigma_m\,T\,v_n$$

- – and the fact that there is no mass transport, $\rho_n v_n + \rho_s v_s = 0$, we obtain pressure

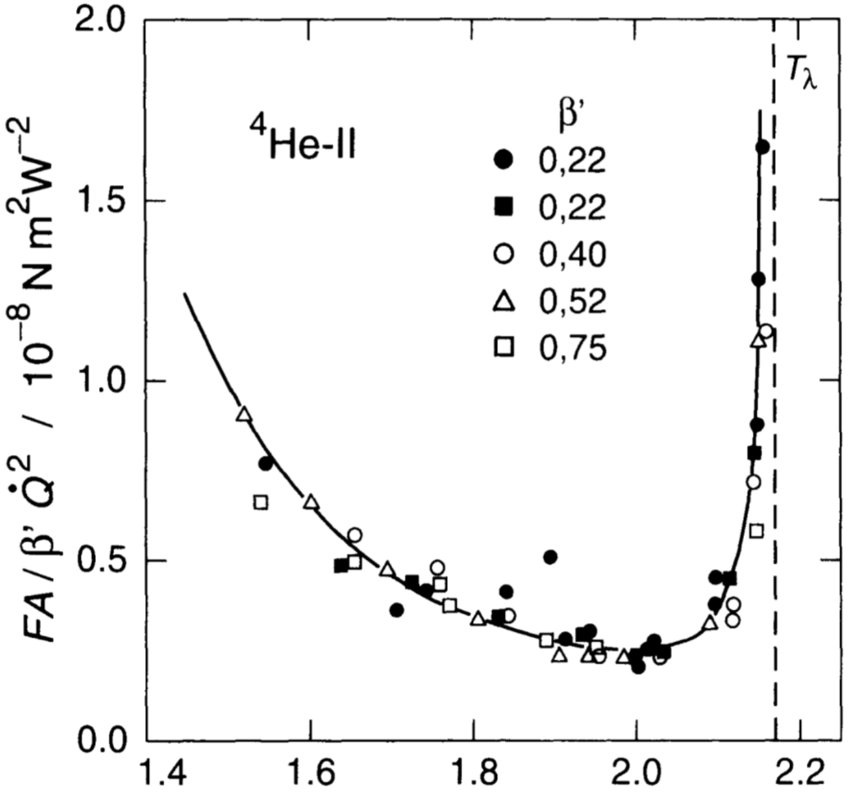
$$\boxed{p = \frac{\rho_n J_h^2}{\rho_s \rho T^2 \sigma_m^2} = \frac{J_h^2}{v_2^2\,\rho C_p T}}$$

I.3.3 Two-Fluid Hydrodynamics

- **two-fluid hydrodynamics: momentum flow associated with heat flow**



$$p = \frac{\rho_n J_h^2}{\rho_s \rho T^2 \sigma_m^2} = \frac{J_h^2}{v_2^2\, \rho C_p T}$$



temperature T (K)

Enns, Hunklinger,
Low Temperature Physics, Springer (2000)

Contents Part I: Quantum Liquids

Contents:

I.3.4 Excitation Spectrum of Superfluid ^{4}He

- important question: Does the formation of a condensate automatically result in superfluidity?
  Does an ideal Bose-Einstein condensate show superfluidity as superfluid ^{4}He?

Answer: *Structure of the excitation spectrum is crucial for observability of superfluidity*
(Lev Landau, 1947)

– **note**:

  ➢ we have used a phenomenological two-fluid model to describe superfluid current density in ^{4}He-II

$$\mathbf{J}_\rho(\mathbf{r}) = n_0 \underbrace{\frac{\hbar}{m}\nabla\theta(\mathbf{r})}_{v_s} = n_0 \mathbf{v}_s(\mathbf{r})$$

– **still unknown**:

  ➢ is there a maximum value for the superfluid velocity $\mathbf{v}_s$?

  ➢ if yes, what determines the value of $\mathbf{v}_s$?

  ➢ is there a qualitative difference between a condensate of a non-interacting Bose gas and an interacting Bose gas ?

I.3.4 Excitation Spectrum of Superfluid ^{4}He

- **Gedanken experiment by Lev Landau**

  – what happens, if we move an object (e.g. sphere) of mass M at velocity $\mathbf{v}_i$ through the condensate/superfluid ?

  → at $T = 0$, the deceleration of the object is possible only by the generation of excitations

  → at which velocity can the moving object generate an **excitation** of energy $\boldsymbol{E_p}$ and momentum $\mathbf{p} = \hbar\mathbf{k}$?

  – with velocity difference $v_i - v_f$ of object we obtain (*energy and momentum conservation*):

$$E_p = \frac{1}{2} M \left(v_i^2 - v_f^2\right) \qquad\qquad \mathbf{p} = M \left(\mathbf{v}_i - \mathbf{v}_f\right)$$

  – elimination of v_f yields

$$E_p = \mathbf{p} \cdot \mathbf{v}_i - \frac{1}{2M} p^2 = \mathbf{p} \cdot \mathbf{v}_i - \mathcal{O}\left(\frac{1}{M}\right)$$

⇨

$$v_L = \min\left\{\frac{E_p}{|\mathbf{p}|}\right\}$$

Landau critical velocity

v_L = smallest v_i for momentum $\mathbf{p}$ of the excitation parallel to $\mathbf{v}_i$

- ➢ there may be a minimum value of $\mathbf{v}_i$ for which the equation can be satisfied
- ➢ mass of object can be large → we can neglect 2nd term on rhs

I.3.4 Excitation Spectrum of Superfluid ^{4}He

$$E_p = \mathbf{p}\cdot\mathbf{v}_i - \frac{1}{2M}p^2 = \mathbf{p}\cdot\mathbf{v}_i - \mathcal{O}\left(\frac{1}{M}\right)$$

$$v_L = \min\left\{\frac{E_P}{|\mathbf{p}|}\right\}$$

- **Gedanken experiment by Lev Landau**
  - for free bosons: $E_p = p^2/2m$ (m = mass of the boson)

$$\frac{p^2}{2m} = \mathbf{p}\cdot\mathbf{v}_i$$

equation always satisfied for cone of momentum vectors $|p| = 2mv_i\cos\vartheta$, where $\vartheta = \measuredangle(\mathbf{p},\mathbf{v}_i)$

➔ object (e.g. a sphere) moving in superfluid can always transfer momentum to superfluid and is thereby decelerated ➔ $\boldsymbol{v_L = 0}$

ideal Bose-Einstein condensate of free (non-interacting) bosons with dispersion $E_p = p^2/2m$ does not show superfluidity !!

I.3.4 Excitation Spectrum of Superfluid ^{4}He

- **motivation for Landau critical velocity**

$$E_p = \mathbf{p}\cdot\mathbf{v}_i - \frac{1}{2M}p^2 \simeq \mathbf{p}\cdot\mathbf{v}_i$$

$$v_L = \min\left\{\frac{E_P}{|\mathbf{p}|}\right\}$$

  - deceleration of moving object by generation of excitations with energy E_p only possible if $\frac{p^2}{2m} = \mathbf{p}\cdot\mathbf{v}_i$ can be satisfied

$$\Rightarrow\ E_p = \hbar\omega = \underbrace{\frac{\partial E_p}{\partial p}}_{\mathbf{v}_i}\cdot\mathbf{p} = \hbar\mathbf{v}_i\cdot\mathbf{k} = \hbar v_i k\cos\vartheta \le \hbar v_i k$$

$$v_i \ge \frac{\omega}{k} = v_{\mathrm{ph}}$$

v_{ph} = Phasengeschwindigkeit

$$v_L = \min(v_{\mathrm{ph}})$$

v_{L} = **Landau critical velocity**

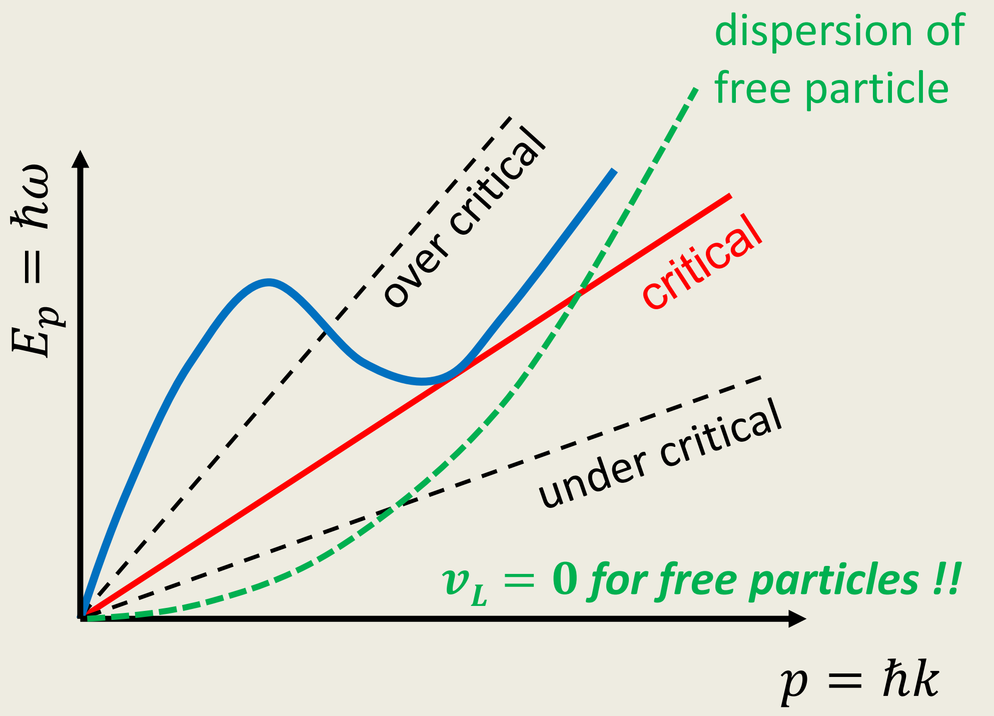
$\min(v_{\mathrm{ph}}) = 0$ for free bosons

no deceleration of object, if $v_i < v_{\mathrm{L}}$

Laudau:

in order to have a superfluid, we need a dispersion relation differing significantly from that of free bosons !

how to obtain the blue dispersion curve ??



$v_L = 0$ for free particles !!

I.3.4 Excitation Spectrum of Superfluid ^{4}He

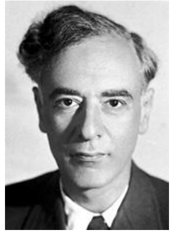
dispersion relation must deviate from that of free bosons, $E_p = p^2/2m$, as in this case $v_L = 0$!!

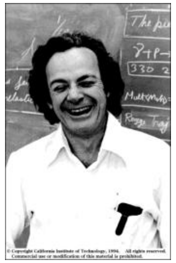
- excitation spectrum of superfluid ^{4}He: historical aspects (1)

Nikolai Bogoliubov (1947):

→ interaction effects drastically change the nature of elementary excitations
collective excitations *(many particles move at the same time)*

→ even for weak repulsive interaction: $E_p = c\,|\mathbf{p}|$ (phonon-like dispersion)





Lev Landau (1947) and ***Richard Feynman*** (1953):

→ discussion of situation for superfluid ^{4}He (stronger interaction)

→ postulation of the following excitation spectrum (phonons and rotons):

(i) small p: $\lim_{p\to 0} E_p = c_s|\mathbf{p}|$ *phonon-like collective excitations*

(ii) higher p: $\lim_{|p|\to p_0} E_p = \Delta_{\rm rot} + \frac{(|\mathbf{p}| - \mathrm{p}_0)^2}{2m_{\rm rot}}$ *roton-like collective excitations*
(finite energy gap $\Delta_{\rm rot}$ for excitation)

expression „roton“ used by Feynman due to analogy with „smoke ring“, since it is connected with a forward motion of a particle accompanied by a ring of back-flowing particles

➔ ***however: detailed nature of rotons still not clarified***

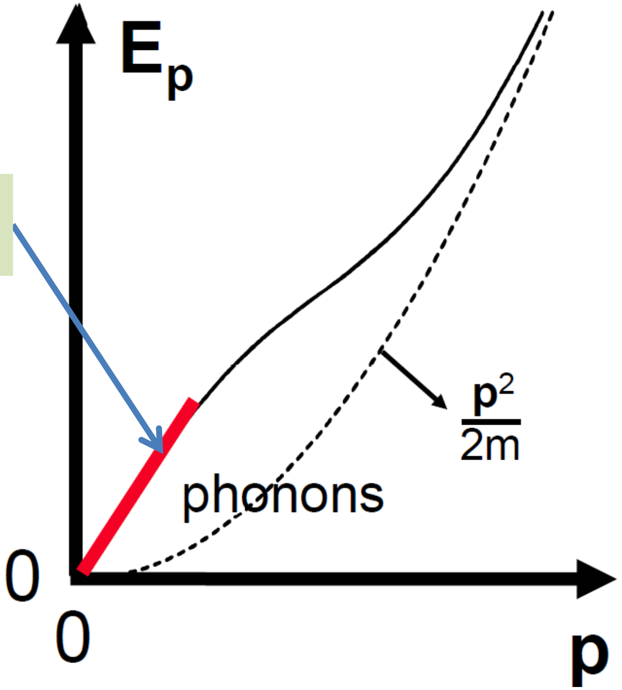
I.3.4 Excitation Spectrum of Superfluid ^{4}He

- excitation spectrum of superfluid ^{4}He: historical aspects (2)

interacting Bose gas
→ *finite stiffness*
→ *solid like excitations*

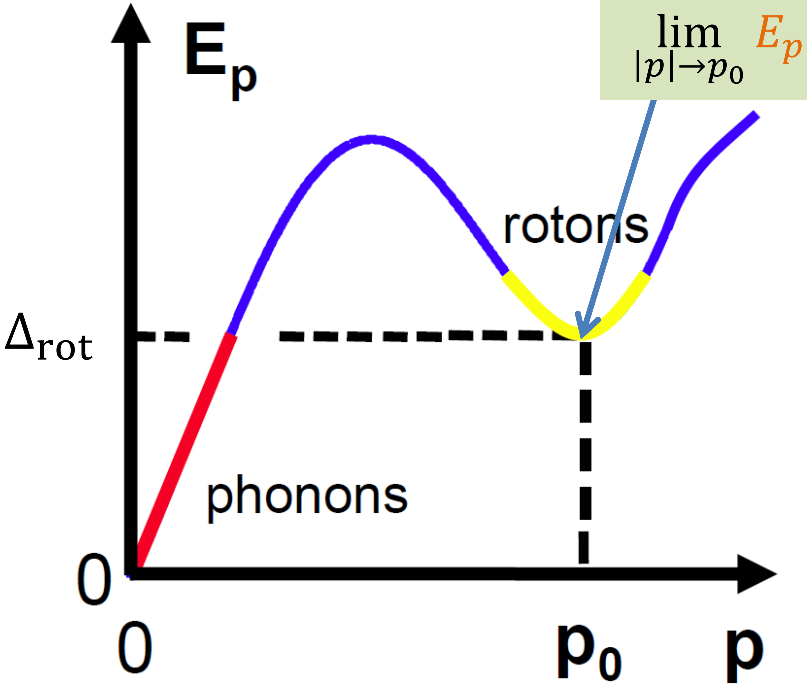
weak interactions:
phonons
[Bogoliubov, 1947]

$$\lim_{p\to 0} E_p = c_s|\mathbf{p}|$$



strong interactions:
phonons and rotons
[Landau, 1947, Feynman & Cohen, 1957]

$$\lim_{|p|\to p_0} E_p = \Delta_{\text{rot}} + \frac{(|\mathbf{p}| - \text{p}_0)^2}{2m_{\text{rot}}}$$



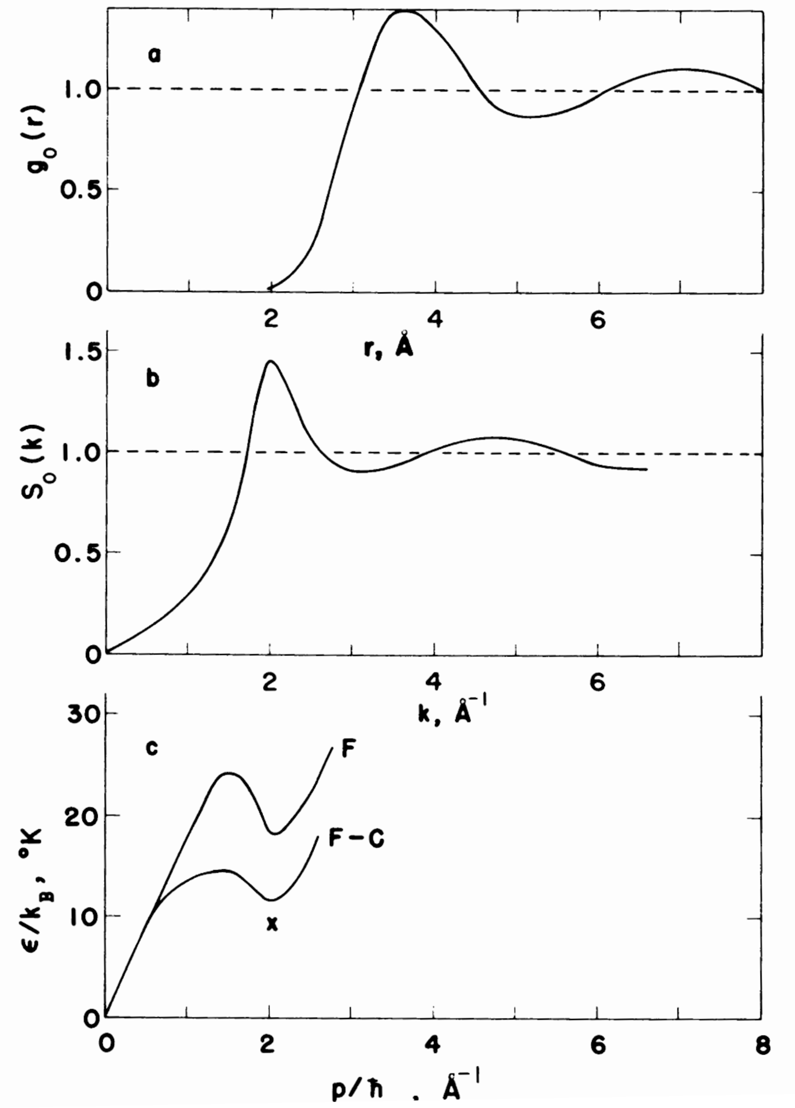
I.3.4 Excitation Spectrum of Superfluid ^{4}He

- excitation spectrum of superfluid ^{4}He: historical aspects (3)

Richard P. Feynman:
(from: „The beat of a different drum“ Jagdish Mehra, 1994)

„I cannot remember, how it happened. I was walking along the street... and zing! I understood it!“ ...

... „But of course, that is just what liquids must do. If you measure X-ray diffraction, then because of the spatial structure of the liquid, *which is almost like a solid,* there will be a maximum corresponding to the first diffraction ring of the X-ray pattern! That was a terriffic moment!“

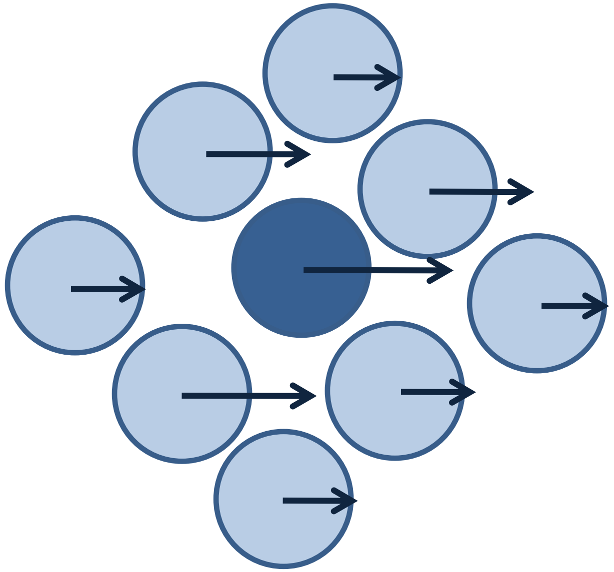


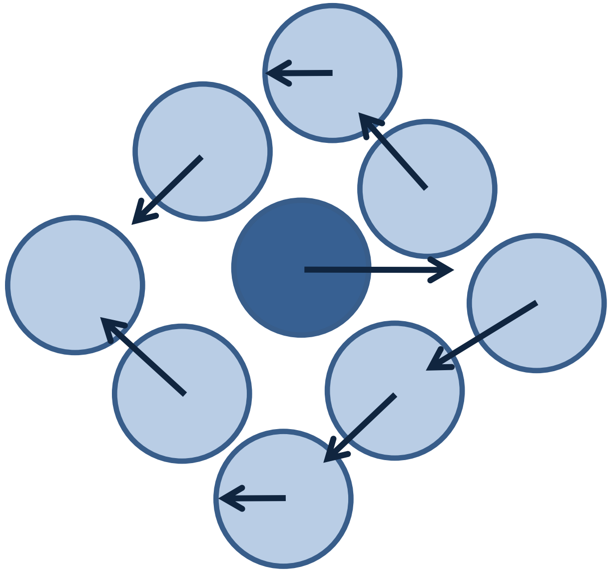
I.3.4 Excitation Spectrum of Superfluid ^{4}He

- excitation spectrum of superfluid ^{4}He: plausibility considerations

physical interpretation of the phonon and roton part of the excitation spectrum

phonon





roton

- wavelength $\lambda = 2\pi/k \gg$ atomic distance
- coupled motion of group of atoms (like for acoustic phonon in solid)

- wavelength $\lambda = 2\pi/k \simeq$ interatomic distance
- central atom moves forward while closely packed neighbors move (backward) out of the way in circular motion (like smoke ring)

in both cases we have collective excitations with dispersion $E_p(k)$ ➔ quasiparticles

I.3.4 Excitation Spectrum of Superfluid ^{4}He

- **excitation spectrum of superfluid ^{4}He: measurement by inelastic scattering experiments**

– **experimental study**:

inelastic neutron scattering
(coupling to density)

– **experimental parameter**:

$$\frac{\Delta_{\rm rot}}{k_{\rm B}} = 8.61\ \mathrm{K},\ \frac{p_0}{\hbar} = 1.92\ \mathrm{Å}^{-1},\ m_{\rm rot} = 0.14\ m_{\rm He}$$

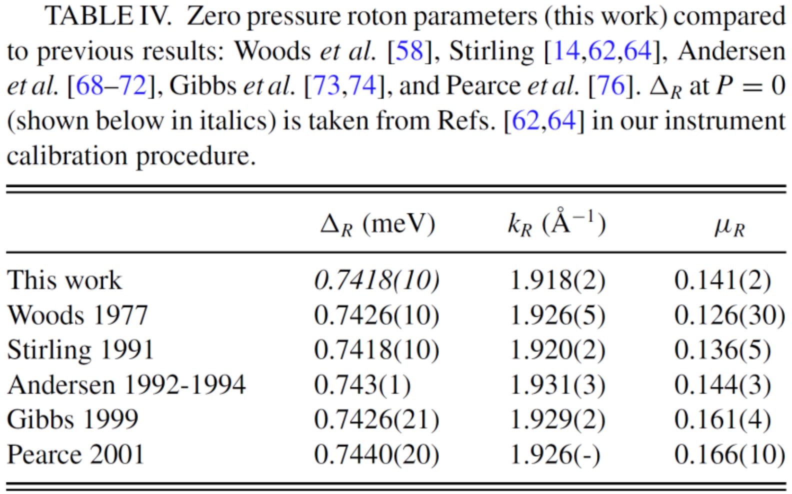
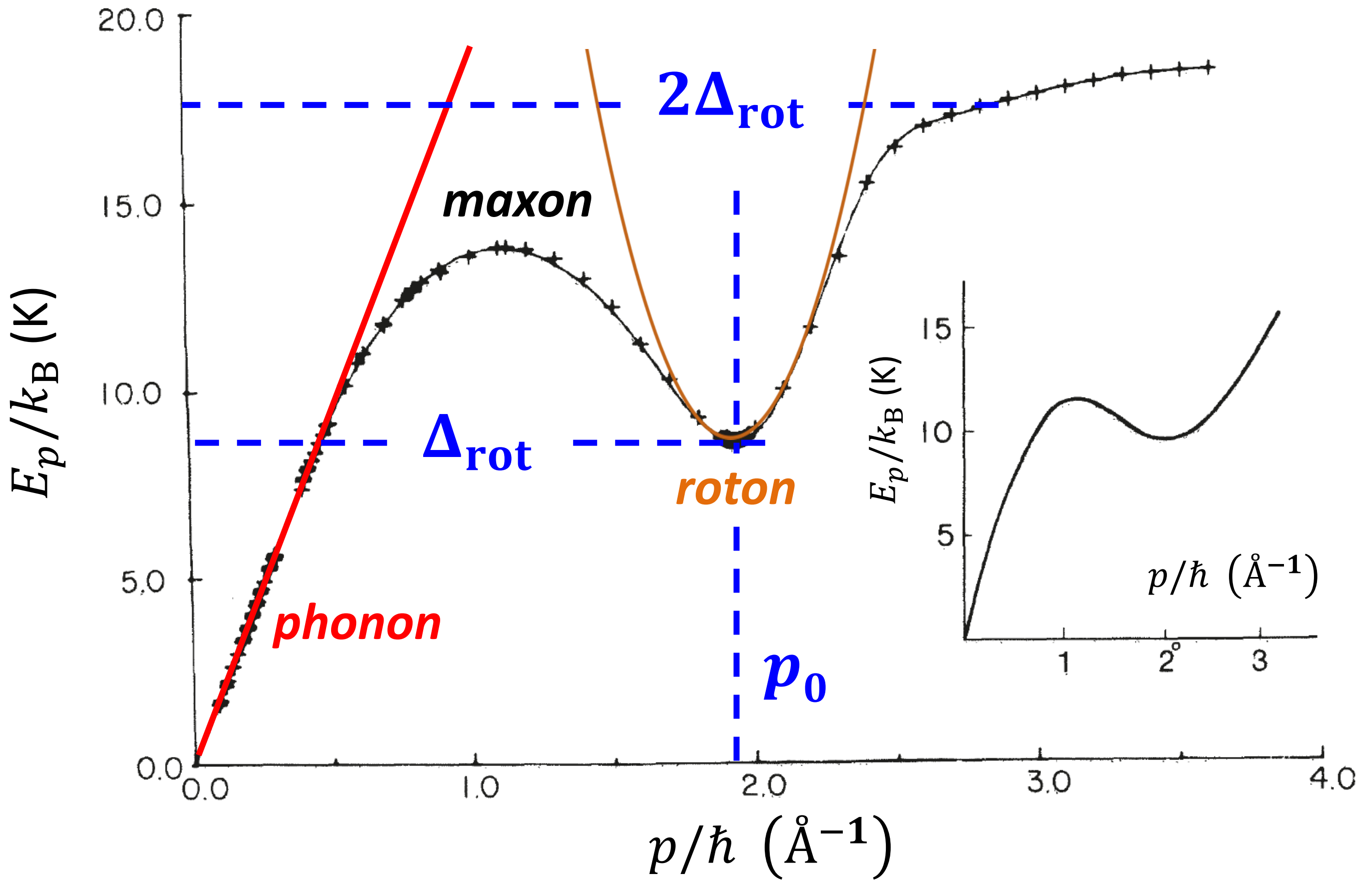


TABLE IV. Zero pressure roton parameters (this work) compared to previous results: Woods *et al.* [58], Stirling [14,62,64], Andersen *et al.* [68–72], Gibbs *et al.* [73,74], and Pearce *et al.* [76]. Δ_R at $P = 0$ (shown below in italics) is taken from Refs. [62,64] in our instrument calibration procedure.

	Δ_R (meV)	k_R (Å^{-1})	μ_R
This work	*0.7418(10)*	1.918(2)	0.141(2)
Woods 1977	0.7426(10)	1.926(5)	0.126(30)
Stirling 1991	0.7418(10)	1.920(2)	0.136(5)
Andersen 1992-1994	0.743(1)	1.931(3)	0.144(3)
Gibbs 1999	0.7426(21)	1.929(2)	0.161(4)
Pearce 2001	0.7440(20)	1.926(-)	0.166(10)

H. Godfrin et al., Phys. Rev. B **103**, 104516 (2021)

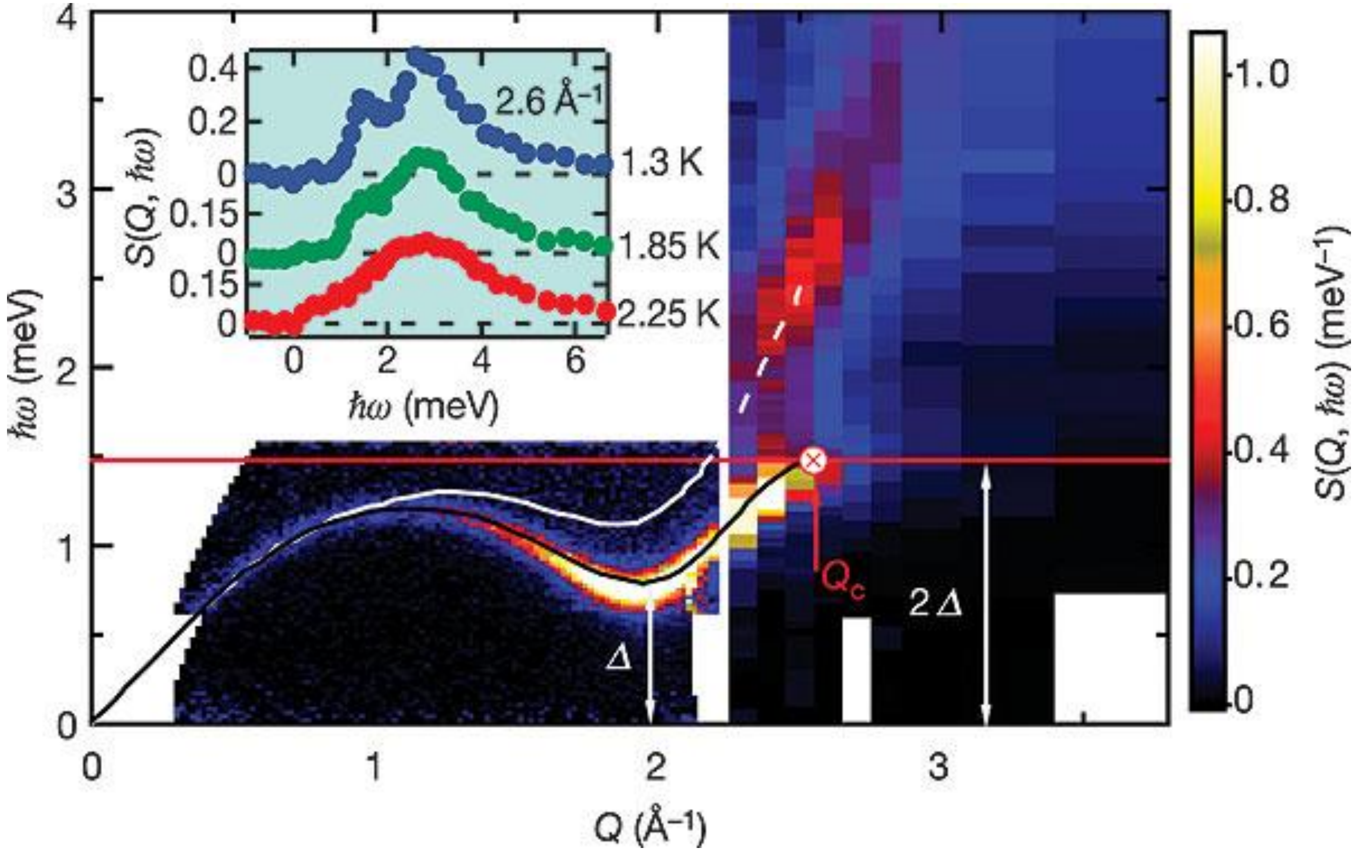


R. J. Donnelly, J. A. Donnelly, R. N. Hills,
J. Low Temp. Phys. **44**, 471 (1981).

Inset:
L. D. Landau, J. Phys. U.S.S.R. **11**, 91 (1947).

I.3.4 Excitation Spectrum of Superfluid ^{4}He

- **excitation spectrum of superfluid ^{4}He: measurement by inelastic scattering experiments**



Liquid helium excitation spectrum S(Q, $\hbar\omega$) *from inelastic neutron* scattering measurements.

Main panel, excitation spectrum in 4He for 1.5 < T < 1.8 K. Data for wave vector Q ≥ 2.3 Å^{-1} are reproduced from ref. 13, data at smaller Q are from C.L.B. and S.-H. Lee, unpublished results. Solid black line, dispersion from ref. 13; red circle with cross, spectrum termination point at Q = Q_c and $\hbar\omega = 2\Delta$. White line, Feynman–Cohen bare dispersion in absence of decays; horizontal red line at $\hbar\omega = 2\Delta$, onset of two-roton states for $\hbar\omega \geq 2\Delta$. Inset, excitations near termination point, at Q = 2.6 Å^{-1} ≈ Q_c, for several temperatures.

Quasiparticle breakdown in a quantum spin liquid
Matthew B. Stone, Igor A. Zaliznyak, Tao Hong, Collin L. Broholm and Daniel H. Reich
Nature **440**, 187-190 (9 March 2006)

I.3.4 Excitation Spectrum of Superfluid ^{4}He

- **excitation spectrum of superfluid ^{4}He: qualitative understanding**

periodic crystal:

$\boldsymbol{k}$ and $\boldsymbol{k} + \boldsymbol{G}$ are equivalent

liquid:

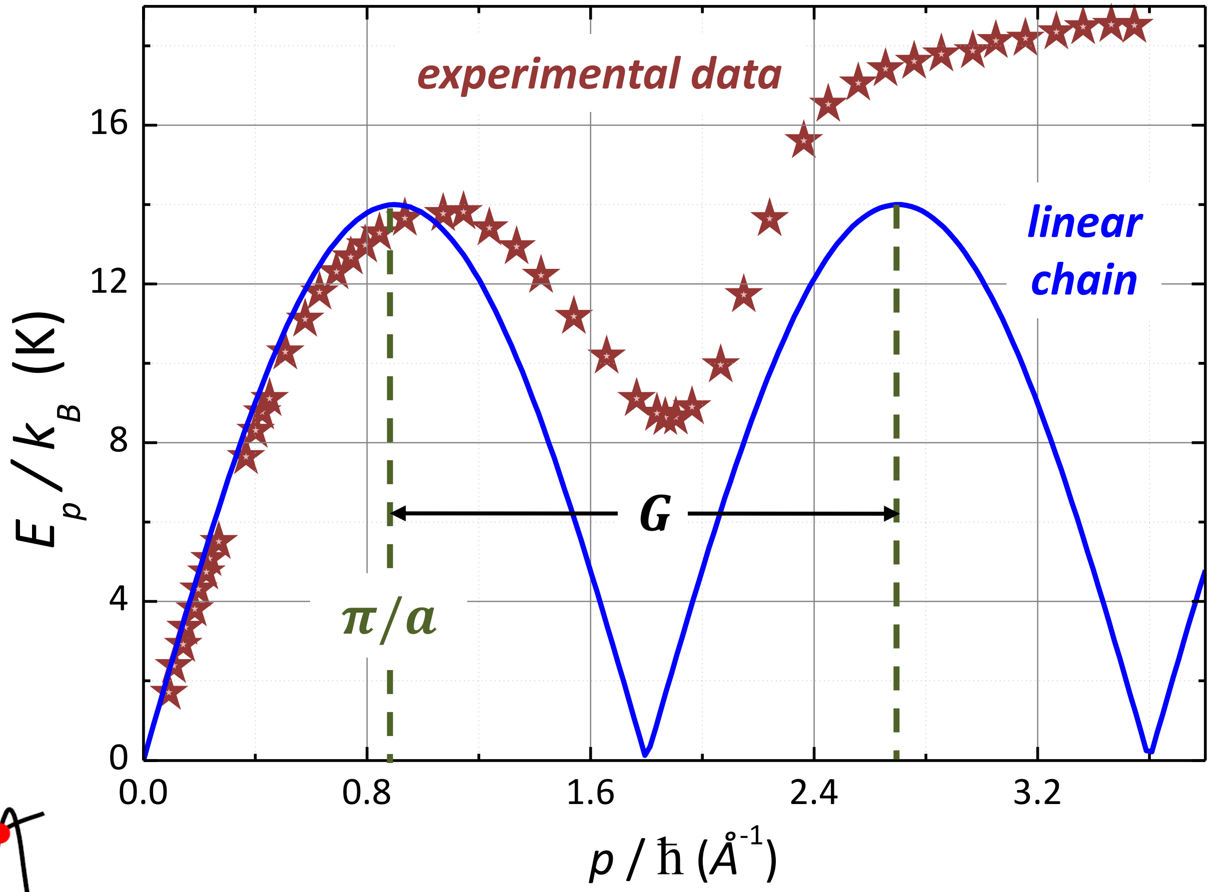
only short distance ordering

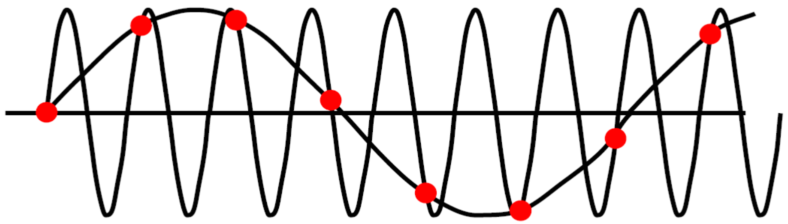
➔ a becomes smeared out

➔ also $G = 2\pi/a$ is smeared out

$a = 3.6$ Å ➔ $2\pi/a \approx 1.8$ Å^{-1}

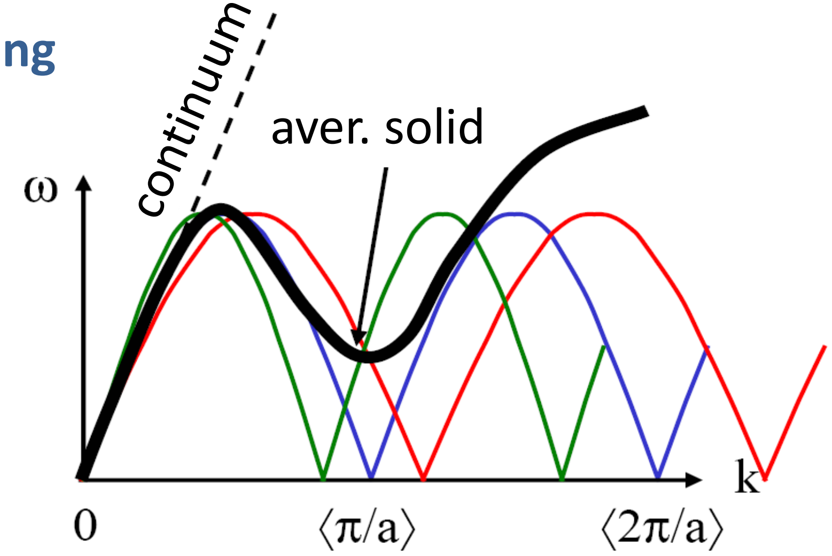
(*reasonable agreement*)





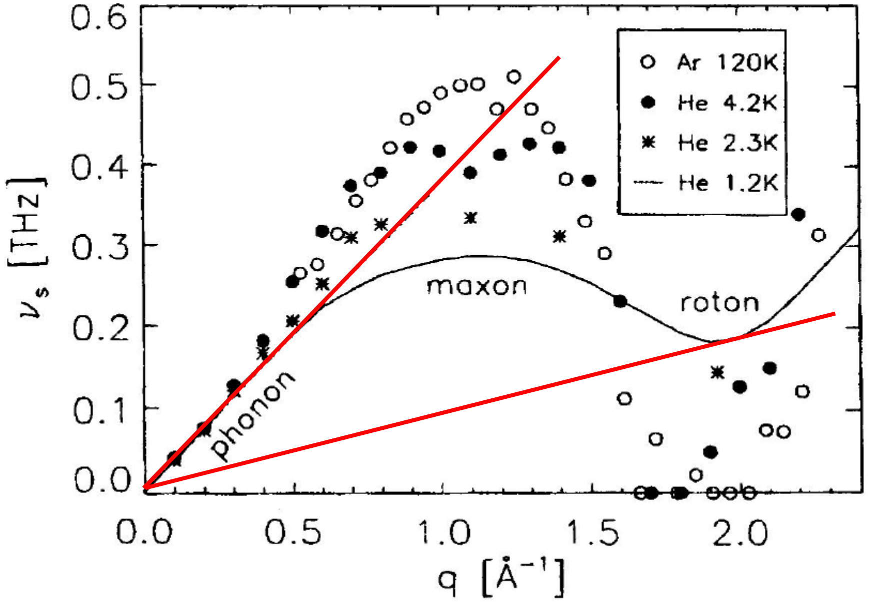
I.3.4 Excitation Spectrum of Superfluid ^{4}He

- **excitation spectrum of superfluid ^{4}He: qualitative understanding**



- *smearing of $G = 2\pi/a$*
  → *modelling by average solid + continuum*
- *other liquids:*
  → *rotons strongly damped*
- *superfluid He:*
  → *no damping, since there are no single particle excitations*
- *critical velocities:*
  → $v_L = 238$ ***m/s for phonons***
  → $v_L \approx 60$ ***m/s for rotons***

measured v_L usually smaller
→ *excitation of vortices*



I.3.4 Excitation Spectrum of Superfluid ^{4}He

- **measurement of Landau critical velocity**

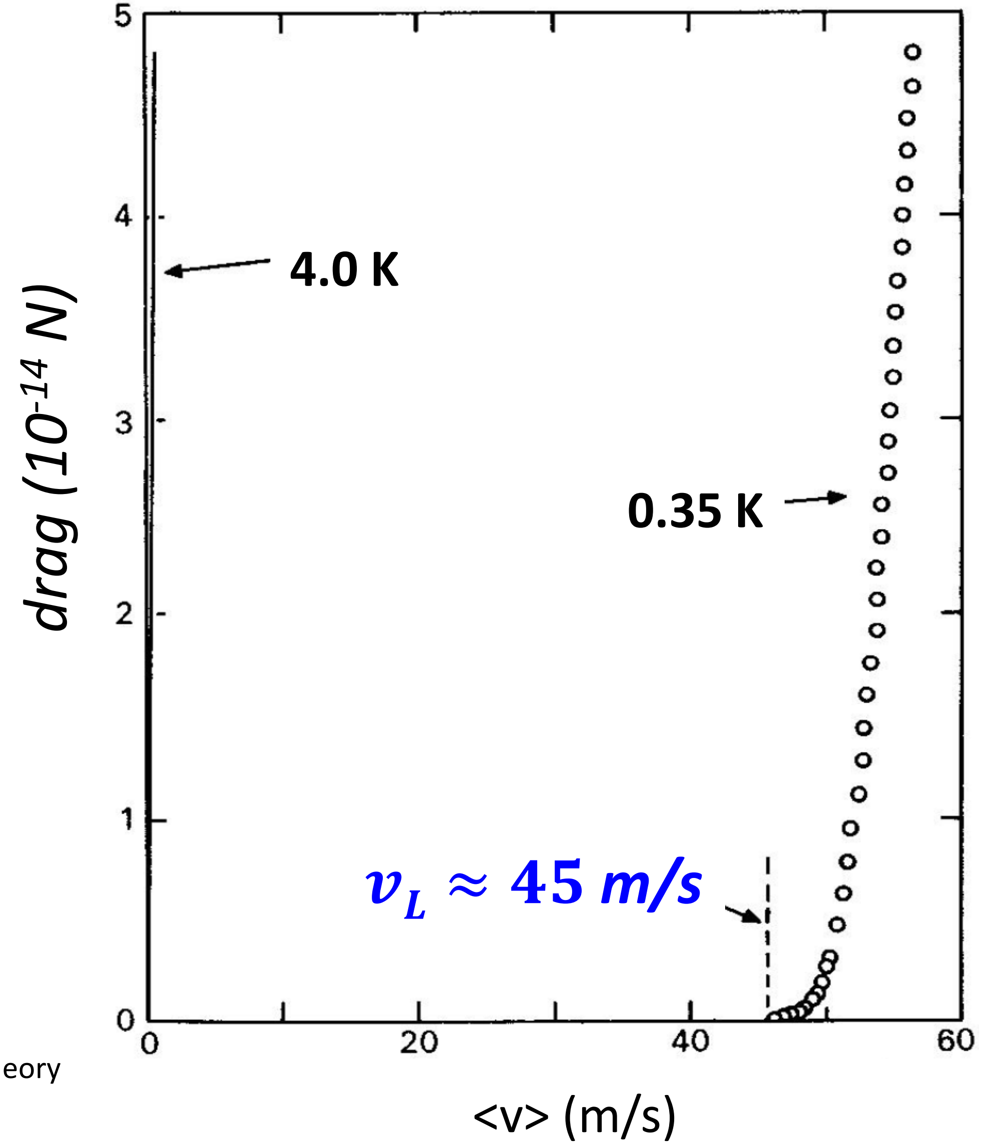
measurement of drag
on ions in ^{4}He-II

➔ *no generation of excitations for $v < v_L$*

➔ *no friction*

key result:

there is a finite Landau critical velocity in ^{4}He-II!

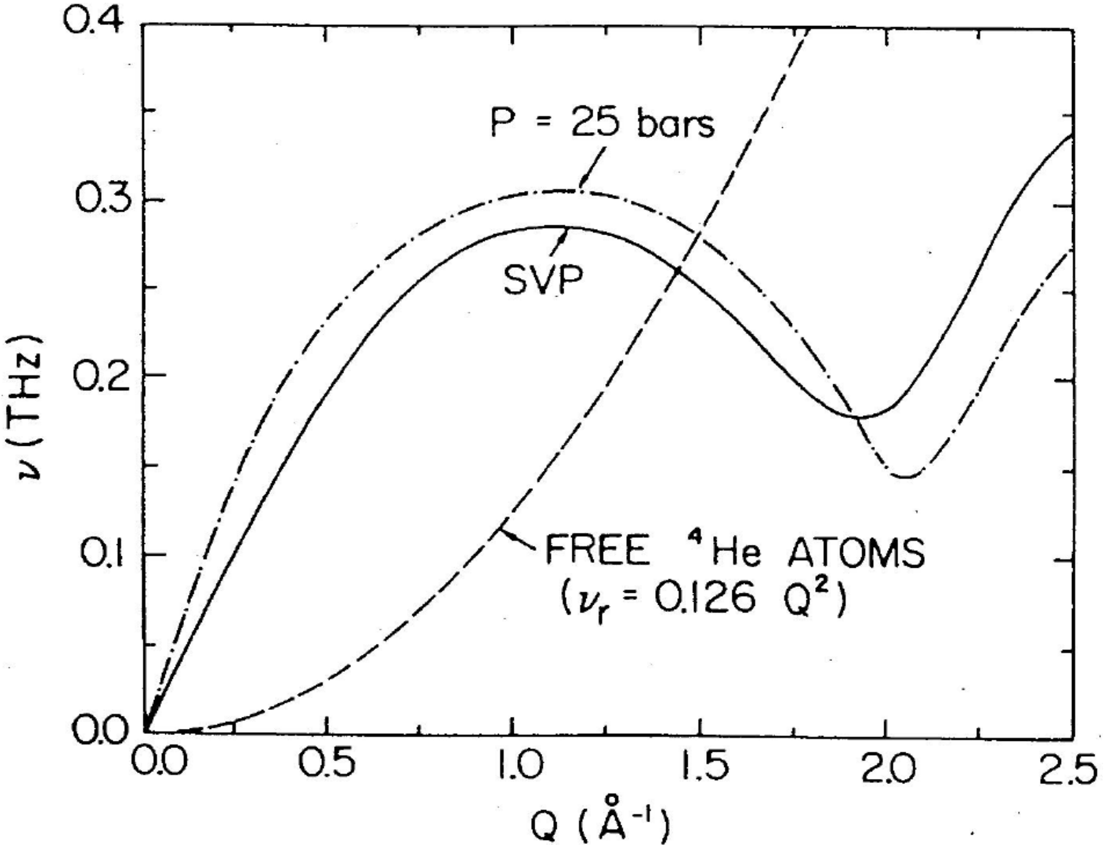


The breakdown of superfluidity in liquid 4He : an experimental test of Landau's theory
D. R. Allum, Peter V. E. McClintock, A. Phillips, R. M. Bowley
Philosophical Transactions of the Royal Society of London A **284**, 179-224 (1977).

I.3.4 Excitation Spectrum of Superfluid ^{4}He

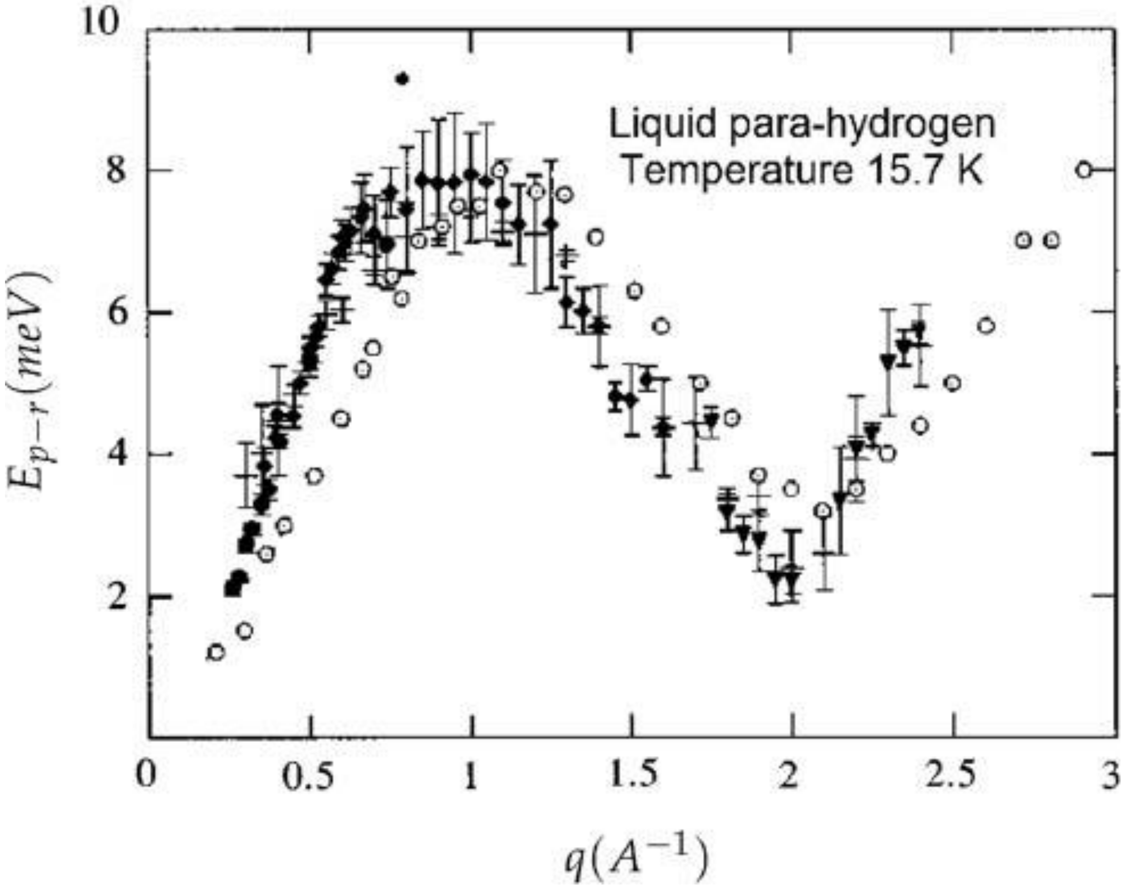
- **excitation spectrum of superfluid ^{4}He: pressure dependence**

- average distance *a* becomes smaller
- He becomes harder
  → ***sound velocity increases***
- ordering increases
  → ***roton gap decreases***



I.3.4 Excitation Spectrum of Superfluid ^{4}He

- **excitation spectrum of superfluid ^{4}He: pressure dependence**



The experimental INS phonon–roton curve for molecular para-hydrogen at $T = 15.7$ K. The filled symbols (inverted triangles, diamonds and crosses) depict data measured at different spectrometers and the open circles with a dot show the simulation results.

J. Dawidowski, F. J. Bermejo, M. L. Ristig, B. Fak, C. Cabrillo, et al., Phys. Rev. **B 60**, 15154 (1999).

I.3.4 Excitation Spectrum of Superfluid ^{4}He

- **excitation spectrum of superfluid ^{4}He: pressure dependence**

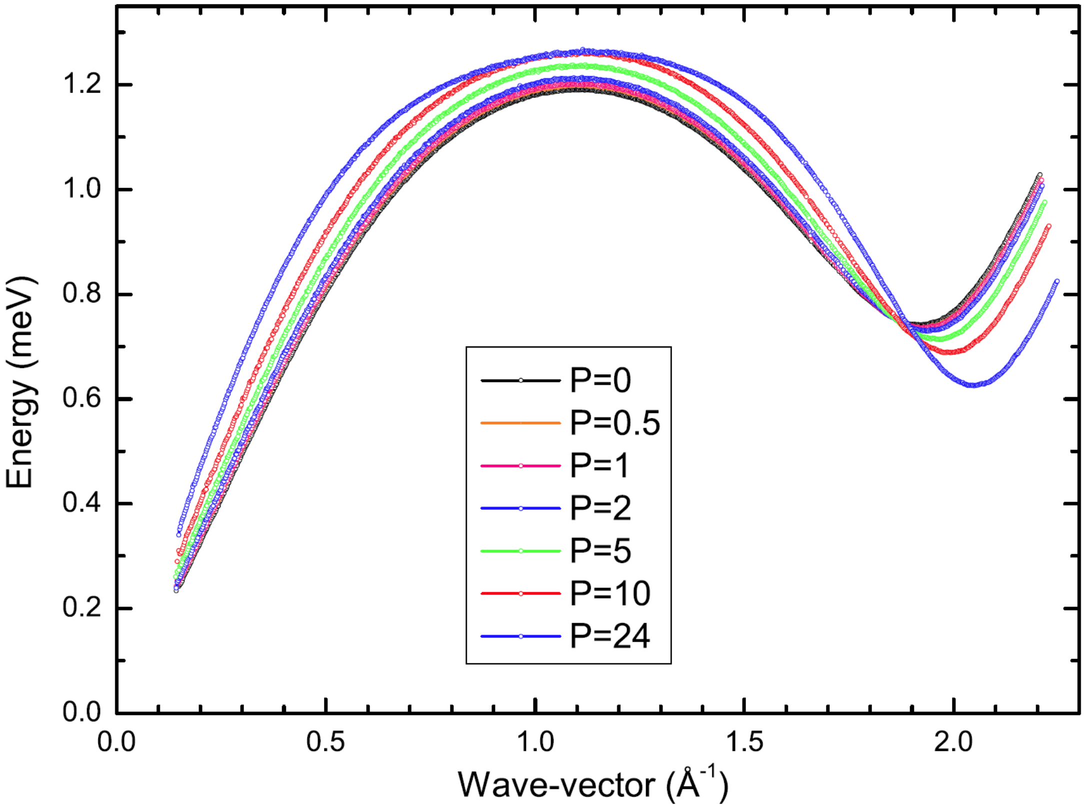
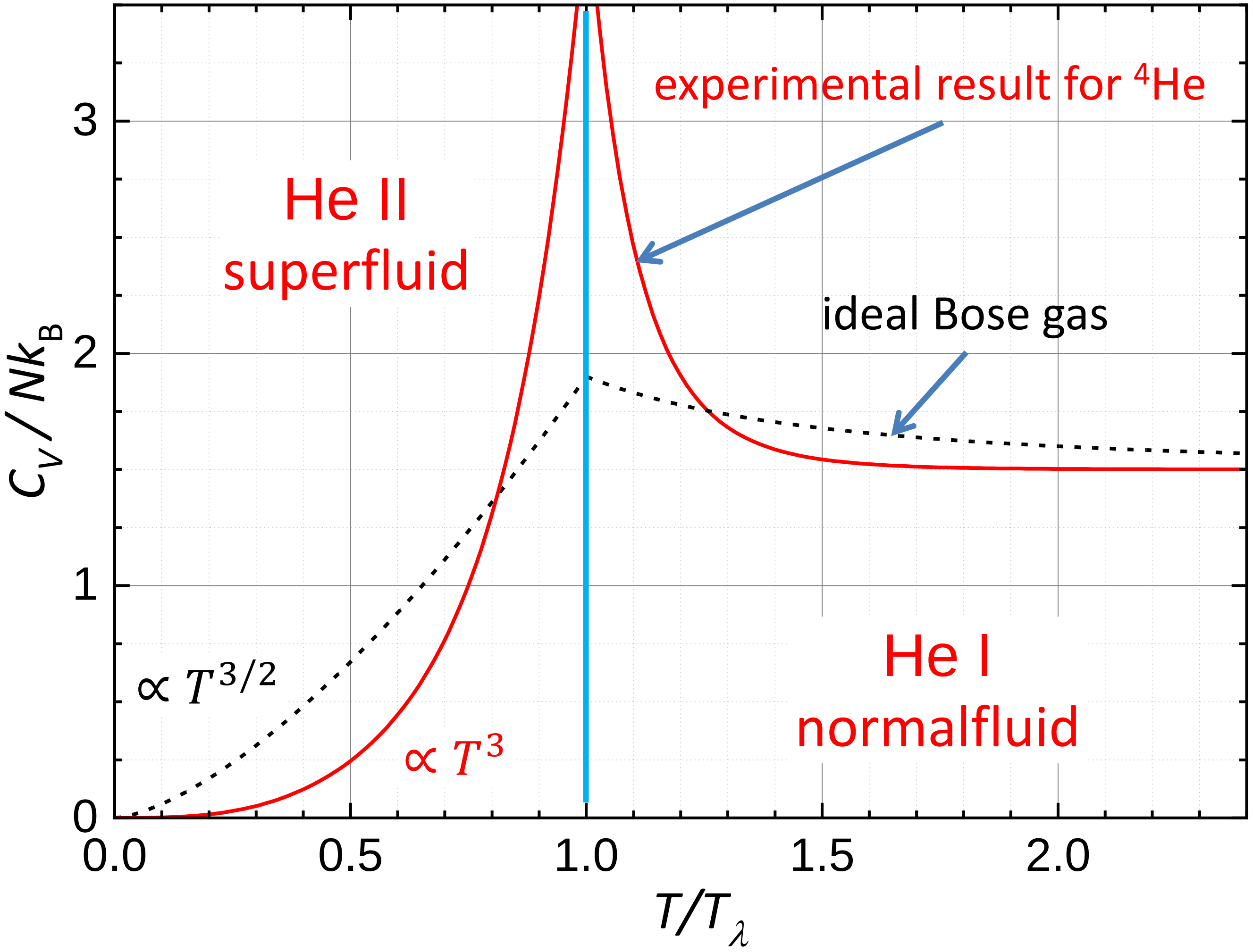


FIG. 17. Dispersion curves $\epsilon(k)$ measured for several pressures in the 0 to 24 bar range. The individual data points are represented by small circles (best seen on-line). Accurate values for the pressures are given in Table I.

H. Godfrin, K. Beauvois, A. Sultan, E. Krotscheck, J. Dawidowski, B. Fåk, and J. Ollivier
Phys. Rev. B **103**, 104516 (2021)

I.3.4 Excitation Spectrum of Superfluid ^{4}He

- **specific heat of ^{4}He revisited**



deviations from ideal Bose gas are expected due to finite interactions
→ ***collective excitations: phonons, rotons***

$C_V(T)$ determined by freeze out of phonons and rotons at $T < T_\lambda$

2nd order phase transition is expected for interacting Bose gas

➔ however, no jump of C_V at $T = T_\lambda$ is observed, as expected for 2nd order phase transition

I.3.4 Excitation Spectrum of Superfluid ^{4}He

- **specific heat of ^{4}He revisited: temperature dependence below T_λ**

I. **$T < 0.6$ K:**

in this temperature regime the dominant excitations are long wave length phonons

$$C_{V,ph} \propto T^3$$ *(corresponds to Debye model in solid state physics)*

$$\frac{C_V}{M} = \frac{2}{15}\frac{\pi^2 k_B^4}{\hbar^3 v_s^3}\frac{V}{M}\,T^3$$ single crystal, for single acoustic phonon branch

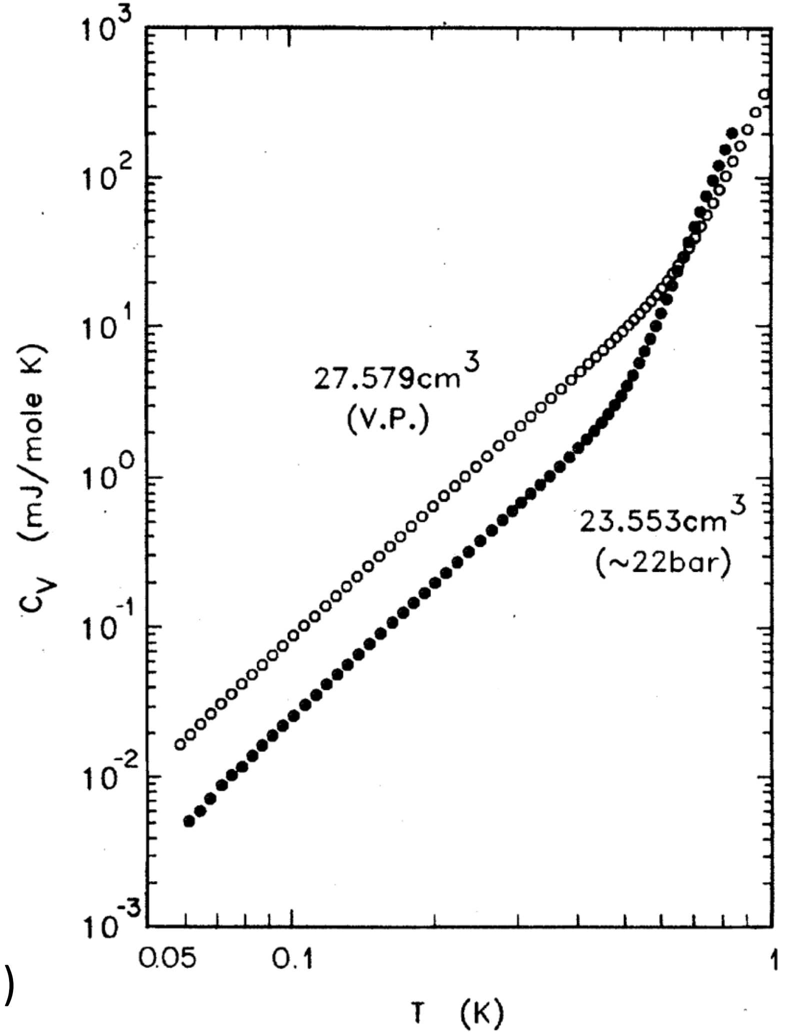
$$\frac{C_{V,ph}}{M} = \frac{2}{15}\frac{\pi^2 k_B^4}{\hbar^3 v_s^3}\frac{V}{M}\,T^3$$ phononic specific heat of ^{4}He-II

thermal conductivity:

$$\kappa = \frac{1}{3} C_V v\,\ell \simeq \frac{1}{3} C_V v\; f(d) \propto T^3$$

if $\ell \simeq d$ (sample dimension, Casimir regime)

absolute value of κ depends on details:
e.g., $\kappa \propto d^3$ for flow through narrow slit of width d



Specific Heat and Phonon Dispersion of liquid ^{4}He
D. S. Greywall, Phys. Rev. B **18**, 2127 (1978);
see also Phys. Rev. B **21**, 1329 (1979)

I.3.4 Excitation Spectrum of Superfluid ^{4}He

- **specific heat of ^{4}He revisited: temperature dependence below T_λ**

II. $\mathbf{0.6\ K < \boldsymbol{T} < 1.2\ K}$:

in this temperature regime the dominant excitations are rotons,
number N_R of rotons increases with T → thermal activation

$$C_{V,R} \propto N_R \propto \frac{1}{\sqrt{T}} \exp\left(-\frac{\Delta_{\mathrm{rot}}}{k_{\mathrm{B}}T}\right)$$

III. $\mathbf{1.2\ K < \boldsymbol{T} < \boldsymbol{T_\lambda}}$:

- lifetime broadening of roton states comparable to Δ_{rot}
- additional excitations
- more complicated behavior

I.3.4 Excitation Spectrum of Superfluid ^{4}He

- **normal fluid density**

  – within two-fluid model, the normal component can be considered as formed by the excitations out of the ground state (quasiparticles)

  ➔ **normal fluid density is determined by phonon and roton density**

  – temperature dependence of phonon and roton density is given by

$$\rho_{n,\mathrm{ph}} = \frac{1}{V}\int D_{\mathrm{ph}}(E)\, n_{\mathrm{ph}}(E,T)\,\mathrm{d}E$$

$$\rho_{n,\mathrm{rot}} = \frac{1}{V}\int D_{\mathrm{rot}}(E)\, n_{\mathrm{rot}}(E,T)\,\mathrm{d}E$$

$n_{\mathrm{ph}}, n_{\mathrm{rot}}$ = occupation probability

$D_{\mathrm{ph}}, D_{\mathrm{rot}}$ = density of states

  – after some math

$$\rho_{n,\mathrm{ph}} = \frac{2\pi^2 k_{\mathrm{B}}^4}{45\hbar^3 v_{\mathrm{ph}}^5}\, T^4$$

$$\rho_{n,\mathrm{rot}} = \frac{2p_0^4}{3\hbar^3}\sqrt{\frac{m_{\mathrm{rot}}}{(2\pi)^3 k_{\mathrm{B}} T}}\,\exp\left(-\frac{\Delta_{\mathrm{rot}}}{k_{\mathrm{B}}T}\right)$$

experimental parameter:

$$\frac{\Delta_{\mathrm{rot}}}{k_{\mathrm{B}}} = 8.61\ \mathrm{K}$$

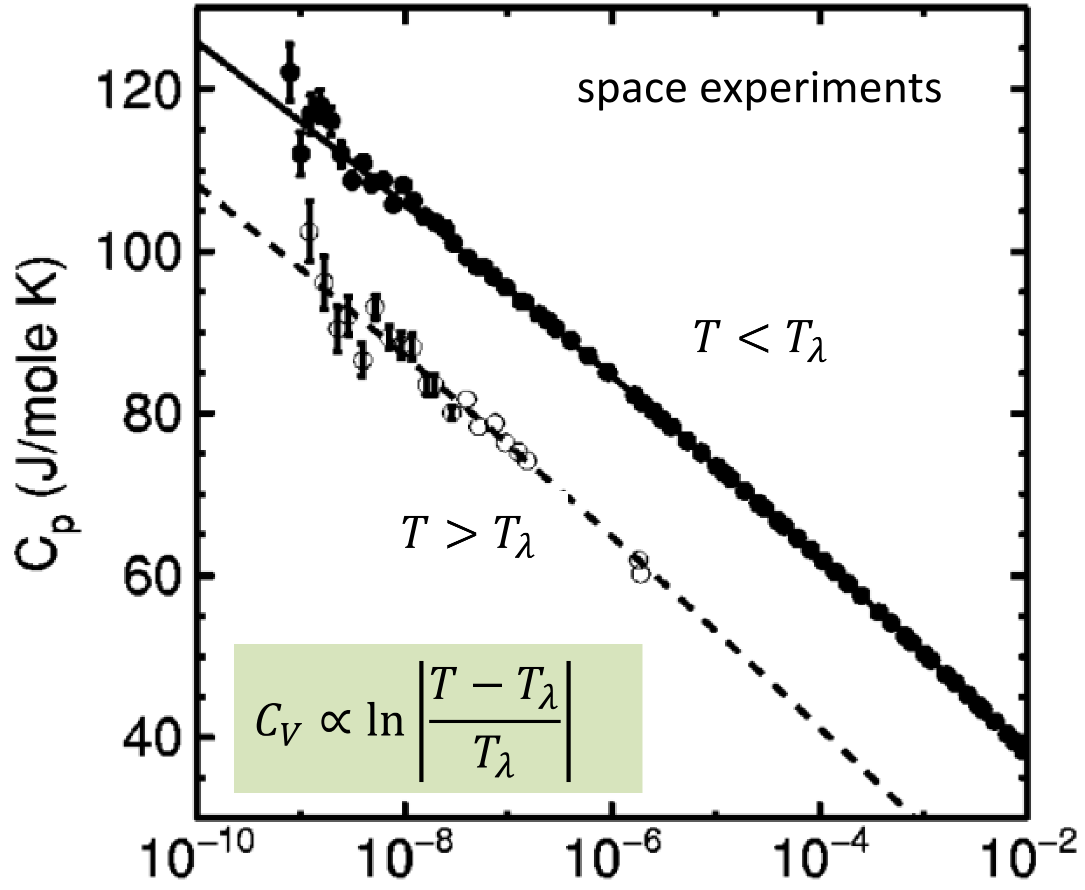
$$\frac{p_0}{\hbar} = 1.92\ \text{Å}^{-1}$$

$$m_{\mathrm{rot}} = 0.15\, m_{\mathrm{He}}$$

$$v_{\mathrm{ph}} = 238\ \mathrm{m/s}$$

I.3.4 Excitation Spectrum of Superfluid ^{4}He

- **specific heat of ^{4}He revisited: temperature dependence close to T_λ**



$$C_V \propto \ln\left|\frac{T-T_\lambda}{T_\lambda}\right|$$

$$\left|\frac{T}{T_\lambda}-1\right|$$

Specific heat of liquid helium in zero gravity very near the lambda point
J. A. Lipa et al., Phys. Rev. B 68, 174518 (2003)

experiment:
second order phase transition with logarithmic singularity

theory of phase transitions:

$$C_p = a^\pm + b^\pm \frac{t^{-\alpha}}{\alpha}\left(1 - c^\pm t^\delta + \cdots\right)$$ with $t = \frac{T}{T_\lambda} - 1$

$\alpha \simeq -0.0127 \pm 0.0003$ (critical exponent), $\delta = 0.529$

$a^\pm, b^\pm, c^\pm =$ constants (agree well with 3D XY universality class)

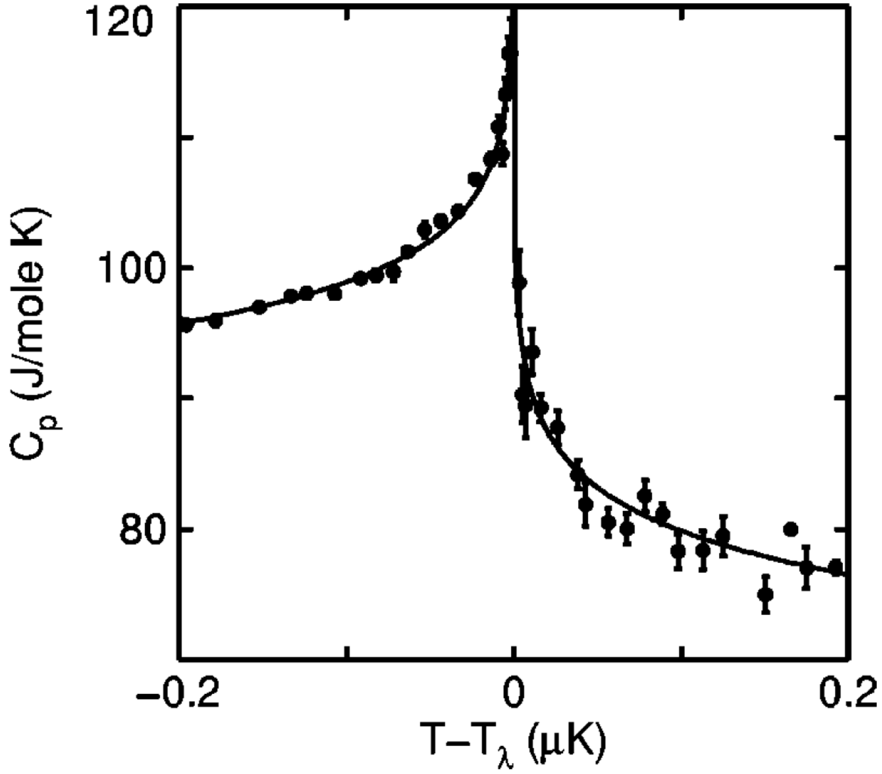
note that $t^{-\alpha} = \exp(-\alpha \ln t) \simeq 1 - \alpha \ln t$, since $\alpha \ln t \ll 1$

$$\Rightarrow \quad C_p \propto b^\pm \frac{t^{-\alpha}}{\alpha} = \frac{b^\pm}{\alpha} - b^\pm \ln t$$

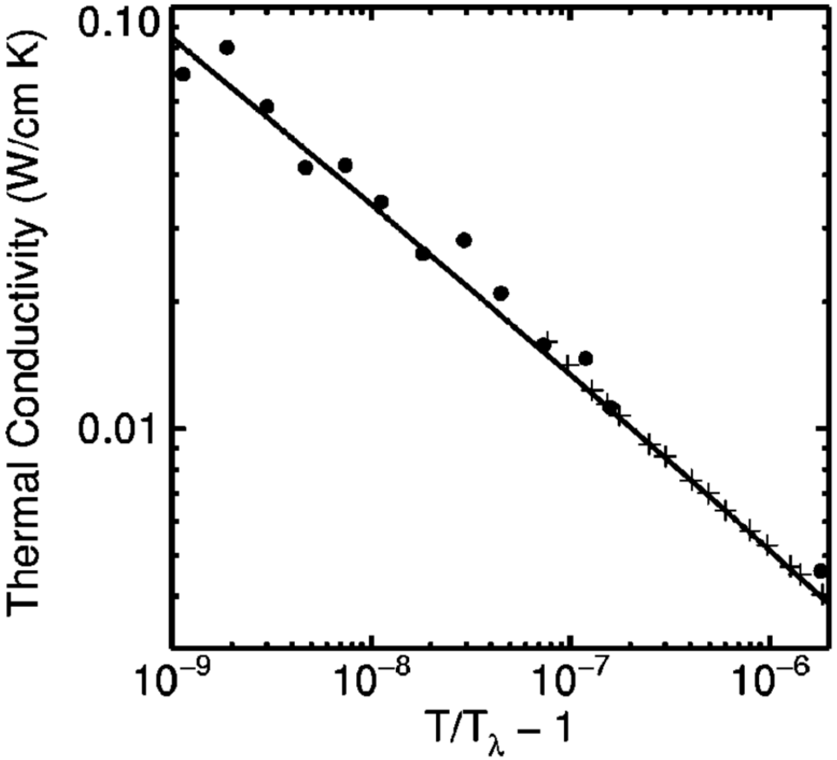
- T_λ depends on pressure, i.e. on the filling height of the He container
  → results in an experimental broadening of the cusp
  ➔ *presicion experiment in space shuttle*
- deviations also appear, if the size of the container becomes smaller than the coherence length, which diverges at T_λ

I.3.4 Excitation Spectrum of Superfluid ^{4}He

- **specific heat and thermal conductivity of ^{4}He revisited: temperature dependence close to T_λ**



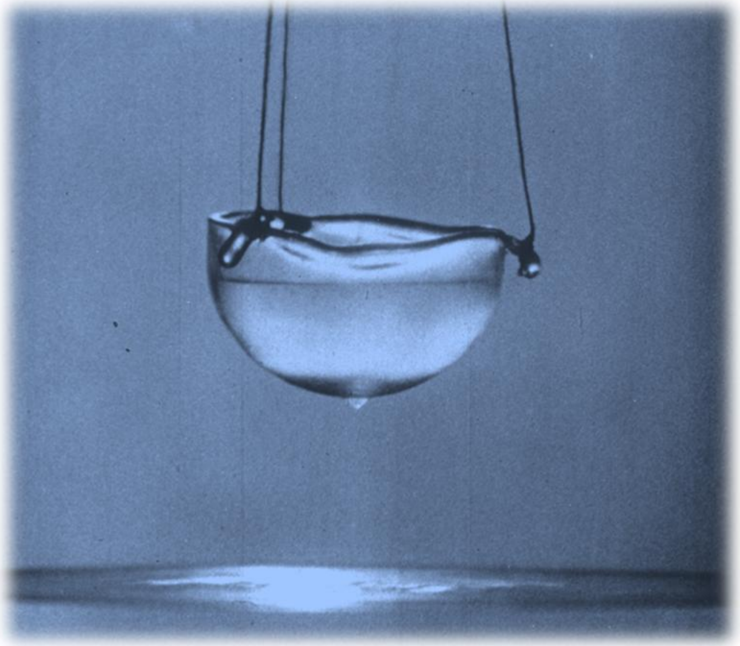
Averaged data close to the transition. Line shows the best-fit function.



Log-log plot of thermal conductivity vs reduced temperature above the lambda point. Filled circles: data, curve: theory model.

Specific heat of liquid helium in zero gravity very near the lambda point
J. A. Lipa et al., Phys. Rev. B 68, 174518 (2003)

Superconductivity and Low Temperature Physics II



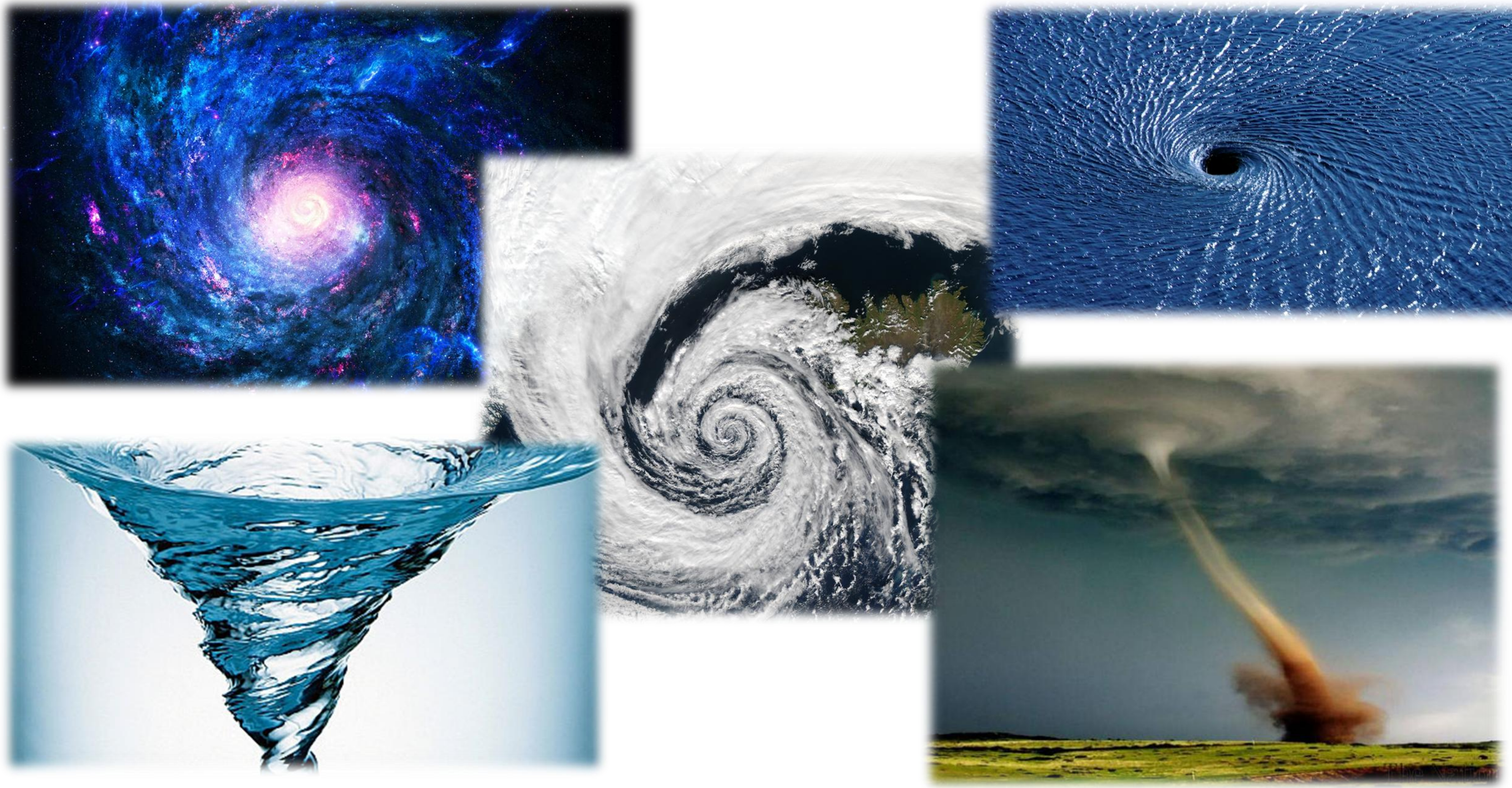
Lecture No. 6

R. Gross

Contents Part I: Quantum Liquids

Contents:

I.4 Vortices



I.4 Vortices

- **vortices in superfluid ^{4}He**

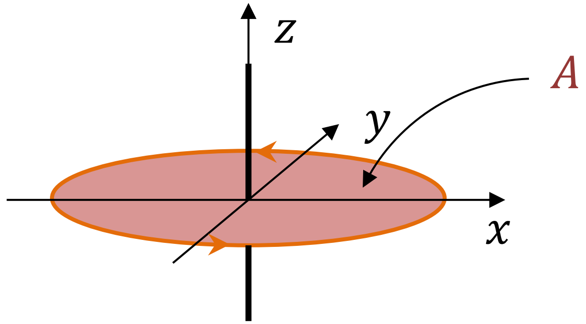
  - in the discussion of the two-fluid model, we assumed the absence of turbulence in $\mathbf{J}_s$: $\Rightarrow \; \boldsymbol{\nabla} \times \mathbf{v}_\mathrm{s} = 0$
  - 1941: ***Landau*** proposed to check this assumption in experiments with superfluid He in rotating containers
  - 1949: ***Onsager*** predicted the appearance of vortices in rotating superfluid He
  - 1953: ***Feynman*** noted that circulation in superfluid ^{4}He should be quantized
  - 1961: first experimental proof of quantization by ***Vinen***

➔ ***quantized vortices in superfluid ^{4}He in the same way as in superconductors (flux lines)***

➔ ***difference: superfluid ^{4}He is not charged, neutral circulation***

I.4.1 Quantization of Circulation

- **vortices – nomenclature and some definitions**



  - ideal vortex in z-direction at position $\mathbf{r} = 0$: $\quad \nabla\times\mathbf{v}(\mathbf{r}) = \Gamma\cdot\delta(x)\cdot\delta(y)$

  - **circulation Γ:**

$$\int_A \nabla\times\mathbf{v}(\mathbf{r})\,\mathrm{d}A \underset{\uparrow}{=} \oint_S \mathbf{v}(\mathbf{r})\cdot\mathrm{d}\mathbf{r} \underset{\uparrow}{=} \Gamma = const.$$

Stokes

Thomson circulation theorem

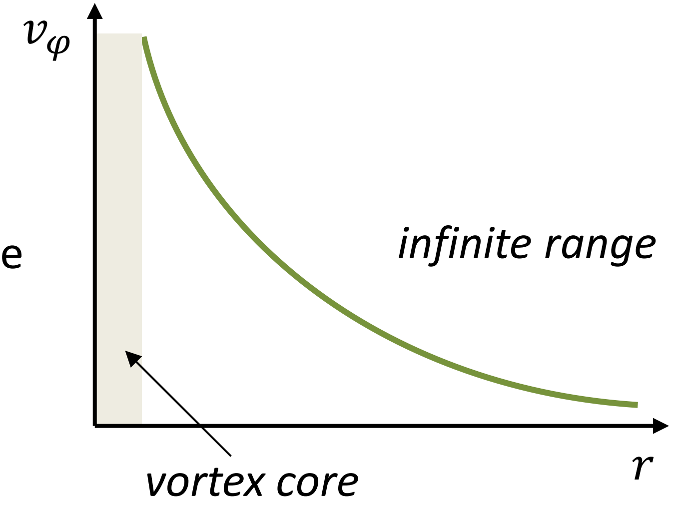
$$\nabla\times\mathbf{v} = \begin{pmatrix} \left[\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right]\hat{x} \\ \left[\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right]\hat{y} \\ \left[\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right]\hat{z} \end{pmatrix}$$

$$\nabla\times\mathbf{v} = \begin{pmatrix} \left[\frac{1}{r}\frac{\partial v_z}{\partial \varphi} - \frac{\partial v_\varphi}{\partial z}\right]\hat{e}_\mathbf{r} \\ \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{e}_\boldsymbol{\varphi} \\ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\varphi) - \frac{\partial v_r}{\partial \varphi}\right]\hat{e}_\mathbf{z} \end{pmatrix}$$

  - if A is a circle, then $\quad \boldsymbol{\Gamma = 2\pi r\cdot v_\varphi}$

    and $\quad \boldsymbol{v_\varphi = \frac{\Gamma}{2\pi r}\,\hat{e}_\varphi}$

    note that $\boldsymbol{\nabla}\times\mathbf{v}_{\boldsymbol{\varphi}} = \mathbf{0}$ apart from the vortex core



➔ ***for superfluid He: circulation Γ is quantized***

I.4.1 Quantization of Circulation

- **description of superfluid component by a macroscopic wave function**

(as done in superconductivity for electronic system: charged superfluid)

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) e^{\imath\theta(\mathbf{r})}, \qquad |\psi_0(\mathbf{r})|^2 = n_0(\mathbf{r}), \qquad N_0 = \int |\psi_0(\mathbf{r})|^2 \, \mathrm{d}^3 r$$

$n_0 = \frac{N_0}{V}$ = particle density in condensate

– phase is related to velocity $\mathbf{v}_s(\mathbf{r})$ of superfluid:

$$\mathbf{J}_\rho(\mathbf{r}) = n_0 \underbrace{\frac{\hbar}{m_{\text{He}}} \nabla\theta(\mathbf{r})}_{\mathbf{v}_s(\mathbf{r})} = n_0 \mathbf{v}_s(\mathbf{r})$$

current density – phase relation (as in superconductivity)

$\mathbf{v}_s$ = superfluid velocity, n_0 = superfluid density

– analogy to superconductors

$$\mathbf{J}_s(\mathbf{r}) = n_s q_s \underbrace{\left[\frac{\hbar}{m_s} \nabla\theta(\mathbf{r}) - \frac{q_s}{m_s} \mathbf{A}(\mathbf{r})\right]}_{\mathbf{v_s}} \qquad \mathbf{v}_s(\mathbf{r}) = \frac{\hbar}{m_s} \nabla\theta(\mathbf{r}) - \frac{q_s}{m_s} \mathbf{A}(\mathbf{r})$$

superfluid He is uncharged: $q_s = 0$

$$\mathbf{v}_s(\mathbf{r}) = \frac{\hbar}{m_{\text{He}}} \nabla\theta(\mathbf{r})$$

I.4.1 Quantization of Circulation

- **circulation of superfluid component**

$$\Gamma = \oint_S \mathbf{v}_s(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \frac{\hbar}{m_{\mathrm{He}}} \underbrace{\oint_S \nabla\theta(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}}_{=2\pi\, n} = \frac{\hbar}{m_{\mathrm{He}}} \cdot 2\pi\, n = \underbrace{\frac{h}{m_{\mathrm{He}}}}_{\kappa} \cdot n$$

➔ circulation is quantized in units of $\boldsymbol{h/m_{\mathrm{He}} = \kappa}$ ***(vorticity)***

➔ quantum number $\boldsymbol{n}$ corresponds to **topological winding number**

(counts the 2π windings along a closed contour)

– generation of vortices by rotation:

angular frequency $\Omega = \Gamma/A \quad \Leftrightarrow \quad$ magnetic field $B = \Phi/A$

Coriolis force $\mathbf{F}_{\mathrm{cor}} = 2m_{\mathrm{He}}\mathbf{v}_s \times \boldsymbol{\Omega} \quad \Leftrightarrow \quad$ **Lorentz force** $\mathbf{F}_{\mathrm{L}} = q_s\mathbf{v}_s \times \mathbf{B}$

$$2m_{\mathrm{He}}\Omega \quad \Leftrightarrow \quad q_s B$$

there is a critical angular frequency Ω_{crit} in analogy to the critical field B_{c1} in superconductors

for circular vortex: $\Gamma = 2\pi r\, v_\varphi \;\Rightarrow\; \Omega_{\mathrm{crit}} = \frac{2\pi R v_{\mathrm{L}}}{\pi R^2} \sim 5\,\frac{1}{s}$ for $R = 10$ cm and $v_{\mathrm{L}} \simeq 45\,\frac{\mathrm{m}}{\mathrm{s}}$

I.4.1 Quantization of Circulation

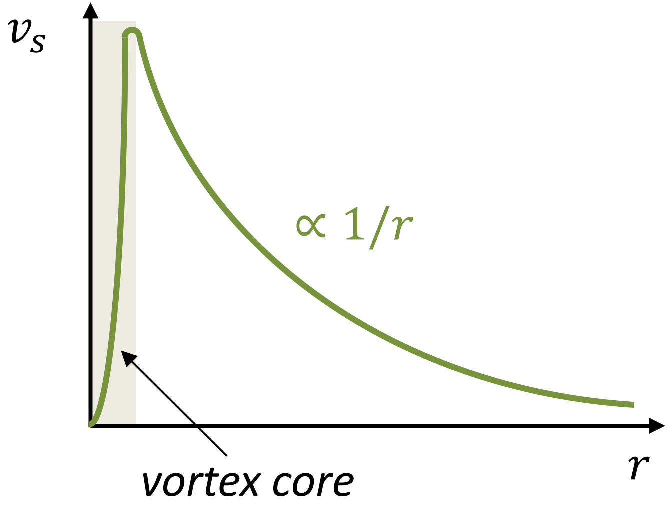
- **consequences of macroscopic quantum description of superfluid**
  - superfluid velocity: $\mathbf{v}_s(\mathbf{r}) = \frac{\hbar}{m_{\mathrm{He}}} \nabla\theta(\mathbf{r}) \quad \Rightarrow \quad \nabla \times \mathbf{v}_s(\mathbf{r}) = \frac{\hbar}{m_{\mathrm{He}}} \underbrace{\nabla \times \nabla\theta(\mathbf{r})}_{=0} = 0$

$$\Longrightarrow \oint \mathbf{v}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \Gamma = 0$$

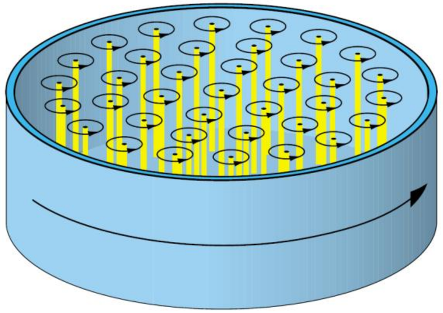
an uncharged superfluid cannot rotate homogeneously

  - **experimental observation:**
    - if we rotate a cylinder filled with superfluid ^{4}He at angular frequency ω, the surface of the superfluid does not stay flat but shows curvature due to centrifugal force

parabolic radial height profile: $z = \frac{\omega^2}{2g} r^2$



    - **apparent contradiction is caused by the presence of vortices**
    - $\nabla \times \mathbf{v}_s(\mathbf{r}) = 0$ is satisfied apart inside the vortex core at $r = 0$:
in vortex core:
macroscopic wave function $|\psi_0(0)|^2 = 0$, phase θ is not defined, superfluid density $J_s = n_0 v_s = 0$



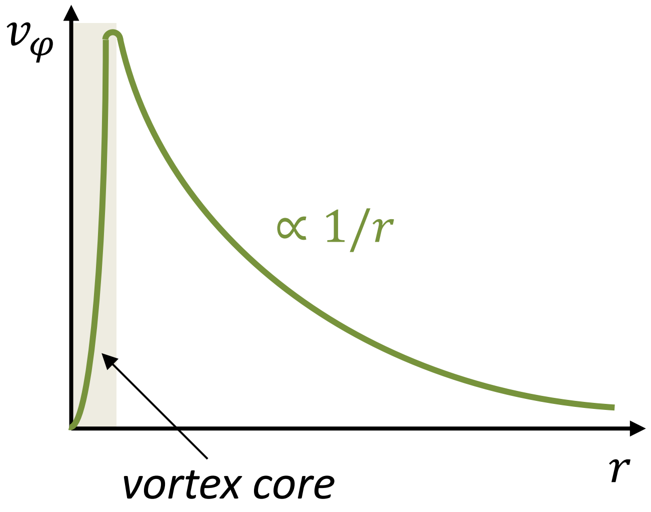
I.4.1 Quantization of Circulation

- **some estimates (1)**

  – magnitude of the superfluid velocity (in circular vortex)

$$\Gamma = \oint \mathbf{v}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \frac{h}{m_{\mathrm{He}}}\, n = 2\pi r\, v_\varphi$$

$$\Rightarrow \quad \boxed{v_\varphi = \frac{h}{m_{\mathrm{He}}} n \frac{1}{2\pi r} = \frac{\kappa}{2\pi r}\, n \simeq \frac{1.5 \times 10^{-8}}{r}\, n \left[\frac{\mathrm{m}}{\mathrm{s}}\right]}$$



  – radius r_{vc} of vortex core (defined by the radius at which v_φ exceeds the Landau velocity)

$$\Rightarrow \quad \boxed{r_{\mathrm{vc}} = \frac{h}{m_{\mathrm{He}}} n \frac{1}{2\pi v_L} = \frac{\kappa}{2\pi v_L}\, n \sim 3\ \text{Å}}$$

for $n = 1$ and $v_{\mathrm{L}} \simeq 45\,\frac{\mathrm{m}}{\mathrm{s}}$ = critical (Landau) velocity for roton formation

  – energy of vortex is proportional to v_φ^2

$$E_{\mathrm{v}} \approx E_{\mathrm{kin}} \propto \kappa^2 \propto n^2$$

➔ it is more favorable to generate two vortices with $n = 1$ instead of a single vortex with $n = 2$

I.4.1 Quantization of Circulation

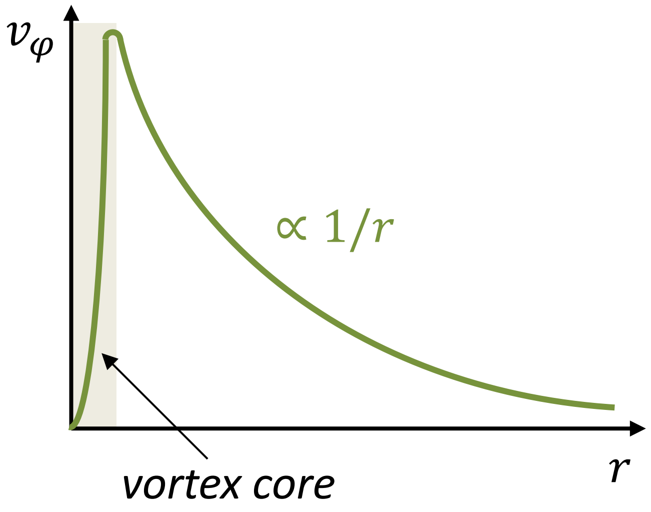
- **some estimates (2)**

  – energy per unit length of vortex

  (circular vortex in cylinder of radius R, ρ is assumed constant)

$$\frac{E_\mathrm{V}}{L} = \frac{1}{2}\rho \int_0^{2\pi} \mathrm{d}\varphi \int_{r_\mathrm{vc}}^{R} v_\varphi^2 \, r\mathrm{d}r = \frac{1}{2} 2\pi\rho \int_{r_\mathrm{vc}}^{R} \left(\frac{\kappa}{2\pi r} n\right)^2 r\mathrm{d}r$$

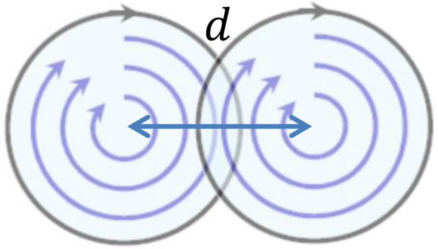
$$\Rightarrow \boxed{\frac{E_\mathrm{V}}{L} = \frac{1}{2}\frac{\rho}{2\pi}\kappa^2 n^2 \ln\left(\frac{R}{r_\mathrm{vc}}\right) = \frac{1}{2}\rho\frac{(\hbar/m_\mathrm{He})^2}{2\pi} n^2 \ln\left(\frac{R}{r_\mathrm{vc}}\right)}$$



  – energy of two parallel vortices at distance $d \gg r_\mathrm{vc}$

$$\Rightarrow \boxed{\frac{E_\mathrm{V}(d)}{L} = \frac{1}{2}\frac{\rho}{2\pi} 2\kappa^2 n^2 \ln\left(\frac{R^2}{r_\mathrm{vc} d}\right)}$$

vortices repel each other as energy increases with decreasing d
(decreasing mutual cancellation of their velocity fields with decreasing d)

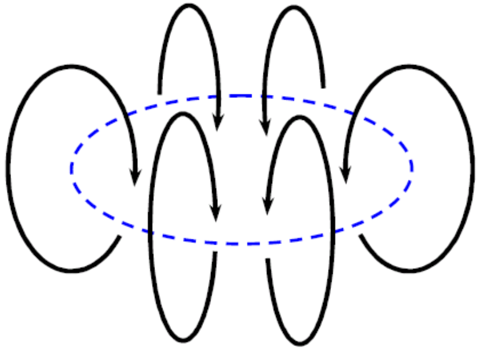


  – anti-parallel vortices attract each other
  (due to the increasing mutual cancellation of their velocity fields with decreasing d)

I.4.1 Quantization of Circulation

- **how many vortices are there in thermal equilibrium ?**

– excitation energy ≈ E_{kin} should be as small as possible for large number:



- minimum v_s	→ $n = 1$
- minimum length	→ vortex rings
- smallest ring	→ atomic dimension

– vortex still macroscopic object

→ $E_{\mathrm{V}} \gg k_{\mathrm{B}}T$ except for $T \sim T_\lambda$

→ momentum is large → critical velocity $v_{\mathrm{L}} = E\,/\,p$ is small for vortex rings

– critical velocity v_{L} is determined by excitation of vortex rings

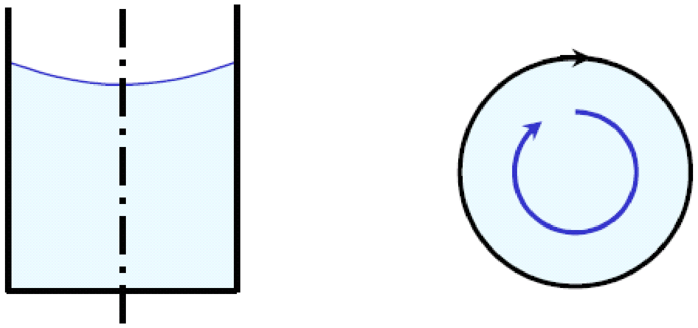
I.4.2 Experimental Study of Vortices

- cooling down a rotating ^{4}He container below T_λ

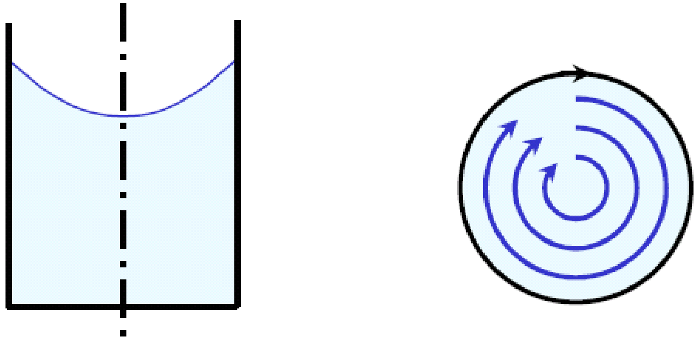
first experiments by Vinen in 1961

The detection of a single quantum of circulation in liquid helium II, W.F. Vinen, *Proc. Roy. Soc. (Lond.)* A **260**,218 (1961).

➢ above T_λ:
rigid rotation

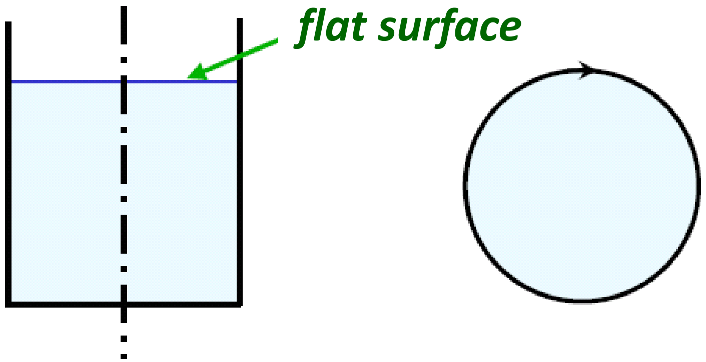


weak circulation



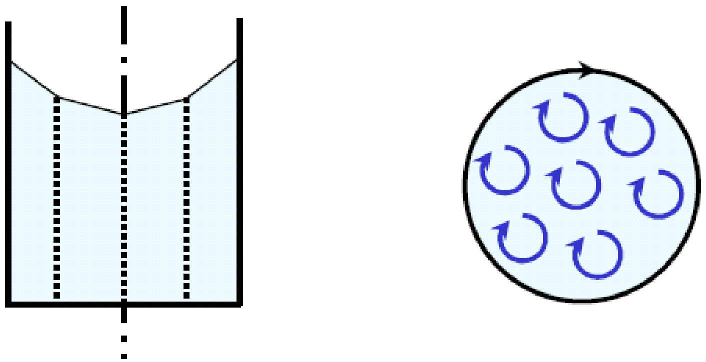
strong circulation

➢ below T_λ:



$\Omega < \Omega_{\text{crit}}$

screening of circulation
„Meißner State"



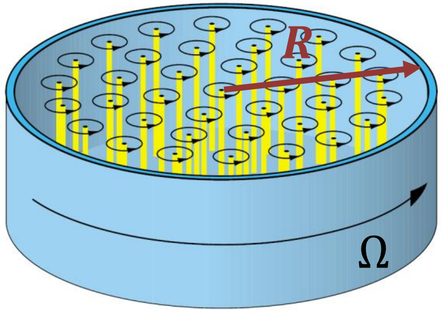
$\Omega > \Omega_{\text{crit}}$

penetration of circulation
„Mixed State"

I.4.2 Experimental Study of Vortices

- **what is the density of vortices inside the container ?**

  – circulation at the edge of the container with radius R:



$$\Gamma = \oint_S \mathbf{v}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = 2\pi R \;\; v(R) = 2\pi R \;\; \Omega R = 2\pi\, R^2\Omega$$

  – circulation must be equal to number of vortices N_V × vorticity κ × area πR^2

$$\Gamma = N_V\, \kappa\, \pi R^2 = 2\pi R^2 \Omega$$

$$\boxed{N_V = \frac{2\Omega}{\kappa} = \frac{2m_{\mathrm{He}}\Omega}{h} = \frac{m_{\mathrm{He}}\Omega}{\pi\hbar}}$$

$$N_{\mathrm{V}} \simeq \frac{20}{\mathrm{mm}^2} \times \Omega\,[\mathrm{Hz}\,]$$

I.4.2 Experimental Study of Vortices

- **rotating cryostats**

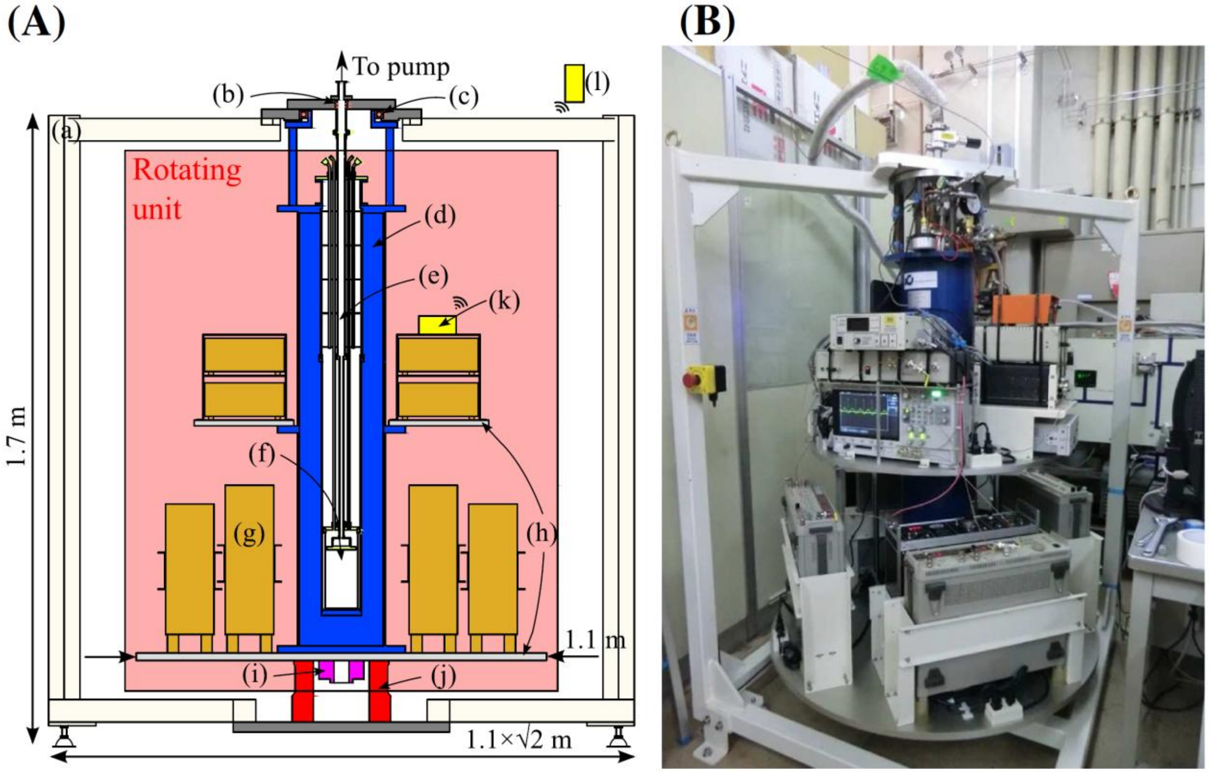


Fig. 1 **a** Schematic vertical section of the rotating cryostat: (*a*) Steel frame, (*b*) rotating o-ring seal, (*c*) ball bearing, (*d*) aluminum/FRP dewar, (*e*) 1K cryostat insert, (*f*) sample space, (*g*) instruments, (*h*) duralumin round tables, (*i*) slip ring for power supply, (*j*) AC servomotor, (*k*) GPIB-USB interface and wireless USB device server and (*l*) wireless LAN access point. **b** Photograph of the rotating cryostat (Color figure online)

A Compact Rotating 1K Cryostat for ^{4}He Studies
Takahiko Makiuchi, Satoshi Murakawa & Keiya Shirahama
J. Low Temp. Phys. **187**, 633-638 (2017)

Helsiki university

http://ltl.tkk.fi/research/applied/rotating3he.html

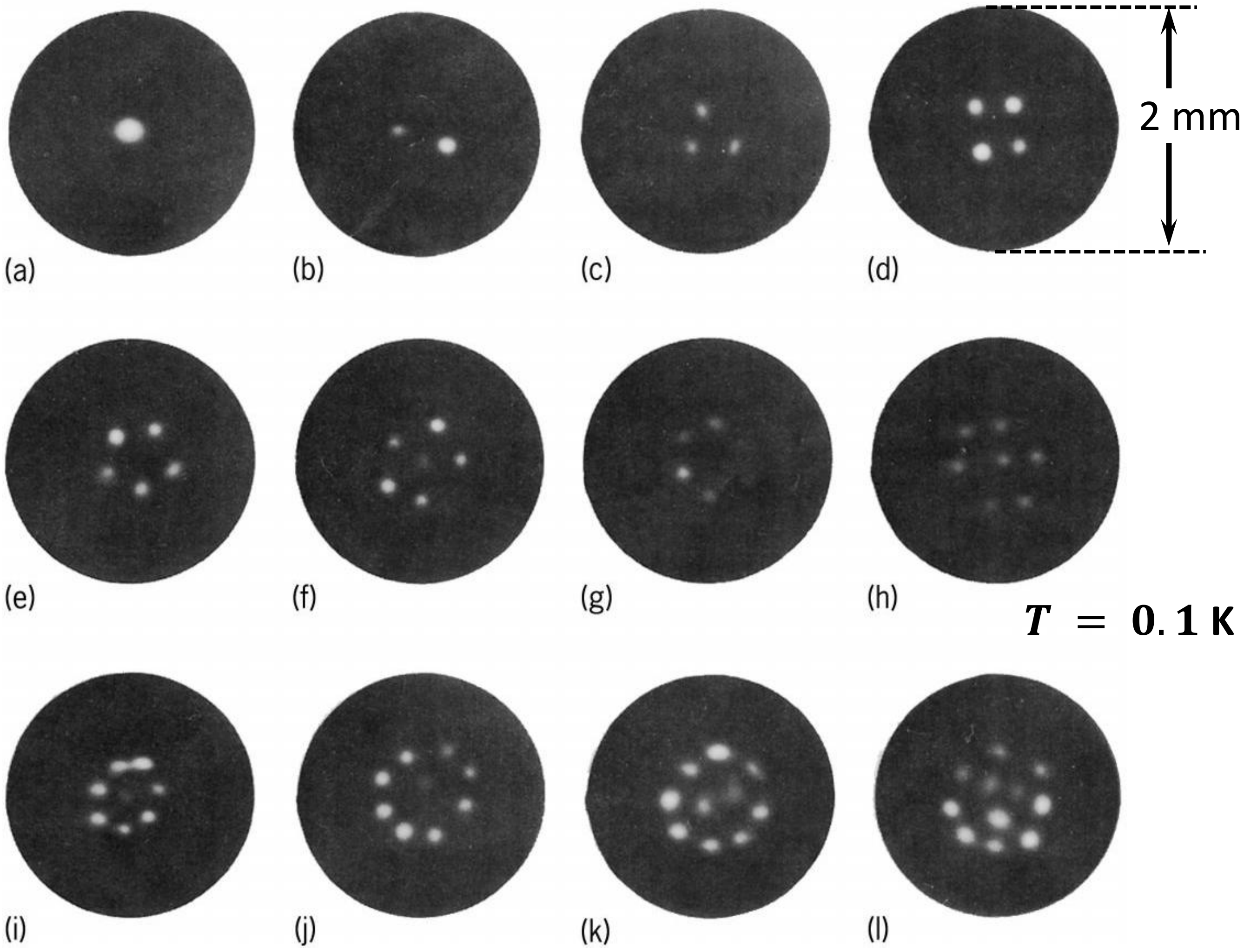
I.4.2 Experimental Study of Vortices

- **imaging of vortices in superfluid ^{4}He**

➢ *He is covered by electrons*
➢ *electrons are trapped in vortex cores*
➢ *on applying an electric field, electrons are extracted to fluorescent screen*

pattern and camera are rotating

pinning of vortices at bottom of container

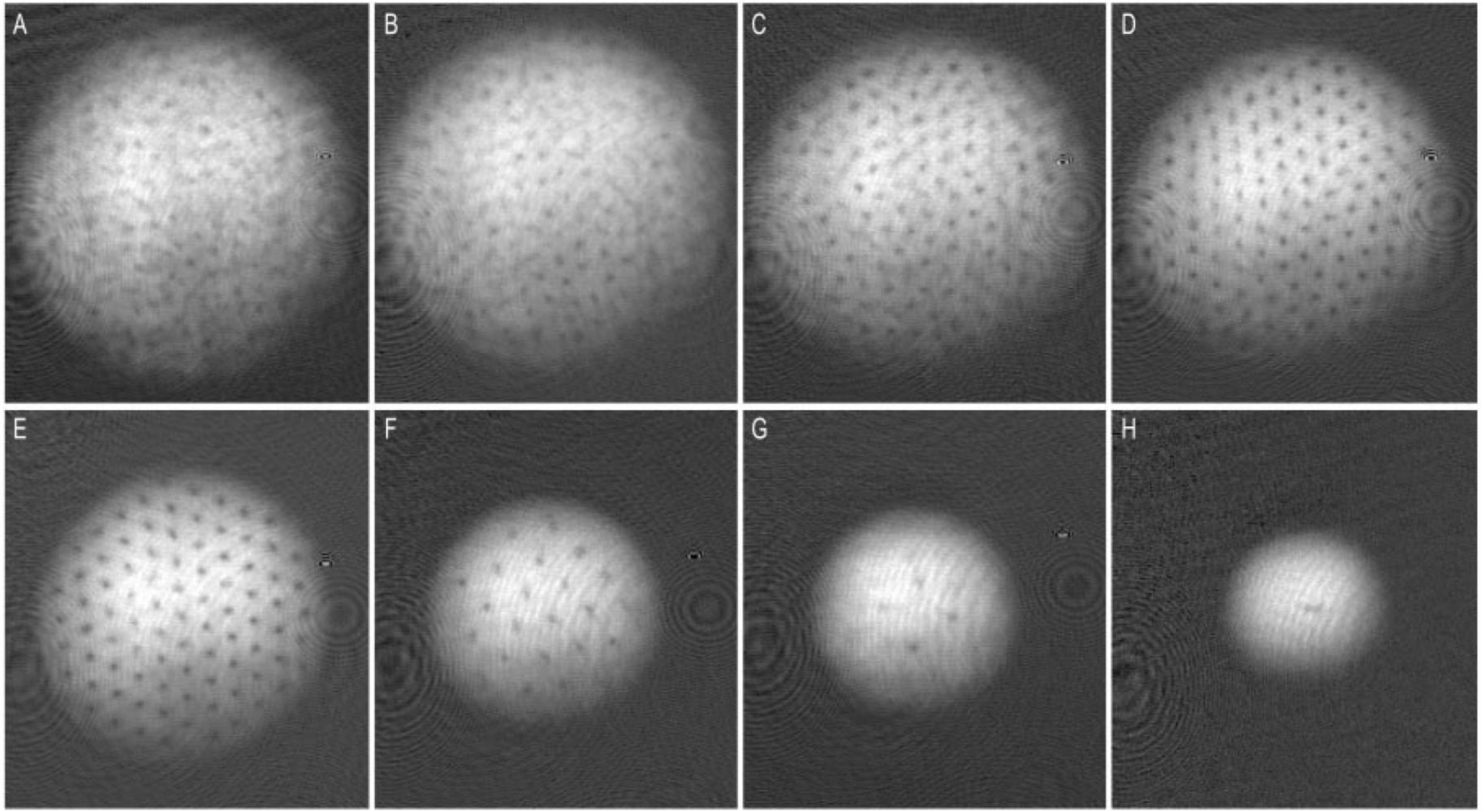


E.J. Yarmchuk, M.J.V. Gordon, R.E. Packard, Phys. Rev. Lett. **43**, 214 (1979)
E.J. Yarmchuk, R.E. Packard, J. Low Temp. Phys. **46**, 479 (1982)

I.4.2 Experimental Study of Vortices

- **imaging of vortices in a Bose-Einstein condensate of Na atoms**

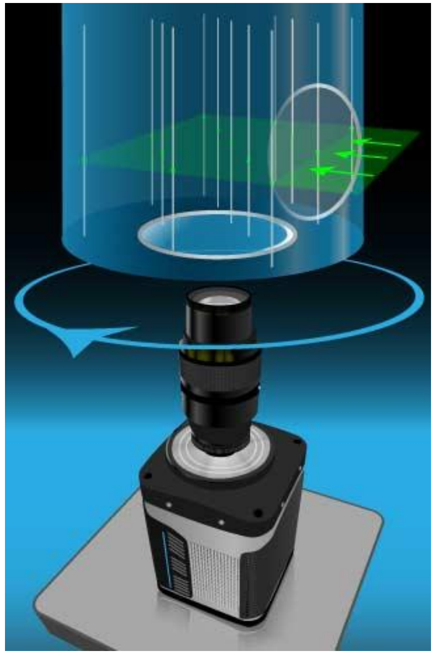
Formation and decay of a vortex lattice. The condensate was rotated for 400 ms and then equilibrated in the stationary magnetic trap for various hold times. (**A**) 25 ms, (**B**) 100 ms, (**C**) 200 ms, (**D**) 500 ms, (**E**) 1 s, (**F**) 5 s, (**G**) 10 s, and (**H**) 40 s. The decreasing size of the cloud in (E) to (H) reflects a decrease in atom number due to inelastic collisions. The field of view is about 1 mm by 1.15 mm.



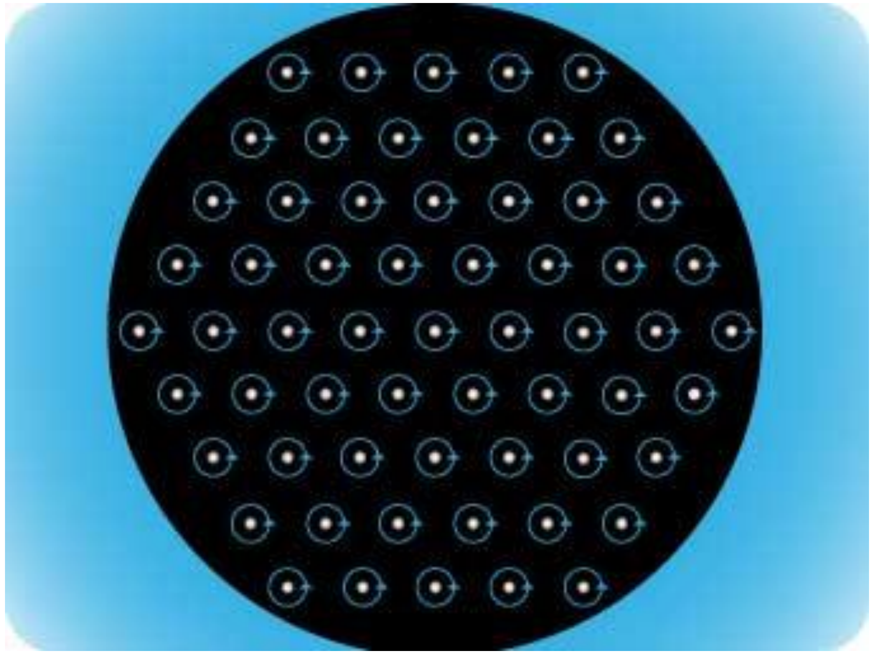
Observation of Vortex Lattices in Bose-Einstein Condensates
W. Ketterle et al., Science **292**, 476 (2001)

I.4.2 Experimental Study of Vortices

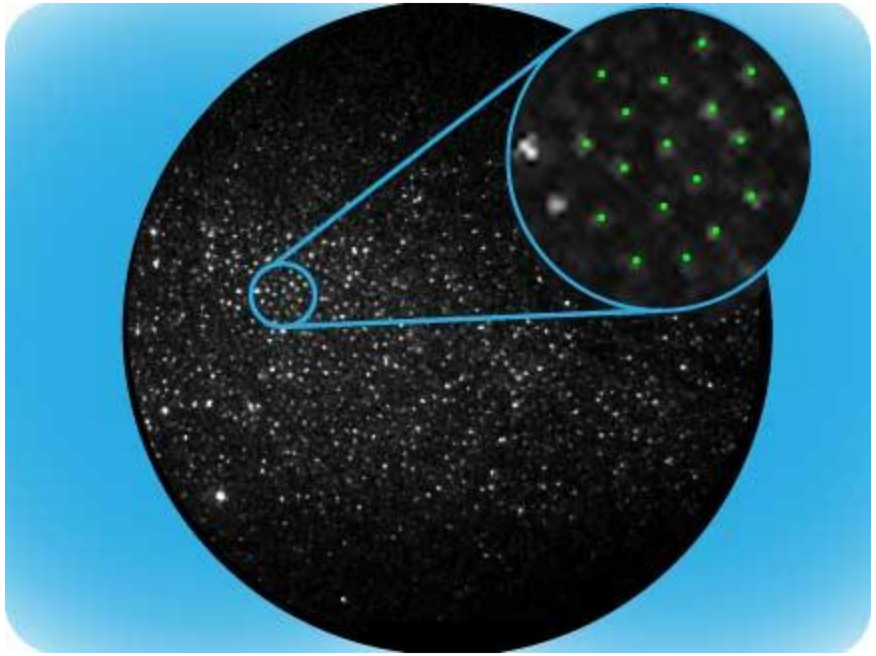
- **imaging of vortices**



A sheet of laser light illuminates frozen hydrogen particles trapped on vortex cores, which are then captured on video.



Quantized vortices arrange themselves in a triangular lattice pattern.



A still image captures the vortex lattice along the axis of rotation in a vat of superfluid helium. A perfect lattice is not observed due to waves and other boundary effects.

https://engage.aps.org/dfd/resources/virtual-pressroom/2009-pressroom/language-papers-2009/tornadoes

I.4.2 Experimental Study of Vortices

- **generation of persistent current**
  - rotate a normal fluid at $T > T_\lambda$
  - cool down the rotating normal fluid to $T < T_\lambda$ ➔ the rotation is frozen in and does not decay

- **generation of circulation jumps or phase slip events**
  - start with a non-rotating fluid at $T < T_\lambda$: $\kappa = 0, \;\; n = 0$
  - rotate the cylindrical vessel containing the fluid:

    ➔ normal fluid rotates at the speed of the vessel

    ➔ superfluid stays in rest
  - reduce the temperature:

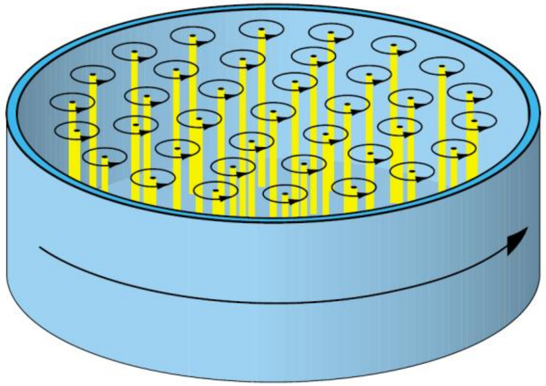
    ➔ more and more particles of the normal fluid enter the superfluid

    ➔ the lost angular momentum is transferred to the superfluid, which starts to rotate

    ➔ the circulations and the vorticity κ do not change continuously but in jumps of h/m_{He}

    ➔ jumps are associated with **additional vortex** corresponding to phase change of 2π called **phase slip**

    ➔ phase slips correspond to **transition between two quantum states** with different quantum number n



Contents Part I: Quantum Liquids

Contents:

I.5 ^{3}Helium

- **basic properties of ^{3}He**

^{4}He	$p^+\uparrow$ $p^+\downarrow$	$n\uparrow$ $n\downarrow$	$e^-\uparrow$ $e^-\downarrow$	**boson**
^{3}He	$p^+\uparrow$ $p^+\downarrow$	$n\uparrow$	$e^-\uparrow$ $e^-\downarrow$	**fermion**

– ^{3}He has spin ½ ➔ ***Fermion***

– fluid ^{3}He: ***interacting Fermi gas*** → Fermi liquid (*ideal non-interacting Fermi gas with finite interaction*)
→ theoretical description by Landau (1956-1958)

L.D. Landau, Soviet. Phys. JETP 3, 920 (1957)
L.D. Landau, Soviet. Phys. JETP 5, 101 (1957)

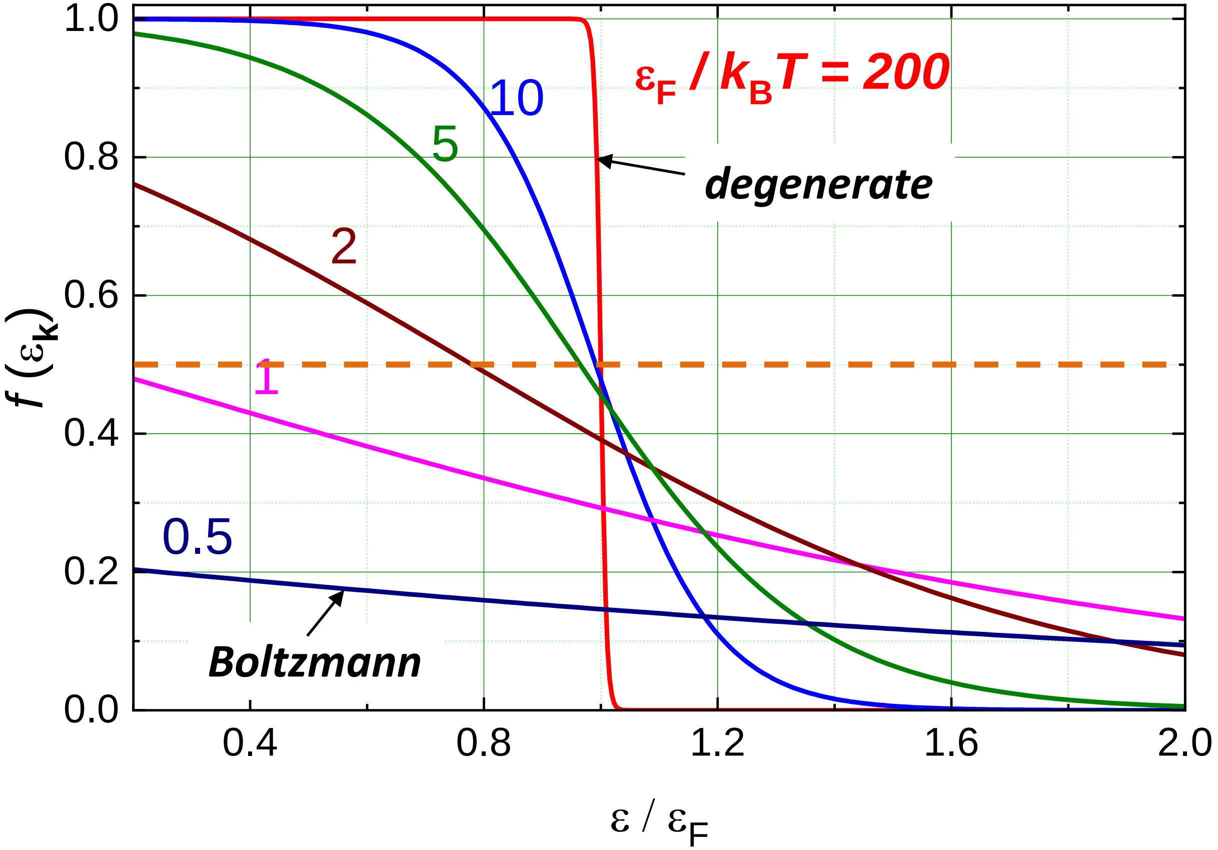
– solid ^{3}He shows nuclear magnetism

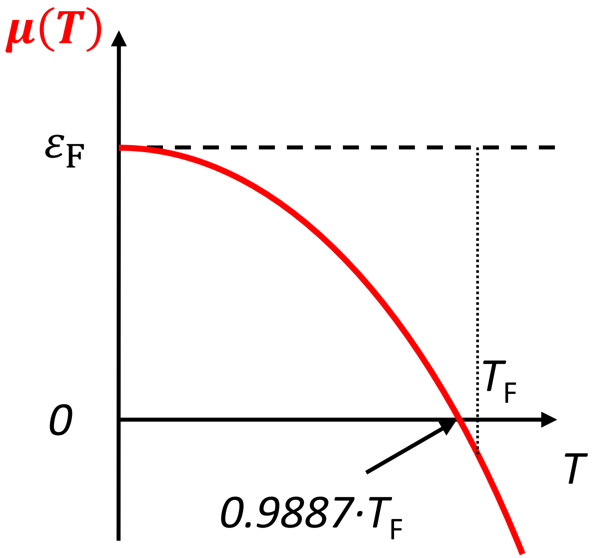
➔ **due to different quantum statistics:**
liquid ^{3}He has completely different properties than liquid ^{4}He

I.5.1 Normal Fluid ^{3}Helium

- **first approximation:** description by an ***ideal Fermi gas*** (*non-interacting Fermions*)
  - Fermi-Dirac distribution:

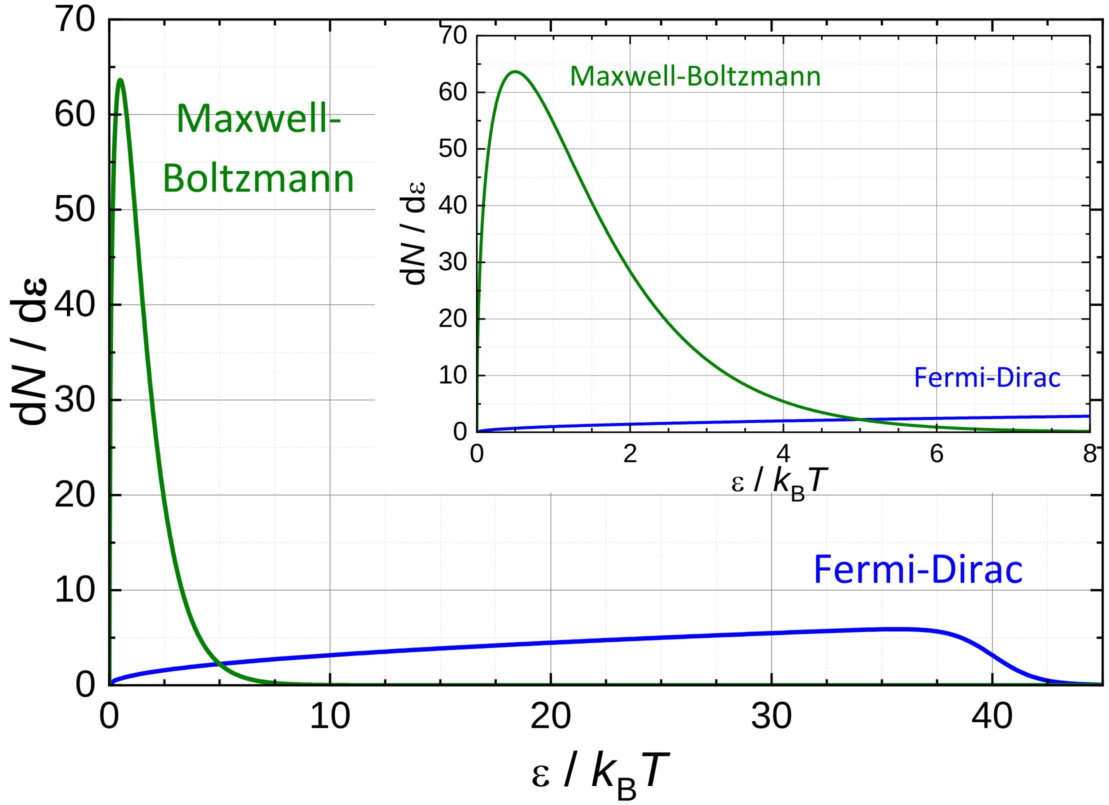
$$f(\varepsilon_{\mathbf{k}}) = \frac{1}{\exp\left(\frac{\varepsilon_{\mathbf{k}} - \mu}{k_B T}\right) + 1}$$

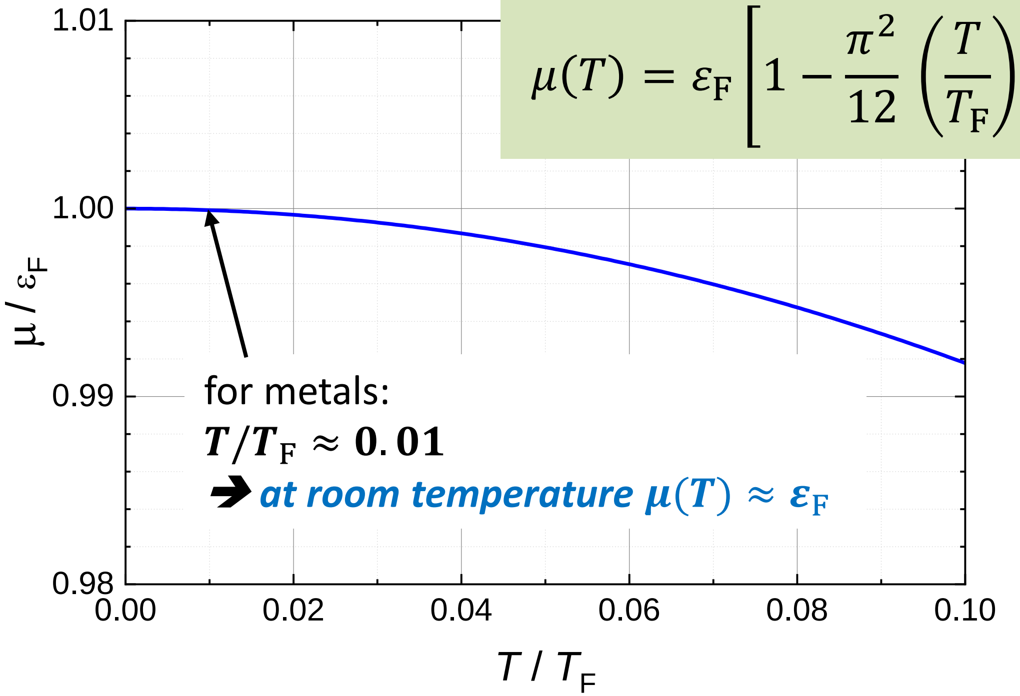




I.5.1 Normal Fluid ^{3}Helium

- **ideal (non-interacting) Fermi gas**





$$\mu(T) = \varepsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{T}{T_F} \right)^2 \right]$$

for metals:
$T/T_F \approx 0.01$
➔ ***at room temperature*** $\mu(T) \approx \varepsilon_F$

$$N = \int D(\varepsilon) f(\varepsilon)\, \mathrm{d}\varepsilon$$

$$\frac{\mathrm{d}N}{\mathrm{d}\varepsilon} = D(\varepsilon) f(\varepsilon)$$

I.5.1 Normal Fluid ^{3}Helium

- **ideal (non-interacting) Fermi gas**

$$\varepsilon_\mathrm{F} = \frac{\hbar^2 k_\mathrm{F}^2}{2m} = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$$ ***Fermi energy***

$$T_\mathrm{F} = \varepsilon_\mathrm{F}/k_\mathrm{B}$$ ***Fermi temperature***

$$\lambda_\mathrm{F} = 2\pi/k_\mathrm{F}$$ ***Fermi wavelength***

$$v_\mathrm{F} = \frac{p_\mathrm{F}}{m} = \frac{\hbar k_\mathrm{F}}{m}$$ ***Fermi velocity***

N = particle number
n = particle density [1/m³]

$m = m_\mathrm{He}$ is large → expected Fermi temperature (energy) is small: $T_\mathrm{F} \simeq 3.8$ K (for $\rho = 0.055 \frac{\mathrm{g}}{\mathrm{cm}^3}$
$n = 1.10 \times 10^{22} \mathrm{cm}^{-3}$
at boiling point)

comparison to electron gas in metal:

electrons in metals:

$n \approx 10^{23}$ cm^{-3} is large, $m = m_e$ is small
→ $T_\mathrm{F} \approx 10^5$ K is large

^{3}He:

$n \approx 10^{22}$ cm^{-3} is slightly smaller, $m = m_\mathrm{He}$ is large
→ $T_\mathrm{F} \approx$ few K is small

$$T_\mathrm{F} = \frac{\hbar^2 k_\mathrm{F}^2}{2m} = \frac{\hbar^2}{2mk_\mathrm{B}}(3\pi^2 n)^{2/3}$$

R. Gross © Walther-Meißner-Institut (2004 - 2024)

I.5.1 Normal Fluid ^{3}Helium

- **ideal (non-interacting) Fermi gas: specific heat**
  - **classical:** $$C_V^{\text{class}} = \left.\frac{\partial\langle U\rangle}{\partial T}\right|_V = N\cdot 2\cdot 3\cdot\frac{1}{2}k_\mathrm{B} = 3Nk_\mathrm{B}$$ **(Dulong-Petit)**

    spin

    kinetic degrees of freedom

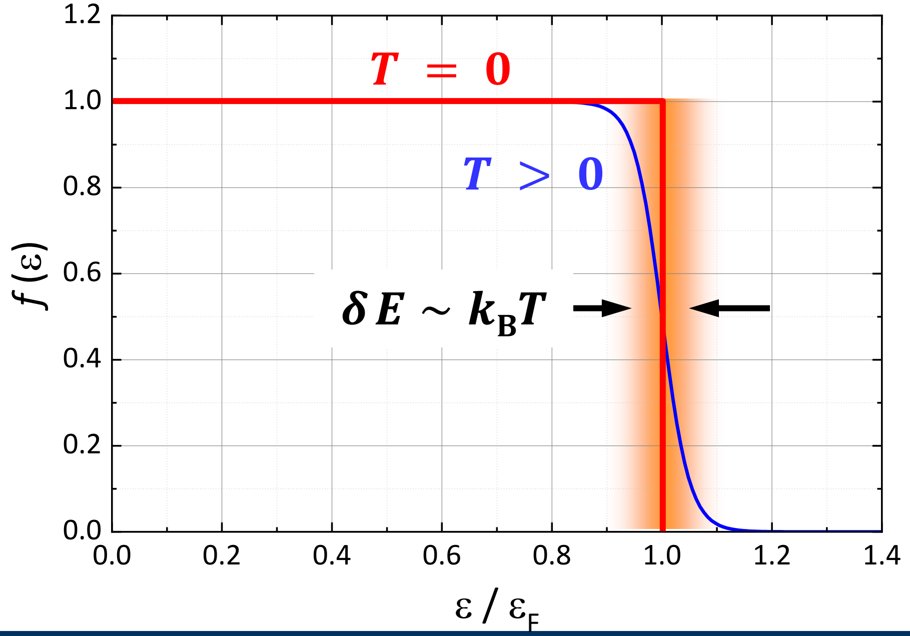
    contribution per degree of freedom (equipartition theorem)
  - **quantum mechanical:** $$U = \int_0^\infty \varepsilon\, D(\varepsilon) f(\varepsilon)\,\mathrm{d}\varepsilon$$

Sommerfeld expansion $$U = U(T=0) + (k_\mathrm{B}T)^2\frac{\pi^2}{6}\,D(\varepsilon_\mathrm{F})$$

$$\frac{D(\varepsilon_\mathrm{F})}{V} = \frac{3}{2}\frac{n}{\varepsilon_\mathrm{F}}$$

$$\frac{C_V^{\text{qm}}}{V} = \frac{\pi^2}{2}nk_\mathrm{B}^2\frac{T}{\varepsilon_\mathrm{F}} = \gamma\cdot T$$

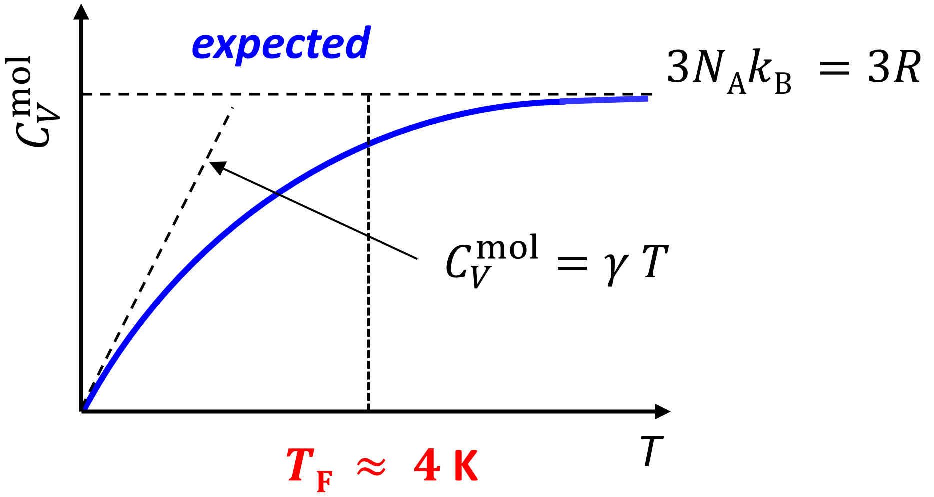
Sommerfeld coefficient $$\gamma = \frac{\pi^2}{3}k_\mathrm{B}^2\frac{D(\varepsilon_\mathrm{F})}{V} = \frac{\pi^2}{2}\frac{nk_\mathrm{B}^2}{\varepsilon_\mathrm{F}}$$

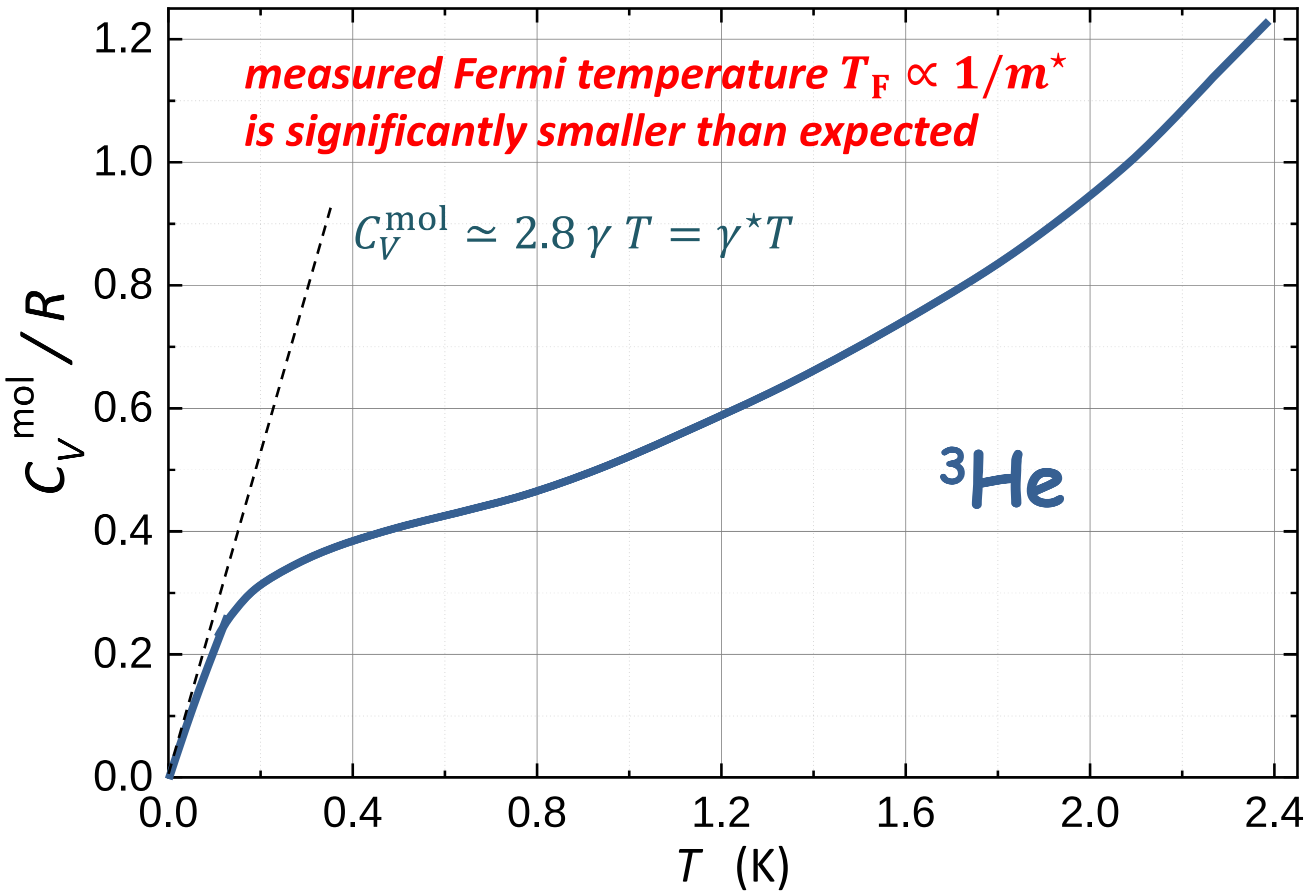


- *only small fraction* $\simeq T/T_\mathrm{F}$ *of particles in energy interval* $k_\mathrm{B}T$ *around* E_F *contributes to specific heat*
- *energy change per particle* $\approx k_\mathrm{B}T$
- *total energy change* $\approx k_\mathrm{B}T\cdot Nk_\mathrm{B}\left(\frac{T}{T_\mathrm{F}}\right) \propto D(E_\mathrm{F})T^2$
- $C_V \propto D(E_\mathrm{F})T$

I.5.1 Normal Fluid ^{3}Helium

- **ideal (non-interacting) Fermi gas: specific heat**





- interacting Fermi gas → ***Fermi liquid*** *(Landau)*
- increased effective mass $m^\star$

  → *smaller Fermi temperature* $T_F \propto 1/m^\star$

  $m^\star/m_{\mathrm{He}} \simeq 2.8$ *(@ p =1 bar)*

  $\simeq 5.5$ *(@ p = 30 bar)*

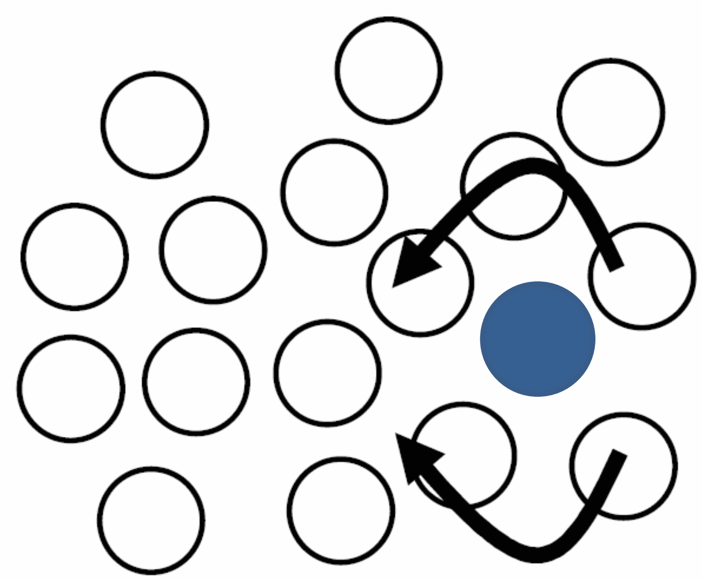
- *additional contribution due to **spin fluctuations (paramagnons)***

  → *strong repulsive interaction at short distance* → *antisymmetric orbital wavefunction*

  → *symmetric spin wavefunction* → *parallel orientation of neighboring spins*

I.5.1 Normal Fluid ^{3}Helium

- **intuitive argument for increased effective mass:**



interaction causes „backflow“ of particles

➔ increased effective mass m^*

➔ reduced Fermi temperature $T_\mathrm{F} \propto 1/m^\star$

I.5.1 Normal Fluid ³Helium

- **magnetic susceptibility:** **nuclear spin $I = ½$ determines magnetic properties**
  - $T > T_\mathrm{F}$: ***non-degenerate Fermi gas*** ➔ magnetic susceptibility follows **Curie law**: $\chi \propto C/T$

$$\chi = \mu_0 \left(\frac{\partial M}{\partial B_\mathrm{ext}}\right)_{T,V} = \frac{\mu_0 n \mu_\mathrm{eff}^2}{k_\mathrm{B} T} = \frac{C}{T} \qquad \mu_\mathrm{eff} = g_n \mu_n$$

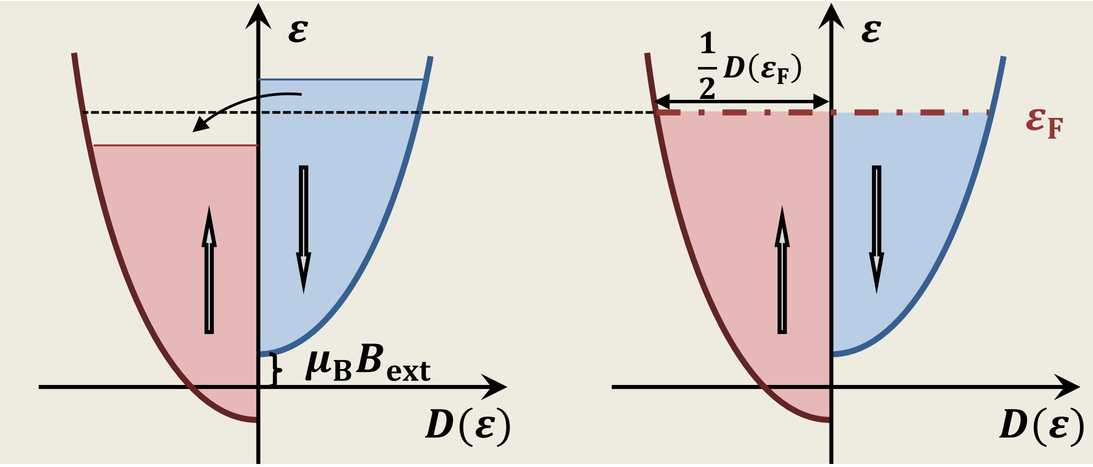
g_n = *Landé factor*

μ_n = *magnetic moment of ³He*

  - $T < T_\mathrm{F}$: ***degenerate Fermi gas*** ➔ temperature independent **Pauli spin susceptibility** (only fraction T/T_F contributes)

for electron liquid: $$\chi_P = \mu_0 \left(\frac{\partial M}{\partial B_\mathrm{ext}}\right)_{T,V} = \mu_0 \mu_\mathrm{B}^2 \frac{D(\varepsilon_\mathrm{F})}{V} = \frac{3}{2}\frac{\mu_0 n \mu_\mathrm{B}^2}{k_\mathrm{B} T_\mathrm{F}} \simeq const.$$

for ³He liquid: $$\boxed{\chi_P = \mu_0 \mu_\mathrm{eff}^2 \frac{D(\varepsilon_\mathrm{F})}{V}}$$ with $\mu_\mathrm{eff} = g_n \mu_n$



$$M = \frac{\mu_\mathrm{eff}\,\Delta \mathrm{N}}{\mathrm{V}} = \mu_\mathrm{eff}\,\Delta \mathrm{n}$$

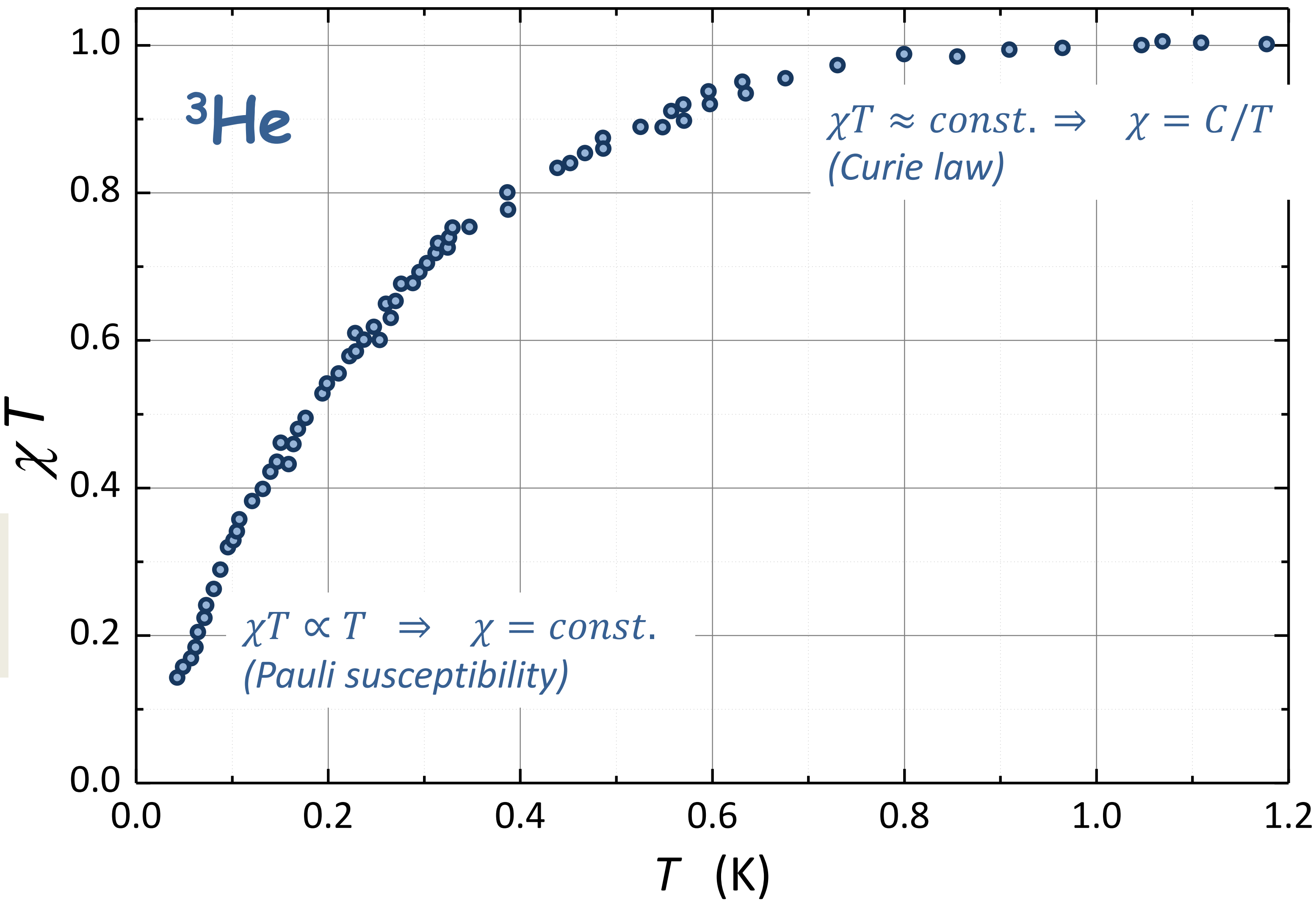
$$\Delta n \approx \mu_\mathrm{eff}\, B_\mathrm{ext} \cdot D(\varepsilon_\mathrm{F})/V$$

$$\chi = \mu_0 \mu_\mathrm{eff}^2 D(\varepsilon_\mathrm{F})/V$$

$$D(\varepsilon_\mathrm{F}) \propto m^\star$$

I.5.1 Normal Fluid ^{3}Helium

- **magnetic susceptibility**



degenerate

non-degenerate

cannot be measured for an electron gas system in a metal since $T_\mathrm{F} \gg$ melting temperature

I.5.1 Normal Fluid ^{3}Helium

- **transport properties**

  – description by ***kinetic gas theory***:

  → viscosity: $\eta = \frac{1}{3}\rho\, v\, \ell$ (ℓ = mean fee path)

  → diffusion coefficient: $D = \frac{1}{3} v\, \ell$ (self-diffusion)

  → thermal conductivity: $\kappa = \frac{1}{3} C_V\, v\, \ell$

  – classical gas: v_{th} = thermal velocity $v_{\mathrm{th}} = \sqrt{2k_{\mathrm{B}}T/m}$ (Maxwell-Boltzmann statistics)

  – Fermi gas: v_{F} = Fermi velocity $v_{\mathrm{F}} = \frac{\hbar}{m}(3\pi^2 n)^{1/3}$ (Fermi statistics)

  – Fermi gas, with $\ell = v_{\mathrm{F}}\tau$

$$\eta = \frac{1}{3}\rho\tau v_{\mathrm{F}}^2 \propto \frac{1}{T^2}$$

$$D = \frac{1}{3}\tau\, v_{\mathrm{F}}^2 \propto \frac{1}{T^2}$$

$$\kappa = \frac{1}{3}C_V\tau\, v_{\mathrm{F}}^2 \propto \frac{1}{T}$$

$$\boldsymbol{\tau^{-1} \propto \left(\frac{T}{T_{\mathrm{F}}}\right)^2}$$

particle-particle scattering rate

I.5.1 Normal Fluid ^{3}Helium

- comparison of properties of liquid ^{3}He with those of ideal (non-interacting) Fermi gas

	^{3}He	Fermi Gas	ratio
$C_V^{\mathrm{mol}}/N_\mathrm{A}k_\mathrm{B}T$	**2.78**	**1.0**	**2.78**
v_F (m/s)	**56**	**157**	**2.80**
$\chi/\mu_0\mu_{\mathrm{eff}}^2$ $(\mathrm{Jm}^3)^{-1}$	**$3.3\cdot10^{51}$**	**$3.6\cdot10^{50}$**	**9.2**

significant deviations due to finite interactions → *Landau Fermi liquid*
→ *effective mass*

mole volume: $V_{\mathrm{mol}} = 36.84$ cm³ @ $p = 0$ bar, $T = 0$ K → $n = 1.63 \times 10^{28}$ m^{-3}
mass: $m_{3\mathrm{He}} = 3.016$ amu $= 5.006 \times 10^{-27}$ kg

Contents Part I: Quantum Liquids

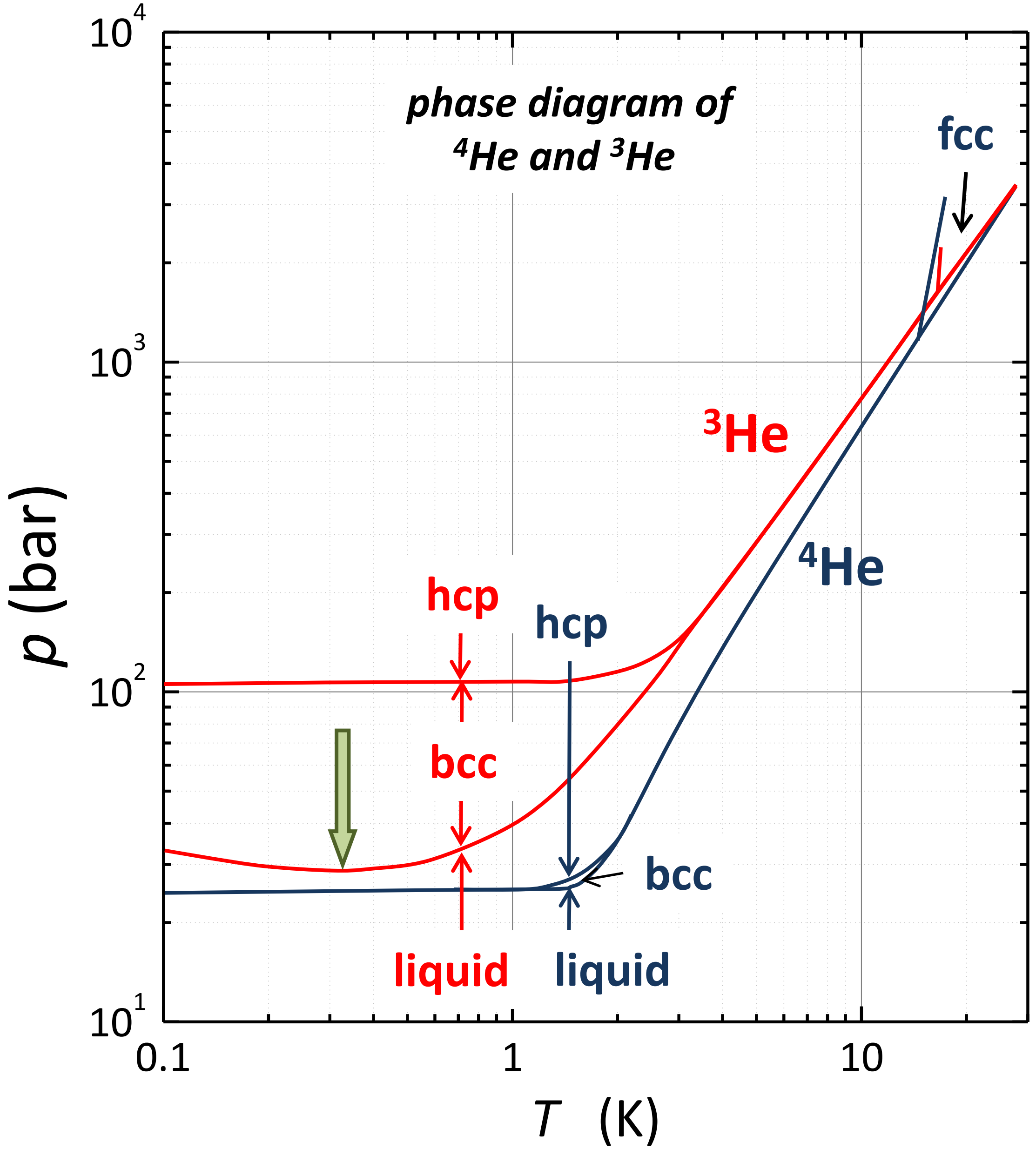
Contents:

I.5.2 Solid ^{3}He and Pomerantchuk Effect

- comparison of phase diagrams of ^{3}He and ^{4}He

  - ^{4}He mostly hcp, small region with bcc,
    at high T: fcc
  - ^{3}He shows bcc-hcp transition at 100 bar,
    at high T: fcc
  - ^{3}He shows minimum in melting curve at
    $T = 0.32\ \mathrm{K}$

➔ can be used for cooling of ^{3}He
Pomerantchuk effect



Isaak Jakowlewitsch Pomerantschuk
(*Исаак Яковлевич Померанчук*;
geb: 20. Mai 1913 in Warschau;
gest: 14. Juli 1966 in Moskau)
russischer Physiker.

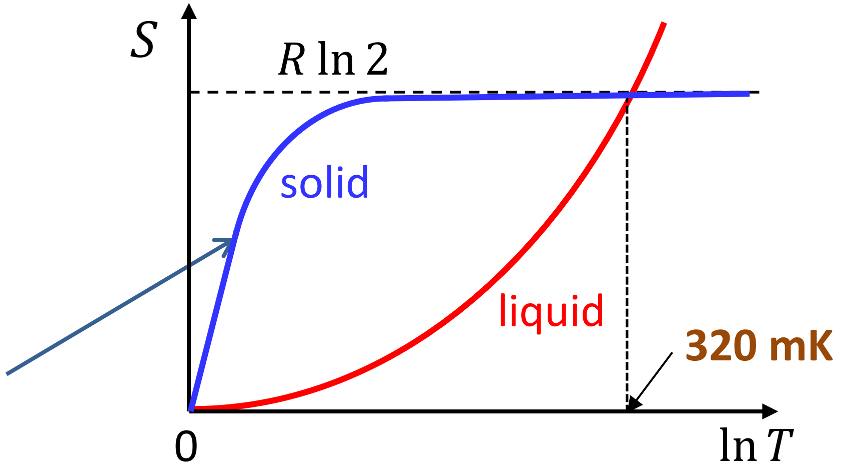
I.5.2 Solid ^{3}He and Pomerantchuk Effect

- Clausius-Clapeyron equation: $\left.\frac{\partial p_m}{\partial T}\right|_{\text{melting}} = \frac{S_{\text{liq}} - S_{\text{sol}}}{V_{\text{liq}} - V_{\text{sol}}} = \frac{\Delta S_m}{\Delta V_m} = \frac{\Delta Q_m}{\Delta V_m T}$ and $V_{\text{liq}} > V_{\text{sol}}$

for $T < 0.32$ K: $\left.\frac{\partial p_m}{\partial T}\right|_{\text{melting}} < 0 \quad \Rightarrow \quad S_{\text{liq}} < S_{\text{sol}}$ ➔ *disorder larger in solid than in liquid phase ??*

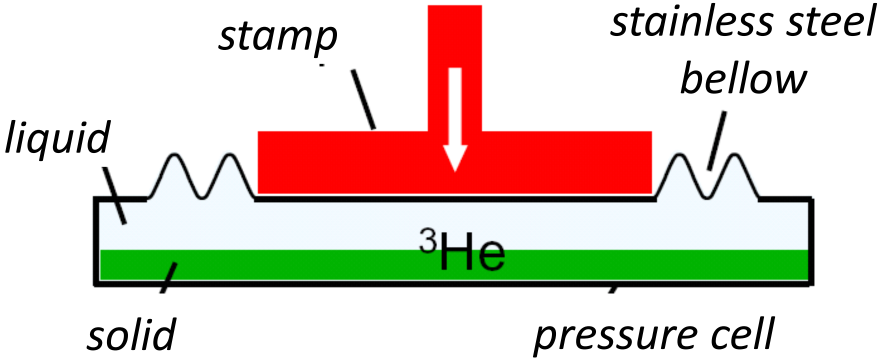
- **explanation**:
  - → **solid phase**: atoms are ordered,
  - → spins are disordered and determine entropy: $S_{\text{solid}} = N_A k_B \ln 2 = R \ln 2$
    at low T: antiparallel ordering of spins, S decreases towards zero
  - → **liquid phase**: atoms are spatially disordered, but ordering in k-space (Fermi liquid)
  - → entropy of Fermi liquid:

$$S_{\text{liquid}} = \int_0^T \frac{C_V}{T'} \mathrm{d}T' \propto \frac{T}{T_F}$$



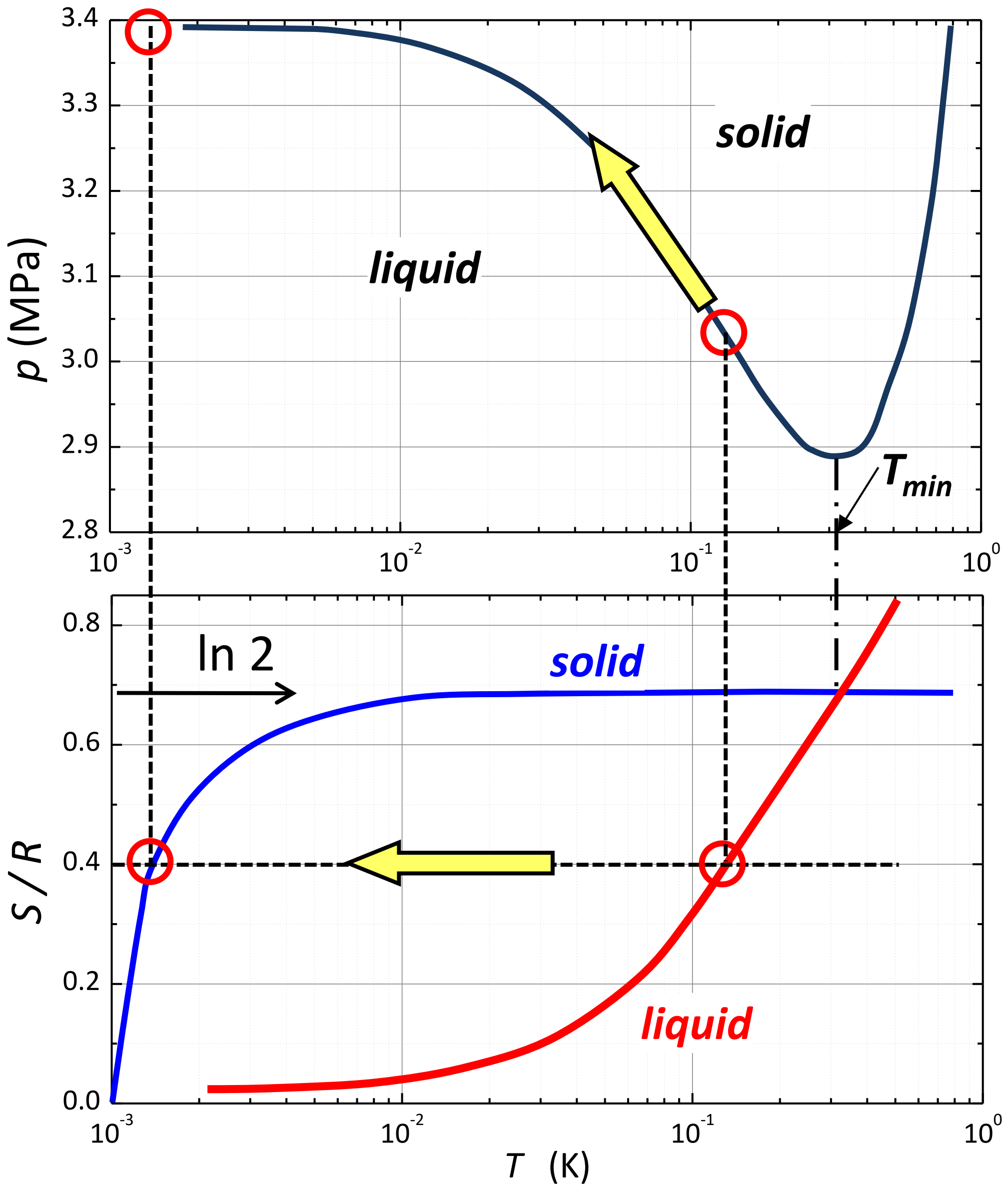
I.5.2 Solid ^{3}He and Pomerantchuk Effect

- **Pomerantchuk cooling**



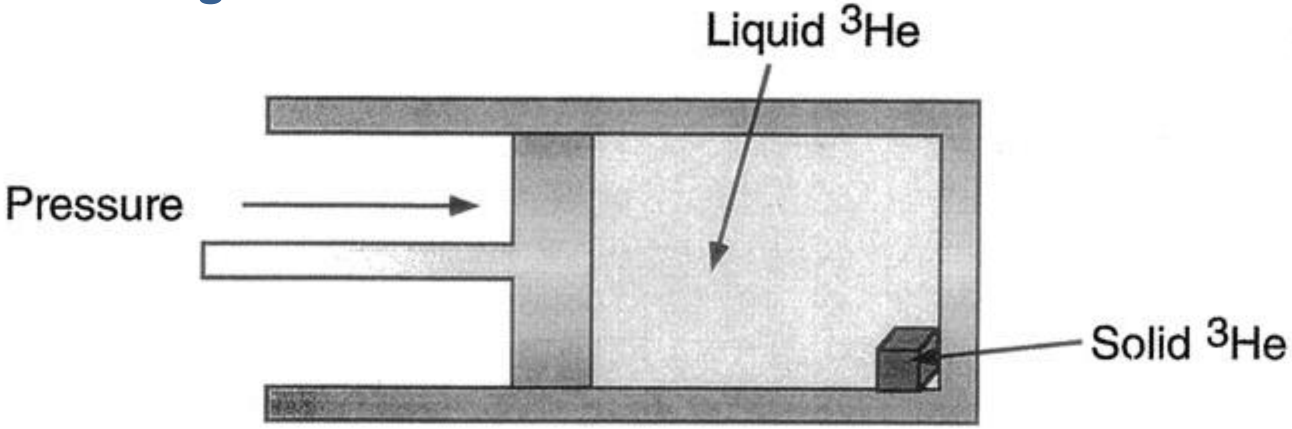
  - precooling to $T < T_\mathrm{min}$
  - adiabatic compression
    ➔ *solidification and cooling*
  - lowest T: ≈ 1.5 mK

    ➔ *limitation due to antiparallel spin ordering in solid ^{3}He*



I.5.2 Solid ^{3}He and Pomerantchuk Effect

- **Pomerantchuk cooling**



**Richardson,
Nobel Prize Lecture (1996)**

$$\frac{\Delta V}{V} \simeq 5\ \%$$

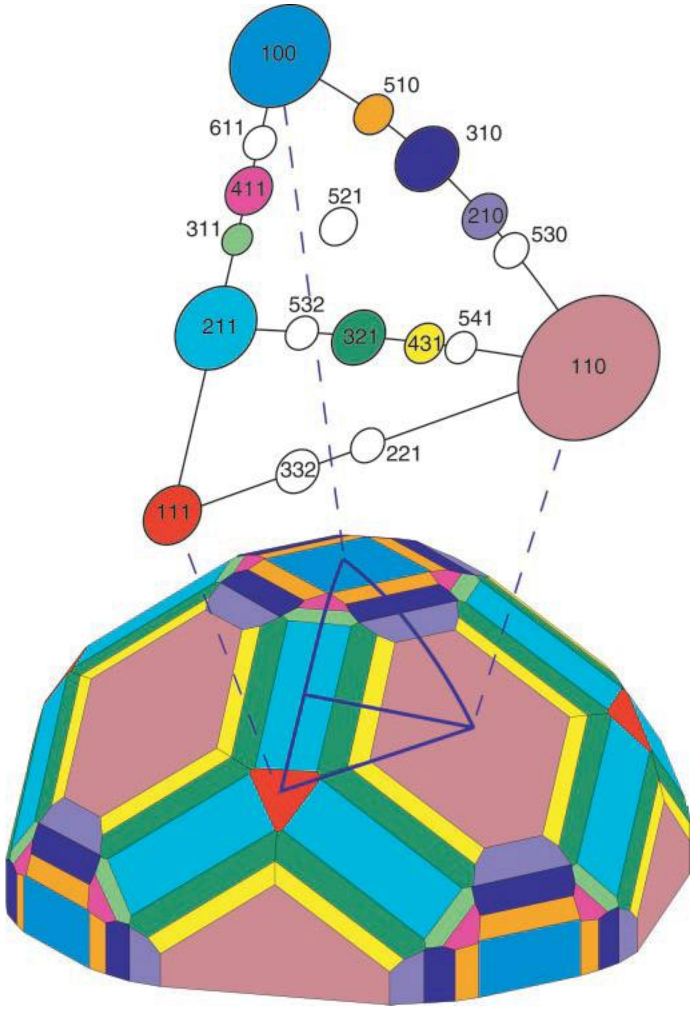
volume change for complete conversion of liquid to solid

Pomeranchuk's suggestion for cooling a melting mixture of ^{3}He:

The solid phase has a higher entropy than the liquid at low temperatures. As the liquid-solid mixture is compressed, heat is removed from the liquid phase as solid crystallites form. The fractional change of volume required to completely convert liquid into solid is approximately 5%. Unlike melting water, the solid phase forms at the hottest part of the container.

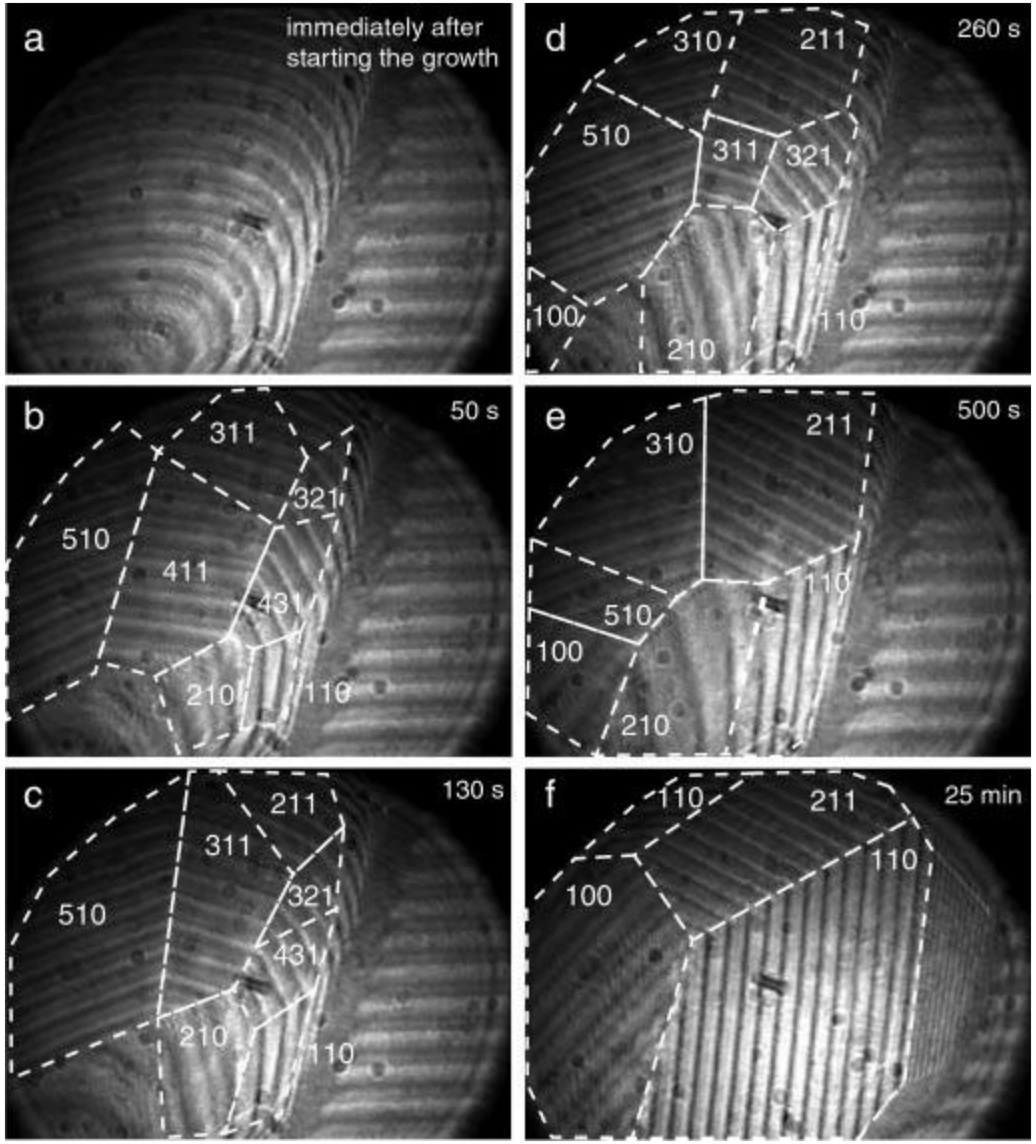
I.5.2 Solid ^{3}He and Pomerantchuk Effect

- **solid ^{3}He: single crystals**



computer-generated shape of a bcc crystal presented together with an elementary patch of a crystal habit where the facets are labeled with Miller indices.

H. Alles et al., PNAS **99**, 1796-1800 (2002)



Sequence of interferograms of a growing ^{3}He crystal at T = 0.55 mK. The dashed white lines outline the identified facets marked with Miller indices.

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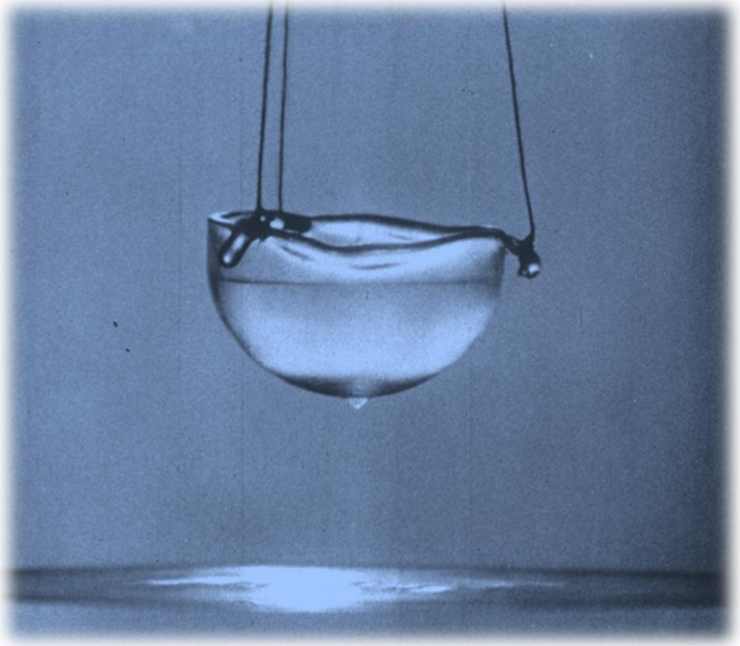
Superconductivity and Low Temperature Physics II



Lecture No. 7

R. Gross
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Contents Part I: Quantum Liquids

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I.3 Superfluid ^{4}He

I.4 Vortices

I.5 ^{3}He

I.6 ^{3}He / ^{4}He mixtures

I.5.3 Superfluid ^{3}He

- **history of superfluid ^{3}He**

  - ***initial believe***: no superfluid ^{3}He due to Pauli's exclusion principle for fermions
    → no Bose-Einstein condensation of ^{3}He, no superfluid phase

  - ***after development of BCS theory***: ***pairing of ^{3}He atoms to Cooper pairs ??***
    → problem: liquid ^{3}He is system of strongly interacting particles
    → formation of Cooper pairs out of „naked" ^{3}He atoms not possible (nature of exchange boson ??)
    → Landau: weakly interacting quasiparticles in Fermi liquid

  - ***1963***: critical pairing temperature of about 100 mK was expected

  - ***1971:*** discovery of superfluid ^{3}He at $T < 2.7$ mK by ***Osheroff***, ***Richardson*** and ***Lee*** (used Pomerantschuk cooling)
    (*Nobel Prize in Physics 1996 „for their discovery of superfluid ^{3}He"*)
    D.D. Osheroff, R.C. Richardson, D.M. Lee, Phys. Rev. Lett. **28**, 855 (1972),
    D.D. Osheroff, W.J. Gully, R.C. Richardson, D.M. Lee, Phys. Rev. Lett. **29**, 920 (1972)

  - ***initially:*** two superfluid phases **A** and **B**
    later: three phases (in magnetic field additional **A$_1$** phase)

  - experimental result: pairs have spin $S = 1$, ***spin triplet pairs***
    → pairs with $S = 1$ must have $L = 1$ (see below)
    → occupation probability for same position is negligible *(reduction of short length repulsion)*

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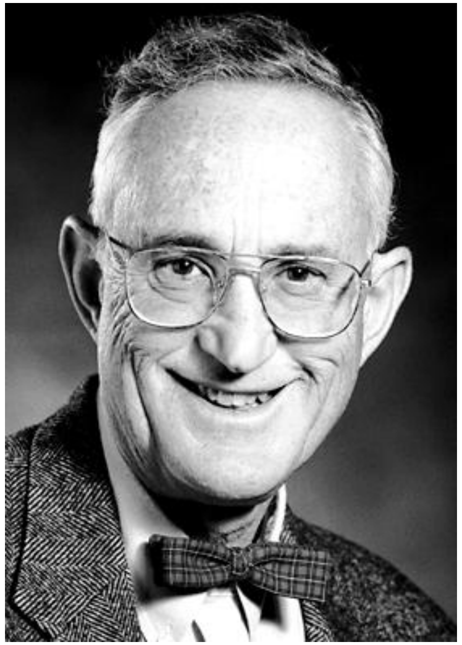
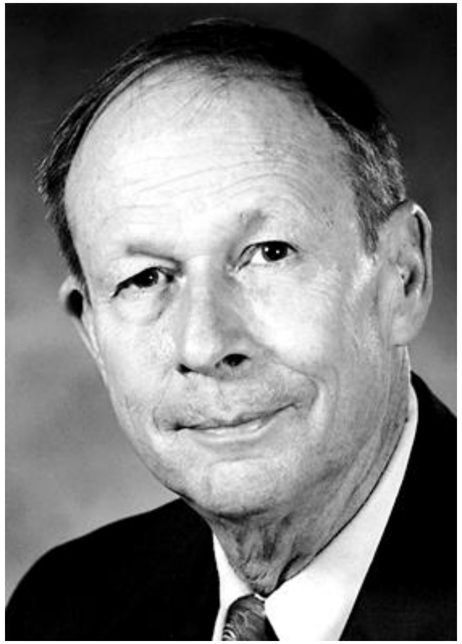
I.5.3 Superfluid ^{3}He

- discovery of superfluid ^{3}He in 1971

Douglas D. Osheroff,
Stanford University,
Stanford, California, USA



David M. Lee,
Cornell University, Ithaca,
New York, USA



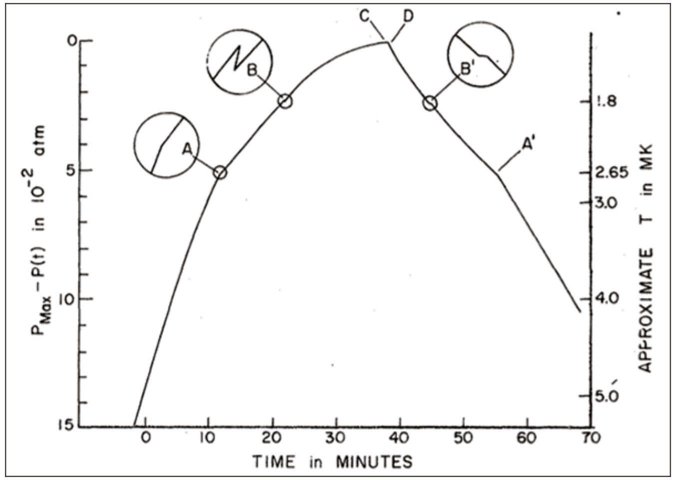
Robert C. Richardson,
Cornell University, Ithaca,
New York, USA

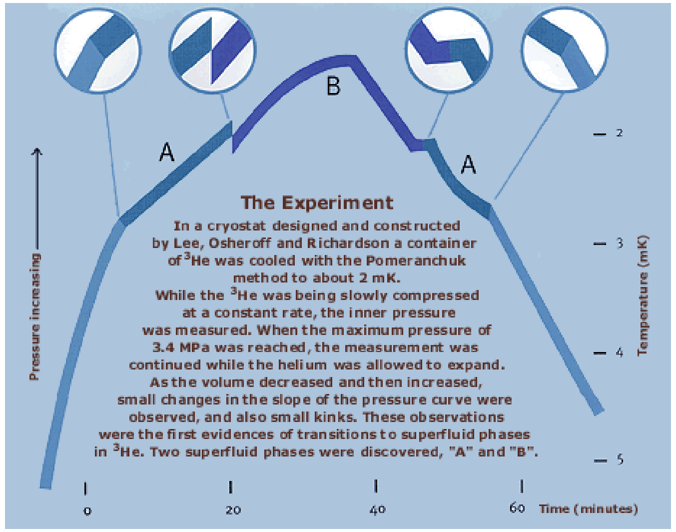
The Nobel Prize in Physics 1996

"for their discovery of superfluidity in helium-3"

I.5.3 Superfluid ^{3}He

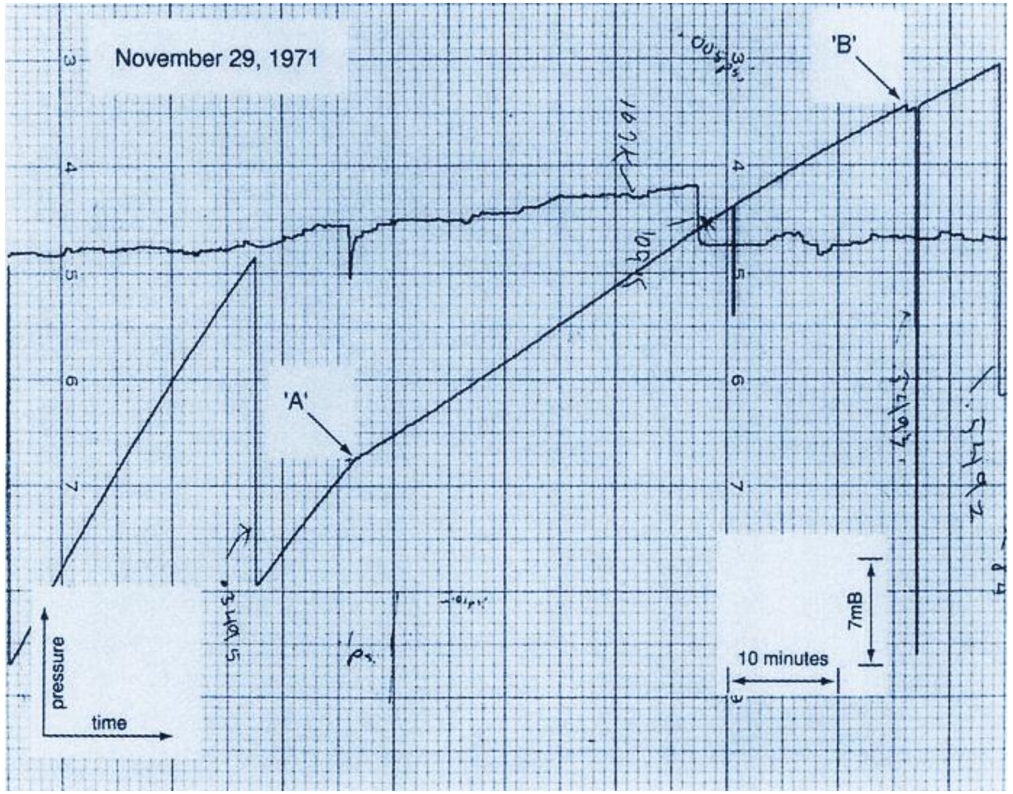
- **discovery of superfluid ^{3}He in 1971**





The Experiment

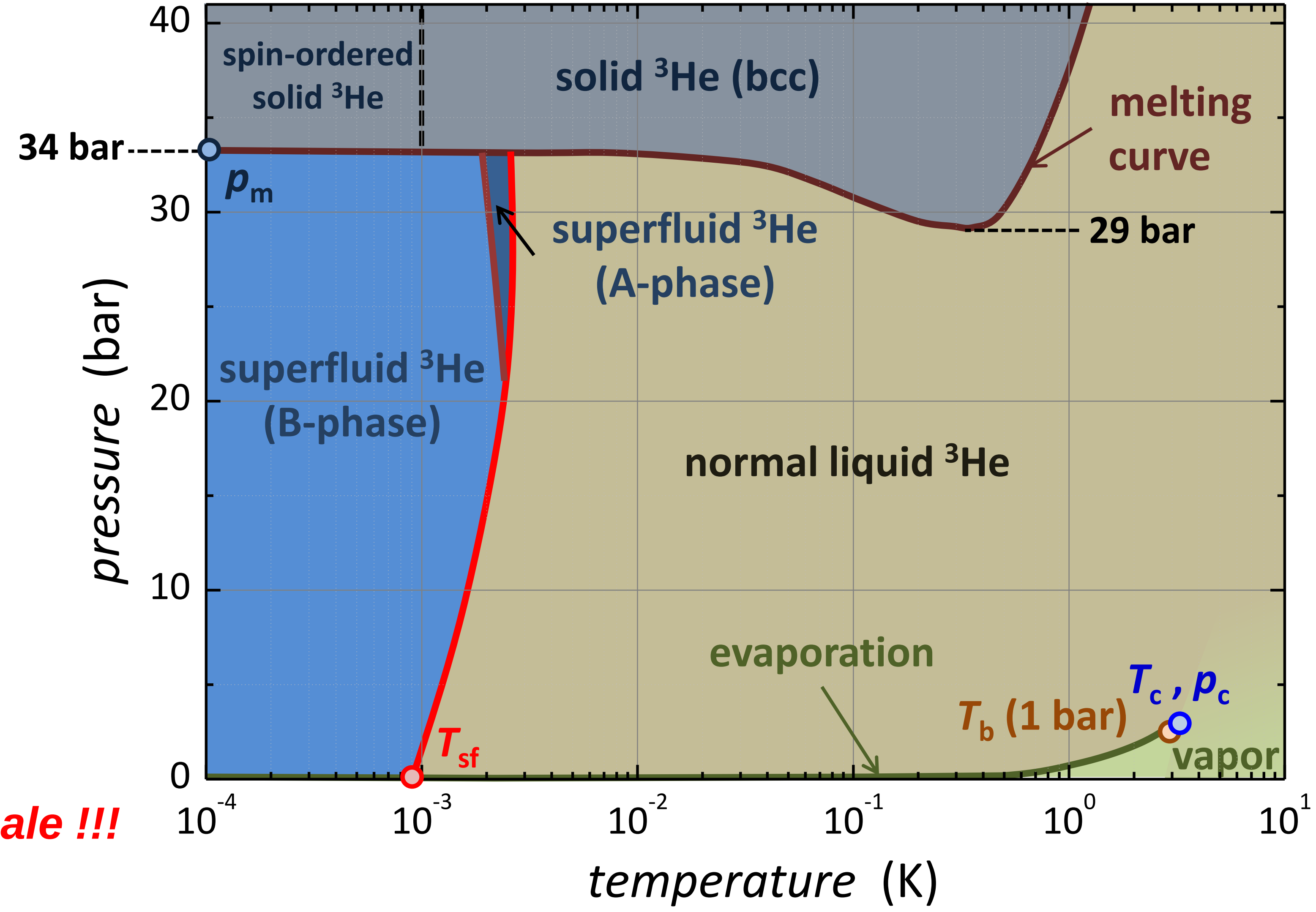
In a cryostat designed and constructed by Lee, Osheroff and Richardson a container of ^{3}He was cooled with the Pomeranchuk method to about 2 mK. While the ^{3}He was being slowly compressed at a constant rate, the inner pressure was measured. When the maximum pressure of 3.4 MPa was reached, the measurement was continued while the helium was allowed to expand. As the volume decreased and then increased, small changes in the slope of the pressure curve were observed, and also small kinks. These observations were the first evidences of transitions to superfluid phases in ^{3}He. Two superfluid phases were discovered, "A" and "B".



- cool down of ^{3}He via Pomeranchuk effect
- observation of melting curve
  → ***anomalies give hint to phase transitions***

source: Nobel Prize Lecture Lee/Osheroff (1996)

I.5.3 Superfluid ^{3}He: phase diagram for $B = 0$



log T scale !!!

big difference in value of T_{sf} compared to ^{4}He

– normal/superfluid transition is dominated by quantum effects

– quantum statistics matters!! (bosons vs. fermions)

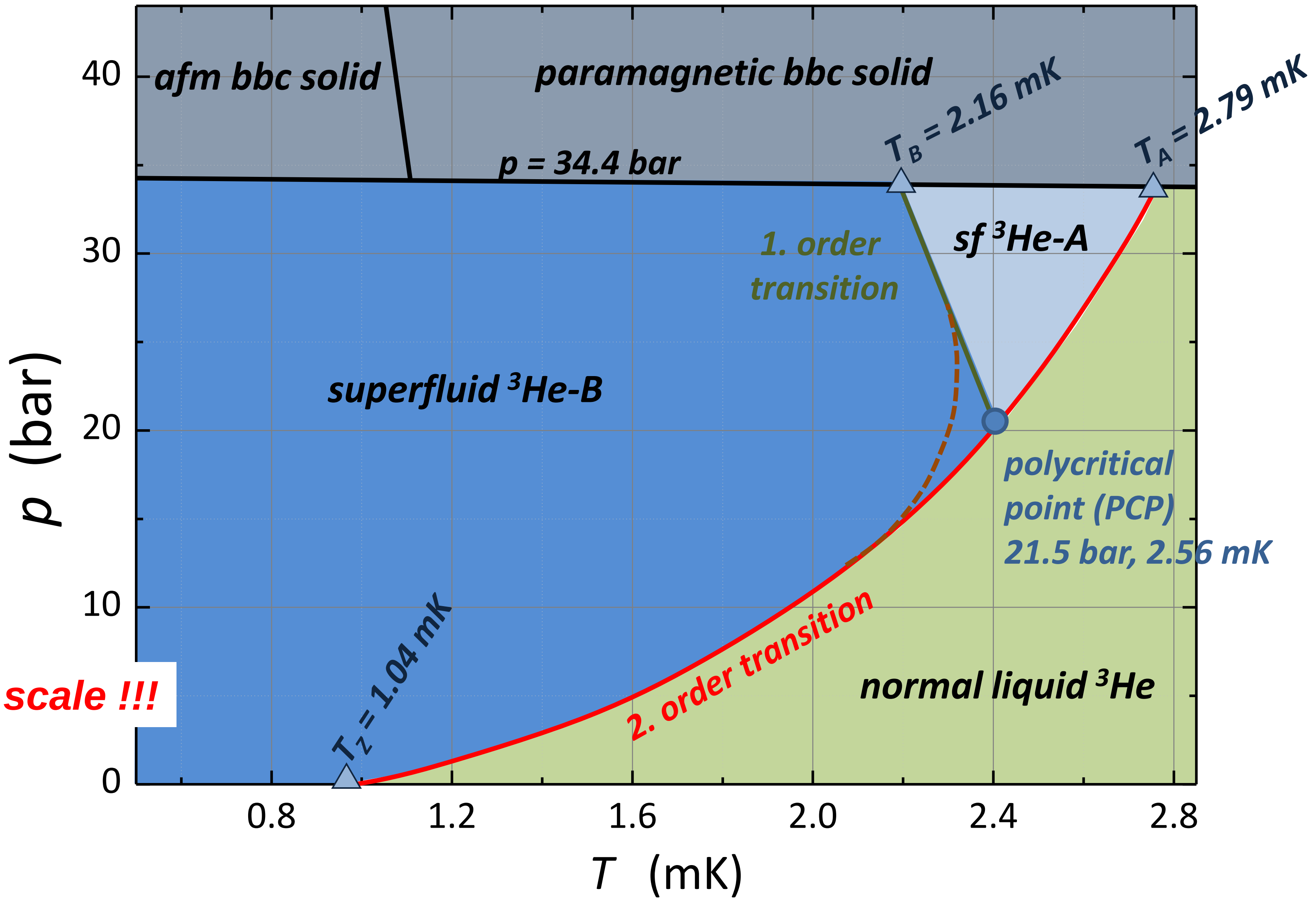
– **nature of superfluid phase in ^{4}He and ^{3}He is different**

- phase diagram of ^{3}He: note the logarithmic temperature scale. There are ***two superfluid phases of ^{3}He, A and B***.
- the line within the solid phase indicates a transition between spin-ordered and spin disordered structures (at low and high temperatures, respectively).

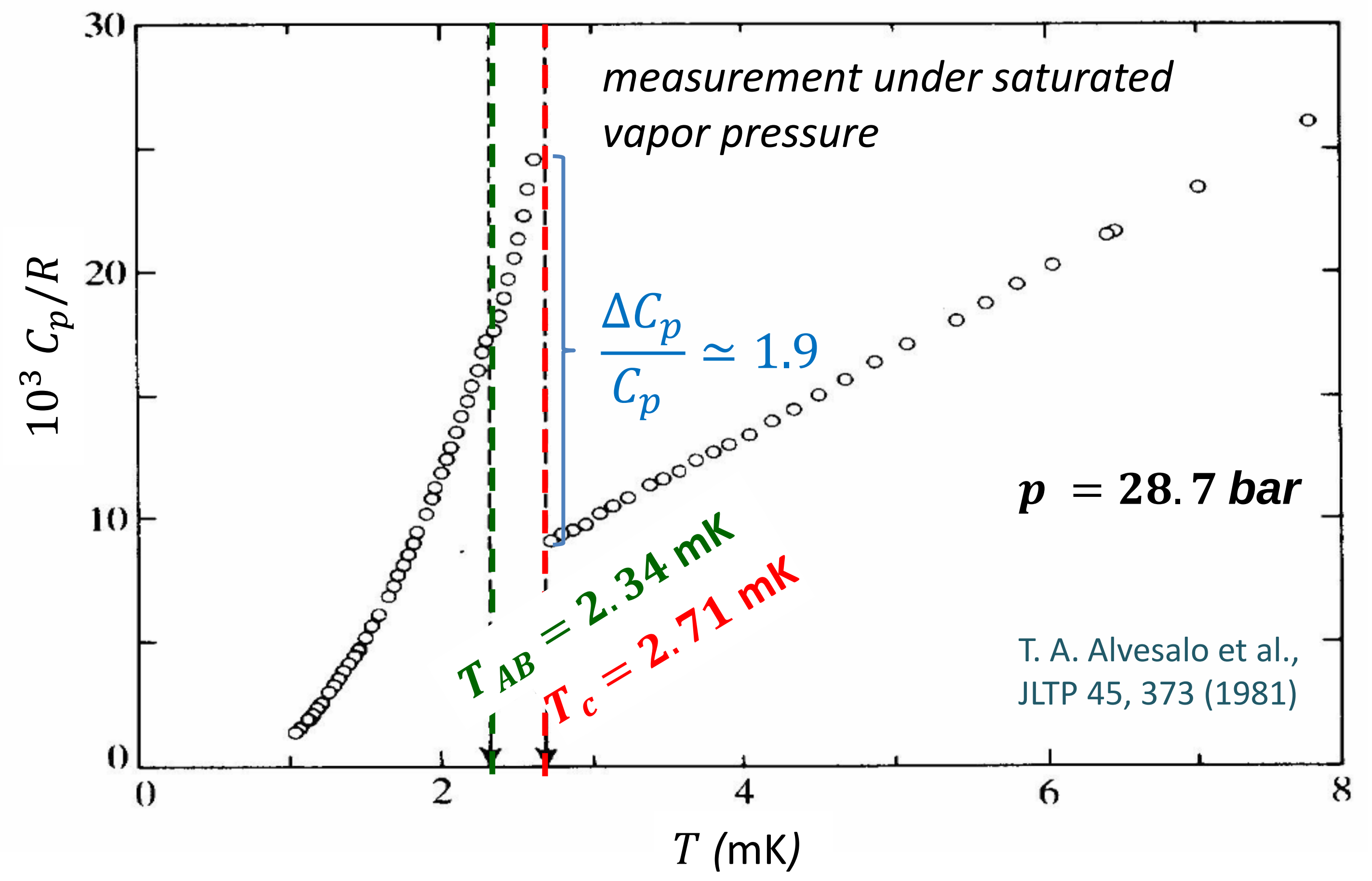
I.5.3 Superfluid ^{3}He: phase diagram for $B = 0$

pressure increases pairing interaction

linear T scale !!!



I.5.3 Superfluid ^{3}He: specific heat



T. A. Alvesalo et al., JLTP 45, 373 (1981)

$\Delta C_p / C_p \simeq 2$
@ *p = 30 bar*

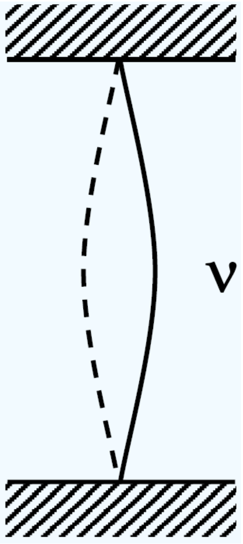
$\Delta C_p / C_p \simeq 1.4$
@ *p = 1 bar*

weak coupling BCS theory:
$\Delta C_p / C_p = 1.43$

- T_c: 2. order phase transition → jump of C_p as for superconductor
- T_{AB}: 1. order phase transition → small change of temperature dependence of C_p
  → hysteresis, latent heat
- PCP: first and second order phase transition merge

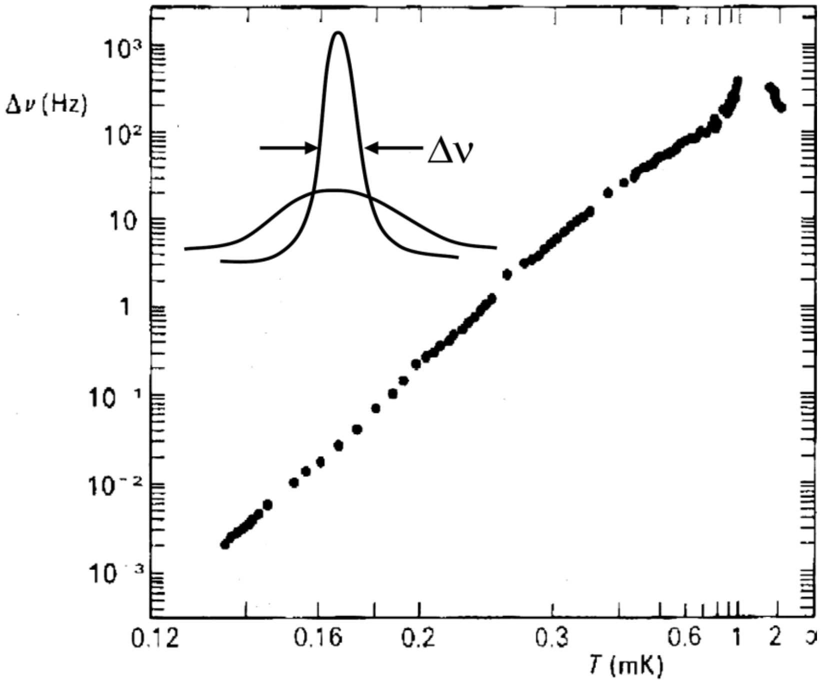
I.5.3 Superfluid ^{3}He: proof of superfluidity

- use of ***vibrating wire***, measurement of quality factor $\nu / \Delta\nu$

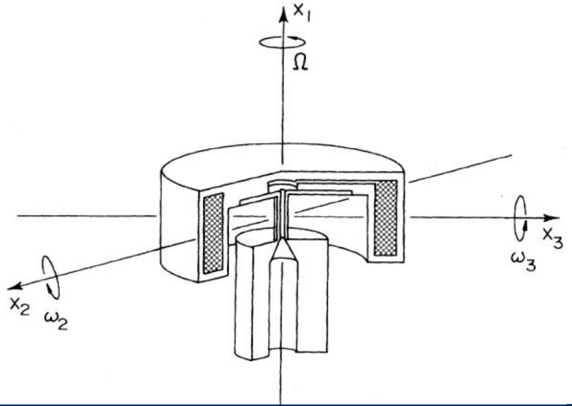


viscosity changes by 5 orders of magnitude

normal fluid freezes out



- ***other experiments:***
  - → *persistent flow experiments*:
    - viscosity of B phase changes at least by 12 orders of magnitude
    - persistent flow in A phase decays slowly (appearence of textures)
    - small critical velocity: 1 to 100 mm/s



Persistent-Current Experiments on Superfluid He 3 -B and He 3 -A
J. P. Pekola, J. T. Simola, K. K. Nummila, O. V. Lounasmaa, R. E. Packard,
Phys. Rev. Lett. **53**, 70 (1984)

I.5.3 Superfluid ^{3}He: pairing

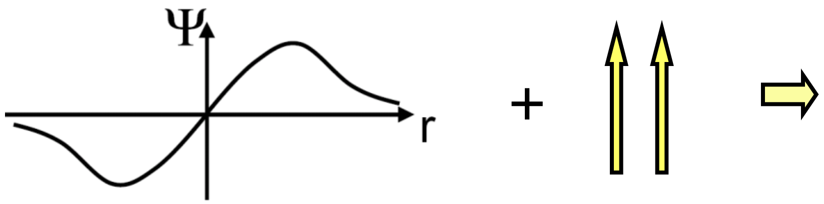
- ^{3}He is ***strongly interacting Fermi liquid*** (different to electrons in metals)
  → exchange interaction favors parallel spin alignment
  → Fermi statistics favors anti-parallel spin orientation

- ^{3}He forms Cooper pairs with spin $S = 1$ (***spin triplet pairs***)
  in contrast to electrons in metals: $S = 0$ (***spin singlet pairs pairs***)

- note the general requirement: ***antisymmetric pair wave function*** for fermions

symmetric spin wave function ➔ ***antisymmetric*** orbital wave function ($L = 1,3,\ldots$)
antisymmetric spin wave function ➔ ***symmetric*** orbital wave function ($L = 0,2,\ldots$)

examples: (i) Cooper pairs in metallic superconductors ($L = 0, S = 0$) ➔ ***s-wave, spin-singlet pairs***
(ii) Cooper pairs in high T_c superconductors ($L = 2, S = 0$) ➔ ***d-wave, spin-singlet pairs***
(iii) Cooper pairs in superfluid ^{3}He ($L = 1, S = 1$) ➔ ***p-wave, spin-triplet pairs***



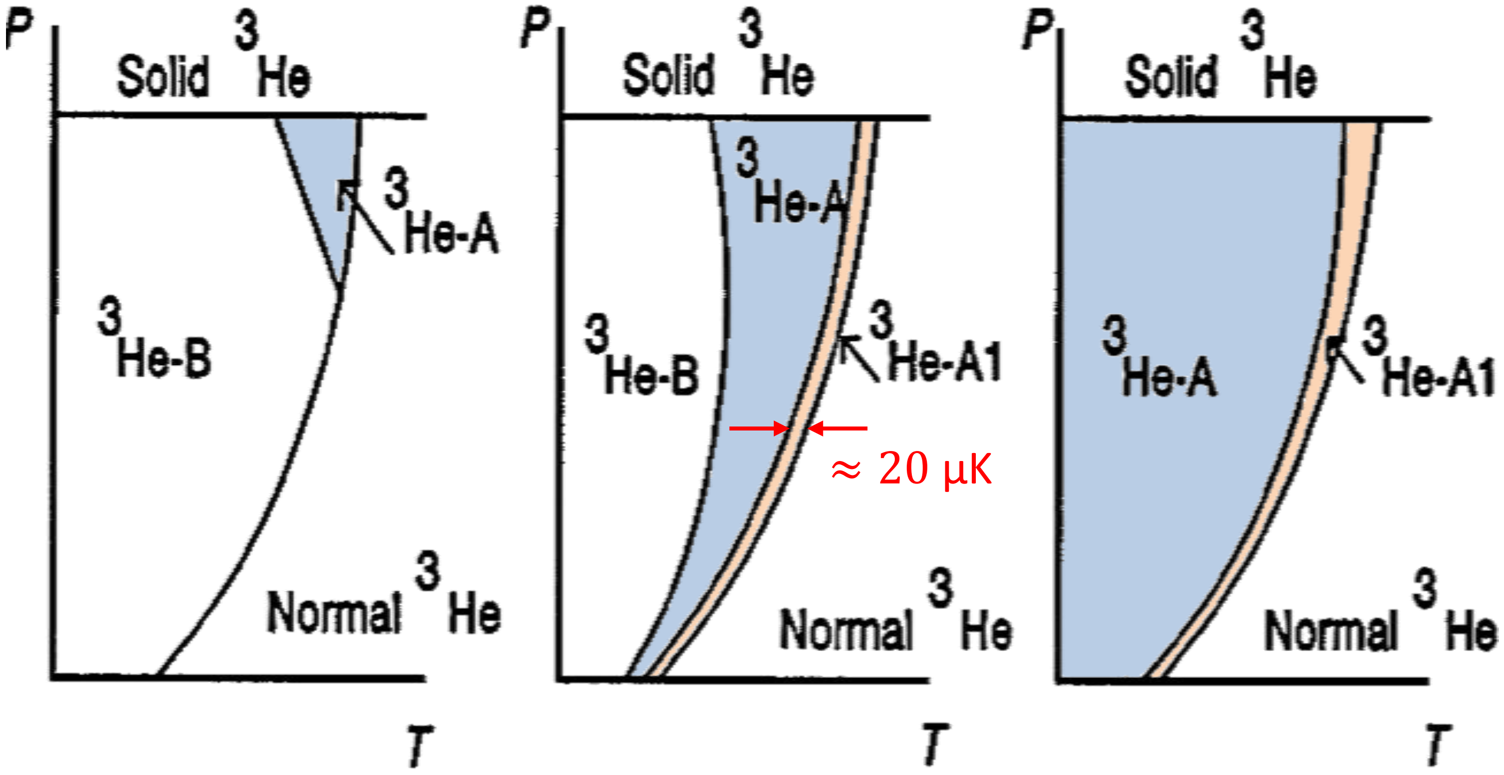
complex behavior due to
3 x 3 combinations for L_z / S_z

p-type orbital wave function minimizes hard core repulsion

I.5.3 Superfluid ^{3}He: in $B \neq 0$

spin triplet pairs have magnetic moment → ***phase diagram depends on magnetic field***

$B = 0$ $0 < B < 0.6\,\mathrm{T}$ $B > 0.6\,\mathrm{T}$

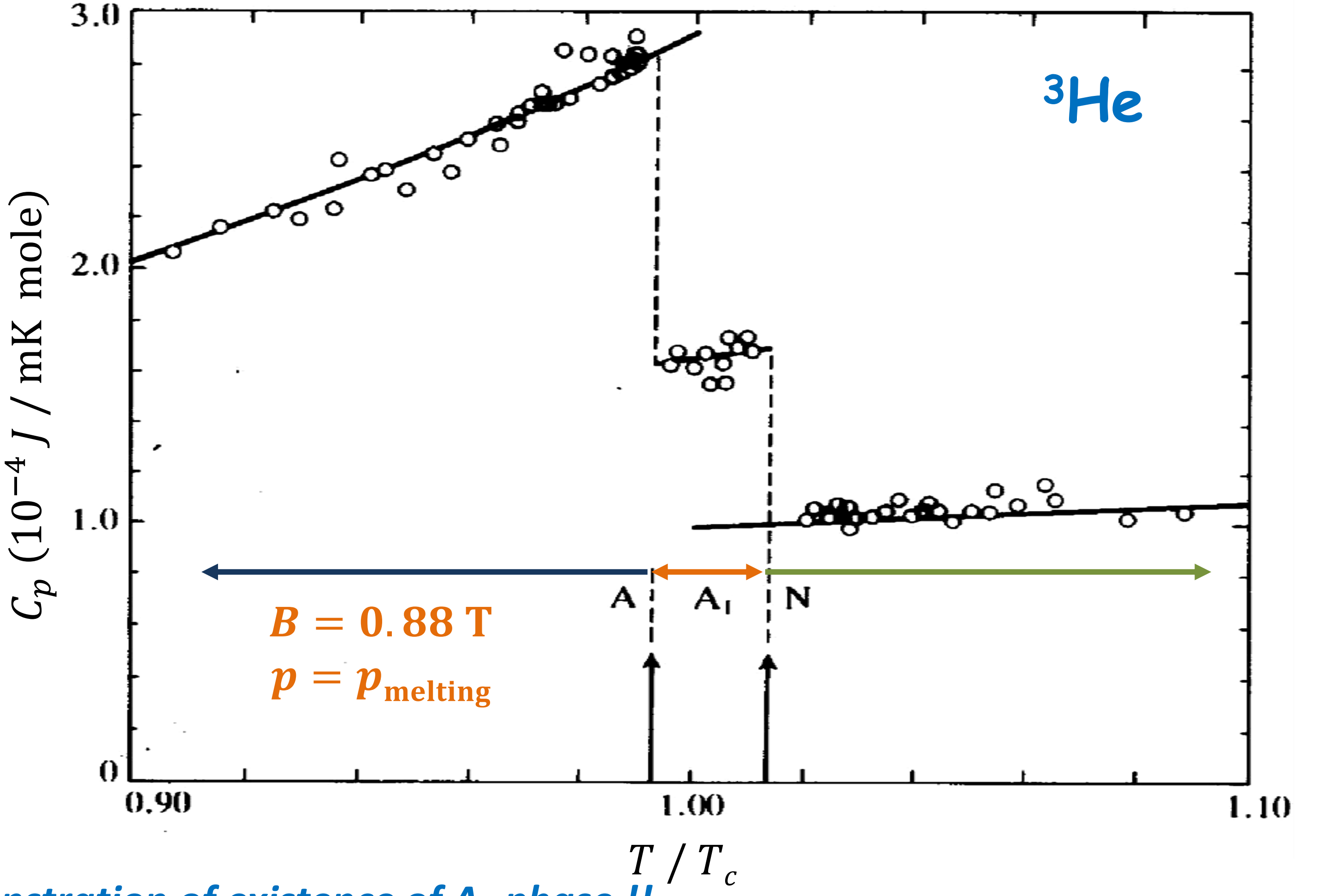


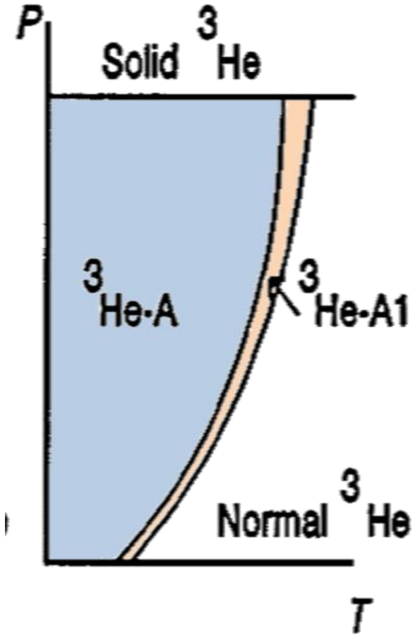
-new phase A_1
-phase A gains,
-phase B looses

-phase B disappears

I.5.3 Superfluid ^{3}He: in $B \neq 0$

- **specific heat**



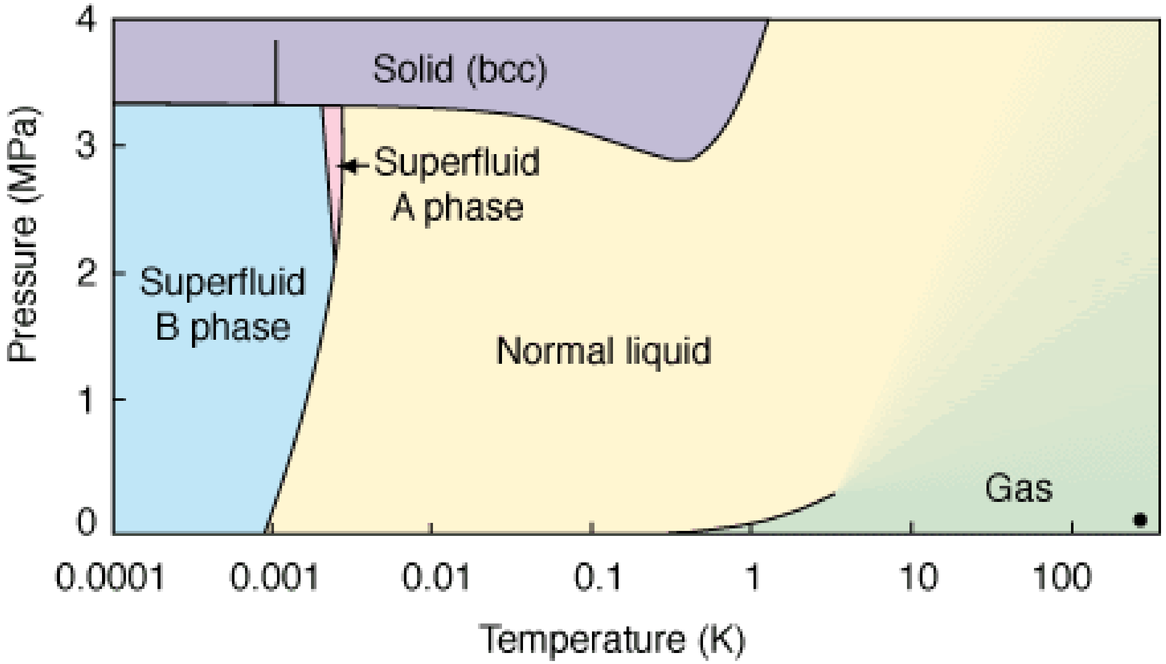


clear demonstration of existence of A_1 phase !!

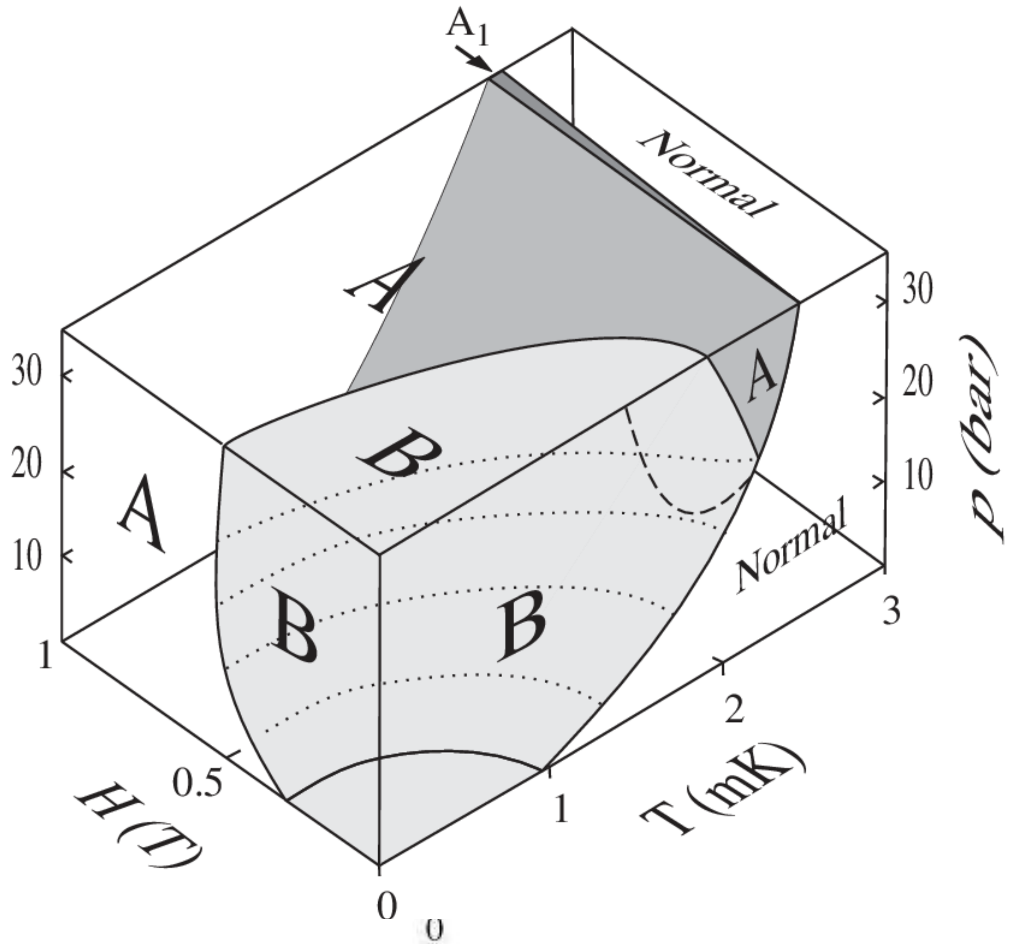
W.P. Halperin et al.,
Phys. Rev. B**13**, 2124 (1974)

I.5.3 Superfluid ^{3}He: in $B \neq 0$

- **phase diagram of ^{3}He**



source: http://ltl.tkk.fi/research/theory/helium.html



- important: Fermi temperature $T_F \sim 4$ K is much larger than transition temperature $T_c \sim 1$ mK into superfluid state
- normal liquid just above T_c represents a ***degenerate Fermi gas***
- interactions cannot be neglected: ***Fermi gas ➔ Fermi liquid***

I.5.3 Superfluid ^{3}He: Fermi liquid description

- **description of liquid ^{3}He by a Fermi liquid (interacting Fermi gas)**

  Landau: what happens, if we increase the interaction strength in a non-interacting systems continuously?

➢ continuous increase of interactions

$$\mathcal{H}^{(\alpha)} = \sum_{i=1}^{N} -\frac{\hbar^2}{2m}\nabla_i^2 + \frac{\alpha}{2}\sum_{i\neq j} V(\mathbf{r}_i - \mathbf{r}_j)$$

$\alpha = 0$ (no interactions)
$\alpha = 1$ (fully interacting system)

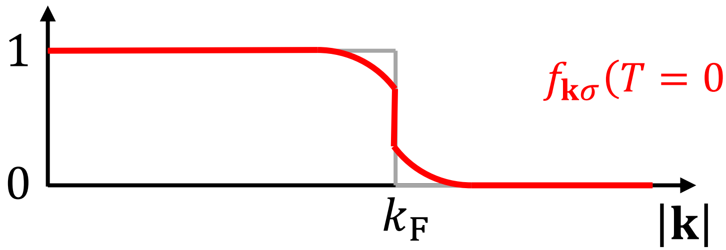
– we obtain a particular ground state wave function $\Psi_0^{(\alpha)}$ for each value of α

$$\mathcal{H}^{(\alpha)}\Psi_0^{(\alpha)}(\mathbf{r}_i,\sigma_1,\dots,\mathbf{r}_N,\sigma_N) = E_0^{(\alpha)}\Psi_0^{(\alpha)}(\mathbf{r}_i,\sigma_1,\dots,\mathbf{r}_N,\sigma_N)$$

Landau's argument of *adiabatic continuity* (wave function changes continuously, when switching on interaction):

$$\Psi_0^{(\alpha=0)} \Rightarrow \Psi_0^{(\alpha)} \Rightarrow \Psi_0^{(\alpha=1)}$$

impact on momentum distribution $f_{\mathbf{k}\sigma}(T=0)$ (discontinuity at k_F stays but step height changes continuously to smaller value Z<1)



$$f_{\mathbf{k}\sigma}(T=0) = \left\langle \Psi_0^{(\alpha)} \middle| c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} \middle| \Psi_0^{(\alpha)} \right\rangle$$

k_F stays unchanged: Luttinger theorem

I.5.3 Superfluid ^{3}He: Fermi liquid description

- **description of liquid ^{3}He by a Fermi liquid (interacting Fermi gas)**

extension of Landau's argument of *adiabatic continuity* to *excited states*

- non-interacting system: addition/removal of one particle with momentum $\mathbf{k}$ and spin σ to ground state $|\Psi_0^{(\alpha=0)}\rangle$

$$|\Psi_{\mathbf{k}\sigma}^{(\alpha=0)}\rangle = c_{\mathbf{k}\sigma}^\dagger |\Psi_0^{(\alpha=0)}\rangle \qquad \tilde{\varepsilon}_{\mathbf{k}} = \frac{\hbar^2(k^2 - k_F^2)}{2m} \qquad f_{\mathbf{k}\sigma}(T=0) = \begin{cases} 0 & \text{for } k < k_F \\ 1 & \text{for } k > k_F \end{cases}$$

- interacting system: addition/removal of one particle changes energies of all other particles due to interaction

quasiparticle energy has similar k-dependence: $\varepsilon_{\mathbf{k}} = \frac{\hbar^2(k^2 - k_F^2)}{2m^*}$ $\qquad m^*$ = effective mass

for k close to k_F: $\varepsilon_{\mathbf{k}} \simeq \left.\frac{\partial \varepsilon_{\mathbf{k}}}{\partial k}\right|_{k=k_F} (|\mathbf{k}| - k_F) \simeq \frac{\hbar k_F}{m^*}\hbar(|\mathbf{k}| - k_F) = \hbar(|\mathbf{k}| - k_F)\, v_F$

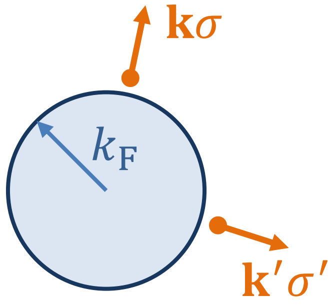
I.5.3 Superfluid ^{3}He: Fermi liquid description

- **description of liquid ^{3}He by a Fermi liquid (interacting Fermi gas)**

➢ total energy: $E = \underbrace{E_0}_{\text{ground state}} + \underbrace{\sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}} \delta n_{\mathbf{k}\sigma}}_{\text{quasiparticle excitations}} + \underbrace{\frac{1}{2} \sum_{\mathbf{k}\sigma,\mathbf{k}'\sigma'} \eta(\mathbf{k}\sigma,\mathbf{k}'\sigma')\, \delta n_{\mathbf{k}\sigma} \delta n_{\mathbf{k}'\sigma'}}_{\text{interaction energy}}$

➢ two particle energy: $\varepsilon_{\mathbf{k}} + \varepsilon_{\mathbf{k}'} + \underbrace{\eta(\mathbf{k}\sigma,\mathbf{k}'\sigma')} \neq$ sum of single particle energies $\varepsilon_{\mathbf{k}}, \varepsilon_{\mathbf{k}'}$

describes effective interaction between two quasiparticles:
is usually split in parts $\eta_1(\mathbf{k},\mathbf{k}')$ and $\eta_2(\mathbf{k},\mathbf{k}')$ for singlet and triplet contribution



➢ at low T: $|\mathbf{k}|, |\mathbf{k}'| \simeq k_\mathrm{F}$ and due to spherical symmetry $\eta(\mathbf{k},\mathbf{k}') \propto \mathbf{k}\cdot\mathbf{k}' = k_\mathrm{F}^2 \cos\theta$

➔ we can express $\eta_1(\mathbf{k},\mathbf{k}')$ and $\eta_2(\mathbf{k},\mathbf{k}')$ by Legendre polynoms $P_\ell(\cos\theta)$

$$g(\varepsilon_\mathrm{F})\eta_1(\mathbf{k},\mathbf{k}') = \sum_\ell F_\ell \; P_\ell(\cos\theta) \qquad g(\varepsilon_\mathrm{F})\eta_2(\mathbf{k},\mathbf{k}') = \sum_\ell Z_\ell \; P_\ell(\cos\theta)$$

$g(\varepsilon_\mathrm{F})$ = density of states (states per energy interval) ➔ F_ℓ, Z_ℓ are dimensionless

Theory by Leggett:

➢ F_0, F_1 are large and positive ➔ $m^* = m(1 + F_1/3) \simeq 2.8$ with $F_1 \simeq 5.4$ (increases with pressure)

➢ $Z_0 \simeq -3$ and Z_1 is small ➔ magnetic susceptibility $\chi = \frac{m^*}{m} \frac{\chi_0}{1+(Z_0/4)}$

➔ $\chi \simeq 9.2\, \chi_0$: with $m^\star \simeq 2.8$ and $Z_0 \simeq -2.78$ → He is close to ferromagnetic instability

I.5.3 Superfluid ^{3}He: pairing interaction

– interaction of ^{3}He atoms due to van der Waals potential $V(\mathbf{r}_i - \mathbf{r}_j) = V(\mathbf{r})$

$$V(\mathbf{k},\mathbf{k}') = \int \mathrm{e}^{-\imath(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}\, V(\mathbf{r})\, \mathrm{d}^3 r$$

pairing potential in k-space

- at low T: $|\mathbf{k}|, |\mathbf{k}'| \simeq k_\mathrm{F}$ and due to spherical symmetry $V(\mathbf{k},\mathbf{k}') \propto \mathbf{k}\cdot\mathbf{k}' = k_\mathrm{F}^2 \cos\theta$

  ➔ we can express $V(\mathbf{k},\mathbf{k}')$ by Legendre polynoms $P_\ell(\cos\theta)$

$$V(\mathbf{k},\mathbf{k}') = \sum_\ell \frac{2\ell+1}{2}\; V_\ell P_\ell(\cos\theta)$$

with $V_\ell = \frac{1}{4\pi}\int V(\hat{\mathbf{n}}(\theta,\varphi), \hat{e}_z)\, P_\ell(\cos\theta)\sin\theta\, \mathrm{d}\theta \mathrm{d}\varphi$

$\frac{2\ell+1}{2} = \int P_\ell^2(\cos\theta)\sin\theta\,\mathrm{d}\theta$ (normalization)

- calculation of V_ℓ for van der Waals potential (e.g. by Leggett 1975):

  V_0 = strongly repulsive, V_1, V_2 = weakly attractive ➔ cannot explain pairing in superfluid ^{3}He

- attractive interaction results from exchange of **virtual spin fluctuations**

one ^{3}He atom polarizes neighboring nuclear spins which then result in an effective attractive interaction for 2nd ^{3}He atom

$$V(\mathbf{k},\mathbf{k}') \propto \frac{1}{g(\varepsilon_\mathrm{F})}\,\frac{\chi_0}{1+(Z_0/4)}\;\hat{\mathbf{s}}_\mathbf{k}\cdot\hat{\mathbf{s}}_{\mathbf{k}'}$$

(Leggett 1975)

$\hat{\mathbf{s}}_\mathbf{k}\cdot\hat{\mathbf{s}}_{\mathbf{k}'} = +1/4$ for triplet
$\hat{\mathbf{s}}_\mathbf{k}\cdot\hat{\mathbf{s}}_{\mathbf{k}'} = -3/4$ for singlet

I.5.3 Superfluid ^{3}He: superfluid phases

formalism of second quantization is used (>1927, Dirac, Fock, Jordan et al.)

- starting point: BCS Hamiltonian

BCS Hamiltonian for N interacting electrons

$$\mathcal{H}_{\mathrm{BCS}} = \underset{\text{spin}}{\sum_{\sigma}} \sum_{i=1}^{N} \left(\underbrace{-\frac{\hbar^2}{2m}\nabla_i^2}_{\text{kinetic energy}} + \underbrace{V_{\mathrm{ext}}(\mathbf{r})}_{\text{potential energy}} \right) + \frac{1}{2} \sum_{\sigma} \sum_{i,j=1}^{N} \underbrace{V_{\mathrm{int}}(\mathbf{r}_i - \mathbf{r}_j)}_{\text{interaction energy}}$$

insertion of field operators and intergation over volume ➔ FT of $\mathcal{H}_{\mathrm{BCS}}$ into k-space
(see R. Gross, A. Marx, „Festkörperphysik", 4. Auflage, appendix H.2)

$$\mathcal{H}_{\mathrm{BCS}} = \underbrace{\sum_{\sigma}\sum_{\mathbf{k}} (\varepsilon_{\mathbf{k}} - \mu)\, \hat{c}^{\dagger}_{\mathbf{k}\sigma} \hat{c}_{\mathbf{k}\sigma}}_{\text{energy of non-interacting free electron gas}} + \underbrace{\frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} \sum_{\mathbf{k},\mathbf{k}'}^{N} V_{\alpha\beta\gamma\delta}(\mathbf{k},\mathbf{k}')\, \hat{c}^{\dagger}_{\mathbf{k}'\alpha} \hat{c}^{\dagger}_{-\mathbf{k}'\beta} \hat{c}_{-\mathbf{k}\gamma} \hat{c}_{\mathbf{k}\delta}}_{\text{interaction energy}}$$

$$\varepsilon_{\mathrm{k}} - \mu = \frac{\hbar^2 (k - k_{\mathrm{F}})^2}{2m}$$

$$V(\mathbf{k},\mathbf{k}') = \frac{1}{\Omega} \int_{\Omega} V(\mathbf{r})\, \mathrm{e}^{\imath(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} \mathrm{d}V$$

factor $\frac{1}{2}$ avoids double counting

operator describes scattering from state $(-\mathbf{k}\gamma, \mathbf{k}\delta)$ into $(\mathbf{k}'\alpha, -\mathbf{k}'\beta)$ by exchange of exchange boson with wave vector $(\mathbf{k} - \mathbf{k}')$

I.5.3 Superfluid ³He: superfluid phases

- **symmetry of the pair wavefunction**

  - **important**: pair consistst of two fermions ➔ ***total wavefunction must be antisymmetric: minus sign for particle exchange***

$$\Psi(\mathbf{r}_1,\boldsymbol{\sigma}_1,\mathbf{r}_2,\boldsymbol{\sigma}_2) = \underbrace{\frac{1}{\sqrt{V}}\, \mathrm{e}^{i\,\mathbf{K}_s\cdot\mathbf{R}_s}}_{\substack{\text{center of mass motion}\\ \text{we assume } \mathbf{K}_s = 0}} \underbrace{f(\mathbf{q},\mathbf{r})}_{\substack{\text{orbital}\\ \text{part}}} \underbrace{\chi(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)}_{\substack{\text{spin}\\ \text{part}}} = -\Psi(\mathbf{r}_2,\boldsymbol{\sigma}_2,\mathbf{r}_1,\boldsymbol{\sigma}_1)$$

$$\mathbf{R}_s = (\mathbf{r}_1 + \mathbf{r}_2)/2$$

$$\mathbf{r} = (\mathbf{r}_1 - \mathbf{r}_2)$$

$$\mathbf{K}_s = (\mathbf{k} + \mathbf{k}')/2$$

$$\mathbf{q} = (\mathbf{k} - \mathbf{k}')$$

  - possible ***spin wavefunctions*** $\chi(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)$ for electron pairs

$$S = \begin{cases} 0 \quad m_s = 0 \qquad \chi^a = \dfrac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow) \\ 1 \quad m_s = \begin{cases} -1 & \chi^s = \downarrow\downarrow \\ 0 & \chi^s = \dfrac{1}{\sqrt{2}}(\uparrow\downarrow+\downarrow\uparrow) \\ +1 & \chi^s = \uparrow\uparrow \end{cases} \end{cases}$$

→**singlet pairing**, antisymmetric spin wavefunction
symmetric orbital function:
L = 0, 2, … (s, d, ….)

→**triplet pairing**, symmetric spin wavefunction
antisymmetric orbital function:
L = 1, 3, … (p, f, ….)

I.5.3 Superfluid ^{3}He: superfluid phases

- **BCS mean-field approximation**

pairing amplitude:

$$g_{\alpha\beta}(\mathbf{k}) \equiv \langle \hat c_{-\mathbf{k}\alpha}\hat c_{\mathbf{k}\beta}\rangle \neq 0$$

$$g^*_{\alpha\beta}(\mathbf{k}) \equiv \langle \hat c^\dagger_{\mathbf{k}\alpha}\hat c^\dagger_{-\mathbf{k}\beta}\rangle \neq 0$$

$$\hat c_{-\mathbf{k}\alpha}\hat c_{\mathbf{k}\beta} = \underbrace{\langle \hat c_{-\mathbf{k}\alpha}\hat c_{\mathbf{k}\beta}\rangle}_{g_{\alpha\beta}(\mathbf{k})} + \underbrace{\hat c_{-\mathbf{k}\alpha}\hat c_{\mathbf{k}\beta} - \langle \hat c_{-\mathbf{k}\alpha}\hat c_{\mathbf{k}\beta}\rangle}_{\delta g_{\alpha\beta}(\mathbf{k})} \qquad \hat c^\dagger_{\mathbf{k}\alpha}\hat c^\dagger_{-\mathbf{k}\beta} = \underbrace{\langle \hat c^\dagger_{\mathbf{k}\alpha}\hat c^\dagger_{-\mathbf{k}\beta}\rangle}_{g^*_{\alpha\beta}(\mathbf{k})} + \underbrace{\hat c^\dagger_{\mathbf{k}\alpha}\hat c^\dagger_{-\mathbf{k}\beta} - \langle \hat c^\dagger_{\mathbf{k}\alpha}\hat c^\dagger_{-\mathbf{k}\beta}\rangle}_{\delta g^*_{\alpha\beta}(\mathbf{k})}$$

- make use of **pairing potential** $\Delta_{\alpha\beta}(\mathbf{k}) \equiv -\sum_{\mathbf{k}',\gamma,\delta} V_{\alpha\beta\gamma\delta}(\mathbf{k},\mathbf{k}')\, g_{\alpha\beta}(\mathbf{k}')$ $\qquad \Delta^*_{\alpha\beta}(\mathbf{k}) \equiv -\sum_{\mathbf{k}',\gamma,\delta} V_{\alpha\beta\gamma\delta}(\mathbf{k},\mathbf{k}')\, g^*_{\alpha\beta}(\mathbf{k}')$

four component function due to two values α, β of spin

spin structure

orbital structure

⇒

$$\boxed{\mathcal{H}_{\mathrm{BCS}} - \mathcal{N}\mu = \sum_{\mathbf{k},\sigma} (\varepsilon_{\mathbf{k}} - \mu)\, c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k},\alpha,\beta} \left[\Delta_{\alpha\beta}(\mathbf{k})\, \hat c^\dagger_{\mathbf{k}\alpha}\hat c^\dagger_{-\mathbf{k}\beta} + \Delta^*_{\alpha\beta}(\mathbf{k})\, \hat c_{\mathbf{k}\alpha}\hat c_{-\mathbf{k}\beta}\right]}$$

- with $\alpha, \beta = \uparrow, \downarrow$: two fermions with $\mathbf{s} = $ ½ and $\alpha, \beta = m_s = \uparrow, \downarrow$

$$g(\mathbf{k}) \equiv \begin{pmatrix} \langle \hat c_{-\mathbf{k}\uparrow}\hat c_{\mathbf{k}\uparrow}\rangle & \langle \hat c_{-\mathbf{k}\uparrow}\hat c_{\mathbf{k}\downarrow}\rangle \\ \langle \hat c_{-\mathbf{k}\downarrow}\hat c_{\mathbf{k}\uparrow}\rangle & \langle \hat c_{-\mathbf{k}\downarrow}\hat c_{\mathbf{k}\downarrow}\rangle \end{pmatrix} \qquad g^*(\mathbf{k}) \equiv \begin{pmatrix} \langle \hat c^\dagger_{-\mathbf{k}\uparrow}\hat c^\dagger_{\mathbf{k}\uparrow}\rangle & \langle \hat c^\dagger_{-\mathbf{k}\uparrow}\hat c^\dagger_{\mathbf{k}\downarrow}\rangle \\ \langle \hat c^\dagger_{-\mathbf{k}\downarrow}\hat c^\dagger_{\mathbf{k}\uparrow}\rangle & \langle \hat c^\dagger_{-\mathbf{k}\downarrow}\hat c^\dagger_{\mathbf{k}\downarrow}\rangle \end{pmatrix} \qquad \Delta(\mathbf{k}) = \begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} \qquad \Delta^*(\mathbf{k}) = \begin{pmatrix} \Delta^*_{\uparrow\uparrow}(\mathbf{k}) & \Delta^*_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta^*_{\downarrow\uparrow}(\mathbf{k}) & \Delta^*_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix}$$

4 components covering all spin orientations before and after interaction

I.5.3 Superfluid ^{3}He: superfluid phases

- **diagonalization of mean-field Hamiltonian by Bogoliubov transformation**

  – BCS energy gap equation:

    (Nambu, 1962)

$$\begin{array}{c} \begin{array}{cc} \overbrace{\hspace{9em}}^{\text{particles}} & \overbrace{\hspace{9em}}^{\text{pairs}} \end{array} \\ \begin{pmatrix} \varepsilon_{\mathbf{k}}-\mu & 0 & \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ 0 & \varepsilon_{\mathbf{k}}-\mu & \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \\ \Delta^*_{\uparrow\uparrow}(\mathbf{k}) & \Delta^*_{\uparrow\downarrow}(\mathbf{k}) & -(\varepsilon_{\mathbf{k}}-\mu) & 0 \\ \Delta^*_{\downarrow\uparrow}(\mathbf{k}) & \Delta^*_{\downarrow\downarrow}(\mathbf{k}) & 0 & -(\varepsilon_{\mathbf{k}}-\mu) \end{pmatrix} \\ \begin{array}{cc} \underbrace{\hspace{9em}}_{\text{pairs}} & \underbrace{\hspace{9em}}_{\text{holes}} \end{array} \end{array} \begin{pmatrix} u_{\mathbf{k}\uparrow n} \\ u_{\mathbf{k}\downarrow n} \\ v_{\mathbf{k}\uparrow n} \\ v_{\mathbf{k}\downarrow n} \end{pmatrix} = E_{\mathbf{k}n} \begin{pmatrix} u_{\mathbf{k}\uparrow n} \\ u_{\mathbf{k}\downarrow n} \\ v_{\mathbf{k}\uparrow n} \\ v_{\mathbf{k}\downarrow n} \end{pmatrix}$$

- 4 x 4 matrix equation instead of 2 x 2 matrix equation for spin singlet superconductor
- diagonalization by suitable Bogoliubov transformation
- two positive eigenvalues $E_{\mathbf{k}n}$ $(n = 1,2)$ and two negative eigenvalues $-E_{\mathbf{k}n}$
- general gap equation allows for both spin singlet and spin triplet pairing

  – symmetry properties of pairing amplitude and pairing potential (follows from anti-commutation relation of fermionic operators)

$$g_{\alpha\beta}(\mathbf{k}) = \langle \hat{c}_{-\mathbf{k}\alpha}\, \hat{c}_{\mathbf{k}\beta} \rangle = -\langle \hat{c}_{\mathbf{k}\beta}\, \hat{c}_{-\mathbf{k}\alpha} \rangle = -g_{\beta\alpha}(-\mathbf{k})$$

$$\Delta_{\alpha\beta}(\mathbf{k}) = -\Delta_{\beta\alpha}(-\mathbf{k})$$

I.5.3 Superfluid ^{3}He: superfluid phases

- **representation of the energy gap matrix $\Delta_{\alpha\beta}(\mathbf{k})$ by a scalar gap function $\Delta(\mathbf{k})$ and a vector $\mathbf{d}(\mathbf{k})$**

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \imath\,[\Delta(\mathbf{k})\,\mathbf{I} + \mathbf{d}(\mathbf{k})\cdot\boldsymbol{\sigma}\,]\,\sigma_y$$

scalar gap function

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

d-vector

Pauli spin matrix $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\boldsymbol{\sigma} = (\sigma_x, \sigma_y\,, \sigma_z)$ = vector of Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -d_x(\mathbf{k}) + \imath d_y(\mathbf{k}) & \Delta(\mathbf{k}) + d_z(\mathbf{k}) \\ -\Delta(\mathbf{k}) + d_z(\mathbf{k}) & d_x(\mathbf{k}) + \imath d_y(\mathbf{k}) \end{pmatrix}$$

$$\mathbf{d} = \hat{\mathbf{d}}\;|\mathbf{d}|\;\mathrm{e}^{\imath\phi}$$

direction of spin wave function

superfluid density

phase factor

- amplitude $|\mathbf{d}|$ ($\mathbf{d}$ is vector with complex components):
  pair density $|d|^2 = d_x^2 + d_y^2 + d_z^2 = n_s$
- direction $\hat{\mathbf{d}} = \mathbf{d}/|\mathbf{d}|$:
  spatial orientation of spin wave function: $\hat{\mathbf{d}} \cdot \mathbf{S} = 0$

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I.5.3 Superfluid ^{3}He: superfluid phases

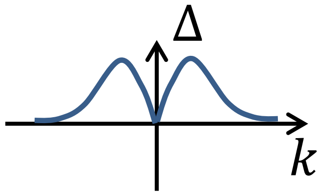
$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -d_x(\mathbf{k}) + \imath d_y(\mathbf{k}) & \Delta(\mathbf{k}) + d_z(\mathbf{k}) \\ -\Delta(\mathbf{k}) + d_z(\mathbf{k}) & d_x(\mathbf{k}) + \imath d_y(\mathbf{k}) \end{pmatrix}$$

- **parity consideration**
  - with $g_{\alpha\beta}(\mathbf{k}) = -g_{\beta\alpha}(-\mathbf{k})$ and $\Delta_{\alpha\beta}(\mathbf{k}) = -\Delta_{\beta\alpha}(-\mathbf{k})$ we obtain

orbital structure of the pairing potential

$$\Delta(\mathbf{k}) = \Delta(-\mathbf{k})$$

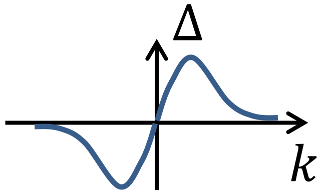
scalar component is **even** with respect to inversion $\mathbf{k} \rightarrow -\mathbf{k}$



$L = 0,2,4, \ldots$

$$\mathbf{d}(\mathbf{k}) = -\mathbf{d}(-\mathbf{k})$$

vector component is **odd** with respect to inversion $\mathbf{k} \rightarrow -\mathbf{k}$



$L = 1,3,5, \ldots$

  - anti-symmetry of pair wave function requires
    - spin singlet ($S = 0$):

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} 0 & \Delta(\mathbf{k}) \\ -\Delta(\mathbf{k}) & 0 \end{pmatrix}$$

    - spin triplet ($S = 1$):

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -d_x(\mathbf{k}) + \imath d_y(\mathbf{k}) & +d_z(\mathbf{k}) \\ +d_z(\mathbf{k}) & d_x(\mathbf{k}) + \imath d_y(\mathbf{k}) \end{pmatrix}$$

I.5.3 Superfluid ^{3}He: superfluid phases

- **representation of the scalar function $\Delta(\mathbf{k})$ and vector $\mathbf{d}(\mathbf{k})$**

  – we can restrict ourselves to wave vectors near the Fermi level
  ➔ due to spherical symmetry, we can use spherical harmonics Y_{lm} to represent

$$d_i(\mathbf{k}) = \sum h_{ilm} Y_{lm}(\theta_\mathbf{k}, \varphi_\mathbf{k}) \qquad i = x, y, z$$

- ➢ spin singlet: $l = 0,2,4, \ldots$
- ➢ spin triplet: $l = 1,3,5, \ldots$

- ➢ in general, one has to sum up over all angular momentum quantum numbers l
- ➢ practically, one can restrict oneself to a particular quantum number (dominating pairing channel, all other channels have much smaller T_c)
- ➢ for superfluid ^{3}He, the **dominating pairing channel is the $l = 1$ channel** mediated by spin fluctuations

⇨

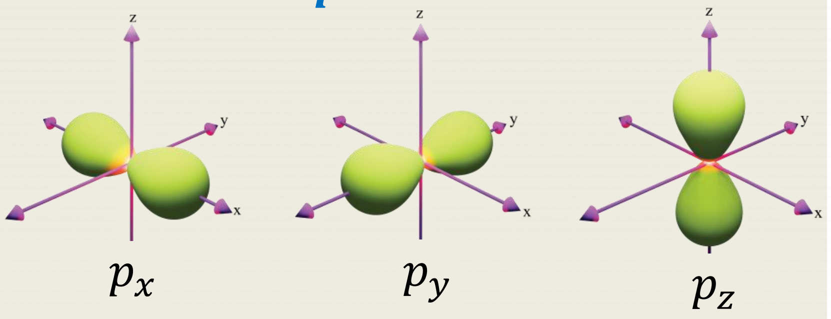
$$d_i(\mathbf{k}) = \sum h_{im}\, p_j(\theta_\mathbf{k}, \varphi_\mathbf{k})$$

$$p_x(\theta_\mathbf{k}, \varphi_\mathbf{k}) = \sqrt{3/4\pi}\cos\varphi_\mathbf{k}\sin\theta_\mathbf{k}$$

$$p_y(\theta_\mathbf{k}, \varphi_\mathbf{k}) = \sqrt{3/4\pi}\sin\varphi_\mathbf{k}\cos\theta_\mathbf{k}$$

$$p_z(\theta_\mathbf{k}, \varphi_\mathbf{k}) = \sqrt{3/4\pi}\cos\theta_\mathbf{k}$$

p-orbitals



$h_{im} = 3 \times 3$ matrix as $i = x, y, z$ and $m = -1, 0, +1$

complex matrix elements have to be determined by solving BCS gap equation

I.5.3 Superfluid ^{3}He

- **superfluid phases of ^{3}He**

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -d_x(\mathbf{k}) + \imath d_y(\mathbf{k}) & +d_z(\mathbf{k}) \\ +d_z(\mathbf{k}) & d_x(\mathbf{k}) + \imath d_y(\mathbf{k}) \end{pmatrix}$$

$$d_i(\mathbf{k}) = \sum h_{im}\, p_m(\theta_{\mathbf{k}}, \varphi_{\mathbf{k}})$$

$$h_{im} = \begin{pmatrix} h_{xx} & h_{xy} & h_{xz} \\ h_{yx} & h_{yy} & h_{yz} \\ h_{zx} & h_{zy} & h_{zz} \end{pmatrix}$$

  - **Anderson-Brinkman-Morrel (ABM or A) phase**

$$h_{im} = h \begin{pmatrix} 1 & \imath & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

➔ $d_x(\mathbf{k}) = p_x + \imath p_y$

$$|\psi\rangle_{\text{ABM}} = \left(p_x + \imath p_y\right) |\downarrow\downarrow\rangle - \left(p_x - \imath p_y\right) |\uparrow\uparrow\rangle$$

linear combination of $S_z = \pm 1$

(i) $B = 0$: $|h_{xx}| = |h_{xy}|$ ➔ $\langle S_z \rangle = 0$

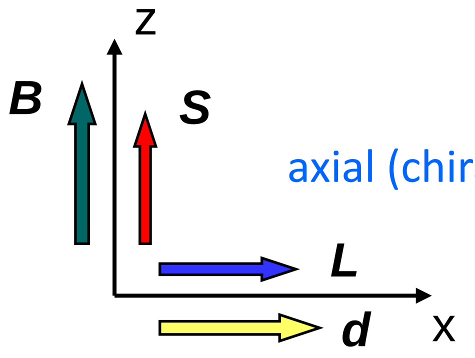
(ii) $B \neq 0$: $|h_{xx}| \neq |h_{xy}|$ ➔ $\langle S_z \rangle \neq 0$ (anisotropic, paramagnetic)

I.5.3 Superfluid ^{3}He

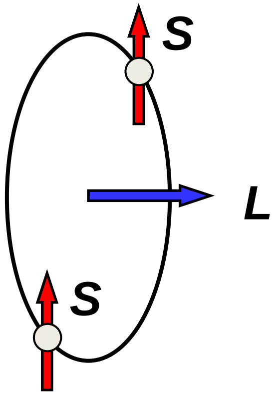
– **Anderson-Brinkman-Morrel (ABM or A) phase**

$$|\psi\rangle_{\mathrm{ABM}} = (p_x + \imath p_y)\,|\downarrow\downarrow\rangle - (p_x - \imath p_y)\,|\uparrow\uparrow\rangle$$

- ➢ both spins are always parallel
- ➢ all paired atoms are moving in plane perpendicular to $\mathbf{L}$

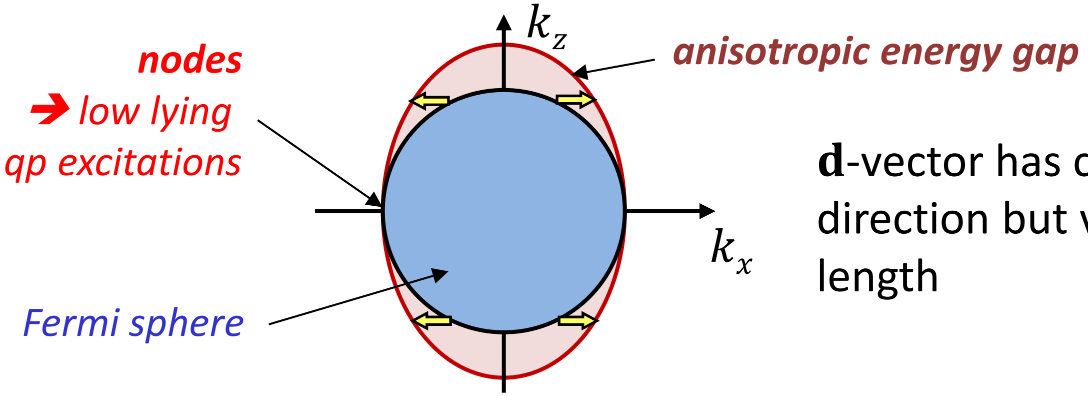


axial (chiral) state

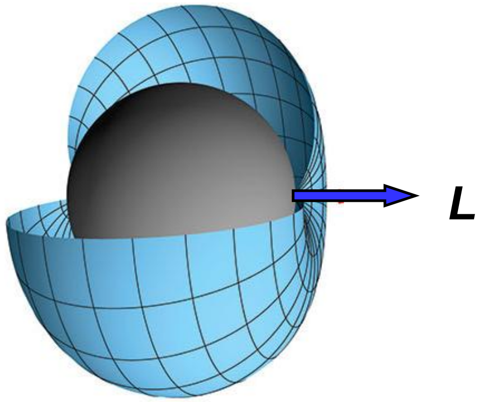


note that $\mathbf{L} \,||\, \mathbf{S}$ *would result in repulsive spin-orbit interaction*

- ➢ unpaired atoms moving in **L** direction do not perturb ➔ no energy gap in **L**-direction



$\mathbf{d}$-vector has constant direction but varying length



I.5.3 Superfluid ^{3}He

- **superfluid phases of ^{3}He**

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -d_x(\mathbf{k}) + \imath d_y(\mathbf{k}) & +d_z(\mathbf{k}) \\ +d_z(\mathbf{k}) & d_x(\mathbf{k}) + \imath d_y(\mathbf{k}) \end{pmatrix}$$

$$d_i(\mathbf{k}) = \sum h_{im}\, p_m(\theta_{\mathbf{k}}, \varphi_{\mathbf{k}})$$

$$h_{im} = \begin{pmatrix} h_{xx} & h_{xy} & h_{xz} \\ h_{yx} & h_{yy} & h_{yz} \\ h_{zx} & h_{zy} & h_{zz} \end{pmatrix}$$

  - **Balian-Werthamer (BW or B) phase**

$$h_{im} = h \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

➔ $d_x(\mathbf{k}) = p_x,\ d_y(\mathbf{k}) = p_y,\ d_z(\mathbf{k}) = p_z,$

$$|\psi\rangle_{\mathrm{BW}} = (p_x + \imath p_y)\,|\downarrow\downarrow\rangle - (p_x - \imath p_y)|\uparrow\uparrow\rangle + p_z(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

linear combination of $S_z = -1, 0, +1$ (all spin states are represented equally)

$|h_{xx}| = |h_{yy}| = |h_{zz}| \neq 0$ ➔ $\langle S_z \rangle = 0$ (isotropic, non-magnetic)

$B \neq 0$: $S_z = 0$ component is non-magnetic ➔ $\chi_{\mathrm{BW}} = \frac{2}{3}\chi_{\mathrm{ABM}}$

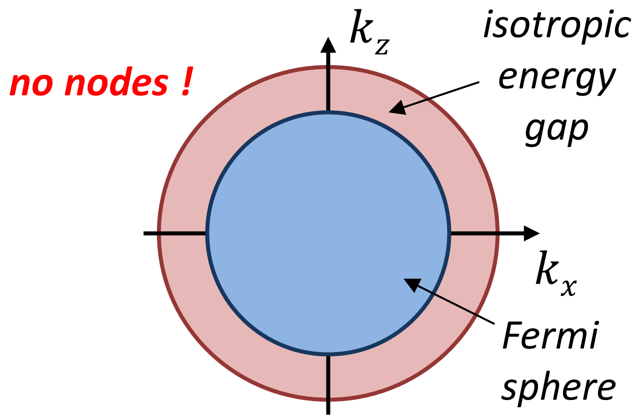
BW phase energetically less favorable than ABM phase in applied magnetic field !!

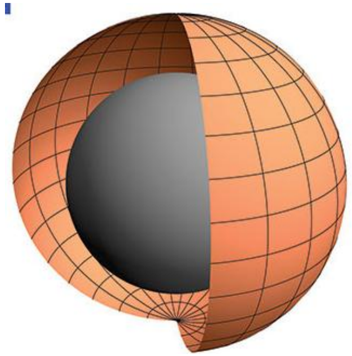
I.5.3 Superfluid ^{3}He

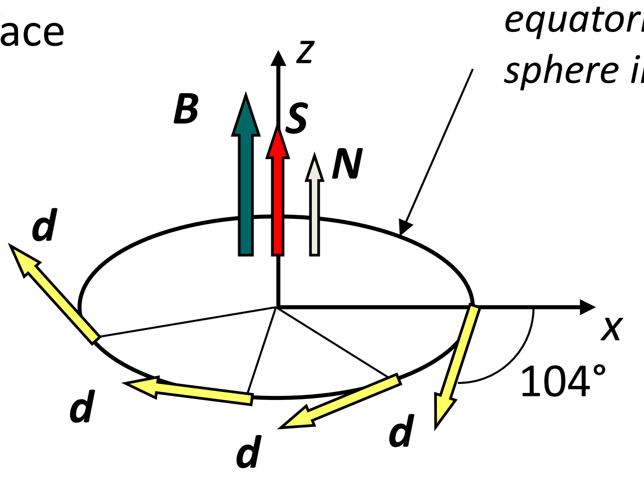
– **Balian-Werthamer (BW or B) phase**

$$|\psi\rangle_{\mathrm{BW}} = \left(p_x + ıp_y\right) | \downarrow\downarrow\rangle - \left(p_x - ıp_y\right) | \uparrow\uparrow\rangle + p_z(| \uparrow\downarrow\rangle + | \downarrow\uparrow\rangle)$$

- $\mathbf{d}$-vector has constant length all over the Fermi surface
  ➔ constant magnitude of the energy gap



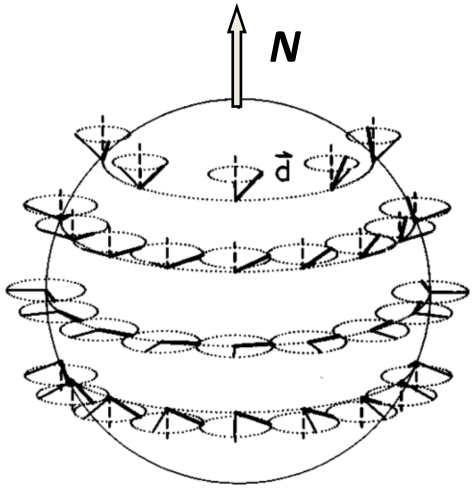




$\mathbf{S}, \mathbf{L}, \mathbf{N} =$ *expectation values for spin, orbital momentum and texture*
($\mathbf{N}$ accounts for effects related to relative orientation of $\mathbf{d}$ and $\mathbf{L}$)

pseudo-isotropic state
(constant gap as s-wave superconductor)

$$C_V(T) \propto \exp(-\Delta(T)/k_{\mathrm{B}}T)$$



d-vectors (thick lines) are rotated by 104° about a vector defining the radial directions (thin lines) for all points on the Fermi sphere.
D.M. Lee, Reviews of Modern Physics **69**, 645 (1997)

I.5.3 Superfluid ^{3}He

- **superfluid phases of ^{3}He**

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -d_x(\mathbf{k}) + \imath d_y(\mathbf{k}) & +d_z(\mathbf{k}) \\ +d_z(\mathbf{k}) & d_x(\mathbf{k}) + \imath d_y(\mathbf{k}) \end{pmatrix}$$

$$d_i(\mathbf{k}) = \sum h_{im}\, p_m(\theta_\mathbf{k}, \varphi_\mathbf{k})$$

$$h_{im} = \begin{pmatrix} h_{xx} & h_{xy} & h_{xz} \\ h_{yx} & h_{yy} & h_{yz} \\ h_{zx} & h_{zy} & h_{zz} \end{pmatrix}$$

  - **A$_1$ phase (only for $B > 0$)**

$$h_{im} = h \begin{pmatrix} 1 & \imath & 0 \\ \imath & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

➔ $d_x(\mathbf{k}) = p_x + \imath p_y$

➔ $d_y(\mathbf{k}) = \imath p_x + p_y$

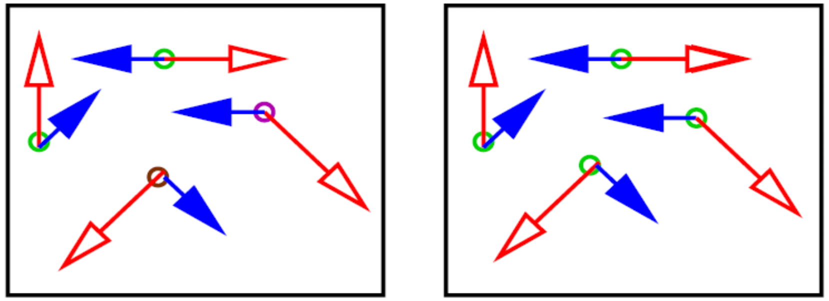
$$|\psi\rangle_{\mathrm{A}_1} = 2\left(p_x + \imath p_y\right) |\uparrow\uparrow\rangle$$

only state $|\uparrow\uparrow\rangle$ contributes: $\langle S_z \rangle = 1$ (“ferro”magnetic)

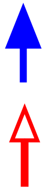
➔ **long-range ordered magnetic liquid**

I.5.3 Superfluid ^{3}He: broken symmetries

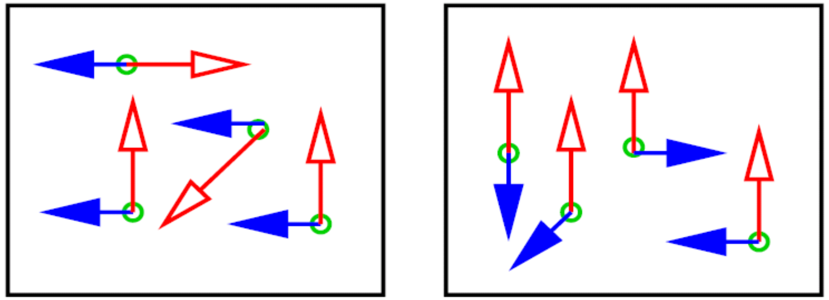
disordered state
(isotropic with respect to orientation of both degrees of freedom)



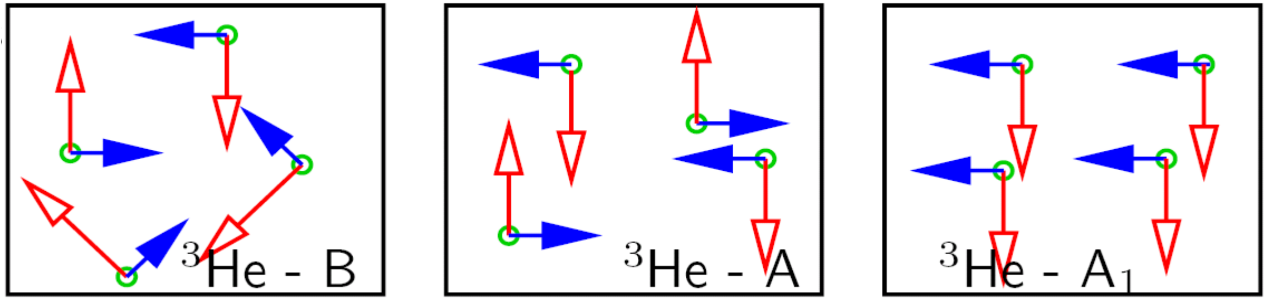
broken gauge symmetry
“superfluid”



broken gauge and rotational symmetry in spin space
“ferromagnetic superfluid”



broken gauge and rotational symmetry in orbital space
“ferroorbital superfluid”



only symmetry related to relative orientation is broken

rotational symmetry is broken separately in spin and orbital space

rotational symmetry is broken separately in spin and orbital space

Contents Part I: Quantum Liquids

Contents:

I.6 ^{3}He/^{4}He mixtures

- **why interesting**:

➔ model system for testing Fermi liquid theories
(^{3}He is fermion, can be diluted in passive ^{4}He background → change of interaction strength)

➔ technical application in cooling machines
H.E. Hall, P.J. Ford and K. Thomson, Cryogenic **6** (1966) 80
B.S. Neganov, N.S. Borisov and M.Yu. Liburg, Zh. Exp. Teor. Fiz. **50** (1966) 1445.

- **experimental observations**:

➔ specific heat vs. T curve of ^{4}He is shifted to lower T on adding ^{3}He, but does not change its shape

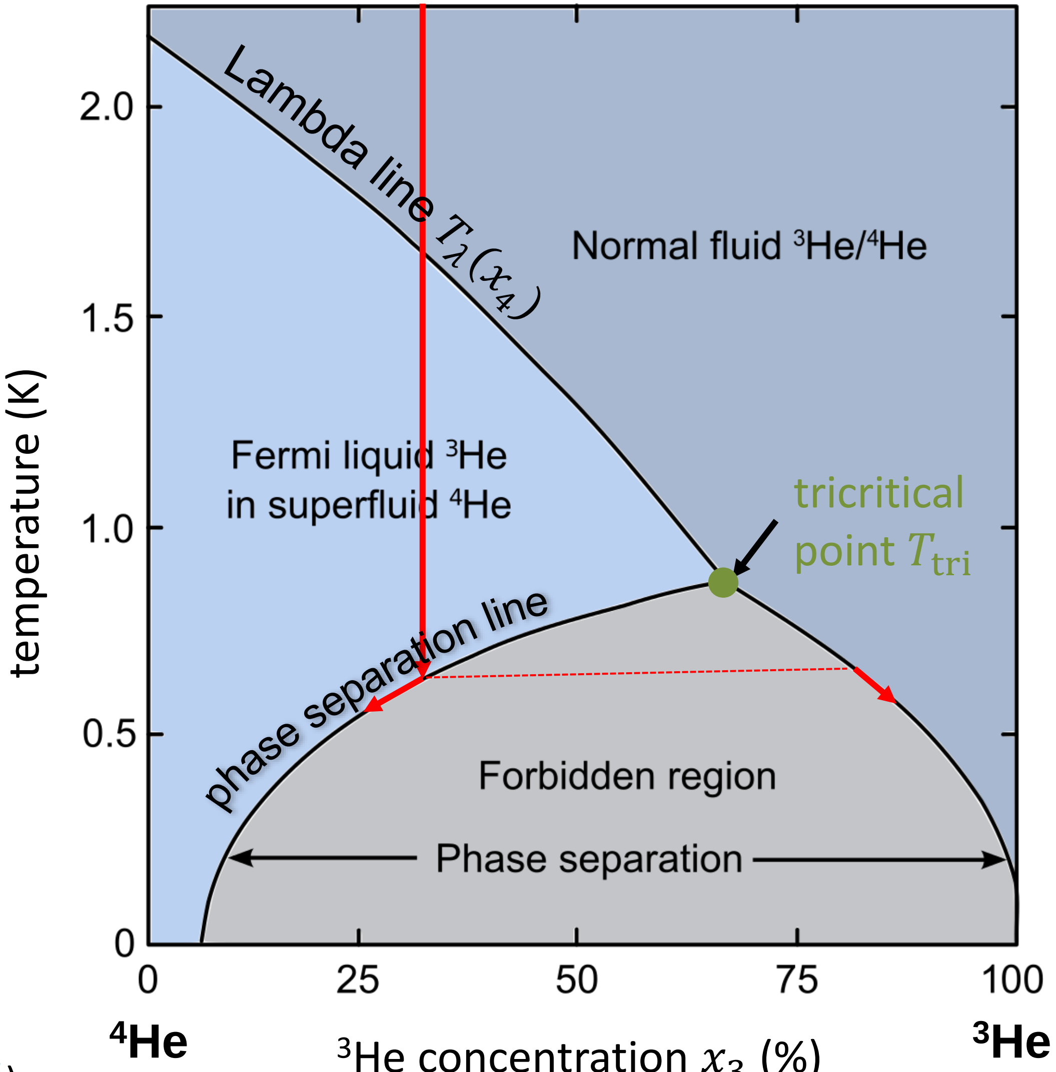
➔ phase separation of ^{4}He and ^{3}He at low T

I.6 ^{3}He/^{4}He mixtures: phase diagram

- high T:
  ^{3}He reduces $T_\lambda \propto n_4$ (^{4}He concentration)
- tricritical point $\boldsymbol{T_{\mathrm{tri}} = 0.87\ \mathrm{K}}$
- $T < T_{\mathrm{tri}} = 0.87$ K:
  ➔ ***phase separation***
  ^{3}He rich phase: lighter
  ^{3}He poor phase: heavier
- $T \to 0$:
  ➔ ***incomplete phase separation***
  6.5% of ^{3}He still dissolved in ^{4}He
- concentrations: $x_3 = \frac{n_3}{n_3+n_4}$; $x_4 = 1 - x_3 = \frac{n_4}{n_3+n_4}$
- limiting concentrations:

$$x_4 = aT^{3/2}\mathrm{e}^{-b/T}\ (a = 0.85\ \mathrm{K}^{-3/2}, b = 0.56\ \mathrm{K}^{-1})$$

$$x_3 = 0.0648\,(1 + cT^2 + dT^3)\quad (c = 8.4\ \mathrm{K}^{-2}, d = 9.4\mathrm{K}^{-3})$$



I.6 ^{3}He/^{4}He mixture as diluted Fermi liquid

- at $T_{\text{phase sep}} < T < T_\lambda(x_4)$:

→ ^{4}He forms superfluid with negligible amounts of excitations (phonons, rotons)

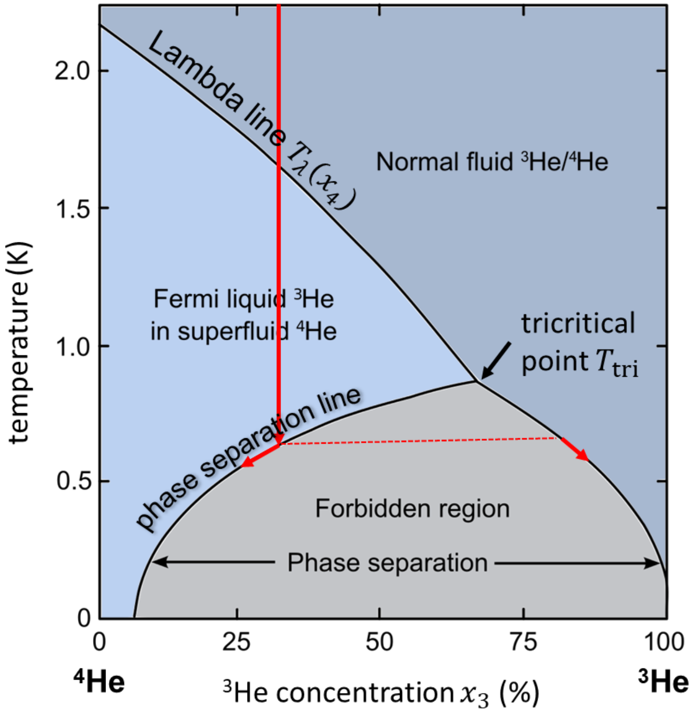
→ viscosity, entropy and specific heat go to zero:

→ ***^{4}He = inert superfluid background***

- ^{3}He obeys Fermi statistics

→ Fermi temperature is about 1 K for pure ^{3}He

→ dissolving ^{3}He in ^{4}He reduces the Fermi temperature



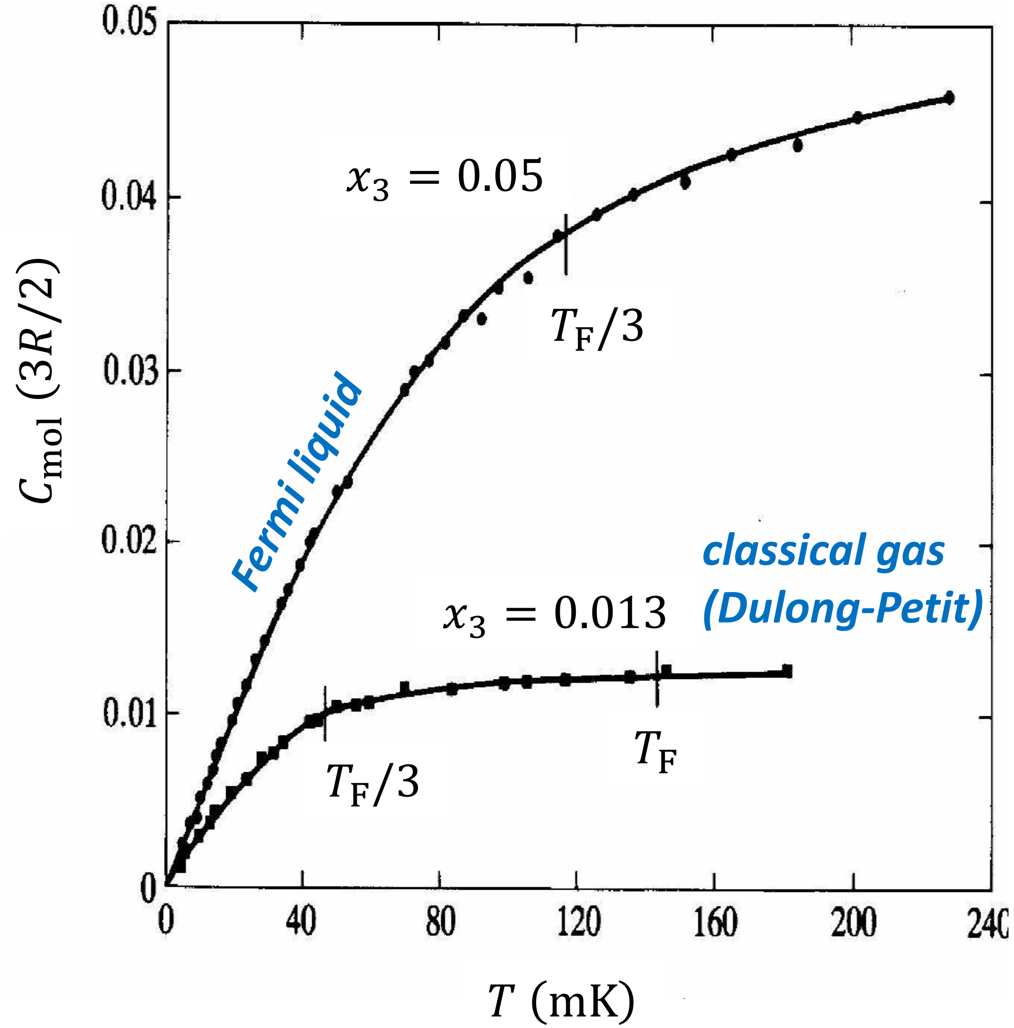
$$T_F = \frac{\hbar^2}{2m^*_{3\text{He}}}(3\pi^2 n_{3\text{He}})^{2/3} \propto x_3^{2/3}$$

$m^*_{3\text{He}}$: effective mass of ^{3}He dissolved in ^{4}He

$n_{3\text{He}} = \frac{N_{3\text{He}}}{V} = \frac{x_3 N}{V}$: density of ^{3}He atoms

I.6 ^{3}He/^{4}He mixture as diluted Fermi liquid

- **specific heat**



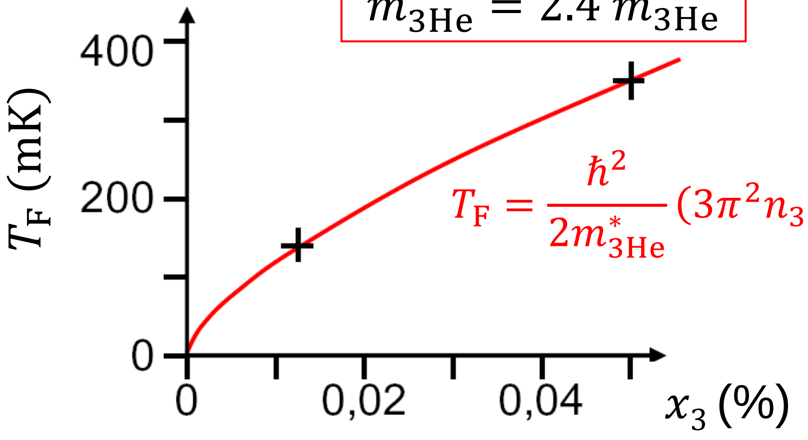
- Fermi gas ($T \ll T_F$) $$C_{\mathrm{mol}} = \frac{\pi^2}{2} R \frac{T}{T_F}$$

- classical gas ($T > T_F$) $$C_{\mathrm{mol}} = \frac{3}{2} N_A k_B = \frac{3}{2} R$$

- T_F increases with x_3

interaction with ^{4}*He*

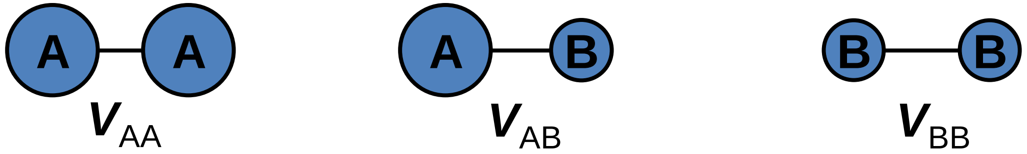
$$m^*_{3\mathrm{He}} = 2.4\, m_{3\mathrm{He}}$$



$$T_F = \frac{\hbar^2}{2m^*_{3\mathrm{He}}}(3\pi^2 n_{3\mathrm{He}})^{2/3} \propto x_3^{2/3}$$

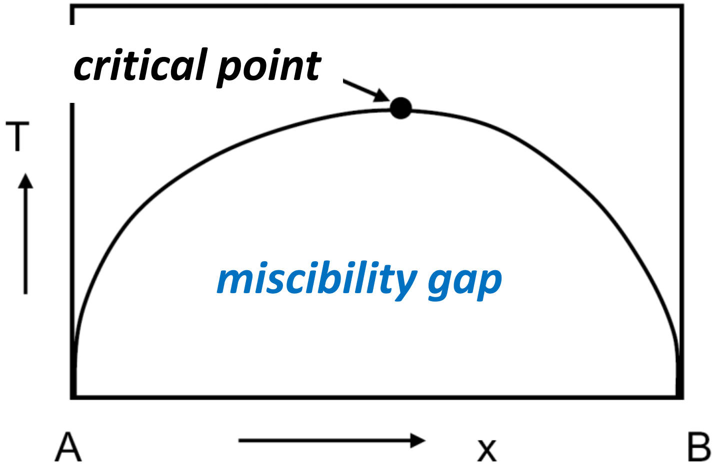
I.6 ^{3}He/^{4}He mixtures: miscibility gap in binary systems

- bonding:



- $V_{AB} > \frac{1}{2}(V_{AA} + V_{BB})$ ***complete miscibility*** (e.g. water and alcohol)
- $\frac{1}{2}(V_{AA} + V_{BB}) > V_{AB}$ ***phase separation*** (e.g. water and petrol)

➔ ***optimization of binding total energy U***
complete phase separation @ T = 0



increasing mixing with increasing *T*:

$$F = U - TS \rightarrow min$$

binding energy
thermal motion

➔ ***minimization of free energy***

I.6 ^{3}He/^{4}He mixtures: finite solubility of ^{3}He in ^{4}He

$L_3 = U_{\mathrm{vap}} - U_{\mathrm{liq}} + p\Delta V$
$p\Delta V$ = work done against ambient pressure

- **binding energy of ^{3}He in ^{3}He (V_{33}) and ^{4}He (V_{34}):**

i. **single ^{3}He atom in liquid ^{3}He**: average binding energy is given by the latent heat of evaporation L_3:

binding energy for single ^{3}He atom: $\epsilon_{3,c} = -\frac{L_3}{N_A} = \mu_{3,c}$

$\mu_{3,c}$ = *chemical potential of pure (concentrated) liquid*

ii. **single ^{3}He atom in liquid ^{4}He**:
binding energy for single ^{3}He atom: $\epsilon_{3,d}(x_3 \to 0) = \mu_{3,d}(x_3 \to 0)$

$\mu_{3,d}$ = *chemical potential of dilute phase*

– $|\epsilon_{3,d}(0)| > |\epsilon_{3,c}|$ **or vice versa ?**

^{3}He has smaller mass → larger zero point fluctuations → occupies larger volume

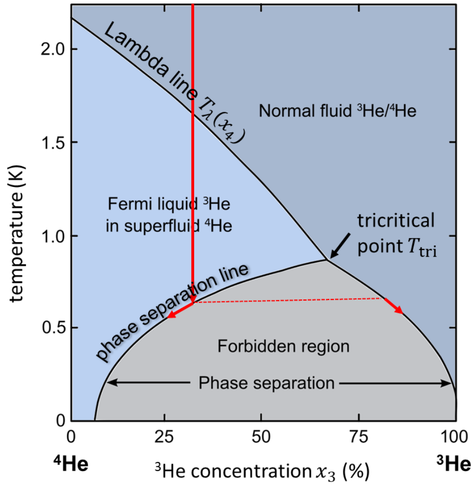
→ binding of ^{3}He is larger in ^{4}He than in ^{3}He due to larger density of ^{4}He

→ $|\epsilon_{3,d}(0)| > |\epsilon_{3,c}|$ (it is energetically more favorable to add a ^{3}He atom to the ^{4}He superfluid than to pure ^{3}He)

→ corresponds to case $V_{\mathrm{AB}} > \frac{1}{2}(V_{\mathrm{AA}} + V_{\mathrm{BB}})$ → ***complete miscibility expected, why miscibility only up to 6.5% ??***

I.6 ^{3}He/^{4}He mixtures: finite solubility of ^{3}He in ^{4}He

- **questions**



  – **why can‘t we dissolve more than 6.5% of ^{3}He in ^{4}He at $T = 0$?**

    we have to take into account two effects:

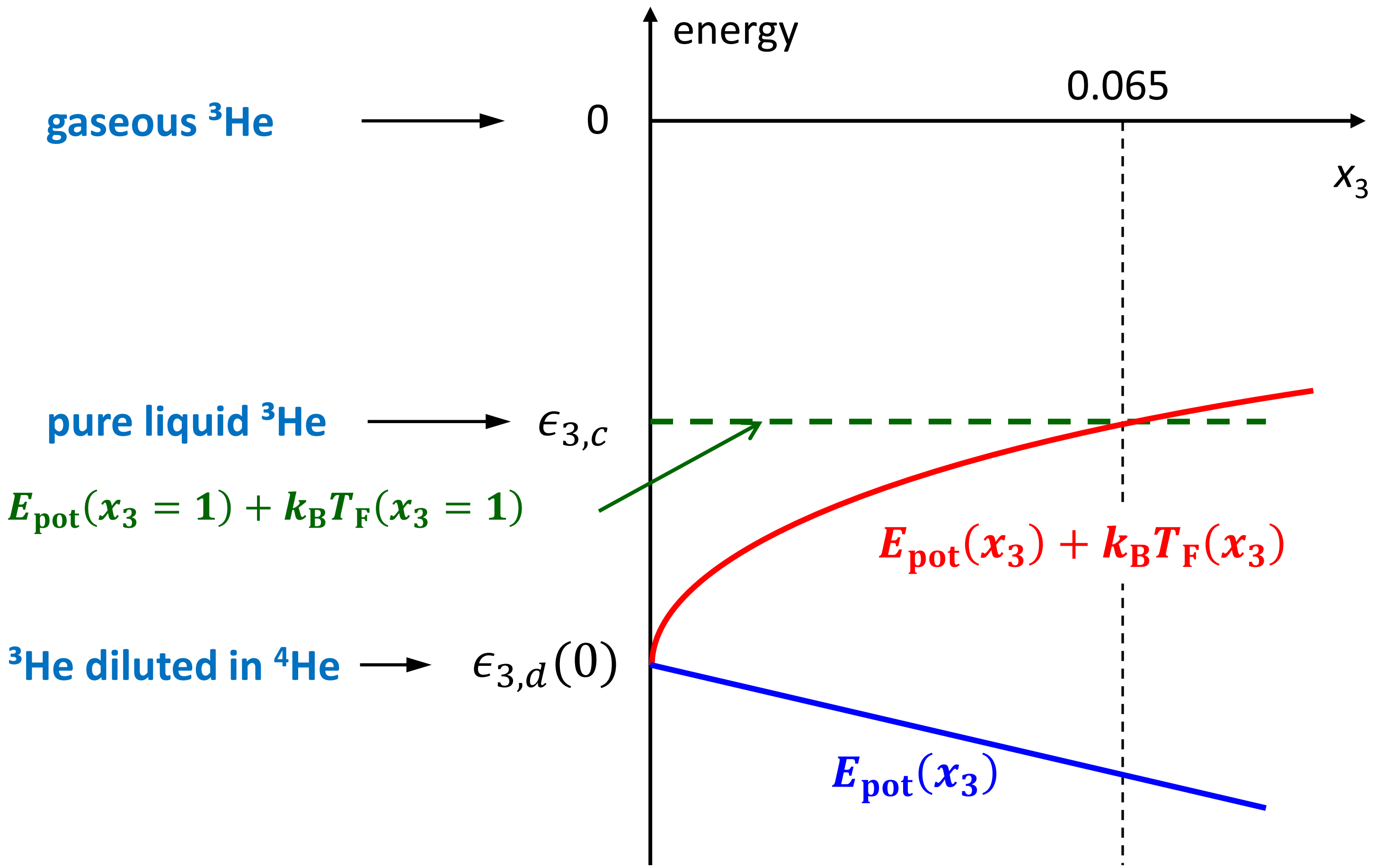
    i. ^{3}He forms degenerate Fermi liquid
       → T_{F} increases with x_3
       → for $x_3 > 6.5\%$, the increase of Fermi energy exceeds the gain in binding energy

    ii. effective attraction between two ^{3}He atoms (*magnetic and volume effect*)
       → $|\epsilon_{3,d}(x_3)| > |\epsilon_{3,d}(x_3 = 0)|$

  – **why don‘t we have a complete phase separation into pure ^{3}He and pure ^{4}He at $T = 0$?**

    incomplete phase separation results in finite disorder, violation of 3rd law of thermodynamic ?
       → no: *we have* ***degenerate Fermi gas, ordering in k-space***

I.6 ^{3}He/^{4}He mixtures: finite solubility of ^{3}He in ^{4}He



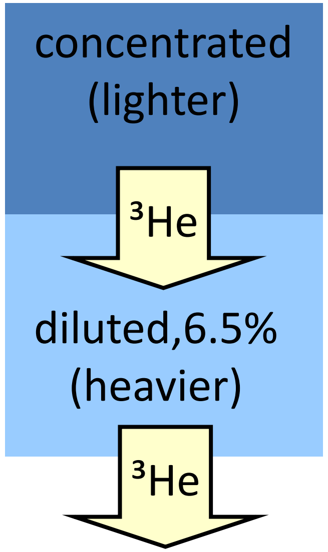
for $x_3 > 6.5\%$:

$$\left|E_\mathrm{pot}(x_3) + k_\mathrm{B} T_\mathrm{F}(x_3)\right| < |\epsilon_{3,c}| = |E_\mathrm{pot}(x_3 = 1) + k_\mathrm{B} T_\mathrm{F}(x_3 = 1)|$$

➔ **separation of pure ^{3}He**

I.6 ^{3}He/^{4}He mixtures: cooling effect

- ^{3}He/^{4}He dilution refrigeration
  - operation principle:
    - → remove ^{3}He atoms from the dilute phase below $T_{\mathrm{tri}} = 0.87$ K
    - → transport of ^{3}He atoms across phase boundary to maintain equilibrium concentration
    - → corresponds to evaporation of ^{3}He from concentrated phase
    - → ***cooling effect as the latent heat of evaporation is removed***



concentrated (lighter)

^{3}He

diluted,6.5% (heavier)

^{3}He

- for Fermi liquid:

$$\rightarrow \boldsymbol{C}_{\mathrm{V,concentrated}} < \boldsymbol{C}_{\mathrm{V,diluted}}\ (\boldsymbol{x_3 = 0.065}) \qquad (C_V \propto T/T_{\mathrm{F}},\ \ T_{\mathrm{F}} \propto n_{3\mathrm{He}}^{2/3})$$

- with $U = Q = \int_0^T C_V(T')\,dT' = \frac{\gamma}{2}\,T^2$ we therefore obtain ($C_V = \gamma T, \gamma =$ Sommerfeld coefficient):

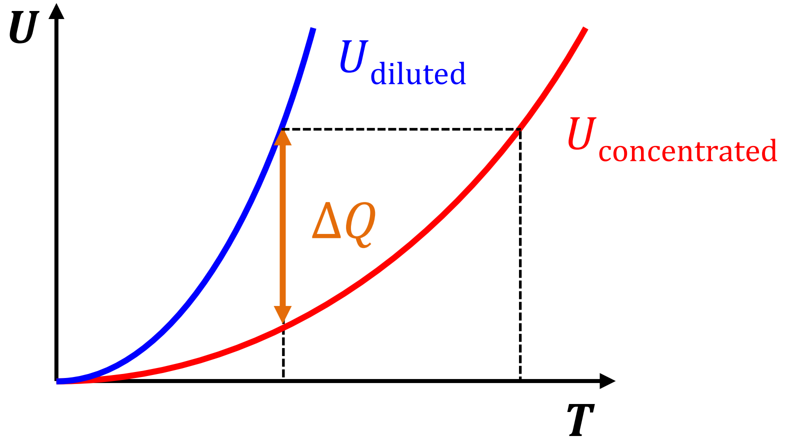
$$\boldsymbol{U_{\mathrm{concentrated}}(T) < U_{\mathrm{diluted}}(T)},\quad S_{\mathrm{con}}(T) < S_{\mathrm{dil}}(T)$$

- on transition across phase boundary:

$$dG = 0 = dU - TdS \;\Rightarrow\; dU = TdS = dQ$$

→ removal of heat: $\boldsymbol{\Delta Q = T\Delta S = T\,[S_{\mathrm{dil}}(T) - S_{\mathrm{con}}(T)]}$

➔ ***cooling effect***



→ ***^{3}He/^{4}He dilution refrigerator***

I.6 ^{3}He/^{4}He mixtures: cooling effect

- **^{3}He/^{4}He dilution refrigeration: cooling power**
  - assumption: ***one mole of ^{3}He crosses boundary between diluted and concentrated phase***
  - removed heat: $\Delta Q = T\Delta S = T\,[S_{\text{dil}}(T) - S_{\text{con}}(T)]$
  - cooling power: $\Delta\dot{Q} = \left(\frac{\Delta Q}{n_{3\text{He}}}\right)\cdot \dot{n}_{3\text{He}} \Rightarrow$ $\Delta\dot{Q} = \dot{n}_{3\text{He}} T\,[s_{\text{dil}}(T) - s_{\text{con}}(T)]$ $\quad s_{\text{dil}} = \frac{S_{\text{dil}}}{n_{3\text{He}}}, s_{\text{con}} = \frac{S_{\text{con}}}{n_{3\text{He}}}$
  - since there is no volume change: $\mathrm{d}Q = \mathrm{d}U = T\mathrm{d}S = C_V \mathrm{d}T,\quad U(T) = \int_0^T C_V dT',\quad S(T) = \int_0^T \frac{C_V}{T'} dT'$

➢ with $C_{\text{mol}} = \frac{\pi^2 N_A k_B^2 T}{2\varepsilon_F}$ and $\varepsilon_F = \frac{\hbar^2}{2m_{3\text{He}}^*}\left(\frac{3\pi^2 N_{3\text{He}}}{V}\right)^{2/3} = \frac{\hbar^2}{2m_{3\text{He}}^*}\left(\frac{3\pi^2 N_A}{V_{\text{mol}}}\right)^{2/3}$ *(standard expressions for Fermi liquid)*

we obtain the entropy

$$\boxed{S_{\text{mol}} = \frac{\pi^2 N_A k_B^2 m_{3\text{He}}^*}{\hbar^2}\left(\frac{V}{3\pi^2 N_{3\text{He}}}\right)^{2/3} T \propto \frac{T}{n_{3\text{He}}}}$$

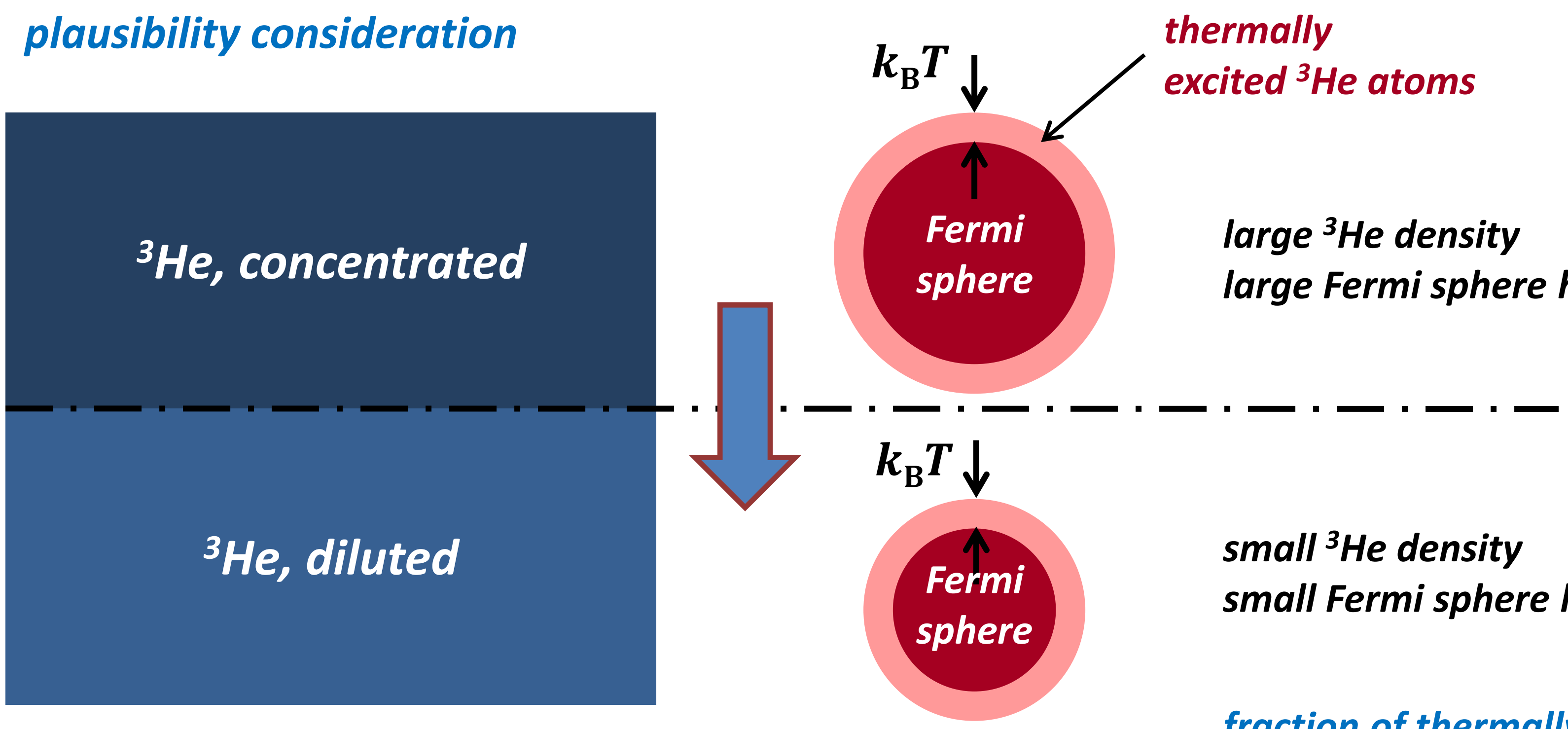
⇨ $S_{\text{con}} < S_{\text{dil}}$ (as $n_{3,\text{con}} > n_{3,\text{dil}}$)

⇨ $\dot{Q}(T) = 84\,\dot{n}_{3\text{He}} T^2$[W] (cooling power ∝ ^{3}He throughput $\dot{n}_3$ [mol/s])

I.6 ^{3}He/^{4}He mixtures: cooling effect

- ^{3}He/^{4}He dilution refrigeration: plausibility consideration

plausibility consideration



large ^{3}He density
large Fermi sphere high T_F

small ^{3}He density
small Fermi sphere low T_F

fraction of thermally excited ^{3}He atoms increases ($\propto T/T_\mathrm{F}$)

entropy increases going from concentrated to diluted phase

removed heat: $dQ = TdS$